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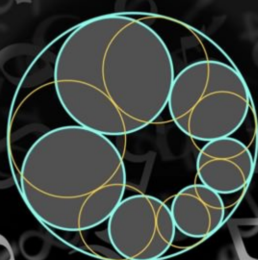

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On sums of distinct powers of 3 and 4

Maximilian F. Hasler and Giuseppe Melfi

In 1996 Erdős conjectured that the set $\Sigma(\mathrm{Pow}(\{3, 4\}), 1)$ defined as the sums of distinct powers of 3 and distinct powers of 4 has positive asymptotic density. We investigate some structure properties of this set. We also prove some asymptotic estimates for its counting function $P_{\{3,4\}}(x)$. In particular we prove that $P_{\{3,4\}}(x) \gg x^{0.97777}$, improving an old estimate of Melfi.

1. Introduction

Let $a_1, a_2, \ldots, a_k$ be a finite nondecreasing sequence of positive integers and let s be a nonnegative integer. Let $\Sigma(\mathrm{Pow}(\{a_1, a_2, \ldots, a_k\}, s))$ be the set of all terms that are sums of an arbitrary number of terms of the type a_i^r for distinct pairs (i, r) with $r \geq s$ and $i = 1, \ldots, k$, and including the empty sum.

Define also the counting function

$$P_{\{a_1,a_2,\ldots,a_k\}}(x) = \#\{n \in \Sigma(\mathrm{Pow}(\{a_1, a_2, \ldots, a_k\}, 0)) : n \leq x\}.$$

In [Burr et al. 1996], where the same notation as above is proposed, among other things the following question is raised: does the set of sums of distinct powers of 3 and distinct powers of 4 have positive asymptotic density? Erdős [1997] (see also [OEIS 2019]) conjectured a positive answer to this question in 1996. So far, the best result in terms of asymptotic behavior [Melfi 2001] for the sums of distinct powers of 3 and distinct powers of 4 is $P_{\{3,4\}}(x) \gg x^{0.965}$.

As pointed out in [Melfi 2001, Proposition 1], the problem of studying the asymptotic behavior of $\Sigma(\mathrm{Pow}(\{3, 4\}, 1))$ is not fundamentally different from the problem of studying the asymptotic behavior of $\Sigma(\mathrm{Pow}(\{3, 4\}, 0))$: indeed, if $P_{\{3,4\},1}(x)$ is the counting function of $\Sigma(\mathrm{Pow}(\{3, 4\}, 1))$, one has $P_{\{3,4\},1}(x) \leq P_{\{3,4\}}(x) \leq 4P_{\{3,4\},1}(x)$.

At the origin of the study of this problem there is the trivial remark that

$$\Sigma(\mathrm{Pow}(\{2\}, 0)) = \Sigma(\mathrm{Pow}(\{3, 3\}, 0)) = \mathbb{N}.$$

So, in terms of asymptotic density and other asymptotic properties, the most natural nontrivial case appears to be $\Sigma(\mathrm{Pow}(\{3, 4\}, 0))$.

In Section 2 we will focus on some structural properties of $\Sigma(\mathrm{Pow}(\{3, 4\}, 0))$, in particular by defining a special function whose properties are strictly related to the arithmetic structure of $\Sigma(\mathrm{Pow}(\{3, 4\}, 0))$.

In Section 3 we will prove

$$P_{\{3,4\}}(x) \gg x^{0.97777}.$$

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We also give an upper bound for the lower asymptotic density as follows:

$$\liminf_{x\to\infty} \frac{P_{\{3,4\}}(x)}{x} \le \frac{1015}{1458} \simeq 0.69616.$$

In Section 4, we conclude with some general remarks on the possible developments and on other closely related questions.

2. Some preliminary tools

This section is mainly dedicated to the definition of an ad hoc function $k : [1, 4/3] \to \mathbb{R}$ and to the study of its properties. This function will appear as one of the main tools in the proof of the main result in the next section.

Let $D = [1, 4/3]$, and $c \in D$. If

$$1 \le c < \frac{3^9}{4^7} = \frac{19683}{16384} \simeq 1.201355,$$

let

$$A_c = \bigcup_{\substack{\alpha\in\Sigma(1,3,9,\dots,3^9)\\ \beta\in\Sigma(1,4,16,\dots,4^7)}} \left[\alpha+\beta c,\ \alpha+\beta c+\frac{1}{2}+\frac{c}{3}\right]. \tag{1}$$

Here, according to [Burr et al. 1996], $\Sigma(x, y, z, \dots) := \{0, x\} + \{0, y\} + \{0, z\} + \cdots$. As will become clear in Section 3, the choice of powers of 3 up to 3^9 for α and powers of 4 up to 4^7 for β comes from the fact that $[3^9, 3^{10}] \subseteq [c4^7, c4^8]$.

If $3^9/4^7 \le c \le 4/3$, let

$$A_c = \bigcup_{\substack{\alpha\in\Sigma(1,3,9,\dots,3^8)\\ \beta\in\Sigma(1,4,16,\dots,4^6)}} \left[\alpha+\beta c,\ \alpha+\beta c+\frac{1}{2}+\frac{c}{3}\right]. \tag{2}$$

In this case $[3^8, 3^9] \subseteq [c4^6, c4^7]$.

Definition 1. We define $k : D \to \mathbb{R}$ as

$$k(c) = \min\left\{\frac{1}{x}\int_0^x \mathbb{1}_{A_c}(t)\,dt : x \in [0, \max A_c]\right\}. \tag{3}$$

Here, $\mathbb{1}_{A_c}$ is the characteristic function of the set A_c, so the integral (3) over t is equal to the measure of the set $A_c \cap [0, x]$.

Lemma 2. *We have*:

(a) *$k(c)$ is a continuous function on $D \setminus \{3^9/4^7\}$.*

(b) *$k(c)$ is piecewise either of the form $A + Bc$, or of the form $p + q/c$, with $A, B, p, q \in \mathbb{Q}$.*

Proof. (a) Let $c \in D$. The set A_c is a union of intervals, and its connected components are of the form

$$\left[\alpha+\beta c, \alpha'+\beta' c+\frac{1}{2}+\frac{c}{3}\right]. \tag{4}$$

The value (or the values) of $x_{\min}$ that define $k(c)$ must correspond to a value in A_c which is a left border of a connected component of A_c. In order to minimize the expression in the infimum of (3), $x_{\min}$ must be

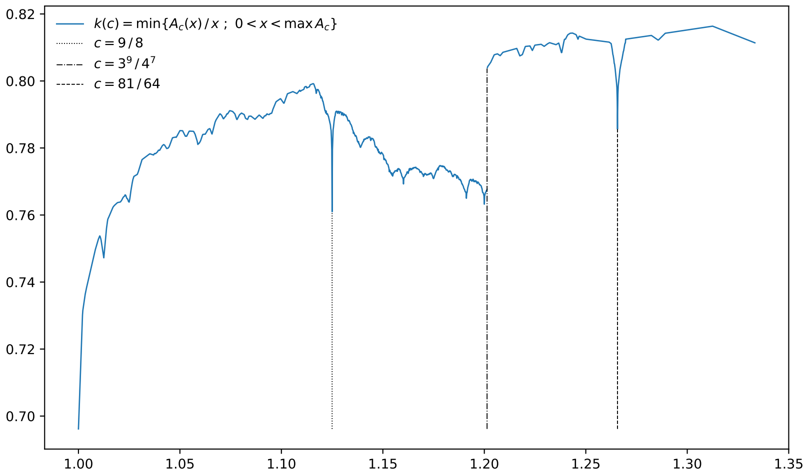

Figure 1. $k(c)$ for $1 \leq c \leq 4/3$.

preceded by a relatively large interval not belonging to A_c. Computations show (see also Table 1) that $x_{\min}$ is always among the five values $\{3^4, 3^5, c\,4^3, c\,4^6, c\,4^7\}$. Let us consider the following five functions $k_i : D \to \mathbb{R}$:

$$k_i(c) = \frac{1}{3^i}\int_0^{3^i} \mathbb{1}_{A_c}(t)\,dt \qquad \text{for } i = 4, 5,$$

$$k_i(c) = \frac{1}{c\,4^i}\int_0^{c4^i} \mathbb{1}_{A_c}(t)\,dt \quad \text{for } i = 3, 6, 7.$$

Each k_i is a continuous function on $D \setminus \{3^9/4^7\}$. Since $k(c) = \min_{3\leq i\leq 7} k_i(c)$, it follows that k is a continuous function on $D \setminus \{3^9/4^7\}$ as well.

The discontinuity at $c_0 = 3^9/4^7$ comes just from the fact that the definition of A_c abruptly changes at that point, according to (1)–(2). Indeed, k cannot be prolonged to a continuous function in $c_0 = 3^9/4^7$: we can compute

$$\lim_{c\to c_0^-} k(c) = p_0 + \frac{q_0}{c_0} = \frac{7736399}{10077696} \simeq 0.76767537$$

(with $(p_0, q_0) = (239/1536, 24095/2^{15})$ in the last interval to the left of c_0) and

$$\lim_{c\to c_0^+} k(c) = k(c_0) = \frac{533311}{663552} \simeq 0.80372149;$$

see also Figure 1.

(b) Let us consider the set $E = \{c_1, \dots, c_N\}$ of all values $1 < c_1 < c_2 < \cdots < c_N = 4/3$ of $c \in [1, 4/3]$ for which

$$\alpha + \beta\,c = \alpha' + \beta'\,c + \frac{1}{2} + \frac{c}{3} \tag{5}$$

or

$$\alpha + \beta\,c = \alpha' + \beta'\,c \tag{6}$$

for some $\alpha, \alpha' \in \Sigma(1, 3, \dots, 3^9)$ and $\beta, \beta' \in \Sigma(1, 4, \dots, 4^7)$.

Values c_j satisfying (5) are values for which two distinct intervals of the form $[\alpha+\beta c,\ \alpha+\beta c+1/2+c/3]$ may merge or split when one considers the family of their unions in a neighborhood of c_j. Moreover, if c_j and c_{j+1} satisfy both (5) and (6), the Lebesgue measure on the union of two such intervals increases or decreases linearly in $[c_j, c_{j+1}]$. Note also that $c_j \in \mathbb{Q}$ for each $j = 1, \dots, N$.

Let $c \in [1, 4/3] \setminus E$ and j be such that $c_j < c < c_{j+1}$. Let us consider the restricted function

$$k_4|_{[c_j,c_{j+1}]} : [c_j,\ c_{j+1}] \to \mathbb{R}, \qquad c \mapsto \frac{1}{3^4}\int_0^{3^4} \mathbb{1}_{A_c}(t)\,dt.$$

For a certain $B_4 \in \mathbb{Q}$ we have $k_4|_{[c_j,c_{j+1}]}(c) = k_4|_{[c_j,c_{j+1}]}(c_j) + B_4(c-c_j)$, with $k_4|_{[c_j,c_{j+1}]}(c_j) \in \mathbb{Q}$ as well. This is because as long as $c_j < c < c_{j+1}$, each of the connected components of $A_c \cap [0,\ 3^4]$ has a Lebesgue measure that decreases or increases linearly with c. The same argument shows that there is $B_5 \in \mathbb{Q}$ so that $k_5|_{[c_j,c_{j+1}]}(c) = k_5|_{[c_j,c_{j+1}]}(c_j) + B_5(c-c_j)$, with $k_5|_{[c_j,c_{j+1}]}(c_j) \in \mathbb{Q}$ as well.

Let $c \in [1, 81/64]$ and $c_j < c < c_{j+1}$. Let us consider

$$k_3|_{[c_j,c_{j+1}]} : [c_j,\ c_{j+1}] \to \mathbb{R}, \qquad c \mapsto \frac{1}{c\,4^3}\int_0^{c4^3} \mathbb{1}_{A_c}(t)\,dt.$$

Let

$$A'_c = \bigcup_{\substack{\alpha\in\Sigma(1,3,9,27)\\ \beta\in\Sigma(1,4,16)}} \left[\alpha+\beta c, \alpha+\beta c+\frac{1}{2}+\frac{c}{3}\right].$$

Since $1+3+3^2+3^3+c+4c+16c+1/2+c/3 < 4^3c < 3^4$, we have

$$\int_0^{c4^3} \mathbb{1}_{A_c}\,dt = \mu(A'_c),$$

where μ denotes here the Lebesgue measure in $\mathbb{R}$. Note that as long as c varies in $[c_j,\ c_{j+1}]$, the measure of A'_c decreases or increases linearly with c. Therefore, there are $p_3, q_3 \in \mathbb{Q}$ such that $k_3|_{[c_j,c_{j+1}]}(c) = p_3 + q_3/c$.

Similarly, for $c \in [3^9/4^7, 4/3]$, there are $p_6, q_6 \in \mathbb{Q}$ such that $k_6|_{[c_j,c_{j+1}]}(c) = p_6 + q_6/c$, and for $c \in [0,\ 3^9/4^7]$, there are $p_7, q_7 \in \mathbb{Q}$ such that $k_7|_{[c_j,c_{j+1}]}(c) = p_7 + q_7/c$.

By computations (see Table 1), for $1 \le c \le 3^9/4^7$,

$$\min\{k_5(c),\ k_7(c)\} \le \min\{k_3(c),\ k_4(c),\ k_6(c)\},$$

for $3^9/4^7 \le c \le 3^4/4^3$,

$$\min\{k_3(c),\ k_5(c),\ k_6(c)\} \le \min\{k_4(c),\ k_7(c)\},$$

and for $3^4/4^3 \le c \le 4/3$,

$$\min\{k_4(c),\ k_6(c)\} \le \min\{k_3(c),\ k_5(c),\ k_7(c)\}.$$

Therefore, the function $k(c) = \min\{k_i(c) : 3 \le i \le 7\}$ is piecewise either of the form $A + Bc$ or of the form $p + q/c$, with $A, B, p, q \in \mathbb{Q}$. □

Remark. The type of the function k and its piecewise parameters can be explicitly computed in each interval of the form $[c_j, c_{j+1}]$. However, for some j, the form of $k|_{[c_j,c_{j+1}]}$ could be neither entirely of the form $A + Bc$ with A and B constant throughout $[c_j, c_{j+1}]$, nor entirely of the form $p + q/c$ with p and q constant throughout $[c_j, c_{j+1}]$. In these cases it could be piecewise of different types. For example, it

i	$c_{(i)}$	$x_{\min}$ on $[c_{(i)}, c_{(i+1)}[$	piecewise $k(c)$-type on $[c_{(i)}, c_{(i+1)}[$
0	$1 = 1$	$3^5 = 243$	$k_5(c) = A + B\,c$
1	$(\sqrt{1317648910247977} - 35239219)/950272 \simeq 1.115696$	$c\,4^7$	$k_7(c) = p + q/c$
2	$(11226431 - \sqrt{103351134926209})/950272 \simeq 1.115738$	$3^5 = 243$	$k_5(c) = A + B\,c$
3	$(\sqrt{51408122101177} - 6109675)/950272 \simeq 1.115755$	$c\,4^7$	$k_7(c) = p + q/c$
4	$3^9/4^7 = 19683/16384 \simeq 1.201355$	$3^5 = 243$	$k_5(c) = A + B\,c$
5	$(3\sqrt{15889} - 219)/128 \simeq 1.243396$	$c\,4^3$	$k_3(c) = p + q/c$
6	$4131/3272 \simeq 1.262531$	$c\,4^6$	$k_6(c) = p + q/c$
7	$(2080353 - 3\sqrt{429430429945})/90112 \simeq 1.269806$	$3^4 = 81$	$k_4(c) = A + B\,c$
8	$4/3 \simeq 1.333333$	–	–

Table 1. List of the eight subintervals in $[1, 4/3]$ on which $x_{\min}$ has the same expression. On each of these subintervals, the function k is piecewise linear (affine) or piecewise a translated equilateral hyperbola.

could be $k|_{[c_j, c_{j+1}]} = \min\{A + B\,c,\ p + q/c\}$, with a value $c^* \in\,]c_j, c_{j+1}[$ for which $A + B\,c^* = p + q/c^*$. When such a case occurs, c^* is an algebraic number of degree 2, at which the value of $x_{\min}$ switches from one form to another, see Table 1 for the complete list of such cases.

Lemma 3. *Let $k : D \to \mathbb{R}$ be defined as above. We have*

$$\min\{k(c) : c \in D\} = k(1) = \frac{1}{243}\int_0^{243} \mathbb{1}_{A_1}(t)\,dt = \frac{1015}{1458} \simeq 0.69616.$$

Proof. In a suitable right neighborhood of $c = 1$, the function k is a piecewise linear (affine) function. In particular for $1 \le c \le 513/512 \simeq 1.00195$ we have $x_{\min} = 243$, and $k(c) - k(1) = (c-1)12440/729$; this is because as long as

$$3 + 1 + (64 + 16 + 4 + 1)c + \frac{1}{2} + \frac{c}{3} < 81 + 9$$

(or, simply $85c + c/3 < 85 + 1/2$), there are no further intersections of intervals of the form $[(\alpha + \beta c), (\alpha + \beta c) + 1/2 + c/3]$ in $[0, 243]$, apart from the ones already accounted for $c = 1$, and each of the connected components of A_c in $[0, 243]$ grows linearly with c. In particular, since the only positive integers not exceeding 243 that are not in $\Sigma(\mathrm{Pow}(\{3, 4\}, 0))$ are 62, 63, 143, 144 and the 36 integers between 207 and 242, we have

$$k(1) = \int_0^{243} \mathbb{1}_{A_1}(t)\,dt = \frac{5}{6}\cdot\frac{243-2-2-36}{243} = \frac{1015}{1458}.$$

The slope $k'(1) = 12440/729$ is merely issued from an analogous computation.

From Lemma 2 and the complete computation of k, it turns out that, for $c \in]1, 4/3]$, we have $k(c) > k(1)$ (see also Figure 1). □

3. The main results

Let $B = \{3, 4, 9, 16, 27, 64, 81, 243, 256, 729, 1024, \ldots\}$ be the increasing sequence of powers of 3 and 4. At each pair of consecutive odd terms (indeed two consecutive powers of 3), we will make a "cut" to split up B into subsequences B_n, $n \geq 0$, which consist of runs of (either 7 or 9) consecutive terms of B of alternating parity, more precisely of the form

(i) $B_n = \{3^{r_n}, 4^{\ell_n}, 3^{r_n+1}, 4^{\ell_n+1}, 3^{r_n+2}, 4^{\ell_n+2}, 3^{r_n+3}, 4^{\ell_n+3}, 3^{r_n+4}\}$ or

(ii) $B_n = \{3^{r_n}, 4^{\ell_n}, 3^{r_n+1}, 4^{\ell_n+1}, 3^{r_n+2}, 4^{\ell_n+2}, 3^{r_n+3}\}$,

with $\min B_{n+1} = 3 \max B_n$. We will call "cycle" any such subset B_n, of either type. Let $n > 0$ and $c_n = 4^{\ell_n}/3^{r_n} \in]1, 4/3[$ so that $4^{\ell_n} = c_n\, 3^{r_n}$. Note that $c_n \neq 3^4/4^3$ and $c_n \neq 3^9/4^7$.

If $1 < c_n < 3^9/4^7 \simeq 1.201$, then B_n and B_{n+1} are both cycles of type (i); if $3^9/4^7 < c_n < 81/64 \simeq 1.266$ then B_n is of type (i) and B_{n+1} is of type (ii); if $81/64 < c_n < 4/3$, then B_n is of type (ii) and B_{n+1} is of type (i).

Theorem 4. *Let $P_{\{3,4\}}(x)$ be the counting function of* $\Sigma(\mathrm{Pow}(\{3, 4\}), 0)$. *We have*

$$P_{\{3,4\}}(x) \gg x^{0.97777}.$$

Proof. The proof is inspired from [Melfi 2001], where the terms of the sequence of integers $\{3^{r_n}\}_{n\in\mathbb{N}}$ have been used as "milestones" to extrapolate the main result. Here a different set of milestones is used, and the iteration step is done over two cycles. Moreover, the function k in its full complexity will play a central role.

Let $\varepsilon > 0$. Let n be an even positive integer. Let $d_n = \sum_{j=0}^{r_n-1} 3^j + \sum_{i=0}^{\ell_n-1} 4^i$. Note that $d_n < 3^{r_n}$, and that d_n is the largest integer in $\Sigma(\mathrm{Pow}(\{3, 4\}), 0)$ with this property. Suppose that $P_{\{3,4\}}(x) \geq ax$ for every $x \leq d_n$ and for a suitable $a > 0$. Note that $a > 0$ is possible because $\{0, 1\} \subset \Sigma(\mathrm{Pow}(\{3, 4\}), 0)$.

Our aim is to find the best possible $b \leq a$ such that $P_{\{3,4\}}(x) \geq bx$ for every $x \leq d_{n+2}$. We analyze the interval $[3^{r_n}, d_{n+2}]$ by studying suitable overlapping subintervals.

Note that $3^{r_n} < 4^{\ell_n} < 3^{r_n+1}$, and $4^{\ell_n} = c_n\, 3^{r_n}$ with $1 < c_n < 4/3$. Note also that c_n is always of the form $4^\ell/3^r$, with $\ell \geq 8$. In particular $c_n \neq 19683/16384$. If $1 < c_n < 81/64$, B_n is a cycle of the form (i); if $81/64 < c_n \leq 4/3$, B_n is of the form (ii). Note that

$$d_n = \left(\frac{1}{2} + \frac{c_n}{3}\right) \cdot 3^{r_n} - \frac{5}{6}.$$

If $1 < c_n < 19683/16384$, every integer expressible in $[3^{r_n}, d_{n+2}]$ is of the form

$$\sum_{j=0}^{9} \varepsilon_{3,j} 3^{r_n+j} + \sum_{i=0}^{7} \varepsilon_{4,i} 4^{\ell_n+i} + x$$

for a suitable integer $x \in \Sigma(\mathrm{Pow}(\{3, 4\}), 0)$ with $0 \leq x \leq d_n$, and $\varepsilon_{3,j}, \varepsilon_{4,i} \in \{0, 1\}$.

In particular, all positive integers in $\Sigma(\mathrm{Pow}(\{3, 4\}), 0)$ not exceeding d_{n+2} are in a union of (possibly overlapping) 2^{18} intervals

$$A'_{c_n} = \bigcup_{\substack{\alpha\in\Sigma(1,3,9,\ldots,3^9)\\ \beta\in\Sigma(1,4,16,\ldots,4^7)}} [(\alpha + \beta c_n) \cdot 3^{r_n},\ (\alpha + \beta c_n) \cdot 3^{r_n} + d_n].$$

Similarly, if $19683/16384 < c_n < 4/3$, all positive integers in $\Sigma(\mathrm{Pow}(\{3,4\},0))$ not exceeding d_{n+2} are in a union of 2^{16} intervals:

$$A'_{c_n} = \bigcup_{\substack{\alpha\in\Sigma(1,3,9,\dots,3^8)\\ \beta\in\Sigma(1,4,16,\dots,4^6)}} [(\alpha+\beta c_n)\cdot 3^{r_n},\ (\alpha+\beta c_n)\cdot 3^{r_n}+d_n].$$

Whatever the connected component of D to which c_n belongs, in each interval of the form $[(\alpha+\beta c_n)\cdot 3^{r_n}, (\alpha+\beta c_n)\cdot 3^{r_n}+d_n]$ the number of integers that are in $\Sigma(\mathrm{Pow}(\{3,4\}),0)$ is at least ad_n, and for $0\le x\le d_n$, the number of integers in $[(\alpha+\beta c_n)\cdot 3^{r_n},\ (\alpha+\beta c_n)\cdot 3^{r_n}+x]$ that are in $\Sigma(\mathrm{Pow}(\{3,4\}),0)$ is at least ax. Note also that $(3^{-r_n}A'_{c_n})\cap\mathbb{N} = A_{c_n}\cap\mathbb{N}$. In particular, for sufficiently large n,

$$P_{\{3,4\}}(d_{n+2}) \ge a\,(1-\varepsilon)\int_0^{\max A_{c_n}} \mathbb{1}_{A_{c_n}}(t)\,dt.$$

Assuming $P_{\{3,4\}}(x)\ge a\,x$ for every $x\le d_n$, we have

$$P_{\{3,4\}}(x) \ge a\,k(c_n)\,(1-\varepsilon)\,x \quad \text{for every } x\le d_{n+2}.$$

Let $D' = [0, \log(4/3)]$, and

$$g: D'\to\mathbb{R}, \quad t\mapsto g(t) := k(e^t).$$

Note, among other things, that g is continuous on $D'\setminus\{\log(19683/16384)\}$.

Let $c'_n = \log c_n = \ell_n\log 4 - r_n\log 3$. We have $0 < c'_n < \log(4/3)$. Since $\log 4/\log 3\notin\mathbb{Q}$, we have that c'_n is uniformly distributed in $[0,\log(4/3)]$.

Let $d_n\le x < d_{n+2}$. It is easy to check that

$$n = \left(\frac{1}{\log 3} - \frac{1}{\log 4}\right)\log x + \kappa,$$

with $|\kappa| < 5$. Hence

$$\begin{aligned} P_{\{3,4\}}(x) &\gg x\prod_{i=1}^{n/2}(g(c'_{2i})(1-\varepsilon)) = x\left(\exp\left(\frac{2}{n}\sum_{i=1}^{n/2}\log(g(c'_{2i})(1-\varepsilon))\right)\right)^{n/2} \\ &\gg x\exp\left\{\left(\frac{1}{2\log(4/3)}\int_0^{\log(4/3)}\log(g(u)(1-\varepsilon))\,du\right)\left(\frac{1}{\log 3}-\frac{1}{\log 4}\right)\log x\right\}. \end{aligned}$$

We have

$$\tau = \frac{1}{\log(4/3)}\int_0^{\log(4/3)}(-\log g(u))\,du \simeq 0.2353664,$$

and $1-(\tau/2)(1/(\log 3)-1/(\log 4)) = \gamma > 0.97777$ (the code for computing this integral numerically is shared at http://github.com/m-f-h/SumPow34). Thus, for sufficiently small ε we have $P_{\{3,4\}}(x)\gg x^{0.97777}$. □

Another related result that follows from the arguments developed above is the following.

Proposition 5. *Let $P_{\{3,4\}}(x)$ be the counting function of $\Sigma(\mathrm{Pow}(\{3,4\}),0)$. We have*

$$\liminf_{x\to\infty}\frac{P_{\{3,4\}}(x)}{x} \le k(1) = \frac{1015}{1458} \simeq 0.69616. \tag{7}$$

Proof. To prove (7), for every $\varepsilon > 0$ we have to find infinitely many integers x such that $P_{\{3,4\}}(x)/x < k(1)+\varepsilon$. Since $\log(4/3)\notin\mathbb{Q}$, for every $\delta>0$ there exist infinitely many m and l such that $3^m < 4^l < (1+\delta)3^m$.

For such a sequence $\{m_n\}_{n\in\mathbb{N}}$, let us consider the sequence $\{3^{m_n+5}-1\}_{n\in\mathbb{N}}$. For the sake of simplicity let $x_n = 3^{m_n+5}-1$. By Lemma 2, k is continuous. Therefore $\lim_{n\to\infty} P_{\{3,4\}}(x_n)/x_n \le k(1+\delta)$. In particular, for sufficiently small $\delta > 0$ we have $k(1+\delta) < k(1)+\varepsilon$, and by Lemma 3, inequality (7) follows. □

4. Concluding remarks

The technique of the proof of Theorem 4 reaches its limits in its present form. We developed an iteration step that includes two consecutive cycles B_n and B_{n+1}. This required a considerable amount of computations, and the possibility of performing an iteration step that includes three or more consecutive cycles appears to be out of present computation capabilities in terms of computation time. So it appears very difficult to improve the estimate of $P_{\{3,4\}}(x)$ with these techniques.

On the other hand, this technique could be applied to other sets of sums of distinct powers of a finite set of positive integers.

A strictly related question was asked by Burr, Erdős, Graham and Li [Burr et al. 1996]. Suppose that $a_1, \ldots, a_k$ are positive integers with

$$\sum \frac{1}{\log a_i} > \frac{1}{\log 2}. \tag{8}$$

Does $\Sigma(\mathrm{Pow}(\{a_1, \ldots, a_k\}), 1)$ have positive asymptotic density? Positive upper asymptotic density? As shown in [Melfi 2001], for $k > 1$, condition (8) is a necessary condition for $\Sigma(\mathrm{Pow}(\{a_1, \ldots, a_k\}), 1)$ to have positive upper asymptotic density. On the other hand it is not difficult to find examples for which the same condition is not sufficient: as already shown on [Melfi 2001], $\Sigma(\mathrm{Pow}(\{3, 9, 81\}), 1)$ has zero asymptotic density, but verifies (8). We may summarize these considerations by stating the following open question that generalizes the Erdős conjecture on $\Sigma(\mathrm{Pow}(\{3, 4\}), 1)$:

Open Question 6. Does $\Sigma(\mathrm{Pow}(\{a_1, \ldots, a_k\}), 1)$ have positive asymptotic density if (8) holds and $\gcd(a_i, a_j) = 1$ for every pair of terms?

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