Sharp Steklov upper bound for submanifolds of revolution

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Abstract

In this note, we find a sharp upper bound for the Steklov spectrum on a submanifold of revolution in Euclidean space with one boundary component.

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1 Introduction

The Steklov eigenvalues of a smooth, compact, connected Riemannian manifold (M, g) of dimension $n \geq 2$ with smooth boundary Σ are the real numbers σ for which there exists a nonzero harmonic function $f : M \to \mathbb{R}$ which satisfies $\partial_\nu f = \sigma f$ on the boundary Σ. Here and in what follows, ∂_ν is the outward-pointing normal derivative on Σ. The Steklov eigenvalues form a discrete sequence $0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \nearrow \infty$, where each eigenvalue is repeated according to its multiplicity.

Recently, relationships between the geometry of the boundary Σ and the Steklov spectrum have been investigated extensively. For $n \geq 3$, in [3], the authors show that fixing only the geometry of the boundary while letting the Riemannian metric inside M be arbitrary does not suffice to control the Steklov eigenvalues: they can be as large or as small as one wishes. On the other hand, it was shown in [5, 7, 10] that fixing g in a neighborhood of Σ has a much stronger influence on the spectrum.

In [4], the authors consider n-dimensional submanifolds of revolution (M, g) of the Euclidean space $\mathbb{R}^{n+1}$ with one boundary component $\mathbb{S}^{n-1} \subset \mathbb{R}^n \times \{0\}$. They show the sharp lower bound

$$\sigma_k(M) \geq \sigma_k(\mathbb{B}^n), \quad \text{for each } k \in \mathbb{N},$$

where $\mathbb{B}^n$ denotes the submanifold of revolution given by the n-dimensional Euclidean ball. For $n \geq 3$, in the case of equality for one of the eigenvalues σ_k, $k \geq 1$, M has to be isometric to $\mathbb{B}^n$. The authors also find an upper bound for σ_k on a submanifold of revolution in Euclidean space with one boundary component, but it is not sharp. In particular, the authors proved this bound by using quasi-isometry. Note that, for $n = 2$, all submanifolds of revolution with boundary S^1 are isospectral (see [4], Proposition 1.10).

The goal of this work is to investigate sharp upper bounds for submanifolds of revolution M in Euclidean space $\mathbb{R}^{n+1}$ with one boundary component $\mathbb{S}^{n-1} \subset \mathbb{R}^n \times \{0\}$. We denote by

$$0 = \sigma_{(0)}(M) < \sigma_{(1)}(M) < \sigma_{(2)}(M) < \ldots$$

the *distinct* (counted without multiplicity) eigenvalues of the submanifold of revolution M. Now we state our main result.

Theorem 1. *Let $M \subset \mathbb{R}^{n+1}$ be an n-dimensional submanifold of revolution with one boundary component isometric to the round sphere $\mathbb{S}^{n-1}$ of radius one. Then for $n \geq 3$, we have for each $k \geq 1$,*

$$\sigma_{(k)}(M) < k + n - 2.$$

Moreover, this bound is sharp. For each $\epsilon > 0$ and each $k \geq 1$, there exists a submanifold of revolution M_ϵ such that $\sigma_{(k)}(M_\epsilon) > k + n - 2 - \epsilon$.

However, the inequality is strict: for each k, there do not exist submanifolds of revolution M such that $\sigma_{(k)}(M) = k + n - 2$.

We do not know if similar result holds for a revolution manifold with two boundary components. At least, we cannot apply our method easily on such manifolds (see Remark 8).

Note that such bounds exist for abstract revolution metrics on the ball $\mathbb{B}^n$ if we impose bounds on the curvature of (M, g) (see [8], [9]). Roughly speaking, in [8], the author considers the Steklov problem on a ball with rotationally invariant metric under the assumption that the radial curvature is bounded from below (or bounded from above) by some real number and proves a two-sided bound for the Steklov eigenvalues. For warped product manifolds with only one boundary component, the author in [9] has obtained a lower bound (upper bound) for the Steklov eigenvalues under the hypothesis that the manifold has nonnegative (nonpositive) Ricci curvature and strictly convex boundary.

Theorem 1 will be a consequence of the study of the mixed Steklov-Dirichlet and Steklov-Neumann spectra on an annulus and Proposition 7 which states that given a submanifold of revolution M_1 with one boundary component, it is always possible to construct another submanifold of revolution M_2 with larger Steklov eigenvalues.

The rest of the paper is organized as follows. In Section 2, we present the Steklov and mixed Steklov problems. In Section 3, we consider the specific situation of submanifolds of revolution of Euclidean space with one boundary component. Finally, in Section 4, we give the proof of Theorem 1.

2 Some general facts about Steklov and mixed problems

Let (M, g) be a compact Riemannian manifold with boundary Σ. The Steklov eigenvalues of (M, g) can be characterized by the following variational formula

$$\sigma_j(M) = \min_{E \in \mathcal{H}_j} \max_{0 \neq f \in E} R_M(f), \quad j \geq 0, \tag{1}$$

where $\mathcal{H}_j$ is the set of all $(j+1)$-dimensional subspaces in the Sobolev space $H^1(M)$ and

$$R_M(f) = \frac{\int_M |\nabla f|^2 dV_M}{\int_\Sigma |f|^2 dV_\Sigma}$$

is the Rayleigh quotient.

In order to obtain bounds for $\sigma_j(M)$, we will compare the Steklov spectrum with the spectra of the mixed Steklov-Dirichlet or Steklov-Neumann problems on domains $A \subset M$ such that $\Sigma \subset \partial A$. Let $\partial_{int} A$ denote the intersection of the boundary of A with the interior of M. Also, we suppose that it is smooth.

The mixed Steklov-Neumann problem on A is the eigenvalue problem

$$\Delta f = 0 \text{ in } A,$$
$$\partial_\nu f = \sigma f \text{ on } \Sigma, \qquad \partial_\nu f = 0 \text{ on } \partial_{int} A,$$

where ν denotes the outward-pointing normal to ∂A. The eigenvalues of this mixed problem form a discrete sequence

$$0 = \sigma_0^N(A) \leq \sigma_1^N(A) \leq \sigma_2^N(A) \leq \cdots \nearrow \infty,$$

and for each $j \geq 0$, the j^{th} eigenvalue is given by

$$\sigma_j^N(A) = \min_{E \in \mathcal{H}_j(A)} \max_{0 \neq f \in E} \frac{\int_A |\nabla f|^2 \, dV_A}{\int_\Sigma |f|^2 \, dV_\Sigma},$$

where $\mathcal{H}_j(A)$ is the set of all $(j+1)$-dimensional subspaces in the Sobolev space $H^1(A)$.

The mixed Steklov-Dirichlet problem on A is the eigenvalue problem

$$\Delta f = 0 \text{ in } A,$$
$$\partial_\nu f = \sigma f \text{ on } \Sigma, \qquad f = 0 \text{ on } \partial_{int} A.$$

The eigenvalues of this mixed problem form a discrete sequence

$$0 < \sigma_0^D(A) \leq \sigma_1^D(A) \leq \cdots \nearrow \infty,$$

and the j^{th} eigenvalue is given by

$$\sigma_j^D(A) = \min_{E \in \mathcal{H}_{j,0}(A)} \max_{0 \neq f \in E} \frac{\int_A |\nabla f|^2 \, dV_A}{\int_\Sigma |f|^2 \, dV_\Sigma},$$

where $\mathcal{H}_{j,0}(M)$ is the set of all $(j+1)$-dimensional subspaces in the Sobolev space $H_0^1(A) = \{u \in H^1(A) : u = 0 \text{ on } \partial_{int} A\}$.

For each $j \in \mathbb{N}$, comparisons between the variational formulae give the following bracketing:

$$\sigma_j^N(A) \leq \sigma_j(M) \leq \sigma_j^D(A). \tag{2}$$

Note in particular that for $j = 0$, we have

$$0 = \sigma_0^N(A) = \sigma_0(M) < \sigma_0^D(A).$$

3 Submanifolds of revolution of Euclidean space

A compact submanifold of revolution M of dimension n with one boundary component is a revolution metric on the n-dimensional ball. It can be seen as the warped product $[0, L] \times \mathbb{S}^{n-1}$ with the Riemannian metric

$$g(r, p) = dr^2 + h^2(r) g_0(p),$$

where $(r, p) \in [0, L] \times \mathbb{S}^{n-1}$, g_0 is the canonical metric on the $(n-1)$-dimensional sphere of radius one and $h \in C^\infty([0, L])$ satisfies $h > 0$ on $[0, L[$, $h'(L) = 1$ and $h(L) = 0$. If we suppose that the boundary is the round sphere of radius one, we also have $h(0) = 1$. Moreover, the fact that M is an n-dimensional submanifold of revolution of Euclidean space implies

$$1 - r \leq h(r) \leq 1 + r.$$

For more details, see [4].

3.1 Steklov spectrum and eigenfunctions of a submanifold of revolution

The Steklov spectrum and the eigenfunctions of a submanifold of revolution with one connected boundary component are very well explained in Proposition 8 of [9]. Before proceeding further, we would like to mention that by Laplace-Beltrami operator, we mean $\Delta = -\operatorname{div}\ \operatorname{grad}$, which is positive, whereas in [9], the author considers $\Delta = \operatorname{div}\ \operatorname{grad}$. This explains the difference of the signs in the following.

Proposition 2. *Let M be a submanifold of revolution in Euclidean space $\mathbb{R}^{n+1}$ with one boundary component $\mathbb{S}^{n-1} \subset \mathbb{R}^n \times \{0\}$. Then each eigenfunction f of the Steklov problem on M can be written as $f(r,p) = u(r)v(p)$, where v is a spherical harmonic of the sphere $\mathbb{S}^{n-1}$ of degree $k \geq 0$, i.e.,*

$$\Delta v = \lambda_{(k)} v \text{ on } \mathbb{S}^{n-1},$$

where $\lambda_{(k)} = k(n-2+k)$ and u is a nontrivial solution of the equation

$$\frac{1}{h^{n-1}}\frac{d}{dr}(h^{n-1}\frac{d}{dr}u) - \frac{1}{h^2}\lambda_{(k)}u = 0 \tag{3}$$

on $(0,L)$ satisfying the condition $u(L) = 0$. The Steklov eigenvalue $\sigma_{(k)}$ has the same multiplicity as $\lambda_{(k)}$.

The proof of this proposition is similar to Proposition 8 of [9]. But for the convenience of the reader, we are giving details here.

Proof. Equation (3) comes from the fact that

$$\Delta(u(r)v(p)) = -\frac{1}{h^{n-1}}\frac{d}{dr}(h^{n-1}\frac{d}{dr}u)v(p) + \frac{u}{h^2}\Delta_{\mathbb{S}^{n-1}}v(p).$$

If v is an eigenfunction associated to a Laplace eigenvalue λ on the sphere $\mathbb{S}^{n-1}$, we obtain

$$\Delta(u(r)v(p)) = -\frac{1}{h^{n-1}}\frac{d}{dr}(h^{n-1}\frac{d}{dr}u)v(p) + \frac{u}{h^2}\lambda v(p),$$

and, because f is harmonic, we have

$$-\frac{1}{h^{n-1}}\frac{d}{dr}(h^{n-1}\frac{d}{dr}u) + \lambda\frac{u}{h^2} = 0.$$

The condition $u(L) = 0$ comes from the fact that f has to be smooth at the point where $r = L$. We take $u(0) = 1$ to have a unique solution of (3).

If we take $\{v_k\}, k = 0, 1, \cdots$, a set of spherical harmonics on $\mathbb{S}^{n-1}$ and arrange v_k such that v_0 is of degree zero, $\{v_k\}_{k=1}^n$ are of degree one, etc, then $\Delta v_k = \lambda_{(m(k))}v_k$ on $\mathbb{S}^{n-1}$, where $m(k)$ is the degree of v_k and $\lambda_{(m)} = m(n-2+m)$. Let $u_0 = 1$ and $u_k, k \geq 1$ is a nontrivial solution of

$$-\frac{1}{h^{n-1}}\frac{d}{dr}(h^{n-1}\frac{d}{dr}u) + \lambda_{(k)}\frac{u}{h^2} = 0, \quad r \in (0,L)$$

with $u_k(L) = 0$ and $u_k(0) = 1$. Now any eigenfunction f of Steklov problem on M can be written as $f = \Sigma_{j=0}^{\infty}u_j(r)v_j(p)$.

We know that $\sigma_{(0)}(M)$ has multiplicity one which is same as the multiplicity of Laplace eigenvalue $\lambda_{(0)}(\mathbb{S}^{n-1})$. Let f be an eigenfunction associated to $\sigma_{(1)}(M)$, then $\int_{\mathbb{S}^{n-1}} f = 0$ and $f = \Sigma_{j=1}^{\infty}u_j(r)v_j(p)$. If $j \geq n+1$, we have

$$|\nabla(u_jv_j)|^2 = u_j'^2(r) + \lambda_{(m(j))}\frac{u_j^2(r)}{h^2(r)} > u_j'^2(r) + \lambda_{(1)}\frac{u_j^2(r)}{h^2(r)} = |\nabla(u_jv_1)|^2$$

Using the min-max principle for $\sigma_{(1)}(M)$, we get

$$\sigma_{(1)}(M) = \frac{\sum_{j=1}^{\infty}\int_0^L |\nabla(u_j v_j)|^2 h^{n-1}(r)dr}{\sum_{j=1}^{\infty} u_j^2(0)} > \frac{\sum_{j=1}^{\infty}\int_0^L |\nabla(u_j v_1)|^2 h^{n-1}(r)dr}{\sum_{j=1}^{\infty} u_j^2(0)} \geq \sigma_{(1)}(M).$$

This shows that f must be of the form $u_1 v_i$, where v_i is an eigenfunction associated to $\lambda_{(1)}(\mathbb{S}^{n-1})$. Thus the multiplicity of $\sigma_{(1)}(M)$ is the same as the multiplicity of Laplace eigenvalue $\lambda_{(1)}(\mathbb{S}^{n-1})$. By repeating the similar arguments, we can prove that the multiplicity of $\sigma_{(k)}$ is equal to the multiplicity of $\lambda_{(k)}$, $k \geq 0$. □

We will also need a result similar to Proposition 2 for the mixed Steklov-Dirichlet and Steklov-Neumann spectra on a revolution manifold M of dimension n. We write M as the warped product $[L_1, L_2] \times \mathbb{S}^{n-1}$ with the Riemannian metric $g(r,p) = dr^2 + h^2(r)g_0(p)$, where $(r,p) \in [L_1, L_2] \times \mathbb{S}^{n-1}$. We take the Steklov condition at the boundary $r = L_1$ and the Dirichlet or Neumann condition at $r = L_2$. Similar to proposition 2, each eigenfunction f of both the mixed Steklov problem on M can be written as $f(r,p) = u(r)v(p)$, where v is a spherical harmonic of the sphere $\mathbb{S}^{n-1}$ of degree $k \geq 0$ and u is a solution of the equation

$$\frac{1}{h^{n-1}}\frac{d}{dr}(h^{n-1}\frac{d}{dr}u) - \frac{1}{h^2}\lambda_{(k)}u = 0$$

on $[L_1, L_2]$. For the Steklov-Dirichlet problem, we have $u(L_2) = 0$ and for the Steklov-Neumann problem, we have $u'(L_2) = 0$. We denote by $\sigma^D_{(0)}(M) < \sigma^D_{(1)}(M) < \sigma^D_{(2)}(M) < \cdots$ and $0 = \sigma^N_{(0)}(M) < \sigma^N_{(1)}(M) < \sigma^N_{(2)}(M) < \cdots$ the *distinct* (counted without multiplicity) eigenvalues of the mixed Steklov-Dirichlet problem and the mixed Steklov-Neumann problem, respectively on M. Then using similar arguments used in Proposition 2, we can also conclude the following proposition.

Proposition 3. *On a submanifold of revolution $M \subset \mathbb{R}^{n+1}$, the multiplicity of the k^{th} mixed Steklov-Dirichlet eigenvalue $\sigma^D_{(k)}$ (counted without multiplicity) and the k^{th} mixed Steklov-Neumann eigenvalue $\sigma^N_{(k)}$ (counted without multiplicity) has the same multiplicity as $\lambda_{(k)}$, the k^{th} eigenvalue (counted without multiplicity) of the round sphere $\mathbb{S}^{n-1}$.*

In Subsections 3.2 and 3.3, we calculate the eigenvalues of the mixed Steklov problems on annular domains and these calculations are directly inspired by those in [6], Section 3.

3.2 The mixed Steklov-Dirichlet eigenvalues on annular domains

Proposition 4. *Let B_1 and B_L be the balls in $\mathbb{R}^n$, $n \geq 3$, centered at the origin of radius one and L, respectively. Consider the following eigenvalue problem on $\Omega_0 = B_L \setminus \overline{B_1}$*

$$\begin{array}{rcll} \Delta f &=& 0 & \text{in } B_L \setminus \overline{B_1}, \\ f &=& 0 & \text{on } \partial B_L, \\ \frac{\partial f}{\partial \nu} &=& \sigma^D f & \text{on } \partial B_1. \end{array} \qquad (4)$$

Then for $0 \leq k < \infty$, the k^{th} eigenvalue (counted without multiplicity) of eigenvalue problem (4) is

$$\sigma^D_{(k)}(\Omega_0) = \frac{k}{L^{2k+n-2} - 1} + \frac{(k+n-2)\,L^{2k+n-2}}{L^{2k+n-2} - 1}.$$

Also

$$\lim_{L \to \infty} \sigma^D_{(k)}(\Omega_0) = k + n - 2.$$

Proof. The eigenfunctions of (4) are of the form $f_k(r,p) = u(r)v(p)$, where v is an eigenfunction for the k^{th} eigenvalue of the sphere $\mathbb{S}^{n-1}$ and u is a real-valued function defined on the interval $[1, L]$. For $f_k(r,p)$ to be an eigenfunction corresponding to the k^{th} eigenvalue (counting without multiplicity) of the mixed Steklov-Dirichlet problem on Ω_0, u should satisfy the following

$$\begin{aligned}
&u(r) = ar^k + br^{-k+2-n}, \text{ for any nonnegative integer } k,\\
&u(L) = 0, u'(1) = -\sigma^D_{(k)} u(1).
\end{aligned}$$

Since $u'(r) = kar^{k-1} - (n+k-2)br^{-(n+k-1)}$ for $k > 0$, conditions $u(L) = 0$ and $u'(1) = -\sigma^D_{(k)} u(1)$ give

$$\begin{aligned}
aL^k + bL^{-k+2-n} &= 0,\\
ka + (-k+2-n)b &= -\sigma^D_{(k)}(a+b).
\end{aligned}$$

Eliminating a and b, we obtain

$$L^{2k+n-2}(\sigma^D_{(k)} - k + 2 - n) = k + \sigma^D_{(k)}$$

and

$$(L^{2k+n-2} - 1)\sigma^D_{(k)} = k + (n+k-2)L^{2k+n-2}.$$

This gives the desired result. □

3.3 The mixed Steklov-Neumann eigenvalues on annular domains

Proposition 5. *Let B_1 and B_L be the balls in $\mathbb{R}^n$, $n \geq 3$, centered at the origin of radius one and L, respectively. Consider the following eigenvalue problem on $\Omega_0 = B_L \setminus \overline{B_1}$*

$$\begin{array}{rcll}
\Delta f &=& 0 & \text{in } B_L \setminus \overline{B_1},\\
\frac{\partial f}{\partial \nu} &=& 0 & \text{on } \partial B_L,\\
\frac{\partial f}{\partial \nu} &=& \sigma^N f & \text{on } \partial B_1.
\end{array} \tag{5}$$

Then for $0 \leq k < \infty$, the k^{th} eigenvalue (counted without multiplicity) of eigenvalue problem (5) is

$$\sigma^N_{(k)}(\Omega_0) = k\frac{(n+k-2)(L^{(n+2k-2)} - 1)}{kL^{(n+2k-2)} + (n+k-2)},$$

and

$$\lim_{L\to\infty} \sigma^N_{(k)}(\Omega_0) = k + n - 2.$$

Proof. Note that the eigenfunctions $f_k(r,p)$ of (5) can be expressed as $f_k(r,p) = u(r)v(p)$, where v is an eigenfunction for the k^{th} eigenvalue of the sphere $\mathbb{S}^{n-1}$ and u is a real-valued function defined on $[1, L]$. If the function u corresponds to the k^{th} eigenvalue (counting without multiplicity) of the mixed Steklov-Neumann problem on Ω_0, then

$$\begin{aligned}
&u(r) = ar^k + br^{-k+2-n}, \text{ for any nonnegative integer } k,\\
&u'(L) = 0, u'(1) = -\sigma^N_{(k)} u(1).
\end{aligned}$$

These conditions give

$$\begin{aligned}
akL^{k-1} - b(n+k-2)L^{-(k+n-1)} &= 0,\\
ka + (-k+2-n)b &= -\sigma^N_{(k)}(a+b).
\end{aligned}$$

By eliminating a and b, we obtain

$$-(k+\sigma_{(k)}^N)(n+k-2)L^{-(k+n-1)} + kL^{k-1}(n+k-2-\sigma_{(k)}^N) = 0$$

and

$$\sigma_{(k)}^N(kL^{k-1} + (n+k-2)L^{-(k+n-1)}) = k(n+k-2)(L^{k-1} - L^{-(k+n-1)}).$$

Multiplying by $L^{(n+k-1)}$ to get

$$\sigma_{(k)}^N(kL^{(n+2k-2)} + (n+k-2)) = k(n+k-2)(L^{(n+2k-2)} - 1),$$

and

$$\sigma_{(k)}^N = k\frac{(n+k-2)(L^{(n+2k-2)}-1)}{kL^{(n+2k-2)} + (n+k-2)}.$$

□

4 Proof of the main theorem

4.1 Comparison of submanifolds of revolution

Recall that for an n-dimensional submanifold of revolution M of Euclidean space $\mathbb{R}^{n+1}$ with one boundary component $\mathbb{S}^{n-1} \subset \mathbb{R}^n \times \{0\}$ the induced Riemannian metric may be written as

$$g(r,p) = dr^2 + h^2(r)g_0(p),$$

where g_0 is the canonical metric of $\mathbb{S}^{n-1}$, $r \in [0, L]$ and $h(0) = 1$, $h(L) = 0$, $h(r) > 0$ if $0 < r < L$, $h'(L) = 1$ and $1 - r \leq h(r) \leq 1 + r$.

Proposition 6. *Let $M = [0, L] \times \mathbb{S}^{n-1}$, $n \geq 3$, be a Riemannian manifold with metric $g_i = dr^2 + h_i^2(r)g_{\mathbb{S}^{n-1}}$, $i = 1, 2$. Moreover suppose that $h_1(0) = h_2(0) = 1$ and $h_1(r) \leq h_2(r)$. Consider the mixed Steklov-Neumann problem on M (Steklov at $r = 0$ and Neumann at $r = L$). Then $\sigma_k^N(M, g_1) \leq \sigma_k^N(M, g_2)$ for all $k \in \mathbb{N} \cup \{0\}$.*

Proof. The Rayleigh quotient of a function $f(r, p)$ defined on M is given by

$$R_{g_i}(f) = \frac{\int_0^L \int_{\mathbb{S}^{n-1}} \left(\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{h_i^2(r)} \|\bar{\nabla} f\|^2 \right) h_i^{n-1}(r) dr \; dv_{g_{\mathbb{S}^{n-1}}}}{\int_{\mathbb{S}^{n-1} \times \{0\}} f^2(0,p) dv_{g_{\mathbb{S}^{n-1}}}},$$

where $\bar{\nabla}$ is the exterior derivative in the direction of $\mathbb{S}^{n-1}$. Since $n \geq 3$, the condition $h_1(r) \leq h_2(r)$ gives that $R_{g_1}(f) \leq R_{g_2}(f)$. Hence, we have $\sigma_k^N(M, g_1) \leq \sigma_k^N(M, g_2)$ for all $k \in \mathbb{N} \cup \{0\}$. □

Proposition 7. *For any submanifold of revolution $(M_1, g_1) \subset \mathbb{R}^{n+1}$, with boundary $\mathbb{S}^{n-1} \times \{0\}$, there exists a submanifold of revolution $(M_2, g_2) \subset \mathbb{R}^{n+1}$ with the same boundary such that, for all $k \geq 1$, $\sigma_{(k)}(M_2) > \sigma_{(k)}(M_1)$.*

Proof. Note that M_1 will be of the form $[0, L_1] \times \mathbb{S}^{n-1}$ with metric $g_1 = dr^2 + h_1^2(r) g_{\mathbb{S}^{n-1}}$, where h_1 satisfies $h_1(0) = 1$, $|h_1'(r)| \leq 1$ and $h_1(L_1) = 0$. The condition $|h_1'(r)| \leq 1$ gives $1 - r \leq h_1(r) \leq 1 + r$. Consider a submanifold of revolution $M_2 = [0, L_2] \times \mathbb{S}^{n-1}$ with metric $g_2 = dr^2 + h_2^2(r) g_{\mathbb{S}^{n-1}}$, where $L_2 = 2L_1 + 2$ and

$$h_2(r) = \begin{cases} 1 + r, & \text{if } r \leq L_1, \\ L_2 - r, & \text{if } L_1 + 1 \leq r \leq L_2. \end{cases}$$

For $L_1 \leq r \leq L_1 + 1$, we just ask that h joins $h(L_1)$ and $h(L_1 + 1)$ smoothly. Since $h_2(L_2) = 0$, submanifold of revolution M_2 has only one boundary component. Note that $h_1(r) \leq h_2(r)$ for $r \leq L_1$. Further, $h_1(r)$ can not be identically equal to $1 + r$ on $[0, L_1]$ as h_1 is continuous on $[0, L_1]$ and $h_1(L_1) = 0$. Thus, there exist points in $(0, L_1)$ such that $h_2(r) > h_1(r)$ at those points.
Now for arbitrary small $\epsilon > 0$, consider the mixed Steklov-Neumann problem on $\tilde{M} = [0, L_1 - \epsilon] \times \mathbb{S}^{n-1}$ with two metrics g_1 and g_2. Then from Proposition 3 and Proposition 6, we get $\sigma_{(k)}^N(\tilde{M}, g_1) < \sigma_{(k)}^N(\tilde{M}, g_2)$ for all $k \geq 1$. The strict inequality follows from Proposition 6 applied to eigenfunctions of $(\tilde{M}, g_1)$ and from the existance of points in $\tilde{M}$ on which $h_2(r) > h_1(r)$.
Recall that because of the bracketing,

$$\sigma_{(k)}(M_2, g_2) \geq \sigma_{(k)}^N(\tilde{M}, g_2), \quad k \in \mathbb{N} \cup \{0\}.$$

Using the method of Anné (see [2], Theorem 2, and [1] for a less general but easiest version of the result), we have that as $\epsilon \to 0$, $\sigma_{(k)}^N(\tilde{M}, g_1) \to \sigma_{(k)}(M_1, g_1)$. As a consequence, we get $\sigma_{(k)}(M_2, g_2) > \sigma_{(k)}(M_1, g_1)$. □

Next we prove Theorem 1 by using Proposition 7.

Proof. Note that M will be of the form $[0, L] \times \mathbb{S}^{n-1}$ with metric $g = dr^2 + h^2(r) g_{\mathbb{S}^{n-1}}$, where h satisfies $h(0) = 1$, $|h'(r)| \leq 1$ and $h(L) = 0$.
Proposition 7 already shows that it is always possible to construct a submanifold of revolution with the same boundary but strictly larger Steklov eigenvalues. Moreover, Proposition 7 gives the existence of a sequence of submanifolds of revolution $M_i = [0, L_i] \times \mathbb{S}^{n-1}$, $1 \leq i < \infty$, with boundary $\mathbb{S}^{n-1} \times \{0\}$ and metric $g_i = dr^2 + h_i^2(r) g_{\mathbb{S}^{n-1}}$ (h_i and L_i are constructed as in Proposition 7) such that

$$\sigma_{(k)}(M) < \sigma_{(k)}(M_1) < \sigma_{(k)}(M_2) < \cdots .$$

Also, for $i \geq 2$,

$$\sigma_{(k)}^N(A_i) \leq \sigma_{(k)}(M_i) \leq \sigma_{(k)}^D(A_i),$$

where A_i is an annular domain with inner radius one and outer radius $1 + L_{i-1}$, and it is a neighborhood of the boundary of M_i.
Moreover, we have $L_i \to \infty$ as $i \to \infty$ since $L_i = 2L_{i-1} + 2$. Note that for $k > 0$,

$$\lim_{i \to \infty} \sigma_{(k)}^D(A_i) = \lim_{i \to \infty} \sigma_{(k)}^N(A_i) = k + n - 2.$$

This shows $\lim_{i \to \infty} \sigma_{(k)}(M_i) = k + n - 2$. Combining this with the fact that $\sigma_{(k)}(M_i)$ is an increasing sequence proves the theorem. □

Remark 8. *The method developed in this paper is not easily applicable for a revolution manifold with two boundary components. As in this case, unlike Proposition 2, corresponding to each eigenfunction of $\lambda_{(k)}$ we may have two eigenfunction of Steklov problem and it becomes difficult to compare $\sigma_{(k)}^D$ and $\sigma_{(k)}^N$ with $\sigma_{(k)}$. For example, take a revolution manifold with two boundary components isometric* *to the sphere of radius* 1*. Then corresponding to the eigenvalue* $\lambda_{(0)} = 0$*, there are two Steklov eigenfunctions, first is the constant function* $u_1 = 1$*, and other is a function* u_2 *taking the value* 1 *on one boundary component and* -1 *on the other component. The Steklov eigenfunction* u_2 *may be associated with the first nonzero Steklov eigenvalue. Particularly, in case of cylinder* $[0, L] \times \mathbb{S}^{n-1}$*, corresponding to each eigenfunction associated with* $\lambda_{(k)}$*, we get two eigenfunctions of Steklov problem associated with different eigenvalues. Also, for* L *large enough,* u_2 *corresponds to the first nonzero Steklov eigenvalue, which is not the case if* L *is small enough (see Lemma* 2.1 *in [4]).*

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