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SOME PHILOSOPHICAL IMPLICATIONS OF GÖDEL'S THEOREM

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Some internal dynamics of formalism

If we reflect on the complex history of formalism, we can see that the acceptance of a more or less substantial degree of formalism was a feature common to a lot of trends of modern mathematics. This explains why what is termed 'modern mathematics' (practically speaking the mathematics of our century) is so deeply characterized by 'abstract' theories, by 'formal' procedures and tools, by stress being laid on 'structures', and by an indifference towards 'intuition'. All this (which has been programmatically expressed and concretely realized e.g. in the famous *Eléments de mathématique* of the Bourbaki group) has even led to a strictly formal reconstruction - by means of abstract axiomatic systems analogous to those of algebra and topology - of traditionally constructive and intuitive mathematical theories, such as arithmetic and analysis. Moreover, the considerable success of formalization in mathematics strongly encouraged the extension of this methodology to many other fields. R. Carnap (1966), for instance, developed formalism as a critical methodology for a linguistic treatment of *philosophy*, while axiomatizations of *physical* theories were promoted especially regarding quantum mechanics. In this field not only were formal axiomatizations of various kinds soon elaborated at the view to coping with the non-'visualizability' of micro-phenomena, but the special logical calculi of many-valued logics were used in order to cope with some well known difficulties related to the uncertainty principle[1]. But one could also mention mathematical *psychology*, where different formal axiom systems have been proposed e.g. by P. Suppes (1969), for a precise analysis of human behaviour. Also the elaboration of the notion of preference in the field of *economic* and *social* activity has received this kind of treatment by J. von Neumann and O. Morgenstern (1944), not to mention the attempts to apply this methodology in the field of *juridical* science (see Klug 1951). But we are not interested in multiplying these examples: we want rather to explore the main motivations which encouraged this general acceptance of the 'formal way of thinking'.

Some of these motivations have something of a 'negative' flavour, in the sense that they derive from a kind of frustration in the attempt to attain truth by gaining command of the 'content' of intuitive knowledge, this content being at the same time both the configuration of a meaning and the relation to a referent. This unreliability of intuition, however, must be further analyzed; and if we take into account the fact that it emerged in the field of mathematics, it appears that the mistrust regards *intellectual intuition*, (possibly) leaving *sensory* intuition untouched. But if this is so, we immediately see that the origin of this phenomenon goes back to a level deeper than the history of non-Euclidean geometry. Indeed one should at least consider Kant, the transition of whose thought from the so-called pre-critical to the critical stage is characterized by the fact that in the *Critique of Pure Reason* (1781) intuition is confined to the senses and denied to the intellect, which was not the case at least until Kant's famous dissertation of 1770[2]. But Kant himself was, in this sense, the terminus of a longer itinerary, the first part of which had been travelled by other philosophers (especially the British empiricists), and even more significantly by the founders of modern science. When Galilei was rejecting the grasping of the

1 This trend originated with Birkhoff/von Neumann (1936), and was taken up again e.g. by v. Weizsäcker (1955), Mittelstaedt (1968) and Scheibe (1964), and in more recent years has given rise to a rich branch of studies known as *quantum logic*.

2 *De mundi sensibilis atque intelligibilis forma et principiis dissertatio.*

'essences' of natural substances as a «desperate enterprise»[3], he was actually expressing a mistrust in the possibility of having an intellectual intuition. And when modern science, following his conceptual revolution, accepted that natural *laws* should constitute the immutable structure of physical reality, it attributed to these laws the character of being *forms* of the phenomena, in this way breaking the identification of 'form' and 'essence' which had governed philosophy from the time of the Greeks. This amounts to saying that 'formal knowledge' (though in a sense which was not at all identical with 'formalistic knowledge') was considered to be reliable and rigorous. In this sense, Kant's 'Copernican Revolution' and transcendental philosophy may be seen as a powerful generalization of knowledge as such, and even of all forms of the activity of reason - a generalization of that view which modern science had advocated for a more restricted field. Indeed, transcendental philosophy tries to determine the *a priori* formal conditions of every judgment, be it cognitive, moral, or aesthetic.

It is interesting to note that, when a crisis of certitude affected the *physical* sciences at the end of the 19th and beginning of the 20th century (a crisis which concerned the validity of the 'formal' concepts and laws of physics themselves), an issue was found again in an intensification of the formal point of view, i.e. in a formalization of the second level, which was very close to a 'formalistic' perspective. It would lead us too far to explore this development, but this means that we could just as well have used the case of the physical sciences (instead of that of mathematics) as an illustration of the rise of modern formalism. However this path would have been much less transparent and would have needed a much more elaborate and sophisticated analysis. Along it we would have arrived essentially at the same conclusion concerning the 'negative' motivations which promoted the pre-eminence of formalism: a mistrust with regard to both truth and the 'intensional' sense of 'meaning'.

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3 This claim is made in the third letter to M. Welser on the sunspots. See Galilei (1929-1939), vol. V, pp. 187-188.

But there were also several 'positive' reasons for the success of formalism, which might be condensed into one fundamental feature: the satisfaction of the requirement of generality. Generality resides in form and not in content: on this point a remarkable continuity exists in the history of philosophy. So too when 'form' was equated with 'essence' was it stressed that generality (or even better 'universality') was connected with form, while individuality was the specific contribution of 'matter'. The whole history of philosophy might be reconstructed in the light of the different ways in which the complex relation between form and matter has been conceived (the distinction between form and 'content' being nothing but a particular instance of this dichotomy). We shall certainly not sketch such a reconstruction here, but satisfy ourselves with hinting only at a few salient aspects of this question.

The first deserving mention is probably the Aristotelian invention of *formal logic*, the aim of which was to find general rules capable of granting the correctness of inferential discourse, i.e. discourse in which the consequences 'necessarily follow' from the premisses. In spite of the fact that this inquiry was moved by the desire to grant the discourse permanent *truth*, it turned out that, in order for this permanence to be really granted with necessity, one was to disregard the 'accidental' fact of the propositions involved being true (and consequently disregard also their *meaning*). This means that, in the case of a correct inference, the truth of the conclusion can be stated not so much because of the truth of the premisses, but *by force* of the cogency of the link, a cogency which remains intact if the link is applied to false premisses, but which *necessarily* ensures the transmission of truth, when it is applied to true premisses. This is why the treatment of the *Prior Analytics* is not only 'formal' in this sense, but also largely symbolic. Here we have a double level of universality: formal logical rules are valid not only for whatever 'content' or meaning the propositions might have, but also for whatever their truth-value might be, which means that they also apply to discourses in which the truth of the premisses is uncertain. Of course, in the *Posterior Analytics* Aristotle fully recovers the requirements of a knowledge endowed with uncontroversial truth, and proposes the structure of the 'classical' axiomatic method which we outlined above. Within this structure, the 'formal' tool only plays the role of an instrument, while an intellectual intuition is required to establish the absolute truth of the premisses (but then the point of view is no longer formal).

As a second prominent example we shall mention again the philosophy of Kant, in which the most central tenet is perhaps that no universality or necessity can be linked to what is empirical, so that we must inevitably look for something *a priori* whenever we want to understand or justify any universal judgment of our reason. However, it turns out that for Kant the term «empirical» is applied to any 'content' whatever, while *a priori* is equivalent to *formal*, so that *a priori* or transcendental *forms* are proposed for identifying all possible kinds of universals, from those of sensory knowledge (space and time) to those of intellectual knowledge (the categories), and the moral law (the categorical imperative).

A remarkable quality of the universality granted by *form* is that it has nothing to do with the 'equivocation' or 'ambiguity' which is the price that most common sense notions pay when they are applied to different referents. On the contrary, in spite of their being applicable to (or interpretable on) a potentially infinite number of single cases, all formal notions are strictly 'univocal', and if one looks for the reason for this prima facie astonishing property, one finds that it is because they are free from the ambiguities which are *implicit* in the meanings of many intuitive concepts. Therefore the first result of formalization, which is so decisive that it deserves being chosen as its defining characteristic, is total *explicitness*. It is this requirement that suggests, as a practical measure, that we leave meaning aside, since meaning may be charged with hidden, uncontrolled and implicit elements that could produce errors in the long run. A classical example is the *quaternio terminorum* in a syllogism, which occurs when a certain concept is taken with two different meanings in two different propositions, so that the terms really involved are four instead of three, thus making the conclusion incorrect. If two different symbols are applied to denote the two different meanings (which corresponds to a process of formalization), the error becomes patent and may be avoided. This feature explains first of all why formalization turned out to be such a powerful tool in making rigorous all kinds of disciplines (mathematical or not). It did this by permitting the elimination of ambiguities via the analysis and separation of several components in the meanings of basic notions, thereby making explicit presuppositions which were tacit in the immediate intuitions of several traditional disciplines. (In this regard consider e.g. that it was only with Pasch (1882) that the ordering properties of the points on a line - which had remained hidden in the Euclidean no less than in the non-Euclidean axiomatizations of geometry - were made explicit and axiomatized, so that the consequences of admitting or not admitting them could also be investigated). Secondly, this also explains the reason for the multi-valency of the formalisms: their being uncommitted to any *particular* meaning (which is quite different from being devoid of any possible meaning) leaves them open to a plurality of *distinct* (and therefore unambiguous) interpretations.

We may have said enough to present some of the most important factors which have determined the high ranking of formal thinking in our time. Of course, several more might be investigated, such as the impressive power of *unification* with which it is endowed, but it would lead us too far if we should continue this inquiry.

The justification of formalism

We have seen that the predominance of the formalistic attitude in the field of foundational research (especially regarding mathematics) was a result of a multitude of factors (some of a 'negative' and some of a 'positive' character), none of which in itself demanded that mathematical 'content' be fully discarded, but rather that it be kept in the background in order not to interfere with the obtaining of rigour. In fact, the working mathematician is used to availing himself of a special kind of intuition (he 'sees - so to speak - with the eyes of his mind' numbers, functions, structures, sets, large cardinals, abstract spaces and their properties), and has therefore a natural tendency to consider them as 'existing' at least in some vaguely determinable sense. This impression is far from being a simple naive belief, and could be rigorously analyzed and justified by resorting to skilful distinctions which have been proposed in the history of philosophy, from the Medieval doctrines of the *ens rationis*, to the investigations of Meinong[4] in our century. This feeling is so spontaneous and deeply rooted that one really has to be 'forced' to abandon or even reject it. Now the reason which many mathematicians felt to be 'forcing' them to take this step was the explosion of contradictions. Before that moment, formalization and symbolization (including the use of symbolic logic) were mainly conceived as *instrumental*, but not as *indispensable* for foundational research. This means that formal and symbolic axiomatizations were primarily intended either to provide extremely powerful tools for the logical *analysis* of concepts, assumptions and proofs occurring in mathematical theories; or to show how the content of a certain mathematical theory could be expressed by means of quite different axiomatizations (i.e. of axiomatic systems based on different sets of 'primitive notions'); or finally to reduce to the smallest possible number the primitive notions and assumptions necessary for reconstructing the content of a certain theory.

This was the chief interest of G. Peano and his school, to which we owe the famous conception of mathematics as a collection of *hypothetico-deductive* systems, as well as the idea

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4 It would be out of place to go into details here concerning Meinong's doctrine of the objects of thinking. Let us simply recall that, according to him, there are different kinds of *objects*, which are characterized by the different mental acts through which we perceive them. So the objects of sense perception are different from those of thinking, but the latter are no less 'objective' than the former: they are 'apprehended' through thinking, but not constituted by it. Meanings and judgements are examples of this second kind of object: according to his terminology, they 'subsist' (*bestehen*), while individual things and qualities 'exist'. In this sense, objects of thinking may be *real* without 'existing' in the technical sense given by Meinong, and mathematical objects are such. In other words, an intellectual investigation is not bound to the empirical existence of its objects, no *existential presupposition* is needed in order for its objects to be given and 'real' in an exactly specifiable sense. The object *as such* is indifferent to the character of existence. *Theory of the Object* (see Meinong 1904) is probably the work where the most central claims of his doctrine are to be found.

that axioms provide 'implicit definitions' of their primitive concepts, and the first characterization of this kind of definition[5]. In this context the chief metatheoretical problem taken into consideration and widely investigated was that of the *independence* of concepts and axioms, while those of *consistency* and *completeness* were hardly perceived. But this was true even of Hilbert, whose famous book *The Foundations of Geometry*, first published in 1899 (ten years after the first writings of Peano in this field), is formal without being 'formalistic' (no symbolization is used and not even the logic employed is made explicit), and pays great attention to problems of independence, while consistency is only marginally treated, and completeness is not considered at all. Moreover, he says of his axioms that they «express certain fundamental homogeneous facts of our intuition», while stressing that 'the exact and complete description of these relations for the purposes of mathematics» follows from the axioms and nothing else[6]. But this is no wonder: at that time he was still quite Kantian, and in the introduction of his book he writes that «the problem in question (i.e. of setting up axioms for geometry and investigating the way in which they are connected) amounts to logical analysis of our intuition of space», while this intention was made clear again by his placing at the head of the introduction, as a motto for the whole book, these famous words from the *Critique of Pure Reason*: «Thus all human knowledge begins with intuitions, goes from there to concepts, and ends with ideas». It was only at a later stage that he fully concentrated upon the formalisms which are used in the formulation of mathematics, but even at that time he certainly did not intend to *reduce* the essence of mathematics to formalisms, as could easily be shown[7]. But why was Hilbert

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5 Besides several articles devoted by Peano to the problem of definition, one should mention at least the contributions of Pieri (1899) and Padoa (1900).

6 See the first section of the first chapter of Hilbert (1899).

7 Hilbert certainly considered formalism to be an indispensable passage through which the consistency of even the most audacious mathematical theories was hopefully to be proved. But the 'content' of these theories was by no means trivialized, not even the «Paradise (of the infinite) which Cantor has created and from which nobody will ever chase us» (see Hilbert 1926) At the beginning of *The Foundations of Mathematics* (see Hilbert-Bernays 1934-39) an important distinction is drawn between 'concrete' (*inhaltlich*) and 'formal' axiomatics, where the former are by no means considered

(and not only Hilbert) led to concentrate almost entirely on formalisms? The reason is that no other way seemed to be suitable for establishing the consistency of mathematics. Indeed, the problem of consistency had already emerged in the discussions concerning non-Euclidean geometries, and was solved by an *indirect* proof, i.e. by constructing 'Euclidean models' of these geometries. This move contained several implicit requirements of which the geometers who constructed these models (Beltrami, Cayley, Klein, Poincaré) were unaware, and which became clear only in the much more sophisticated metamathematics of our century. Yet their basic intuition was correct: if we are able to *interpret* the axioms of a non-Euclidean geometry so that they become true of some structure of geometrical entities which, in spite of being 'artificial' still obeys the axioms of Euclidean geometry, then we can say that, in case a contradiction should follow from the axioms of the non-Euclidean geometry, this would be reflected in the model, and therefore it would follow also from the Euclidean axioms. Hence, *if* Euclidean geometry is consistent, non-Euclidean geometry must also be so. The obvious question is then: is Euclidean geometry consistent? One could adopt the same strategy and construct a model of this geometry in real number analysis (by using the familiar tools of analytic geometry), but in this way the problem would simply be shifted, as it would be again if one should go on and 'reduce' it to the consistency, say, of elementary arithmetic or of set theory (i.e. of two theories which have been indicated as possible grounds for the 'construction' of all number systems, and therefore of the whole of mathematics). In short, one must in the end be able to prove the consistency of one of the basic mathematical theories *directly*. But which one, and in what way?

Traditional mathematics did not know of and *could not* know of this problem, for it was considered to be based upon *immediately true* axioms. Since the use of correct inference necessarily implies that correctly deduced consequences share the

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spurious. Finally it is certainly not insignificant that in the very years when he was working on his formalistic programme, Hilbert published a book on «intuitive geometry» (see Hilbert/Cohn-Vossen 1932).

truth of their premisses, it follows that the consequences of such axioms must be necessarily true. Therefore a contradiction, which is always *false*, can never be deduced from true axioms, and the problem does not even exist. But modern mathematics, as we have seen, is characterized by a distrust of the possibility of affirming the *immediate truth* of any axiom system, so that the traditional path was barred. On the other hand, by considering axioms as purely formal, i.e neither true nor false, it was even more fundamentally impossible to prove their consistency by resorting to the notion of truth and truth consequences. This was the reason why the only hope seemed to many to reside in the possibility of taking a given axiom system - conceived as a combination of pure symbols devoid of meaning and truth and submitted only to material transformation rules - and trying to show its *formal* consistency, i.e. that a contradiction could not be *proved* in it.

Hilbert's programme

This displacement of the conceptual center of gravity was drastic, since it meant that the warranty of consistency was no longer to be looked for by exploring the nature of the axioms, but the nature and structure of the mathematical proofs. This move was explicitly taken by Hilbert when he characterized as *Beweistheorie* (or *proof theory)* his new approach to the problem of consistency. Since a proof is formal, and moreover since it was now intended to apply to purely formal systems of axioms, a complete formalization was required: this means that not only the specifically mathematical axioms, but also the logical calculus used for producing the proofs, be explicitly and symbolically given. Symbolism becomes now not only useful, but strictly necessary: if our task is that of investigating proofs, they must be susceptible of inspection, and therefore be given in a materially testable way, which only symbolization can afford. Another obstacle may also be avoided: a consistency proof for an axiomatic system had traditionally been believed to be impossible because it seemed to entail an infinite task (our having deduced a hundred or a thousand theorems from the axioms without finding any contradiction does not ensure that we shall never find one). But a *formal* law already known to Medieval logicians is that from a contradictory set of premisses *whatever* conclusion may be correctly deduced, and this law is preserved in standard logical calculi. This means that a consistency proof may reduce to showing that even *one single* arbitrarily selected (and particularly simple) formula, such as 0=1, cannot be formally derived in our axiom system. Now, the axioms are *finite* in number (and each of them consists of a *finite* sequence of symbols), and the transformation rules of the logical calculus used in the formalization (which permit the construction of proofs) are *also finite*. Thus it seems reasonable to hope that by manipulating the axioms according to ways of reasoning resembling those of the combinatorial calculus (which are confined to the consideration of the finite grouping and permutations of symbols) one might be able to show that that particular formula could never be derived. This non-hypothetical reasoning, which is halfway between the material manipulation and the intuitive perception of graphic symbols, was called by Hilbert the *finitistic* method, or «finitary inference» (*finites Schliessen*), and it was prescribed as the method to follow to obtain a direct proof of the consistency of formalisms. We have thus found the answer to the question 'how' to provide this proof.

The second question concerns which axiomatic system should be submitted to this kind of examination. In principle any system whatever could be chosen, but considering that the aim was that of providing a 'foundation' for mathematics, and that the investigations performed in the 19th century had already shown many relationships among the various mathematical theories, it was reasonable to choose the simplest of the known systems. Thus *elementary arithmetic* was considered, with the hope of being able to prove its consistency directly, and then extending this consistency to the other basic axiomatic systems of mathematics (essentially to analysis and set theory). This was in short the celebrated 'Hilbert Programme' which was first announced in the early years of the century, took a precise and very explicit shape around 1920, and was thereafter developed by Hilbert in cooperation with several disciples and collaborators[8].

Gödel's theorem

In spite of its reasonableness, Hilbert's programme failed to meet with the success expected: after some partial positive results, which concerned certain 'weakened' formal systems of elementary arithmetic, in 1931 K. Gödel (1931) proved a theorem which is deservedly considered one of the most outstanding results in the whole of 20th century science, and whose impact on the philosophical discussion was probably equalled only by Heisenberg's uncertainty principle in quantum mechanics. This theorem expresses the intrinsic impossibility of realizing Hilbert's programme in its original form. Indeed, Gödel proved that, if a formal system is *assumed* to be consistent (for if it is inconsistent anything may be proved in it), and is powerful enough to formalize elementary arithmetic, then it is impossible to prove its consistency by means of tools which are formalizable in the system itself. In a shorter form it may be said that any formal system satisfying the very minimal requirements just mentioned is unable to prove its own consistency 'internally'. Now, since the 'finitistic methods' admitted by Hilbert's programme are certainly formalizable in any axiomatic system for elementary arithmetic (and *a fortiori* in any more powerful mathematical system whatever), it was clear that Hilbert's programme had been condemned. This does not mean that some 'modified Hilbert's

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8 This programme was first made known by Hilbert in a paper presented in 1904 at the International Congress of Mathematics in Heidelberg (See Hilbert 1905), where he even used the neologism *metamathematical* to characterize his method for investigating the consistency of a totally symbolized axiomatic system). This proposal remained uninfluential and was almost not understood at that time. Hilbert took it up again and developed it in a succession of papers between 1922 and 1928. (See Hilbert 1922, 1923, 1926, 1928); and some of his outstanding disciples, such as W. Ackermann, J. von Neumann and P. Bernays contributed to the effort of applying it concretely. In spite of the (at least partial) failure of this programme due to Gödel's theorem, the spirit of Hilbert's metamathematics and proof theory continued in a clearly defined line of foundational research, which also found expression in certain standard handbooks such as Hilbert-Bernays (1934-39) and Schütte (1960).

programme' could not work, but only by resorting to more powerful methods which, though not 'finitistic', could nevertheless be considered 'sure' or reliable. Since that time these methods have come to be called 'constructive', and by using them the consistency of arithmetic (and some - but not all - more complex mathematical theories) was actually proved[9].

A very extensive literature has been devoted to the critical analysis of Gödel's result, and we are certainly not able to mention its various aspects here. However a few comments must be made in connection with the specific questions addressed in this paper. First of all we have to note that this result meant a definitive refutation of the *formalistic* view of mathematics, a view which claims that mathematical theories are 'nothing but' formal axiomatic structures devoid of meaning and truth. This claim, as we have seen, is tenable only if formalisms are able to provide at least the internal proof of their own consistency, which is not the case. A consistency proof is only possible by resorting to some tools or methods belonging to some other more powerful system (practically speaking, set theory), and this replicates the procedure of 'discharging' the consistency of one system onto that of some other, a strategy we recognize from the case of non-Euclidean geometries. This of course only expresses relative *consistency*, while proof of *absolute consistency* was being sought. But since the need to use 'external' tools is common to every formal system, not even the most powerful of them (e.g. set theory) could escape this condition. This means that, to the extent that a strictly formal view is accepted, only relative consistency can be established in mathematics - i.e. one establishing mutual relations among different mathematical theories - while it is impossible to obtain an absolute consistency proof either for one single theory or for mathematics as a whole.

One could assume another attitude, and say that the tools accepted for proving the consistency of, say, arithmetic, are

9 Let us only mention that already in 1936 G. Gentzen was able to obtain a consistency proof of elementary arithmetic by using as a metatheoretical tool (i.e. as a 'materially' accepted procedure) the transfinite induction of Cantor's set theory up to a 'constructively' definable ordinal number (the 'first number'). See Gentzen (1934). In more recent years this constructivist proof theory has experienced a rich variety of developments.

certainly formalizable in some mathematical theory, but are nevertheless sure and reliable *in themselves*, quite independently of that accidental circumstance. This position corresponds to the attitude Hilbert adopted towards his finitary methods, but how should we consider it? It undeniably expresses a return to *intuition* and, in this sense, equally disclaims that programme of eliminating intuition which we have seen to be at the very root of the formalistic view. This trend is even more clearly perceivable in a practice which has become very usual for granting consistency after Gödel's result: the practice of finding a *model* in which the axioms are satisfied, i.e. are *true*. This strategy was encouraged after Tarski (1936) had shown a rigorous way of interpreting formal languages, and has been developed in that important branch of mathematical logic known as *model theory*. In this case we are in the presence of a vindication of the 'classical' view: if the axioms are 'true' of something then we feel safe, since no contradiction may be deduced from true sentences. Of course, one can still remark that, after all, models are constructed by using methods that are formalizable in the last analysis within set theory, so that they ultimately depend on it. But if we accept this view, we are back again to the impossibility of proving consistency.

There is another way out, when Hilbert decided to accept the finitary methods as unquestionable, he was explicitly underlining their material and empirical nature, and this enables us to say intellectual intuition (a position which we have already met). Unfortunately such 'materially intuitive' tools proved to be insufficient. But now we may ask whether the 'constructive' methods advocated by the supporters of proof theory after Hilbert can be credited with being 'materially intuitive', and this is at least highly debatable.

Other remarks are related to a consideration of the methods used in Gödel's proof (as well as in analogous metatheoretical investigations). These methods consist in mapping the symbols of the formal system into the natural numbers, so that all the metatheoretical properties of the system are automatically mapped into concrete numerical properties. In this way these metatheoretical properties considered to be 'material' and no longer formalistic (up to the point at which the formally undecidable sentence of Gödel's proof turns out to be universally true in arithmetic). This means that not only must the material symbols of the formal system be accepted as objects of our sensory intuition in performing such an investigation, but so must the natural numbers with their properties; and this too oversteps the strict limitations of the formalistic approach.

As a conclusion of this survey we must say that the programme of employing a strictly formalistic conception of mathematics proved to be unworkable, and that the necessity of relying upon some kind of intuition - related to meaning and reference - was recognized, in spite of the reasons which had led to mistrusting these factors in the past.

The adequacy of formalism

What we have being considering in the above section are sometimes called the internal limitations of formalism, and may be seen as disclaiming the pretension of the 'self-sufficiency' of formalism. However, a set of interesting problems remains after this renunciation and concerns the adequacy of formalism with regard to those tasks which it has often been allotted, and may be summarized by saying that formalism should free us of the servitude of meaning and reference. As we have seen in our historical account, this has to a great extent been a consequence of our having constructed *artificial* symbolic languages according to the model of abstract algebraic calculi, which are totally free of any *reference* to particular objects. This fact was soon expressed by saying that these calculi are 'devoid of *meaning*', and in such a way an identification of meaning and reference was implicitly, and even unconsciously, introduced. Moreover, since formalization appeared to be an excellent tool for eliminating the logical difficulties which seemed to nestle in several familiar 'intuitions', and since these intuitions may be qualified as 'meanings' (to the extent that meaning is often understood as being the content of some intellectual representation), the idea that formalisms are abstract constructions without meaning was strongly reinforced, this time without implicitly identifying meaning and reference. As a result, the undeniable success of formalization was easily attributed to the fact of its having eliminated the encumbering burden of meaning and reference. But meaning and reference had always been considered the basic conditions for producing any true discourse, so that their elimination also eliminated *truth* from the formalized discourse, as we have actually seen.

The first impression one might have is that the elimination of truth was to involve total destruction, leaving only a desert, rather than the more solid knowledge which was aimed at. But the actual situation was not that negative: indeed the realm of the abstract is not a vacuum, and the systems of symbols can be 'understood', which means that they remain endowed with some kind of meaning after all. This meaning is at the same time purely intellectual and essentially operational, and since it has to be attributed to symbolic structures which are considered as 'languages', it may be reasonably qualified as a 'syntactic' meaning. What the formal point of view discards is the meaning that we could call 'eidetic', and that corresponds (as this terminology expresses through its links with a long philosophical tradition going from Plato and Aristotle to Husserl) to the 'intension' of concepts and propositions, to a mental content or representation.

These remarks show that the move towards formalization was by no means a loss, but rather a gain of *intelligibility*: the abstract turns out to be more intelligible than the intuitive, than the 'concrete', and the intellectual effort which is demanded for overstepping the eidetic intuitions is compensated by a higher degree of clarity, by an elimination of ambiguities and inconsistencies. However, one might fear that these advantages are nothing but the consequence of a terrible impoverishment, of a closure of the intellect upon itself. Of course, once this closure is in operation, everything is clear: indeed how could the intellect not 'intellige' its own constructions? But what could be the point of such an 'intellection'? What kind of knowledge would it provide?

The point of departure had been an attempt to reach a more reliable form of knowledge of reality, while the point of arrival risks being knowledge of certain of our intellectual games. It is therefore clear that the problem is that of not making intelligibility so banal as to reduce it to emptiness, and this concretely means that a formalistic programme - if it is stripped of its possible polemic and iconoclastic shadings - appears as a challenge to *replace* the traditional features of reference, meaning and truth with something that is able to play their role in spite of being confined within the limits of what we have called 'syntactic' meaning.

Two quite different ways could be, and have been, adopted to escape the making of intelligibility banal: we have already mentioned them, and we may now add that they could be labelled as the 'realistic' and the 'idealistic' solutions respectively. The first is usually expressed by saying: formal systems do not possess any meaning, but may *receive* one, and even more than one, through suitable interpretations. In order to be fully consistent with the formalistic point of view, such an interpretation must be conceived extensionally, and not intensionally, i.e. the interpretation should provide referents, sets of referents, sets of ordered n-tuples of referents, etc. to be directly connected with the symbolic expressions of the formal language, without the mediation of any intensional or eidetic meaning. We could say that according to this view there is, apart from the syntactic meaning, only a 'referential meaning', which is sufficient to avoid banality and also to recover the notion of truth via the notion of 'satisfaction'. The second solution is usually expressed by saying: the formulae of the formal systems *determine* the meaning of their primitive notions contextually, and these notions become at the same time the objects of the discourse. We have called this position idealistic, because it shares with idealism the basic tenet that there are no things or referents 'outside' thought, that they are determined by thought and nothing more. One could say that in the formalistic context it is said of language and not of thought; however, since this language is not seen as simply a heap of material signs, but rather as a structure endowed with 'syntactic meaning', the real situation is that this meaning is claimed to be able to exhaust the roles of the eidetic and referential meanings, and this escapes banality only within the idealistic claim that there is no reality other than that constructed through thinking.

Let us now try to evaluate the success of these two strategies in avoiding the use of eidetic meaning. Following the 'realistic' approach, the adequacy of a formalism should be determined on the basis of its ability to fully characterize a universe of 'given' referents or objects. Now certain well known theorems of mathematical logic show that this ability is not very great. On the one hand we know since the time of Gödel's theorem that even such a simple theory as elementary arithmetic is 'semantically incomplete', i.e. that *any* formal system designed with the aim of describing the structure of the natural numbers will be unable to capture *all* their properties (since there are propositions about these numbers which are true of them, but which are not provable in the formal system). On the other hand the 'isomorphism theorem' of mathematical logic proves that if a formal system admits a model in a given universe of referents, it automatically admits an infinity of other models isomorphic with the first. This fact indicates that we can at best formally characterize only the *structure* of our domain of referents. Still this happens to be a rather exceptionally fortunate situation, for it is known already from a theorem of Skolem[10], that in general formal systems are

10 The theorem, technically known in the literature under the name of «Löwenheim-Skolem», since it was partially anticipated by L. Löwenheim in 1914, says that if a set of formulae of first-order logic is simultaneously satisfiable in any non-empty domain (i.e. if the set has a model at all), then it is simultaneously satisfiable in a denumerable domain (i.e. it has a denumerable model). Since an axiom system is a set of formulae, it follows that every first order theory that has a model has a denumerable model (see Skolem 1919). This result already implies that if our intention with our axiomatization were that of characterizing a domain of objects having a finite, or a more than denumerable cardinality, we would not succeed in our effort, since that axiomatization would also be satisfied in a domain with a denumerable cardinality (it is also provable that it would have a model of any arbitrary infinite cardinality). By developing this line of thought, Skolem (1933, 1934) proved that if a first order theory of arithmetic (with identity) has its intended model, then it also has a normal model that is not isomorphic with its intended model. This was the origin of what has been called *non-standard arithmetic*. A. Robinson, by using similar methods for a formal system of the theory of real numbers, has developed a theory of *non-standard analysis* (see Robinson 1966).

not 'categorical', i.e. that they admit different models which are not even mutually isomorphic, and this is tantamount to saying that they cannot even univocally characterize the 'structure' of a given domain of referents (in the sense that what they say about this domain is equally valid for some non-standard or unintended models). It would be out of place to enter here into technical details concerning the strategies for recovering categoricity at least to certain degrees; and in any case it would appear that these strategies imply giving up completeness, so that these two features, which should *jointly* grant the ability of a formal system to perform referentially, are hardly compatible, therefore making such a performance not really adequate.

Coming now to the 'idealistic' strategy, its adequacy would consist, as we have seen, in a kind of capacity of the formal system to 'produce' a domain of referents, in order that it not be simply an empty language, but one which, at least in a minimal sense, speaks 'about' something. This requirement is imposed by the fact that a formal system which is supposed or even intended to express a certain specific theory (e.g. elementary arithmetic) in a strict sense. Indeed even in mathematics a difference is admitted (except by logicists) between 'logical truths' and 'mathematical truths', and if one agrees to qualify as logically true those propositions which are true in all possible models (or in all 'possible worlds'), one should say that mathematically true propositions are true only in some possible worlds. Our problem may now receive the following more precise formulation: given a set of propositions satisfying the minimal 'formal' requirement of being consistent, can we be sure that there exists at least one possible world in which these propositions are true? We know that several mathematicians have been and are of this opinion, including some who were not ready simply to identify mathematical existence with consistency (let us only mention Poincaré, who was a pre-intuitionist in certain respects while nevertheless being inclined to accept the view that every consistent set of mathematical propositions admits a model).

What does mathematical logic tell us about this claim? It says that if we can prove, for whatever formal language, that *every* consistent set M of sentences of that language admits a model, then it is possible to formulate in this language a logical calculus that is 'semantically complete'. This result already shows how the above claim is untenable: if we really maintain that *every* consistent set of mathematical sentences has a model, we should conclude that *every* logical calculus is semantically complete, while it is known that semantical completeness is a property which does not hold in general for logical calculi beyond the level of first-order logic. This conclusion is already interesting because it shows at any rate that consistency and truth are not totally identifiable even in mathematics, since there is at least some possibility of discriminating between them. A 'realist' would say that all that is obvious, for it was actually too pretentious to suggest that consistency be a sufficient warranty for the 'existence' of objects. However, we shall try to be fair and see whether there is at least a sense in which a consistent set of sentences may 'create' its own model, without pretending this to be a 'creation *ex nihilo*', but rather a construction obtained by using the ingredients provided by the language itself in which the consistent set of sentences is formulated. This is clearly quite a generous concession, but it is only given this concession that it is possible to prove that every consistent set of sentences formulated in the language of *first-order* logic possesses a model (this model being constructed out of the sentences' linguistic ingredients), while this is no longer the case if the axioms are formulated in a higher-order language[11]. This fact is of

11 One could object that, by using the same techniques introduced by Henkin for the construction of the model of any consistent set of first-order sentences, it is also possible to obtain the same result for consistent sets of sentences of higher order, as is shown e.g. in Henkin (1949, 1950). But this fact entails that semantic completeness also holds for second order logic and for the theory of types. This is contrary to what is commonly said in textbooks of mathematical logic, and contrary to the fact that the semantic incompleteness of second order logic derives directly, e.g., from Gödel's completeness theorem for first-order logic with his incompleteness theorem for elementary arithmetic and the categoricity of second order arithmetic. Without going into technicalities, it is clear that semantic completeness and the existence of a model for any consistent set of sentences are understood in a *particular and partially different* sense in these enlarged versions, so that they are not incompatible with the preceding claims and results. In particular, the domains of objects which are introduced are *non-standard*, i.e. they respect the condition that transformation rules preserve the truth from the premisses to the consequences in such models, which, however, are not maximal. Moreover the condition of extensionality

importance especially if we keep in mind the philosophically relevant point of the 'idealistic' claim, i.e. that the referents of a formal theory, such as the natural numbers, are created by the formal system itself. Now, according to the result just mentioned, the warranty for the existence of such referents would depend not so much on the skilful choice of consistent set of axioms, but basically on the expressive power of the language used. This is already strange, but it becomes more puzzling if we consider that this warranty decreases with the improvement of the richness and expressiveness of the language. This fact contradicts the spirit of any idealistically inspired approach, for which the mark of 'truth' or soundness is typically represented by the comprehensiveness of a system, and its capability of embracing the widest variety of properties, relations and their levels (which corresponds to an increase in the expressive power and the complexity of the language)[12].

However, this is still less important than another fact. In the construction of the said model, the usual methods are applied which characterize the Tarskian procedure for defining interpretations and the satisfaction of the expressions of a formalized language, and these methods presuppose that a certain domain of objects be given which is distinct from the language that will be interpreted on it and 'speak about' it. Now this

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is not preserved. Let us note that the criteria which preside over the construction of such models and make reasonable the weakening of the standard semantical conventions are dictated by the goal of assuring the strictest possible correspondence (which however cannot in general be one-to-one) between the components of the language and those of the 'linguistic' domain on which they are interpreted. In this way the components of the language replicate the features of the 'linguistic' model obtained for the consistent sets of first-order language.

12 It would be instructive, but would take us too far, to investigate the strict links which exist between the idealistic thesis of the 'wholeness' of truth, with its consequent doctrine of 'internal relations', and the idea of the self-containing nature of formal systems, which 'internally' determines all the meaning and truth (these notions being taken 'formally') of all their components. This idealistic doctrine has known a variety of developments and modifications, from Hegel to Hamelin, Bonsaquet, Bradley, Gentile, Joachim and others. An account of it may be found e.g. in Ewing (1934). A heated polemics was engaged in against this line of thought in the first decades of our century by Moore and Russell, whose criticism of the 'axiom of internal relations' was expressed particularly in Russell (1906, 1910). However, a significant family resemblance with this doctrine may be found in the 'coherence theory of truth' recently elaborated by N. Rescher. See in particular Rescher (1973).

distinction is eliminated in the construction of such a model, since the domain of objects is constituted by the set of closed terms of the language, by a kind of double 'self-reference': every term is taken as the metalinguistic name of itself, and every sentence is declared to be 'true' of the terms it contains simply if and only if it belongs to the consistent set M[13]. Hence one should at least be conscious of the very particular situation here, which we might describe as follows. We can take the above result to mean that, at least in the case of first-order languages, every consistent set M of sentences describes a 'possible world'. Then we ask: «what possible world?», and the answer we receive is: «the world described by M, of course». The situation seems quite ironical, but actually depends on the fact that the 'referential' feature of truth, which is contained in the Tarskian methods, is reduced to a pure appearance, so that no possibility of 'falsifying' the sentence of M really exists. In fact, since they do not speak about any independent structure of objects, such sentences are protected against any extra-linguistic or referential refutation, and they are also protected against the only remaining source of 'linguistic' refutation, i.e. contradiction, since the set M is *assumed* to be consistent.

At this point one might ask why such a result is taken seriously in mathematical logic. There are at least two reasons. The first is that the procedure of interpreting a language in terms of itself, while certainly 'artificial', is not absurd, and is adopted with profit in certain branches of abstract mathematics (e.g. when as a representation of a group one takes the permutations on a domain whose elements are the elements of the group itself):

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13 It is perhaps worth mentioning that models of consistent sets of first order sentences may be constructed using as a domain of objects not the ingredients of the language, but e.g. natural numbers. However this does not really change the situation, for in the construction of these models the strategy of numerically 'coding' the individual variables of the sentences is usually followed, so that the model is constructed using these coding numbers. This move does not imply any reference to the structure of natural numbers proper, since, when the sentences are interpreted, they are again 'declared' true if and only if they belong to M and not if they express some *numerically* true property in a real sense. This is possible thanks to the extensionalist semantics, which makes properties and relations in the model entirely dependent upon the language, without any 'reference' to the structure of the domain of objects. For a more detailed discussion of these issues see Agazzi (1978).

what matters is to understand what this procedure actually means. The second reason is that, after all, even this 'artificial' model is not granted for logical languages of a higher order, and this fact is strictly bound to such an important requirement as the 'semantic completeness' of the logical calculi, which does not hold in general, in a proper sense, beyond first-order logic.

We can derive two morals from the above reflections. The first is that formalisms may perform a kind of weak truth-granting role (which we may call 'truth by consistency') at the level of first-order logic, while they are not generally in the position of performing even this role at higher levels. The second is that the existence of these results seems to impose a moderation upon the formalistic trends in mathematics and upon a certain excess of importance given to axiomatizations of all possible kinds, from which (under the assumption that they are consistent) many curious theorems are deduced. The question is: are these axiom systems really mathematical? In order to show that they are, one should show that they can apply to some recognized 'mathematical objects' or (if one prefers to remain ontologically uncommitted) to be of help in the solution of 'serious' mathematical problems (the requirement of seriousness being expressed by the fact that such problems are not just 'internal' to the formalism which is intended to solve them). This is the gist of the reaction of certain outstanding mathematicians of our time against 'gratuitous' abstractions and 'meaningless and uninteresting' axiomatizations.

Formal and non-formal

The different reasons for the inadequacy of the radical formalism which we have considered above may probably be clarified through an analysis of the way in which a language may be used in speaking about a given domain of objects. The first condition is, obviously, that these objects be there, and this is not realized through a speech act, but through the *presence* of these objects to our thought. We shall here call this presence 'phenomenological', in order to cover by this term all possible 'ways' of being present, and to suspend any judgment about the ontological status of what is present. It also seems legitimate to call this phenomenological situation *truth*, for it is such that an object, simply by being present, offers to thought an irrefutable (and perhaps the only irrefutable) witness of itself. Therefore the phenomenological truth is 'unstable' and 'private': it is unstable because it does not allow us to remain within the realm of truth when we leave the immediate presence; it is private because the presence of certain objects is such only for that individual for which they are actually and instantaneously present. Now it can be said that a fundamental function of language is that of overcoming these shortcomings, enabling us to 'preserve' in some way truth also outside the instant of its immediateness, and to make it *intersubjective*. In such a way the characteristic of truth, which is intrinsic to the phenomenological situation, is transferred to language, as is shown by the fact that the most common (and most appropriate) use of the term "true" concerns the propositions of a language, and is primarily attributed to them when they 'denote' a state of the objects which is phenomenologically present.

But language itself can enter the sphere of presence, and this under two aspects. On the one hand it is present with its structures and ways of functioning; on the other its 'denoting' (i.e. its referring to another sector of presence) is also present. Formal logic may be seen as belonging to this stage and, more specifically, to the investigation of those functions of the language capable of leading from propositions denoting a presence to other propositions also expected to denote a presence. This is tantamount to saying that logical inference itself develops within the phenomenological sphere (i.e. so-called 'logical evidence' is a special kind of phenomenological evidence). And thanks to this, by means of a logically evident inference we can bring a proposition which does not phenomenologically denote a presence into the domain of truth, provided we can derive it from propositions which are phenomenologically evident.

This quick description actually involves several delicate steps. First of all, the language must rely upon tools for 'retaining' the presence of objects even when they are actually no longer there, and these tools are the *meanings*, which in this way appear to be 'extracted' from the *referential situation* (which is the situation of presence), but do not coincide with it, since they hold even outside this situation. However, owing to this original link, they may be seen as a permanent and open 'possibility of referring'. It is worth noting that meanings are already only partially 'faithful' with regard to any particular phenomenological presence or referential situation they might denote. For example, the concept 'man' does not contain all the details of every single man who might be denoted through it, but its not doing so is part of the condition for it to denote men who do not share all these details.

Concepts or meanings are mental entities and, as such, are private. In order to become public they must be associated with the expressions of a language, and this determines the passage to the *formal* level, «formal» being here understood (in conformity with an already explained view) as the realization of conditions for explicitness and non-ambiguity, so that the correct application of these conditions should enable man to understand what another 'means' by using a certain expression. This stage entails the creation of a complex structure, since the compactness of the 'presence' is not only analyzed through a complex net of mutually interwoven meanings, but the language itself must then replicate to some extent the complexity of this meaning-structure in order to express it. This is why the *semantic* performance of the language (i.e. its ability to express meaning) necessarily presupposes its possessing a certain *syntactic* structure. On the other hand a syntactic structure, the different components of which can be seen as having the 'possibility of expressing a meaning' in a sense analogous to that of meanings having the possibility of denoting referents. Meaningful sentences which also denote a presence are said to be true, and a conspicuous part of syntax, as we have said, consists in exploring the structure of the domain of true sentences.

Let us now consider some consequences of certain technical results which have been surveyed in the preceding sections. Certain of them say that a formal system (i.e. a language according to our last remarks) on the one hand says 'more', and on the other says 'less' than what is true of its intended models (or referents). This means that no language can express the 'presence' completely adequately. This is already the case with any private use of the language (my linguistic description is unfaithful with respect to what is present to my thought). Moreover, an interlocutor receiving my linguistic communication will translate it to denote a presence for him, and an additional inadequacy will be added. All this is tantamount to saying that absolute truth (which we have seen to coincide with the situation of phenomenological presence) is not intersubjective. Another limitation we have seen is that in general formal logical calculi cannot deductively cover the whole domain of the true sentences expressible in a formal language (semantic incompleteness).

These facts may be restated by saying that it is not possible to fully characterize a structure of objects, through a language, or to fully master even that portion of 'truth' which the language is able to express about these objects through the deductive operation of the language. The first fact indicates the inadequacy of the semantic dimension with respect to the referential or phenomenological one; the second indicates the inadequacy of the syntactic dimension with respect to the semantic one. Moreover, it must be stressed that these conclusions could be reached within the phenomenological presence in which the analysis of the denotation and of the syntactic structure of the language were performed. This links with what we have noted when speaking of the impossibility of escaping 'intuitive' reasoning of different kinds in metatheoretical investigations.

These remarks are by no means intended to discredit formalism. They only reject the opposed pretensions of 'reducing' one level to another, on the one hand, and of 'disconnecting' them on the other. But once these mistaken attitudes are avoided, the great merits and advantages of formalism become clear. Indeed, the fact that we have spoken of 'inadequacies' may suggest the idea of a negative appreciation, but we have actually seen *why* it was not only reasonable but necessary to pay this price in order to overstep the authentic limitations of the original situation of phenomenological truth. Two of these reasons have already been indicated: proceeding towards formalization allows one to overcome the instability and privacy of the original truth. But something else may be added. In the first place, the advancement in the construction of a more and more complex net of meanings, which is made possible partly by the increasing formal complexity of languages, corresponds to the achievment of higher levels of universality. Therefore the trend towards formalization does not express a 'flight' from reference and meaning, but a search for universality, which allows for a unification of our experience.

From this point of view, formalization is the latest expression of our aims at unifying the core of our rational understanding of reality, and has been the constant driving force of philosophy since the earliest times. This effort has to do with the fact that the immediate 'presence' is instantaneous while at the same time we feel ourselves to be in the presence of 'one' reality; and this challenges us to find 'general' tools for connecting the instantaneous witnesses of the presence, so as to express the wholeness of reality. Formalization, as we have seen, offers a very efficient means to this end.

A second reason is perhaps even deeper. The maturation of human knowledge (both individual and collective) corresponds to the awareness that the content of the immediate presence is only a part of the much richer dimensions of reality, so that the great cognitive adventure of man consists in going behind immediateness to enlarge the domain of truth. But then it is clear that the elaboration of instruments consisting precisely in 'detaching' us from immediateness is of paramount importance. These instruments belong to the sphere which we could broadly call that of *abstraction*, and they are to be used very delicately, since we are not sure where they will lead us, once we leave (as we *must*) the sound ground of the immediate presence. From this point of view it is undeniable that the development of formalization in the history of mankind corresponds to a powerful process of *controlled and reliable use of abstraction*, the eloquent results of which may be found in the different sciences and in the great amount of knowledge they have reached by overstepping immediateness (and it is not by chance that this process has been accelerating with the increase of the use of formal tools in most recent times).

The impulse towards formalization, as we have seen, has been especially powerful in the last two centuries, when it has also expressed the manifestation of the free (though not fully arbitrary) creativity of the human mind. From this point of view, the formal way of thinking is a manifestation of the typically 'modern' awareness of the complexity of knowledge, and of the great role which the subject plays in it. The active intervention of the subject in the construction of knowledge has been stressed in several ways and it has sometimes led to the annihilation of the idea of an object of knowledge. Here again we can see in the process of formalization a very valuable expression of many ways in which this constructive aspect of human subjectivity intervenes in the knowing process; and investigations on the 'limitations' of formalism may also be understood as quite objective indications of the impossibility of eliminating the objects of knowledge. This is perhaps one of the most significant contributions we have reached on the clarification of the intimate complementarity of these two aspects, just to give an example: it is often (and correctly) claimed that a mathematician must first intuitively 'see' a theorem before formally proving it. However it is equally true that what he 'sees' will not become a theorem until it has been formally proved. More generally, we must recognize that our knowledge of reality has known astonishing advance, e.g. in the domain of the natural sciences, thanks to the application of more and more abstract conceptualizations and formal tools.

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