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Abstraction and Nominalization in Leśniewski's Ontology

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1. Introduction

In this paper I intend to examine certain features that characterize the logicist construction that can be performed within the categorial and expansive framework provided by Leśniewski's Ontology, by putting these features in relation with the question of procedures of abstraction and nominalization. The latter will be placed in the problematic framework of classical logicism, relative to which the treatment of this question acquires all its relevance, given the fully effective resolution that Ontology makes possible.

As we have shown, Ontology, an extensional calculus of names of higher order, allows constructing Peano arithmetic, while conceding no more than an axiom of infinity to pure logic, and the development of this construction is not impeded by the difficulties met by the classical logicist approach[1]. Elaborated in the 1920s, Ontology historically responded to the problem of logical foundations of Mereology, the theory of collective classes that Leśniewski conceived following his discovery of Russell's antinomy in 1911. Starting from that date he concentrated all his efforts on a resolution *strictu sensu* of the contradiction. His work resulted in the development of three

[1] For a detailed presentation of the construction, see Gessler, Joray, Degrange (2005); cf. also Canty (1967) and Joray's paper on this volume.

theories: Mereology, Ontology and Protothetic – this last system being the propositional calculus on which Ontology is based[2]. The structural order of these theories is contrary to the chronological order of their conception. Mereology was conceived the first, in direct reaction to Russell's antinomy. Rejecting the usual definitions of class (or set) that he judged as not being consistent with the ordinary intuition of the concrete character of classes and of the individuals that these represent, Leśniewski introduced a collective definition of the class and showed that it is not subject to any antinomy similar in spirit to Russell's.

I need to emphasize through the evocation of the Leśniewskian program founded on the antinomy that without doubt, one of his major successes is having managed to reconcile the point of view of the class as collection and that of the class as extension of concept, conciliation that was inconceivable in the theoretical constructions elaborated by the tenants of classical logicism. Let us listen to Frege expressing his opinion on this subject, in his beautiful 1895 paper, directed against Schröder's algebra of logic, in which he reproaches the latter the fact of having privileged a purely extensional conception of classes, i.e. classes as collections of objects. This conception is not acceptable for Frege, both for its technical insufficiencies – as it does not allow seizing naturally the concept of empty class, and leads to the assimilation of an individual to the class that is composed only by him – and for its theoretical insufficiencies – as the concepts take logical precedence of their extension [Frege 1895: 456]. He writes:

> The complete difference, and indeed incompatibility, between *these conceptions of classes* is concealed ad first. Thus there arises a cruder conception of classes and extensions, side by side with a subtler one, the only that can be used in logic. (Frege 1895: 452-53)

The difference that Frege indicates, by qualifying it as incompatibility, expresses however fully the Leśniewskian program: starting from the adoption of a collective conception of classes, motivated by Leśniewski as possessing an intuitive legitimacy, as opposed to the

[2] For a detailed presentation of these three theories, see Miéville (2001, 2004) ; Gessler (2005).

concept of class (or set) as an abstract object, makes this one compatible with a language of pure logic, and in the end, clarifies the very reasons of the antinomy by actually solving it.

Before entering the heart of the subject, I add some words on Ontology. This theory, pinned above under the name of extensional calculus of names of higher order, constitutes a free, universal logic, and ontologically neutral. As I implied, its historical mission was to provide a foundational language for Mereology. This language was intended to meet two requirements: to be free from any antinomy of the Russell type and to avoid what is precisely not acceptable with Mereology, that is, introducing classes as abstract entities; in other words, it had to offer a treatment of the extensional dimension without introducing classes. This program will be realized on the basis of what is called the theory of semantic categories, theory that gives the general structure of the language in which the system of Ontology is formulated. The expressions of Ontology belong to distinct categories, which distinguish themselves through the fact that an expression belonging to a certain category cannot substitute for an expression of another category without losing the well formed character of the starting element where the substitution takes place. Formally, the stake of the theory of semantic categories is the same as for a theory of types, with the major difference that the tools function without an *ad hoc* installation. Besides, these categories are semantical in the sense that Ontology is an interpreted system, in which the formalism is under complete dependence of the semantic interpretation, the latter being anchored in the intuitions claimed by Leśniewski.

There are two categories recognized in the language as basic categories: that of propositions *S* and that of names *N*; the two systems of Protothetic and Ontology developed in parallel from this distinction. The other categories are functorial categories or derivated from the two basic categories. I will specify hereafter how any category can be introduced in the system. For the moment, I would like to expand on three major points that configure the paradigm of Ontology and that constitute the keys of the success of the logicist construction that can be performed starting from them.

2. The keys of success

The first point concerns the analysis of the singular proposition. The elementary proposition of Ontology has the form "*a* is *b*", formally "*a* ε *b*". The terms *a* and *b* belong to the semantic category of names, being thus either singular, or empty, or plural. The copula is thus analyzed as a formator functor of a proposition with two nominal arguments, which can be represented categorially as *S/NN*.

A proposition of the form "*a* ε *b*" is true if and only if the name *a* is a singular name and if the object designated by it belongs to the extension of objects indicated by the name *b*. These are the truth conditions that are formalized in the unique axiom of Ontology.

$$\lfloor ab \rfloor \lceil a\varepsilon b \equiv \lfloor \exists c \rfloor \lceil c\varepsilon a \rceil \wedge \lfloor cd \rfloor \lceil c\varepsilon a \wedge d\varepsilon a \supset c\varepsilon d \rceil \wedge \lfloor c \rfloor \lceil c\varepsilon a \supset c\varepsilon b \rceil \rceil$$

The axiom is a general sentence of the form $\lfloor \ldots \rfloor \lceil \ldots \rceil$, the characters $\lfloor$, $\rfloor$, $\lceil$ and $\rceil$ marking the quantifiers. Quantification, as I will specify hereafter, is of a nature different from that which operates in standard logic. This axiom is read:

Whatever the names *a* and *b*, a is *b* if and only if:

1) there is at least one *c* that is *a*
(the name *a* denotes)

2) for all *c* and *d*, if *c* is *a* and *d* is *a*, then *c* is *d*
(the name *a* denotes at most one individual)

3) for all *c*, if *c* is *a* then *c* is *b*
(any object denoted by the name *a* is also denoted by the name *b*)

In other words, the name *a* is neither an empty, nor a plural name – i.e. a singular name – and the object that it denotes belongs to the extension of the name *b*.

It is crucial to understand here that it is by no means possible, when referring to the extension of a name, to associate it an object that would be the distributive class of the objects that it denotes. This logical treatment of names allows disqualifying the concept of class by limiting the concept of extension to its pure extensional dimension, be it empty, singular or multiple. That is the manner in which is neutralized, at this level of primitive Ontology, the antinomian feature that in the classical theories affects at the same time the question of the relation between extension as a multiplicity and extension as a unit. This question does simply not need to be considered any longer.

We will see hereafter that the fact that one can provide for the nominalization of the extension of a name is due to the categorial analysis and to the formal treatment of higher order. We will next measure all the importance of the analysis of the proposition that departs form the Fregean tradition in function/argument, and allows thus to escape all the thorny problems that accompanied the doctrines of functional symbols as incomplete symbols.

The second point refers to the ontological neutrality of Ontology. The source of this neutrality lies in the methods of interpretation of the quantification. Neither referential, nor substitutional, the quantify-cation is of categorial nature. It applies to variables of any semantic category, be it propositional, nominal or functorial, while its interpretation eludes the question of ontological commitment regarding the existence of objects constituting the possible significances of the related variables. Where there is the question of semantic categories, there is by no means a question of ontological categories. To each category is associated a quantification domain, which must be understood as the possibilities of extensional significances falling under the category in question. The forms $\lfloor v \rfloor \lceil A(v) \rceil$ and $\lfloor \exists v \rfloor \lceil A(v) \rceil$ must be read thus: "whatever the extensional significance allotted to the variable v – respectively for any extensional significance allotted to the variable v – it is the case that A(v)", *given the semantic category of variable v*. Without going into details, let us retain that Ontology achieves the purpose of a universal language, escaping the constraint of the adoption of a material of abstract entities, and being free both of any commitment, as of any ontological implication[3].

The third point refers to the constructive dimension of Ontology. This one is given by an internal adjustment to language of the definition procedure. This adjustment arises under the form of an inferential

[3] For more details concerning quantification in Ontology, see Simons (1985), Miéville (1999), Joray (1999, 2005).

rule of definition allowing the treatment of explicit definitions, not as metalinguistic abbreviations, but as theses, by inserting them in the system through their equivalential formulation, expressed by using the biconditional operator '≡'. This explains why the unique primitive functor of the propositional calculus on which Ontology is based – Protothetic – is the biconditional, of category *S/SS*. The definition procedure ensures the language a categorial expressivity that is potentially infinite. It makes indeed possible, on the basis of the primitive significances contained in the axioms, the introduction of functors of any category built from the basic categories *S* and *N*.

The definition procedure rests on two distinct inferential directives: one of proposition type (inherited from Protothetic), the other of ontological or nominal type (necessitated with the introduction of the category of names and related to the primitive functor epsilon 'ε'). I give below a schematic presentation of both types, using each time two different notations (a conventional one and a "contextual" one, which was Leśniewski's original):

- Definition of propositional type (Dfs)

$\lfloor v_1 \ldots v_n \rfloor \lceil f(v_1 \ldots) \ldots [\ldots v_n] \equiv F_{v1 \ldots vn} \rceil$ *conventional*

$\lfloor v_1 \ldots v_n \rfloor \lceil \equiv (f(v_1 \ldots) \ldots [\ldots v_n]\ F_{v1 \ldots vn}) \rceil$ *contextual*

- Definition of nominal type (Dfn)[4]

$\lfloor v_1 \ldots v_n a \rfloor \lceil a\ \varepsilon\ g(v_1 \ldots) \ldots [\ldots v_n] \equiv .\ a \varepsilon a \wedge E_{av1 \ldots vn} \rceil$ *conventional*

$\lfloor v_1 \ldots v_n a \rfloor \lceil \equiv (\varepsilon\{a\ g(v_1 \ldots) \ldots [\ldots v_n]\} \wedge (\varepsilon\{aa\} E_{av1 \ldots vn}) \rceil$ *contextual*

Some remarks on these schemas are necessary. First, why two different writings? Being an expansive system, allowing the introduction

[4] In this type, the left hand formula is of form "*a* ε *definiendum*". As a singular sentence, it requires the subject term *a* to be a singular one. This explains the surprising conjunctive form of the right hand formula "a ε a ∧ *definiens*": the first conjunct of this formula "a ε a" only expresses the condition of singularity concerning the term *a*, the second conjunct being the *definiens* strictly speaking.

of new symbols by definitions, Ontology cannot be grounded on a given list of symbols for its variables and constants, which would be presemantically determined. The recognition of the different categories of the signs occurring in formulae is governed by a contextual approach of syntax: a prefixed writing using distinct parenthesizings that mark the various categories. These parenthesizings, qualified by the word "contexts", are formally characterized by the shape of the brackets and the number of argument places they delimit. In what concerns the defining diagrams, '(--)' and '{--}' are the primitive contexts. The first was associated to the category *S/SS* of the biconditional in the axiomatic basis of Protothetic, the second one to the category *S/NN* of epsilon, in the axiom of Ontology.

The other brackets in the defining diagrams are in dotted lines. This is to signal that their shape will depend on the number and respective categories of the arguments on which the defined functor will operate. If this functor is designed to belong to a category already present in the system, then it will have to be followed by the context corresponding to this category. If, by contrast, it is designed to belong to a new category, then a new context will need to be chosen.

As for the arguments of the functor to be defined, they can be divided in one or more contexts. In the first case (where the functor is followed by just one context), the definition is called *regular*, in the second case (where the functor is followed by more than one context), it is called a *parametric* definition. The functor introduced by a parametric definition is a *multi-link*-functor, i.e. a functor forming functor. At last, the functor to be defined can also not have any context; in this case the definition is said to be an *absolute* one. Here you have some examples. I only specify the contextual writing for the first two.

Df1: $\lfloor a \rfloor \lceil\, !\{a\} \equiv \lfloor \exists b \rfloor \lceil b\ \varepsilon\ a \rceil \rceil$ *Dfs*, !, *S/N*

$\lfloor a \rfloor \lceil \equiv(\, !\{a\} \lfloor \exists b \rfloor \lceil \varepsilon\{b\ a\} \rceil \rceil$

!{a} is read: "the name *a* denotes".

Df2: $\lfloor ab \rfloor \lceil \subset\{ab\} \equiv \lfloor c \rfloor \lceil c\varepsilon a \supset c\varepsilon b \rceil \rceil$ *Dfs*, ⊂, *S/NN*

$\lfloor ab \rfloor \lceil \equiv(\subset\{ab\}\ \lfloor c \rfloor \lceil \supset (\ \varepsilon\ \{ca\}\ \ \varepsilon\ \{cb\}) \rceil\,) \rceil$

This is the definition of inclusion for names; $\subset\{ab\}$ is read: "the name *a* is included in the name *b*", that is, "the extension of the name *a* is included in the name *b*".

$$\text{Df3:}\quad \lfloor ab \rfloor \lceil \approx\{ab\} \equiv \lfloor c \rfloor \lceil c\varepsilon a \equiv c\varepsilon b \rceil \rceil \qquad Dfs, \approx, S/NN$$

$\approx$ represents the *extensional nominal identity*; $\approx\{ab\}$ is read: "the names *a* and *b* have the same extension".

$$\text{Df4:}\quad \lfloor ab \rfloor \lceil \approx\langle a\rangle\{b\} \equiv a \approx b \rceil \qquad Dfs, \approx, (S/N)/N$$

This definition is the parametric version of the functor defined above. The element $\approx\langle a\rangle$, with *a* as parameter, followed by the context {-}, whose argument is of category *N*, is of category *S/N*; whereas the functor $\approx$, associated to the context $\langle$-$\rangle$, whose argument is of category *N*, is of category *(S/N)/N*.

The choice of these last two definitions is not innocuous. We will actually find them next, during our discussion on the act of abstraction and nominalization, and we will then be able to judge the major rôle that the procedure of parameterization plays in Ontology.

$$\text{Df5:}\quad \lfloor a \rfloor \lceil a\ \varepsilon\ \Lambda \equiv.\ a\varepsilon a \wedge \sim(a\varepsilon a) \rceil \qquad Dfn, \Lambda, N$$

Definition Df5 introduces the empty or contradictory name.

$$\text{Df6:}\quad \lfloor abc \rfloor \lceil a\ \varepsilon \cap \lceil bc \rceil \equiv.\ a\varepsilon a \wedge a\varepsilon b \wedge a\varepsilon c \rceil \qquad Dfn, \cap, N/NN$$

$$\text{Df7:}\quad \lfloor a\alpha\beta \rfloor \lceil \cap\backslash\alpha\beta/\{a\} \equiv.\ \alpha\{a\} \wedge \beta\{a\} \rceil \qquad Dfs, \cap, (S/N)/(S/N)(S/N)$$

These definitions are two analogues of logical product: nominal intersection and predicative intersection. The last definition is equally parametric. The functor $\cap\backslash\alpha\beta/$ is of category *S/N*; the defined functor $\cap$, associated to the context $\backslash$--$/$, is of category *(S/N)/(S/N)(S/N)*. We will read thus $\cap\backslash\alpha\beta/\{a\}$: "the name *a* forms/is the intersection of predicates α and β". $\cap\backslash\alpha\beta/$ represents thus the intersection of predicates α and β.

The following three definitions are analogous to the first three, for the category *S/N*:

$$\text{Df8:}\quad \lfloor \alpha \rfloor \lceil ![\alpha] \equiv \lfloor \exists a \rfloor \lceil \alpha\{a\} \rceil \rceil \qquad Dfs, !, S/(S/N)$$

$$\text{Df9:}\quad \lfloor \alpha\beta \rfloor \lceil \subset[\alpha\beta] \equiv \lfloor a \rfloor \lceil \alpha\{a\} \supset \beta\{a\} \rceil \rceil \qquad Dfs, \subset, S/(S/N)(S/N)$$

Df10: $\lfloor\alpha\beta\rfloor\lceil{\approx}[\alpha\beta] \equiv \lfloor a\rfloor\lceil\alpha\{a\} \equiv \beta\{a\}\rceil\rceil$ *Dfs*, ≈, *S/(S/N)(S/N)*

Notice that the choice of the same symbols as in the precedent definitions is not accompanied by any ambiguity, since the contexts allow distinguishing one constant from the other. It is the same with the parametric definitions Df4 and Df7.

3. The elements of the logicist construction

Let us now focus on the determinations and positive incidences on the logicist construction of the above described three characteristic features of Ontology (the analysis of proposition, the categorial nature of quantification and the definition procedure). I have selected four issues, the list being of course not exhaustive.

1) Peano arithmetic is built without resorting to the concepts of class or set, not even as linguistic conveniences, like in the *Principia Mathematica*. The cardinal number is defined as the property of a name. It is then of category *S/N* and the definition to which we arrived is the following:

$$\lfloor\alpha\rfloor\lceil Cn[\alpha] \equiv \lfloor\exists a\rfloor\lceil\alpha \approx \infty\langle a\rangle\rceil\rceil$$

I leave aside for the moment the comments related to the formal expression of this thesis, and provide only an intuitive reading:

> α is a cardinal number if and only if there is a name *a* such that α expresses *being equinumeric to* a.

2) The definition procedure is limited to the use of explicit definitions adjusted within the language, as we have emphasized above. Logicism is thus equipped with a really constructive dimension. It is a construction, an Ontology among the other possible ones, and its development is entirely regulated by the formal tools of the adopted logic.

3) Except for the presence of an axiom of infinity, this logicism is ontologically neutral. The analysis of the concept of number is in this way released from the yoke of abstract entities and from the need of dealing with the question of the nature of numbers.

4) The stratification of language is managed without making recourse to the concept of systematic ambiguity, on which rests type theory in the *Principia Mathematica*. If, as in *Principia*, there is an arithmetic at "each floor", the modes of formalization allow apprehending the question on the hierarchy of types in its generating movement, and finding isomorphism not by the external requisite that it is seen and it arises from a systematic analogy, but by the fact that it proves itself and it functions at the interior of the language, under the cut of the formal modes of this language. This stratification is managed by the possibility of reproducing the axiom of Ontology at each floor, by previously defining the so-called *higher-order epsilons*. For example, on the basis of the functor of extensional identity $\approx$ defined above (Df3), the following definition of a higher epsilon can be written, of category *S/(S/N)(S/N)*:

$$\mathrm{Df}(\varepsilon_{\approx})\text{: } \lfloor\alpha\beta\rfloor \lceil \varepsilon[\alpha\beta] \equiv. \lfloor\exists a\rfloor \lceil \alpha\{a\} \wedge \beta\{a\}\rceil \wedge \lfloor ab\rfloor \lceil \alpha\{a\} \wedge \alpha\{b\} .\supset \approx\{ab\}\rceil\rceil$$

> This definition can be read: for any functor α and β of category *S/N*, α is β if and only if for some name *a*, α and β are verified by *a* and, for any name *a* and *b*, if α is verified by *a* and by *b*, then *a* and *b* have the same extension.

Afterwards we can derive the structural equivalent of the axiom of Ontology, in which variables of category *S/N* replace the nominal variables, and the defined superior epsilon of category *S/(S/N)(S/N)* replaces the primitive epsilon of category *S/NN*. Let the following thesis:

$$\mathrm{Ax}(\varepsilon_{\approx})\text{: } \lfloor\alpha\beta\rfloor \lceil \varepsilon[\alpha\beta] \equiv$$

$$\lfloor\exists\gamma\rfloor \lceil \varepsilon[\gamma\alpha]\rceil \wedge \lfloor\gamma\rfloor \lceil \varepsilon[\gamma\alpha] \supset \varepsilon[\gamma\beta]\rceil \wedge \lfloor\gamma\delta\rfloor \lceil \varepsilon[\gamma\alpha] \wedge \varepsilon[\delta\alpha] .\supset \varepsilon[\gamma\delta]\rceil\rceil$$

We can consequently dispose of the structural analogue of all the theses susceptible to be registered in the primitive Ontology. For example, for the definition Df1: $\lfloor a\rfloor \lceil !\{a\} \equiv \lfloor\exists b\rfloor \lceil b\varepsilon a\rceil\rceil$, we can derive, on the basis of superior functor ! of category *S/N* introduced with the definition Df6, the following thesis: $\lfloor\alpha\rfloor\lceil ![\alpha] \equiv \lfloor\exists\beta\rfloor \lceil\beta\varepsilon\alpha\rceil\rceil$.

What do we retain from here? The theory allows the treatment of higher entities as pseudo-names, without reification, and without returning them to a statute of logical fictions, as is in the *Principia Mathematica.* As each language layer managed by a higher epsilon imitates by its isomorphism the nominal layer managed by the primitive epsilon, the process of nominalization of the superior entities is thus validated by their pseudo-nominal representation.

Having expressed here the accomplishment and the success of Ontology in what regards nominalization, I rephrase the considerations by anchoring them in the difficulties met on this subject by classical logicism, to plunge them next in the theoretical framework of Ontology, and to examine them from a structural point of view.

4. Where we escape from the misfortunes of classical logicism

Russell writes in the *Principles*, aiming to describe the relationship between extension as a multiplicity and extension as a unit:

> [...] without a single object to represent an extension, mathematics crumbles. (1903: 489)

This quotation alone expresses the threshold at which hit – and even broke – classical logicism. It is the concept of class as one – or Frege's *Werthverlauf* – that is found in the heart of the antinomy, and that led Russell, for want of anything better, to the theory of logical fictions, carried by the technique of contextual definitions. It is with such a technique, attesting that a certain type of expression functioning seemingly as a unit of significance is in fact an incomplete symbol, that Russell faced the crucial problem of nominalization of the class as many. While classes are treated as linguistic conveniences, as fictitious objects, there is nothing that resists nominalization. The difficulty in cause is thus eliminated. Here I only evoke, without developing it, this crucial problem that prevented classical logicism from accomplishing in the way its founders had dreamed.

Let us now consider the issue within Ontology. How is the process of nominalization regulated? How can an extension or a number represent? I will approach the answer to these questions through two theses of Ontology. The first is the ontological expression of Frege's

Law V, which admits the logical equivalence between the identity of extensions and the formal equivalence of concepts. Carrying in germ the contradiction that was brought to daylight by the discovery of Russell's antinomy, this axiom can be expressed in the following manner, where I make use of the usual symbol for the class abstractor:

Law V: $\hat{x}F(x) = \hat{x}G(x) \equiv (\forall x)(F(x) \equiv G(x))$

The corresponding thesis in Ontology is[5]:

Th1: $\lfloor ab \rfloor \lceil {\approx}\langle a \rangle \approx {\approx}\langle b \rangle \ .{\equiv}.\ a \approx b \rceil$

I leave for later the comments concerning the formal expression of this thesis that makes use of the functors defined in section 2, and for now I give its following reading:

> Two names determine identical *expressions of extension* if and only if they are coextensive.

The second thesis is the ontological version of Hume's Principle, expressing the identity criterion for numbers. Considering the principle as an implicit definition of "the number of", neo-Fregeans showed that the essence of Frege's construction could be rephrased, without the famous Law V, on the unique basis of classical second-order logic expanded with Hume's Principle.

Hume's Principle: $(\forall FG)(\text{Number}(F) = \text{Number}(G)\ .{\equiv}.\ F \infty G)$

The ontological analogue of this principle is the following thesis:

Th2: $\lfloor ab \rfloor \lceil \infty\langle a \rangle \approx \infty\langle b \rangle \ .{\equiv}.\ a \infty b \rceil$

As before, I restrain for now to a reading, which will however allow shedding some light on the definition given above for the cardinal number:

> The cardinal number of *a* is identical to the cardinal number of *b* if and only if *a* is equinumeric to *b*.

[5] This thesis highlights the three available significances relative to the symbol $\approx$ (Df3, Df4, Df10). To be sure that this triple use is legitimate and that all confusion is removed, it suffices to rewrite the thesis in full contextual notation: $\lfloor ab \rfloor \lceil {\equiv}(\ {\approx}[{\approx}\langle a \rangle\ {\approx}\langle b \rangle]\ {\approx}\{ab\})\rceil$.

For better understanding of the proposed readings, let us consider more in detail the functors that come into play in these theses.

• Th1: $\lfloor ab \rfloor \lceil \approx\langle a\rangle \approx \approx\langle b\rangle .\equiv. a \approx b \rceil$

i) $\approx$, occurring in "a $\approx$ b", is the functor of nominal extensional identity. Being of category *S/NN*, it was introduced into the preceding section, by definition Df3: $\lfloor ab \rfloor \lceil a \approx b .\equiv. \lfloor c \rfloor \lceil c\ \varepsilon\ a \equiv c\ \varepsilon\ b \rceil$. It is a reflexive, symmetrical and transitive relation.

ii) $\approx\langle - \rangle$ is the parametric version of the functor $\approx$, introduced by definition Df4: $\lfloor ab \rfloor \lceil \approx\langle a\rangle\{b\} .\equiv. a \approx b \rceil$. Let's take a look at this definition. Taking into account the significance of the *definiens* of this definition "a $\approx$ b", we can read the *definiendum* "$\approx\langle a\rangle\{b\}$" as "the *b* form/are the extension of *a*", i.e. "the names *a* and *b* have the same extension". Consequently $\approx\langle a\rangle$, of category *S/N*, can be assimilated to *being the extension of the name a*. And we can, under the authority of this categorial analysis, call $\approx\langle a\rangle$, *the extension of the name a*.

iii) $\approx$, occurring between the elements of the left hand expression "$\approx\langle a\rangle \approx \approx\langle b\rangle$" of the thesis is the functor of extensional identity between predicates of category *S/N*. It was defined with Df10: $\lfloor \alpha\beta \rfloor \lceil \approx[\alpha\beta] \equiv \lfloor a \rfloor \lceil \alpha\{a\} \equiv \beta\{a\} \rceil\rceil$.

• Th2: $\lfloor ab \rfloor \lceil \infty\langle a\rangle \approx \infty\langle b\rangle .\equiv. a \infty b \rceil$

i) the symbol ∞ appearing in "a ∞ b" is that of the relation of equinumericity between names. This relation is defined in a similar way with what is done within a classical framework between classes or sets: two names *a* and *b* are equinumeric if and only if there is a one-one relation between them, *a* represents the domain of the relation, and *b* its co-domain. In the language of Ontology and our arithmetic construction, one have[6]:

$$\lfloor ab \rfloor \lceil a \infty b \equiv \lfloor \exists R \rfloor \lceil \mathit{OneOne}(R) \wedge \mathit{Dom}(R)\{a\} \wedge \mathit{Cdom}(R)\{b\} \rceil \rceil$$

The functor ∞ has the properties of reflexivity, symmetry and transitivity.

ii) $\infty\langle a \rangle$: it is the parametric version of the relation of nominal equinumericity ∞. We introduce it with this definition:

$$\lfloor ab \rfloor \lceil \infty\langle a \rangle\{b\} \;.\!\equiv\!.\; a \infty b \rceil$$

According to a categorial reading similar with the one done on the parametric version of the extensional identity, $\infty\langle a \rangle$, of category *S/N*, expresses the cardinal number of *a*, in the categorial sense of *being the cardinal number of a*.

iii) The relation $\approx$, in the first member of the thesis "$\infty\langle a \rangle \approx \infty\langle b \rangle$", is – like in Th1 – that of extensional identity between elements of category *S/N*.

Let us subject now the preceding considerations to a more precise examination. First of all, a remark must be done concerning the identity expressed with the functor $\approx$, of category *S/(S/N)(S/N)*, and introduced with definition Df10. We notice that the logical paradigm of Ontology allows avoiding the absolute conception of identity as relating exclusively to objects. In the expression "$\alpha \approx \beta$", $\approx$ is not a referential identity, but an extensional identity between predicates of category *S/N*. It is by no means an identity expressing a relationship between abstract objects starting from the functors α and β, i.e., in Fregean language, the extensions of concepts. As we know, such writing is rejected by Frege as it does not reflect the character of

[6] Taking into account the following definitions:
- being a one-one relation between singular names:
$\lfloor R \rfloor \lceil \mathit{OneOne}(R) \equiv \lfloor abc \rfloor \lceil R\{ac\} \wedge R\{bc\} .\vee. R\{ca\} \wedge R\{cb\} :\supset a \varepsilon b \rceil \rceil$
- being the domaine of a relation of category *S/NN*:
$\lfloor Ra \rfloor \lceil \mathit{Dom}(R)\{a\} \equiv \lfloor b \rfloor \lceil \lfloor \exists c \rfloor \lceil R\{bc\} \equiv b \varepsilon a \rceil \rceil \rceil$
- being the co-domaine of a relation of category *S/NN*:
$\lfloor Ra \rfloor \lceil \mathit{Cdom}(R)\{a\} \equiv \lfloor b \rfloor \lceil \lfloor \exists c \rfloor \lceil R\{cb\} \equiv b \varepsilon a \rceil \rceil \rceil$

non-saturation of the functions. According to his analysis, when mathematicians use this – incorrect – notation, they use in an implicit way the fundamental Law V and the possibility of passing from the assertion of identity between the values taken by two functions on the arguments, to the assertion of identity between suits of values. In Ontology, such an expression will be understood as expressing "*the* functor α and *the* functor β have the same extension", in the same way in which will be read, for example, ![α] (*cf.* Df8), "*the* functor α denotes", although α and β do not have the statute of a name in either expression. There is no impossibility here of naming the functions, while eluding any problem of reification. In addition, as I previously emphasized, the nominalization of superior entities is completely legitimated in Ontology by the possibility of raising the axiom to any higher category. It is thus possible to reproduce by the definition approach the arithmetic of a given category towards superior categories. This fact shows, in addition, that the constructed arithmetic is indifferent to the nature of entities to which the numbers apply.

It is equally necessary to insist on the fact that ontological Th2 does not say the same thing as Hume's Principle, which is an implicit definition. In Ontology abstraction does not define, but it nominalizes through the procedure of parameterization that authorizes the disconnection from a certain linguistic element, categorically autonomous. Moreover, by contrast with Hume's Principle, cardinal numbers are not objects to be counted as elements of the universe.

In the light of the previous affirmations, let us now turn towards the definition of cardinal number (given in section 3):

$$\lfloor\alpha\rfloor\lceil \mathrm{Cn}[\alpha] \equiv \lfloor\exists a\rfloor\lceil \alpha \approx \infty\langle a\rangle\rceil\,\rceil$$

This definition introduces the functor Cn, *being a cardinal number*, of category *S/(S/N)*, a cardinal number being of category *S/N*. We can introduce a similar thesis to define *being an extension*. Say:

$$\lfloor\alpha\rfloor\lceil \mathrm{Ext}[\alpha] \equiv \lfloor\exists a\rfloor\lceil \alpha \approx \approx\langle a\rangle\rceil\,\rceil$$

This definition is read: "α is an extension if and only if there is a name *a* such that α expresses being the extension of *a*".

Let us consider now, in a parallel manner, some theses that are associated with these parametric functors expressing the extension of a name and the cardinal number of a name.

$\lfloor a \rfloor \lceil \infty\langle a\rangle\{a\}\rceil$ $\lfloor a \rfloor \lceil \approx\langle a\rangle\{a\}\rceil$

$\lfloor a \rfloor \lceil\ !\ [\infty\langle a\rangle]\rceil$ $\lfloor a \rfloor \lceil\ !\ [\approx\langle a\rangle]\rceil$

$\lfloor a \rfloor \lceil \lfloor \exists\alpha \rfloor \lceil \alpha \approx \infty\langle a\rangle\rceil\rceil$ $\lfloor a \rfloor \lceil \lfloor \exists\alpha \rfloor \lceil \alpha \approx \approx\langle a\rangle\rceil\rceil$

These theses reveal that to any name, be it empty, singular or plural, is associated a function, of category *S/N*, that expresses the cardinal number of *a*. Similarly, it can be associated a function that expresses the extension of *a*. This result is corroborated to the following theses, using the defined functors for *being a cardinal number* (Cn) and for *being an extension* (Ext).

$\lfloor a \rfloor \lceil \mathrm{Cn}\ [\infty\langle a\rangle]\rceil$ $\lfloor a \rfloor \lceil \mathrm{Ext}\ [\approx\langle a\rangle]\rceil$

They are read: "any name – whether it denotes or not – determines a cardinal number", and "any name – whether it denotes or not – determines an expression of extension". Let us notice that the latter thesis fulfils the first requisite of Russell's theory of classes i.e. each propositional function with an argument must determine a class which can be regarded as the collection of all the arguments satisfying the property in question[7].

In conclusion, the parametric functors $\infty\langle a\rangle$ and $\approx\langle a\rangle$ are thus *façons de parler* the extension and the number of a name. Both of category *S/N*, these functors provide the nominalization of the cardinal number, respectively, the extension of the name *a*. In what regards the question of knowing how to legitimate this process of nominalization, I have already answered. This legitimating concerns epsilons of a higher order, qualified thus for being functors of category *S/CC*, where $C \neq N$, and which, allowing to derive the equivalent of the axiom for their category, are also paraphrasable by "is", as proposition forming functors for pseudo-names. As each layer of language managed by a superior epsilon imitates, so to speak, the nominal layer

7 See (Russell 1919: 184f) and (Whitehead & Russell 1927: 76f).

managed by the primitive epsilon, the process of nominalization of superior entities is therefore validated by their pseudo-nominal representation.

We will also retain the enlargement of the concept of name that Ontology calls. This concept is thus able to characterize any expression designating a linguistic entity of the language, either being intended to designate names in the strict sense (of category *N*, empty, singular or plural), or "names" of functors, functors of functors, etc., whatever the degree of categorial complexity of the functors in question.

5. Conclusion

That is how Ontology unties inherent difficulties within the classical conceptual framework and relative to the procedure of abstraction and of nominalization. Thanks to its analysis of proposition, Ontology, as we saw, reconciles the concept of name and that of function by a formal and categorial adjustment of the linguistic process of nominalization, without having the inconvenient of a complication of writing to which Frege was subject in his object/concept distinction and analysis of the language that marks this distinction. It also shows that its categorial and expansive tools refute Frege, who exploited the natural language to show, without possible compromise, that numbers are objects. Moreover, it has the merit, while allowing the functions to be named in the same manner as any entity of language, of advantages such as naturalness and simplicity, if one contemplates the practice of mathematics and an enlarged interpretation of the concept of name in adequacy with the linguistic process of objectivation. And finally, it shows that the logicist thought can find rest while developing outside of any dogmatic realism or razor of Occam that multiply the logical fictions.

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