

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/375159511>

On invariant distributions of Feller Markov chains with applications to dynamical systems with random switching

Preprint · October 2023

DOI: 10.13140/RG.2.2.17691.03360

CITATIONS

0

READS

54

2 authors, including:



Michel Benaim

Université de Neuchâtel

133 PUBLICATIONS 4,304 CITATIONS

SEE PROFILE

On invariant distributions of Feller Markov chains with applications to dynamical systems with random switching

Michel Benaïm¹ and Oliver Tough²

October 31, 2023

Abstract

We introduce simple conditions ensuring that invariant distributions of a Feller Markov chain on a compact Riemannian manifold are absolutely continuous with a lower semi-continuous, continuous or smooth density with respect to the Riemannian measure. This is applied to Markov chains obtained by random composition of maps and to piecewise deterministic Markov processes obtained by random switching between flows.

Contents

1	Introduction	2
2	Notation, hypotheses and basic results	3
2.1	On Assumption 2.2: a uniqueness result	7
3	Random maps	8
3.1	Expansion volume rates and spectral radius	13
3.2	Application to random maps	22

¹Institut de Mathématiques, Université de Neuchâtel, Switzerland.

²Department of Mathematical Sciences, University of Bath, United Kingdom

4	Piecewise deterministic Markov processes	25
4.1	A discrete kernel associated to $(Z_t)_{t \geq 0}$	26
4.2	Invariant distributions	29
4.3	Smooth invariant distributions on the torus	32
4.4	Smooth invariant distributions under fast switching	37

1 Introduction

The aim of this paper is to propose and discuss simple conditions guaranteeing that the invariant distributions of a Feller Markov chain on a compact space satisfy certain regularity properties, such as having lower semi-continuous, continuous or smooth densities with respect to a reference measure.

Our initial motivation comes from *piecewise deterministic Markov processes* (PDMPs) generated by random switching between deterministic flows. The ergodic properties of these type of processes have been the focus of much attention in the last decade and conditions ensuring existence, uniqueness, and absolute continuity (with respect to a reference Riemannian measure) of invariant measures, are now well understood ([2], [7], [8], [14], [11], [6]). Concerning the regularity (continuity, smoothness) of these densities, some partial results have been obtained in dimension one by Bakhtin, Hurth and Mattingly in [1], Bakhtin, Hurth, Lawley and Mattingly in [3] and [4] for specific systems in dimension two, and by the present authors in [12] for systems under “sufficiently fast” switching. Also worth mentioning is Löcherbach’s beautiful article [21] on certain PDMPs with jumps, in which techniques (similar to those in [3]) are used to prove regularity. However, beyond these cases, the problem remains largely open. One of our principal goals is to revisit these questions, and to provide a simple and general framework allowing - in particular - for the results of [3] and [12] to be extended.

The general idea of the paper can be roughly described as follows. Suppose P is a Feller Markov kernel on some compact metric space M and that $\mathcal{C}(M)$ is a convex cone of measures embedded in some Banach space E . For instance, if M is a Riemannian manifold, $\mathcal{C}(M)$ can be chosen to be the set of measures having a C^r ($r \geq 0$) density with respect to the Riemannian measure, and $E = C^r(M)$.

Suppose that $P = Q + \Delta$ where Q, Δ are sub-Markov kernels such that Q maps the set of probability measures into $\mathcal{C}(M)$ and Δ maps $\mathcal{C}(M)$ into itself. Then, it is not hard to show that if the spectral radius of Δ (seen as

an operator on E) is < 1 , invariant distributions of P lie in $\mathcal{C}(M)$.

The paper explores and develops this idea. Section 2 sets the general framework, notation and hypotheses. Here we state and prove our general results, such as the aforementioned Theorem 2.8, along with other results ensuring absolute continuity of the invariant distributions and lower semi-continuity of their densities (Theorems 2.6 and 2.11).

Section 3 considers the situation where P is induced by a random iterative system on a compact Riemannian manifold and provides conditions ensuring that the decomposition $P = Q + \Delta$ holds with $\mathcal{C}(M)$ the set of measures having a density (respectively a lower semi continuous, of C^r density) with respect to the Riemannian measure. In the specific case where $\Delta = \delta_\phi$ with ϕ a local diffeomorphism, Δ is nothing but the Ruelle transfer operator of ϕ and its spectral radius can be estimated in terms of certain topological (or measure-theoretic) invariants for ϕ . This is done in subsection 3.1 and applied to specific examples in subsection 3.2.

Section 4 is devoted to PDMPs, as described above. We prove that under certain Hörmander conditions, there are finitely many ergodic measures that are absolutely continuous with respect to the Riemannian measure and whose densities are lower semi continuous (Theorem 4.5). If the Hörmander condition only holds at an accessible point, such a measure is unique (Theorem 4.4). In subsection 4.3 we consider the situation of two transverse vector fields on the torus, and give a precise condition (involving the switching rates and the Floquet exponents of the linearly stable periodic orbits of the vector fields) ensuring that the invariant measures have a C^k density (Theorem 4.6). This result relies on the spectral radius estimate of the Ruelle transfer operator given in section 3.1 and substantially extends the results in [3]. The last section 4.4 is devoted to general PDMPs under fast switching. We show how our approach provides for a short proof that under fast switching and a certain Hörmander condition, invariant densities are C^r .

2 Notation, hypotheses and basic results

Let M be a compact metric space equipped with its Borel sigma field $\mathcal{B}(M)$.

We let $\mathcal{M}(M)$ (respectively $\mathcal{P}(M)$) denote the set of non negative (respectively, probability) measures over M .

A *convex cone* of a measures is a set $\mathcal{C}(M) \subset \mathcal{M}(M)$ such that $\alpha\mu + \beta\nu \in \mathcal{C}(M)$, for all $\mu, \nu \in \mathcal{C}(M)$, and all $\alpha, \beta \geq 0$.

Example 2.1 Suppose that M is a Riemannian manifold with Riemannian measure m . Examples of convex cones in $\mathcal{M}(M)$ include:

- $\mathcal{M}_{ac}(M) \subset \mathcal{M}(M)$, the set of measures which are absolutely continuous with respect to m ;
- $\mathcal{M}_{ac}^{ls}(M) \subset \mathcal{M}_{ac}(M)$, the subset which have a lower semi-continuous density;
- $\mathcal{M}_{ac}^r(M) \subset \mathcal{M}_{ac}(M)$, $r \geq 0$, the subset which have a C^r density.

A *bounded kernel* on M is a family $Q = \{Q(x, \cdot)\}_{x \in M}$ with $Q(x, \cdot) \in \mathcal{M}(M)$ such that for all $A \in \mathcal{B}(M)$, the mapping $x \rightarrow Q(x, A)$ is measurable, and $\sup_{x \in M} Q(x, M) < \infty$. We say that Q is *non-degenerate* if $Q(x, M) > 0$ for all $x \in M$; *sub-Markov* if $\sup_{x \in M} Q(x, M) \leq 1$; and *Markov* if $Q(x, \cdot) \in \mathcal{P}(M)$ for all $x \in M$.

We let $B(M)$ (respectively $C^0(M)$) denote the Banach space of bounded measurable (respectively continuous) real valued functions on M , endowed with the uniform norm $\|f\|_0 = \sup_{x \in M} |f(x)|$.

A bounded kernel Q induces a bounded operator on $B(M)$ defined by

$$Qf(x) = \int_M f(y)Q(x, dy),$$

for all $f \in B(M)$. We call it *Feller* if it maps $C^0(M)$ into itself. It also induces an operator on $\mathcal{M}(M)$ defined by

$$\mu Q(A) = \int \mu(dx)Q(x, A),$$

for all $\mu \in \mathcal{M}(M)$ and $A \in \mathcal{B}(M)$.

If Q is Markov, we let $\text{Inv}(Q)$ denote the set of *invariant probability measures* of Q ; that is the set of $\mu \in \mathcal{P}(M)$ such that $\mu Q = \mu$. If Q is Markov and Feller, then $\text{Inv}(Q)$ is a nonempty convex compact (for the weak* topology) subset of $\mathcal{P}(M)$ (see e.g [10], Corollary 4.21).

From now on, we let P denote a Markov Feller kernel and $\mathcal{C}(M)$ a convex cone of measures. Our standing assumption is given by the following assumption.

Assumption 2.2 (Standing assumption) *The kernel P may be decomposed into $P = Q + \Delta$, whereby:*

- (i) Q is a non-degenerate Feller sub-Markov kernel and Δ is a (possibly degenerate) sub-Markov kernel;
- (ii) $\mathcal{M}(M)Q := \{\mu Q : \mu \in \mathcal{M}(M)\} \subset \mathcal{C}(M)$;
- (iii) $\mathcal{C}(M)\Delta := \{\mu\Delta : \mu \in \mathcal{C}(M)\} \subset \mathcal{C}(M)$.

In our applications, $\mathcal{C}(M)$ will be, like in Example 2.1, a set of measure having certain regularity properties. In words, Assumption 2.2 means that Q "creates" regularity, whilst Δ "preserves" regularity.

It follows from Assumption 2.2 that Δ is Feller and that

$$\sup_{x \in M} \Delta(x, M) := \rho < 1.$$

In particular,

$$(I - \Delta)^{-1} := \sum_{k \geq 0} \Delta^k$$

is also a Feller kernel and

$$(I - \Delta)^{-1}(x, M) \leq 1 - \rho.$$

Here $I = \Delta^0 = \{\delta_x(\cdot)\}_{x \in M}$.

The following result is a straightforward consequence of Assumption 2.2, and will be used repeatedly.

Lemma 2.3 *Let $\Pi \in \text{Inv}(P)$. Then, under Assumption 2.2 (i),*

$$\Pi = \Pi Q (I - \Delta)^{-1} = \sum_{k \geq 0} \Pi Q \Delta^k.$$

Proof: This follows directly from the equation $\Pi = \Pi P \Leftrightarrow \Pi(I - \Delta) = \Pi Q$.
□

Example 2.4 Suppose that $Q(x, dy) = \pi(dy)$ with $\pi \in \mathcal{M}(M)$. Lemma 2.3 shows that

$$\text{Inv}(P) = \{\pi(I - \Delta)^{-1}\}.$$

We say that $\mathcal{C}(M)$ is *stable by monotone convergence* if for every sequence $(\mu_n)_{n \geq 0}$ with $\mu_n \in \mathcal{C}(M)$ and $\mu_n \leq \mu_{n+1}$, $\mu = \lim_{n \rightarrow \infty} \mu_n$ lies in $\mathcal{C}(M)$. Here, $\mu = \lim_{n \rightarrow \infty} \mu_n$ simply means that $\mu(A) := \lim_{n \rightarrow \infty} \mu_n(A) \in [0, \infty]$ for all $A \in \mathcal{B}(M)$.

Remark 2.5 The sets $\mathcal{M}_{ac}(M)$ and $\mathcal{M}_{ac}^{ls}(M)$ as defined in Example 2.1 are stable by monotone convergence.

A first useful (and immediate) consequence of Lemma 2.3 is the next result.

Theorem 2.6 *Assume Assumption 2.2 holds with $\mathcal{C}(M)$ stable by monotone convergence. Then $\text{Inv}(P) \subset \mathcal{C}(M)$.*

Corollary 2.7 *Suppose M is a Riemannian manifold. Assume Assumption 2.2 holds with $\mathcal{C}(M) = \mathcal{M}_{ac}^{ls}(M)$. Then:*

- (i) $\text{Inv}(P) \subset \mathcal{C}(M)$;
- (ii) *If $\mu, \nu \in \text{Inv}(P)$ are ergodic, either $\mu = \nu$ or there exist nonempty disjoint open sets U, V such that $\mu(U) = \nu(V) = 1$. In particular, if M is connected and an invariant distribution has full support, then it is the unique invariant distribution of P .*

Proof: (i) follows from Proposition 2.6 and Remark 2.5. We now turn to (ii). By ergodicity either $\mu = \nu$ or μ and ν are mutually singular. By Proposition 2.6, $\mu(dx) = h(x)m(dx)$ and $\nu(dx) = g(x)m(dx)$ with h, g lower semi-continuous. Set $U = \{x \in M : h(x) > 0\}$ and $V = \{x \in M : g(x) > 0\}$. Then U and V are open and $\mu(dx) \geq \frac{h(x)}{g(x)}1_V(x)\nu(dx)$. So, if μ and ν are mutually singular, h has to be zero on V . \square

Another useful (and immediate) consequence of Lemma 2.3 is given by the next result.

Theorem 2.8 *Assume Assumption 2.2 holds with $\mathcal{C}(M)$ a closed subset of some Banach space $(E, \|\cdot\|_E)$. Assume furthermore that one of the two following conditions holds:*

- (i) *For all $\mu \in \mathcal{C}(M)$, $\sum_{k \geq 0} \|\mu \Delta^k\|_E < \infty$;*
- (ii) *$\mu \rightarrow \mu \Delta$ extends to a bounded operator on E , whose resolvent set contains 1.*

Then $\text{Inv}(P) \subset \mathcal{C}(M)$.

Remark 2.9 In the following sections, this theorem will be used when M is a Riemannian manifold, $\mathcal{C}(M) = \mathcal{M}_{ac}^r(M)$, and E is the Banach space of bounded signed measures whose density is C^r (naturally identified with $C^r(M)$ equipped with the C^r norm).

Remark 2.10 A sufficient practical condition ensuring condition (i) (and also (ii)) in Theorem 2.8 is that $\mu \rightarrow \mu\Delta$ extends to a bounded operator on E whose spectral radius,

$$\mathcal{R}(\Delta, E) = \lim_{n \rightarrow \infty} \|\Delta^n\|_E^{1/n},$$

is strictly less than 1.

2.1 On Assumption 2.2: a uniqueness result

It is often the case that a Markov kernel P doesn't satisfy the standing assumption, Assumption 2.2, but that some power of P, P^k (for some $k \geq 1$), or its a -resolvent

$$R_a = (1 - a) \sum_{k \geq 0} a^k P^k$$

(for some $0 < a < 1$), does. Since

$$\text{Inv}(R_a) = \text{Inv}(P) \subset \text{Inv}(P^k),$$

the conclusions of the previous theorems remain valid in these cases.

The next theorem illustrates this idea. Let P be a Feller Markov kernel which doesn't necessarily satisfy the standing assumption. A point $p \in M$ is called *accessible* (for P) if for every neighbourhood U of p and every $x \in M$, $R_a(x, U) > 0$ (for some, hence all $0 < a < 1$). The set of points which are accessible for P is the, possibly empty, compact set

$$\Gamma_P = \bigcap_{x \in M} \text{supp}(R_a(x, \cdot)).$$

Point p is called a *weak Doeblin* point, if there exists a neighbourhood V of p , a non-trivial measure $\pi \in \mathcal{M}(M)$ and $0 < a < 1$, such that $R_a(x, dy) \geq \pi(dy)$ for all $x \in V$. The measure π is called a *minorizing* measure.

Theorem 2.11 *Let $\mathcal{C}(M)$ be a convex cone stable by monotone convergence. Suppose that $\mathcal{C}(M)P \subset \mathcal{C}(M)$ and that P possesses an accessible weak Doeblin point with a minorizing measure $\pi \in \mathcal{C}(M)$. Then P has a unique invariant probability measure Π and $\Pi \in \mathcal{C}(M)$.*

Proof: By assumption, there exists an open set V such that $R_a(x, dy) \geq \pi(dy)$ for all $x \in V$ and $R_a(x, V) > 0$ for all $x \in M$. Thus, by the Feller continuity of P (hence R_a), compactness and the Portmanteau theorem, $R_a(x, V) \geq \delta > 0$ for all $x \in M$ and some $\delta > 0$. It follows that $R_a^2(x, dy) \geq \delta\pi(dy)$. By Theorem 2.6 and Example 2.4 applied to R_a^2 , we get that $\text{Inv}(P) \subset \text{Inv}(R_a^2) = \{\Pi\} \subset \mathcal{C}(M)$. \square

3 Random maps

We suppose here that M is a compact d -dimensional connected Riemannian manifold. For $k \geq 0$, we let $C^k(M)$ denote the space of C^k functions $\rho : M \rightarrow \mathbb{R}$, equipped with the C^k topology (see e.g [18], Chapter 2). We let $\|\cdot\|_{C^k(M)}$ denote a norm on $C^k(M)$ making $C^k(M)$ a Banach space. We let $C^k(M, M)$ be the space of C^k maps from M into itself, equipped with the C^k topology and associated Borel σ -field.

We now let $r \geq 1$, and let ν be a probability measure on $C^r(M, M)$. Consider the chain on M induced by the random iterative system

$$X_{k+1} = \varphi_{k+1}(X_k),$$

where $(\varphi_k)_{k \in \mathbb{N}}$ is a family of i.i.d random variables, independent of X_0 , having distribution ν .

The kernel of this chain can then be written

$$P^\nu f(x) = \int_{C^r(M, M)} f(\varphi(x)) \nu(d\varphi), \quad (1)$$

and is clearly Feller. For further reference, we call this kernel the *kernel induced by ν* .

Throughout this section we shall take $P := P^\nu$, and assume that ν may be written as

$$\nu := (1 - a)\nu_0 + a\nu_1,$$

whereby ν_0, ν_1 are two probability measures over $C^r(M, M)$ and $0 < a < 1$. Thus we can write $P = Q + \Delta$ with $Q = (1 - a)P^{\nu_0}$ and $\Delta = aP^{\nu_1}$, where P^{ν_0}, P^{ν_1} are defined like P^ν with ν_0, ν_1 in place of ν . We furthermore assume that ν_0, ν_1 satisfy the following hypotheses 3.1 and 3.3 below. These are natural hypotheses ensuring that the standing assumption, Assumption 2.2,

holds true with $\mathcal{C}(M)$ being one of the sets $\mathcal{M}_{ac}(M)$, $\mathcal{M}_{ac}^{ls}(M)$ or $\mathcal{M}_{ac}^{r-1}(M)$ as defined in Example 2.1. To be concise, Assumption 3.1 assumes that ν_0 is the image measure of a finite dimensional C^r density by a submersion, while Assumption 3.3 assumes that ν_1 is supported by local diffeomorphisms.

Assumption 3.1 (Standing assumption 1 for RDS) *There exist $n \geq d$, a smooth n -dimensional manifold Θ with smooth Riemann measure $d\theta$, a C^r probability density function $h_0 : \Theta \rightarrow \mathbb{R}_+$ with compact support $\text{supp}_0(h)$, and a C^r map*

$$\begin{aligned} \Phi &: M \times \Theta \mapsto M, \\ (x, \theta) &\rightarrow \Phi(x, \theta) = \Phi_\theta(x) \end{aligned}$$

such that:

(i) ν_0 is the image measure of $h_0(\theta)d\theta$ by the map $\theta \rightarrow \Phi_\theta$. That is

$$P^{\nu_0}(f)(x) = \int_{\Theta} f(\Phi_\theta(x))h_0(\theta)d\theta.$$

(ii) For all $x \in M$ and $\theta \in \text{supp}(h_0)$ $\partial_\theta \Phi(x, \theta) : T_\theta \Theta \mapsto T_{\Phi(x, \theta)} M$ is surjective.

Proposition 3.2 *Assume Assumption 3.1. Then, there exists a C^r map $q : M \times M \mapsto \mathbb{R}_+$ such that*

$$P^{\nu_0}(x, dy) = q(x, y)m(dy).$$

In particular, $\mathcal{M}(M)Q \subset \mathcal{M}_{ac}^r(M)$.

Proof: We assume for notational convenience that $\Theta = \mathbb{R}^n$, but the proof easily extends to the general case.

Claim: For all $x^* \in M$ and $\theta^* \in \text{supp}(h_0)$, there exist neighbourhoods $U(= U(x^*, \theta^*))$ of x^* and $V(= V(x^*, \theta^*))$ of θ^* such that for every C^r function $\eta : \mathbb{R}^n \mapsto \mathbb{R}$ with compact support $\text{supp}(\eta) \subset V$, there exists a C^r map $q_\eta : M \times M \rightarrow \mathbb{R}_+$ with the property that

$$\int_{\mathbb{R}^n} f(\Phi(x, \theta))h(\theta)\eta(\theta)d\theta = \int_M q_\eta(x, y)f(y)m(dy)$$

for all $x \in U$, and $f \in B(M)$.

We assume for the time being that the claim is proven. Fix $x^* \in M$. We extract from the family $\{V(x^*, \theta^*), \theta^* \in \text{supp}(h_0)\}$ a covering of $\text{supp}(h_0)$ by open sets $V_i = V(x^*, \theta_i), i \in I$, with I finite. Set $U = \bigcap_{i \in I} U(x^*, \theta_i)$. Using a partition of unity subordinate to $\{V_i\}_{i \in I}$, h_0 can be written as $h_0 = \sum_{i \in I} h_0 \eta_i$ where η_i is smooth with compact support in V_i , $0 \leq \eta_i$, and $\sum_{i \in I} \eta_i = 1$. It then follows from the claim that for all $x \in U$,

$$P^{\nu_0}(x, dy) = \sum_{i \in I} q_i(x, y) m(dy),$$

where $q_i : M \times M \rightarrow \mathbb{R}_+$ is C^r . This proves the lemma.

Proof of the claim: After a permutation of the canonical basis of \mathbb{R}^n we can assume that $\theta = (\theta_1, \theta_2) \in \mathbb{R}^d \times \mathbb{R}^{n-d}$ where $\partial_{\theta_1} \Phi(x^*, \theta^*)$ has rank d . Thus, by the inverse function theorem, there exist open neighbourhoods U' of x^* and $V = V_1 \times V_2$ of $\theta^* = (\theta_1^*, \theta_2^*)$ such that the map

$$H : (\theta_1, \theta_2, x) \rightarrow (\Phi(x, \theta), \theta_2, x)$$

is a C^r diffeomorphism from $V \times U'$ onto its image $W = H(V \times U')$. Its inverse is then given by $(y, \theta_2, x) \rightarrow (\psi(y, \theta_2, x), \theta_2, x)$, where $\psi : W \mapsto V_1$ is C^r .

Let U be a relatively compact neighbourhood of x^* with $\bar{U} \subset U'$, and let $\eta : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a C^r function with compact support $\text{supp}(\eta) \subset V$. Set $K = \text{supp}(\eta) \times \bar{U}$ and let $\tilde{k}(x, y, \theta_2)$ be a C^r function which coincides with

$$(\eta h_0)(\psi(y, \theta_2, x), \theta_2) |\det \partial_y \psi(y, \theta_2, x)|$$

on $H(K)$ and is zero outside W . We define $q_\eta : M \times M \mapsto \mathbb{R}_+$ by

$$q_\eta(x, y) = \int \tilde{k}(x, y, \theta_2) d\theta_2.$$

Then q_η is C^r and by the change of variable formula,

$$\int f(\Phi(x, \theta)) (\eta h_0)(\theta) g(x) d\theta m(dx) = \int q_\eta(x, y) g(x) f(y) m(dx) m(dy)$$

for every continuous function g with support contained in U . This proves the claim. \square

We define $\text{Diff}_{\text{loc}}^r(M) \subset C^r(M, M)$ to be the (open) set of maps $\varphi \in C^r(M, M)$ for which $D\varphi(x) : T_x M \mapsto T_{\varphi(x)} M$ is invertible at every point $x \in M$.

We let $\varphi \in \text{Diff}_{\text{loc}}^r(M)$. Since the space M is connected, $\varphi^{-1}(y)$ is nonempty, finite, and its cardinality doesn't depend on y for all $y \in M$. We denote this cardinality by $\text{deg}(\varphi)$.

We let $J(\varphi, x) > 0$ denote the *Jacobian* of φ at x with respect to m . If the tangent spaces $T_x M$ and $T_{\varphi(x)} M$ are equipped with orthonormal bases, then

$$J(\varphi, x) = |\det D\varphi(x)|.$$

The *transfer* or *Ruelle-Perron-Frobenius* operator induced by φ is the operator \mathcal{L}_φ acting on $L^1(m)$ or $C^{r-1}(M)$, defined by

$$\mathcal{L}_\varphi(\rho)(y) = \sum_{\{x \in \varphi^{-1}(y)\}} \frac{\rho(x)}{J(\varphi, x)}. \quad (2)$$

The fact that $\mathcal{L}_\varphi(\rho)$ maps $C^{r-1}(M)$ into itself easily follows from the inverse function theorem. Indeed, for all $y \in M$, there exist an open neighbourhood U of y and C^r diffeomorphisms $\psi_i : U \mapsto \psi_i(U)$, $i = 1, \dots, \text{deg}(\varphi)$ such that for all $z \in U$,

$$\mathcal{L}_\varphi(\rho)(z) = \sum_{i=1}^{\text{deg}(\varphi)} \frac{\rho(\psi_i(z))}{J(\varphi, \psi_i(z))}.$$

This expression also shows that \mathcal{L}_φ is a bounded operator on $C^{r-1}(M)$. We let

$$\|\mathcal{L}_\varphi\|_{C^{r-1}(M)} = \sup_{\{\rho : \|\rho\|_{C^{r-1}(M)} \leq 1\}} \|\mathcal{L}_\varphi(\rho)\|_{C^{r-1}(M)}$$

denote its operator norm.

For $0 \leq k \leq r - 1$, we let

$$\mathcal{R}(\mathcal{L}_\varphi, C^k(M)) = \lim_{n \rightarrow \infty} \|(\mathcal{L}_\varphi)^n\|_{C^k(M)}^{1/n} \quad (3)$$

be the spectral radius of \mathcal{L}_φ on $C^k(M)$.

Assumption 3.3 (Standing assumption 2 for RDS)

$$\nu_1(\text{Diff}_{\text{loc}}^r(M)) = 1.$$

Proposition 3.4 *Assume Assumption 3.3. If $\mu \in \mathcal{M}_{ac}(M)$ has density ρ , then $\mu P^{\nu_1} \in \mathcal{M}_{ac}(M)$ and its density is given by*

$$y \rightarrow \mathcal{L}_{\nu_1}(\rho)(y) := \int_{\text{Diff}_{loc}^r(M)} (\mathcal{L}_\varphi \rho)(y) \nu_1(d\varphi).$$

This density is lower semi-continuous whenever ρ is. In particular $\mathcal{C}(M)\Delta \subset \mathcal{C}(M)$ with $\mathcal{C}(M) = \mathcal{M}_{ac}^{ls}(M)$.

If in addition

$$\int_{\text{Diff}_{loc}^r(M)} \|\mathcal{L}_\varphi\|_{C^{r-1}(M)} \nu_1(d\varphi) < \infty,$$

then \mathcal{L}_{ν_1} is a bounded operator on $C^{r-1}(M)$ and

$$\|\mathcal{L}_{\nu_1}\|_{C^{r-1}(M)} \leq \int_{\text{Diff}_{loc}^r(M)} \|\mathcal{L}_\varphi\|_{C^{r-1}(M)} \nu_1(d\varphi).$$

In particular $\mathcal{C}(M)\Delta \subset \mathcal{C}(M)$ with $\mathcal{C}(M) = \mathcal{M}_{ac}^{r-1}(M)$.

Proof: For all $f \in B(M)$,

$$\begin{aligned} \int_M P^{\nu_1}(f)(x) \rho(x) m(dx) &= \int_M \left(\int_{\text{Diff}_{loc}^r(M)} f(\varphi(x)) \nu_1(d\varphi) \right) \rho(x) m(dx) \\ &= \int_{\text{Diff}_{loc}^r(M)} \left(\int_M f(\varphi(x)) \rho(x) m(dx) \right) \nu_1(d\varphi) \\ &= \int_{\text{Diff}_{loc}^r(M)} \left(\int_M f(x) \mathcal{L}_\varphi(\rho)(x) m(dx) \right) \nu_1(d\varphi) \\ &= \int_M f(x) \left(\int_{\text{Diff}_{loc}^r(M)} (\mathcal{L}_\varphi \rho)(x) \nu_1(d\varphi) \right) \rho(x) m(dx). \end{aligned}$$

The second and last equalities follow from Fubini's theorem, and the third one follows from the change of variable formula. This proves the first assertion. If ρ is lower semi-continuous, so is $\mathcal{L}_\varphi \rho$. Thus, if $y_n \rightarrow y$,

$$\liminf_{n \rightarrow \infty} \int \mathcal{L}_\varphi \rho(y_n) \nu_1(d\varphi) \geq \int \liminf_{n \rightarrow \infty} \mathcal{L}_\varphi \rho(y_n) \nu_1(d\varphi) \geq \int \mathcal{L}_\varphi \rho(y) \nu_1(d\varphi)$$

by Fatou's Lemma. This shows that $\frac{d\mu P^{\nu_1}}{dm}$ is lower-semicontinuous.

We now prove the last statement. For all $\rho \in C^{r-1}(M)$, the mapping $\mathcal{L}_{(\cdot)}\rho : \text{Diff}_{\text{loc}}^r(M) \rightarrow C^{r-1}(M)$, $\varphi \mapsto \mathcal{L}_\varphi\rho$ is continuous, hence measurable. It is then Bochner measurable (see [16], Theorem 2, Section 1, Chapter 2) and the condition that $\int_{\text{Diff}_{\text{loc}}^r(M)} \|\mathcal{L}_\varphi(\rho)\|_{C^{r-1}(M)} \nu_1(d\varphi) < \infty$ makes it Bochner integrable ([16], Theorem 2, Section 2, Chapter 2). Properties of Bochner integrals ([16], Theorem 4, Section 2, Chapter 2) imply that

$$\left\| \int_{\text{Diff}_{\text{loc}}^r(M)} \mathcal{L}_\varphi(\rho) \nu_1(d\varphi) \right\|_{C^{r-1}(M)} \leq \int_{\text{Diff}_{\text{loc}}^r(M)} \|\mathcal{L}_\varphi(\rho)\|_{C^{r-1}(M)} \nu_1(d\varphi).$$

This concludes the proof. \square

We recall that $P = P^\nu$ is given by (1). Corollary 2.7 and Theorem 2.8 applied to the present setting, combined with Propositions 3.2 and 3.4, imply the following.

Theorem 3.5 *Assume Hypotheses 3.1 and 3.3. Then, $\text{Inv}(P) \subset \mathcal{M}_{ac}^{ls}(M)$. If $\mu \in \text{Inv}(P)$ has full support, then $\text{Inv}(P) = \{\mu\}$.*

Theorem 3.6 *Assume Hypotheses 3.1 and 3.3. If*

$$\int_{\text{Diff}_{\text{loc}}^r(M)} \|\mathcal{L}_\varphi\|_{C^{r-1}(M)} \nu_1(d\varphi) < \infty,$$

and $1/a$ is in the resolvent set of \mathcal{L}_{ν_1} (on $C^{r-1}(M)$), then $\text{Inv}(P) \subset \mathcal{M}_{ac}^{r-1}(M)$.

3.1 Expansion volume rates and spectral radius

In this subsection and the following, we consider the case where

$$\nu_1 = \delta_\phi$$

for some $\phi \in \text{Diff}_{\text{loc}}^r(M)$, $r \geq 1$, so that \mathcal{L}_{ν_1} is the transfer operator \mathcal{L}_ϕ . When ϕ is an *expanding* map (see the definition below), the spectral properties of \mathcal{L}_ϕ have been well understood since the seminal work of Ruelle [23]. We refer the reader to the excellent monograph [5] for a comprehensive introduction to the subject.

When ϕ is non-expanding, it is still possible to give simple sufficient conditions ensuring that $\frac{1}{a}$ lies in the resolvent of \mathcal{L}_ϕ , so that Theorem 3.6 applies.

This is the object of the next proposition, Proposition 3.10. Before stating this proposition we introduce certain quantities that will naturally appear in the estimate of the spectral radius of \mathcal{L}_ϕ : the *expansion rate* and the *expansion volume rates* of ϕ .

Let K be a nonempty, compact and forward invariant set (i.e $\phi(K) \subset K$). The *expansion constant* of ϕ at x is the positive number

$$EC(\phi, x) = \inf_{v \in T_x M, \|v\|_x=1} \|D\phi(x)v\|_{\phi(x)}.$$

Here $\|\cdot\|_x$ stands for the Riemannian norm on $T_x M$. Following Hirsch [19], define the (logarithmic) *expansion rate* of ϕ at K as

$$\mathcal{E}(\phi, K) = \lim_{n \rightarrow \infty} \frac{1}{n} \log(\min_{x \in K} EC(\phi^n, x)),$$

where the limit exists by subadditivity. The *expansion rate* of ϕ is defined as

$$\mathcal{E}(\phi) = \mathcal{E}(\phi, M).$$

We let $\text{Inv}(\phi)$ and $\text{Inv}_{\text{erg}}(\phi)$ respectively denote the set of invariant (respectively ergodic) probability measures for ϕ . By a theorem of Schreiber [24],

$$\mathcal{E}(\phi) = \inf_{\mu \in \text{Inv}_{\text{erg}}(\phi)} \Lambda_1(\mu), \quad (4)$$

where $\Lambda_1(\mu)$ stands for the smallest Lyapunov exponent for (ϕ, μ) .

For all $k \geq 0$, we analogously define the *k-expansion volume rate* of Φ at K as

$$\mathcal{EV}_k(\phi, K) = \lim_{n \rightarrow \infty} \frac{1}{n} (\min_{x \in K} [\log(J(\phi^n, x)) + k \log(EC(\phi^n, x))]),$$

and the *k-expansion volume rate* of Φ as

$$\mathcal{EV}_k(\phi) = \mathcal{EV}_k(\phi, M). \quad (5)$$

Again, these limits exist by subadditivity.

The following characterization easily follows from a beautiful result due to Schreiber [25] on the growth rates of sub-additive functions.

Proposition 3.7 *The k-expansion volume rate of Φ is given by*

$$\mathcal{EV}_k(\phi) = \inf_{\mu \in \text{Inv}_{\text{erg}}(\phi)} ((k+1)\Lambda_1(\mu) + \Lambda_2(\mu) + \dots + \Lambda_d(\mu)), \quad (6)$$

where $\Lambda_1(\mu) \leq \dots \leq \Lambda_d(\mu)$ are the Lyapunov exponents of (ϕ, μ) counted with their multiplicity.

Proof: Let $F : M \times \mathbb{N} \rightarrow \mathbb{R}$ be defined as

$$F(x, n) = -\log J(\phi^n, x) - kEC(\phi^n, x).$$

Then F is continuous in x and subadditive with respect to ϕ , meaning that

$$F(x, n+1) \leq F(x, n) + F(\phi(x), 1).$$

This directly follows from the properties $J(\phi^{n+1}, x) = J(\phi^n, \phi(x))J(\phi, x)$ and $EC(\phi^{n+1}, x) \geq EC(\phi^n, \phi(x))EC(\phi, x)$. Therefore, by Theorem 1 in [25],

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\sup_{x \in M} \frac{1}{n} F(x, n) \right) &= \inf_{n > 0} \left(\sup_{x \in M} \frac{1}{n} F(x, n) \right) \\ &= \sup_{\mu \in \text{Inv}_{erg}(\phi)} \inf_{n > 0} \frac{1}{n} \int_M F(n, x) \mu(dx). \end{aligned}$$

For all $\mu \in \text{Inv}_{erg}(\phi)$ we have that

$$\begin{aligned} &\frac{1}{n} \int_M F(n, x) \mu(dx) \\ &= -\frac{1}{n} \sum_{k=0}^{n-1} \int_M \log(J(\phi, \phi^k(x))) \mu(dx) - k \frac{1}{n} \int_M \log(EC(\phi^n, x)) \mu(dx) \\ &= -\int_M \log(J(\phi, x)) \mu(dx) - k \frac{1}{n} \int_M \log(EC(\phi^n, x)) \mu(dx). \end{aligned}$$

The first term on the right hand side is equal to $-(\Lambda_1(\mu) + \dots + \Lambda_d(\mu))$ by the multiplicative ergodic theorem, and the second term converges to $-k\Lambda_1(\mu)$. \square

Remark 3.8 We let

$$\omega_\phi(x) = \bigcap_{n \geq 0} \overline{\{\phi^k(x) : k \geq n\}}$$

be the *omega limit set* of x ,

$$BC(\phi) = \overline{\{x \in M : x \in \omega_\phi(x)\}}$$

the *Birkhoff center* of ϕ , and

$$MC(\phi) = \overline{\bigcup_{\mu \in \text{Inv}_{erg}(\phi)} \text{supp}(\mu)}$$

the *minimal center of attraction* of ϕ . By the Poincaré recurrence theorem (see e.g. [22], Chapter 1), $MC(\phi) \subset BC(\phi)$. Thus, equalities (4) and (6) imply that

$$\mathcal{E}(\phi) = \mathcal{E}(\phi, BC(\phi)) = \mathcal{E}(\phi, MC(\phi))$$

and

$$\mathcal{E}\mathcal{V}_k(\phi) = \mathcal{E}\mathcal{V}_k(\phi, BC(\phi)) = \mathcal{E}\mathcal{V}_k(\phi, MC(\phi)).$$

These properties prove to be useful to compute or estimate the expansion and expansion volume rates in certain cases (see Examples 3.17 and 3.18 below).

Remark 3.9 We have that

$$d\mathcal{E}(\phi) \leq \mathcal{E}\mathcal{V}_0(\phi) \leq \log(\deg(\phi)).$$

The first inequality follows from identities (4) and (6), while the second follows from the second statement in the next proposition.

Note that this has the consequence that

$$d\mathcal{E}(\phi) \leq \mathcal{E}\mathcal{V}_0(\phi) \leq 0$$

when ϕ is a diffeomorphism. Observe also that if $\mathcal{E}\mathcal{V}_0(\phi) \leq 0$, then $k \mapsto \mathcal{E}\mathcal{V}_k(\phi)$ is nonincreasing.

We recall (see equation (3)) that for all $0 \leq k \leq r-1$, $\mathcal{R}(\mathcal{L}_\phi, C^k(M))$ is the spectral radius of \mathcal{L}_ϕ on $C^k(M)$.

Proposition 3.10 *We have the following:*

- (i) if $\mathcal{E}(\phi) > 0$, then $\mathcal{R}(\mathcal{L}_\phi, C^{r-1}(M)) = 1$;
- (ii) if $\mathcal{E}(\phi) \leq 0$, then

$$1 \leq \mathcal{R}(\mathcal{L}_\phi, C^{r-1}(M)) \leq \deg(\phi) \max_{0 \leq k \leq r-1} e^{-\mathcal{E}\mathcal{V}_k(\phi)}.$$

Remark 3.11 The first assertion of this proposition is a direct consequence of the seminal work of Ruelle ([23]). Some details are given below.

Some of Ruelle's results have been extended by Campbell and Latushkin in [13] to the situation where ϕ is no longer expanding but is a covering map (i.e a local diffeomorphism as in the present setting). They compute the

essential spectral radius of the transfer operator and provide an upper bound for the spectral radius in $C^0(M)$ (in the present setting) given by

$$\begin{aligned} & \exp \left(\sup_{\mu \in \text{InVerg}(\phi)} \left[H(\mu) - \int_M \log(J(\phi, x)) \mu(dx) \right] \right) \\ &= \exp \left(- \inf_{\mu \in \text{InVerg}(\phi)} \left[(\Lambda_1(\mu) + \dots + \Lambda_d(\mu)) - H(\mu) \right] \right), \end{aligned} \quad (7)$$

where $H(\mu)$ is the measure-theoretic entropy of (ϕ, μ) . They claim (see [13, Theorem 1]) that this upper bound is also an upper bound for the spectral radius in $C^r(M)$ for $r \geq 1$. Although this result is true when ϕ is expanding, it cannot be true when ϕ is not expanding, as shown by the following simple example. The error in their proof comes from the fact that they rely on estimates (given in [23]) which are valid only for expanding maps.

The estimate given in Proposition 3.10, (ii), provides a correct estimate well-suited to non expanding maps.

Example 3.12 We take $M = S^1 = \mathbb{R}/\mathbb{Z}$, and suppose that ϕ is a smooth, orientation preserving diffeomorphism with two fixed points, 0 and $1/2$, such that ϕ coincides with

$$x \mapsto \frac{x}{\alpha}$$

on a neighbourhood of 0, whereby $\alpha > 1$ and $\phi'(1/2) > 1$. The ergodic measures of ϕ are the Dirac measures $\delta_0, \delta_{1/2}$, and for all $k \geq 0$,

$$\mathcal{E}\mathcal{V}_k(\phi) = -\ln(\alpha)(k+1) < 0.$$

Thus, by Proposition 3.10, $\mathcal{R}(\mathcal{L}_\phi, C^r(M)) \leq \alpha^{r+1}$ for all $0 \leq r < \infty$. We now let $\rho(x) = \sin(2\pi x)$ if r is odd, and $\rho(x) = \cos(2\pi x)$ if r is even. Then

$$\|(\mathcal{L}_{\phi^n}(\rho))\|_{C^r(M)} := \sum_{k=0}^r \|(\mathcal{L}_{\phi^n}(\rho))^{(k)}\|_0 \geq |(\mathcal{L}_{\phi^n}(\rho))^{(r)}(0)| = \alpha^{n(r+1)}.$$

This implies that

$$\mathcal{R}(\mathcal{L}_\phi, C^r(M)) = \alpha^{r+1} \quad \text{for all } 0 \leq r < \infty. \quad (8)$$

This simple example shows that the inequality in Proposition 3.10 can be an equality, for any r .

The measure-theoretic entropy for any Dirac mass is 0, whence we see that the Campbell-Latushkin upper bound in (7) is precisely α . However, the authors claim in [13, Theorem 1] that this same upper bound for the C^r spectral radius holds for all $0 \leq r < \infty$, which cannot be true for any $r \geq 1$ by (8).

Proof of Proposition 3.10

Step 1. If $\mathcal{E}(\phi) > 0$, then $\inf_{x \in M} EC(\phi^n, x) \geq \theta > 1$ for some $n \geq 1$ and some $\theta > 1$. Thus, replacing ϕ by ϕ^n , we can assume that $EC(\phi, x) \geq \theta > 1$. This conditions means that ϕ is *expanding*. Then, by a theorem due to Ruelle [23], Theorem 3.6 (ii), (see also [5], Theorem 2.6) $R = \mathcal{R}(\mathcal{L}_\phi, C^{r-1}(M))$ is an eigenvalue of \mathcal{L}_ϕ associated to a positive eigenfunction ρ . Since $\int_M \rho dm = \int_M (\mathcal{L}_\phi \rho) dm$, R must be 1. This proves the first assertion.

Step 2. We now prove the left hand side inequality of assertion (ii). Suppose for the sake of contradiction that $\mathcal{R}(\mathcal{L}_\phi, C^{r-1}(M)) < 1$. Then

$$\lim_{n \rightarrow \infty} \|\mathcal{L}_\phi^n\|_{C^{r-1}(M)} = 0,$$

so that $\lim_{n \rightarrow \infty} \|\mathcal{L}_\phi^n 1\|_0 = 0$ in particular. On the other hand, $\int_M \mathcal{L}_\phi^n 1 dm = \int_M 1 dm = m(M) > 0$. This is a contradiction.

Step 3. Our last goal is to prove the right hand side inequality of assertion (ii). It is first convenient to specify a norm on $C^k(M)$, $k \geq 0$.

Throughout, \mathbb{R}^d is equipped with the Euclidean norm. For all $k \geq 1$, let $L_{sym}^k(\mathbb{R}^d)$ be the vector space of k -linear symmetric forms on \mathbb{R}^d . If $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a linear map and $L \in L_{sym}^k(\mathbb{R}^d)$, $A^*L \in L_{sym}^k(\mathbb{R}^d)$ is defined by $A^*L(u_1, \dots, u_k) = L(Au_1, \dots, Au_k)$. The norm of $L \in L_{sym}^k(\mathbb{R}^d)$ is defined as $\|L\| = \sup\{|L(u_1, \dots, u_k)| : u_i \in \mathbb{R}^d, \|u_i\| \leq 1\}$.

We consider $U \subset \mathbb{R}^d$ open and $f \in C^k(U) := \{f : U \rightarrow \mathbb{R}, C^k\}$. The k -th derivative of f is a continuous mapping $D^k f : U \rightarrow L_{sym}^k(\mathbb{R}^d)$. The following lemma will be used below. It follows from classical rules in differential calculus. Its verification is left to the reader.

Lemma 3.13 *Let $k \geq 1$, and U, V open subsets of \mathbb{R}^d .*

(i) *Let $g \in C^k(U)$. There exists a continuous function $C_g : U \rightarrow \mathbb{R}_+$ (depending on g and k) such that for all $f \in C^k(U)$ and $x \in U$,*

$$\|D^k(gf)(x) - g(x)D^k f(x)\| \leq C_g(x) \left(|f(x)| + \sum_{i=1}^{k-1} \|D^i f(x)\| \right).$$

(ii) *Let $\Psi : U \rightarrow V$ be a C^k map. There exists a continuous function $C'_\Psi : U \rightarrow \mathbb{R}_+$ (depending on Ψ and k) such that for all $f \in C^k(V)$ and*

$x \in U$,

$$\begin{aligned} & \|D^k(f \circ \Psi)(x) - D\Psi(x)^* D^k f(\Psi(x))\| \\ & \leq C'_\Psi(x) \left(|f(\Psi(x))| + \sum_{i=1}^{k-1} \|D^i f(\Psi(x))\| \right). \end{aligned}$$

We now define a norm on $C^k(M)$. Let W be the open ball in \mathbb{R}^d centered at the origin with radius 2 and let V be the open ball centered at the origin with radius 1.

By the compactness of M there exists an atlas $\{\alpha, \mathcal{O}_\alpha\}_{\alpha \in \aleph}$ with \aleph finite such that:

- (i) α maps \mathcal{O}_α diffeomorphically onto an open set in \mathbb{R}^d containing \overline{W} ;
- (ii) the open sets $\mathcal{O}'_\alpha = \alpha^{-1}(V)$, $\alpha \in \aleph$, cover M .

If $\rho \in C^k(M)$ and $1 \leq j \leq k$, we set

$$|\rho|_j = \sup_{\alpha \in \aleph, x \in \overline{V}} \|D^j(\rho \circ \alpha^{-1})(x)\|$$

and

$$\|\rho\|_k = \|\rho\|_0 + \sum_{j=1}^k |\rho|_j.$$

It is not hard to verify that $\|\cdot\|_k$ is a norm on $C^k(M)$ inducing the C^k topology.

Lemma 3.14 *Let $k \geq 1$ and let $L : C^k(M) \rightarrow C^k(M)$ be a bounded operator. Suppose that there exist sequences $(a_n)_{n \geq 0}$, $(b_n)_{n \geq 0}$, $a_n \geq 0$, $b_n \geq 0$ such that for all $n \geq 0$ and $\rho \in C^k(M)$,*

$$|L^n \rho|_k \leq a_n |\rho|_k + b_n \|\rho\|_{k-1}.$$

Then

$$\mathcal{R}(L, C^k(M)) \leq \max \left(\mathcal{R}(L, C^{k-1}(M)), \limsup_{n \rightarrow \infty} a_n^{1/n} \right).$$

Proof: For all $\delta > 0$, we set

$$\|\rho\|_{k,\delta} = \|\rho\|_{k-1} + \delta |\rho|_k.$$

Note that $\|\rho\|_{k,\delta}$ and $\|\rho\|_k$ are equivalent norms. In particular we have

$$\mathcal{R}(L, C^k(M)) = \lim_{n \rightarrow \infty} \|L^n\|_{k,\delta}^{1/n} \leq \|L\|_{k,\delta}$$

for all $\delta > 0$.

We now fix $A > \limsup_{n \rightarrow \infty} a_n^{1/n}$ and $R > \mathcal{R}(L, C^{k-1}(M))$. Then, for some $n \geq 0$ sufficiently large and all $\delta > 0$,

$$\begin{aligned} \|L^n \rho\|_{k,\delta} &\leq \|L^n \rho\|_{k-1} + \delta[a_n |\rho|_k + b_n \|\rho\|_{k-1}] \\ &\leq R^n \|\rho\|_{k-1} + \delta[A^n |\rho|_k + b_n \|\rho\|_{k-1}] \\ &\leq \max(R^n + \delta b_n, A^n) \|\rho\|_{k,\delta}. \end{aligned}$$

Thus

$$\mathcal{R}(L^n, C^k(M)) \leq \|L^n\|_{k,\delta} \leq \max(R^n + \delta b_n, A^n).$$

Since $\delta > 0$ is arbitrary, this shows that

$$\mathcal{R}(L^n, C^k(M)) \leq \max(R^n, A^n).$$

Thus,

$$\mathcal{R}(L, C^k(M)) = \mathcal{R}(L^n, C^k(M))^{1/n} \leq \max(A, R).$$

This concludes the proof. \square

Lemma 3.15 (i) $\mathcal{R}(\mathcal{L}_\phi, C^0(M)) \leq \deg(\phi)e^{-\mathcal{E}\mathcal{V}_0(\phi)}$;

(ii) For all $1 \leq k \leq r-1$, \mathcal{L}_ϕ satisfies the assumptions of Lemma 3.14 with

$$\limsup_{n \rightarrow \infty} a_n^{1/n} \leq \deg(\phi)e^{-\mathcal{E}\mathcal{V}_k(\phi)}$$

Proof: Throughout this proof we set

$$j_\phi(x) = \frac{1}{J(\phi, x)}.$$

(i) : By the definition of \mathcal{L}_ϕ ,

$$\|\mathcal{L}_\phi(\rho)\|_0 \leq \deg(\phi) \sup_{x \in M} j_\phi(x) \|\rho\|_0$$

for all $\rho \in C_0(M)$. Thus, replacing ϕ by ϕ^n , we obtain that

$$\|\mathcal{L}_\phi^n(\rho)\|_0 \leq \mathbf{deg}(\phi)^n \sup_{x \in M} j_{\phi^n}(x) \|\rho\|_0,$$

whence the result follows from the definition of $\mathcal{E}\mathcal{V}_0(\phi)$.

(ii) : To shorten notation we firstly consider the case where $\mathbf{deg}(\phi) = 1$, so that ϕ is a diffeomorphism with inverse ψ . Then $\mathcal{L}_\phi(\rho) = (\rho \circ \psi)(j_\phi \circ \psi)$. Our first goal is to bound

$$|\mathcal{L}_\phi(\rho)|_k = \sup_{x \in \bar{V}, \alpha \in \aleph} \|D^k(\mathcal{L}_\phi(\rho) \circ \alpha^{-1})(x)\|.$$

We let $\alpha \in \aleph$ and $\bar{x} \in \bar{V}$, and choose $\beta \in \aleph$ such that $\psi(\alpha^{-1}(\bar{x})) \in \mathcal{O}'_\beta$ (recall that the family $\{U'_\beta\}$ cover M).

Set $U = \alpha(\psi^{-1}(\mathcal{O}'_\beta) \cap \mathcal{O}_\alpha)$, $f = \rho \circ \beta^{-1} : V \rightarrow \mathbb{R}$, $g = j_\phi \circ \psi \circ \alpha^{-1} : U \rightarrow \mathbb{R}$ and $\Psi = \beta \circ \psi \circ \alpha^{-1} : U \rightarrow V$. Then on U we have

$$\mathcal{L}_\phi(\rho) \circ \alpha^{-1} = (f \circ \Psi)g.$$

Hence, relying on Lemma 3.13, one can find a smaller neighbourhood of \bar{x} , $U_{\bar{x}} \subset U$ and a constant $C(\phi, \bar{x})$ (depending on ϕ and \bar{x}) such that for all $x \in U_{\bar{x}}$

$$\begin{aligned} & \|D^k(\mathcal{L}_\phi(\rho) \circ \alpha^{-1})(x) - g(x)D\Psi(x)^*D^k f(x)\| \\ & \leq C(\phi, \bar{x}) \left(|f(\Psi(x))| + \sum_{i=1}^{k-1} \|Df^i(\Psi(x))\| \right) \\ & \leq C(\phi, \bar{x}) \|\rho\|_{k-1}. \end{aligned}$$

We take constants $0 < c, c' < \infty$ (depending only upon the atlas $\{\alpha, \mathcal{O}_\alpha\}$) such that for all $\alpha \in \aleph, x \in \alpha^{-1}(\bar{W})$ and $u \in T_x M$ we have

$$c' \|u\|_x \leq \|D\alpha(x)u\| \leq c \|u\|_x.$$

Thus, defining $c'' = c/c'$, for all $x \in U_{\bar{x}}$ we have that

$$\begin{aligned} & \|g(x)D\Psi(x)^*D^k f(x)\| \leq g(x) \|D^k f(x)\| \|D\Psi(x)\|^k \\ & \leq c'' g(x) \|D^k f(x)\| \|D\psi(\alpha^{-1}(x))\|_{\alpha^{-1}(x)}^k \\ & = c'' j_\phi(\psi \circ \alpha^{-1}(x)) \|D^k f(x)\| EC(\phi, \psi \circ \alpha^{-1}(x))^{-k} \\ & \leq c'' \|D^k f(x)\| \sup_{y \in M} j(\phi, y) EC(\phi, y)^{-k}. \end{aligned}$$

Finally, since \bar{V} can be covered by finitely many neighbourhoods of the form $U_{\bar{x}}$, we obtain that

$$|\mathcal{L}_\phi(\rho)|_k \leq c'' |\rho|_k \sup_{y \in M} \left[j(\phi, y) EC(\phi, y)^{-k} \right] + c_\phi \|f\|_{k-1},$$

where c'' depends only upon the atlas $\{\alpha, \mathcal{O}_\alpha\}$ and c_ϕ depends on ϕ . Replacing ϕ by ϕ^n gives

$$|\mathcal{L}_{\phi^n}(\rho)|_k \leq c'' |\rho|_k \sup_{y \in M} \left[j(\phi^n, y) EC(\phi^n, y)^{-k} \right] + c_{\phi^n} \|f\|_{k-1}.$$

This proves the desired result.

The proof for $\deg(\phi) > 1$ is similar, with the inverse of ϕ be replaced by the $\deg(\phi)$ local inverses. \square

The proof of the right hand side inequality of Proposition 3.10 (ii) now easily follows from lemmas 3.14 and 3.15.

3.2 Application to random maps

We recall that $P = P^\nu$, as defined in the beginning of the present section.

Theorem 3.16 *We assume Assumption 3.1 and that $\nu_1 = \delta_\phi$ for some $\phi \in \text{Diff}_{\text{loc}}^r(M)$.*

(i) *If $\mathcal{E}(\phi) > 0$, then $\text{Inv}(P) \subset \mathcal{M}_{ac}^{r-1}(M)$ for all $a < 1$.*

(ii) *If $\mathcal{E}(\phi) \leq 0$, then $\text{Inv}(P) \subset \mathcal{M}_{ac}^{r-1}(M)$ for all $a < \min_{k=0, \dots, r-1} \frac{e^{\mathcal{E}\mathcal{V}_k(\phi)}}{\deg(\phi)}$.*

Proof: Theorem 3.16 follows from Theorem 3.6 and Proposition 3.10 \square

As an illustration of this last result, consider two examples where ϕ is a diffeomorphism, so that $\min_{k=0, \dots, r-1} \frac{e^{\mathcal{E}\mathcal{V}_k(\phi)}}{\deg(\phi)} = e^{\mathcal{E}\mathcal{V}_{r-1}(\phi)}$, and where $\mathcal{E}\mathcal{V}_{r-1}(\phi)$ can be easily expressed.

Example 3.17 Suppose that ϕ is a C^r diffeomorphism on M such that for all $x \in M$,

$$\omega_\phi(x) \subset \text{Fix}(\phi) := \{p \in M : \phi(p) = p\}.$$

One can, for instance, imagine that $\phi = \Phi^1$ is the time one map of a flow $\{\Phi^t\}$ induced by a C^r , $r \geq 1$, gradient vector field $F = -\nabla V$ on M (or more generally a vector field having a strict Lyapounov function).

Here $BC(\phi) = \text{Fix}(\phi)$, so that by Remark 3.8,

$$\mathcal{E}\mathcal{V}_{r-1}(\phi) = \mathcal{E}\mathcal{V}_{r-1}(\phi, \text{Fix}(\phi)) = \inf_{p \in \text{Fix}(\phi)} \log(J(\phi, p)) + (r-1)\Lambda_1(p)$$

and

$$\mathcal{E}(\phi) = \mathcal{E}(\phi, \text{Fix}(\phi)) = \inf_{p \in \text{Fix}(F)} \Lambda_1(p).$$

Here

$$J(\phi, p) = \log(|\det D\phi(p)|)$$

and

$$\Lambda_1(p) = \min\{\log(|z|) : z \text{ is an eigenvalue of } D\phi(p)\}.$$

Note that, in case ϕ is the time one map of the flow induced by $F = -\nabla V$, $\text{Fix}(\phi) = \text{Eq}(F) = F^{-1}(0)$, $J(\phi, p) = \text{div}_p(F) = -\Delta V(p)$ and $\Lambda_1(p)$ is the smallest eigenvalue of the Hessian of $-V$ at p .

Example 3.18 We suppose here that $M = S^2$ and that $\phi = \Phi^1$ where $\{\Phi^t\}$ is induced by a C^r vector field F . We no longer assume that F is gradient-like but will assume that $\text{Eq}(F)$ is finite.

If $p \in \text{Eq}(F)$ we let

$$\Lambda_1(p) \leq \Lambda_2(p)$$

denote the real part of the eigenvalues of $DF(p)$. Note that

$$\text{div}_p(F) = \Lambda_1(p) + \Lambda_2(p).$$

Given $T > 0$, a T -periodic orbit is an orbit $\gamma = \{\Phi^t(p), t \in \mathbb{R}\}$ such that $\Phi^T(p) = p$ and $\Phi^t(p) \neq p$ for all $0 < t < T$. We let $\text{Per}_T(F)$ denote the set of such orbits and $\text{Per}(F) = \cup_{T>0} \text{Per}_T(F)$.

If $\gamma \in \text{Per}_T(F)$ and $p \in \gamma$, $D\Phi^T(p)$ has two (possibly equal) eigenvalues (that depend only of γ): 1 (corresponding to eigenvector $F(p)$) and $J(\Phi^T, p)$. We let

$$\{\Lambda_1(\gamma), \Lambda_2(\gamma)\} = \left\{0, \frac{\log(J(\Phi^T, p))}{T}\right\}$$

denote the logarithms of these eigenvalues, with the convention that $\Lambda_1(\gamma) \leq \Lambda_2(\gamma)$. A periodic orbit, γ , is said to be *linearly stable* if $\Lambda_1(\gamma) < 0$. We let $\text{Per}_-(F)$ denote the set of linearly stable periodic orbits. Note that, although $\text{Per}(F)$ may be uncountable, $\text{Per}_-(F)$ is finite.

In the following lemma, Lemma 3.19, we implicitly identify an equilibrium point, p , with the orbit $\{p\} = \{\Phi^t(p) : t \in \mathbb{R}\}$. Again, combined with Theorem 3.16, this gives simple conditions on a ensuring the smoothness of invariant distributions.

Lemma 3.19 *Suppose that F has finitely many equilibria. Let μ be an ergodic probability measure for ϕ . Then $\int \log(J(\phi, x))\mu(dx) = \Lambda_1(\gamma) + \Lambda_2(\gamma)$ and $\Lambda_1(\mu) = \Lambda_1(\gamma)$ for some equilibrium or periodic orbit γ . In particular,*

$$\mathcal{E}\mathcal{V}_{r-1}(\phi) = \min_{\gamma \in \text{Eq}(F) \cup \text{Per}_-(F)} r\Lambda_1(\gamma) + \Lambda_2(\gamma)$$

and

$$\mathcal{E}(\phi) = \min_{\gamma \in \text{Eq}(F) \cup \text{Per}_-(F)} \Lambda_1(\gamma).$$

Proof: By the Poincaré recurrence theorem and Birkhoff ergodic theorem, there exists a set $\Omega \subset M$, with $\mu(\Omega) = 1$, such that $x \in \omega_\phi(x)$ (Poincaré) and $\frac{1}{n} \sum_{k=0}^{n-1} \delta_{\phi^k(x)} \Rightarrow \mu$ (Birkhoff) for all $x \in \Omega$. Here \Rightarrow stands for weak* convergence.

We take $p \in \Omega$. We claim that p is either a periodic point (i.e lies in a periodic orbit) or an equilibrium point for $\{\Phi^t\}$. Clearly $\omega_\phi(p) \subset \omega_{\{\Phi^t\}}(p)$, the omega limit set of p for $\{\Phi^t\}$. Such a set is *internally chain recurrent* for $\{\Phi^t\}$. Therefore, by a result proved in [9], Theorem 1.1, every point in $\omega_{\{\Phi^t\}}(p)$ is either periodic or belongs to an *orbit cycle*. An orbit cycle is a finite sequence $\Gamma = \gamma_1, \dots, \gamma_m$ of orbits such that the alpha limit set of γ_i (for $\{\Phi^t\}$) is an equilibrium e_{i-1} and its omega limit set is an equilibrium e_i , with $e_0 = e_m$. Therefore, because $p \in \omega_{\{\Phi^t\}}(p)$, p is either a periodic or an equilibrium point. This proves the claim.

If p is an equilibrium, then $\mu = \delta_p$, $\int \log(J(\phi, y))\mu(dy) = \text{div}_p(F) = \Lambda_1(p) + \Lambda_2(p)$, and $\Lambda_1(\mu) = \Lambda_1(p)$. If p is T -periodic for $\{\Phi^t\}$ and $T = N/K$ is rational, then $\mu = \frac{1}{N} \sum_{i=0}^{N-1} \delta_{\phi^i(p)}$ with $\phi^N(p) = \phi^{TK}(p) = p$. Thus

$$\begin{aligned} \int \log(J(\phi, y))\mu(dy) &= \frac{1}{TK} \log(J(\Phi^T, p)^K) \\ &= \frac{1}{T} \log(J(\Phi^T, p)) = \Lambda_1(\gamma) + \Lambda_2(\gamma). \end{aligned}$$

If T is irrational, then $\mu = \frac{1}{T} \int_0^T \delta_{\Phi^s(p)} ds$ and again we have that

$$\begin{aligned} \int \log(J(\phi, y))\mu(dy) &= \frac{1}{T} \int_0^T \log(J(\phi, \Phi^s(p))) ds \\ &= \frac{1}{T} \int_0^T \int_0^1 \text{Tr}(DF(\Phi^{s+u}(p))) du ds = \frac{1}{T} \int_0^1 \int_0^T \text{Tr}(DF(\Phi^{s+u}(p))) du ds \\ &= \frac{1}{T} \int_0^T \text{Tr}(DF(\Phi^u(p))) du = \Lambda_1(\gamma) + \Lambda_1(\gamma). \end{aligned}$$

□

4 Piecewise deterministic Markov processes

We let E be a finite set and $\{F_i\}_{i \in E}$ be a family of C^r ($r \geq 1$) vector fields on M where M is, as before, a d -dimensional compact connected Riemannian manifold.

We set $\mathbf{M} = M \times E$. Then \mathbf{M} can be viewed as a d -dimensional compact manifold with $\text{card}(E)$ components. A map $g : \mathbf{M} \mapsto \mathbb{R}$ is C^k if $x \rightarrow g(x, i) = g_i(x)$ is C^k for all $i \in E$. A map $g : \mathbf{M} \mapsto \mathbb{R} \cup \{\infty\}$ is lower semi-continuous if g_i is lower semi-continuous for all $i \in E$. The Riemannian measure on \mathbf{M} is given by $\mathbf{m} = m \otimes \sum_{i \in E} \delta_i$, whereby m is the Riemannian measure on M . The sets $\mathcal{M}_{ac}(\mathbf{M})$, $\mathcal{M}_{ac}^{ls}(\mathbf{M})$ and $\mathcal{M}_{ac}^r(\mathbf{M})$ are defined accordingly.

We let $(Z_t = (X_t, I_t))_{t \geq 0}$ be a continuous time Feller Markov process living on \mathbf{M} whose infinitesimal generator \mathcal{A} acts on functions $g \in C^1(\mathbf{M})$ according to the formula

$$\mathcal{A}g(x, i) = \langle F_i(x), \nabla g_i(x) \rangle_x + \sum_{j \in E} \alpha_{ij}(x)(g_j(x) - g_i(x)),$$

whereby:

- (i) $\alpha_{ij}(x) \geq 0$ and (for convenience) $\alpha_{ii}(x) = 0$ for all $i, j \in E$;
- (ii) The matrix $(\alpha_{ij}(x))_{i, j \in E}$ is irreducible and C^{r-1} in x .

For further reference, we sometimes call the data $\{\{F_i\}_{i \in E}, (\alpha_{ij}(x))_{i, j \in E}\}$ the *characteristics* of $(Z_t)_{t \geq 0}$.

An alternative pathwise description of the process is as follows. The component $(X_t)_{t \geq 0}$ is a solution to the differential equation

$$\frac{dX_t}{dt} = F_{I_t}(X_t),$$

while $(I_t)_{t \geq 0}$ is a jump process whose jump rates depends on (X_t) ,

$$\mathbb{P}(I_{t+s} = j | \sigma(Z_u, u \leq t), I_t = i) = \alpha_{ij}(X_t)s + o(s).$$

In words, starting from (x, i) , X_t follows the ode induced by F_i and switches to the ode induced by F_j at rate $\alpha_{ij}(X_t)$. Then X_t follows the ode induced by F_j until it switches to the ode induced by F_k at rate $\alpha_{jk}(X_t)$, and so on.

This type of process falls under the broader category of *piecewise deterministic Markov processes*, introduced by Davis [15]. Their ergodic properties have been the focus of much attention in the last decade ([2], [7] [8], [1], [11], [3]).

4.1 A discrete kernel associated to $(Z_t)_{t \geq 0}$

In order to use the results of the preceding sections, we firstly introduce a (discrete time) Markov kernel P whose invariant distributions are linked to the invariant distributions of $(Z_t)_{t \geq 0}$.

We let $\{\Phi_i^t\}_{t \in \mathbb{R}}$ denote the flow induced by F_i . We fix $\alpha > 0$ sufficiently large so that

$$\sup_{x \in M} \sum_{j \in E} \alpha_{ij}(x) < \alpha. \quad (9)$$

Set $A_{ij}(x) = \frac{\alpha_{ij}(x)}{\alpha}$ for $i \neq j$ and $A_{ii}(x) = 1 - \sum_{j \neq i} A_{ij}(x)$. Let A, K and P be the Markov operators on \mathbf{M} respectively defined by

$$Ag(x, i) = \sum_j A_{ij}(x)g(x, j), \quad (10)$$

$$Kg(x, i) = \int_0^\infty \alpha e^{-\alpha t} g(\Phi_t^i(x), i) dt \quad (11)$$

and

$$P = KA \quad (12)$$

Remark 4.1 The Kernel P is the kernel of a discrete time chain $(X_n, I_n)_{n \geq 0}$ living on \mathbf{M} whose dynamics can be described as follows. Starting from $(x, i) \in \mathbf{M}$, we pick a random variable T having an exponential distribution with parameter α , and set $X_1 = \Phi_i^T(x)$. We then choose $I_1 = j$ with probability $A_{ij}(X_1)$.

Invariant distributions of the Markov kernel P and invariant distributions of the Markov process $(Z_t)_{t \geq 0}$ are linked by the following result proved in [7, Proposition 2.4 and Lemma 2.6].

Proposition 4.2 *We let $(Z_t)_{t \geq 0}$ be the piecewise-deterministic Markov process having characteristics $\{\{F_i\}_{i \in E}, (\alpha_{ij}(x))_{i,j \in E}\}$. The mapping $\mu \rightarrow \mu K$ maps homeomorphically $\text{Inv}(P)$ (respectively $\text{Inv}_{\text{erg}}(P)$, the set of ergodic probability measures of P) onto the set of invariant (respectively ergodic) probability measures for $(Z_t)_{t \geq 0}$. Its converse homeomorphism is given by $\mu \mapsto \mu A$. Moreover we have that $\text{supp}(\mu) = \text{supp}(\mu K)$ for all $\mu \in \text{Inv}(P)$.*

By Liouville's formula, the transfer operator of Φ_i^t (see Section 3) is given by

$$\mathcal{L}_{\Phi_i^t}(\rho)(x) = \rho(\Phi_i^{-t}(x)) \exp \left[- \int_0^t \text{div}(F_i)(\Phi_i^{-s}(x)) ds \right] \quad (13)$$

for $\rho \in L^1(m)$, where $\text{div}(F_i)$ denotes the divergence of F_i on M . We also set

$$\mathcal{L}_i(\rho)(x) = \int_0^\infty \alpha e^{-\alpha t} \mathcal{L}_{\Phi_i^t}(\rho)(x) dt \quad (14)$$

for $\rho \in L^1(m)$. Observe that, using the notation of Proposition 3.4, $\mathcal{L}_i := \mathcal{L}_{\nu_1}$ where ν_1 is the measure on $\text{diff}_{\text{loc}}^r(M)$ given by $\nu_1 = \int_0^\infty \alpha e^{-\alpha t} \delta_{\Phi_i^t} dt$.

Associated to K is the transfer operator defined on $L^1(\mathbf{m})$ by

$$\mathcal{K}\rho(x, i) = \mathcal{L}_i \rho_i(x).$$

The purpose of the next lemma is twofold. Firstly, it will be used to show that P satisfies Assumption 2.2, (iii), with $\mathcal{C}(\mathbf{M})$ one of the sets $\mathcal{M}_{\text{ac}}(\mathbf{M})$, $\mathcal{M}_{\text{ac}}^{\text{ls}}(\mathbf{M})$ or $\mathcal{M}_{\text{ac}}^r(\mathbf{M})$. Secondly, it shows that the mapping $\mu \rightarrow \mu K$ in Proposition 4.2 preserve these sets.

Lemma 4.3 *Suppose that $\mu \in \mathcal{M}_{\text{ac}}(\mathbf{M})$ has density ρ with respect to \mathbf{m} . Then we have the following:*

(i) μA has density $A^t \rho$ given by

$$A^t \rho(x, i) = \sum_j \rho_j(x) A_{ji}(x).$$

If ρ is lower semi-continuous or C^k with $0 \leq k \leq r - 1$, then so is $A^t \rho$.

(ii) μK has a density given by $\mathcal{K}\rho$. If ρ is lower semi-continuous, then so is $\mathcal{K}\rho$.

(iii) If we furthermore assume that

$$\alpha > \max_{i \in E} \log \left(\mathcal{R}(\mathcal{L}_{\Phi_i^1}, C^{r-1}(M)) \right), \quad (15)$$

then \mathcal{K} is a bounded operator on $C^{r-1}(\mathbf{M})$ and

$$\mathcal{R}(\mathcal{K}, C^{r-1}(\mathbf{M})) \leq \frac{\alpha}{\alpha - \max_{i \in E} \log \left(\mathcal{R}(\mathcal{L}_{\Phi_i^1}, C^{r-1}(M)) \right)}.$$

Proof: (i) is immediate to verify and (ii) easily follows from Proposition 3.4.

We now turn to (iii). By classical results (see [17, Chapter V, Corollary 4.1] for example), $(t, x) \mapsto \Phi_i^t(x)$ is C^r . The form of $\mathcal{L}_{\Phi_i^t}$ (see equation (13)) and the fact that $\operatorname{div}(F_i)$ is C^{r-1} imply that

$$\sup_{0 \leq t \leq 1} \|\mathcal{L}_{\Phi_i^t}\|_{C^{r-1}(M)} \leq C$$

for some constant $C < \infty$ (depending on r). For $t \geq 0$, we write $t = n + s$ for $n \in \mathbb{N}$ and $0 \leq s \leq 1$. Thus

$$\mathcal{L}_{\Phi_i^t} = \mathcal{L}_{\Phi_i^1}^n \circ \mathcal{L}_{\Phi_i^s}.$$

Therefore for all $\varepsilon > 0$ there exists another constant $C' < \infty$ such that for all $t \geq 0$ we have

$$\|\mathcal{L}_{\Phi_i^t}\|_{C^{r-1}(M)} \leq C' e^{n(\log(R_i) + \varepsilon)} \leq C' e^{t(\log(R_i) + \varepsilon)}, \quad (16)$$

where R_i stands for $\mathcal{R}(\mathcal{L}_{\Phi_i^1}, C^{r-1}(M))$. Proposition 3.4 then implies that \mathcal{L}_i is a bounded operator on $C^{r-1}(M)$. We likewise have that \mathcal{K} is a bounded operator on $C^{r-1}(\mathbf{M})$.

We now establish the upper bound on the spectral radius. Note that for all $n \in \mathbb{N}$ we have

$$\mathcal{K}^n \rho(x, i) = \mathbb{E}(\mathcal{L}_{\Phi_i^{S_n}}(\rho_i)(x)),$$

where $S_n = T_1 + \dots + T_n$ and $\{T_i\}_{i \geq 1}$ is a sequence of independent random variables having an exponential distribution with parameter α . Thus

$$\|\mathcal{K}^n \rho\|_{C^{r-1}(M)} \leq \max_{i \in E} C' \mathbb{E}[e^{S_n(\log(R_i) + \varepsilon)}] = \max_{i \in E} C' \left(\mathbb{E}[e^{T_1(\log(R_i) + \varepsilon)}] \right)^n.$$

This proves the result. \square

4.2 Invariant distributions

Let $C_{pc}(\mathbb{R}_+, E)$ be the set of piecewise continuous functions $J : \mathbb{R}_+ \rightarrow E$. Given $J \in C_{pc}(\mathbb{R}_+, E)$, we let $t \rightarrow \Phi^t(x, J)$ denote the solution to the non autonomous differential equation

$$\frac{dx}{dt} = F_{J(t)}(x), \quad (17)$$

with initial condition $x(0) = x$. For all $x \in M$, we define

$$\gamma^+(x) = \{\Phi^t(x, J) : t \geq 0 \text{ and } J \in C_{pc}(\mathbb{R}_+, E)\}.$$

We let Γ be the possibly empty, compact connected set defined by

$$\Gamma = \bigcap_{x \in M} \overline{\gamma^+(x)}.$$

Connectedness (as well as other topological properties of Γ) are proved in [7, Proposition 3.11] (see also the erratum [8]). By Proposition 3.13 in [7] we have

$$\Gamma_P = \Gamma \times E,$$

where Γ_P is *the accessible set* (as defined in Section 2.1) of the kernel P given by (12).

We let $r_{max} \in \{1, 2, \dots\} \cup \{\infty\}$ be the maximal r such that all the F_i s are C^r . We define $\mathbf{F}_0 := \{F_i : i \in E\}$ and inductively, for all $n = 1, \dots, r_{max} - 1$, $\mathbf{F}_n = \mathbf{F}_{n-1} \cup \{[F, G] : F \in \mathbf{F}_0, G \in \mathbf{F}_{n-1}\}$, where $[F, G]$ is the Lie bracket of F and G .

We let $n \leq r_{max} - 1$. Inspired by the terminology used in [7] (see also [10, Chapter 6]), we say that a point $p \in M$ satisfies the *n-weak bracket condition* if $\mathbf{F}_n(p) := \{G(p) : G \in \mathbf{F}_n\}$ spans $T_p M$. We say that p satisfies the *weak bracket condition* if it satisfies the *n-weak bracket condition*, for some $n \leq r_{max} - 1$.

It was proved in [2] (for $\alpha_{ij}(x)$ constant over x) and in [7] that for C^∞ vector fields (i.e $r_{max} = \infty$), the existence of a point $p \in \Gamma$ at which the weak bracket condition holds implies that (Z_t) has a unique invariant distribution which is absolutely continuous with respect to \mathbf{m} . The next theorem also shows that its density is lower semi-continuous. A first version of this result, when $\alpha_{ij}(x)$ is constant over x , was proved in [12].

Theorem 4.4 *Assume there exists a point $p \in \Gamma$ at which the weak bracket condition holds. Then (Z_t) has a unique invariant probability measure Π which is absolutely continuous with respect to \mathbf{m} and whose density ρ is lower semi continuous. In addition, $\text{supp}(\Pi) = \Gamma \times E$ and for all $i \in E$,*

$$\text{supp}(\rho_i) := \overline{\{x \in M : \rho_i(x) > 0\}} = \Gamma.$$

Proof: We let $(p, i_0) \in \Gamma \times E = \Gamma_P$. By Theorems 4.1 and 4.4 in [7], (p, i_0) is a weak Doeblin point (as defined in Section 2.1) of P with a minorizing measure given by

$$\pi(dx di) = c \mathbf{1}_{\mathcal{V} \times E}(x, i) \mathbf{m}(dx di),$$

for some nonempty open set $\mathcal{V} \subset M$ and $c > 0$. This shows that $\pi \in \mathcal{M}_{ac}^{ls}(\mathbf{M})$. Therefore, by Theorem 2.11, P has a unique invariant distribution μ having a lower semi-continuous density h . By Proposition 4.2 and Lemma 4.3, $\Pi = \mu K$ is the unique invariant distribution of $(Z_t)_{t \geq 0}$ and its density, $\rho = \mathcal{K}h$, is lower semi-continuous. Also μ and Π have the same support.

Basic properties of the accessible set (see [10, Proposition 5.8 (iv)], for example) imply that $\text{supp}(\mu)$ (hence $\text{supp}(\Pi)$) is equal to Γ_P . Clearly $\text{supp}(\Pi) \subset \text{supp}(\rho)$. Conversely, if $\rho_i(x) > \delta > 0$, by the lower semi-continuity of ρ , there exists a ball $B(x, \varepsilon)$ such that $\rho_i(y) > \delta$ for all $y \in B(x, \varepsilon)$. Thus $\Pi(B(x, \varepsilon) \times \{i\}) > 0$. This proves the converse inclusion $\text{supp}(\rho) \subset \text{supp}(\Pi)$. \square

The next result considers the situation where Γ is empty but the weak bracket condition holds everywhere. It relies on the preceding result combined with ideas and results from [11].

Theorem 4.5 *We assume that the weak bracket condition holds at every point $p \in M$. Then $(Z_t)_{t \geq 0}$ has finitely many ergodic probability measures Π^1, \dots, Π^k . These are absolutely continuous with respect to \mathbf{m} , with lower semi-continuous densities ρ^1, \dots, ρ^k . For each $j = 1, \dots, k$, the support of Π^j can be written $\text{supp}(\Pi^j) = \Gamma^j \times E$, where Γ^j is a compact connected set. Furthermore, for all $i \in E$,*

$$\text{supp}(\rho_i^j) := \overline{\{x \in M : \rho_i^j(x) > 0\}} = \Gamma^j.$$

Proof: The proof uses some results and ideas from control theory. For consistency with the terminology used in [7], we phrase it using differential inclusions. We let

$$\text{co}(F)(x) = \left\{ \sum_{i \in E} p_i F_i(x) : p_i \geq 0, \sum_{i \in E} p_i = 1 \right\} \in T_x M$$

be the convex hull of the family $\{F_i(x)\}_{i \in E}$. A solution to the differential inclusion

$$\dot{\eta} \in \text{co}(F)(\eta) \quad (18)$$

is an absolutely continuous function $\eta \in C^0(\mathbb{R}, M)$ which satisfies $\dot{\eta}(t) \in \text{co}(F)(\eta(t))$ for almost all $t \in \mathbb{R}$. Such a differential inclusion induces a set-valued dynamical system defined as

$$\Psi_t(x) = \{\eta(t) : \eta(0) = x \text{ and } \eta \text{ is solution to (18)}\}.$$

We refer the reader to [7] for background and references. For $I \subset \mathbb{R}$, we set $\Psi_I(x) = \bigcup_{t \in I} \Psi_t(x)$. We call a set $C \subset M$, a *compact invariant control set* if C is nonempty, compact and $C = \overline{\Psi_{[0, \infty)}(x)}$ for all $x \in C$. This is consistent with the terminology used in control theory (see, for instance, [11, Definition 2.4 and Theorem 2.2]). The set Γ previously defined is, when it exists, a compact invariant control set. This follows, for instance, [7, Proposition 3.11]. Under the present assumption that the weak bracket conditions holds at every point $p \in M$, there are, by [11, Corollary 2.13], finitely many compact invariant control sets $\Gamma^1, \dots, \Gamma^k$. Furthermore we have the following:

- (i) for all $j \in \{1, \dots, k\}$ $\overline{\text{Int}(\Gamma^j)} = \Gamma^j$;
- (ii) for each $x \in M$, there exists $j \in \{1, \dots, k\}$ such that $\gamma^+(x) \cap \text{Int}(\Gamma^j) \neq \emptyset$;
- (iii) for each $j \in \{1, \dots, k\}$ and $x \in \Gamma^j$, $\text{Int}(\Gamma^j) \subset \gamma^+(x)$.

It follows from (i), (iii) and the definition of a compact invariant control set, that $\Gamma^j = \bigcap_{x \in \Gamma^j} \overline{\gamma^+(x)}$. The proof of Theorem 4.5 then applies verbatim to P restricted to Γ^j . This proves that P restricted to Γ^j has a unique, hence ergodic for P , invariant distribution Π^j with density ρ^j enjoying the properties stated in the theorem.

To establish that the Π^j s are the only ergodic probability measures, it suffices to show that every $\mu \in \text{Inv}(P)$ is supported by $\bigcup_{j=1}^k \Gamma^j$. It easily follows from (ii) that $W = \bigcup_{j=1}^k \text{Int}(\Gamma^j)$ is accessible for P , that is $R_a(x, W) > 0$ for all $x \in M$ (this can, for instance, be deduced from the support theorem, [7, 8, Theorem 3.4]). By the Feller continuity of R_a (inherited from the Feller continuity of P), the Portmanteau theorem and the compactness of M , we have that $R_a(x, W) \geq \delta > 0$ for all $x \in M$, for some $\delta > 0$. Since $R_a(y, M \setminus \overline{W}) = 0$ for all $y \in \overline{W}$ one obtains that (one may compare this to

[11, Theorem 4.7])

$$\begin{aligned}\mu(M \setminus \overline{W}) &= \mu R_a^2(M \setminus \overline{W}) = \int_{M \setminus \overline{W}} \mu R_a(x, dy) R_a(y, M \setminus \overline{W}) \\ &\leq (1 - \delta) \mu R_a(M \setminus \overline{W}) = (1 - \delta) \mu(M \setminus \overline{W}).\end{aligned}$$

We therefore obtain that $\mu(M \setminus \overline{W}) = 0$. \square

4.3 Smooth invariant distributions on the torus

This section is motivated by the work of Bakhtin, Hurth, Lawley and Mattingly [3]. It retrieves and substantially extends their main result (see Remark 4.8).

Here we assume that $M = \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ is the two dimensional flat torus, $E = \{1, 2\}$, and that the vector fields F_1, F_2 are C^r with $r \geq 2$, and transverse everywhere - that is $\{F_1(p), F_2(p)\}$ span $T_p \mathbb{T}^2$ for all p . In particular F_1, F_2 never vanish. Moreover we assume that the jump rates are constant, that is

$$\alpha_{12}(x) = \alpha_{12} > 0, \text{ and } \alpha_{21}(x) = \alpha_{21} > 0.$$

Using the notation introduced in Example 3.18, we let $\text{Per}_-(F_i)$ denote the (possibly empty) finite set of linearly stable periodic orbits of F_i . For $\gamma \in \text{Per}_-(F_i)$ we let $\Lambda_{1,i}(\gamma) < 0$ denote the non-zero Floquet exponent of γ .

We shall establish here the following result.

Theorem 4.6 *We let $1 \leq k \leq r$. Assume that for all $i = 1, 2$ and $\gamma \in \text{Per}_-(F_i)$*

$$\min(\alpha_{12}, \alpha_{21}) > -k \Lambda_{1,i}(\gamma).$$

Then, (Z_t) has finitely many ergodic probability measures (see Theorem 4.5), each of which has a C^{k-1} density with respect to \mathbf{m} .

Corollary 4.7 *Suppose that F_1, F_2 have no periodic orbits. Then (Z_t) has a unique invariant distribution and its density is C^{r-1} .*

Proof: A fixed-point-free C^2 flow with no periodic orbits on \mathbb{T}^2 has dense orbits (see the proof of Proposition 4.13). The accessible set is then \mathbb{T}^2 and uniqueness follows (see e.g Theorem 4.4). The C^{r-1} continuity follows from Theorem 4.6. \square

Remark 4.8 Using ideas inspired by Malliavin calculus, Bakhtin, Hurth, Lawley and Mattingly in [3], give a proof of Corollary 4.7, (when $r = \infty$ and $\alpha_{12} = \alpha_{21}$) in the particular case where each of the flows induced by F^1 and F^2 possesses an invariant probability measure with an everywhere positive C^∞ density. This, it should be noted, is a strong assumption.

Proof of Theorem 4.6

The idea of the proof is to show that P^n (for n sufficiently large) satisfies the standing assumption, Assumption 2.2, and the assumptions of Theorem 2.8. We assume here that F_1, F_2 are C^r with $r \geq 1$. The assumption that $r \geq 2$ will be required in Proposition 4.13.

We let $(X_n, I_n)_{n \geq 0}$ be the discrete-time Markov chain with kernel P (see Remark 4.1), and define $\tau = \min\{k \geq 1 : I_k \neq I_0\}$ to be the first switching time. For $n \geq 2$ and $1 \leq k \leq n - 1$ we set

$$P_{n,k}(f)(x, i) = \mathbb{E}[f(X_n, I_n) \mathbf{1}_{\tau=k} | (X_0, I_0) = (x, i)],$$

and

$$\Delta_{n,n}(f)(x, i) = \mathbb{E}[f(X_n, I_n) \mathbf{1}_{\tau \geq n} | (X_0, I_0) = (x, i)].$$

Clearly we have that

$$P^n f = \sum_{k=0}^{n-1} P_{n,k} f + \Delta_{n,n} f.$$

We now decompose the (matrix) operator A as $A = S + \bar{S}$ where S (corresponding to switching) is defined by

$$Sf(x, i) = A_{ij} f(x, j) \text{ with } j = 3 - i,$$

and \bar{S} (corresponding to not switching) is given by $\bar{S}f(x, i) = A_{ii} f(x, i)$. It is readily seen that

$$P_{n,k} = (K\bar{S})^{k-1} K S P^{n-k} = (K\bar{S})^{k-1} [K S K] A P^{n-k-1}.$$

This simply express the fact that the first switch occurs at time k . We likewise have that

$$\Delta_{n,n} = (K\bar{S})^{n-1} P.$$

In the next three lemmas, we use the following convenient notation. We denote $\mathcal{C}(\mathbf{M}) = \mathcal{M}_{ac}^{r-1}(\mathbf{M})$, and if $\mu \in \mathcal{C}(\mathbf{M})$ has density ρ , then $\|\rho\|_{C^{r-1}(\mathbf{M})}$ is denoted by $\|\mu\|_{\mathcal{C}(\mathbf{M})}$. We also assume that the parameter α that occurs in the definitions of A and K satisfies inequality (15).

Lemma 4.9 *We suppose that F_1, F_2 are transverse at every point $p \in \mathbb{T}^2$. For all $\varepsilon > 0$, KSK can be decomposed as $KSK = Q + \Delta$ where Q, Δ are Feller sub-Markov kernels and satisfy:*

- (i) $\mathcal{M}(\mathbf{M})Q \subset \mathcal{C}(\mathbf{M})$;
- (ii) $\mathcal{C}(\mathbf{M})\Delta \subset \mathcal{C}(\mathbf{M})$;
- (iii) $\|\mu\Delta\|_{\mathcal{C}(\mathbf{M})} \leq \varepsilon\|\mu\|_{\mathcal{C}(\mathbf{M})}$ for all $\mu \in \mathcal{C}(\mathbf{M})$.

Proof: We set $j = 3 - i$ for $i \in \{1, 2\}$. We note that

$$KSKf(x, i) = A_{ij} \int_{\mathbb{R}_+^2} f(\Phi_j^t \circ \Phi_i^s(x), j) \alpha^2 e^{-\alpha(t+s)} dt ds.$$

For all $n > 1$, we let $\eta_n : \mathbb{R} \mapsto \mathbb{R}_+$ be a C^∞ function such that $\eta_n = 1$ on $[\frac{1}{n}, n]$, $\eta_n = 0$ on $\mathbb{R} \setminus [\frac{1}{2n}, 2n]$, and $0 \leq \eta_n \leq 1$. We set

$$Qf(x, i) = A_{ij} \int_{\mathbb{R}_+^2} f(\Phi_j^t \circ \Phi_i^s(x), j) \alpha^2 e^{-\alpha(t+s)} \eta_n(t) \eta_n(s) dt ds$$

and $\Delta = KSK - Q$. The assumption that F_1, F_2 are transverse makes the map $(t, s) \in \mathbb{R}_+^2 \rightarrow \Phi_j^t \circ \Phi_i^s(x) \in \mathbb{T}^2$ a submersion for all $x \in \mathbb{T}^2$. Indeed, denoting $y = \Phi_i^s(x)$, we have that

$$\left(\frac{\partial}{\partial t} \Phi_j^t \circ \Phi_i^s(x), \frac{\partial}{\partial s} \Phi_j^t \circ \Phi_i^s(x) \right) = (D\Phi_j^t(y)F_j(y), D\Phi_j^t(y)F_i(y)).$$

Proposition 3.2 implies that condition (i) is satisfied. For the second assertion, we proceed as in the proof of Lemma 4.3 (iii). For all $\mu \in \mathcal{C}(\mathbf{M})$ we have that

$$\begin{aligned} \|\mu\Delta\|_{\mathcal{C}(\mathbf{M})} &\leq \|\mu\|_{\mathcal{C}(\mathbf{M})} \left[\int_{\mathbb{R}} \max_{i=1,2} \|\mathcal{L}_{\Phi_i^t}\|_{C^{r-1}(M)} \alpha e^{-\alpha t} (1 - \eta_n(t)) dt \right]^2 \\ &\leq \left[\int_{\mathbb{R}} C' e^{-\beta t} (1 - \eta_n(t)) dt \right]^2, \end{aligned}$$

for some constant $C', \beta > 0$ (see equation (16)). For n sufficiently large, the right hand term can be made arbitrary small, by monotone convergence. \square

Lemma 4.10 *We assume that F_1, F_2 are transverse at every point $p \in \mathbb{T}^2$. Then for all $n \geq 2, k = 1, \dots, n-1$, and $\varepsilon > 0$, $P_{n,k}$ can be decomposed into $P_{n,k} = Q_{n,k} + \Delta_{n,k}$, where $Q_{n,k}, \Delta_{n,k}$ are Feller sub-Markov kernels and satisfy:*

- (i) $\mathcal{M}(\mathbf{M})Q_{n,k} \subset \mathcal{C}(\mathbf{M})$;
- (ii) $\mathcal{C}(\mathbf{M})\Delta_{n,k} \subset \mathcal{C}(\mathbf{M})$;
- (iii) for all $\mu \in \mathcal{C}(\mathbf{M})$, $\|\mu\Delta_{n,k}\|_{\mathcal{C}(\mathbf{M})} \leq \varepsilon\|\mu\|_{\mathcal{C}(\mathbf{M})}$.

Proof: With Q, Δ as in Lemma 4.9, we set

$$Q_{n,k} = (K\bar{S})^{k-1}QAP^{n-k-1}, \quad \Delta_{n,k} = (K\bar{S})^{k-1}\Delta AP^{n-k-1}.$$

Then we have

$$P_{n,k} = (K\bar{S})^{k-1}[KSK]AP^{n-k-1} = Q_{n,k} + \Delta_{n,k}.$$

Since $\mathcal{M}(\mathbf{M})$ and $\mathcal{C}(\mathbf{M})$ are invariant under the operators K, A, \bar{S}, P , assertion (i) and (ii) follow directly from Lemma 4.9. We likewise have $\|\mu\Delta_{n,k}\|_{\mathcal{C}(\mathbf{M})} \leq \varepsilon\|\mathcal{K}\|_{C^{r-1}(\mathbf{M})}^{n-2}\|\mu\|_{\mathcal{C}(\mathbf{M})}$, by Lemma 4.9. Replacing ε by $\varepsilon/\|\mathcal{K}\|_{C^{r-1}(\mathbf{M})}^{n-2}$, we obtain (iii). \square

Making α larger if necessary, we assume as we may that

$$\alpha > \log\left(\mathcal{R}(\mathcal{L}_{\Phi_i^1}, C^{r-1}(M))\right).$$

Lemma 4.11 *We have the following:*

- (i) $\mathcal{C}(\mathbf{M})\Delta_{n,n} \subset \mathcal{C}(\mathbf{M})$;
- (ii) For all $\varepsilon > 0$, there exists $C < \infty$ such that for all $\mu \in \mathcal{C}(\mathbf{M})$ and $n \geq 2$, we have

$$\|\mu\Delta_{n,n}\|_{\mathcal{C}(\mathbf{M})} \leq C \left[\frac{\alpha - \min(\alpha_{12}, \alpha_{21})}{\alpha - \max_{i=1,2} \log\left(\mathcal{R}(\mathcal{L}_{\Phi_i^1}, C^{r-1}(M))\right)} \right]^n e^{n\varepsilon} \|\mu\|_{\mathcal{C}(\mathbf{M})}.$$

Proof: We firstly observe that K and \bar{S} commute, so that $\Delta_{n,n} = \bar{S}^{n-1}K^n A$. We therefore have that

$$\begin{aligned} \|\mu\Delta_{n,n}\|_{\mathcal{C}(\mathbf{M})} &\leq \|\bar{S}\|^{n-1} \|\mathcal{K}^n\|_{C^{r-1}(\mathbf{M})} \|A^t\| \|\mu\|_{\mathcal{C}(\mathbf{M})} \\ &= \max\left(1 - \frac{\alpha_{12}}{\alpha}, 1 - \frac{\alpha_{21}}{\alpha}\right)^{n-1} \|\mathcal{K}^n\|_{C^{r-1}(\mathbf{M})} \|A^t\| \|\mu\|_{\mathcal{C}(\mathbf{M})} \end{aligned}$$

for all $\mu \in \mathcal{C}(\mathbf{M})$, whence the result follows from Lemma 4.3 (iii). \square

Theorem 4.12 *Suppose that F_1, F_2 are $C^r, r \geq 1$, transverse at every point $p \in \mathbb{T}^2$, and that*

$$\min(\alpha_{12}, \alpha_{21}) > \max_{i=1,2} \log \left(\mathcal{R}(\mathcal{L}_{\Phi_i^1}, C^{r-1}(M)) \right).$$

Then every ergodic measure for (Z_t) has a C^{r-1} density with respect to \mathbf{m} .

Proof: Using the notation of the proceeding lemmas, we write $P^n = Q_n + \Delta_n$, whereby $Q_n = \sum_{k=0}^{n-1} Q_{n,k}$ and $\Delta_n = \sum_{k=0}^n \Delta_{n,k}$. Then (Q_n, Δ_n) satisfies the standing assumption, Assumption 2.2, and for n sufficiently large, there exists $0 \leq \theta < 1$ such that $\|\mu \Delta_n\|_{\mathcal{C}(\mathbf{M})} \leq \theta \|\mu\|_{\mathcal{C}(\mathbf{M})}$ for all $\mu \in \mathcal{C}(\mathbf{M})$. Theorem 4.12 then follows from Theorem 2.8 \square

We then obtain Theorem 4.6 as a consequence of Theorem 4.12 and the next proposition, Proposition 4.13, combined with the estimates given by Proposition 3.10.

For a C^1 flow $\{\Phi^t\}$ we define the expansion rate and k -expansion volume rate of Φ to be the expansion rate and k -expansion volume rate of the time one map Φ^1 , which we denote as $\mathcal{E}(\Phi)$ and $\mathcal{E}\mathcal{V}_k(\Phi)$ respectively.

Proposition 4.13 *We let F be a C^2 vector field on \mathbb{T}^2 with no equilibria (i.e. $\text{Eq}(F) = F^{-1}(0) = \emptyset$) and let $\{\Phi^t\}$ be the induced flow. Then*

$$\mathcal{E}(\Phi) = \min_{\{\gamma \in \text{Per}_-(F)\}} \Lambda_1(\gamma),$$

and

$$\mathcal{E}\mathcal{V}_k(\Phi) = (k+1) \min_{\{\gamma \in \text{Per}_-(F)\}} \Lambda_1(\gamma)$$

for all $k \geq 0$, with the convention that the right hand sides are 0 whenever $\text{Per}_-(F) = \emptyset$.

Proof: By Propositions 14.2.2 and 14.2.4 in Katok and Hasselblat, [20], a fixed-point-free C^2 flow on \mathbb{T}^2 must enjoy one of the following two properties:

- (a) either all recurrent points are periodic;

(b) or there exists a closed transversal and every orbit crosses this transversal. Furthermore, the return map to this transversal is a C^2 circle diffeomorphism $f : S^1 \mapsto S^1$ which, by the Denjoy Theorem ([20, Theorem 12.1.1]), is topologically conjugate to an irrational rotation.

If F has no periodic orbit then we are in case (b). We then have that $\mathcal{E}(\Phi) \leq 0$ by Remark 3.9. We now assume for contradiction that $\mathcal{E}(\Phi) < -\lambda < 0$. Then, by [24, Corollary 2], there exists two distinct points $x, y \in \mathbb{T}^2$ such that $\limsup_{t \rightarrow \infty} \frac{\log(d(\Phi^t(x), \Phi^t(y)))}{t} < -\lambda$. This implies that the return map f has two distinct points $\theta, \alpha \in S^1$ such that $d(f^n(\theta), f^n(\alpha)) \rightarrow 0$ as $n \rightarrow \infty$. However f is topologically conjugate to a rotation and a rotation is an isometry, whence we obtain a contradiction.

If F has periodic orbits, then we are in case (a). We let μ be an ergodic probability measure for Φ^1 . By the Poincaré recurrence theorem and Birkhoff's theorem, there exists a point p , recurrent for Φ^1 , such that

$$\frac{1}{n} \sum_{k=1}^n \delta_{\Phi^k(p)} \Rightarrow \mu.$$

By (a), p is T -periodic for $\{\Phi^t\}$, for some $T > 0$. Thus, reasoning as in Example 3.18, either $\mu = \frac{1}{N} \sum_{i=0}^{N-1} \delta_{\Phi^i(p)}$ for some $N \in \mathbb{N}$ (if T is rational) or $\mu = \frac{1}{T} \int_0^T \delta_{\Phi^s(p)} ds$ (if T is irrational). In both cases, $\Lambda_1(\mu)$ equals the Floquet exponent $\Lambda_1(\gamma)$ of the periodic orbit. The result then follows from Schreiber's theorem (equation (4)). \square

Remark 4.14 The fact that $\mathcal{E}(\Phi) = 0$ when F has no periodic orbit answers a question raised by Moe Hirsch in [19]. An affirmative answer to this question is given in the introduction of Schreiber's paper [24], but the proof and the assumptions are not detailed in the paper. The result does actually directly follow from Schreiber's results as shown above, at least for C^2 flows. The question is open for C^1 flows.

4.4 Smooth invariant distributions under fast switching

We return here to the general model of a PDMP (as described in the beginning of Section 4), but under the assumption that the rate matrix $(\alpha_{ij}(x))_{i,j \in E}$ takes the simple form

$$\alpha_{ij} = p_j \alpha \mathbf{1}_{i \neq j} \tag{19}$$

where $\alpha > 0, p_j > 0$ and $\sum_{j \in E} p_j = 1$.

We shall prove here the following result.

Theorem 4.15 *Let $(Z_t)_{t \geq 0}$ be the PDMP corresponding to the characteristics $(\{F_i\}_{i \in E}, (\alpha_{ij})_{i,j \in E})$, where α_{ij} is given by (19). Suppose that the 1-Bracket condition holds at every point $x \in M$. Then, there exists $\alpha^* > 0$ such that for all $\alpha \geq \alpha^*$ the ergodic measures of (Z_t) (see Theorem 4.5) have a C^{r-1} density with respect to \mathbf{m} .*

A version of this result (for slightly more general switching rates but under the assumption that there exists an accessible point), was established by the present authors in [12]. However, the proof given here is simpler and provides a good illustration of our general method.

We firstly observe that invariant distributions of $(Z_t)_{t \geq 0}$ can be described in terms of invariant distributions of a discrete-time Markov chain living on M .

We let \bar{P} be the Markov kernel on M (not on \mathbf{M}) defined by

$$\bar{P}(f)(x) = \sum_{i \in E} p_i \int_0^\infty \alpha e^{-\alpha t} f(\Phi_t^i(x)) dt. \quad (20)$$

Recall that K is the operator given by (11). For $\mu \in \mathcal{P}(M)$, we let $(\mu \otimes p) \in \mathcal{P}(\mathbf{M})$ denote the probability measure defined by $\mu \otimes p(dx \times \{i\}) = p_i \mu(dx)$.

The next proposition is similar to Proposition 4.2.

Proposition 4.16 *Let $(Z_t)_{t \geq 0}$ be the PDMP corresponding to the characteristics $(\{F_i\}_{i \in E}, (\alpha_{ij})_{i,j \in E})$, where α_{ij} is given by (19). Then the set of invariant distributions of $(Z_t)_{t \geq 0}$ is given by*

$$\{(\mu \otimes p)K : \mu \in \text{Inv}(\bar{P})\}.$$

Proof: We set $A_{ij} = p_j$ for all i, j and let $P = KA$ (see equation (12)). By Proposition 4.2 it suffices to show that $\text{Inv}(P) = \{\mu \otimes p : \mu \in \text{Inv}(\bar{P})\}$. We let $\mu \in \text{Inv}(\bar{P})$ and $f \in \mathcal{B}(\mathbf{M})$. Then we have that

$$\begin{aligned} (\mu \otimes p)Pf &= \sum_{i \in E} p_i \int_M \left(\int_{\mathbb{R}_+} \sum_j p_j f_j(\Phi_t^i(x)) \alpha e^{-\alpha t} dt \right) \mu(dx) \\ &= \sum_{j \in E} p_j \int_M \int_{\mathbb{R}_+} [\sum_{i \in E} p_i f_j(\Phi_t^i(x))] \alpha e^{-\alpha t} \mu(dx) \\ &= \sum_{j \in E} p_j \mu(\bar{P}f_j) = \sum_{j \in E} p_j \mu f_j = (\mu \otimes p)f. \end{aligned}$$

Thus we have that $\mu \otimes p \in \text{Inv}(P)$.

Conversely, we let $\nu \in \text{Inv}(P)$ and $f \in \mathcal{B}(\mathbf{M})$. Then we have $\nu K(Af) = \nu f$, that is

$$\int_M \left(\sum_{j \in E} p_j f_j(x) \right) \sum_{i \in E} \nu K(dx \times i) = \nu f.$$

This shows that $p \otimes \mu = \nu$, whereby

$$\mu(dx) = \sum_{i \in E} (\nu K)(dx \times i).$$

On the other hand, the relation $\nu K A = \nu$ implies $(\nu K) A K f = \nu K f$ for all $f \in \mathcal{B}(\mathbf{M})$. If one applies this relation to the map f such that $f_i(x) = g(x)$ for all i , it follows that $\mu \bar{P} g = \mu g$. Thus we have that $\mu \in \text{Inv}(\bar{P})$. \square

Proof of Theorem 4.15

In light of Proposition 4.16, it suffices to consider invariant distributions of the operator \bar{P} given by equation (20). To highlight the influence of the switching rate parameter α , we rewrite \bar{P} as \bar{P}_α .

We let $n \geq 1$, $\mathbf{i} = (i_1, \dots, i_n) \in E^n$, and $h : (\mathbb{R}_+^*)^n \mapsto [0, 1]$ be a C^∞ function. Set $p_{\mathbf{i}} = p_{i_1} \cdots p_{i_n}$ and let $\bar{P}_{\alpha, \mathbf{i}, h}$ denote the sub-Markovian operator on M defined by

$$\bar{P}_{\alpha, \mathbf{i}, h} f(x) = p_{\mathbf{i}} \int_{\mathbb{R}_+^n} f(\Phi_{i_n}^{t_n/\alpha} \circ \cdots \circ \Phi_{i_1}^{t_1/\alpha}(x)) e^{-|t|} h(t) dt,$$

whereby $|t| = t_1 + \dots + t_n$. If $h \equiv 1$, we write $\bar{P}_{\alpha, \mathbf{i}}$ for $\bar{P}_{\alpha, \mathbf{i}, h}$. Clearly we have that

$$\bar{P}_{\alpha, \mathbf{i}} = \bar{P}_{\alpha, \mathbf{i}, 1-h} + \bar{P}_{\alpha, \mathbf{i}, h}$$

and

$$\bar{P}_\alpha^n = \sum_{\mathbf{i} \in E^n} \bar{P}_{\alpha, \mathbf{i}}.$$

Lemma 4.17 (i) *If $\mu \in \mathcal{M}_{ac}(M)$ has density $\rho \in L^1(m)$, then $\mu \bar{P}_{\alpha, \mathbf{i}, h}$ has density $p_{\mathbf{i}} \mathcal{L}_{\alpha, \mathbf{i}, h}(\rho)$, whereby*

$$\mathcal{L}_{\alpha, \mathbf{i}, h}(\rho) = \int_{\mathbb{R}_+^n} \mathcal{L}_{\Phi_{i_n}^{t_n/\alpha} \circ \cdots \circ \Phi_{i_1}^{t_1/\alpha}}(\rho) e^{-|t|} h(t) dt.$$

Moreover if we have $\alpha > \max_{i \in E} \log \left(\mathcal{R}(\mathcal{L}_{\Phi_i^1}, C^{r-1}(M)) \right)$, then $\mathcal{L}_{\alpha, \mathbf{i}, h}$ is a bounded operator on $C^{r-1}(M)$.

(ii) We have that

$$\limsup_{\alpha \rightarrow \infty} \|\mathcal{L}_{\alpha, \mathbf{i}, h}\|_{C^{r-1}(M)} \leq \int_{\mathbb{R}_+^n} e^{-|t|} h(t) dt.$$

Proof: The proof of (i) is similar to the proof of Lemma 4.3 (iii) (itself relying on Proposition 3.4). Its verification is left to the reader.

(ii). We let $\eta(t_1, \dots, t_n) = \|\mathcal{L}_{\Phi_{i_n}^{t_n}}\|_{C^{r-1}(M)} \dots \|\mathcal{L}_{\Phi_{i_1}^{t_1}}\|_{C^{r-1}(M)}$. By the C^r continuity of the map $(t, x) \mapsto \Phi_i^t(x)$ (see, for example, [17, Chapter V, Corollary 4.1]), $\Phi_i^t \rightarrow \Phi_i^0$ as $t \rightarrow 0$ in the C^r topology. From this and the formula (13), it follows that $\limsup_{t \rightarrow 0_{\mathbb{R}^n}} \eta(t) \leq 1$. Therefore for all $\varepsilon > 0$, there exists some $\varepsilon > 0$ such that $\eta(t) \leq 1 + \varepsilon$ for all $t \in \mathbb{R}_+^n$ such that $|t| \leq \delta$. Thus we have

$$\begin{aligned} \|\mathcal{L}_{\alpha, \mathbf{i}, h}\|_{C^{r-1}(M)} &\leq \int_{\mathbb{R}_+^n} \eta(t/\alpha) e^{-|t|} h(t) dt \\ &\leq (1 + \varepsilon) \int_{\mathbb{R}_+^n} e^{-|t|} h(t) \mathbf{1}_{|t| \leq \alpha \delta} dt + \int_{\mathbb{R}_+^n} \eta(t/\alpha) h(t) e^{-|t|} \mathbf{1}_{|t| \geq \alpha \delta} dt. \end{aligned}$$

When $\alpha \rightarrow \infty$, the first term on the right goes to $1 + \varepsilon$ while the second term goes to 0. This follows from the fact that $\eta(t) \leq C' e^{\beta|t|}$ for some $\beta > 0$ and $C' < \infty$, by equation (16). This concludes the proof. \square

Proposition 4.18 *We suppose that there exist $n \geq 1, \mathbf{i} = (i_1, \dots, i_n) \in E^n$ and $U \subset (\mathbb{R}_+^*)^n$ a nonempty open set such that:*

- (i) $\frac{1}{\alpha} U \subset U$ for all $\alpha \geq 1$;
- (ii) for all $x \in M$, the map $(t_1, \dots, t_n) \rightarrow \Phi_{i_n}^{t_n} \circ \dots \circ \Phi_{i_1}^{t_1}(x)$ is a submersion on U .

Then, there exists $\alpha^* \geq 1$ such that for all $\alpha \geq \alpha^*$ $\text{Inv}(\bar{P}_\alpha) \subset \mathcal{M}_{ac}^{r-1}(M)$.

Proof: We let $h : (\mathbb{R}_+^*)^n \rightarrow [0, 1]$ be a C^∞ nonzero function with compact support in U . We set $Q_\alpha = \bar{P}_{\alpha, \mathbf{i}, h}$ and $\Delta_\alpha = \bar{P}_{\alpha, \mathbf{i}, 1-h} + \sum_{\mathbf{j} \in E^n \setminus \{\mathbf{i}\}} \bar{P}_{\alpha, \mathbf{j}}$ and take

$\mathcal{C}(M) = \mathcal{M}_{ac}^{r-1}(M)$. By Proposition 3.2, $\mathcal{M}(M)Q_\alpha \subset \mathcal{C}(M)$ for all $\alpha \geq 1$. By Lemma 4.17, $\mathcal{C}(M)\Delta_\alpha \subset \mathcal{C}(M)$ and

$$\limsup_{\alpha \rightarrow \infty} \|\Delta_\alpha\|_{\mathcal{C}(M)} \leq [1 - p_i \int_{(\mathbb{R}_+^*)^n} e^{-(t_1 + \dots + t_n)} h(t_1, \dots, t_n) dt_1 \dots dt_n] < 1,$$

whereby $\|\Delta_\alpha\|_{\mathcal{C}(M)}$ represents $\|p_i \mathcal{L}_{i,\alpha,h} + \sum_{j \neq i} p_j \mathcal{L}_{j,\alpha,1}\|_{C^{r-1}(M)}$. The proposition then follows from Theorem 2.8. \square

By [12, Proposition 5.1], the 1-Bracket condition implies that the assumptions of Proposition 4.18 are satisfied. This concludes the proof of Theorem 4.15. \square

Acknowledgement: The work of MB, and partially that of OT, was funded by the grant 200020-219913 from the Swiss National Foundation. The work of OT was also partially funded by the EPSRC MathRad programme grant EP/W026899/.

Conflict of Interest : None.

References

- [1] Y. Bakhtin, T. Hurth, and J.C. Mattingly, *Regularity of invariant densities for 1d-systems with random switching*, Nonlinearity **28** (2015), 3755–3787.
- [2] Yuri. Bakhtin and Tobias. Hurth, *Invariant densities for dynamical systems with random switching*, Nonlinearity (2012), no. 10, 2937–2952.
- [3] Yuri Bakhtin, Tobias Hurth, Sean D. Lawley, and Jonathan C. Mattingly, *Smooth invariant densities for random switching on the torus*, Nonlinearity **31** (2018), no. 4, 1331–1350. MR 3816636
- [4] ———, *Singularities of invariant densities for random switching between two linear ODEs in 2D*, SIAM J. Appl. Dyn. Syst. **20** (2021), no. 4, 1917–1958. MR 4322095
- [5] Viviane Baladi, *Positive transfer operators and decay of correlations*, Advanced Series in Nonlinear Dynamics, vol. 16, World Scientific Publishing Co., Inc., River Edge, NJ, 2000. MR 1793194

- [6] M. Benaïm, T. Hurth, and E. Strickler, *A user-friendly condition for exponential ergodicity in randomly switched environments*, Electronic Communications in Probability **23** (2018), no. 44, 1–12.
- [7] M. Benaïm, S. Le Borgne, F. Malrieu, and P.-A. Zitt, *Qualitative properties of certain piecewise deterministic Markov processes*, Ann. Inst. Henri Poincaré Probab. Stat. **51** (2015), no. 3, 1040–1075.
- [8] ———, *Erratum: Qualitative properties of certain piecewise deterministic Markov processes*, Ann. Inst. Henri Poincaré Probab. Stat. **55** (2019), no. 4.
- [9] Michel Benaïm and Morris W. Hirsch, *Chain recurrence in surface flows*, Discrete Contin. Dynam. Systems **1** (1995), no. 1, 1–16. MR 1355862
- [10] Michel Benaïm and Tobias Hurth, *Markov chains on metric spaces, a short course*, Universitext, vol. 99, Springer, Cham, 2022.
- [11] M Benaïm, F Colonijs, and R Lettau, *Supports of invariant measures for piecewise deterministic Markov processes*, Nonlinearity **30** (2017), no. 9, 3400.
- [12] Michel Benaïm and Oliver Tough, *Regularity of the stationary density for systems with fast random switching*, 2023.
- [13] J. Campbell and Latushkin Y., *Sharp estimates in Ruelle theorems for matrix transfer operators*, Comm. Math. Phys. **185** (1997), 379–396.
- [14] Bertrand Cloez and Martin Hairer, *Exponential ergodicity for Markov processes with random switching*, Bernoulli **21** (2015), no. 1, 505–536. MR 3322329
- [15] M. H. A. Davis, *Markov models and optimization*, Monographs on Statistics and Applied Probability, vol. 49, Chapman & Hall, London, 1993.
- [16] J. Diestel and J. J. Uhl, Jr., *Vector measures*, Mathematical Surveys, vol. No. 15, American Mathematical Society, Providence, RI, 1977, With a foreword by B. J. Pettis. MR 453964
- [17] Philip Hartman, *Ordinary differential equations*, second ed., Birkhäuser, Boston, Mass., 1982. MR 658490

- [18] Morris W. Hirsch, *Differential topology*, Graduate Texts in Mathematics, No. 33, Springer-Verlag, New York-Heidelberg, 1976. MR 0448362
- [19] ———, *Asymptotic phase, shadowing and reaction-diffusion systems*, Differential equations, dynamical systems, and control science, Lecture Notes in Pure and Appl. Math., vol. 152, Dekker, New York, 1994, pp. 87–99. MR 1243195
- [20] Anatole Katok and Boris Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995, With a supplementary chapter by Katok and Leonardo Mendoza. MR 1326374
- [21] E. Löcherbach, *Absolute continuity of the invariant measure in piecewise deterministic Markov processes having degenerate jumps*, Stochastic Process. Appl. **128** (2018), no. 6, 1797–1829. MR 3797644
- [22] Ricardo Mañé, *Ergodic theory and differentiable dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 8, Springer-Verlag, Berlin, 1987, Translated from the Portuguese by Silvio Levy. MR 889254
- [23] David Ruelle, *The thermodynamic formalism for expanding maps*, Comm. Math. Phys. **125** (1989), no. 2, 239–262. MR 1016871
- [24] Sebastian J. Schreiber, *Expansion rates and Lyapunov exponents*, Discrete Contin. Dynam. Systems **3** (1997), no. 3, 433–438. MR 1444204
- [25] ———, *On growth rates of subadditive functions for semiflows*, J. Differential Equations **148** (1998), no. 2, 334–350. MR 1643183