Laplacians in Riemannian Geometry:

a Spectral Comparison through Discretization

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Laplacians in Riemannian Geometry : a Spectral Comparison through Discretization

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Résumé

Keywords: Laplacian, rough Laplacian, Hodge Laplacian, combinatorial Laplacian, discrete magnetic Laplacian, eigenvalues, discretization.

Mots-clés: laplacien, laplacien brut, laplacien de Hodge, laplacien combinatoire, laplacien magnétique discret, valeurs propres, discrétisation.

Le but de cette thèse est d'étudier le spectre de laplaciens apparaissant en géométrie riemannienne (laplacien sur les fonctions, laplacien de Hodge sur les formes différentielles et laplacien brut sur un fibré vectoriel) au travers de la discrétisation. Plus précisément, il s'agit de comparer uniformément le spectre de ces laplaciens à des laplaciens discrets (laplacien combinatoire associé à un graphe, laplacien combinatoire associé à un complexe de cochaînes de Čech et généralisation du laplacien magnétique discret) agissant sur des espaces vectoriels de dimension finie construits grâce à la discrétisation.

La comparaison spectrale se veut uniforme dans le sens qu'elle est valable pour une famille de variétés riemanniennes fixée par un certain nombre de paramètres géométriques dont dépendra exclusivement la comparaison. Enonçons le résultat principal obtenu dans le cadre du laplacien agissant sur les fonctions d'une variété riemannienne compacte à valeurs réelles. Dans ce cas, la ε-discrétisation considérée est un graphe construit sur un sous-ensemble ε-séparé maximal de la variété. Le laplacien discret associé n'est alors rien d'autre que le laplacien combinatoire standard agissant sur les fonctions des sommets du graphe à valeurs réelles.

Théorème 1 *Soient n, $\kappa \geq 0$, $r_0 > 0$ et $0 < \varepsilon < \frac{1}{2}r_0$. Alors, il existe des constantes c_1, $c_2 > 0$ ne dépendant que de n, κ et ε telles que pour toute variété riemannienne (M^n, g) compacte, connexe telle que $Ricci(M, g) \geq -(n-1)\kappa g$ et $Inj(M, g) \geq r_0$ et pour toute ε-discrétisation X de M,*

$$c_1\lambda_k(X) \leq \lambda_k(M) \leq c_2\lambda_k(X)$$

pour tout $k < |X|$.

où $\lambda_k(M)$ et $\lambda_k(X)$ désignent les $k^{\text{èmes}}$ valeurs propres de M et de X respectivement.

Ce théorème est alors généralisé aux laplaciens susmentionnés. En outre, plusieurs applications de ces comparaisons spectrales ainsi obtenues sont présentées. Citons en particulier la minoration de la première valeur propre non-nulle du laplacien de Hodge, en fonction du volume de la variété (et des paramètres fixés lors de la discrétisation) et l'encadrement de la première valeur propre non-nulle du laplacien brut sur les fibrés vectoriels plats, en fonction de l'holonomie.

Introduction

This thesis is organized in three parts that reproduce as accurately as possible three manuscripts that have already been published or at least submitted for publication. The aim of this introduction is to motivate and to present shortly the origine of this work and the content of the papers. Hence, it might seem slightly redundant with the introductions of the papers (for a more precise description of each part, see these respective introductions). The three manuscripts have been collected in chronological order even if maybe Part II should have been switched for Part III for contents' reasons.

Context. The purpose of the thesis is to study the spectrum of the Laplacian through the discretization. More precisely, it is to link the spectrum of the Laplacian to the spectrum of some combinatorial Laplacian acting on a finite dimensional space associated to a process of discretization. Let us clarify what is meant by the terms "Laplacian", "combinatorial Laplacian" and "discretization".

This work deals with the following Laplacians (appearing in Riemannian geometry and referred as "smooth" Laplacians):

- the Laplacian acting on functions on a compact Riemannian manifold defined by $\Delta = -div(grad)$,
- the rough Laplacian associated to a Riemannian connection ∇ on a vector bundle over a compact Riemannian manifold given by $\overline{\Delta} = \nabla^*\nabla$,
- the Hodge Laplacian acting on differential forms on a compact Riemannian manifold defined by $\Delta = dd^* + d^*d$.

Obviously, the second and the third Laplacian mentioned above generalize in a certain way the Laplacian acting on functions. Moreover, the Hodge Laplacian and the rough Laplacian are linked by the Weitzenboeck formula given by $\Delta = \overline{\Delta} + \mathcal{R}$, where $\mathcal{R}$ is a tensor field built from the curvature tensor (see for instance [57], Appendix 5).

The "corresponding" combinatorial Laplacians (referred also as "discrete" Laplacians) we will need in the sequel are:

- the combinatorial Laplacian acting on real-valued functions on a graph,
- the discrete magnetic Laplacian associated to a Harper operator acting on $\mathbb{R}^n$-valued functions on a graph (which is a particular case of the "twisted" Laplacian),
- the combinatorial Laplacian acting on Čech cochains associated to an open cover and constructed from the coboundary operator.

Discretization is a powerful means to compare the spectrum of a "smooth" Laplacian to the spectrum of a "discrete" Laplacian. Roughly speaking, discretizing a compact Riemannian manifold consists in constructing a finite subset of the manifold with respect to some mesh of discretization and then endowing this finite subset with a structure, for instance a structure of graph, an open cover or a triangulation. Then, there are essentially two research directions concerned with discretization.

The first one we should describe here briefly, but which will not be discussed in this work, consists in fixing a compact Riemannian manifold and discretizing it finer and finer. The aim is to establish the convergence of the spectrum of the "discrete" Laplacian associated to the discretization to the spectrum of the "smooth" Laplacian, as the mesh of the discretization goes to zero. In this context, Dodziuk and Patodi have shown in [27] that for a fixed compact Riemannian manifold M, we can approximate the spectrum of the Hodge Laplacian (on differential forms) by the spectrum of the combinatorial Laplacian associated to finer and finer triangulations. Their method is inspired by the finite element method used to solve partial differential equations numerically. An underlying question that has not been solved in Dodziuk and Patodi's work is to express how good might be the approximation with respect to the geometric quantities involved. In [31], Fujiwara gives a partial answer for the Laplacian acting on functions (in his result, the curvature is only involved in the rate of convergence and not in the approximation). Aubry announced in his thesis [1] a more precise result for the Hodge Laplacian but the result has not been published yet. As far as I know, the general question of finer and finer discretizations has neither been solved nor stated for the rough Laplacian (or other differential operators) on vector bundles.

The second kind of discretization problem and the one developed in this thesis should be described as a "uniform" discretization in the following sense. We fix a family of compact Riemannian manifolds (or of Riemannian vector bundles) given by a set of geometric parameters. Then, we choose a suitable (with respect to the parameters) mesh of discretization and discretize any manifold of the family with this same mesh of discretization. The goal is then to establish a uniform comparison of the spectra of the "smooth" Laplacian and the "discrete" one. The comparison is said uniform in the sense that it depends only on the set of parameters and on the mesh of discretization and not on the manifolds.

Before stating more precisely the results, let us point out here that in Hirani's thesis [39], the author develops a theory of discrete exterior calculus. His goal is to construct discrete operators as the Hodge star, the wedge product or the exterior derivative, in a given space of cochains (a discrete analogue of differential forms) and establish classical results as Stokes' Theorem in this discrete context. Even if the space of cochains might be produced by a discretization process, the goal of Hirani's thesis was to stay in the discrete ambient as far as possible and construct a theory of calculus, independent of the smooth one, that parallels the smooth theory. It would be interesting to establish a convergence or a comparison of this theory to the smooth theory.

Statements. Let us now state more precisely the main results concerning the comparison of spectra.

In Part I, for $\kappa \geq 0$, $r_0 > 0$ and $n \geq 1$, we consider $\mathcal{M}(n, \kappa, r_0)$ the set of all connected compact n-dimensional Riemannian manifolds with Ricci curvature bounded below by $-(n-1)\kappa$ and injectivity radius bounded below by r_0. For a compact Riemannian manifold, we define an ε-discretization to be a maximal ε-separated subset endowed with a natural graph structure given by sets of neighbouring points. We denote by $\lambda_k(M)$ the k^{th} positive eigenvalue of the Laplacian acting on functions on M and by $\lambda_k(X)$ the k^{th} positive eigenvalue of the combinatorial Laplacian acting on real-valued functions on the graph associated to X. We have then the following uniform comparison of spectra.

Theorem 1 (Theorem I.3.7) *Let n, $\kappa \geq 0$, $r_0 > 0$. Fix $0 < \varepsilon < \frac{1}{2}r_0$. Then, there exist positive constants c_1, c_2 depending only on n, κ and ε such that for any $M \in \mathcal{M}(n, \kappa, r_0)$ and for any ε-discretization X of M , we have*

$$c_1\lambda_k(X) \leq \lambda_k(M) \leq c_2\lambda_k(X)$$

for any $k < |X|$.

Part II is concerned with the spectrum of rough Laplacians. We keep the same definition of $\mathcal{M}(m, \kappa, r_0)$ and of the ε-discretization as above. Moreover, we denote by $\mathcal{E}(n, k_1, k_2)$ the set of Riemannian vector bundles E with fiber of real rank n, with curvature tensor bounded by k_1 and exterior coderivative of the curvature tensor bounded by k_2. We are interested in comparing the k^{th} eigenvalue of the rough Laplacian $\lambda_k(E)$ to the k^{th} eigenvalue of a discrete operator associated to a discretization X. Here, the problem is to construct canonically such a discrete operator. It turns out that a twisted Laplacian Δ_A added with a potential V (both suitably constructed with respect to the local geometry of the vector bundle) serve the purpose. More precisely, let $\lambda_k(X, A, V)$ denote the k^{th} eigenvalue of $\Delta_A + V$.

Theorem 2 (Theorem II.3.1) *Let m, n be positive integers, κ, k_1, $k_2 \geq 0$ and $r_0 \geq 20\varepsilon > 0$. There exist positive constants c_1, c_2 depending only on m, n, κ, k_1, k_2 and ε such that for any $M \in \mathcal{M}(m, \kappa, r_0)$, any vector bundle $E \in \mathcal{E}(n, k_1, k_2)$ over M satisfying one of the following condition*

I) the curvature of E is harmonic i.e. $d^*R^E = 0$,

II) E is of complex (or quaternionic) rank one

and for any ε-discretization X of E, we can construct a canonical twisted Laplacian Δ_A and a potential V depending only on the local geometry of E such that, for $1 \le k \le n|X|$

$$c_1\lambda_k(X, A, V) \le \lambda_k(E) \le c_2\lambda_k(X, A, V).$$

In particular, if the vector bundle is flat, the potential is zero and Δ_A is a discrete magnetic Laplacian.

Finally, Part III deals with the Hodge Laplacian acting on differential forms. Here, a graph structure does not suffice anymore to control the spectrum of the Laplacian (for topological reasons). Hence, we associate to an ε-discretization X of M an open cover $\mathcal{U}_X$ and consider the Čech cochain complex induced by $\mathcal{U}_X$, denoted by $\{\mathcal{C}^*(\mathcal{U}_X), \delta\}$, where δ is the coboundary operator of the complex. The combinatorial Laplacian comes here naturally with the complex as we can define it (by means of an inner product) by $\Delta = \delta\delta^* + \delta^*\delta$. Moreover, we need a stronger assumption on the curvature to control locally the differential forms. For $\kappa \ge 0$ and $r_0 > 0$ let us denote by $\mathbb{M}(n, \kappa, r_0)$ the set of all connected compact n-dimensional Riemannian manifolds with sectional curvature bounded by κ and injectivity radius bounded below by r_0. We have then the following uniform comparison of spectra.

Theorem 3 (Theorem III.3.1) *Let $n \ge 2$, $\kappa \ge 0$, $r_0 > 0$. Let $1 \le p \le n-1$. There exists a positive constant ρ_0 depending only on n, κ and r_0 such that for $0 < 3\varepsilon < \rho_0$, there exist positive constants c_1, c_2 depending only on n, p, κ and ε such that for any $M \in \mathbb{M}(n, \kappa, r_0)$ and for any ε-discretization X of M, we have*

$$c_1\lambda_{k,p}(X) \le \lambda_{k,p}(M) \le c_2\lambda_{k,p}(X)$$

for any $k \le |\mathcal{C}^p(\mathcal{U}_X)| - \check{b}_p(\mathcal{U}_X) = |\mathcal{C}^p(\mathcal{U}_X)| - b_p(M)$.

where $\lambda_{k,p}(M)$ denotes the k^{th} positive eigenvalue of the Hodge Laplacian on p-forms, $\lambda_{k,p}(X)$ denotes the k^{th} positive eigenvalue of the combinatorial Laplacian on Čech p-cochains and b_p the p^{th} Betti number.

A first approach to prove Theorem 3 could lean on Theorem 2 and Weitzenboeck formula. This was in fact the first motivation to study rough Laplacians through discretization. It turns out that this might not work. The main reason is because parallel forms are harmonic but the converse is not true in general. Hence, the 0-eigenspaces of $\overline{\Delta}$ and Δ can not be related.

It would be interesting to discretize also the additional curvature term $\mathcal{R}$ in order to keep more informations, but it is not clear to us how to do it in a relevant way. From another point of view, the Hodge Laplacian contains a topological information that the rough Laplacian does not contain and this might appear in the discretization.

Applications. The above general statements have shown us how a "smooth" Laplacian can be compared to a "discrete" one. In some sense, this enables to understand how geometry and topology interfere in the spectrum of "smooth" Laplacians or on which local geometric quantities it depends. Moreover, these statements give a new insight to study the spectrum of "smooth" Laplacians. The results suggest to increase the study of "discrete" Laplacians in order to understand better the "smooth" ones. This was already understood and tested by several authors as Brooks [9], [10], Burger [11], Buser [13], Dodziuk and Patodi [27], Kanai [40], [41] and also maybe Chanillo and Trèves [16] and McGowan [50] even if both works are more related to the de Rham Theorem (linking geometry and topology through cohomology) than to the discretization strictly speaking. In this perspective, several applications of the main statements or of the methods developed around the results are presented in each of the three parts. Let us point out some of them and replace them in their context.

As the diameter or the volume of a Riemannian manifold, the spectrum of the Laplacian and in particular the first positive eigenvalue are isometric invariants. Hence, there have been several developments around the subject. A considerable question is to bound below the first positive eigenvalue of the Laplacian. This question has been extensively investigated for the Laplacian acting on functions on a compact Riemannian manifold. Let us recall here a few fundamental results in this context. The first one, due to Li and Yau in the following version in [46] (but see also for instance [6]), gives a lower bound with respect to the diameter of the manifold for manifolds with Ricci curvature bounded below.

Theorem 4 *Let (M^n, g) be a compact n-dimensional Riemannian manifold of diameter d and such that $Ricci(M, g) \geq -(n-1)\kappa$ for $\kappa \geq 0$. We have*

$$\lambda_1(M) \geq \frac{e^{-\left(1+\sqrt{1+4(n-1)^2 d^2 \kappa}\right)}}{2(n-1)d^2}.$$

Another result we want to recall is the Cheeger Inequality (see for instance [17] for a general reference). To that aim, for a compact Riemannian manifold M, we define the Cheeger isoperimetric constant $h(M)$ by $h(M) = \inf_C \left\{ \frac{Vol_{n-1}(C)}{\min\{Vol(M_1), Vol(M_2)\}} \right\}$, where C ranges over all compact codimension one submanifolds of M which divide M into two disjoint connected open submanifolds M_1, M_2 with $\partial M_1 = C = \partial M_2$.

Theorem 5 (Cheeger Inequality) *We have* $\lambda_1(M) \geq \frac{1}{4}h(M)^2$.

Note that the result is valid for any compact Riemannian manifold without any curvature assumption. A similar result is valid for combinatorial Laplacians on graphs (see [47], Proposition 4.2.4).

Cheeger Inequality for combinatorial Laplacians combined with Theorem 1 leads easily to a lower bound such as $\lambda_1(M) \geq \frac{cst}{Vol(M)^2}$ which is much more weaker as Theorem 4 (as we assumed an injectivity radius bounded below) but illustrates well how informations on the spectrum of "discrete" Laplacians may give results for "smooth" Laplacians. Another illustration of this fact is given in Part I as we show that roughly isometric graphs have comparable (in the same sense as before) spectra (see Theorem I.2.1) and as an application (together with Theorem 1) we obtain a simpler proof of a theorem of Brooks on the spectrum of tower of coverings (see Section I.4) and a comparison of spectra for Gromov-Hausdorff close manifolds (see Section I.5).

Going back to the lower bound problem, the situation for the Hodge Laplacian on differential forms is less clear. As far as I know, there is no general Cheeger Inequality valid for differential forms. A general lower bound for $\lambda_{1,p}(M)$ is given in [16] by Chanillo and Trèves for compact Riemannian manifolds with bounded sectional curvature. The lower bound depends on the sectional curvature, the injectivity radius and the volume of the manifold. In [36], Guerini studies the case of Euclidean convexes (the bound depends there on the diameter of the convex). Here, Theorem 3 and a lower bound for $\lambda_{1,p}(X)$ should give a relevant lower bound for $\lambda_{1,p}(M)$. Therefore, in Theorem III.4.4 we give a lower bound (that should be improvable) for the combinatorial Laplacian which leads to the following estimation (compare with Chanillo and Trèves' result and see Remark III.4.3).

Theorem 6 (Theorem III.4.1) *Let* $(M, g) \in \mathbb{M}(n, \kappa, r_0)$ *and* $1 \leq p \leq n-1$. *Then there exist positive constants* c, c' *depending only on* n, p, κ *and* r_0 *such that*

$$\lambda_{1,p}(M) \geq \frac{c}{Vol(M)e^{c' Vol(M)}}$$

where $Vol(M)$ *denotes the volume of* (M, g).

For rough Laplacians, the problematic differs as the holonomy of the vector bundle comes in. In [3], the authors show a lower bound for the first eigenvalue depending on the holonomy. As a corollary of Theorem 2, we obtain a similar lower bound in a more particular case, but also an upper bound involving the holonomy. More precisely, for a flat Riemannian vector bundle E over M with irreducible holonomy, let $\alpha > 0$ be the constant of holonomy defined in [3] i.e. $\alpha > 0$ is such that $\forall x \in M$, $\forall v \in E_x$ there exists a smooth unit speed loop $c_{x,v}$ of length less than two diameters of M such that $|H_{c_{x,v}}(v) - v| \geq \alpha |v|$, where H_c denotes the parallel translation along c.

Theorem 7 (Theorem II.4.1) *Let (E^n, ∇) be a flat Riemannian vector bundle over $M \in \mathcal{M}(m, \kappa, r_0)$ with irreducible holonomy. Then, there exist c_1, $c_2 > 0$ depending only on m, n, κ, r_0 such that*

$$\lambda_1(E) \geq c_1 \frac{\alpha^2}{d(M)^2 c_2^{d(M)}}$$

where $d(M)$ denotes the diameter of M.

Moreover, if there exist $p_0 \in M$, $v_0 \in E_{p_0}$ and α' such that for any loop c at p_0 of length less than $7d(M)$, $|H_c(v_0) - v_0| \leq \alpha' |v_0|$ then, there exists $c_3 > 0$ depending only on n, m, κ and r_0 such that

$$\lambda_1(E) \leq c_3 \alpha'^2.$$

Methods. We don't want to talk too much here about the technical tools developed in the proofs of the comparisons of spectra, but just point out the main similarities and differences between the three kinds of results. We have already seen that a main difference is the choice of a suitable combinatorial Laplacian. Let us emphasize that for the case of functions or of differential forms the combinatorial Laplacian comes in naturally with the discretization process and this is not the case for vector bundles. So that a significant part of the work done in Part II consists in constructing a suitable "discrete" Laplacian as canonical as possible.

A common point of the proofs of Theorems 1, 2 and 3 is the sketch of proof. The general strategy is to compare the spectra of the Laplacians using the variational characterization of the eigenvalues. Hence, a first step is to construct a "smoothing" and a "discretizing" operator in order to link smooth functions, sections or differential forms to functions on graphs or Čech cochains.

The main idea to construct wisely these operators lies in the "intuition" that eigenfunctions associated to "small" eigenvalues are almost constant (for the Laplacian on functions), eigensections associated to "small" eigenvalues are almost parallel (for the rough Laplacian) and eigenforms associated to "small" eigenvalues are almost harmonic (for the Hodge Laplacian). Therefore the "smoothing" and "discretizing" operators construction leans in some sense on these intuitions. Nevertheless, the results obtained comparing "smooth" and "discrete" functions, sections or differential forms concern the eigenvalues of the Laplacian and not the eigen-functions, -sections or -forms strictly speaking.

The smoothing and discretizing operators are suitable to compare "small" eigenvalues. For "large" eigenvalues the problem is easier as it suffices to find upper bounds for the k^{th} eigenvalue of the Laplacians and of the combinatorial Laplacians depending only on the local geometry (i.e. on the set of geometric parameters fixed at the beginning of the problem).

It is useless to precise that as the Laplacians are of different nature, the technical difficulties encountered are really different. For instance, for rough Laplacians we needed fine analysis as Sobolev inequalities or Moser's iteration and so Part II seems quite technical from this point of view. For Hodge Laplacians, the de Rham Theorem is the starting point of the proof and then fine Riemannian geometry is needed to control locally differential forms.

Further developments. To conclude this introduction, we don't want to list here the several questions arising from the present work. But at least, we want to discuss a natural generalization of Part I (and maybe of Part II) that G. Carron pointed out to us. In Part I, the assumption on the injectivity radius is not necessary. The main idea is to replace the graph structure of the discretization by a weighted graph structure exactly as in the work of Coulhon and Saloff-Coste [23] (brought to our knowledge by G. Carron).

More precisely, if X is an ε-discretization of a compact Riemannian manifold (M, g), we provide X with a weight function which associate to a vertex p of the graph, the volume of the ball centered at p of radius ε. This weight function leads naturally to a weighted inner product and so we consider a "weighted" combinatorial Laplacian defined in the same way as the standard combinatorial Laplacian but with respect to the weighted inner product instead of the Euclidean one. Let us denote also by $\lambda_k(X)$ the k^{th} positive eigenvalue of the weighted combinatorial Laplacian. In this context, we should state the following version of Theorem 1.

Theorem 8 *Let n, $\kappa \geq 0$ and $\varepsilon > 0$. Then, there exist positive constants c_1, c_2 depending only on n, κ and ε such that for any compact connected n-dimensional Riemannian manifold (M, g) with Ricci curvature bounded below by $-(n-1)\kappa$ and for any ε-discretization X of M, we have*

$$c_1\lambda_k(X) \leq \lambda_k(M) \leq c_2\lambda_k(X)$$

for any $k < |X|$.

The proof of this theorem is a straightforward adaptation of the proof of Theorem 1. Indeed, the main tools and results developed in [18], that are the starting point of the proof of Theorem 1, can be replaced by completely similar tools and results described in Section 6 of [23].

The other results contained in Part I can also be adapted to weighted graph structures provided that we also adapt the various notions involved. Indeed, a graph endowed with the path metric and provided with a weight function becomes a measured metric space so that every notion that fits for metric spaces has to be adapted in order to fit for measured metric spaces.

In Section I.2, the notion of rough isometry should be replaced by a notion of rough isometry for measured metric spaces. The good definition is the "isométrie à l'infini" considered in [23]. Roughly speaking, two measured metric spaces are roughly isometric if they are roughly isometric as metric spaces (via Φ) and if moreover there exists $C > 0$ such that

$$C^{-1}V_1(p,1) \leq V_2(\Phi(p),1) \leq CV_1(p,1)$$

where $V_i(q,1)$ denotes the volume of a ball centered at q of radius 1 in the respective measured metric spaces. With this definition of roughly isometric measured metric spaces, Theorem I.2.1 holds for weighted finite connected graphs (note that the constants appearing in the theorem will also depend on the weight function and more precisely on the constant C_m defined in [23]). The proof of this version of Theorem I.2.1 is left to the reader as it consists in a straightforward adaptation of the original proof including the tools developed in [18] and used in the proof.

In Section I.4, the Schreier graph appearing in Theorem I.4.1 has to be endowed with a weighted graph structure, for example choose the weight function constant equal to 1. Then, it suffices to compare (as in Theorem I.4.3) the weighted Schreier graph with a weighted discretization of sufficiently small mesh (smaller than the injectivity radius) and then compare (as in Theorem I.4.2) the discretization to the manifold to obtain the result.

In Section I.5, the proof of Theorem I.5.1 can also be adapted, provided we choose, in the proof, discretizations of mesh smaller than the injectivity radius. The example shown in Section I.6 shows that a uniform bound on the injectivity radius is necessary in Theorem I.5.1. But there exists a measured version of Gromov-Hausdorff distance we should replace in the statement of Theorem I.5.1, in order to release the assumption on the injectivity radius. Then, the example in Section I.6 is no more relevant, as the manifolds involved are not close for the measured Gromov-Hausdorff distance.

Finally, as the discretization of vector bundles of Part II is really close to the discretization of functions done in Part I, the question of removing the assumption on the injectivity radius comes naturally. The answer is less obvious. In particular because the construction of the discrete magnetic Laplacian in the case of flat vector bundles is based on the fact that, in a small ball, extending a vector by parallel transport leads to a parallel section. But this doesn't mean that the generalization is not possible. On the contrary, technically it seems possible to generalize it, provided that the technical tools are adapted (in particular, we should use local Poincaré inequalities and the doubling property of the volume to obtain all the technical results, as Sobolev inequality for example). The advantage of such a generalization would be the obtention of a version of Theorem 7 without any dependence on the injectivity radius (note that in this case, the proof of Theorem 7 has to be revised as it relays completely on properties of parallel transport on small balls). A good starting point for the generalization would be also the work of Coulhon and Saloff-Coste [23] and other related works cited there.

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Part I

Discretization of the Laplacian acting on functions

Discretization of Compact Riemannian Manifolds Applied to the Spectrum of Laplacian

This text has already been published at "Annals of Global Analysis and Geometry", see [48]. The reader may encounter a few corrections from the original text.

Abstract. For $\kappa \geq 0$ and $r_0 > 0$ let $\mathcal{M}(n, \kappa, r_0)$ be the set of all connected compact n-dimensional Riemannian manifolds (M^n, g) with $Ricci(M, g) \geq -(n-1)\kappa g$ and $Inj(M) \geq r_0$. We study the relation between the k^{th} eigenvalue $\lambda_k(M)$ of the Laplacian associated to (M^n, g), $\Delta = -div(grad)$, and the k^{th} eigenvalue $\lambda_k(X)$ of a combinatorial Laplacian associated to a discretization X of M. We show that there exist constants c, $C > 0$ (depending only on n, κ and r_0) such that for all $M \in \mathcal{M}(n, \kappa, r_0)$ and X a discretization of M, $c\lambda_k(X) \leq \lambda_k(M) \leq C\lambda_k(X)$ for all $k < |X|$. Then, we obtain the same kind of result for two compact manifolds M and $N \in \mathcal{M}(n, \kappa, r_0)$ such that the Gromov-Hausdorff distance between M and N is smaller than some $\eta > 0$. We show that there exist constants c, $C > 0$ depending on η, n, κ and r_0 such that $c\lambda_k(N) \leq \lambda_k(M) \leq C\lambda_k(N)$ for all $k \in \mathbb{N}$.

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I.1 Introduction

Since the work of Buser in [13], it is known that in order to understand the spectrum of the Laplacian associated to a compact Riemannian manifold, $\Delta = -div(grad)$, it may be very powerful to discretize the manifold. Using this technique, Buser considered manifolds with Ricci curvature and injectivity radius bounded below and gave an uniform estimate of the spectrum, depending only on these bounds (see Theorem 6.2 in [13]). The estimate of the k^{th} eigenvalue turned out to be very precise for large k's (i.e for k larger than a constant proportional to the volume of the manifold). However, for the beginning of the spectrum, the result is not strong enough to decide whether the eigenvalues may be close to zero or not.

Since, this question has been investigated by Brooks ([9], [10]), Burger [11], and Buser himself in [14]. In these papers, the manifolds were especially, closely related to Cayley graphs of groups or to Schreier graphs associated to a family of covering spaces, (in another context see also the work of Kanai [40], [41]).

The point of view that will interest us here is the one taken up by Chavel in his book [18], where the question of discretization is very well explained and where he studies in particular the case of isoperimetric inequalities ([18], Chapter V) and Sobolev inequalities ([18], Chapter VI). This book will be the main reference for this paper.

The purpose of this note is to compare the spectrum of a compact Riemannian manifold (M^n, g) to the spectrum of the combinatorial Laplacian of an associated discretization X (defined as in [18]). More precisely, if $\mathcal{M}(n, \kappa, r_0)$ denotes the set of all compact connected n-dimensional Riemannian manifolds with Ricci curvature and injectivity radius uniformly bounded below (i.e. with $Ricci(M, g) \geq -(n-1)\kappa g$, $\kappa \geq 0$ and $Inj(M) \geq r_0 > 0$, see Definition I.3.6), we will show in Theorem I.3.7 that there exist positive constants c, C (depending only on n, κ, r_0) such that for all manifolds M in $\mathcal{M}(n, \kappa, r_0)$ and X a discretization associated to M, we have

$$c\lambda_k(X) \leq \lambda_k(M) \leq C\lambda_k(X)$$

for all $k < |X|$. Remark that $|X|$ behaves as the volume of M.

After defining precisely in Section I.2 the notion of discretization and the Laplacian related to it, Section I.3 will be concerned with the proof of this result.

In Section I.4, as a corollary of our result, we present a very simple proof of Theorem 1 of Brooks in [9], which says that the first non-zero eigenvalue of a tower of covering spaces of a compact manifold goes to zero if and only if the Cheeger constant of the discretizations associated to the covering spaces does. In fact, we prove that the k^{th} eigenvalue of a tower of covering spaces of a compact manifold goes to zero if and only if the k^{th} eigenvalue of the discretizations associated to the covering spaces does, which implies obviously the result of Brooks (see Theorem I.4.1). Note that our proof avoids integral geometry and some not obvious considerations on the boundary of Dirichlet's fundamental domains.

In Section I.5, we compare the spectrum of two compact Riemannian manifolds $M \in \mathcal{M}(m, \kappa, r_0)$ and $N \in \mathcal{M}(n, \kappa, r_0)$ which are close with respect to the Gromov-Hausdorff distance (see Theorem I.5.1). In particular, as $m = n$ we show in Corollary I.5.2 that we have a uniform control $c\lambda_k(N) \leq \lambda_k(M) \leq C\lambda_k(N)$ for all k and where c, $C > 0$ depend on n, κ, r_0 and on the Gromov-Hausdorff distance between M and N (for the behaviour of the spectrum under convergence of manifolds with respect to the Gromov-Hausdorff distance see Section 7 of [20]).

We conclude this note with Section 6, where we give an example to show that the assumption on the injectivity radius is essential in Theorem I.5.1; the spectra of two manifolds with Ricci curvature bounded below and arbitrarily Gromov-Hausdorff close may strongly differ.

I.2 Spectrum of roughly isometric graphs

Let $X = (V, E)$ be a finite graph, where V denotes the set of vertices and E the set of edges, and consider the path metric on this graph so that it becomes a metric space (see [18] p.140). Denote by $N(v)$ the set of neighbours of $v \in V$, that is to say the set of vertices at distance 1 from v for the path metric. We will refer to $m(v)$ as the number of neighbours of $v \in V$ and to ν_X as an upper bound for $m(v)$ (i.e. ν_X is such that for all $v \in V$, $m(v) \leq \nu_X$).

For such a finite graph, we can define a combinatorial Laplacian as in [47] (Section 4.2) and the spectrum of this Laplacian will be denoted by

$$Spec(X) = \{0 = \lambda_0(X) \leq \lambda_1(X) \leq \ldots \leq \lambda_l(X)\}$$

where $l = |X| - 1$ and $|X|$ denotes the number of vertices of the graph. We have the following variational characterization of $Spec(X)$ (see [8] p.268). For a function $f : V \to \mathbb{R}$, consider the Rayleigh quotient of f,

$$R(f) = \frac{||df||^2}{||f||^2}$$

where $||df||^2 = \sum_{v\in V} |df|^2(v) = \sum_{v\in V} \sum_{w\in N(v)} |f(w) - f(v)|^2$ and $||f||^2 = \sum_{v\in V} f^2(v)$. Then, for any $(k+1)$-dimensional vector subspace $W^{(k+1)}$ of the vector space $\mathcal{F}(V) = \{f : V \to \mathbb{R}\}$, the k^{th} eigenvalue of X satisfies

$$\lambda_k(X) \leq \sup\{R(f) \ : \ f \in W^{(k+1)} \ , \ f \neq 0\}$$

So, if $\Lambda(W)$ denotes the supremum of Rayleigh quotients of non-zero functions in W (i.e $\Lambda(W) = \sup\{R(f) \ : \ f \in W \ , \ f \neq 0\}$) and if E_{k+1} denotes the set of all $(k+1)$-dimensional vector subspaces of $\mathcal{F}(V)$, then we have

$$\lambda_k(X) = \inf\big\{\Lambda(W) \ : \ W \in E_{k+1}\big\}.$$

Moreover, let us recall that a rough isometry is an application between metric spaces $\Phi : (X_1, d_1) \to (X_2, d_2)$ such that there exist some constants $a \geq 1$, $b \geq 0$, $\tau \geq 0$ satisfying

$$\forall x_1, y_1 \in X_1 \ , \quad a^{-1}d_1(x_1, y_1) - b \leq d_2(\Phi(x_1), \Phi(y_1)) \leq a d_1(x_1, y_1) + b$$

and $\cup_{x \in X_1} B(\Phi(x), \tau) = X_2$ (see [18] p.142). The constants a, b and τ will be referred as the constants of rough isometry.

Theorem I.2.1 *Let X and Y be finite, connected graphs. Then, for each rough isometry between X and Y, there exist positive constants c and C depending only on ν_X, ν_Y and on the constants of rough isometry such that*

$$c\lambda_k(Y) \leq \lambda_k(X) \leq C\lambda_k(Y)$$

for all $k < \min\{|X|, |Y|\}$.

Note that the constants c and C are independent of k, $|X|$ and $|Y|$.

Proof: it suffices to prove that it exists $C > 0$ such that

$$\lambda_k(X) \leq C\lambda_k(Y). \tag{1}$$

We proceed in two steps. First, we show that it exists a constant $c' > 0$ independent of k, such that if $\lambda_k(Y) \leq c'$ then (1) is true for some $C > 0$. Let X and Y denote also the set of vertices of the respective graphs. Let $\Phi : X \to Y$ be a rough isometry. To each $f : Y \to \mathbb{R}$, we associate

$$\Phi^* f = f \circ \Phi : X \to \mathbb{R}.$$

It can be shown that, there are positive constants c_1, c_2, c_3, c_4 depending only on ν_X, ν_Y and on the constants of rough isometry such that

$$\begin{aligned} \|\Phi^* f\|^2 &\leq c_1\|f\|^2 &(2)\\ \|d(\Phi^* f)\|^2 &\leq c_2\|df\|^2 &(3)\\ \|f\|^2 &\leq c_3\|df\|^2 + c_4\|\Phi^* f\|^2 &(4)\end{aligned}$$

(see [18], Lemma VI.5.2 and VI.5.4).

Then, consider $f_0, \ldots, f_k : Y \to \mathbb{R}$ eigenfunctions associated to $\lambda_0(Y), \ldots, \lambda_k(Y)$ and the corresponding functions on X, $\Phi^* f_0, \ldots, \Phi^* f_k : X \to \mathbb{R}$. Denote by V the subspace spanned by the f_i's, $V = \langle f_0, \ldots, f_k \rangle$ and by W the corresponding space, $W = \langle \Phi^* f_0, \ldots, \Phi^* f_k \rangle$.
If $\lambda_k(Y) \leq c' = (2c_3)^{-1}$ then W is of dimension $k+1$: indeed, let $g = \sum_{i=0}^k a_i \Phi^* f_i$ with at least one non-zero a_i. In fact, g can be rewritten as $g = \Phi^* f$ with $f = \sum_{i=0}^k a_i f_i$. So, f is a non-zero function of V and satisfies $\|df\|^2 \leq \lambda_k(Y)\|f\|^2$. Then by (4)

$$\|g\|^2 = \|\Phi^* f\|^2 \geq c_4^{-1}(\|f\|^2 - c_3\|df\|^2) \geq c_4^{-1}\|f\|^2(1 - c_3\lambda_k(Y)). \tag{5}$$

In particular, as $\lambda_k(Y) \leq c' = (2c_3)^{-1}$, the function g is not zero and this implies that W is $(k+1)$-dimensional. Moreover, under the same assumption on $\lambda_k(Y)$ and using (3) and (5), we have $R(\Phi^* f) \leq 2c_2c_4R(f)$. Finally, we get

$$\begin{aligned} \lambda_k(X) &\leq \sup\{R(g) \mid g \in W - \{0\}\} \\ &= \sup\{R(\Phi^* f) \mid f \in V - \{0\}\} \\ &\leq 2c_2c_4\lambda_k(Y). \end{aligned}$$

It remains to show that (1) is still true if $\lambda_k(Y) > c' = (2c_3)^{-1}$. But by this assumption, we have $\lambda_k(X) \leq \lambda_k(X)\lambda_k(Y)c'^{-1}$. So, in order to conclude, we need an upper bound on $\lambda_k(X)$. For each $f : X \to \mathbb{R}$, we have

$$\begin{aligned} \|df\|^2 &= \textstyle\sum_{x\in X}\sum_{y\in N(x)} |f(x) - f(y)|^2 \\ &\leq 2\textstyle\sum_{x\in X}\sum_{y\in N(x)} \left(|f(x)|^2 + |f(y)|^2\right) \\ &\leq 4\nu_X\|f\|^2. \end{aligned}$$

This implies that $R(f) \leq 4\nu_X$ for all $f \neq 0$ and $\lambda_k(X) \leq 4\nu_X$. Finally, we get $\lambda_k(X) \leq 4\nu_X c'^{-1}\lambda_k(Y)$. □

I.3 Comparison of spectra between a manifold and its discretization

Let (M^n, g) be a connected, compact Riemannian manifold. Consider the Laplacian associated to M

$$\Delta f = -div(gradf)$$

and denote its spectrum by

$$Spec(M) = \{0 = \lambda_0(M) < \lambda_1(M) \leq \ldots\}.$$

The characterization of the eigenvalues we will use subsequently is given by Rayleigh quotients and Min-Max Theorem (see [57] p.269, Min-Max Theorem). So for $F : M \to \mathbb{R}$ define the Rayleigh quotient of F to be

$$R(F) = \frac{\|dF\|^2}{\|F\|^2} = \frac{\int_M \|dF(x)\|^2 dV(x)}{\int_M F(x)^2 dV(x)}$$

where dV denotes the volume form on (M, g). Then, for any $(k+1)$-dimensional vector subspace $W^{(k+1)}$ of the vector space $\mathcal{C}^\infty(M)$ of smooth functions on M, the k^{th} eigenvalue of M satisfies

$$\lambda_k(M) \leq \sup\{R(F) \ : \ F \in W^{(k+1)} \ , \ F \neq 0\}.$$

So, if $\Lambda(W)$ denotes the supremum of Rayleigh quotients of non-zero functions in W (i.e $\Lambda(W) = \sup\{R(F) \ : \ F \in W \ , \ F \neq 0\}$) and if E_{k+1} denotes the set of all $(k+1)$-dimensional vector subspaces of $\mathcal{C}^\infty(M)$, then we have

$$\lambda_k(M) = \inf\left\{\Lambda(W) \ : \ W \in E_{k+1}\right\}.$$

Now, we associate a graph to a Riemannian manifold following [18] (section V.3.2). Let (M^n, g) be a connected compact n-dimensional Riemannian manifold. A discretization of M, of mesh $\varepsilon > 0$, is a graph $X = (V, E)$ such that the set V of vertices is a maximal ε-separated subset of M (so it verifies that for any v, $w \in V$, $v \neq w$, we have $d(v, w) \geq \varepsilon$ and $\cup_{v \in V} B(v, \varepsilon) = M$). Moreover, the graph structure of X is entirely determined by the collection of neighbours that we define as follows. For each $v \in V$, $w \in V$ is a neighbour of v, if $0 < d(v, w) < 3\varepsilon$ (see [18] p.140). We denote by $N(v)$ the set of neighbours of v.

Furthermore, X is roughly isometric to M (see [18] p.147). So we will use on X the metric induced by M rather than the path metric.

Fix once and for all ε smaller than $\frac{1}{2}Inj(M)$. Denote by $\kappa \geq 0$ a constant such that $Ricci(M, g) \geq -(n-1)\kappa g$. Then, by the Bishop-Gromov comparison theorem, ν_X is bounded above by a constant depending only on n, κ and ε. So we can assume $\nu_X = \nu(n, \kappa, \varepsilon)$. Furthermore, using Croke's Inequality (cf. [18] p.136) and Bishop's comparison theorem, we have

$$\frac{1}{V_{-\kappa}(\varepsilon)} Vol(M) \leq |V| \leq \frac{2^n}{\varepsilon^n c(n)} Vol(M)$$

where $V_{-\kappa}(\varepsilon)$ denotes the volume of the ball of radius ε in the simply connected space of constant sectional curvature $-\kappa$ and of dimension n.

The goal of this section is to compare $Spec(X)$ and $Spec(M)$ using the same idea as in the case of roughly isometric graphs. So we have to associate functions on M to functions on V and vice versa. This leads us to use the smoothing and the discretization applications considered in Chavel's book. Let us recall the main definitions and results from [18] (section VI.5) we will need.

First, to go from the discretization $X = (V, E)$ to the manifold M, we need to smooth discrete functions.

Definition I.3.1 *Let $\{\phi_v\}_{v\in V}$ be a partition of unity subordinate to the cover $\{B(v, 2\varepsilon)\}_{v\in V}$ of M. Then for each $f : V \to \mathbb{R}$, the smoothing of f is defined by $F = \mathcal{S}f : M \to \mathbb{R}$*

$$(\mathcal{S}f)(x) = \sum_{v\in V} \phi_v(x) f(v).$$

Lemma I.3.2 *There exist positive constants c_1 and c_2 depending only on n, κ and on the mesh of the discretization such that*

$$\begin{aligned} ||\mathcal{S}f||^2 &\leq c_1||f||^2, && (6)\\ ||d(\mathcal{S}f)||^2 &\leq c_2||df||^2. && (7) \end{aligned}$$

Proof: see [18], VI.5.2. □

Then, to go in the other direction, we want to discretize smooth functions.

Definition I.3.3 *For each $F : M \to \mathbb{R}$ the discretization of F is defined by $f = \mathcal{D}F : V \to \mathbb{R}$*

$$\mathcal{D}F(v) = \frac{1}{Vol(B(v, 3\varepsilon))} \int_{B(v,3\varepsilon)} F(x)dV(x)$$

where dV denotes the volume form on (M, g).

Lemma I.3.4 *There exist positive constants C_1 and C_2 depending only on n, κ and on the mesh of the discretization such that*

$$\begin{aligned} ||\mathcal{D}F||^2 &\leq C_1||F||^2, && (8)\\ ||d(\mathcal{D}F)||^2 &\leq C_2||dF||^2. && (9) \end{aligned}$$

Proof: see [18], VI.5.1. □

Finally, compose $\mathcal{S}$ and $\mathcal{D}$ and look at how it differs from the identity.

Lemma I.3.5 *There exist positive constants c_3 and C_3 depending only on n, κ and on the mesh of the discretization such that*

$$\|f - \mathcal{DS}f\|^2 \leq c_3 \|df\|^2, \qquad (10)$$
$$\|F - \mathcal{SD}F\|^2 \leq C_3 \|dF\|^2. \qquad (11)$$

Proof: see [18], VI.5.4 and VI.5.5. □

We now state and prove the main technical theorem of this note.

Definition I.3.6 *For $\kappa \geq 0$ and $r_0 > 0$ define $\mathcal{M}(n, \kappa, r_0)$ as the set of all connected, compact n-dimensional Riemannian manifolds (M^n, g) with Ricci curvature and injectivity radius uniformly bounded below i.e. with $Ricci(M, g) \geq -(n-1)\kappa g$ and $Inj(M) \geq r_0$.*

Theorem I.3.7 *There exist positive constants c, C such that for all $M \in \mathcal{M}(n, \kappa, r_0)$ and for any discretization X of M (with mesh $< \frac{1}{2} r_0$), we have for all $k < |X|$*

$$c\lambda_k(X) \leq \lambda_k(M) \leq C\lambda_k(X).$$

The constants c and C depend only on n, κ and on the mesh of X.

In particular, if we fix the mesh equal to $\frac{1}{4} r_0$, then these constants depend only on the local geometry of M i.e. on n, κ and r_0. Note moreover that all constants are independent of k.

Theorem I.3.7 is a direct consequence of Theorem I.3.8 and Theorem I.3.9. In fact, the proof of Theorem I.3.7 goes in two steps as for the discrete case. The first step deals in some sense with small eigenvalues; we show that Theorem I.3.7 is true for eigenvalues smaller than some constant, using Chavel's results (see Theorem I.3.8). The second step of the proof of Theorem I.3.7 consists in showing that Theorem I.3.7 is true even if the eigenvalues are "big", that is to say bigger than the constants appearing in Theorem I.3.8. The proof here is really different from the previous one and uses basic facts on eigenvalues of Laplacian (see Theorem I.3.9).

Theorem I.3.8 *There exist positive constants a, A, c' and C' such that for all $M \in \mathcal{M}(n, \kappa, r_0)$ and for any discretization X of M (with mesh $< \frac{1}{2} r_0$), we have for all $k < |X|$*

i) If $\lambda_k(X) \leq a$, then $\lambda_k(M) \leq c'\lambda_k(X)$.

ii) If $\lambda_k(M) \leq A$, then $\lambda_k(X) \leq C'\lambda_k(M)$.

The constants a, A, c' and C' depend only on n, κ and on the mesh of X.

Proof of i) The idea of the proof is exactly the same as in Theorem I.2.1, using the above lemma in order to bound Rayleigh quotients. Let $M \in \mathcal{M}(n,\kappa,r_0)$ and $X=(V,E)$ a discretization with mesh $< \frac{1}{2}r_0$. Then, consider $f_0, \ldots, f_k : V \to \mathbb{R}$ eigenfunctions associated to the first $k+1$ eigenvalues of X. Denote by W the subspace spanned by these eigenfunctions. Smooth each f_i to obtain $F_0 = \mathcal{S}f_0, \ldots, F_k = \mathcal{S}f_k : M \to \mathbb{R}$ and $\mathcal{S}W$ the subspace spanned by the F_i's.

Then, if $\lambda_k(X) < a = \frac{1}{4}c_3^{-1}$, $\mathcal{S}W$ is $(k+1)$-dimensional. In order to prove this fact, let F be an element of $\mathcal{S}W$, $F=\sum_{i=0}^k a_i F_i$ with at least one non-zero coefficient. In fact, F is the smoothing of a discrete $f \in W$ such that $f = \sum_{i=0}^k a_i f_i$ and by (10), we have $\|f - \mathcal{D}\mathcal{S}f\|^2 \leq c_3 \|df\|^2$. So the norm of F satisfies (by (8))

$$\|F\| \geq C_1^{-\frac{1}{2}} \|\mathcal{D}F\| \geq C_1^{-\frac{1}{2}} (\|f\| - \|f - \mathcal{D}\mathcal{S}f\|) \geq C_1^{-\frac{1}{2}} (\|f\| - c_3^{\frac{1}{2}} \|df\|). \quad (12)$$

But f is a non-zero function belonging to the subspace W, so it satisfies $\|df\| \leq \sqrt{\lambda_k(X)}\|f\|$ and by assumption on $\lambda_k(X)$ and by (12), we get

$$\|F\| \geq \frac{1}{2} C_1^{-\frac{1}{2}} \|f\|. \quad (13)$$

As the eigenfunctions of X are linearly independent, this shows that the dimension of $\mathcal{S}W$ is the same as the dimension of W, that is to say $k+1$. Moreover, under the same assumption on $\lambda_k(X)$ and using (13), we obtain

$$R(\mathcal{S}f) \leq c' R(f)$$

for all $f \in W - \{0\}$. This leads now to the conclusion. Using Min-Max Theorem, we have

$$\begin{aligned} \lambda_k(M) &\leq \sup\{R(F) \mid F \in \mathcal{S}W - \{0\}\} \\ &\leq c' \sup\{R(f) \mid f \in W - \{0\}\} \\ &= c' \lambda_k(X) \end{aligned}$$

and this ends the proof of the theorem. The second part of the proof can be carried out exactly in the same way, because of the symmetry of the results concerning the smoothing and the discretization of functions. So it would not bring more informations to do it here. □

Theorem I.3.9 *Let a and A as in Theorem I.3.8. Then, there exist positive constants c'' and C'' such that for all $M \in \mathcal{M}(n,\kappa,r_0)$ and for any discretization X of M (with mesh $< \frac{1}{2}r_0$), we have for all $k < |X|$*

i) If $\lambda_k(X) > a$, then $\lambda_k(M) \le c''\lambda_k(X)$.

ii) If $\lambda_k(M) > A$, then $\lambda_k(X) \le C''\lambda_k(M)$.

The constants c'' and C'' depend only on n, κ and on the mesh of X.

Proof: i) As $\lambda_k(X) > a$, then $\lambda_k(M) < a^{-1}\lambda_k(X)\lambda_k(M)$. So, it suffices to show that $\lambda_k(M) \le \lambda_1^\kappa(\frac{\varepsilon}{2})$, where $\lambda_1^\kappa(\frac{\varepsilon}{2})$ denotes the first non-zero eigenvalue of the Dirichlet problem on the ball of radius $\frac{\varepsilon}{2}$ in the simply connected space of constant sectional curvature $-\kappa$ and of same dimension as M. We prove this result as follows. For each vertex v_i of X, $i \ge 1$, we can consider $f_i : M \to \mathbb{R}$ the first eigenfunction of the Dirichlet problem for the ball $\overline{B}(v_i, \frac{\varepsilon}{2})$ extended by 0 outside the ball. By Cheng's comparison Theorem $R(f_i) \le \lambda_1^\kappa\left(\frac{\varepsilon}{2}\right)$ (see [17], p.74).

Now, consider V_i the subspace spanned by $f_1, \dots, f_i$. As the balls of radius $\frac{\varepsilon}{2}$ are disjoint, the f_i's are orthogonal and so V_i is of dimension i. We can apply Min-Max Theorem and get for all $k < |X|$

$$\begin{aligned}
\lambda_k(M) &\le \sup\left\{\frac{\sum a_i^2\|df_i\|^2}{\sum a_i^2\|f_i\|^2} \ : \ f = \sum a_i f_i \in V_{k+1}, f \neq 0\right\} \\
&\le \sup\left\{\frac{\sum a_i^2\lambda_1^\kappa(\frac{\varepsilon}{2})\|f_i\|^2}{\sum a_i^2\|f_i\|^2} \ : \ f = \sum a_i f_i \in V_{k+1}, f \neq 0\right\} \\
&\le \lambda_1^\kappa\left(\varepsilon/2\right).
\end{aligned}$$

Finally, we get $\lambda_k(M) < a^{-1}\lambda_1^\kappa(\frac{\varepsilon}{2})\lambda_k(X)$.

ii) As $\lambda_k(M) > A$, then $\lambda_k(X) \le A^{-1}\lambda_k(M)\lambda_k(X)$. We have seen in Theorem I.2.1 that $\lambda_k(X) \le 4\nu_X$. So we get $\lambda_k(X) \le 4A^{-1}\nu_X\lambda_k(M)$. □

I.4 Application to the spectrum of a tower of coverings

As a first application, we will discuss the following theorem.

Theorem I.4.1 *Let (M^n, g) be a compact n-dimensional Riemannian manifold. Let $\{M_i\}_{i\ge1}$ be a family of finite-sheeted covering spaces of M with induced Riemannian metric. Let Γ_i be the Schreier graph of the subgroup $\pi_1(M_i)$ of $\pi_1(M)$. Then, there exist constants c, $C > 0$ such that for all $k < |\Gamma_i|$*

$$c\lambda_k(\Gamma_i) \le \lambda_k(M_i) \le C\lambda_k(\Gamma_i).$$

In particular, for all k

$$\lambda_k(M_i) \to 0 \text{ when } i \to \infty \iff \lambda_k(\Gamma_i) \to 0 \text{ when } i \to \infty.$$

For $k = 1$, it is exactly the result of Brooks (see [9] Theorem1).

We prove the result in two steps. First, we associate to M_i a discretization X_i of sufficiently small mesh, in order to compare $Spec(M_i)$ to $Spec(X_i)$ (as in Section I.3). Secondly, we show that X_i and Γ_i are roughly isometric, which allows us to apply Theorem I.2.1 to $Spec(\Gamma_i)$ and $Spec(X_i)$, so that we obtain the desired result between $Spec(M_i)$ and $Spec(\Gamma_i)$.

The first step is really a direct application of Theorem I.3.7 and is stated in Theorem I.4.2.

Theorem I.4.2 *Let M and $\{M_i\}_{i\geq 1}$ be as in Theorem I.4.1. Let X be a discretization of M (with mesh $< \frac{1}{2}Inj(M)$) and lift it to M_i to obtain a discretization X_i of M_i. Then, there exist positive constants c and C independent of i such that for all $k < |X_i|$*

$$c\lambda_k(X_i) \leq \lambda_k(M_i) \leq C\lambda_k(X_i).$$

In particular, for all k

$$\lambda_k(M_i) \to 0 \text{ when } i \to \infty \iff \lambda_k(X_i) \to 0 \text{ when } i \to \infty.$$

Proof: if $(M^n, g) \in \mathcal{M}(n, \kappa, r_0)$, then $M_i \in \mathcal{M}(n, \kappa, r_0)$, as M_i is provided with the induced Riemannian metric. Moreover, X_i is a discretization of M_i with same mesh of X smaller than $\frac{1}{2}r_0$. So we can apply Theorem I.3.7 to each pair (M_i, X_i) and get constants independent of i. $\square$

In the second step, we have to compare the Schreier graph Γ_i appearing in Theorem I.4.1 to the discretization X_i of M_i appearing in Theorem I.4.2. This is the next result.

Theorem I.4.3 *Let $\{(X_i, \Gamma_i)\}_{i\geq 1}$ be as in Theorems I.4.1 and I.4.2. Then, there exist positive constants c and C independent of i such that for all $k < |\Gamma_i|$*

$$c\lambda_k(\Gamma_i) \leq \lambda_k(X_i) \leq C\lambda_k(\Gamma_i).$$

In particular, for all k

$$\lambda_k(X_i) \to 0 \text{ when } i \to \infty \iff \lambda_k(\Gamma_i) \to 0 \text{ when } i \to \infty.$$

Proof: geometrically, the graph Γ_i corresponds to the lift of a graph Γ in M, where Γ consists of a unique point (see [8] p.254 for a definition of Schreier graphs). As M is compact, Γ and X are roughly isometric (see [18] p.147). Let us call Γ, Γ_i and X, X_i the set of vertices of the respective graphs too and $\Phi : \Gamma \to X$ a rough isometry such that $d(g, \Phi(g)) < \varepsilon$. We can lift Φ and get

$\Phi_i : \Gamma_i \to X_i$ in the following way. If $g \in \Gamma_i$, then by construction of X_i, there exists $\Phi_i(g) = x$ with $\pi_i(x) = \Phi(\pi_i(g))$ and $d(x, g) < \varepsilon$ (where π_i denotes the canonical projection of M_i onto M). Clearly, Φ_i is a rough isometry with same constants of rough isometry as Φ. So we can apply Theorem I.2.1 to each pair (X_i, Γ_i) and get constants independent of i. □

Note that, as any two discretizations of a compact manifold are roughly isometric, then we can replace Γ_i by the lift of any discretization of M and Theorem I.4.1 is still true.

I.5 Gromov-Hausdorff close manifolds have comparable spectra

Another application of Theorem I.3.7 is the following result.

Theorem I.5.1 *Let $(M^m, g_M) \in \mathcal{M}(m, \kappa, r_0)$ and $(N^n, g_N) \in \mathcal{M}(n, \kappa, r_0)$. Suppose that the Gromov-Hausdorff distance between M and N is smaller than $\eta > 0$. Then, there exist positive constants c and C (depending only on η, κ, r_0 and on the dimensions) and $K > 0$ (proportional to the volume of M and N) such that for all $k < K$*

$$c\lambda_k(N) \leq \lambda_k(M) \leq C\lambda_k(N)$$

and there exist c' and C' (depending on the dimensions, η, κ and r_0) such that for all $k \geq K$

$$c'\lambda_k(N)^n \leq \lambda_k(M)^m \leq C'\lambda_k(N)^n.$$

Corollary I.5.2 *Let M and N be two same dimensional compact Riemannian manifolds i.e M, $N \in \mathcal{M}(n, \kappa, r_0)$. Suppose that the Gromov-Hausdorff distance between M and N is smaller than $\eta > 0$. Then, there exist positive constants c and C (depending only on η, n, κ and r_0) such that for all k*

$$c\lambda_k(N) \leq \lambda_k(M) \leq C\lambda_k(N).$$

Proof of Theorem I.5.1: recall (from Section I.3) that if X_M is a discretization of $M \in \mathcal{M}(m, \kappa, r_0)$ of mesh $\varepsilon < \frac{1}{2}r_0$, then

$$\frac{1}{V_{-\kappa}(\varepsilon)}Vol(M) \leq |X_M| \leq \frac{2^m}{\varepsilon^m c(m)}Vol(M).$$

Let X_M be a discretization of $M \in \mathcal{M}(m,\kappa,r_0)$ and X_N a discretization of $N \in \mathcal{M}(n,\kappa,r_0)$ with same mesh $\varepsilon < \frac{1}{2}r_0$. Moreover, choose $\varepsilon < \frac{1}{2}r_0$ such that

$$\min\{|X_M|,|X_N|\} \geq \max\left\{Vol(M)\frac{2^{m+2}}{c(m)r_0^m}, Vol(N)\frac{2^{n+2}}{c(n)r_0^n}\right\}$$

(ε depend only on m, n, κ and r_0).

Then, the proof is done in two steps. First, if $k < \min\{|X_M|,|X_N|\}$, we will apply Theorem I.2.1 and Theorem I.3.7 (as will follow) and second, for $k \geq \min\{|X_M|,|X_N|\}$, we will use a result of Buser (Theorem 6.2 in [13]).

Consider the case $k < \min\{|X_M|,|X_N|\}$. By definition of the Gromov-Hausdorff distance (see [35]) there exist Z a Riemannian manifold and two isometric embeddings $f : M \to Z$, $g : N \to Z$ such that $\cup_{x\in M}B(f(x),\eta) \supset g(N)$ and $\cup_{y\in N}B(g(y),\eta) \supset f(M)$. Then, X_M and X_N are roughly isometric via $\Phi : X_M \to X_N$ defined as follows. For each $x \in X_M$ there exist $x' \in N$ and $x'' \in X_N$ such that $d(f(x),g(x')) < \eta$ and $d(x',x'') < \varepsilon$. Then define $\Phi(x) = x''$ so that $d(f(x),g(\phi(x))) < \eta+\varepsilon$. Then, we clearly have that

$$d(x,y) - 2(\eta+\varepsilon) < d(\Phi(x),\Phi(y)) < d(x,y) + 2(\eta+\varepsilon).$$

Moreover, if $z \in X_N$ then there exist $y \in M$ and $x \in X_M$ such that $d(g(z),f(y)) < \eta$ and $d(x,y) < \varepsilon$. This implies that

$$d(\Phi(x),z) \leq d(g(\Phi(x)),f(x)) + d(f(x),f(y)) + d(f(y),g(z)) < 2(\eta+\varepsilon)$$

and so $\cup_{x\in X_M}B(\Phi(x),2(\eta+\varepsilon)) \supset X_N$. Note that the constants of rough isometry depend only on ε and η. To conclude this first part, it suffices to apply Theorem I.3.7 to (M,X_M) and (N,X_N) and Theorem I.2.1 to (X_M,X_N).

Now, consider the case $k \geq \min\{|X_M|,|X_N|\}$. In this case and by assumption on ε, we can apply Theorem 6.2 of [13] to M and N which leads to the result

$$\left(\frac{k}{Vol(M)}\right)^{\frac{2}{m}} c_1 \leq \lambda_k(M) \leq \left(\frac{k}{Vol(M)}\right)^{\frac{2}{m}} c_2$$

where c_1 and c_2 are constants depending on m, κ and r_0. Similarly

$$\left(\frac{k}{Vol(N)}\right)^{\frac{2}{n}} c_3 \leq \lambda_k(N) \leq \left(\frac{k}{Vol(N)}\right)^{\frac{2}{n}} c_4$$

where c_3 and c_4 are constants depending on n, κ and r_0. Putting both inequalities together, we get that there exist constants c_5 and c_6 depending only on the dimensions, κ and r_0 such that

$$c_5\left(\frac{Vol(M)}{Vol(N)}\right)^2 \lambda_k(N)^n \leq \lambda_k(M)^m \leq c_6\left(\frac{Vol(M)}{Vol(N)}\right)^2 \lambda_k(N)^n.$$

Furthermore, we have seen in Section 3 that there exist constants c_7, c_8, c_9 and c_{10} depending only on the dimensions, κ and r_0 such that

$$c_7|X_M| \leq Vol(M) \leq c_8|X_M|,$$

$$c_9|X_N| \leq Vol(N) \leq c_{10}|X_N|.$$

As we have shown in the first part of the proof the discretizations are roughly isometric (the constants of rough isometry depend only on ε and η). This implies that there exist constants c_{11} and c_{12} depending on ε and η such that

$$c_{11}|X_N| \leq |X_M| \leq c_{12}|X_N|.$$

These last inequalities give us upper and lower bounds (depending only on η, m, n, κ and r_0) for $\frac{Vol(M)}{Vol(N)}$.

Finally, we get that there exist constants c, $C > 0$ depending only on η, m, n, κ and r_0 such that for all k

$$c\lambda_k(N)^n \leq \lambda_k(M)^m \leq C\lambda_k(N)^n.$$

In particular, the constants do not depend on k. □

I.6 An example

Let us now discuss an example in order to show that the assumption on the injectivity radius in Theorem I.5.1 is necessary. Consider M_ε the manifold obtained by taking two hyperbolic cylinders, gluing them together at both ends and "smoothing the angles".

More precisely, for $0 < \varepsilon < 1$ take

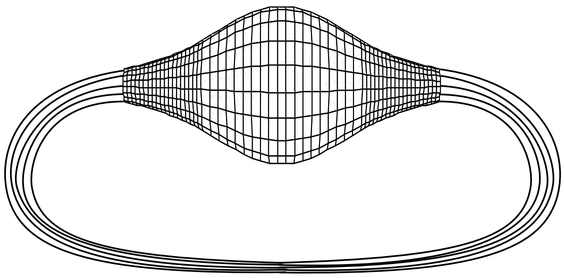
$$M_\varepsilon = [-1-\rho_\varepsilon; 1+\rho_\varepsilon] \times \mathbb{S}^1/_\sim$$

where $\sim$ identifies the ends (i.e. $(\rho_1, t_1) \sim (\rho_2, t_2)$ if and only if $\rho_1 = \rho_2 = \pm(1+\rho_\varepsilon)$ and $t_1 = t_2$) and $\rho_\varepsilon = \operatorname{arccosh}(\varepsilon^{-\frac{1}{2}})$. Provide M_ε with the Riemannian metric

$$ds^2 = d\rho^2 + f(\rho)^2 dt^2$$

where

$$f(\rho) = \begin{cases} \varepsilon\cosh(\rho+\rho_\varepsilon+1) & \text{if } \rho \in [-1-\rho_\varepsilon; -1] \\ \frac{\sqrt{\varepsilon}}{8}\left((1+\sqrt{1-\varepsilon})\rho^4 - 2(1+3\sqrt{1-\varepsilon})\rho^2 + 9+5\sqrt{1-\varepsilon}\right) & \text{if } \rho \in [-1; 1] \\ \varepsilon\cosh(-\rho+\rho_\varepsilon+1) & \text{if } \rho \in [1; 1+\rho_\varepsilon] \end{cases}$$



The manifold M_ε

Then, M_ε has sectional curvature uniformly bounded (i.e independently of ε) and injectivity radius comparable to ε. Moreover, as M_ε admits an involution and if $D_\varepsilon = \{(\rho, t) \ : \ \rho \in [-1-\rho_\varepsilon; 0], t \in \mathbb{R}/\mathbb{Z}\}$, then $\lambda_1(M_\varepsilon)$ is either the first non-zero eigenvalue of D_ε for the Neumann problem $\lambda_1^N(D_\varepsilon)$ or the first eigenvalue of D_ε for the Dirichlet problem $\lambda_1^D(D_\varepsilon)$, where D_ε is provided with the same Riemannian metric as M_ε. But D_ε can be provided with an hyperbolic metric $ds'^2 = d\rho^2 + \varepsilon^2 \cosh^2(\rho + \rho_\varepsilon + 1)dt^2$ and we can easily show that there are positive constants c_1 and c_2 independent of ε such that

$$c_1 ds^2 \le ds'^2 \le c_2 ds^2.$$

Then, applying a result of Dodziuk (see [25] Proposition 3.3), we get a constant c_3 independent of ε such that

$$\lambda_1^{N,D}(D_\varepsilon, ds^2) \ge c_3 \lambda_1^{N,D}(D_\varepsilon, ds'^2).$$

But, we know that for thin hyperbolic cylinders $\lambda_1^{N,D}(D_\varepsilon, ds'^2) \ge \frac{1}{4}$ (see [12] p.35). Finally, we have shown that $\lambda_1(M_\varepsilon) \ge c > 0$ where c is independent on ε.

Moreover, M_ε is Hausdorff-Gromov close to the circle S_ε of length $2\rho_\varepsilon + 2$ and $\lambda_1(S_\varepsilon)$ goes to zero when ε goes to zero too. So

$$\frac{\lambda_1(M_\varepsilon)}{\lambda_1(S_\varepsilon)} \to \infty \text{ when } \varepsilon \to 0.$$

Then, this quotient cannot be bounded and the theorem is not true for the family $(M_\varepsilon, S_\varepsilon)$, because the injectivity radius of M_ε is not uniformly bounded below.

Part II

Discretization of the rough Laplacian

Discretization of Vector Bundles and Rough Laplacian

This text has already been accepted for publication at "The Asian Journal of Mathematics".

Abstract. Let $\mathcal{M}(m,\kappa,r_0)$ be the set of all compact connected m-dimensional manifolds (M,g) such that $Ricci(M,g) \geq -(m-1)\kappa g$ and $Inj(M,g) \geq r_0 > 0$. Let $\mathcal{E}(n,k_1,k_2)$ be the set of all Riemannian vector bundles (E,∇) of real rank n with $|R^E| \leq k_1$ and $|d^*R^E| \leq k_2$. For any vector bundle $E \in \mathcal{E}(n,k_1,k_2)$ with harmonic curvature or with complex rank one, over any $M \in \mathcal{M}(m,\kappa,r_0)$ and for any discretization X of M of mesh $0 < \varepsilon \leq \frac{1}{20}r_0$, we construct a canonical twisted Laplacian Δ_A and a potential V depending only on the local geometry of E and M such that we can compare uniformly the spectrum of the rough Laplacian $\overline{\Delta}$ associated to the connection of E and the spectrum of $\Delta_A + V$. We show that there exist constants c, $c' > 0$ depending only on the parameters of $\mathcal{M}(m,\kappa,r_0)$ and $\mathcal{E}(n,k_1,k_2)$ such that $c'\lambda_k(X,A,V) \leq \lambda_k(E) \leq c\lambda_k(X,A,V)$, where $\lambda_k(\cdot)$ denotes the k^{th} eigenvalue of the considered operators ($k \leq n|X|$). For flat vector bundles, we show that the potential is zero, Δ_A turns out to be a discrete magnetic Laplacian and we relate $\lambda_1(E)$ to the holonomy of E.

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Key words: connection, rough Laplacian, discrete magnetic Laplacian, Harper operator, eigenvalues, discretization, holonomy.

II.1 Introduction

In [48], we have shown that for a family of compact connected manifolds $\mathcal{M}(m,\kappa,r_0)$ with injectivity radius and Ricci curvature bounded below (i.e. $(M,g) \in \mathcal{M}(m,\kappa,r_0)$ if M is a compact connected m-dimensional Riemannian manifold with $Ricci(M,g) \geq -(m-1)\kappa g$ and $Inj(M,g) \geq r_0$), we can compare uniformly the spectrum of the Laplacian acting on functions with the spectrum of the combinatorial Laplacian acting on a graph with fixed mesh constructed on the manifolds. Indeed, we show that there exist positive constants c, c' depending on the parameters of the problem such that for any $M \in \mathcal{M}(m, \kappa, r_0)$ and any discretization X of M (with mesh $\varepsilon < \frac{1}{2}r_0$), the following holds

$$c'\lambda_k(X) \leq \lambda_k(M) \leq c\lambda_k(X) \tag{14}$$

for $k < |X|$, where $\lambda_k(\cdot)$ stands for the k^{th} eigenvalue of the considered Laplacian. This result generalizes in a natural way different works like [9], [11], [14] and [41] that were motivated either by the study of the relation between the fundamental group of a manifold and the spectrum of its finite coverings ([9], [11]) or by the relation between the spectrum of a manifold and its Cheeger isoperimetric constant ([14]) or by the existence of harmonic functions ([41]). More generally, the aim of the discretization is to have an understanding of the spectrum (a global invariant on the manifold) with a minimum of informations about the local geometry of the manifold.

Of course, the problem is interesting for differential operators other than the Laplacian and we may address the following question: does the same kind of comparison hold for other geometric differential operators such that the Laplacian acting on p-forms or the Dirac operator? Most of these operators may be expressed in terms of a connection Laplacian added with a curvature term. In this article, we investigate the case of such a connection (or rough) Laplacian $\overline{\Delta}$ associated to a connection ∇ on a vector bundle. More precisely, the purpose is to establish a uniform comparison of spectra between rough Laplacians on vector bundles and twisted Laplacians on graphs that generalize combinatorial or discrete magnetic Laplacians. The Riemannian vector bundles we are interested in have curvature and exterior coderivative of curvature bounded i.e. we study Riemannian vector bundles E with fiber of real rank n such that $|R^E| \leq k_1$ and $|d^*R^E| \leq k_2$ (denote by $\mathcal{E}(n, k_1, k_2)$ the set of such vector bundles). The main result (Theorem II.3.1) states that there exist positive constants c, c' (depending only on the given parameters) such that for any vector bundle $E \in \mathcal{E}(n, k_1, k_2)$ over any $M \in \mathcal{M}(m, \kappa, r_0)$ satisfying one of the following assumptions

I) the curvature of E is harmonic i.e. $d^*R^E = 0$,

II) E is of complex (or quaternionic) rank one

and for any discretization X of E, we can construct a canonical twisted Laplacian Δ_A and a potential V depending only on the local geometry of E

such that

$$c'\lambda_k(X,A,V) \leq \lambda_k(E) \leq c\lambda_k(X,A,V) \tag{15}$$

for any $k \leq n|X|$, where $\lambda_k(E)$ denotes the k^{th} eigenvalue of the rough Laplacian $\overline{\Delta}$ and $\lambda_k(X,A,V)$ the k^{th} eigenvalue of $\Delta_A + V$.

The case of flat vector bundles is especially enlightening. Indeed, if E is flat, we show that the potential V is zero and that Δ_A is a discrete magnetic Laplacian. This particular case shows how the construction of Δ_A is strongly related to the holonomy of E. This fact is emphasized by Theorem II.4.1 which relates the holonomy (in the sense of [3]) to the first eigenvalue of Δ_A and therefore of $\overline{\Delta}$. In order to understand the problem of non-flat vector bundles, go back to the case of functions. Recall that for functions we needed to establish correspondances between functions on the manifold and functions on the graph. To that aim and in particular to associate smooth functions to functions on the graph, we had to extend locally such a function in a constant way and then smooth it (with a partition of unity). The question of extending locally is a central problem for the case of vector bundles. It turns out that extending by parallel transport is really efficient for flat vector bundles as it produces parallel sections. But, as soon as the curvature comes in, parallel transport is not convenient anymore and we need to construct a finer way to extend locally a section. In fact, the obstruction to extend in a parallel manner is double: the holonomy plays the role of a global obstruction to extend as parallel as possible and locally the curvature plays the same role. The twisted Laplacian will precisely render the holonomy of the vector bundle, while the potential will take into account the local non-flat geometry.

The paper is organized as follows. In Section II.2, we introduce the notations, we define the general notion of twisted Laplacian on a graph and recall the main properties of the discretization of a manifold (that will coincide with the notion of discretization of vector bundles). Section II.3 is devoted to the proof of the main result (Theorem II.3.1). The main difficulty is to construct a suitable twisted Laplacian (see Section II.3.1). From a geometric point of view, the problem is the dependence on the local geometry of the Laplacian and the potential to have enough informations to estimate the spectrum of the vector bundle. Technically, we need fine analysis on vector bundles like Sobolev inequalities for sections to achieve the construction. The particular case of flat vector bundles can be kept in mind as the ground example during the reading. In this case, the proofs can be done easier (we can avoid the technical tools described in Section II.3.1). Nevertheless, this case already contains the essential information for Δ_A as it shows how the holonomy is related to Δ_A (see Section II.4). For non-flat vector bundles, Δ_A does not suffice anymore to control the rough Laplacian, so that we have to introduce a potential V which takes care of the curvature locally. The generalization of the flat case is then done for two different cases (see assumptions I) and II)), for rank one vector bundles and for vector bundles with harmonic curvature. These two cases are really of different nature. This appears all along Section II.3 and this begins with the construction of $\Delta_A + V$ (in Section II.3.2) which differs according to the assumptions I) or II). In Section II.4, we establish the relationship between the holonomy and the first eigenvalue of the rough Laplacian for flat vector bundles. The part of Theorem II.4.1 that bounds from below the first eigenvalue in terms of the holonomy can be generalized easily to vector bundles with harmonic curvature. But this will not be done here. This result is in fact due to Ballmann, Brüning and Carron in a more general setting (see [3]). Finally, we collect some more technical proofs in the appendix to make easier the reading, even if the results are not of minor importance for the paper.

II.2 Settings

II.2.1 Rough Laplacian

In this section, we recall basic facts on the rough Laplacian (for a general reference see [5], [53] or [54] for instance). Let (M, g) be a compact connected m-dimensional Riemannian manifold without boundary and with volume form denoted by dV. Moreover, let (E, ∇) be a Riemannian vector bundle with n-dimensional fiber over M i.e. E is a vector bundle over M endowed with a smooth metric $\langle \cdot, \cdot \rangle$ and a compatible connection ∇. On the set $\Gamma(E)$ of smooth sections of E, denote by $(\cdot, \cdot)$ the L^2-inner product endowed by $\langle \cdot, \cdot \rangle$ and g. Recall that the connection extends to p-tensors on M with values in E and that we define ∇^* to be the adjoint of ∇ with respect to the L^2-inner product. The rough Laplacian (or connection Laplacian) acting on $\Gamma(E)$ is then defined by $\overline{\Delta} = \nabla^* \nabla$. The spectrum of $\overline{\Delta}$ is discrete and non-negative and will be denoted

$$Spec(E) = \{\lambda_1(E) \leq \lambda_2(E) \leq \ldots \leq \lambda_k(E) \leq \ldots\}.$$

The Rayleigh quotient of a non-zero section s is defined by $R(s) = \frac{\|\nabla s\|^2}{\|s\|^2}$, where $\|\cdot\|$ denotes the L^2-norm associated to the L^2-inner product defined above. Later we will need the following variational characterizations of $Spec(E)$ known as min-max and max-min theorems. For any $k \geq 1$,

$$\begin{aligned}\lambda_k(E) &= \min_{\Omega^k} \max\{R(s) : s \in \Omega^k \setminus \{0\}\} \\ &= \max_{\Omega^{k-1}} \min\{R(s) : s \in \Omega^{k-1} \setminus \{0\},\ s \perp \Omega^{k-1} \text{ w.r.t } (\cdot,\cdot)\}\end{aligned}$$

where Ω^k (resp. Ω^{k-1}) ranges over all k-dimensional (resp. $(k-1)$-dim.) subspaces of $\Gamma(E)$.

II.2.2 Twisted Laplacian

Let $\Gamma = (X, E(X))$ be a finite connected graph endowed with the path metric. For $p \in X$ denote by $N(p)$ the set of vertices at distance 1 from p and by $m(p)$ the number of such vertices. In order to generalize the combinatorial Laplacian (see [47] for a definition) and the discrete magnetic Laplacian (see [49] for a definition), let us consider the set of functions on X with values in $\mathbb{R}^n$ i.e. $\mathcal{F}(X) = \{f : X \to \mathbb{R}^n\}$, provided with the inner product $(f, g) = \sum_{p \in X} f(p) \cdot g(p)$, where $\cdot$ denotes the Euclidean inner product of $\mathbb{R}^n$.

Definition II.2.1 *For any $p \in X$ and $q \in N(p)$ assume that $A(p, q) : \mathbb{R}^n \to \mathbb{R}^n$ is a given linear transformation. The **twisted Laplacian** associated to A is the operator $\Delta_A : \mathcal{F}(X) \to \mathcal{F}(X)$ defined by*

$$\Delta_A f(p) = \frac{1}{2} \sum_{q \in N(p)} \left(\mathbb{I} + A^t(p, q) A(p, q)\right) f(p) - \left(A(q, p) + A^t(p, q)\right) f(q).$$

Remark II.2.2 *If for any p, q, the operator $A(p, q)$ is the identity, then Δ_A is the combinatorial Laplacian.*

Remark II.2.3 *If $A(p, q)$ belongs to $O(n)$ and $A^t(p, q) = A(q, p)$, then $\Delta_A f(p) = m(p) f(p) - \sum_{q \in N(p)} A(q, p) f(q)$. In this case the twisted Laplacian is usually called **discrete magnetic Laplacian** or Laplacian associated to the Harper operator A.*

Let us introduce the space of functions $\mathcal{F}(X \times X) = \{F : X \times X \to \mathbb{R}^n\}$ and provide it with the inner product given by $(F, G) = \frac{1}{2} \sum_{p \in X} \sum_{q \in X} F(p, q) \cdot G(p, q)$.

Lemma II.2.4 *Let $A(p, q)$ be as in Definition II.2.1 and Δ_A the twisted Laplacian associated to A. Let $D_A : \mathcal{F}(X) \to \mathcal{F}(X \times X)$ be defined by*

$$D_A f(p, q) = \begin{cases} f(q) - A(p, q) f(p) & \text{if } p \in N(q), \\ 0 & \text{otherwise.} \end{cases}$$

Then, for any f,$g \in \mathcal{F}(X)$, we have $(\Delta_A f, g) = (D_A f, D_A g)$.

Proof: let f, $g \in \mathcal{F}(X)$. Then, we have

$$\begin{aligned}(\Delta_A f, g) &= \frac{1}{2}\sum_{p\in X}\sum_{q\in N(p)} (f(p) - A(q,p)f(q)) \cdot g(p) \\ &\quad -\frac{1}{2}\sum_{p\in X}\sum_{q\in N(p)} (f(q) - A(p,q)f(p)) \cdot A(p,q)g(p) \\ &= \frac{1}{2}\sum_{p\in X}\sum_{q\in X} D_A f(q,p)\cdot g(p) + \frac{1}{2}\sum_{p\in X}\sum_{q\in X} D_A f(p,q)\cdot D_A g(p,q) \\ &\quad -\frac{1}{2}\sum_{p\in X}\sum_{q\in X} D_A f(p,q)\cdot g(q) \;=\; (D_A f, D_A g). \qquad \square\end{aligned}$$

A direct consequence of this lemma is that Δ_A is symmetric and non-negative, so it admits a non-negative spectrum. If $V : \mathcal{F}(X) \to \mathcal{F}(X)$ is a non-negative potential, then the spectrum of $\Delta_A + V$ is characterized by min-max theorem as follows

$$\forall 1 \le k \le n|X|, \qquad \lambda_k(X, A, V) = \min_{W^k} \max\{R(f) : f \in W^k \setminus \{0\}\}$$

where W^k ranges over all k-dimensional vector subspaces of $\mathcal{F}(X)$ and $R(f)$ is the Rayleigh quotient of f defined by $R(f) = \frac{\|D_A f\|^2 + (Vf,f)}{\|f\|^2}$.

II.2.3 Discretization of vector bundles

In this section, we define the notion of discretization of a vector bundle.

Definition II.2.5 *Let (E, ∇) be a Riemannian vector bundle over (M, g) a compact connected Riemannian manifold with $\partial M = \emptyset$. An ε-**discretization** of E is a discretization of M of mesh $\varepsilon > 0$.*

The discretization of a manifold (of mesh ε) is defined as in [18] (Section V.3.2). Let us recall the definition and the properties of such a discretization. Let (M, g) be a compact connected m-dimensional Riemannian manifold. A discretization of M, of mesh $\varepsilon > 0$, is a maximal ε-separated subset X of M provided with a graph structure given by the sets $N(p) = \{q \in X \mid 0 < d(p,q) < 3\varepsilon\}$, for any $p \in X$. In other words, X is such that for any distinct p, $q \in X$, $d(p,q) \ge \varepsilon$ and $\bigcup_{p\in X} B(p,\varepsilon) = M$. Moreover, pq is an edge if and only if $0 < d(p,q) < 3\varepsilon$. Denote by $m(p)$ the number of elements of $N(p)$.

Remark II.2.6 *Let us remark that if $B(p,\rho)$ is a ball in M with radius $\rho < \frac{1}{2}Inj(M)$, then the volume $V(p,\rho)$ of the ball $B(p,\rho)$ is bounded below* *by a constant depending only on ρ and m (this is Croke's Inequality, see for instance in [18] p.136). Moreover, if M has Ricci curvature bounded below by $-(m-1)\kappa$ then the volume of a ball of radius R is bounded above by a constant depending only on m, κ and R (this follows from Bishop's comparison theorem, see for instance [18], p.126). These bounds will be used frequently in the sequel.*

Choose ε smaller than $\frac{1}{2}Inj(M)$. Denote by $\kappa \geq 0$ a constant such that $Ricci(M,g) \geq -(m-1)\kappa g$. Then, using Remark II.2.6 we can show that $m(p)$ is bounded above by a constant ν_X depending only on m, κ and ε and that $\frac{1}{V_{-\kappa}(\varepsilon)}Vol(M) \leq |X| \leq \frac{2^m}{\varepsilon^m c(m)}Vol(M)$, where $V_{-\kappa}(\varepsilon)$ denotes the volume of the ball of radius ε in the simply connected space of constant sectional curvature $-\kappa$ and of dimension m.

II.3 Spectra comparison for rough Laplacian and twisted Laplacian

In this section, we will establish the comparison between the spectra of the rough Laplacian and a twisted Laplacian. Let us state the main result.

Theorem II.3.1 *Let m, n be positive integers, κ, k_1, $k_2 \geq 0$ and $r_0 \geq 20\varepsilon > 0$. There exist positive constants c, c' depending only on m, n, κ, k_1, k_2 and ε such that for any $M \in \mathcal{M}(m,\kappa,r_0)$, any vector bundle $E \in \mathcal{E}(n,k_1,k_2)$ over M satisfying one of the following condition*

I) the curvature of E is harmonic i.e. $d^*R^E = 0$,

II) E is of complex (or quaternionic) rank one

and for any ε-discretization X of E, we can construct a canonical twisted Laplacian Δ_A and a potential V depending only on the local geometry of E such that, for $1 \leq k \leq n|X|$

$$c'\lambda_k(X,A,V) \leq \lambda_k(E) \leq c\lambda_k(X,A,V).$$

In particular, if the vector bundle is flat, the potential is zero and Δ_A is a discrete magnetic Laplacian.

Roughly speaking, the basic idea of the proof is the same as to prove the theorem of comparison of spectra between the Laplacian acting on functions and the combinatorial Laplacian ([48], Theorem 3.7). But a first fundamental difference between the functions and the vector bundles cases is the construction of the twisted Laplacian. Indeed, in [48] the combinatorial Laplacian appearing in Theorem 3.7 is canonically associated to the graph that discretizes the manifold. For vector bundles, such a canonical Laplacian on graphs does not obviously exist. Hence, a first step of the proof consists in constructing a suitable twisted Laplacian Δ_A and a potential V (Section II.3.2) that will depend only on the local geometry. The construction of $\Delta_A + V$ differs according to the assumptions I) and II). We will work with balls centered on X and for both cases the construction of Δ_A relies essentially on changes of bases from a ball to a neighboring ball, but for vector bundles satisfying II) the definition of Δ_A is slightly harder. A more significant difference is the construction of the potential V. For rank one vector bundles, V involves only the first eigenvalue of balls (with Neumann boundary condition), while in the other case, we will distinguish "small" eigenvalues of balls from "large" eigenvalues. In rank one vector bundles the n first eigenvalues (of such a ball) are the same and correspond to the minimum of the energy, so that it will make easier the estimating of V.

After defining the twisted Laplacian and the potential, we follow the same way of proof as for the case of functions, but the underlying analysis is much more difficult. For instance, we need to establish some Sobolev inequalities for sections that requires fine tools of analysis as Moser's iteration and Sobolev inequalities for functions (Lemma II.5.1 in Appendix). The definition of the smoothing operator $\mathcal{S}$ and the discretizing operator $\mathcal{D}$ generalizes in some sense the similar operators defined by Chavel in [18] (Sections VI.5.1 and VI.5.2). Similarly, we establish norms estimations for these operators $\mathcal{S}$ and $\mathcal{D}$ (Propositions II.3.18 and II.3.21) in order to compare Rayleigh quotients of sections with Rayleigh quotients of functions on the discretization. Then, min-max theorem leads to the result for "small" eigenvalues. It suffices moreover to have upper bounds on the respective spectra (Lemma II.3.23) to compare "large" eigenvalues and conclude the proof of Theorem II.3.1 (Section II.3.6).

II.3.1 Local extension

In this section we define a way to extend a section as parallel as possible. In the case of flat vector bundles parallel transport is the suitable tool, because of the lemma below. Let $\tau_{x,p}$ denotes the parallel transport from E_p to E_x along the minimizing geodesic joining p to x (for $d(p,x) < \frac{1}{2} Inj(M)$).

Lemma II.3.2 *Let (E, ∇) be a flat Riemannian vector bundle over a Riemannian manifold (M, g). Let $p \in M$ and $B(p, r)$ the ball centered at p of* *radius* $r < \frac{1}{2}Inj(M)$. *Then for any* $v \in E_p$, *the section* σ *over* $B(p, r)$ *defined by* $\sigma(x) = \tau_{x,p}v$ *is parallel.*

Proof: see [28] Section 2.2.1. □

In the non-flat case, extending by parallel transport is not strong enough for our purpose, because we need to control the covariant derivative of such extended sections. More precisely, we want to extend in an energy minimizing way. This means that we have to take into account local small eigenvalues. Hence, we introduce eigensections of the Neumann problem on balls which give an obstruction to extension in a parallel way. Such eigensections on balls associated to small eigenvalues are almost parallel (Lemma II.3.3) and will provide a good way to extend sections. Nevertheless, it may happen that there are no (or only a few) small eigensections on a ball. In this case, parallel transport will be good enough to extend as we will see.

Lemma II.3.3 *Let* $(E, \nabla) \in \mathcal{E}(n, k_1, k_2)$ *over* $(M, g) \in \mathcal{M}(m, \kappa, r_0)$. *For* $0 < r < \frac{1}{2}r_0$ *and* $p \in M$, *let* $\sigma : B(p, r) \to E$ *be a section such that* $\overline{\Delta}\sigma = \lambda\sigma$ *for a constant* $\lambda \geq 0$. *Let* $0 < \theta < 1$. *Then there exist* $0 < c(m) \leq s \leq 1$ *and* c, $c' > 0$ *depending on an upper bound for* λ *and on* m, n, κ, r, k_1, k_2 *and* θ *such that*

$$\begin{aligned} \|\sigma\|_{\infty,\theta r} &\leq c\|\sigma\|_{2,r}, \\ \|\nabla\sigma\|_{\infty,\theta r} &\leq c'\|\nabla\sigma\|_{2,r}^{s} \end{aligned}$$

where $\|\cdot\|_{q,\rho}$ *denotes the* L^q*-norm on the ball centered at* p *of radius* ρ *(*c' *depends on* $c\|\sigma\|_{2,r}$ *too).*

Moreover, there exists $c'' > 0$ *depending on* c, c' *and* r *such that*

$$|\sigma(x) - \tau_{x,p}\sigma(p)| \leq c''\|\nabla\sigma\|_{2,r}^{s}$$

for all $x \in B(p, \theta r)$. *If* $k_2 = 0$ *i.e. if* E *is of harmonic curvature, then* $s = 1$ *in the previous inequalities.*

Proof: the idea is to use a Moser iteration to prove the statement. The more technical part of the argument is carried out in the appendix (see Lemma II.5.1). In order to use Lemma II.5.1, let $\delta > 0$ and $u_\delta : B(p, r) \to \mathbb{R}$ defined by $u_\delta = \sqrt{|\sigma|^2 + \delta}$. Then in one hand $\Delta(u_\delta^2) = 2u_\delta\Delta u_\delta - 2|du_\delta|^2$ and in the other hand $\Delta(u_\delta^2) = 2\langle\sigma, \overline{\Delta}\sigma\rangle - 2|\nabla\sigma|^2$ which implies that

$$u_\delta\Delta u_\delta \leq \langle\sigma, \overline{\Delta}\sigma\ \rangle = \lambda|\sigma|^2 \leq \lambda u_\delta^2.$$

We can then apply Lemma II.5.1 to u_δ and we get that $\|u_\delta\|_{\infty,\theta r} \leq c\|u_\delta\|_{2,r}$. Then let $\delta \to 0$ to obtain the first claim.

For the second inequality, let $\delta > 0$ and $v_\delta : B(p,r) \to \mathbb{R}$ defined by $v_\delta(x) = \sqrt{|\nabla\sigma(x)|^2 + \delta}$. Then

$$\Delta(v_\delta^2) = 2v_\delta \Delta v_\delta - 2|dv_\delta|^2 = 2\langle \nabla\sigma, \overline{\Delta}(\nabla\sigma)\rangle - 2|\nabla\nabla\sigma|^2.$$

But we have that $|\nabla\nabla\sigma|^2 - |dv_\delta|^2 \geq 0$ and therefore

$$v_\delta \Delta v_\delta \leq \langle \nabla\sigma, \overline{\Delta}(\nabla\sigma)\rangle = \langle \nabla\sigma, \overline{\Delta}(\nabla\sigma) - \nabla(\overline{\Delta}\sigma)\rangle + \lambda|\nabla\sigma|^2.$$

By a commuting argument (see [2], Lemma 2.3) we have for a local orthonormal frame $\{X_i\}_{i=1,\dots,m}$ of M

$$\langle \nabla\sigma, \overline{\Delta}(\nabla\sigma) - \nabla(\overline{\Delta}\sigma)\rangle =$$
$$\lambda|\nabla\sigma|^2 - \langle \nabla_{Ric(\cdot)}\sigma, \nabla\sigma\rangle - 2\sum_{i=1}^{m}\langle R^E(X_i,\cdot)\nabla_{X_i}\sigma, \nabla\sigma\rangle + \langle (d^*R^E)\sigma, \nabla\sigma\rangle$$

and as $Ricci(M,g) \geq -(m-1)\kappa g$, $|R^E| \leq k_1$ and $|d^*R^E| \leq k_2$ we then get

$$\langle \nabla\sigma, \overline{\Delta}(\nabla\sigma) - \nabla(\overline{\Delta}\sigma)\rangle \leq \left(\lambda + (m-1)\kappa + 2n^2k_1\right)|\nabla\sigma|^2 + n^2k_2|\sigma||\nabla\sigma|.$$

By the first part of the proof, we obtain that on $B(p,\theta r)$

$$\langle \nabla\sigma, \overline{\Delta}(\nabla\sigma) - \nabla(\overline{\Delta}\sigma)\rangle \leq$$
$$\left(\lambda + (m-1)\kappa + 2n^2k_1\right)|\nabla\sigma|^2 + n^2k_2c\|\sigma\|_{2,r}|\nabla\sigma|$$

and this implies (on $B(p,\theta r)$)

$$\Delta v_\delta \leq \left(\lambda + (m-1)\kappa + 2n^2k_1\right)v_\delta + n^2k_2c\|\sigma\|_{2,r}.$$

If $\theta' < \theta$ we can apply Lemma II.5.1 to v_δ and let $\delta \to 0$ to obtain

$$\|\nabla\sigma\|_{\infty,\theta' r} \leq c'\|\nabla\sigma\|_{2,\theta r}^s \leq c'\|\nabla\sigma\|_{2,r}^s. \tag{16}$$

Note that if $k_2 = 0$, then $s = 1$ and c' does not depend on $c\|\sigma\|_{2,r}$. The two first inequalities in the statement are then true for any θ' such that $0 < \theta' < \theta < 1$. So rename θ' by θ to obtain the statement.

Finally, recall that if γ is the minimizing geodesic joining p to $x \in B(p,\theta r)$ of length l $(< \theta r)$, then $|\sigma(x) - \tau_{x,p}\sigma(p)| \leq \int_0^l \left|\nabla_{\dot\gamma(t)}\sigma(\gamma(t))\right| dt \leq l\|\nabla\sigma\|_{\infty,\theta r}$. Using (16) leads to the result. □

From now on, let $E \in \mathcal{E}(n,k_1,k_2)$ over $M \in \mathcal{M}(m,\kappa,r_0)$ and fix $\varepsilon \leq \frac{1}{20}r_0$. Let X be an ε-discretization of E. Let $\sigma_k^p : B(p,10\varepsilon) \to E$ be the eigensection associated to the k^{th} eigenvalue $\lambda_k(p)$ of $\overline{\Delta}$ on $B(p,10\varepsilon)$ with Neumann boundary condition such that $\int_{B(p,10\varepsilon)}\langle \sigma_k^p, \sigma_l^p\rangle dV = \delta_{kl}V(p,10\varepsilon)$.

Remark II.3.4 *If E is flat $\lambda_1(p) = \ldots = \lambda_n(p) = 0$ and the σ_k^p's give a local orthonormal frame over $B(p, 10\varepsilon)$.*

Remark II.3.5 *If $n = 2$ (resp. $n = 4$) and E is of complex (resp. quaternionic) rank one, then $\lambda_1(p) = \ldots = \lambda_n(p)$. Indeed, the section $i\sigma_1^p$ (resp. $i\sigma_1^p$, $j\sigma_1^p$, $k\sigma_1^p$ where i, j, k are the quaternions with $i^2 = j^2 = k^2 = -1$) satisfies $\nabla(i\sigma_1^p) = i\nabla\sigma_1^p$ which implies that $i\sigma_1^p$ is a $\lambda_1(p)$-eigensection orthogonal to σ_1^p. Hence, we can choose σ_k^p such that for any x in $B(p, 10\varepsilon)$, $\langle \sigma_k^p(x), \sigma_l^p(x)\rangle = 0$ for any $1 \le k \le n$, $1 \le l \le n$, $k \ne l$.*

Lemma II.3.6 *Let $0 \le \alpha < \frac{1}{n+1}$. There exists $\delta > 0$ depending only on α, m, n, k_1, k_2, κ, ε such that if $\lambda_k(p) \le \delta$ then $\forall\, 1 \le i, j \le k$ and $\forall x \in B(p, 8\varepsilon)$ $|\langle \sigma_i^p(x), \sigma_j^p(x)\rangle - \delta_{ij}| \le \alpha$. In particular, if $\lambda_k(p) \le \delta$, then $\{\sigma_1^p(x), \ldots, \sigma_k^p(x)\}$ spans a k-dimensional vector subspace of E_x, for any $x \in B(p, 8\varepsilon)$.*

To prove this lemma, let us recall a basic fact of linear algebra (the proof of the fact is left to the reader). Let V be an n-dimensional vector space provided with an inner product $\langle\cdot,\cdot\rangle$. If $\{v_1, \ldots, v_n\} \subseteq V$ is such that $|\langle v_i, v_j\rangle - \delta_{ij}| \le \alpha < \frac{1}{n+1}$, then $\{v_1, \ldots, v_n\}$ is a basis of V. Moreover for any $v = \sum_{i=1}^n a_i v_i$, we have $(1-\alpha(n+1))\sum_{i=1}^n a_i^2 \le \|v\|^2 \le (1+\alpha(n+1))\sum_{i=1}^n a_i^2$. Such a basis will be referred as an **almost orthonormal basis**.

Proof of Lemma II.3.6: let $f_{ij}(x) = \langle \sigma_i^p(x), \sigma_j^p(x)\rangle$ and denote by m_{ij} its mean over $B(p, 10\varepsilon)$, then

$$m_{ij} = \frac{1}{V(p, 10\varepsilon)} \int_{B(p,10\varepsilon)} f_{ij}\, dV = \delta_{ij}.$$

A result of Kanai ensuring the existence of $c_K > 0$ depending only on ε and κ (see [18], Lemma VI.5.5) and the assumption $\lambda_k(p) \le \delta$ imply

$$0 \le \int_{B(p,10\varepsilon)} |f_{ij} - \delta_{ij}|\, dV \le c_K \int_{B(p,10\varepsilon)} |df_{ij}|\, dV \le c_K V(p, 10\varepsilon)\sqrt{\delta}. \quad (17)$$

Moreover,

$$\begin{aligned} \inf_{x \in B(p,\frac{\varepsilon}{2})} \{|f_{ij}(x) - \delta_{ij}|\} V\left(p, \frac{\varepsilon}{2}\right) &\le \int_{B(p,\frac{\varepsilon}{2})} |f_{ij}(x) - \delta_{ij}|\, dV(x) \\ &\le c_K V(p, 10\varepsilon)\sqrt{\delta}. \quad (18) \end{aligned}$$

The last inequality follows from (17). Hence (18) implies that there exists $p' \in M$, $d(p, p') \le \frac{\varepsilon}{2}$, such that

$$|\langle \sigma_i^p(p'), \sigma_j^p(p')\rangle - \delta_{ij}| \le 2c_K \frac{V(p, 10\varepsilon)}{V(p, \frac{\varepsilon}{2})}\sqrt{\delta} \le c\sqrt{\delta}.$$

We conclude then as follows

$$\begin{aligned}
&\left|\langle\sigma_i^p(x),\sigma_j^p(x)\rangle-\delta_{ij}\right|\le\\
&\quad\left|\langle\sigma_i^p(x),\sigma_j^p(x)\rangle-\langle\tau_{x,p'}\sigma_i^p(p'),\tau_{x,p'}\sigma_j^p(p')\rangle\right|+\left|\langle\sigma_i^p(p'),\sigma_j^p(p')\rangle-\delta_{ij}\right|\\
&\qquad\le\left|\langle\sigma_i^p(x),\sigma_j^p(x)\rangle-\langle\tau_{x,p'}\sigma_i^p(p'),\tau_{x,p'}\sigma_j^p(p')\rangle\right|+c\sqrt{\delta}. \quad (19)
\end{aligned}$$

For any $x\in B(p,8\varepsilon)$ the minimizing geodesic $\overline{xp'}$ stays in $B(p,9\varepsilon)$, so we can write

$$\begin{aligned}
\left|\langle\sigma_i^p(x),\sigma_j^p(x)\rangle-\langle\sigma_i^p(p'),\sigma_j^p(p')\rangle\right| &\le 9\varepsilon\|d\langle\sigma_i^p,\sigma_j^p\rangle\|_{\infty,9\varepsilon}\\
&\le 9\varepsilon\left(\|\nabla\sigma_i^p\|_{\infty,9\varepsilon}\|\sigma_j^p\|_{\infty,9\varepsilon}+\|\sigma_i^p\|_{\infty,9\varepsilon}\|\nabla\sigma_j^p\|_{\infty,9\varepsilon}\right)\\
&\le 9\varepsilon c'\left(\|\nabla\sigma_i^p\|_{2,10\varepsilon}^s\|\sigma_j^p\|_{2,10\varepsilon}+\|\sigma_i^p\|_{2,10\varepsilon}\|\nabla\sigma_j^p\|_{2,10\varepsilon}^s\right)
\end{aligned}$$

where we used Lemma II.3.3 in the last inequality. By definition of the σ_i^p's and by assumption on $\lambda_i(p)$ we get

$$|\langle\sigma_i^p(x),\sigma_j^p(x)\rangle-\langle\sigma_i^p(p'),\sigma_j^p(p')\rangle|\le c''\sqrt{\delta^s}. \quad (20)$$

Finally, (19) and (20) imply that for a sufficiently small δ we have

$$|\langle\sigma_i^p(x),\sigma_j^p(x)\rangle-\delta_{ij}|\le\left(c\sqrt{\delta}+c''\sqrt{\delta^s}\right)\le\alpha<\frac{1}{n+1}$$

and this ends the proof. □

Definition II.3.7 *Fix once and for all* $0<\alpha<\frac{1}{n+1}$*. Let* δ *be given by Lemma II.3.6. For* $p\in X$*, define then* $\boldsymbol{\mu(p)}$ *as the largest integer such that* $\lambda_{\mu(p)}(p)\le\delta$*.*

Remark II.3.8 *If the vector bundle is flat,* $\mu(p)=n$*, for any* $p\in X$*.*

For $p\in X$, we want to extend a section in a neighborhood of p as parallel as possible and taking care of local small eigenvalues as said before. So let us define the **local extension** that associates to a vector in E_p a local section over $B(p,10\varepsilon)$. Consider $E_{\mu(p)}$ the $\mu(p)$-dimensional vector subspace of E_p spanned by $\{\sigma_1^p(p),\ldots,\sigma_{\mu(p)}^p(p)\}$. Let $E_{\mu(p)}^\perp$ be the orthogonal complement of $E_{\mu(p)}$ in E_p and choose $\{e_{\mu(p)+1}^p,\ldots,e_n^p\}$ an orthonormal basis of $E_{\mu(p)}^\perp$. By construction, $\{e_1^p=\sigma_1^p(p),\ldots,e_{\mu(p)}^p=\sigma_{\mu(p)}^p(p),e_{\mu(p)+1}^p,\ldots,e_n^p\}$ is an almost orthonormal basis of E_p. We extend this basis on $B(p,10\varepsilon)$ by

$$e_i^p(x):=\begin{cases}\sigma_i^p(x) & \text{if } i\le\mu(p),\\ \tau_{x,p}e_i^p & \text{otherwise}\end{cases}$$

and we define the local extension of $v=\sum_{i=1}^n v_ie_i^p$ by $\sum_{i=1}^n v_ie_i^p(x)$.

Remark II.3.9 *If E is flat, the local extension corresponds to the extension by parallel transport along radial geodesics. In this case, it suffices to choose any orthonormal basis $\{e_1^p, \ldots, e_n^p\}$ of E_p and extend it radially to obtain $\{e_1^p(x), \ldots, e_n^p(x)\}$.*

Lemma II.3.10 *For any $x \in B(p, 8\varepsilon)$, $\{e_1^p(x), \ldots, e_n^p(x)\}$ is an almost orthonormal basis of E_x.*

Proof: if $\mu(p) = 0$ the claim is clearly true. If $\mu(p) = n$ the claim follows from Lemma II.3.6. Hence suppose $1 \leq \mu(p) \leq n-1$. By Lemma II.3.6 $\langle e_1^p(x), \ldots, e_{\mu(p)}^p(x)\rangle$ is $\mu(p)$-dimensional and as parallel translation preserves the inner product $\langle e_{\mu(p)+1}^p(x), \ldots, e_n^p(x)\rangle$ is $(n-\mu(p))$-dimensional. So we have to show that there exists $c > 0$ such that

$$|\langle e_i^p(x), e_j^p(x)\rangle| \leq c < \frac{1}{n+1}\,, \qquad \forall 1 \leq i \leq \mu(p) < j \leq n.$$

Let us prove this estimate. As $e_j^p(p)$ and $\sigma_i^p(p)$ are orthogonal, we have

$$\begin{aligned} \left|\langle e_j^p(x), e_i^p(x)\rangle\right| &= \langle e_j^p(p), \tau_{p,x}\sigma_i^p(x) - \sigma_i^p(p)\rangle \\ &\leq \left|e_j^p(p)\right| \cdot |\sigma_i^p(x) - \tau_{x,p}\sigma_i^p(p)| = |\sigma_i^p(x) - \tau_{x,p}\sigma_i^p(p)|\,. \end{aligned}$$

By Lemma II.3.3 $|\sigma_i^p(x) - \tau_{x,p}\sigma_i^p(p)| \leq c'\sqrt{\delta s}$. Hence $\left|\langle e_j^p(x), e_i^p(x)\rangle\right| \leq c'\sqrt{\delta s}$. Then, readjust δ if necessary to obtain $\left|\langle e_j^p(x), e_i^p(x)\rangle\right| \leq c < \frac{1}{n+1}$. □

Remark II.3.11 *For the sequel, let δ' denote a constant, $0 < \delta' < 1$, such that $(1-\delta')\sum_{i=1}^n v_i^2 \leq |\sum_{i=1}^n v_i e_i^p(x)|^2 \leq (1+\delta')\sum_{i=1}^n v_i^2$, for any $x \in B(p, 8\varepsilon)$.*

Lemma II.3.12 *There exists a positive constant c depending only on n, k_1, ε such that for any $p \in X$ and any $\mu(p) < i \leq n$, $\|\nabla e_i^p\|_{\infty, 9\varepsilon} \leq c$.*

Proof: let $x \in B(p, 9\varepsilon)$ and consider γ the minimizing geodesic from p to x of length l ($l < 9\varepsilon$) and $\{X_1 = \dot\gamma(t), X_2, \ldots, X_n\}$ an orthonormal basis of E_x with $\nabla_{X_i} X_j = 0$. Then

$$|\nabla e_i^p(x)|^2 = \sum_{j=1}^n \left|\nabla_{X_j} e_i^p(x)\right|^2 \leq \sum_{j=1}^n \left(\int_0^l \left|\nabla_{\dot\gamma(t)}\nabla_{X_j} e_i^p(x)\right| dt\right)^2$$

but $|R^E(\dot\gamma(t), X_j)e_i^p| = |\nabla_{\dot\gamma(t)}\nabla_{X_j} e_i^p| \leq k_1$. Therefore $|\nabla e_i^p(x)|^2 \leq k_1^2 l^2 n$ and this concludes the proof. □

II.3.2 Construction of the twisted Laplacian

The construction of Δ_A differs according to the assumptions done on E. However, the basic idea is the same in all cases and relies on the fact that A has to express the holonomy. So let us consider $p,q \in X$, $p \in N(q)$ and let $x \in B(p, 8\varepsilon) \cap B(q, 8\varepsilon)$. Then define $a(p,q)_{ij}(x)$ by

$$e_j^p(x) = \sum_{i=1}^{n} a(p,q)_{ij}(x) e_i^q(x) \qquad \forall j = 1, \ldots, n$$

where e_i^p, e_j^q are defined in Section II.3.1. We define $A(p,q) : \mathbb{R}^n \to \mathbb{R}^n$ on the canonical basis $\{e_1, \ldots e_n\}$ of $\mathbb{R}^n$ by $A(p,q)e_j = \sum_{i=1}^n A(p,q)_{ij} e_i$, where $A(p,q)_{ij}$ is defined as follows.

If E is of harmonic curvature then define $A(p,q)_{ij}$ by

$$A(p,q)_{ij} = a(p,q)_{ij}(q)$$

If E is of complex (or quaternionic) rank one then define $A(p,q)_{ij}$ by

$$A(p,q)_{ij} = \frac{1}{V_{pq}} \int_{B_{pq}} a(p,q)_{ij}(x) dV(x)$$

where B_{pq} is the ball centered at the mid-point of p and q of radius 5ε and V_{pq} denotes its volume. Note that $B_{pq} \supseteq B(p, 3\varepsilon) \cup B(q, 3\varepsilon)$.

Remark II.3.13 *In the canonical basis of* $\mathbb{R}^n$*, we can write*

$$D_A f(p,q) = \sum_{i=1}^{n} D_A f(p,q)_i e_i = \sum_{i=1}^{n} \left(f_i(q) - \sum_{j=1}^{n} A(p,q)_{ij} f_j(p) \right) e_i$$

Remark II.3.14 *If* E *is of harmonic curvature, we have by definition* $e_j^p(q) = \sum_{i=1}^n A(p,q)_{ij} e_i^q(q)$, $\forall j = 1, \ldots, n$.

Remark II.3.15 *If* E *is flat,* $a(p,q)_{ij}(x)$ *is constant and so for* $j = 1, \ldots, n$ *and for any* $x \in B(p, 8\varepsilon) \cap B(q, 8\varepsilon)$, $e_j^p(x) = \sum_{i=1}^n A(p,q)_{ij} e_i^q(x)$. *Moreover, in this case* $A(p,q)A(p,q)^t = Id$ *and* $A(p,q)^t = A(q,p)$. *So that* Δ_A *is a discrete magnetic Laplacian.*

If E is of harmonic curvature let $V : \mathcal{F}(X) \to \mathcal{F}(X)$ be defined by

$$(Vf)(p) = \sum_{i \le \mu(p)} \lambda_i(p) f_i(p) e_i + \sum_{i > \mu(p)} f_i(p) e_i.$$

If E is of complex (or quaternionic) rank one let $V : \mathcal{F}(X) \to \mathcal{F}(X)$ be defined by

$$(Vf)(p) = \left(\lambda_1(p) + \sum_{q \in N(p)} \lambda_1(q) \right) f(p). \tag{21}$$

Remark II.3.16 *If the vector bundle is flat, then we have $V = 0$.*

II.3.3 Smoothing operator

Definition II.3.17 *Let $\{\psi_p\}_{p \in X}$ be a partition of unity subordinate to the cover $\{B(p, 2\varepsilon)\}_{p \in X}$. Define the* ***smoothing operator*** *$\mathcal{S} : \mathcal{F}(X) \to \Gamma(E)$ by*

$$(\mathcal{S}f)(x) = \sum_{p \in X} \psi_p(x) \left(\sum_{i=1}^{n} f_i(p) e_i^p(x) \right)$$

where $f(p) = \sum_{i=1}^n f_i(p) e_i$.

Proposition II.3.18 *There exist constants c_0, c_1, c_2 and $\Lambda > 0$ depending only on m, n, k_1, k_2, κ and ε such that*

i) $\forall f \in \mathcal{F}(X)$, $\|\mathcal{S}f\|^2 \le c_0 \|f\|^2$,

ii) $\forall f \in \mathcal{F}(X)$, $\|\nabla(\mathcal{S}f)\|^2 \le c_1 \left(\|D_A f\|^2 + (Vf, f)\right)$,

iii) $\forall f \in \mathcal{F}(X)$ *with* $\|D_A f\|^2 + (Vf, f) \le \Lambda \|f\|^2$, $\|\mathcal{S}f\|^2 \ge c_2 \|f\|^2$ *holds.*

Proof: for the first inequality note that $\{B(p, \varepsilon)\}_{p \in X}$ covers M. Hence

$$\begin{aligned} \|\mathcal{S}f\|^2 &\le \sum_{q \in X} \int_{B(q,\varepsilon)} \left| \sum_{p \in B(q,3\varepsilon) \cap X} \psi_p(x) \sum_{i=1}^{n} f_i(p) e_i^p(x) \right|^2 dV(x) \\ &\le (1+\delta') \sum_{q \in X} V(q, \varepsilon) \sum_{p \in B(q,3\varepsilon) \cap X} |f(p)|^2 \le (1+\delta') c \|f\|^2. \end{aligned}$$

In order to prove *ii)* fix $q \in X$ and let $x \in B(q, \varepsilon)$. Then as $\{\psi_p\}_{p \in X}$ is a partition of unity, we have $\sum_{p \in X} d\psi_p = 0$, so that we can write

$$\begin{aligned} \nabla(\mathcal{S}f)(x) &= \sum_{p \in B(q,3\varepsilon) \cap X} \psi_p(x) \Big(\sum_{i=1}^{n} f_i(p) \nabla e_i^p(x) \Big) + \\ &\quad \sum_{p \in N(q)} d\psi_p(x) \Big(\sum_{i=1}^{n} f_i(p) e_i^p(x) - \sum_{i=1}^{n} f_i(q) e_i^q(x) \Big). \end{aligned} \tag{22}$$

Then, Lemma II.3.12 implies

$$\int_{B(q,\varepsilon)} \left| \sum_{p\in B(q,3\varepsilon)\cap X} \psi_p(x)\Big(\sum_{i=1}^n f_i(p)\nabla e_i^p(x)\Big)\right|^2 dV(x) \leq$$

$$n \sum_{p\in B(q,3\varepsilon)\cap X} \left(\sum_{i\leq\mu(p)} f_i(p)^2 \int_{B(q,\varepsilon)} |\nabla e_i^p(x)|^2 dV(x) + c \sum_{i>\mu(p)} f_i(p)^2 \right)$$

$$\leq c' \sum_{p\in B(q,3\varepsilon)\cap X} (Vf)(p)\cdot f(p). \quad (23)$$

To estimate the second term of (22), we need the following lemma.

Lemma II.3.19 *There exists a positive constant c depending only on m, n, k_1, k_2, κ and ε such that*

$$\int_{B(q,\varepsilon)} \left|\sum_{i=1}^n f_i(p)e_i^p(x) - \sum_{i=1}^n f_i(q)e_i^q(x)\right|^2 \leq$$

$$c\left(|D_A f(q,p)|^2 + (Vf)(p)\cdot f(p) + (Vf)(q)\cdot f(q)\right).$$

Proof: see Appendix II.5.1. □

Hence by (23), (22) and Lemma II.3.19 we get

$$\int_{B(q,\varepsilon)} |\nabla(\mathcal{S}f)(x)|^2 dV(x) \leq$$

$$c'' \sum_{p\in B(q,3\varepsilon)\cap X} \Big(|D_A f(q,p)|^2 + (Vf)(p)\cdot f(p) + (Vf)(q)\cdot f(q)\Big).$$

Then summing on $q\in X$ implies the claim.

To prove the third part of Proposition II.3.18, define $(\mathcal{S}_q f)(x) = \sum_{i=1}^n f_i(q)e_i^q(x)$ for x in $B(q,\frac{\varepsilon}{2})$. Then, by Lemma II.3.19 we get

$$\int_{B(q,\frac{\varepsilon}{2})} |(\mathcal{S}f)(x) - (\mathcal{S}_q f)(x)|^2\, dV(x) =$$

$$\int_{B(q,\frac{\varepsilon}{2})} \left| \sum_{p\in N(q)} \psi_p(x) \sum_{j=1}^n \left(f_j(p)e_j^p(x) - f_j(q)e_j^q(x)\right)\right|^2 dV(x) \leq$$

$$c \sum_{p\in N(q)} \left(|D_A f(q,p)|^2 + (Vf)(p)\cdot f(p) + (Vf)(q)\cdot f(q)\right). \quad (24)$$

As the balls of radius $\frac{\varepsilon}{2}$ centered on X are disjoint, we can write

$$\begin{aligned}\|\mathcal{S}f\|^2 &\geq \sum_{q\in X}\int_{B(q,\frac{\varepsilon}{2})} |(\mathcal{S}_q f(x) - \mathcal{S}f(x)) - \mathcal{S}_q f(x)|^2\, dV(x)\\ &\geq \sum_{q\in X}\int_{B(q,\frac{\varepsilon}{2})} |\mathcal{S}_q f(x)|^2\, dV(x)\\ &\quad -2\sum_{q\in X}\int_{B(q,\frac{\varepsilon}{2})} |\mathcal{S}_q f(x)|\,|\mathcal{S}f(x) - \mathcal{S}_q f(x)|\, dV(x).\end{aligned}$$

By construction, $(1-\delta')|f(q)|^2 \leq |\mathcal{S}_q f(x)|^2 \leq (1+\delta')|f(q)|^2$ and by Cauchy-Schwarz inequality combined with (24), we get

$$\begin{aligned}&\sum_{q\in X}\int_{B(q,\frac{\varepsilon}{2})} |\mathcal{S}_q f(x)|\,|\mathcal{S}f(x) - \mathcal{S}_q f(x)|\, dV(x)\\ &\qquad\qquad \leq c'(1+\delta')\|f\|\sqrt{\|D_A f\|^2 + (Vf,f)}.\end{aligned}$$

Hence, $\|\mathcal{S}f\|^2 \geq (1-\delta')c''\|f\|^2 - 2c'(1+\delta')\|f\|\sqrt{\|D_A f\|^2 + (Vf,f)}$. Choose $\Lambda > 0$ sufficiently small so that if f satisfies $\|D_A f\|^2 + (Vf,f) \leq \Lambda\|f\|^2$, then

$$\|\mathcal{S}f\|^2 \geq \|f\|^2\left((1-\delta')c'' - 2c'(1+\delta')\sqrt{\Lambda}\right) \geq \frac{(1-\delta')c''}{2}\|f\|^2.$$

This concludes the proof of Proposition II.3.18. □

II.3.4 Discretizing operator

Definition II.3.20 *Define the* ***discretizing operator*** $\mathcal{D} : \Gamma(E) \to \mathcal{F}(X)$ *by*

$$(\mathcal{D}s)(p) = \sum_{i=1}^{n} \frac{1}{V(p,3\varepsilon)}\int_{B(p,3\varepsilon)} s_i^p(x)dV(x)e_i$$

where $s(x) = \sum_{i=1}^{n} s_i^p(x)e_i^p(x)$ *for* x *in* $B(p,3\varepsilon)$.

Proposition II.3.21 *There exist constants* c_0', c_1', c_2' *and* $\Lambda' > 0$ *depending only on* m, n, κ, k_1, k_2 *and* ε *such that*

i) $\forall s \in \Gamma(E)$, $\|\mathcal{D}s\|^2 \leq c_0'\|s\|^2$,

ii) $\forall s \in \Gamma(E)$, $\|D_A(\mathcal{D}s)\|^2 + (V(\mathcal{D}s), \mathcal{D}s) \leq c_1'\|\nabla s\|^2$, *iii)* $\forall s \in \Gamma(E)$ *such that* $\|\nabla s\|^2 \leq \Lambda' \|s\|^2$, $\|\mathcal{D}s\|^2 \geq c_2' \|s\|^2$ *holds.*

Proof: the first point follows directly from the following inequality

$$|\mathcal{D}s(p)|^2 \leq c \int_{B(p,3\varepsilon)} \sum_{i=1}^n |s_i^p(x)|^2 dV(x) \leq c(1-\delta')^{-1} \int_{B(p,3\varepsilon)} |s(x)|^2 dV(x).$$

To prove the second point, we first prove that

$$\|D_A(\mathcal{D}s)\|^2 + (V(\mathcal{D}s), \mathcal{D}s) \;\; \leq \;\; c \left(\|\nabla s\|^2 + \sum_{p \in X} \left(\widetilde{V} s\right)(p) \right) \tag{25}$$

where if **E is of harmonic curvature** then

$$\left(\widetilde{V} s\right)(p) = \left(\sum_{i \leq \mu(p)} \lambda_i(p) \int_{B(p,3\varepsilon)} |s_i^p|^2 dV + \sum_{i > \mu(p)} \int_{B(p,3\varepsilon)} |s_i^p|^2 dV \right)$$

and if **E is of complex (or quaternionic) rank one**

$$\left(\widetilde{V} s\right)(p) = \left(\lambda_1(p) + \sum_{q \in N(p)} \lambda_1(q) \right) \int_{B(p,3\varepsilon)} |s|^2 dV$$

and s is written locally as $s(x) = \sum_{i=1}^n s_i^p(x) e_i^p(x)$ for $x \in B(p, 8\varepsilon)$. First, $|(\mathcal{D}s)(p)_j|^2 \leq c \int_{B(p,3\varepsilon)} |s_j^p(x)|^2 dV(x)$ implies obviously

$$(V(\mathcal{D}s), \mathcal{D}s) \;\; \leq \;\; \sum_{p \in X} c' \left(\widetilde{V} s\right)(p). \tag{26}$$

Secondly, for p and $q \in N(p)$ let us introduce $B'_{pq} \subseteq B(p, 3\varepsilon) \cap B(q, 3\varepsilon)$ the ball centered at the mid-point of p and q of radius ε and V'_{pq} its volume. Then

$$|D_A(\mathcal{D}s)(q,p)|^2 = \sum_{i=1}^n \left(\frac{1}{V'_{pq}} \int_{B'_{pq}} \left| \mathcal{D}s(p)_i - \sum_{j=1}^n A(q,p)_{ij} \mathcal{D}s(q)_j \right| dV(y) \right)^2$$

$$\leq \;\; 3 \sum_{i=1}^n \left(\frac{1}{V'_{pq}} \int_{B'_{pq}} |\mathcal{D}s(p)_i - s_i^p(y)| \, dV(y) \right)^2 \tag{27}$$

$$+ \;\; 3 \sum_{i=1}^n \left(\frac{1}{V'_{pq}} \int_{B'_{pq}} \left| \sum_{j=1}^n A(q,p)_{ij} \left(s_j^q(y) - \mathcal{D}s(q)_j \right) \right| dV(y) \right)^2 \tag{28}$$

$$+ \;\; 3 \sum_{i=1}^n \left(\frac{1}{V'_{pq}} \int_{B'_{pq}} \left| s_i^p(y) - \sum_{j=1}^n A(q,p)_{ij} s_j^q(y) \right| dV(y) \right)^2 . \tag{29}$$

We estimate each of the three terms separately.

By a result of Kanai (see [18], Lemma VI.5.5), there exists $c_K > 0$ depending only on ε and κ such that

$$\frac{1}{V'_{pq}}\int_{B'_{pq}} |\mathcal{D}s(p)_i - s_i^p(y)|\, dV(y) \;\; \leq \;\; c_K \int_{B(p,3\varepsilon)} |ds_i^p(y)|\, dV(y).$$

Moreover

$$\sqrt{1-\delta'}\, |ds_i^p(y)| \leq \left| \sum_{j=1}^{n} ds_j^p(y) e_j^p(y) \right| = \left| \nabla s(y) - \sum_{j=1}^{n} s_j^p(y) \nabla e_j^p(y) \right|. \qquad (30)$$

Therefore

$$\sqrt{1-\delta'} \int_{B(p,3\varepsilon)} |ds_i^p(y)|\, dV(y) \leq$$
$$\left(V(p,3\varepsilon) \int_{B(p,3\varepsilon)} |\nabla s(y)|^2 dV(y) \right)^{\frac{1}{2}} + n \sum_{j=1}^{n} \|\nabla e_j^p\|_{2,3\varepsilon} \|s_j^p\|_{2,3\varepsilon}$$

so that we obtain by Lemma II.3.12 and by construction of e_j^p

$$\sum_{i=1}^{n} \left(\int_{B(p,3\varepsilon)} |ds_i^p(y)|\, dV(y) \right)^2 \leq c \int_{B(p,3\varepsilon)} |\nabla s(y)|^2 dV(y) + c\widetilde{V} s(p).$$

We have then the following upper bound for (27)

$$\sum_{i=1}^{n} \left(\frac{1}{V'_{pq}} \int_{B'_{pq}} |\mathcal{D}s(p)_i - s_i^p(y)|\, dV(y) \right)^2$$
$$\leq c_K^2 c \left(\int_{B(p,3\varepsilon)} |\nabla s(y)|^2 dV(y) + \left(\widetilde{V} s\right)(p) \right). \quad (31)$$

By the same kind of arguments as for (27) and using that $\sum_{i,j=1}^{n} |A(q,p)_{ij}|^2$ is bounded above by a uniform constant, we can bound (28) as follows

$$\sum_{i=1}^{n} \left(\frac{1}{V'_{pq}} \int_{B'_{pq}} \left| \sum_{j=1}^{n} A(q,p)_{ij} \left(s_j^q(y) - \mathcal{D}s(q)_j \right) \right| dV(y) \right)^2 \leq$$
$$c' \left(\int_{B(q,3\varepsilon)} |\nabla s(y)|^2 dV(y) + \left(\widetilde{V} s\right)(q) \right). \quad (32)$$

The last term (29) is then bounded by the following lemma

Lemma II.3.22 *There exists a positive constant c depending only on m, n, k_1, k_2, κ and ε such that*

$$\sum_{i=1}^{n}\left(\int_{B'_{pq}}\left|s_i^p(y)-\sum_{j=1}^{n}A(q,p)_{ij}s_j^q(y)\right|dV(y)\right)^2 \leq c\left(\left(\widetilde{V}f\right)(p)+\left(\widetilde{V}f\right)(q)\right).$$

Proof: see Appendix II.5.2. □

Finally, (31), (32) and Lemma II.3.22 imply that

$$\begin{aligned}|D_A(\mathcal{D}s)(p,q)|^2 &\leq c''\left(\int_{B(p,3\varepsilon)}|\nabla s(y)|^2dV(y)+\int_{B(q,3\varepsilon)}|\nabla s(y)|^2dV(y)\right)\\ &\quad + c''\left(\left(\widetilde{V}s\right)(p)+\left(\widetilde{V}s\right)(q)\right).\end{aligned}$$

Taking the sum over p and q leads to

$$\|D_A(\mathcal{D}s)\|^2 \leq c'''\left(\|\nabla s\|^2+\sum_{p\in X}\left(\widetilde{V}s\right)(p)\right) \tag{33}$$

so that (33),(26) imply (25). In order to conclude the proof of point *ii*) of this lemma, we have to show that there exists $c>0$ such that

$$\sum_{p\in X}\left(\widetilde{V}s\right)(p)\leq c\|\nabla s\|^2. \tag{34}$$

Fix $q\in X$, let $B=B(q,10\varepsilon)$, $V(B)$ its volume. Let $(\cdot,\cdot)_B$ and $\|\cdot\|_B$ the L^2-inner product respectively the L^2-norm on E restricted to B. We are going to show that there exists $c>0$ such that

$$\left(\widetilde{V}s\right)(q)\leq c\sum_{p\in B(q,3\varepsilon)\cap X}\|\nabla s\|^2_{B(p,10\varepsilon)}. \tag{35}$$

Then (34) is a direct consequence of (35). To prove (35) we have to consider separately the cases E is of complex (or quaternionic) rank one and E is of harmonic curvature.

Assume E is of rank one. The proof of (35) in this case is much easier than in the other case as the potential involves only the first eigenvalue of the ball. Recall that $\lambda_1(q) \leq \frac{\|\nabla s\|_B^2}{\|s\|_B^2}$ for any non-zero s. Therefore and as $B(q,3\varepsilon) \subseteq B(p,10\varepsilon)$ for any $p \in N(q)$

$$\left(\widetilde{V}s\right)(q) \leq \|s\|^2_{B(q,3\varepsilon)} \sum_{p\in B(q,3\varepsilon)\cap X} \frac{\|\nabla s\|^2_{B(p,10\varepsilon)}}{\|s\|^2_{B(p,10\varepsilon)}} \leq \sum_{p\in B(q,3\varepsilon)\cap X} \|\nabla s\|^2_{B(p,10\varepsilon)}$$

and this concludes the first case.

Assume E is of harmonic curvature. If $y \in B$, write $s(y)$ as a sum of orthogonal sections (with respect to $(\cdot,\cdot)_B$) $s(y) = \widetilde{s}(y) + s^\perp(y)$ with $\widetilde{s}(y) = \sum_{j\leq\mu(q)} \frac{(s,e_j^q)_B}{V(B)} e_j^q(y)$. We have the following properties of the decomposition.

$$\begin{aligned} (s^\perp, e_j^q)_B &= 0\,,\ \forall j \leq \mu(q), & (\nabla s^\perp, \nabla\widetilde{s})_B &= 0, \\ \|s\|_B^2 &= \|s^\perp\|_B^2 + \|\widetilde{s}\|_B^2, & \|\nabla s\|_B^2 &= \|\nabla s^\perp\|_B^2 + \|\nabla\widetilde{s}\|_B^2, \\ \|\widetilde{s}\|_B^2 &= \sum_{j\leq\mu(q)} \frac{(s,e_j^q)_B^2}{V(B)}, & \|\nabla\widetilde{s}\|_B^2 &= \sum_{j\leq\mu(q)} \frac{(s,e_j^q)_B^2}{V(B)}\lambda_j(q). \end{aligned}$$

Consider then two cases. First assume $\|s^\perp\|_B^2 = 0$. Then $s(y) = \widetilde{s}(y)$ which means that if $y \in B(p,10\varepsilon)$

$$s_j^q(y) = \begin{cases} 0 & \text{if } j > \mu(q), \\ \frac{(s,e_j^q)_B}{V(B)} & \text{if } j \leq \mu(q). \end{cases}$$

Therefore

$$\begin{aligned} \left(\widetilde{V}s\right)(q) &= \left(\sum_{j\leq\mu(q)} \lambda_j(q) \int_{B(q,3\varepsilon)} |s_j^q|^2 dV + \sum_{j>\mu(q)} \int_{B(q,3\varepsilon)} |s_j^q|^2 dV \right) \\ &= V(q,3\varepsilon) \sum_{j\leq\mu(q)} \frac{(s,e_j^q)_B^2}{V(B)^2}\lambda_j(q) \leq c\|\nabla\widetilde{s}\|_B^2. \end{aligned}$$

Moreover as $s^\perp$ is zero $\|\nabla\widetilde{s}\|_B^2 = \|\nabla s\|_B^2$ and so in this case (35) is verified.

For the second case, assume $\|s^\perp\|_B^2 \neq 0$. Then apply max-min theorem to $s^\perp$ to obtain $\lambda_{\mu(q)+1}(q) \leq \frac{\|\nabla s^\perp\|_B^2}{\|s^\perp\|_B^2}$ and by definition of $\mu(q)$ this implies that

$$\delta\|s^\perp\|_B^2 \leq \|\nabla s^\perp\|_B^2. \tag{36}$$

Moreover, let us rewrite $s^\perp$ as follows, for $y \in B(q,8\varepsilon)$

$$s^\perp(y) = \sum_{j\leq\mu(q)} \left(s_j^q(y) - \frac{(s,e_j^q)_B}{V(B)} \right) e_j^q(y) + \sum_{j>\mu(q)} s_j^q(y) e_j^q(y).$$

As $\{e_j^q(y)\}$ is an almost orthonormal basis, we obtain for $y \in B(q, 8\varepsilon)$

$$\sum_{j\le\mu(q)} \left| s_j^q(y) - \frac{(s,e_j^q)_B}{V(B)} \right|^2 + \sum_{j>\mu(q)} |s_j^q(y)|^2 \;\le\; (1-\delta')^{-1}|s^\perp(y)|^2.$$

In particular, this implies

$$\sum_{j>\mu(q)} \int_{B(q,3\varepsilon)} |s_j^q(y)|^2 dV(y) \;\le\; (1-\delta')^{-1}\|s^\perp\|_B^2 \tag{37}$$

and

$$\sum_{j\le\mu(q)} \lambda_j(q) \int_{B(q,3\varepsilon)} |s_j^q(y)|^2 dV(y) \le$$
$$2\sum_{j\le\mu(q)} \lambda_j(q) \int_{B(q,3\varepsilon)} \left| s_j^q(y) - \frac{(s,e_j^q)_B}{V(B)} \right|^2 dV(y) + 2\sum_{j\le\mu(q)} \frac{(s,e_j^q)_B^2}{V(B)} \lambda_j(q)$$
$$\le \frac{2\delta}{1-\delta}\|s^\perp\|_B^2 + 2\|\nabla\widetilde{s}\|_B^2. \tag{38}$$

Then (37) and (38) imply that $\left(\widetilde{V}s\right)(q) \le c\left(\|s^\perp\|_B^2 + \|\nabla\widetilde{s}\|_B^2\right)$. Use (36) together with this inequality to obtain (35) and therefore (34). Finally (25) together with (34) imply *ii*).

To prove *iii*) consider the following sum. By the work of Buser (Lemma 5.1 in [13]), there exists $c_B > 0$ depending only on m, κ and ε such that

$$\sum_{i=1}^n \int_{B(p,3\varepsilon)} |\mathcal{D}s(p)_i - s_i^p(x)|^2 \, dV(x) \;\le\; c_B \sum_{i=1}^n \int_{B(p,3\varepsilon)} |ds_i^p(x)|^2 \, dV(x).$$

Moreover, using (30) we obtain

$$\sum_{i=1}^n \int_{B(p,3\varepsilon)} |\mathcal{D}s(p)_i - s_i^p(x)|^2 \, dV(x) \le$$
$$\frac{2nc_B}{1-\delta'} \left(\int_{B(p,3\varepsilon)} |\nabla s(y)|^2 dV(y) + n\sum_{j=1}^n \|\nabla e_j^p\|_{\infty,3\varepsilon}^2 \|s_j^p(y)\|_{2,3\varepsilon}^2 \right). \tag{39}$$

Therefore, from (39) we obtain

$$\begin{aligned}
|\mathcal{D}s(p)|^2 &\geq c \int_{B(p,3\varepsilon)} \sum_{i=1}^{n} |(s_i^p(x) - \mathcal{D}s(p)_i) - s_i^p(x)|^2 \, dV(x) \\
&\geq c \int_{B(p,3\varepsilon)} \sum_{i=1}^{n} |s_i^p(x)|^2 dV(x) - 2c \int_{B(p,3\varepsilon)} \sum_{i=1}^{n} |s_i^p(x)||\mathcal{D}s(p)_i - s_i^p(x)| dV(x) \\
&\geq c' \|s\|^2_{B(p,3\varepsilon)} - c'' \|s\|_{B(p,3\varepsilon)} \left(\|\nabla s\|^2_{B(p,3\varepsilon)} + \sum_{j=1}^{n} \|\nabla e_j^p\|^2_{\infty,3\varepsilon} \|s_j^p\|^2_{2,3\varepsilon} \right)^{\frac{1}{2}} \qquad (40)
\end{aligned}$$

Assume E is of harmonic curvature and combine Lemma II.3.3 and Lemma II.3.12 with (40) to obtain

$$|\mathcal{D}s(p)|^2 \geq c' \|s\|^2_{B(p,3\varepsilon)} - c'' \|s\|_{B(p,3\varepsilon)} \left(\|\nabla s\|^2_{B(p,3\varepsilon)} + \left(\widetilde{V} s \right)(p) \right)^{\frac{1}{2}}.$$

Moreover, by (35) $\left(\widetilde{V} s\right)(p)$ is bounded above by $c \sum_{q \in B(p,3\varepsilon) \cap X} \|\nabla s\|^2_{B(q,10\varepsilon)}$. Then, taking the sum over $p \in X$ produces new c', $c'' > 0$ such that

$$\|\mathcal{D}s\|^2 \quad \geq \quad c' \|s\|^2 - c'' \|s\| \|\nabla s\|.$$

Finally, if $\|\nabla s\|^2 \leq \Lambda' \|s\|^2$, we get $\|\mathcal{D}s\|^2 \geq \|s\|^2 (c' - c'' \sqrt{\Lambda'})$. Choose then Λ' suitably to conclude the proof of the proposition in this case.

Assume E is of rank one. If $\lambda_1(p) \leq \delta$, by Lemma II.3.3, $\|\nabla e_j^p\|^2_{\infty,3\varepsilon} \leq c\lambda_1^s(p)$. If $\lambda_1(p) > \delta$, by Lemma II.3.12 $\|\nabla e_j^p\|^2_{\infty,3\varepsilon} \leq c \leq c\delta^{-1}\lambda_1(p)$. Therefore, (40) can be changed in (with new constants c, c', c'')

$$|\mathcal{D}s(p)|^2 \geq \begin{cases} (c' - c\lambda_1^{\frac{s}{2}}(p)) \|s\|^2_{B(p,3\varepsilon)} - c'' \|s\|_{B(p,3\varepsilon)} \|\nabla s\|_{B(p,3\varepsilon)} & \text{if } \lambda_1(p) \leq \delta, \\ c' \|s\|^2_{B(p,3\varepsilon)} - c'' \|s\|_{B(p,3\varepsilon)} \|\nabla s\|_{B(p,10\varepsilon)} & \text{otherwise.} \end{cases}$$

By choosing δ smaller, we can assume that if $\lambda_1(p) \leq \delta$, $c' - c\lambda_1(p)^{\frac{s}{2}} \geq c''' > 0$. This implies that (for any values of $\lambda_1(p)$)

$$|\mathcal{D}s(p)|^2 \quad \geq \quad c''' \|s\|^2_{B(p,3\varepsilon)} - c'' \|\nabla s\|_{B(p,3\varepsilon)} \|s\|_{B(p,10\varepsilon)}.$$

Then, take the sum over $p \in X$ to obtain for $\|\nabla s\| \leq \Lambda' \|s\|$

$$\|\mathcal{D}s\|^2 \quad \geq \quad c''' \|s\|^2 - c'' \|\nabla s\| \|s\| \quad \geq \quad \|s\|^2 (c''' - c'' \sqrt{\Lambda'})$$

and conclude choosing Λ' suitably. □

II.3.5 Upper bounds

Lemma II.3.23 *Let m, n, k_1, k_2, κ, r_0, ε be as before. Then there exist positive constants c_3 and c_3' depending only on m, n, k_1, k_2, κ, ε so that for any vector bundle $E \in \mathcal{E}(n,k_1,k_2)$ over any $M \in \mathcal{M}(m,\kappa,r_0)$, for any X ε-discretization of E and for $\Delta_A + V$ constructed in Section II.3.2, we have*

i) $\lambda_k(E) \le c_3$, $\forall k \le n|X|$,

ii) $\lambda_k(X,A,V) \le c_3'$, $\forall k \le n|X|$.

Proof: *i)* Let p_i be a vertex of X and consider $f_i : M \to \mathbb{R}$ the first eigenfunction of the Dirichlet problem for the ball centered at p_i of radius $\frac{\varepsilon}{2}$ extended by zero. By Cheng's comparison theorem $\frac{\|df_i\|^2}{\|f_i\|^2} \le \lambda_1\left(\frac{\varepsilon}{2},\kappa\right)$ (where $\lambda_1\left(\frac{\varepsilon}{2},\kappa\right)$ denotes the first non-zero eigenvalue of the Dirichlet problem on the ball of radius $\frac{\varepsilon}{2}$ in the simply connected space of constant sectional curvature $-\kappa$ and of same dimension as M). Define then the sections $\sigma_j^i(x) = f_i(x)e_j^{p_i}(x)$ for $1 \le i \le |X|$, and $1 \le j \le n$. Then $\{\sigma_j^i \mid 1 \le i \le |X|, 1 \le j \le n\}$ spans a vector subset W of $\Gamma(E)$ of dimension $n|X|$ as $\{e_j^{p_i}\}_{j=1,\dots,n}$ is an almost orthonormal frame. Moreover

$$\nabla\sigma_j^i(x) = df_i(x)e_j^{p_i}(x) + f_i(x)\nabla e_j^{p_i}(x)$$

hence by construction of $e_j^{p_i}$ and Lemma II.3.3 and Lemma II.3.12, we have

$$\|\nabla\sigma_j^i\|^2 \le c\left(\|df_i\|^2 + \|f_i\|^2\right)$$

so that by definition of the f_i's

$$\|\nabla\sigma_j^i\|^2 \le c\|f_i\|^2\left(1 + \lambda_1\left(\frac{\varepsilon}{2},\kappa\right)\right).$$

By min-max theorem we get then

$$\lambda_k(E) \le c' \max\left\{\frac{\sum_{i,j} a_{ij}^2\|\nabla\sigma_j^i\|^2}{\sum_{i,j} a_{ij}^2\|\sigma_j^i\|^2}\right\} \le c'c\left(1 + \lambda_1\left(\frac{\varepsilon}{2},\kappa\right)\right).$$

This concludes the first part of the lemma.

ii) Let $f \in \mathcal{F}(X)$. As $A(p,q)$ is a change of almost orthonormal bases we have

$$\begin{aligned}\|D_Af\|^2 + (Vf,f) &= \frac{1}{2}\sum_{p\in X}\sum_{q\in N(p)} |f(q) - A(p,q)f(p)|^2 + \sum_{p\in X}(Vf)(p)\cdot f(p) \\ &\le c\sum_{p\in X}\sum_{q\in N(p)}\left(|f(p)|^2 + |f(q)|^2\right) + \max\{\delta,1\}\|f\|^2 \\ &\le (2c\nu_X + \max\{\delta,1\})\,\|f\|^2.\end{aligned}$$

Therefore, $R(f) \le 2c\nu_X + \max\{\delta,1\}$, $\forall f \in \mathcal{F}(X)\setminus\{0\}$ and this implies $\lambda_k(X,A,V) \le 2c\nu_X + \max\{\delta,1\}$, $\forall k \le n|X|$. □

II.3.6 Conclusion

Proof of Theorem II.3.1: by symmetry of the results concerning the smoothing and the discretizing, it suffices to deduce $\lambda_k(E) \leq c\lambda_k(X, A, V)$. The proof proceeds in two steps.

First, assume that k is such that $\lambda_k(X, A, V) \geq \Lambda$, for Λ given by Proposition II.3.18 iii). Then, Lemma II.3.23 *i*) leads to $\lambda_k(E) \leq c_3\Lambda^{-1}\lambda_k(X, A, V)$.This is the required inequality.

Secondly, assume that k is such that $\lambda_k(X, A, V) \leq \Lambda$. Let W_k be the k-dimensional vector subspace of $\mathcal{F}(X)$ spanned by $f_i : X \to \mathbb{R}^n$, $i = 1, \ldots, k$, $\lambda_i(X, A, V)$-eigenfunction of Δ_A chosen so that $(f_i, f_j) = \delta_{ij}|X|$. By min-max theorem, $\lambda_k(X, A, V) = \max\{R(f) : f \in W_k \setminus \{0\}\}$. Let then $\mathcal{S}W_k$ be the vector subspace of $\Gamma(E)$ spanned by the $\mathcal{S}f_i$'s i.e. $\mathcal{S}W_k = \langle \mathcal{S}f_1, \ldots, \mathcal{S}f_k\rangle = \{\mathcal{S}f \mid f \in W_k \setminus \{0\}\}$. As $\lambda_k(X, A, V) \leq \Lambda$, for any non-zero function f in W_k, we have $\|D_Af\|^2 + (Vf, f) \leq \Lambda\|f\|^2$. Hence, by Proposition II.3.18 *iii*), for any f in W_k, $\|\mathcal{S}f\|^2 \geq c_2\|f\|^2$ holds. In particular, $\mathcal{S}f$ is the zero function if and only if f is zero which means that $\mathcal{S}W_k$ is k-dimensional. So we can apply min-max theorem to $\mathcal{S}W_k$ and obtain

$$\lambda_k(E) \leq \max\{R(\mathcal{S}f) \mid f \in W_k \setminus \{0\}\}.$$

Moreover, by Proposition II.3.18 *ii*) and *iii*) we obtain that $R(\mathcal{S}f) \leq \frac{c_1}{c_2}R(f)$ for any non-zero f in W_k, which leads to

$$\lambda_k(E) \leq \frac{c_1}{c_2}\max\{R(f) \mid f \in W_k \setminus \{0\}\} = \frac{c_1}{c_2}\lambda_k(X, A, V).$$

This concludes the proof. □

II.4 Estimation of the first non-zero eigenvalue for a flat vector bundle

Let (E^n, ∇) be a flat Riemannian vector bundle with irreducible holonomy over $M \in \mathcal{M}(m, \kappa, r_0)$. We recall the definition of the constant related to the holonomy given by Ballmann, Brüning and Carron in [3]. If c is a unit speed loop, denote by H_c its holonomy. Then there exists $\alpha > 0$ such that $\forall x \in M$, $\forall v \in E_x$ there exists a smooth unit speed loop $c_{x,v}$ of length less than two diameters of M such that

$$|H_{c_{x,v}}(v) - v| \geq \alpha|v|. \tag{41}$$

The following theorem shows that if E has significant holonomy, then the first eigenvalue of $\overline{\Delta}$ can not be too small. Conversely, if there exists v in E_x which has a small holonomy, then the first eigenvalue is not too large.

Theorem II.4.1 *Let (E^n, ∇) be a flat Riemannian vector bundle over $M \in \mathcal{M}(m, \kappa, r_0)$ with irreducible holonomy. Then there exist c, c′ > 0 depending only on m, n, κ, r_0 such that*

$$\lambda_1(E) \geq c' \frac{\alpha^2}{d(M)^2 c^{d(M)}}$$

where $d(M)$ denotes the diameter of M.

Moreover, if there exist $p_0 \in M$, $v_0 \in E_{p_0}$ and α' such that for any loop c at p_0 of length less than $7d(M)$, $|H_c(v_0) - v_0| \leq \alpha'|v_0|$ then, there exists $c'' > 0$ depending only on n, m, κ and r_0 such that

$$\lambda_1(E) \leq c''\alpha'^2.$$

The first part of the theorem is in fact due to Ballmann, Brüning and Carron (see [3]). We present here a more conceptual proof that relies on the fact that the discrete magnetic Laplacian associated to a discretization of a flat bundle is strongly related to the holonomy of the vector bundle.

Proof: let $\varepsilon = \frac{1}{100} r_0$ and let X be an ε-discretization of E. Then by Theorem II.3.1 there exist Δ_A a discrete magnetic Laplacian and $c > 0$ such that $\lambda_1(E) \geq c\lambda_1(X, A)$. So it suffices to prove the statement for $\lambda_1(X, A)$. Let $f \in \mathcal{F}(X)$ such that $\Delta_A f = \lambda f$. Let $p_0 \in X$ and $v_0 = \sum_{i=1}^n f_i(p_0) e_i^{p_0} \in E_{p_0}$. By (41), there exists a smooth unit speed loop $c_0 : [0, l] \to M$ at p_0 of length $l \leq 2d(M)$ and $|H_{c_0}(v_0) - v_0| \geq \alpha|v_0|$. Let $N \in \mathbb{N}$ such that $N\frac{\varepsilon}{2} \leq l < (N + 1)\frac{\varepsilon}{2}$ and consider a partition of $[0, l]$, $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = l$ such that $\frac{\varepsilon}{2} \leq t_j - t_{j-1} \leq \varepsilon$. By definition of X, $\forall j = 1, \ldots, N-1$, $\exists p_j \in X$ such that $d(p_j, c_0(t_j)) < \varepsilon$. Moreover, let $p_N = p_0 \in X$. Note that $d(p_{j-1}, p_j) < 3\varepsilon$. Consider then the piecewise geodesic loop $\overline{c}_0$ at p_0 passing through all p_j, $j = 1, \ldots N-1$ (i.e $\overline{c}_0$ joins p_{j-1} to p_j via the minimizing geodesic $p_{j-1}p_j$). Note that $\overline{c}_0$ is of length less than $3N\varepsilon \leq 12d(M)$. Moreover, as E is flat, the holonomy of c_0 is the same as the holonomy of $\overline{c}_0$. More precisely, parallel translation from $c_0(t_{j-1})$ to $c_0(t_j)$ along c_0 is the same as parallel translation along minimizing geodesics from $c_0(t_{j-1})$ to p_{j-1}, then from p_{j-1} to p_j and finally from p_j to $c_0(t_j)$. Hence $H_{c_0}(v) = H_{\overline{c}_0}(v)$ for any $v \in E_{p_0}$. So that we obtain

$$|H_{\overline{c}_0}(v_0) - v_0| \geq \alpha|v_0| = \alpha|f(p_0)|.$$

Consider then $v_j = \sum_{i=1}^{n} f_i(p_j)e_i^{p_j} \in E_{p_j}$. By triangle inequality and as parallel transport is an isometry, we obtain easily the following inequality

$$\alpha|f(p_0)| \leq \sum_{j=1}^{N} |\tau_{p_j,p_{j-1}} v_{j-1} - v_j|.$$

Moreover, by construction of D_A we have

$$\begin{aligned} |\tau_{p_j,p_{j-1}} v_{j-1} - v_j| &= \left| \sum_{i=1}^{n} f_i(p_{j-1})\tau_{p_j,p_{j-1}} e_i^{p_{j-1}} - \sum_{i=1}^{n} f_i(p_j)e_i^{p_j} \right| \\ &= \left| \sum_{i=1}^{n} \left(\sum_{k=1}^{n} A(p_{j-1},p_j)_{ik} f_k(p_{j-1}) - f_i(p_j) \right) e_i^{p_j} \right| \\ &= |D_A f(p_{j-1},p_j)|. \end{aligned}$$

This implies that $\alpha|f(p_0)| \leq |D_A f(p_0,p_1)| + \ldots + |D_A f(p_{N-1},p_N)|$. We have shown that for any $p_0 \in X$, there exists a piecewise geodesic loop $\overline{c}_0 = \{p_0, p_1, \ldots, p_N\}$ of length less than $12d(M)$ such that

$$\alpha^2 |f(p_0)|^2 \leq 4\frac{d(M)}{\varepsilon} \left(|D_A f(p_0,p_1)|^2 + \ldots + |D_A f(p_{N-1},p_N)|^2 \right)$$

and $d(p_{j-1},p_j) < 3\varepsilon$. The goal is to apply this last inequality to $\|f\|^2$. To that end, we need to find an upper bound for the number of loops of the kind $\{p, q, \ldots, p\}$ that can pass through $p \in X$ and $q \in N(p)$ and of length less than $12d(M)$. This upper bound on the length of the loop implies that such a loop can pass through at most $P \leq 12\frac{d(M)}{\varepsilon}$ points of X. Therefore, there are at most ν^{P-1} loops of the kind $\{p, q, \ldots, p\}$ and each of these loops is suitable for P points in X. Hence, we obtain

$$\begin{aligned} \alpha^2 \|f\|^2 &\leq P\nu^{P-1} 8\frac{d(M)}{\varepsilon} \|D_A f\|^2 \\ &\leq 72\frac{d(M)^2}{\varepsilon^2} \nu^{12\frac{d(M)}{\varepsilon}} \|D_A f\|^2. \end{aligned}$$

This leads then to the conclusion of the first part $\alpha^2 \frac{\varepsilon^2}{72 d(M)^2 \nu^{12\frac{d(M)}{\varepsilon}}} \leq \lambda$.

To prove the second part of the theorem let $\varepsilon = \frac{1}{100} r_0$ and X be an ε-discretization of E such that $p_0 \in X$. Recall that X is the set of vertices of a finite connected graph G. Then construct a spanning tree S of G (see [8], Section I.2) as follows. Let $X_i = \{p \in X \mid d_G(p, p_0) = i\}$ where d_G denotes the path metric on G. Note that if q is in X_i then there exists q' in

X_{i-1} which is joined by an edge to q. Let then S be the subgraph of G with vertices set X and edges set $E(S) = \{qq' \mid q \neq p_0\}$. We have constructed a spanning tree S of G.

By construction of S, for any p in X there exists a unique curve c_p in S joining p to p_0 (i.e. c_p is a piecewise geodesic curve $\{p, \ldots, p_0\}$ such that two consecutive points of X in c_p are joined in S). Moreover the length of such a c_p is bounded above by $3d(M)$. Now, choose in E_{p_0} an orthonormal basis $\{e_1^{p_0}, \ldots, e_n^{p_0}\}$ and define an orthonormal basis $\mathcal{B}_p$ of E_p by $\mathcal{B}_p = \{e_i^p = \tau_{c_p} e_i^{p_0}\}_{i=1,\ldots,n}$, where τ_{c_p} denotes parallel transport along c_p from p_0 to p. Then $e_i^p(x) = \tau_{x,p} e_i^p$ gives a local orthonormal frame made of parallel sections. Hence, consider the discrete magnetic Laplacian Δ_A associated to this choice of bases (constructed as in Section II.3.2) which satisfies $\lambda_1(E) \leq c\lambda_1(X, A)$ by Theorem II.3.1. So that it suffices to prove the result for the first eigenvalue of Δ_A. By min-max theorem $\lambda_1(X, A) \leq R(f)$ for any non-zero function on X. So consider $f : X \to \mathbb{R}^n$ defined by $f(p) = \sum_{i=1}^n v_i e_i$ where the v_i's are the coordinates of v_0 in the basis $\mathcal{B}_{p_0}$. If p and q are neighboring points in X such that $d(p, p_0) \leq d(q, p_0)$ and $p \in c_q$, then we have $\tau_{q,p} e_j^p = e_j^q$. Hence in this case $A(p, q)_{ij} = \delta_{ij}$ and so $D_A f(p, q) = 0$. In the other case i.e. if $p \in N(q)$, $d(p, p_0) \leq d(q, p_0)$ and p is not on c_q, consider the loop c at x_0 going from x_0 to p via c_p, from p to q via the minimizing geodesic pq and from q to x_0 via c_q^{-1}. Then c is of length less than $7d(M)$ and by assumption

$$|H_c(v_0) - v_0| \leq \alpha' |v_0|. \tag{42}$$

But, we have $H_c(v_0) = \tau_{c_q}^{-1} \tau_{q,p} \tau_{c_p} v_0$ and

$$\langle H_c(v_0), e_i^{p_0} \rangle = \left\langle \sum_{j=1}^n \tau_{q,p} e_j^p, e_i^q \right\rangle = \sum_{j=1}^n A(p, q)_{ij} v_j.$$

Combining this last equality with (42) we obtain $\alpha' |v_0| \geq |D_A f(p, q)|$. Finally, computing $\|D_A f\|^2$ leads to

$$\|D_A f\|^2 \leq \frac{1}{2} \alpha'^2 \nu \|f\|^2.$$

So that the second part of the theorem follows. □

II.5 Appendix: technical tools

The following lemma is a generalization of Lemma 11.1 in [45] and a local version of Lemma 0.1 of [56].

Lemma II.5.1 *Let $M \in \mathcal{M}(m, \kappa, r_0)$ and u a non-negative function on the ball $B(p, R)$, with $R < \frac{1}{2}r_0$, such that $\Delta u \leq \alpha u + \beta$. Let $0 < \theta < 1$. Then there exist c_1, c_2, $c_3 > 0$ (depending only on m, n, κ, R, α and β) and $0 < c(m) < s \leq 1$ such that*

$$\|u\|_{\infty,\theta R} \leq \left(\left(c_1 + c_2 \frac{1}{(1-\theta)^2}\right)^{c_3} \|u\|_{2,R}\right)^s$$

where $\|u\|_{\infty,\theta R} = \sup\{u(x) \mid x \in B(p, \theta R)\}$, and $\|u\|^q_{q,R} = \int_{B(p,R)} u^q(x) dV(x)$.

Note that, if $\beta = 0$ then $s = 1$ (see [45], Lemma 11.1).

Proof: the proof combines the proof given in [45] (Lemma 11.1) and Lemma 0.1 of [56] . Let $u : B(p, R) \to \mathbb{R}$, $u \geq 0$ such that $\Delta u \leq \alpha u + \beta$. Let $\nu = \frac{m}{2}$ if $m \geq 3$ and $\nu = 2$ otherwise. Let μ be such that $\frac{1}{\mu} + \frac{1}{\nu} = 1$. For $0 < \rho < \rho + \sigma < R$, let $\phi_{\rho,\sigma}$ be the Lipschitz cut-off function depending only on the distance to p given by

$$\phi_{\rho,\sigma}(r) = \phi(r) = \begin{cases} 0 & \text{on } B(p,R) \setminus B(p, \rho+\sigma), \\ \frac{\rho+\sigma+r}{\sigma} & \text{on } B(p,\rho+\sigma) \setminus B(p,\rho), \\ 1 & \text{on } B(p,\rho). \end{cases}$$

Then for an arbitrary constant $a \geq 1$, we have

$$\|u^{2a}\|_{\mu,\rho} \leq \|\phi u^a\|^2_{2\mu}.$$

As the injectivity radius of M is bounded below ($Inj(M) \geq r_0 > 0$) and the Ricci curvature too ($Ricci(M, g) \geq -(m-1)\kappa g$) Sobolev embeddings for complete manifolds are valid and we can apply the Sobolev inequalities to $\|\phi u^a\|^2_{2\mu}$ (see [37], Theorem 3.3). More precisely, there exists a constant $c_s > 0$ depending only on m, κ and r_0 such that

$$\|\phi u^a\|^2_{2\mu} \leq c_s \left(\|d(\phi u^a)\|^2_2 + \|\phi u^a\|^2_2\right).$$

Replacing c_s by CR^2, we can rewrite the inequality as

$$\|\phi u^a\|^2_{2\mu} \leq CR^2 \left(\|d(\phi u^a)\|^2_2 + \|\phi u^a\|^2_2\right).$$

Therefore,

$$\|u^{2a}\|_{\mu,\rho} \leq CR^2 \left(\|d(\phi u^a)\|^2_2 + \|\phi u^a\|^2_2\right).$$

However

$$\int_M |d(\phi u^a)|^2 dV \leq a \int_M \phi^2 u^{2a-1} \Delta u dV + \int_M |d\phi|^2 u^{2a} dV$$

(see [45], p.81). Hence using the assumption on Δu and $u \geq 0$ we obtain

$$\|u^{2a}\|_{\mu,\rho} \leq CR^2 \left(\int_M \phi u^{2a} dV + a\alpha \int_M \phi^2 u^{2a} dV \right.$$
$$\left. + a\beta \int_M \phi^2 u^{2a-1} dV + \int_M |d\phi|^2 u^{2a} dV \right)$$

and by construction of ϕ, we obtain

$$\begin{aligned} \|u^{2a}\|_{\mu,\rho} &\leq CR^2 \left(a\alpha + \frac{1}{\sigma^2} + 1 \right) \int_{B(p,\rho+\sigma)} u^{2a} dV + CR^2 a\beta \int_{B(p,\rho+\sigma)} u^{2a-1} dV \\ &\leq CR^2 \left(a\alpha + \frac{1}{\sigma^2} + 1 \right) \|u\|_{2a,\rho+\sigma}^{2a} + CR^2 a\beta V(p,\rho+\sigma)^{\frac{1}{2a}} \|u\|_{2a,\rho+\sigma}^{2a-1}. \end{aligned}$$

Finally, we have shown that for any $a \geq 1$, $0 < \rho < \rho + \sigma < R$, we have

$$\|u\|_{2a\mu,\rho}^{2a} \leq CR^2 \left(a\alpha + \frac{1}{\sigma^2} + 1 \right) \|u\|_{2a,\rho+\sigma}^{2a} +$$
$$CR^2 a\beta V(p,\rho+\sigma)^{\frac{1}{2a}} \|u\|_{2a,\rho+\sigma}^{2a-1}.$$

This was the first step of the proof. Now, we will proceed with a Moser iteration. To that aim, let

$$\begin{aligned} a_0 = 1,\ a_1 &= \frac{m}{m-2} = \mu, \quad \ldots, \quad a_i = \mu^i, \ldots \\ \sigma_0 = \frac{1-\theta}{2} R,\ \sigma_1 &= \frac{1-\theta}{4} R, \quad \ldots, \quad \sigma_i = \frac{1-\theta}{2^{i+1}} R, \ldots \\ \rho_0 = R - \sigma_0,\ \rho_1 &= R - \sigma_0 - \sigma_1, \quad \ldots, \quad \rho_i = R - \sum_{j=0}^{i} \sigma_j, \ldots \end{aligned}$$

and $\rho_{-1} = R$. Observe that $\rho_i > \theta R$ for any i and $\rho_i \to \theta R$ as $i \to \infty$. Moreover, for any A_i, $B_i > 0$

$$(A_i + B_i) \min\{\|u\|_{2a_i,\rho_i+\sigma_i}^{2a_i}, \|u\|_{2a_i,\rho_i+\sigma_i}^{2a_{i-1}}\} \leq$$
$$A_i \|u\|_{2a_i,\rho_i+\sigma_i}^{2a_i} + B_i \|u\|_{2a_i,\rho_i+\sigma_i}^{2a_{i-1}} \leq (A_i + B_i) \|u\|_{2a_i,\rho_i+\sigma_i}^{b_i}$$

where b_i is suitably chosen ($b_i \in \{2a_{i-1}, 2a_i\}$). Now replace above a, ρ, σ by a_i respectively ρ_i, σ_i to obtain

$$\|u\|_{2a_{i+1},\rho_i} \leq \left(CR^2 \left(a_i\alpha + \frac{1}{\sigma_i^2} + 1 + a_i\beta V(p,\rho_{i-1})^{\frac{1}{2a_i}} \right) \right)^{\frac{1}{2a_i}} \|u\|_{2a_i,\rho_{i-1}}^{\frac{b_i}{2a_i}}.$$

Then iterate this inequality to obtain (using Bishop-Gromov comparison theorem, Croke's inequality and $a_i \geq 1$)

$$\|u\|_{\infty,\theta R} \leq c\left(\prod_{i=0}^{\infty}\left(CR^2\left(a_i\alpha + a_i c'\beta + 1\right) + C\frac{R^2}{\sigma_i^2}\right)^{\frac{1}{2a_i}} \|u\|_{2,R}^{\frac{b_0}{2}}\right)^{\prod_{j=1}^{\infty}\frac{b_j}{2a_j}}.$$

By the same argument as in [56], $\prod_{j=0}^{\infty}\frac{b_j}{2a_j}$ converges to $s \in [e^{-(n-2)\frac{\ln(2)}{2}}, 1]$. It remains then to show that $\prod_{i=0}^{\infty}\left(CR^2\left(a_i\alpha + a_i c'\beta + 1\right) + C\frac{R^2}{\sigma_i^2}\right)^{\frac{1}{2a_i}}$ converges too. But we have that $\prod_{i=0}^{\infty} B^{\mu^{-i}} = B^{\frac{\mu}{\mu-1}}$ (as $\mu > 1$) and $\sum_{i=0}^{\infty} i\mu^{-i}$ is finite, therefore

$$\prod_{i=0}^{\infty}\left(CR^2\mu^i\left(\alpha + c'\beta + \mu^{-i}\right) + 4C\frac{4^i}{(1-\theta)^2}\right)^{\frac{1}{2\mu^i}} \leq$$
$$\prod_{i=0}^{\infty}\max\{\mu, 4\}^{\frac{i}{2\mu^i}}\left(CR^2\left(\alpha + c'\beta + 1\right) + C\frac{4}{(1-\theta)^2}\right)^{\frac{1}{2\mu^i}}$$
$$\leq c(\mu)\left(CR^2\left(\alpha + c'\beta + 1\right) + C\frac{4}{(1-\theta)^2}\right)^{\frac{1}{2}\frac{\mu}{\mu-1}}.$$

This implies the claim. □

II.5.1 Proof of Lemma II.3.19

The proof differs according to the assumptions made on E.

Assume E is of harmonic curvature. By Remark II.3.13 and Remark II.3.14, we have

$$\sum_{i=1}^{n} f_i(p)e_i^p(x) - \sum_{i=1}^{n} f_i(q)e_i^q(x) =$$
$$\sum_{i=1}^{n} f_i(p)\left(e_i^p(x) - \tau_{x,p}e_i^p(p)\right) + D_A f(q,p)_i \tau_{x,p}e_i^p(p) + f_i(q)\left(\tau_{x,p}e_i^q(p) - e_i^q(x)\right).$$

By Lemma II.3.3 and as $d^*R^E = 0$, $|e_i^p(x) - \tau_{x,p}e_i^p|^2 \leq c\lambda_i(p)$ for $1 \leq i \leq \mu(p)$ and $|\tau_{x,p}e_i^q(p) - e_i^q(x)|^2 \leq c\lambda_i(q)$ for $1 \leq i \leq \mu(q)$. Moreover if $\mu(q) < i \leq n$, $|\tau_{x,p}e_i^q(p) - e_i^q(x)|^2 \leq 4$. Therefore

$$\left|\sum_{i=1}^{n} f_i(p)e_i^p(x) - \sum_{i=1}^{n} f_i(q)e_i^q(x)\right|^2 \leq$$
$$c'\left(|D_A f(q,p)|^2 + (Vf)(p)\cdot f(p) + (Vf)(q)\cdot f(q)\right)$$

which implies the lemma in this case.

Assume E is of rank one, then

$$\sum_{i=1}^{n} f_i(p)e_i^p(x) - \sum_{i=1}^{n} f_i(q)e_i^q(x) = \sum_{i=1}^{n} D_A f(q,p)_i e_i^p(x) + \sum_{j=1}^{n} f_j(q) \sum_{i=1}^{n} e_i^p(x)\left(A(q,p)_{ij} - a(q,p)_{ij}(x)\right).$$

By definition of $A(q,p)_{ij}$ and by the work of Buser (Lemma 5.1 in [13]) there exists $c_B > 0$ depending only on m, κ and ε such that

$$\int_{B_{pq}} |A(q,p)_{ij} - a(q,p)_{ij}(x)|^2 \, dV(x) \leq c_B \int_{B_{pq}} |da(q,p)_{ij}(x)|^2 \, dV(x).$$

Moreover

$$(1-\delta') \sum_{i=1}^{n} |da(q,p)_{ij}(x)|^2 \leq \left| \sum_{i=1}^{n} da(q,p)_{ij}(x) e_i^p(x) \right|^2 = \left| \nabla e_j^q(x) - \sum_{i=1}^{n} a(q,p)_{ij}(x) \nabla e_i^p(x) \right|^2 \leq c\left(\left|\nabla e_j^q(x)\right|^2 + \sum_{i=1}^{n} |\nabla e_i^p(x)|^2 \right).$$

As the bundle is of rank one, $\lambda_1(p) = \ldots = \lambda_n(p)$. Therefore $\lambda_1(p) \leq \delta$ implies $\int\limits_{B(p,10\varepsilon)} |\nabla e_i^p(x)|^2 \, dV(x) \leq c\lambda_1(p)$. Otherwise $\int\limits_{B(p,10\varepsilon)} |\nabla e_i^p(x)|^2 \, dV(x) \leq c \leq c\delta^{-1}\lambda_1(p)$ by Lemma II.3.12, which implies

$$\int_{B_{pq}} |A(q,p)_{ij} - a(q,p)_{ij}(x)|^2 \, dV(x) \leq c'\left(\lambda_1(p) + \lambda_1(q)\right). \tag{43}$$

Hence

$$\int_{B(q,\varepsilon)} \left| \sum_{i=1}^{n} f_i(p)e_i^p(x) - \sum_{i=1}^{n} f_i(q)e_i^q(x) \right|^2 \leq c''\left(|D_A f(q,p)|^2 + |f(q)|^2 \left(\lambda_1(p) + \lambda_1(q)\right) \right).$$

This concludes the proof of Lemma II.3.19. □

II.5.2 Proof of Lemma II.3.22

The proof differs according to the assumptions made on E.

Assume E is of harmonic curvature. As $\{\tau_{y,p}e_i^p(p)\}_{i=1}^n$ is an almost orthonormal basis and by Remark II.3.14

$$\sum_{i=1}^n \left| s_i^p(y) - \sum_{j=1}^n A(q,p)_{ij} s_j^q(y) \right|^2 \leq$$

$$(1-\delta')^{-1} \left| \sum_{i=1}^n s_i^p(y) \left(\tau_{y,p} e_i^p(p) - e_i^p(y)\right) + \sum_{i=1}^n s_i^q(y) \left(e_i^q(y) - \tau_{y,p} e_i^q(p)\right) \right|^2 .$$

Integrate then over B'_{pq} and apply Lemma II.3.3 to obtain

$$\sum_{i=1}^n \int_{B'_{pq}} \left| s_i^p(y) - \sum_{j=1}^n A(q,p)_{ij} s_j^q(y) \right|^2 dV(y) \leq c \left(\left(\widetilde{V} s\right)(p) + \left(\widetilde{V} s\right)(q) \right).$$

Assume E is of rank one. Recall that $s_i^p(y) = \sum_{j=1}^n a(q,p)_{ij}(y) s_j^q(y)$. Hence

$$s_i^p(y) - \sum_{j=1}^n A(q,p)_{ij} s_j^q(y) = \sum_{j=1}^n \left(a(q,p)_{ij}(y) - A(q,p)_{ij}\right) s_j^q(y).$$

Therefore

$$\int_{B'_{pq}} \left| s_i^p(y) - \sum_{j=1}^n A(q,p)_{ij} s_j^q(y) \right| dV(y) \leq$$

$$\|s\|_{2,3\varepsilon} \sum_{j=1}^n \left(\int_{B'_{pq}} \left| a(q,p)_{ij}(y) - A(q,p)_{ij} \right|^2 \right)^{\frac{1}{2}} dV(y).$$

Finally, as $B'_{pq} \subset B_{pq}$, inequality (43) implies

$$\sum_{i=1}^n \left(\int_{B'_{pq}} \left| s_i^p(y) - \sum_{j=1}^n A(q,p)_{ij} s_j^q(y) \right| dV(y) \right)^2 \leq c \left(\widetilde{V} s\right)(p)$$

and this concludes the proof of Lemma II.3.22. □

Part III

Discretization of the Hodge Laplacian

Discretization of Riemannian Manifolds Applied to the Hodge Laplacian

This text has already been submitted for publication.

Abstract. For $\kappa \geq 0$ and $r_0 > 0$, let $\mathbb{M}(n, \kappa, r_0)$ be the set of all connected compact n-dimensional Riemannian manifolds such that $|K_g| \leq \kappa$ and $Inj(M, g) \geq r_0$. We study the relation between the k^{th} positive eigenvalue of the Hodge Laplacian on differential forms and the k^{th} positive eigenvalue of the combinatorial Laplacian associated to an open cover (acting on Čech cochains). We show that for a fixed sufficiently small $\varepsilon > 0$ there exist positive constants c_1 and c_2 depending only on n, κ, r_0 and ε such that for any $M \in \mathbb{M}(n, \kappa, r_0)$ and for any ε-discretization X of M we have $c_1\lambda_{k,p}(X) \leq \lambda_{k,p}(M) \leq c_2\lambda_{k,p}(X)$ for any $k \leq K$ (K depends on X). Moreover, we find a lower bound for the spectrum of the combinatorial Laplacian and a lower bound for the spectrum of the Hodge Laplacian.

Mathematics Subject Classification (2000): 58J50, 53C20.

Key words: Laplacian, differential form, Čech cohomology, discretization, Whitney form, eigenvalue.

III.1 Introduction

Several works like [9], [11], [13] and more recently [48] have shown that discretizing a Riemannian manifold may be really powerful in order to study the spectrum of the Laplacian acting on functions. The question we want to answer here is "Is there a similar tool for understanding the spectrum of the Hodge Laplacian ($\Delta = dd^* + d^*d$) acting on differential forms?". Part of an answer is given by the de Rham Theorem (saying that the de Rham cohomology of a compact manifold is isomorphic to the singular cohomology and to the Čech cohomology) and several authors have been more or less inspired by this theorem to study the spectrum of Δ. For instance, in [27], Dodziuk and Patodi show that for a fixed compact Riemannian manifold, we can approximate the spectrum of the Hodge Laplacian with the spectrum of a combinatorial Laplacian associated to finer and finer triangulations of the manifold. The main idea in their proof is to associate Čech cochains to smooth forms and vice versa via the integration on simplices and via the Whitney map. Both tools are really crucial in the proof of the de Rham Theorem as they induce the isomorphism between de Rham cohomology and singular cohomology. In [16] and in [50], the authors use another proof of de Rham Theorem due to A. Weil and based on the Čech - de Rham double complexe (see [34]). In [16], Chanillo and Trèves bound from below the smallest non-zero eigenvalue of the Hodge Laplacian on p-forms for a compact Riemannian manifold with bounded sectional curvature, while the purpose of [50] is to study the spectrum of Δ on compact hyperbolic 3-dimensional manifolds. In particular, McGowan develops in [50] a quite general method to bound from below "small" eigenvalues of Δ on compact manifolds (Lemma 2.3 in [50]).

The purpose of this paper is in some sense to improve or to unify these results in the context given by the discretization. More precisely, if M is a compact Riemannian manifold and if X is a discretization of M (in the sense of [18]), we obtain naturally from X a finite open cover $\mathcal{U}_X$ which will be contractible if the mesh of the discretization is sufficiently small. To such an open cover we can associate the complex of Čech cochains naturally endowed with a coboundary operator δ. Moreover, with an inner product on Čech cochains, we can construct the adjoint of δ, namely δ^* and define the following combinatorial Laplacian $\check{\Delta} = \delta\delta^* + \delta^*\delta$.

The main result consists in establishing a uniform comparison between the spectrum of the Hodge Laplacian and the spectrum of such a combinatorial Laplacian. That is to say, if $\mathbb{M}(n, \kappa, r_0)$ denotes the set of compact connected Riemannian manifolds with bounded (by κ) sectional curvature and injectivity radius bounded from below by r_0, we show that there exists a positive constant ρ_0 depending only on n, κ and r_0 such that if we fix $0 < 3\varepsilon < \rho_0$, there exist positive constants c_1 and c_2 depending only on n, p, κ and ε such that for any $M \in \mathbb{M}(n, \kappa, r_0)$ and for any ε-discretization X of M we can compare the k^{th} eigenvalue of Δ on p-forms to the k^{th} eigenvalue of $\check{\Delta}$ on Čech p-cochains (for $1 \leq p \leq n-1$) in the following way

$$c_1\lambda_{k,p}(X) \leq \lambda_{k,p}(M) \leq c_2\lambda_{k,p}(X)$$

for any $k \leq K$ and K depends on X (see Theorem III.3.1 for the precise statement).

As an application of Theorem III.3.1, we obtain a lower bound for the first non-zero eigenvalue of Δ (see Theorem III.4.1) in terms of the volume of the manifold. This result has to be compared with the result obtained by Chanillo and Trèves (Theorem 1.1, in [16]). In their proof, the authors use in a crucial manner a lemma due to Trèves (Lemma A.5 in [59]) which turns out to be false (see Remark III.4.3). In Lemma III.4.2, we state and prove a "weaker" version of Trèves' lemma. A direct corollary of this lemma is a lower bound for the spectrum of the combinatorial Laplacian (see Theorem III.4.4) and so, thanks to Theorem III.3.1, a lower bound for the spectrum of Δ (see Theorem III.4.1).

As another consequence of the proof of Theorem III.3.1, we obtain a version of McGowan's lemma (Lemma 2.3 in [50]) slightly more general as it is concerned with p-forms on compact Riemannian manifolds with bounded sectional curvature, but not so general as it is valid only for contractible open covers (see Lemma III.4.5). Finally, another interesting application of the method developed here concerns Whitney forms. Indeed, Whitney forms come out in [27] as a natural way to smooth Čech cochains. Nevertheless, in order to keep a uniform comparison of the spectra, the results given in [27] on Whitney forms are not useful to our purpose. Hence, we obtain as a corollary of the method, the appropriate results to show that Whitney forms are even so a suitable tool to smooth Čech cochains (see Section III.4.2).

The paper is organized as follows. In Section III.2, we begin by recalling different definitions and properties of differential forms and Čech cochains. In particular, in Section III.2.3, we sketch the proof of the de Rham Theorem due to A. Weil as it will be the starting point of the proof of Theorem III.3.1. Finally, we recall the definition of a discretization and its main properties.

Section III.3 is devoted to the proof of Theorem III.3.1. The basic idea of the proof is to associate a Čech cochain to a differential form via a discretizing operator and vice versa via a smoothing operator, in order to compare "small" eigenvalues. These operators are essentially constructed as in the proof (of A. Weil) of the de Rham Theorem thanks to the Čech - de Rham double complexe. To that aim, we need a few technical results. In particular, we need a normed version of the Poincaré Lemma and a similar result for Čech cochains. This is done in Lemma III.3.2 and in Lemma III.3.5. Moreover, as in [50], it is necessary to bound from below the spectrum of Δ with absolute boundary conditions on finite intersections of open sets of the open cover. To that aim, we show that for a sufficiently small ε, the intersection of balls of radius ε is convex and is quasi-isometric to a Euclidean convex. Thanks to a result of Guerini ([36]) we can then bound from below the spectrum of such intersections (this appears in Section III.2 as properties of the discretization, see Lemma III.2.9 and Lemma III.2.10). Note that Chanillo and Trèves met also this problem and they solve it using a (finite) sequence of open covers and with Lemma 2.2 in [16] (which is a consequence of a normed version of the Poincaré Lemma in the Euclidean setting). For "large" eigenvalues, it suffices to have an upper bound for the k^{th} eigenvalue of Δ and of $\check{\Delta}$ to have the claim.

In Section III.4, we present the consequences of Theorem III.3.1 mentioned above.

Finally, in the appendix we recall the (more or less classical) definition and the properties of Whitney forms. At the end of the appendix, we give the proof of the technical lemma about the Euclidean convexity of the intersection of small balls.

III.2 Settings

In this section, we recall some definitions and basic facts on the Laplacian acting on differential forms and on the Laplacian acting on Čech cochains. For the convenience of the reader and as it is a key tool for the paper, a paragraph is also devoted to the sketch of a classical proof due to A. Weil of the de Rham Theorem (for contractible open covers) relying on the Čech - de Rham double complexe (see for instance Appendix A of [34] or Chapter 3 of [53]). Finally, we define the discretization of a manifold and discuss some of its properties.

III.2.1 Laplacian acting on differential forms

Let (M^n, g) be a compact connected n-dimensional Riemannian manifold without boundary. Denote by $\Lambda^p(M)$ the vector space of smooth differential p-forms, for $0 \le p \le n$. Let $d : \Lambda^p(M) \to \Lambda^{p+1}(M)$ be the exterior differential and $d^* : \Lambda^{p+1}(M) \to \Lambda^p(M)$ its formal adjoint (with respect to the L^2-inner product) the codifferential. Then the Laplacian acting on p-forms is defined by $\Delta : \Lambda^p(M) \to \Lambda^p(M)$, $\Delta = dd^* + d^*d$. The spectrum of Δ is discrete and will be denoted by

$$0 < \lambda_{1,p}(M) \le \lambda_{2,p}(M) \le \ldots \le \lambda_{k,p}(M) \le \ldots$$

where 0 is of multiplicity $b_p(M)$ and the positive eigenvalues are repeated as many times as their multiplicity. Let us recall that half of the spectrum is redundant. That is to say, if $\lambda > 0$ is an eigenvalue of Δ on p-forms and if $E_p(\lambda)$ denotes the λ-eigenspace, then $E_p(\lambda)$ splits as follows $E_p(\lambda) =$

$E_p^{d^*}(\lambda) \oplus E_p^d(\lambda)$ where $E_p^{d^*}(\lambda) = \{\omega \in E_p(\lambda) : d^*\omega = 0\} \subseteq d^*\Lambda^{p+1}(M)$ is the λ-eigenspace of d^*d and $E_p^d(\lambda) = \{\omega \in E_p(\lambda) : d\omega = 0\} \subseteq d\Lambda^{p-1}(M)$ is the λ-eigenspace of dd^*. Moreover, d^* maps $E_p^d(\lambda)$ isomorphically onto $E_{p-1}^{d^*}(\lambda)$ and d maps $E_p^{d^*}(\lambda)$ isomorphically onto $E_{p+1}^d(\lambda)$. Hence, $E_p(\lambda) = E_p^{d^*}(\lambda) \oplus E_{p-1}^{d^*}(\lambda)$. So for our purpose it will be sufficient to study the spectrum of d^*d on coexact forms.

Let $\lambda_{k,p}^{d^*}(M)$ the k^{th} (positive) eigenvalue of $d^*d : d^*\Lambda^{p+1}(M) \to d^*\Lambda^{p+1}(M)$. The following variational characterization of the spectrum of d^*d holds

$$\lambda_{k,p}^{d^*}(M) = \min_{\Sigma^k} \max \left\{ \frac{\|d\omega\|^2}{\|\omega\|^2} : \omega \in \Sigma^k \setminus \{0\} \right\}$$

where Σ^k ranges over all k-dimensional vector subspaces of $d^*\Lambda^{p+1}(M)$ and $\|\cdot\|$ denotes the L^2-norm for differential forms.

III.2.2 Čech cohomology and combinatorial Laplacian

Let M^n be a compact connected n-dimensional manifold. Let $\mathcal{U} = \{U_i\}_{1 \le i \le N}$ be a finite open cover of M. The nerve of $\mathcal{U}$, denoted by $N(\mathcal{U})$, is the simplicial complex whose set of q-simplices is given by

$$S_q(\mathcal{U}) = \{(i_0, \ldots, i_q) : i_0 < \ldots < i_q \text{ and } U_{i_0} \cap \ldots \cap U_{i_q} \neq \emptyset\}$$

for any $q \geq 0$. A Čech q-cochain is an application $c : S_q(\mathcal{U}) \to \mathbb{R}$. Denote by $\mathcal{C}^q(\mathcal{U})$ the set of Čech q-cochains. Let us remark that $\mathcal{C}^q(\mathcal{U})$ is naturally endowed with a vector space structure and let us define a coboundary operator $\delta : \mathcal{C}^q(\mathcal{U}) \to \mathcal{C}^{q+1}(\mathcal{U})$ by

$$\delta c(i_0, \ldots, i_{q+1}) = \sum_{j=0}^{q+1} (-1)^j c(i_0, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{q+1})$$

for any $\{i_0, \ldots, i_{q+1}\} \in S_{q+1}(\mathcal{U})$. Then $\delta \circ \delta = 0$ and the cochain complex $\{\mathcal{C}^q(\mathcal{U}), \delta\}$ gives rise to the Čech cohomology groups of the cover $\mathcal{U}$, $\check{H}^*(\mathcal{U})$. Endow then $\mathcal{C}^q(\mathcal{U})$ with the following scalar product, for any c_1, $c_2 \in \mathcal{C}^q(\mathcal{U})$

$$(c_1, c_2) = \sum_{I \in S_q(\mathcal{U})} c_1(I) c_2(I)$$

and consider $\delta^* : \mathcal{C}^{q+1}(\mathcal{U}) \to \mathcal{C}^q(\mathcal{U})$ the adjoint of δ with respect to $(\cdot, \cdot)$.

Definition III.2.1 *The* ***combinatorial Laplacian*** $\check{\Delta} : \mathcal{C}^q(\mathcal{U}) \to \mathcal{C}^q(\mathcal{U})$ *is defined by* $\check{\Delta} = \delta\delta^* + \delta^*\delta$.

The combinatorial Laplacian is self-adjoint and non-negative by definition. Its spectrum will be denoted by

$$0 < \lambda_{1,q}(\mathcal{U}) \leq \lambda_{2,q}(\mathcal{U}) \leq \ldots \leq \lambda_{L,q}(\mathcal{U})$$

where 0 is of multiplicity $\check{b}_q(\mathcal{U})$ and $L + \check{b}_q(\mathcal{U}) = dim(\mathcal{C}^q(\mathcal{U})) = |S_q(\mathcal{U})|$. As for the Laplacian on differential forms, half of the spectrum is redundant i.e. if $\lambda > 0$ is an eigenvalue of $\check{\Delta}$ on Čech q-cochains and if $\check{E}_q(\lambda)$ denotes the λ-eigenspace, then $\check{E}_q(\lambda) = \check{E}_q^{\delta^*}(\lambda) \oplus \check{E}_{q-1}^{\delta^*}(\lambda)$ where $\check{E}_q^{\delta^*}(\lambda)$ is the λ-eigenspace of $\delta^*\delta$ acting on $\delta^*\mathcal{C}^{q+1}(\mathcal{U})$. So for our purpose it will be sufficient to study the spectrum of $\delta^*\delta$ on $\delta^*\mathcal{C}^{q+1}(\mathcal{U})$ i.e. on coexact Čech cochains. In the sequel, $\lambda_{k,q}^{\delta^*}(\mathcal{U})$ denotes the k^{th} (positive) eigenvalue of $\delta^*\delta : \delta^*\mathcal{C}^{q+1}(\mathcal{U}) \to \delta^*\mathcal{C}^{q+1}(\mathcal{U})$. The following variational characterization holds

$$\lambda_{k,q}^{\delta^*}(\mathcal{U}) = \min_{V^k} \max \left\{ \frac{\|\delta c\|^2}{\|c\|^2} : c \in V^k \setminus \{0\} \right\}$$

where V^k ranges over all k-dimensional vector subspaces of $\delta^*\mathcal{C}^{q+1}(\mathcal{U})$.

III.2.3 De Rham Theorem

Recall that an open cover $\mathcal{U}$ is called contractible if for any $I \in S_q(\mathcal{U})$, $U_I = \bigcap_{i\in I} U_i$ is contractible. The following theorem is due to de Rham.

Theorem III.2.2 *Let (M^n, g) be a compact connected n-dimensional Riemannian manifold without boundary. Let $\mathcal{U}$ be a contractible finite open cover of M. Then the p^{th} group of de Rham's cohomology $H^p(M)$ is isomorphic to $\check{H}^p(\mathcal{U})$.*

Remark III.2.3 *Note that a consequence of the de Rham Theorem is that if $\mathcal{U}$ is a contractible cover, then $b_p(M) = \check{b}_p(\mathcal{U})$.*

Let us introduce now the vector spaces $\mathcal{C}^q(\mathcal{U}, \Lambda^p)$ of q-cochains of p-forms i.e. c is in $\mathcal{C}^q(\mathcal{U}, \Lambda^p)$ if $c(I)$ is a p-form on U_I for any I in $S_q(\mathcal{U})$. Define then the following coboundary operators

$\delta : \mathcal{C}^q(\mathcal{U}, \Lambda^p) \to \mathcal{C}^{q+1}(\mathcal{U}, \Lambda^p)$ defined by

$$\delta c(i_0, \ldots, i_{q+1}) = \sum_{j=0}^{q+1} (-1)^j c(i_0, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{q+1})$$

for any $\{i_0, \ldots, i_{q+1}\} \in S_{q+1}(\mathcal{U})$ and

$$d : \mathcal{C}^q(\mathcal{U}, \Lambda^p) \to \mathcal{C}^q(\mathcal{U}, \Lambda^{p+1}) \text{ defined by } dc(I) = d(c(I))$$

$$
\begin{array}{ccccccccccccc}
 & & \mathcal{C}^0(\mathcal{U}) & \xrightarrow{\delta} & \mathcal{C}^1(\mathcal{U}) & \xrightarrow{\delta} \dots \xrightarrow{\delta} & \mathcal{C}^{q-1}(\mathcal{U}) & \xrightarrow{\delta} & \mathcal{C}^q(\mathcal{U}) & \xrightarrow{\delta} \dots \\
 & & \downarrow i & & \downarrow i & & \downarrow i & & \downarrow i & \\
\Lambda^0(M) & \xrightarrow{r} & \mathcal{C}^0(\mathcal{U},\Lambda^0) & \xrightarrow{\delta} & \mathcal{C}^1(\mathcal{U},\Lambda^0) & \xrightarrow{\delta} \dots \xrightarrow{\delta} & \mathcal{C}^{q-1}(\mathcal{U},\Lambda^0) & \xrightarrow{\delta} & \mathcal{C}^q(\mathcal{U},\Lambda^0) & \xrightarrow{\delta} \dots \\
\downarrow d & & \downarrow d & & \downarrow d & & \downarrow d & & \downarrow d & \\
\Lambda^1(M) & \xrightarrow{r} & \mathcal{C}^0(\mathcal{U},\Lambda^1) & \xrightarrow{\delta} & \mathcal{C}^1(\mathcal{U},\Lambda^1) & \xrightarrow{\delta} \dots \xrightarrow{\delta} & \mathcal{C}^{q-1}(\mathcal{U},\Lambda^1) & \xrightarrow{\delta} & \mathcal{C}^q(\mathcal{U},\Lambda^1) & \xrightarrow{\delta} \dots \\
\downarrow d & & \downarrow d & & \downarrow d & & \downarrow d & & \downarrow d & \\
\vdots & & \vdots & & \vdots & \dots & \vdots & & \vdots & \\
\downarrow d & & \downarrow d & & \downarrow d & & \downarrow d & & \downarrow d & \\
\Lambda^{p-1}(M) & \xrightarrow{r} & \mathcal{C}^0(\mathcal{U},\Lambda^{p-1}) & \xrightarrow{\delta} & \mathcal{C}^1(\mathcal{U},\Lambda^{p-1}) & \xrightarrow{\delta} \dots \xrightarrow{\delta} & \mathcal{C}^{q-1}(\mathcal{U},\Lambda^{p-1}) & \xrightarrow{\delta} & \mathcal{C}^q(\mathcal{U},\Lambda^{p-1}) & \xrightarrow{\delta} \dots \\
\downarrow d & & \downarrow d & & \downarrow d & & \downarrow d & & \downarrow d & \\
\Lambda^p(M) & \xrightarrow{r} & \mathcal{C}^0(\mathcal{U},\Lambda^p) & \xrightarrow{\delta} & \mathcal{C}^1(\mathcal{U},\Lambda^p) & \xrightarrow{\delta} \dots \xrightarrow{\delta} & \mathcal{C}^{q-1}(\mathcal{U},\Lambda^p) & \xrightarrow{\delta} & \mathcal{C}^q(\mathcal{U},\Lambda^p) & \xrightarrow{\delta} \dots \\
\downarrow d & & \downarrow d & & \downarrow d & & \downarrow d & & \downarrow d & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}
$$

Figure 1: The Čech - de Rham double complexe.

for any $I \in S_q(\mathcal{U})$. Then $d \circ d = 0$, $\delta \circ \delta = 0$ and $d \circ \delta = \delta \circ d$. The Čech - de Rham double complex is the following commutative diagram, where r denotes the restriction map to each open of the cover and i the natural injection. The first step in the proof of the de Rham Theorem is to show that the rows (except the first) and the columns (except the first) of this diagram are exact. This is a direct consequence of the Poincaré Lemma (Lemma III.2.4) and Lemma III.2.5.

Lemma III.2.4 *Let $p > 0$. Let $\mathcal{U}$ be a contractible cover. Let $\omega \in \mathcal{C}^q(\mathcal{U}, \Lambda^p)$ such that $d\omega = 0$. Then there exists $\eta \in \mathcal{C}^q(\mathcal{U}, \Lambda^{p-1})$ such that $d\eta = \omega$.*

Proof: see [34], A.6. □

Lemma III.2.5 *Let $q > 0$. Let $c \in \mathcal{C}^q(\mathcal{U}, \Lambda^p)$ such that $\delta c = 0$. Then there exists $b \in \mathcal{C}^{q-1}(\mathcal{U}, \Lambda^p)$ such that $\delta b = c$.*

Proof: see [34], proof of Lemma A.4.1. □

The proof of the de Rham Theorem goes then as follows. Let $\omega \in \Lambda^p(M)$ such that $d\omega = 0$. Let $f_0 = r(\omega) \in \mathcal{C}^0(\mathcal{U}, \Lambda^p)$, then $df_0 = 0 = \delta f_0$ and the system of equations

$$f_0 = df_1 \ , \ \delta f_1 = df_2 \ , \ \delta f_2 = df_3 \ , \ \dots \ , \ \delta f_{p-1} = df_p$$

has a solution with $f_j \in \mathcal{C}^{j-1}(\mathcal{U}, \Lambda^{p-j})$ for $j \geq 1$. Moreover, $\delta(\delta f_p) = 0$, hence $\delta f_p \in \mathcal{C}^p(\mathcal{U})$. The application $\Psi : \{\omega \in \Lambda^p(M) : d\omega = 0\} \to \{c \in \mathcal{C}^p(\mathcal{U}) : \delta c = 0\}$ given by $\Psi(\omega) = \delta f_p$, where f_p is constructed as above, induces an isomorphism in cohomology. In particular, if ω is exact, $\Psi(\omega)$ is also exact i.e. there exists $c \in \mathcal{C}^{p-1}(\mathcal{U})$ such that $\delta c = \Psi(\omega)$ (note that in general $f_p \notin \mathcal{C}^{p-1}(\mathcal{U})$). Naturally, we can construct another application going from closed Čech p-cochains to closed p-forms exactly in the same way and obtain also an isomorphism in cohomology. □

III.2.4 Discretization of a manifold

Let (M^n, g) be a connected compact n-dimensional Riemannian manifold without boundary. Let $\varepsilon > 0$.

Definition III.2.6 *An ε-discretization X of M is a maximal ε-separated subset of M i.e. X is a subset of M satisfying*

(i) $\forall p \neq q \in X$, $d(p, q) \geq \varepsilon$,

(ii) $\mathcal{U}_X = \{B(p, \varepsilon)\}_{p \in X}$ *is an open cover of* M.

Note that as M is compact, X is finite of cardinality $|X|$. So we can number the elements of $X = \{p_1, \ldots, p_{|X|}\}$ and denote $U_i = B(p_i, \varepsilon)$, for $i = 1, \ldots, |X|$. In particular, any discretization of M gives rise to a combinatorial Laplacian $\check{\Delta}$ as defined in Section III.2.2. In the sequel, $\lambda_{k,q}(X)$ will denote the k^{th} eigenvalue of the combinatorial Laplacian associated to the open cover $\mathcal{U}_X$ acting on Čech q-cochains i.e. $\lambda_{k,q}(X) = \lambda_{k,q}(\mathcal{U}_X)$.

Note also that if ε (the mesh of the discretization) is smaller than the convexity radius of M, then $\mathcal{U}_X$ is a contractible open cover and $\check{b}_p(\mathcal{U}_X) = b_p(M)$.

Definition III.2.7 *For $\kappa \geq 0$, $r_0 > 0$ and $n \in \mathbb{N}^*$, we define $\mathbb{M}(n, \kappa, r_0)$ as the set of all connected compact n-dimensional Riemannian manifold (M^n, g) without boundary with uniformly bounded sectional curvature i.e. $|K_g| \leq \kappa$ and injectivity radius bounded below i.e. $Inj(M, g) \geq r_0$.*

Remark III.2.8 *For $n \in \mathbb{N}^*$, $\kappa \geq 0$, $r_0 > 0$ and $0 < 2\varepsilon < r_0$, there exists $\nu(n, \kappa) > 0$ such that, for any $(M, g) \in \mathbb{M}(n, \kappa, r_0)$ and any ε-discretization X of M, the cardinality of $\{j : U_j \cap U_I \neq \emptyset\}$ is bounded above by ν, for any $I \in S_q(\mathcal{U}_X)$. This is a direct consequence of the Bishop-Gromov volume comparison Theorem (see for instance [18], Lemma V.3.1, p.147). Furthermore, by Croke's Inequality and Bishop's comparison Theorem (see [18] p.126 and p.136) we can assert that there exist positive constants c_1, c_2 depending only* *on n, κ and ε such that $c_1 Vol(M) \leq |X| \leq c_2 Vol(M)$. In particular, we obtain that $|\mathcal{S}_q(\mathcal{U}_X)| \leq \frac{\nu^q}{(q+1)!}|X| \leq \frac{\nu^q}{(q+1)!}c_2 Vol(M)$.*

The following lemma shows that in general a sufficiently small ball is quasi-isometric (in the sense of [25], (3.2)) to a Euclidean convex. In particular, this will imply that on intersections of sufficiently small balls we can find a lower bound for the first positive eigenvalue of Δ with absolute boundary condition (see Lemma III.2.10). This is an essential result for the discretization as we will see later.

Lemma III.2.9 *Let $n \in \mathbb{N}^*$, $\kappa \geq 0$ and $r_0 > 0$. There exists a constant $0 < \rho_0 < r_0$ depending only on n, κ and r_0 such that for any $(M, g) \in \mathbb{M}(n, \kappa, r_0)$ and for any $p \in M$, there exist a Euclidean convex $C_p \subseteq \mathbb{R}^n$ and a diffeomorphism $\varphi : C_p \to B(p, \rho_0)$ such that for any $B(q, \rho) \subseteq B(p, \rho_0)$, the ball $B(q, \rho)$ is convex and $\varphi^{-1}(B(q, \rho))$ is a Euclidean convex. Moreover, $(B(q, \rho), g)$ is quasi-isometric to $B(q, \rho)$ endowed with the Euclidean metric induced by φ^{-1} and the constants of quasi-isometry depend only on n, κ and $d(p, q) + \rho$.*

Proof: see Appendix III.5.2. □

Note that the intersection of small balls is a convex with not necessarily smooth boundary. So that it is not obvious that in this case the spectrum of the Laplacian with absolute boundary condition is discrete. In [51], the authors show that the spectrum of the Laplacian with absolute (or relative) boundary condition is discrete even if the boundary is only given by a Lipschitz function (Proposition 5.3 in [51]). Moreover, Theorem 5.1 of [52] implies that the following classical variational characterization of the spectrum is still valid for bounded convex domains i.e. if Ω is a bounded convex domain of M, then the k^{th} eigenvalue of the Laplacian for p-forms on Ω with absolute boundary condition is given by

$$\lambda_{k,p}^{abs}(\Omega) = \min_{\Sigma^k} \max \left\{ \frac{\|d\omega\|^2 + \|\delta\omega\|^2}{\|\omega\|^2} : \omega \in \Sigma^k \setminus \{0\} \text{ such that } i_\nu(\omega) = 0 \right\}$$

where Σ^k ranges over all k-dimensional vector subspaces of $\Lambda^p(\Omega)$ and i_ν is the interior product by ν the outward pointing normal unit vector to the boundary (defined almost everywhere). In particular, the result on quasi-isometric metrics of Dodziuk (Proposition 3.3 of [25]) is valid in this context.

Lemma III.2.10 *Let $n \geq 2$, $\kappa \geq 0$, $r_0 > 0$ and let ρ_0 given by Lemma III.2.9. Let $0 < 3\varepsilon < \rho_0$. Then there exists a positive constant $\mu(n, \kappa, \varepsilon)$*

depending only on n, κ and ε such that for any $(M,g) \in \mathbb{M}(n,\kappa,r_0)$ and for any ε-discretization X of M

$$\lambda_{1,p}^{abs}(U_I) \geq \mu(n,\kappa,\varepsilon)$$

for any $p = 0,\dots,n$ and any $I \in S_q(\mathcal{U}_X)$, $q \geq 0$.

Proof: let $(M,g) \in \mathbb{M}(n,\kappa,r_0)$ and X an ε-discretization of M with $0 < 3\varepsilon < \rho_0$. Fix $p \in X$ and let $q \in X$ such that $B(p,\varepsilon) \cap B(q,\varepsilon) \neq \emptyset$. Then $B(q,\varepsilon) \subseteq B(p,3\varepsilon) \subseteq B(p,\rho_0)$. By Lemma III.2.9, there exists a diffeomorphism φ such that $\varphi^{-1}(B(q,\varepsilon))$ is a Euclidean convex for any $q \in X$ such that $B(q,\varepsilon) \cap B(p,\varepsilon) \neq \emptyset$. In particular, $\varphi^{-1}\left(B(p,\varepsilon) \cap B(q,\varepsilon)\right)$ is an intersection of Euclidean convexes and as such it is a Euclidean convex. Moreover, φ^{-1} restricted to $B(p,3\varepsilon)$ is a quasi-isometry with constants of quasi-isometry depending only on n, κ and ε. Let U_I a non-empty finite intersection of elements of $\mathcal{U}_X$ and $V_I = \varphi^{-1}(U_I)$ the Euclidean convex which is quasi-isometric to U_I via φ i.e. $(\varphi(V_I), (\varphi^{-1})^*(eucl))$ is quasi-isometric to (U_I, g) with constants of quasi-isometry α depending only on n, κ and ε (i.e. $\alpha^{-1}(\varphi^{-1})^*(eucl) \leq g \leq \alpha(\varphi^{-1})^*(eucl)$). Then by Proposition 3.3 of [25], there exist positive constants c_1 and c_2 depending only on α and n such that

$$c_1\lambda_{1,p}^{abs}(U_I, (\varphi^{-1})^*(eucl)) \leq \lambda_{1,p}^{abs}(U_I, g) \leq c_2\lambda_{1,p}^{abs}(U_I, (\varphi^{-1})^*(eucl)). \tag{45}$$

Note that $(U_I, (\varphi^{-1})^*(eucl))$ is a Euclidean convex of diameter bounded above by $d(n,\kappa,\varepsilon)$. Finally, Guerini shows in [36], that the first eigenvalue of the Laplacian with absolute boundary condition on a Euclidean convex with smooth boundary is bounded below by a constant depending on the diameter of the convex. Note that Guerini's proof can be adapted straightforward to obtain the same result for convexes with piecewise smooth boundary. Hence, we obtain that there exists a positive constant $c(n,p)$ such that

$$\lambda_{1,p}^{abs}(U_I, (\varphi^{-1})^*(eucl)) \geq \frac{c(n,p)}{diam(U_I, (\varphi^{-1})^*(eucl))^2} \geq \frac{c(n,p)}{d(n,\kappa,\varepsilon)^2} \tag{46}$$

Finally, (45) and (46) imply the claim. □

III.3 Comparison of spectra

This section is devoted to the proof of the main theorem of the paper. Let us state the result.

Theorem III.3.1 *Let $n \geq 2$, $\kappa \geq 0$, $r_0 > 0$. Let $\rho_0(n,\kappa,r_0)$ be given by Lemma III.2.9 and $0 < 3\varepsilon < \rho_0$. Let $1 \leq p \leq n-1$. Then there exist* *positive constants c_1, c_2 depending only on n, p, κ and ε such that for any $M \in \mathbb{M}(n,\kappa,r_0)$ and for any ε-discretization X of M, we have*

$$c_1\lambda_{k,p}(X) \leq \lambda_{k,p}(M) \leq c_2\lambda_{k,p}(X)$$

for any $1 \leq k \leq |\mathcal{C}^p(\mathcal{U}_X)| - \check{b}_p(\mathcal{U}_X) = |\mathcal{C}^p(\mathcal{U}_X)| - b_p(M)$.

As we have seen before (in Section III.2.1), it will be sufficient to establish the result for the spectrum of d^*d on coexact p-forms and for the spectrum of $\delta^*\delta$ on coexact Čech p-cochains. The proof goes in two steps. First step consists in comparing "small" eigenvalues. We need to construct a discretizing operator that associates to a coexact p-form a coexact Čech p-cochain (see Section III.3.1) and a smoothing operator that goes in the opposite direction (see Section III.3.2), in order to compare their respective Rayleigh quotients. The idea is to proceed as in the proof of the de Rham Theorem and use the Čech - de Rham double complexe. But as we need a control of the norms involved, we have to establish versions of the Poincaré Lemma (Lemma III.2.4) and of Lemma III.2.5 with a suitable control of the norms (see Lemma III.3.2 and Lemma III.3.5). The second step of the proof deals with "large" eigenvalues and is reduced to find upper bounds for the k^{th} eigenvalues involved depending only on the parameters of the problem (see Section III.3.3).

In the sequel, we consider (M,g) in $\mathbb{M}(n,\kappa,r_0)$ and X an ε-discretization with $0 < 3\varepsilon < \rho_0$. Denote by $\mathcal{U}$ the open cover induced by X i.e. $\mathcal{U} = \{U_i = B(p_i,\varepsilon) : i = 1,\ldots,|X|\}$ and fix $1 \leq p \leq n-1$.

III.3.1 From smooth forms to Čech cochains

In this section, we are going to construct

$$\mathcal{D} : d^*\Lambda^{p+1}(M) \to \delta^*\mathcal{C}^{p+1}(\mathcal{U})$$

such that there exist positive constants c_1, c_2 and Λ depending only on n, p, κ and ε such that

$(i)_{\mathcal{D}}$ $\|\delta\mathcal{D}(\omega)\|^2 \leq c_1\|d\omega\|^2$, for any $\omega \in d^*\Lambda^{p+1}(M)$,

$(ii)_{\mathcal{D}}$ $\|\mathcal{D}\omega\|^2 \geq c_2\|\omega\|^2$, for any $\omega \in d^*\Lambda^{p+1}(M)$ satisfying $\|d\omega\|^2 \leq \Lambda\|\omega\|^2$.

To that aim, we need the following version of the Poincaré Lemma. Note that this lemma will be verified in particular by any non-empty intersection of open sets in $\mathcal{U}$ thanks to Lemma III.2.10 (where μ depends on n, κ, ε).

Lemma III.3.2 *Let U be a contractible open set such that $\lambda_{1,p}^{abs,d}(U) \geq \mu > 0$, ($1 \leq p \leq n$). Let ω be a closed L^2-integrable p-form on U i.e. $d\omega = 0$. Then there exists $\eta \in \Lambda^{p-1}(U)$ such that $d\eta = \omega$ and $\|\eta\|^2_{L^2(U)} \leq \frac{2}{\mu}\|\omega\|^2_{L^2(U)}$.*

Proof: we have the following characterization of the first eigenvalue of the Laplacian on exact p-forms (see Proposition 3.1. of [25] or Proposition 2.1. of [50]),

$$\lambda_{1,p}^{abs,d}(U) = \inf_V \sup \left\{ \frac{\|\omega\|^2_{L^2(U)}}{\|\eta\|^2_{L^2(U)}} : \omega \in V \setminus \{0\} \ , \ d\eta = \omega \right\}$$

where V ranges over all 1-dimensional vector subspaces of exact p-forms. If $\omega \in \Lambda^p(U)$ is closed, by the Poincaré Lemma ω is exact. So that we get

$$\mu \leq \lambda_{1,p}^{abs,d}(U) \leq \sup \left\{ \frac{\|\omega\|^2_{L^2(U)}}{\|\eta\|^2_{L^2(U)}} : d\eta = \omega \right\}$$

and hence there exists $\eta \in \Lambda^{p-1}(U)$ such that $d\eta = \omega$ and $\frac{1}{2}\mu \leq \frac{\|\omega\|^2_{L^2(U)}}{\|\eta\|^2_{L^2(U)}}$ which is the claim. □

Remark III.3.3 *Let us introduce the following norm. If* $c \in \mathcal{C}^q(\mathcal{U}, \Lambda^p)$ *let*

$$\|c\|^2 = \sum_{I \in S_q(\mathcal{U})} \|c(I)\|^2_{L^2(U_I)}$$

where $\| \cdot \|_{L^2(U_I)}$ *denotes the* L^2*-norm for* p*-forms on* U_I*. In particular, if* ω *is a* p*-form on* M *and* r *is the restriction to each open of* $\mathcal{U}$*, then there exist positive constants* c_1 *and* c_2 *depending only on* n*,* κ *and* ε *such that* $c_1\|r(\omega)\|^2 \leq \|\omega\|^2 \leq c_2\|r(\omega)\|^2$.

Construction by induction of $\mathcal{D}$

Let $\omega \in d^*\Lambda^{p+1}(M)$. The goal is to construct $\mathcal{D}(\omega) \in \delta^*\mathcal{C}^{p+1}(\mathcal{U})$. The idea is to consider $d\omega$ which is an exact $(p+1)$-form and to construct an exact Čech $(p+1)$-cochain $\delta\mathcal{D}(\omega)$ such that $(i)_\mathcal{D}$ holds. A suitable candidate for $\delta\mathcal{D}(\omega)$ is the Čech cochain given by the proof of the de Rham Theorem and the double complexe. Moreover, the double complexe and the normed version of the Poincaré Lemma give almost directly the inequality $(i)_\mathcal{D}$, whereas $(ii)_\mathcal{D}$ is not a so direct consequence of the construction. Hence, as suggested in [16], we construct an auxiliary p-form thanks to Whitney forms to obtain $(ii)_\mathcal{D}$. We proceed by induction.

First step of induction: define $c_{p+1,0} \in \mathcal{C}^0(\mathcal{U}, \Lambda^{p+1})$ by $c_{p+1,0} = r(d\omega)$ i.e. $c_{p+1,0}(i) = d\omega_{|U_i}$. Then $dc_{p+1,0} = 0 = \delta c_{p+1,0}$ and $W(c_{p+1,0}) = d\omega$, where W is the Whitney map defined in Appendix III.5.1. Then there exist positive constants c_1, c_2 and c_3 depending only on n, p, κ and ε such that the three following assertions hold.

$(a)_1$ There exists $c_{p,0} \in \mathcal{C}^0(\mathcal{U}, \Lambda^p)$ such that $dc_{p,0} = c_{p+1,0}$ and $\|c_{p,0}\|^2 \leq c_1\|d\omega\|^2$.

$(b)_1$ Let $c_{p,1} = \delta c_{p,0}$. We have $\delta c_{p,1} = 0 = dc_{p,1}$ and $\|c_{p,1}\|^2 \leq c_2\|d\omega\|^2$.

$(c)_1$ Let $v^{(1)} = W(c_{p,0}) \in \Lambda^p(M)$. We have $dv^{(1)} = d\omega + W(c_{p,1})$ and $\|v^{(1)}\|^2 \leq c_3\|d\omega\|^2$.

Indeed, $(a)_1$ is a direct consequence of Lemma III.3.2, of the definition of $c_{p+1,0}$ and of Remark III.3.3. Then, clearly $\delta c_{p,1} = 0$ and $dc_{p,1} = \delta dc_{p,0} = \delta c_{p+1,0} = 0$. Moreover, there exists $c(n, \kappa, \varepsilon)$ such that for any cochain $\|\delta b\|^2 \leq c\|b\|^2$ (see (49)) and combined with $(a)_1$ this implies $(b)_1$. Finally, by Lemma III.5.4 $dv^{(1)} = W(c_{p,1}) + W(c_{p+1,0}) = d\omega + W(c_{p,1})$. Moreover, by Lemma III.5.5 and by $(a)_1$, we get $\|v^{(1)}\|^2 \leq cst\|c_{p,0}\|^2 \leq c_3\|d\omega\|^2$.

Induction hypothesis: (for $1 \leq q < p+1$) there exist positive constants c_1, c_2 and c_3 depending only on n, p, κ and ε such that the three following assertions hold.

$(a)_q$ There exists $c_{p+1-q,q-1} \in \mathcal{C}^{q-1}(\mathcal{U}, \Lambda^{p+1-q})$ such that $dc_{p+1-q,q-1} = c_{p+1-(q-1),q-1}$ and $\|c_{p+1-q,q-1}\|^2 \leq c_1\|d\omega\|^2$.

$(b)_q$ Let $c_{p+1-q,q} = (-1)^{q+1}q \cdot \delta c_{p+1-q,q-1}$. We have $\delta c_{p+1-q,q} = 0 = dc_{p+1-q,q}$ and $\|c_{p+1-q,q}\|^2 \leq c_2\|d\omega\|^2$.

$(c)_q$ Let $v^{(q)} = v^{(q-1)} + W(c_{p+1-q,q-1}) \in \Lambda^p(M)$. We have $d\omega = dv^{(q)} + (-1)^q W(c_{p+1-q,q})$ and $\|v^{(q)}\|^2 \leq c_3\|d\omega\|^2$.

Proof: suppose the hypothesis of induction is satisfied for some $1 \leq q \leq p$ and let us show it holds for $q+1$. By $(b)_q$, Lemma III.3.2 and Lemma III.2.10, there exists $c_{p-q,q} \in \mathcal{C}^q(\mathcal{U}, \Lambda^{p-q})$ and $\mu > 0$ such that $dc_{p-q,q} = c_{p+1-q,q}$ and $\|c_{p-q,q}(I)\|^2_{L^2(U_I)} \leq \frac{2}{\mu}\|c_{p+1-q,q}(I)\|^2_{L^2(U_I)}$. Combined with $(b)_q$ this implies that $\|c_{p-q,q}\|^2 \leq \frac{2}{\mu}\|c_{p+1-q,q}\|^2 \leq c_1\|d\omega\|^2$ which is $(a)_{q+1}$. Let us consider now

$$c_{p-q,q+1} = (-1)^q(q+1)\delta c_{p-q,q}$$

then clearly $\delta c_{p-q,q+1} = 0$ and $dc_{p-q,q+1} = (-1)^q(q+1)\delta c_{p+1-q,q} = 0$ by $(b)_q$. Moreover, $\|c_{p-q,q+1}\|^2 \leq cst\|c_{p-q,q}\|^2 \leq c_2\|d\omega\|^2$ by $(a)_{q+1}$. This concludes the proof of $(b)_{q+1}$. Finally, if $v^{(q+1)} = v^{(q)} + W(c_{p-q,q})$ we obtain with $(c)_q$ and Lemma III.5.4 that

$$\begin{aligned} d\omega &= dv^{(q+1)} - d(W(c_{p-q,q})) + (-1)^q W(c_{p+1-q,q}) \\ &= dv^{(q+1)} - (q+1)W(\delta c_{p-q,q}) - (-1)^q W(dc_{p-q,q}) + (-1)^q W(c_{p+1-q,q}) \\ &= dv^{(q+1)} + (-1)^{q+1} W(c_{p-q,q+1}). \end{aligned}$$

Finally, thanks to Lemma III.5.5, $(c)_q$ and $(a)_{q+1}$ we obtain that $\|v^{(q+1)}\|^2 \leq cst(\|v^{(q)}\|^2 + \|c_{p-q,q}\|^2) \leq c_3\|d\omega\|^2$. This concludes the induction.

End of the induction: (for $q = p+1$) we get $c_{0,p+1} \in \mathcal{C}^{p+1}(\mathcal{U}, \Lambda^0)$ such that $dc_{0,p+1} = 0$. This implies in particular that $c_{0,p+1} \in i(\mathcal{C}^{p+1}(\mathcal{U}))$. Moreover by the proof of the de Rham Theorem seen in Section III.2.3, the cochain $c_{0,p+1}$ represents the same cohomology class as $d\omega$ i.e. there exists $\gamma \in \mathcal{C}^p(\mathcal{U})$ such that $i(\delta\gamma) = c_{0,p+1}$.

Definition III.3.4 *We define $\mathcal{D}\omega$ as the unique Čech p-cochain in $\delta^*\mathcal{C}^{p+1}(\mathcal{U})$ such that $i\left(\delta\mathcal{D}(\omega)\right) = c_{0,p+1}$.*

We prove now $(i)_\mathcal{D}$ and $(ii)_\mathcal{D}$. Firstly, by $(b)_{p+1}$ of the induction we get that there exists a constant c_1 depending only on n, p, κ and ε such that

$$\|\delta\mathcal{D}(\omega)\|^2 \leq cst\|c_{0,p+1}\|^2 \leq c_1\|d\omega\|^2$$

and this proves $(i)_\mathcal{D}$. Secondly, by $(c)_{p+1}$ we can write

$$d\omega = dv^{(p+1)} + (-1)^{p+1}W(\delta\mathcal{D}(\omega)) = dv^{(p+1)} + \frac{(-1)^{p+1}}{p+1}dW(\mathcal{D}(\omega)) \qquad (47)$$

where we used Lemma III.5.4 and the fact that $d(i(\mathcal{D}(\omega))) = 0$ in the last equality. Moreover, as ω is coexact, and if $coex(\cdot)$ denotes the coexact part of a form given by the Hodge decomposition, we deduce that

$$\omega = coex(v^{(p+1)}) + \frac{(-1)^{p+1}}{p+1}coex\left(W(\mathcal{D}(\omega))\right).$$

Therefore, by Lemma III.5.5 and using this last equality we obtain

$$\|\mathcal{D}(\omega)\| \geq cst\|W(\mathcal{D}(\omega))\| \geq cst(\|\omega\| - \|v^{(p+1)}\|). \qquad (48)$$

Finally, by $(c)_{p+1}$ there exists C' depending only on n, p, κ and ε such that $\|\mathcal{D}(\omega)\| \geq cst(\|\omega\| - C'\|d\omega\|)$. Let then $\Lambda = \frac{1}{4C'^2}$ so that if $\|d\omega\|^2 \leq \Lambda\|\omega\|^2$ then $\|\mathcal{D}(\omega)\| \geq c_2\|\omega\|$ which is the requested inequality in $(ii)_\mathcal{D}$. □

III.3.2 From Čech cochains to smooth forms

In this section, we are going to construct

$$\mathcal{S} : \delta^*\mathcal{C}^{p+1}(\mathcal{U}) \to d^*\Lambda^{p+1}(M)$$

such that there exist positive constants c_1', c_2' and Λ' depending only on n, p, κ and ε such that

$(i)_{\mathcal{S}}$ $\|d\mathcal{S}(c)\|^2 \leq c_1'\|\delta c\|^2$, for any $c \in \delta^*\mathcal{C}^{p+1}(\mathcal{U})$,

$(ii)_{\mathcal{S}}$ $\|\mathcal{S}c\|^2 \geq c_2'\|c\|^2$, for any $c \in \delta^*\mathcal{C}^{p+1}(\mathcal{U})$ satisfying $\|\delta c\|^2 \leq \Lambda'\|c\|^2$.

The construction of $\mathcal{S}$ is similar to the construction of $\mathcal{D}$. The main difference is that the Whitney map is not the suitable tool to obtain $(ii)_{\mathcal{S}}$. So we have to do a first induction to construct $\mathcal{S}$ and a second induction (slightly different) to prove $(ii)_{\mathcal{S}}$. We begin by adjusting Lemma III.2.5 to our purpose.

Lemma III.3.5 *Let $\mathcal{U}$ be a contractible cover and $\{\varphi_j\}$ a partition of unity subordinated to $\mathcal{U}$. Let $\nu > 0$ such that $|\{j : U_j \cap U_I \neq \emptyset\}| \leq \nu$ for any $I \in S_k(\mathcal{U})$ and any $k = 0, \ldots, n$. Let $c \in \mathcal{C}^q(\mathcal{U}, \Lambda^p)$ $(q \geq 1)$ such that $\delta c = 0$. Then there exists $b \in \mathcal{C}^{q-1}(\mathcal{U}, \Lambda^p)$ such that $\delta b = c$ and there exist positive constants c_1, c_2 depending only on ν and on a bound on $\|d\varphi_j\|_\infty$ such that*

(i) $\|b\|^2 \leq c_1\|c\|^2$

(ii) $\|db\|^2 \leq c_2(\|c\|^2 + \|dc\|^2)$

Proof: a suitable b is given by Lemma A.4.1 in [34] and defined by

$$b(I) = \sum_{j \text{ s.t. } U_j \cap U_I \neq \emptyset} \varphi_j \cdot c(\{j\} \cup I)$$

so that b verifies already $\delta b = c$. Then (i) is an immediate consequence of the definition of b and ν. It remains to show (ii). We have $\|db\|^2 = \sum_{I \in S_{q-1}(\mathcal{U})} \|db(I)\|^2$. Moreover

$$\begin{aligned} \|db(I)\|^2 &= \left\| \sum_{j \text{ s.t. } U_j \cap U_I \neq \emptyset} d\varphi_j \wedge c(\{j\} \cup I) + \varphi_j dc(\{j\} \cup I) \right\|^2 \\ &\leq 2\nu \sum_{j \text{ s.t. } U_j \cap U_I \neq \emptyset} \|d\varphi_j \wedge c(\{j\} \cup I)\|^2 + \|\varphi_j dc(\{j\} \cup I)\|^2 \end{aligned}$$

and this implies the claim. □

Remark III.3.6 *In the sequel, we will consider a partition of unity $\{\varphi_j\}$ subordinated to an open cover made of balls of radius ε, so that we can find a bound on $\|d\varphi_j\|_\infty$ depending only on ε. In particular, this bound will be replaced by a constant depending only on ε.*

Construction by induction of $\mathcal{S}(\cdot)$

Let us now proceed to the construction of $\mathcal{S}$ and to the proof of $(i)_{\mathcal{S}}$. Let $c \in \delta^*\mathcal{C}^{p+1}(\mathcal{U})$. Then δc is an exact Čech $(p+1)$-cochain.
First step of induction: define $c_{0,p+1} \in \mathcal{C}^{p+1}(\mathcal{U}, \Lambda^0)$ by $c_{0,p+1} = i(\delta c)$ i.e. $c_{0,p+1}(I) = \delta c(I)$ for any $I \in S_{p+1}(\mathcal{U})$. Clearly, $\delta c_{0,p+1} = 0 = dc_{0,p+1}$. Then there exist positive constants c_1', c_2' depending only on n, p, κ and ε such that

$(a')_1$ there exists $c_{0,p} \in \mathcal{C}^p(\mathcal{U}, \Lambda^0)$ such that $\delta c_{0,p} = c_{0,p+1}$ and $\|c_{0,p}\|^2 \leq c_1'\|\delta c\|^2$.

$(b')_1$ Let $c_{1,p} = dc_{0,p}$. Then $\delta c_{1,p} = 0$ and $\|c_{1,p}\|^2 \leq c_2'\|\delta c\|^2$.

Indeed, $(a')_1$ is a direct consequence of Lemma III.3.5 as $\delta c_{0,p+1} = 0$ and of (49). The bound on the norm of $dc_{0,p}$ follows also from Lemma III.3.5 as $dc_{0,p+1} = 0$. Finally, we have $\delta c_{1,p} = d\delta c_{o,p} = dc_{0,p+1} = 0$.
Induction hypothesis: (for $1 \leq q < p+1$) there exist positive constants c_1', c_2' depending only on n, p, κ and ε such that

$(a')_q$ there exists $c_{q-1,p+1-q} \in \mathcal{C}^{p+1-q}(\mathcal{U}, \Lambda^{q-1})$ such that $\delta c_{q-1,p+1-q} = c_{q-1,p+1-(q-1)}$ and $\|c_{q-1,p+1-q}\|^2 \leq c_1'\|\delta c\|^2$.

$(b')_q$ Let $c_{q,p+1-q} = dc_{q-1,p+1-q}$. Then $\delta c_{q,p+1-q} = 0$ and $\|c_{q,p+1-q}\|^2 \leq c_2'\|\delta c\|^2$.

Proof: suppose the hypothesis of induction is verified for some $1 \leq q \leq p$ and let us show it holds for $q+1$. By $(b')_q$ and by Lemma III.3.5 there exists $c_{q,p-q} \in \mathcal{C}^{p-q}(\mathcal{U}, \Lambda^q)$ such that $\delta c_{q,p-q} = c_{q,p+1-q}$ and $\|c_{q,p-q}\|^2 \leq cst\|c_{q,p+1-q}\|^2$. Combined with $(b')_q$, this implies $(a)_{q+1}$. Moreover, let us consider $c_{q+1,p-q} = dc_{q,p-q}$. Then, by definition of $c_{q,p+1-q}$ we have $\delta c_{q+1,p-q} = d\delta c_{q,p-q} = dc_{q,p+1-q} = 0$. Finally, by Lemma III.3.5, we have $\|c_{q+1,p-q}\|^2 \leq cst(\|c_{q,p+1-q}\|^2 + \|dc_{q,p+1-q}\|^2)$. As we have $dc_{q,p+1-q} = 0$ and by $(b')_q$, we get $\|c_{q+1,p-q}\|^2 \leq c_2'\|\delta c\|^2$. This concludes the induction.
End of the induction: (for $q = p+1$) we obtain $c_{p+1,0} \in \mathcal{C}^0(\mathcal{U}, \Lambda^{p+1})$ such that $\delta c_{p+1,0} = 0$. This implies that $c_{p+1,0}$ is the restriction of a well-defined $(p+1)$-form and by the de Rham Theorem as δc is exact, the 0-cochain $c_{p+1,0}$ is exact and is the restriction of an exact $(p+1)$-form.

Definition III.3.7 *Let $\mathcal{S}(c) \in d^*\Lambda^{p+1}(M)$ be the unique coexact p-form such that $r(d\mathcal{S}(c)) = c_{p+1,0}$.*

An immediate consequence of the induction is $(i)_{\mathcal{S}}$. Indeed, from $(b')_{p+1}$ and Remark III.3.3 follows that there exists a positive constant c_1' depending only on n, p, κ and ε such that $\|d\mathcal{S}(c)\|^2 \leq c_1'\|\delta c\|^2$.
Let us now proceed to a second induction in order to prove $(ii)_{\mathcal{S}}$. The goal is to construct $b \in \mathcal{C}^p(\mathcal{U})$ such that $\delta b = \pm\delta c$ and $\|b\| \leq cst(\|\mathcal{S}(c)\| + \|\delta c\|)$ where cst is a positive constant depending only on n, p, κ and ε. These are in fact the corresponding equations for (47) and (48) in the discretizing part. In the induction, we will use the $c_{r,s}$ appearing in the construction of $\mathcal{S}$.
First step of induction: define $b_{p,0} = r(\mathcal{S}(c)) - c_{p,0} \in \mathcal{C}^0(\mathcal{U}, \Lambda^p)$. We have $db_{p,0} = c_{p+1,0} - dc_{p,0} = 0$. Then there exist positive constants c_1'', c_2'' depending only on n, p, κ and ε such that

$(a'')_1$ there exists $b_{p-1,0} \in \mathcal{C}^0(\mathcal{U}, \Lambda^{p-1})$ such that $db_{p-1,0} = b_{p,0}$ and $\|b_{p-1,0}\|^2 \leq c_1''(\|\mathcal{S}(c)\| + \|\delta c\|)$.

$(b'')_1$ Let $b_{p-1,1} = \delta b_{p-1,0} + c_{p-1,1}$. Then we have $db_{p-1,1} = 0$ and $\|b_{p-1,1}\| \leq c_2''(\|\mathcal{S}(c)\| + \|\delta c\|)$.

Indeed, as $p \geq 1$ and $db_{p,0} = 0$, by Lemma III.3.2 there exists $b_{p-1,0} \in \mathcal{C}^0(\mathcal{U}, \Lambda^{p-1})$ such that $db_{p-1,0} = b_{p,0}$ and $\|b_{p-1,0}\| \leq cst\|b_{p,0}\|$. By definition of $b_{p,0}$ and by $(a')_{p+1}$ of the previous induction we obtain then $(a'')_1$. Let us consider now $b_{p-1,1} = \delta b_{p-1,0} + c_{p-1,1}$. Then we have $db_{p-1,1} = \delta b_{p,0} + c_{p,1} = -\delta c_{p,0} + c_{p,1} = 0$. Finally, by construction and by (49) $\|b_{p-1,1}\| \leq cst(\|b_{p-1,0}\| + \|c_{p-1,1}\|)$. This last inequality combined with $(a'')_1$ and $(a')_p$ leads to $(b'')_1$.
Induction hypothesis: (for $1 \leq q < p-1$) there exist positive constants c_1'', c_2'' depending only on n, p, κ and ε such that

$(a'')_q$ there exists $b_{p-q,q-1} \in \mathcal{C}^{q-1}(\mathcal{U}, \Lambda^{p-q})$ such that $db_{p-q,q-1} = b_{p-(q-1),q-1}$ and $\|b_{p-q,q-1}\|^2 \leq c_1''(\|\mathcal{S}(c)\| + \|\delta c\|)$.

$(b'')_q$ Let $b_{p-q,q} = \delta b_{p-q,q-1} + (-1)^{q+1}c_{p-q,q}$. Then we have $db_{p-q,q} = 0$ and $\|b_{p-q,q}\| \leq c_2''(\|\mathcal{S}(c)\| + \|\delta c\|)$.

Proof: suppose the induction hypothesis holds for some $1 \leq q \leq p-1$ and let us show it holds for $q+1$. By $(b'')_q$ and Lemma III.3.2 there exists $b_{p-(q+1),q} \in \mathcal{C}^q(\mathcal{U}, \Lambda^{p-(q+1)})$ such that $db_{p-(q+1),q} = b_{p-q,q}$ and $\|b_{p-(q+1),q}\|^2 \leq cst\|b_{p-q,q}\|^2$ and it suffices to use $(b'')_q$ to obtain $(a'')_{q+1}$. Then consider $b_{p-(q+1),q+1} = \delta b_{p-(q+1),q} + (-1)^q c_{p-(q+1),q+1}$. We have

$$\begin{aligned} db_{p-(q+1),q+1} &= \delta b_{p-q,q} + (-1)^q c_{p-q,q+1} \\ &= \delta(\delta b_{p-q,q-1} + (-1)^{q+1}c_{p-q,q}) + (-1)^q \delta c_{p-q,q} \\ &= 0. \end{aligned}$$

Finally, by construction of $b_{p-(q+1),q+1}$ we have

$$\|b_{p-(q+1),q+1}\| \le cst(\|b_{p-(q+1),q}\| + \|c_{p-(q+1),q+1}\|)$$

and with $(a'')_{q+1}$ and $(a')_{p-q}$ we obtain $(b'')_{q+1}$. This ends the induction.
End of the induction: (for $q = p$) we obtain $b_{0,p} \in \mathcal{C}^p(\mathcal{U}, \Lambda^0)$ such that $db_{0,p} = 0$ i.e. $b_{0,p} \in \mathcal{C}^p(\mathcal{U})$ and $\delta b_{0,p} = (-1)^{p+1}\delta c_{0,p} = (-1)^{p+1}c_{0,p+1} = (-1)^{p+1}\delta c$. Hence, $b_{0,p}$ and c have same coexact part and as c is already coexact we obtain by $(b'')_p$, $\|c\| \le \|b_{0,p}\| \le cst(\|\mathcal{S}(c)\| + \|\delta c\|)$. In particular,

$$\|\mathcal{S}(c)\| \ge \frac{1}{cst}\|c\| - \|\delta c\|$$

then let $\Lambda' = \frac{1}{4cst^2}$ so that if $\|\delta c\|^2 \le \Lambda\|c\|^2$ then $\|\mathcal{S}(c)\| \ge c_2'\|c\|$. This ends the proof of $(ii)_{\mathcal{S}}$. □

III.3.3 Upper bounds on the spectra

Lemma III.3.8 *Let (M^n, g) be a compact connected Riemannian manifold and let $\mathcal{U}$ be a finite contractible open cover of M such that there exists $\nu > 0$ such that $|\{j : U_j \cap U_I \ne \emptyset\}| \le \nu$ for any $I \in S_q(\mathcal{U})$ and any $q \ge 0$. Then there exists a positive constant c depending only on ν and p such that $\lambda_{k,q}(\mathcal{U}) \le c$ for any $k = 1, \dots, |S_q(\mathcal{U})| - \check{b}_q(\mathcal{U})$.*

Proof: it suffices to show the result for the spectrum of $\delta^*\delta$ on $\delta^*\mathcal{C}^{p+1}(\mathcal{U})$. We are going to show that there exists a positive constant depending only on ν and p such that for any $b \in \mathcal{C}^p(\mathcal{U})$

$$\|\delta b\|^2 \le cst\|b\|^2 \tag{49}$$

and then the variational characterization of the spectrum of $\delta^*\delta$ will imply the claim. Recall that $\delta b(I) = \sum_{i\in I} \epsilon(i, I\backslash i) b(I\backslash i)$ where $\epsilon(i, I\backslash i)$ denotes the signature of the permutation ordering $\{i\} \cup (I\backslash i)$ to obtain I and $I \in S_{p+1}(\mathcal{U})$. Hence

$$|\delta b(I)|^2 \le (p+2)\sum_{i\in I} |b(I \setminus i)|^2.$$

This implies that

$$\begin{aligned}\|\delta b\|^2 = \sum_{I\in S_{p+1}(\mathcal{U})} |\delta b(I)|^2 &\le (p+2) \sum_{I\in S_{p+1}(\mathcal{U})} \sum_{i\in I} |b(I \setminus i)|^2 \\ &\le (p+2)\nu \sum_{J\in S_p(\mathcal{U})} |b(J)|^2 = (p+2)\nu\|b\|^2\end{aligned}$$

which is the claim. □

Lemma III.3.9 *Let $(M, g) \in \mathbb{M}(n, \kappa, r_0)$ and X an ε-discretization with $0 < \varepsilon \leq r_0$. Let $1 \leq p \leq n-1$. Then there exists a positive constant c' depending only on n, p, κ and ε such that $\lambda_{k,p}(M) \leq c'$ for any $k \leq |S_p(\mathcal{U}_X)| - \check{b}_p(\mathcal{U}_X)$.*

Proof: it suffices to show the result for $k = |S_p(\mathcal{U}_X)| - \check{b}_p(\mathcal{U}_X)$. By a theorem of Abresch (see [22], Theorem 1.12) there exists a Riemannian metric $\tilde{g}$ on M such that

(a) $e^{-\frac{1}{4}} g \leq \tilde{g} \leq e^{\frac{1}{4}} g$

(b) $|\nabla^g - \nabla^{\tilde{g}}| \leq \frac{1}{4}$

(c) $|K_{\tilde{g}}| \leq \tilde{\kappa}(n, \kappa)$ and $|\nabla^{\tilde{g}} R_{\tilde{g}}| \leq K(n, \kappa)$

where $\tilde{\kappa}$ and K depend only on n and κ. By Proposition 3.3. of [25], there exist a positive constant c depending only on $e^{\frac{1}{4}}$ such that

$$\lambda_{k,p}(M, g) \leq c \lambda_{k,p}(M, \tilde{g}).$$

Therefore it suffices to show the claim for $(M, \tilde{g})$. By Remark III.2.8 and by construction of $\tilde{g}$, there exists a positive constant d depending only on n, p, κ, ε such that $|S_p(\mathcal{U}_X)| \leq d Vol(M, \tilde{g})$. Moreover, there exist $\alpha > 0$ depending only on p, n, κ and ε such that if Y is an α-discretization of $(M, \tilde{g})$ then $|Y| \geq |S_p(\mathcal{U}_X)|$ and $\check{b}_p(\mathcal{U}_Y) = \check{b}_p(\mathcal{U}_X)$. Consider then the disjoint balls (for $\tilde{g}$) centered at $y \in Y$ of radius $\frac{\alpha}{2}$. From Proposition 2.3. of [25], on any of these balls there exists a p-form ω_y which is zero on the boundary of the ball, so that we can extend ω_y by zero to obtain a p-form on M also denoted ω_y such that

$$\frac{\|d\omega_y\|_{\tilde{g}}^2 + \|d_{\tilde{g}}^* \omega_y\|_{\tilde{g}}^2}{\|\omega_y\|_{\tilde{g}}^2} \leq \mu(n, p, \kappa, \varepsilon) \tag{50}$$

where $\mu(n, p, \kappa, \varepsilon)$ is a positive constant depending only on n, p, κ and ε. Moreover, we can choose ω_y such that $\|\omega_y\| = 1$.

Let then V the vector subspace of p-forms spanned by $\{\omega_y : y \in Y\}$. By construction, ω_y is orthogonal to ω_x if $x \neq y$. In particular, V is of dimension $|Y|$. Therefore, by the variational characterization of the spectrum, we obtain

$$\lambda_{|Y| - \check{b}_p(\mathcal{U}_Y), p}(M, \tilde{g}) \leq \max \left\{ \frac{\|d\omega\|_{\tilde{g}}^2 + \|d_{\tilde{g}}^* \omega\|_{\tilde{g}}^2}{\|\omega\|_{\tilde{g}}^2} : \omega \in V \setminus \{0\} \right\}. \tag{51}$$

Furthermore, if $\omega = \sum_{y \in Y} a_y \omega_y$, then as the balls centered on Y of radius $\frac{\alpha}{2}$ are disjoint $\|\omega\|_{\tilde{g}}^2 \geq \sum_{y \in Y} a_y^2$ and combined with (50) this implies that

$$\|d\omega\|_{\tilde{g}}^2 \leq \sum_{y \in Y} a_y^2 \|d\omega_y\|_{\tilde{g}}^2 \leq \mu \|\omega\|_{\tilde{g}}^2 \tag{52}$$

and

$$\|d^*_{\tilde{g}}\omega\|^2_{\tilde{g}} \leq \sum_{y\in Y} a_y^2 \|d^*_{\tilde{g}}\omega_y\|^2_{\tilde{g}} \leq \mu\|\omega\|^2_{\tilde{g}}. \tag{53}$$

It suffices then to introduce (52) and (53) in (51) to obtain that

$$\lambda_{|Y|-\check{b}_p(\mathcal{U}_Y),p}(M,\tilde{g}) \leq 2\mu$$

and in particular that $\lambda_{k,p}(M,g) \leq 2c\mu$, for $k \leq |S_p(\mathcal{U}_X)| - \check{b}_p(\mathcal{U}_X)$. $\square$

III.3.4 Proof of the main result

We prove now Theorem III.3.1. We will only proceed to the proof of the inequality $\lambda_{k,p}(M) \leq c_2\lambda_{k,p}(X)$ as the other inequality can be proved in the same way using the corresponding results. Recall it suffices to prove the result for d^*d on coexact forms and for $\delta^*\delta$ on coexact Čech cochains. We proceed in two steps. Let Λ' given by $(ii)_{\mathcal{S}}$.

First step: assume $\lambda^{\delta^*}_{k,p}(X) \geq \Lambda'$. Then, $\lambda^{d^*}_{k,p}(M) \leq \Lambda'^{-1}\lambda^{\delta^*}_{k,p}(X)\lambda^{d^*}_{k,p}(M)$ and by Lemma III.3.9 we obtain $\lambda^{d^*}_{k,p}(M) \leq \Lambda'^{-1}c'\lambda^{\delta^*}_{k,p}(X)$ which is the claim.

Second step: assume now $\lambda^{\delta^*}_{k,p}(X) \leq \Lambda'$. Let us consider $c_1,\ldots,c_k \in \delta^*\mathcal{C}^{p+1}(\mathcal{U},)$ the Čech $\lambda^{\delta^*}_{1,p}(X),\ldots,\lambda^{\delta^*}_{k,p}(X)$-eigencochains such that $(c_i,c_j) = \delta_{ij}$. Denote by V^k the k-dimensional vector subspace of $\delta^*\mathcal{C}^{p+1}(\mathcal{U})$ they span. By the variational characterization of the spectrum we have

$$\lambda^{\delta^*}_{k,p}(X) = \max\left\{\frac{\|\delta c\|^2}{\|c\|^2} : c \in V^k \setminus \{0\}\right\}.$$

Let us consider now $\mathcal{S}V^k$ the vector subspace of $d^*\Lambda^{p+1}(M)$ spanned by $\{\mathcal{S}(c_1),\ldots,\mathcal{S}(c_k)\}$. Then if $\mathcal{S}(c) \in \mathcal{S}V^k$, $\mathcal{S}(c) = \sum_{i=1}^k a_i\mathcal{S}(c_i)$ with $c = \sum_{i=1}^k a_ic_i \in V^k$. So that we have $\|\delta c\|^2 \leq \lambda^{\delta^*}_{k,p}(X)\|c\|^2 \leq \Lambda'\|c\|^2$. Therefore, by $(ii)_{\mathcal{S}}$ we obtain

$$\|\mathcal{S}(c)\|^2 \geq c_2'\|c\|^2 \tag{54}$$

and this says in particular that $\mathcal{S}V^k$ is of dimension k. Using the variational characterization of $\lambda^{d^*}_{k,p}(M)$ we get

$$\begin{aligned}\lambda^{d^*}_{k,p}(M) &\leq \max\left\{\frac{\|d\omega\|^2}{\|\omega\|^2} : \omega \in \mathcal{S}V^k \setminus \{0\}\right\}\\ &= \max\left\{\frac{\|d\mathcal{S}(c)\|^2}{\|\mathcal{S}(c)\|^2} : c \in V^k \setminus \{0\}\right\}.\end{aligned}$$

Finally, (54) and $(i)_{\mathcal{S}}$ imply that $\frac{\|d\mathcal{S}(c)\|^2}{\|\mathcal{S}(c)\|^2} \leq \frac{c_1'}{c_2'} \frac{\|\delta c\|^2}{\|c\|^2}$ so that we obtain

$$\lambda_{k,p}^{d^*}(M) \leq \frac{c_1'}{c_2'} \max \left\{ \frac{\|\delta c\|^2}{\|c\|^2} : c \in V^k \setminus \{0\} \right\} = \frac{c_1'}{c_2'} \lambda_{k,p}^{\delta^*}(X) \tag{55}$$

which concludes the proof. □

III.4 Applications

In this section, we develop several consequences of Theorem III.3.1 or of the methods used to prove Theorem III.3.1.

III.4.1 A lower bound for the spectrum of the Laplacian on differential forms

The goal of this section is to prove the following theorem.

Theorem III.4.1 *Let $(M,g) \in \mathbb{M}(n, \kappa, r_0)$. Let $1 \leq p \leq n-1$. Then there exist positive constants c, c' depending only on n, p, κ and r_0 such that*

$$\lambda_{1,p}(M) \geq \frac{c}{Vol(M) e^{c' Vol(M)}}$$

where $Vol(M)$ denotes the volume of (M,g).

By Theorem III.3.1, it suffices to choose a suitable discretization X of M and prove then a similar result for $\lambda_{1,p}(X)$. To that aim we need the following lemma.

Lemma III.4.2 *Let $A : \mathbb{R}^m \to \mathbb{R}^n$ be a linear operator with matrix coefficients (in the canonical bases) in $\{-1, 0, 1\}$. Suppose there exists an integer k such that any column and any row has at most k non-zero coefficients. Then, there exists $B : \mathbb{R}^n \to \mathbb{R}^m$ such that $ABAv = Av$ for any $v \in \mathbb{R}^m$ and*

$$\|Bu\|^2 \leq nk^{2n} \|u\|^2$$

for any $u \in \mathbb{R}^n$.

Remark III.4.3 *In [59], the author proves a similar result (see Lemma A.5 in [59]) but with a better constant for the matrix norm of B. He asserts that $\|Bu\|^2 \leq c(k)m\|u\|^2$. With the following example we will show that the proof* *of Trèves' result is not correct. Consider the matrix A with m columns and $m-1$ rows given by*

$$A = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & -1 \end{pmatrix}$$

and consider $v = \sum_{i=1}^{m} ie_i$ in $\mathbb{R}^m$. Then $Av = -\sum_{i=1}^{m-1} e_i$ in $\mathbb{R}^{m-1}$. So that $\|Av\|^2 = m-1$. An easy calculation shows that if we choose the $m-1$ first columns of A to span $Im(A)$ and if we consider the map B defined by Trèves, then $BAv = \sum_{i=1}^{m-1} -(m-i)e_i$ in $\mathbb{R}^m$. Hence $\|BAv\|^2 = \frac{(m-1)m(2m-1)}{6} = \frac{m(2m-1)}{6}\|Av\|^2$ which contradicts Lemma A.5 in [59] (here $k=2$). The assertion A.44 in [59] is wrong. It is not clear to us how we can correct this mistake. We think that we should replace k^{2n} by n^l for a suitable l in Lemma III.4.2 but we cannot prove it yet. Let us emphasize that the constant given by Trèves can not be suitable. The result of Trèves would imply a lower bound for the first positive eigenvalue of the combinatorial Laplacian on a graph with n vertices of the kind $\frac{cst}{n}$ (see Theorem III.4.4). But it is a well-known fact that the first positive eigenvalue of the combinatorial Laplacian (for functions) on a cyclic graph with n vertices behaves like $\frac{cst}{n^2}$.

Proof of Lemma III.4.2: let r be the dimension of $Im(A)$. Without lost of generality we can suppose that the r first columns $\{a_1, \dots a_r\}$ of A span $Im(A)$. Then define B as follows (as in Lemma A.5 of [59]). On the orthogonal complement of $Im(A)$ let $B = 0$. Moreover, if $u = Av$ then write u in the basis $\{a_1, \dots, a_r\}$ of $Im(A)$, $u = \sum_{i=1}^{r} u_i a_i$ and define $Bu = \sum_{i=1}^{r} u_i e_i$ where $\{e_i\}$ denotes the canonical basis of $\mathbb{R}^m$. An immediate consequence of the definition of B is that $ABAv = Av$. Moreover, $\|Bu\|^2 = \sum_{i=1}^{r} u_i^2$. Let us show now that

$$u_i^2 \leq k^{2n}\|u\|^2. \tag{56}$$

This will imply $\|Bu\|^2 \leq rk^{2n}\|u\|^2 \leq nk^{2n}\|u\|^2$ which is the claim.
We prove (56) for $i=1$. Let V_1 the vector space spanned by $\{a_2, \dots, a_r\}$ and let $V_1^\perp$ its orthogonal complement in $Im(A)$. Consider $P_1 : Im(A) \to V_1^\perp$ the orthogonal projection onto $V_1^\perp$. We have $P_1(u) = u_1 P_1(a_1)$ so that

$$u_1^2 = \frac{\|P_1(u)\|^2}{\|P_1(a_1)\|^2} \leq \frac{\|u\|^2}{\|P_1(a_1)\|^2}. \tag{57}$$

We can write $P_1(a_1) = a_1 + \alpha_2 a_2 + \ldots + \alpha_r a_r$ with $(P_1(a_1)|a_j) = 0$ for $j = 2, \ldots, r$ and $(P_1(a_1)|a_1) = \|P_1(a_1)\|^2$. In matrix form we obtain

$$\begin{pmatrix} \|a_1\|^2 & (a_1|a_2) & \ldots & (a_1|a_r) \\ (a_1|a_2) & \|a_2\|^2 & \ldots & (a_2|a_r) \\ \vdots & \vdots & \ddots & \vdots \\ (a_1|a_r) & (a_2|a_r) & \ldots & \|a_r\|^2 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha_2 \\ \vdots \\ \alpha_r \end{pmatrix} = \begin{pmatrix} \|P_1(a_1)\|^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and if we call P the matrix $r \times r$ above and Q the submatrix of P obtained by removing the first row and the first column of P we get that

$$\|P_1(a_1)\|^2 = \frac{|\det(P)|}{|\det(Q)|}.$$

As $\{a_1, \ldots a_r\}$ are linearly independent, $\det(P) \neq 0$. Moreover, P is a matrix with integer coefficients so that $|\det(P)| \geq 1$. It remains to find an upper bound for $|\det(Q)|$. So, we are going to prove by induction that the minors of P of size $l \times l$ are bounded above by k^{2l-1}.

The first step of induction asserts that the minors of P of size 1×1 are bounded above by k. This is a direct consequence of the assumption that each column of A has at most k non-zero coefficients. Suppose then that the minors of P of size $l \times l$ are bounded above by k^{2l-1}. Consider then D a minor of P of size $(l+1) \times (l+1)$. Then D can be written as

$$D = \sum_{j=1}^{l+1} c_j D_j$$

where $(c_1, \ldots, c_{l+1})$ is a part of a line of P and D_j is a minor of P of size $l \times l$. By construction of P, the coefficients c_j can be written as follows. There exists $1 \leq J \leq r$ such that

$$c_j = (a_J|a_{i_j}) \text{ for a suitable } i_j$$

so that

$$|D| = \left| \sum_{j=1}^{l+1} (a_J|a_{i_j}) D_j \right| = \left| \sum_{i=1}^{n} (a_J|e_i) \sum_{j=1}^{l+1} (e_i|a_{i_j}) D_j \right|.$$

But by assumption, the i^{th} row of A has at most k coefficients of absolute value 1 and by induction hypothesis we get $|\sum_{j=1}^{l+1} (e_i|a_{i_j}) D_j| \leq k \cdot k^{2l-1}$.

Moreover, by assumption the J^{th} column of A has at most k coefficients of absolute value 1 and with the previous remark this implies

$$|D| \leq k \cdot k \cdot k^{2l-1}$$

and this ends the induction. We apply then the result to $|\det(Q)|$ and we obtain $|\det(Q)| \leq k^{2r-3} \leq k^{2n}$. Finally, we deduce that

$$||P_1(a_1)||^2 \geq \frac{1}{k^{2n}}$$

and combined with (57) this implies (56). □

Theorem III.4.4 *Let $\mathcal{U}$ be a finite open cover of M compact. Let $p \geq 0$. Assume there exists ν such that $|\{j : U_j \cap U_I \neq \emptyset\}| \leq \nu$ for any $I \in S_q(\mathcal{U})$ and $q \geq 0$. Then there exist positive constants $c(\nu, p)$, $c'(\nu, p)$ depending only on ν and p such that*

$$\lambda_{1,p}(\mathcal{U}) \geq \frac{c(\nu,p)}{|\mathcal{U}| \cdot e^{c'(\nu,p)|\mathcal{U}|}} \ .$$

Proof: it suffices to prove the result for $\lambda_{1,p}^{\delta^*}(\mathcal{U})$. By the variational characterization of the spectrum, we have

$$\lambda_{1,p}^{\delta^*}(\mathcal{U}) = \min_V \max \left\{ \frac{||\delta c||^2}{||c||^2} : c \in V \setminus \{0\} \right\}$$

where V ranges over all 1-dimensional vector subspaces of $\delta^* \mathcal{C}^{p+1}(\mathcal{U})$. As in Proposition 3.1 of [25], we can get from the above characterization the following description

$$\lambda_{1,p}^{\delta^*}(\mathcal{U}) = \min_V \max \left\{ \frac{||\delta c||^2}{||b||^2} : \delta b = \delta c \ , \text{ and } \delta c \in V \right\}$$

where V ranges over all 1-dimensional vector subspaces of $\delta \mathcal{C}^p(\mathcal{U})$. In particular, if we consider V that realizes the minimum, then

$$\lambda_{1,p}^{\delta^*}(\mathcal{U}) = \max \left\{ \frac{||\delta c||^2}{||b||^2} : \delta b = \delta c \ , \text{ and } \delta c \in V \right\}. \tag{58}$$

Consider then the canonical basis of $\mathcal{C}^q(\mathcal{U})$ given by

$$\{e_I : S_q(\mathcal{U}) \to \mathbb{R}, I \in S_q(\mathcal{U}) \text{ such that } e_I(J) = \delta_{IJ}\}.$$

In this bases, the matrix of $\delta : \mathcal{C}^p(\mathcal{U}) \to \mathcal{C}^{p+1}(\mathcal{U})$ has coefficients in $\{-1, 0, 1\}$ and has at most $K(\nu, p) = \max\{\nu, p+2\}$ non-zero coefficients by row and by column. Hence we can apply Lemma III.4.2 to δ to obtain that for any $c \in \mathcal{C}^p(\mathcal{U})$, there exists $b \in \mathcal{C}^p(\mathcal{U})$ such that $\delta b = \delta c$ and

$$\|b\|^2 \leq |S_{p+1}(\mathcal{U})| K(\nu,p)^{|S_{p+1}(\mathcal{U})|} \|\delta c\|^2. \tag{59}$$

Finally, if we introduce (59) in (58) and by Remark III.2.8, we obtain

$$\lambda_{1,p}^{\delta^*}(\mathcal{U}) \geq \frac{1}{|S_{p+1}(\mathcal{U})| K(\nu,p)^{|S_{p+1}(\mathcal{U})|}} \geq \frac{c(\nu,p)}{|\mathcal{U}| \cdot e^{c'(\nu,p)|\mathcal{U}|}} . \ \square$$

Proof of Theorem III.4.1: let $(M,g) \in \mathbb{M}(n,\kappa,r_0)$ and X a $\frac{\rho_0}{4}$-discretization of M (where ρ_0 is given by Lemma III.2.9). By Theorem III.3.1, there exists $c_1(n,p,\kappa,r_0) > 0$ such that

$$\lambda_{1,p}(M,g) \geq c_1 \lambda_{1,p}(X). \tag{60}$$

Moreover, by Theorem III.4.4 there exists $c_2(n,p,\kappa,r_0) > 0$ such that

$$\lambda_{1,p}(X) \geq \frac{c_2}{|\mathcal{U}| \cdot e^{|\mathcal{U}|}}. \tag{61}$$

Finally, by Remark III.2.8 there exists $c_3(n,p,\kappa,r_0) > 0$ such that

$$|\mathcal{U}| \leq c_3 Vol(M). \tag{62}$$

To conclude, put (60), (61) and (62) together to obtain that there exist $c(n,p,\kappa,r_0) > 0$, $c'(n,p,\kappa,r_0) > 0$ such that

$$\lambda_{1,p}(M,g) \geq \frac{c}{Vol(M) e^{c' Vol(M)}}$$

and this ends the proof. $\square$

III.4.2 Whitney forms: a natural way of smoothing

As suggested in [27], a candidate for the smoothing operator should be given by Whitney forms in the following way. Let

$$\tilde{\mathcal{S}} : \delta^* \mathcal{C}^{p+1}(\mathcal{U}) \to d^* \Lambda^{p+1}(M) \ , \ c \mapsto \tilde{\mathcal{S}}(c) = coex(W(c))$$

where W is the Whitney map (see Appendix III.5.1). The results of Dodziuk and Patodi in [27] concerning Whitney forms can not be used in our context as their approximations (obtained thanks to the heat kernel) involve the manifold itself. More precisely, the constants there depend on the volume of the manifold.

Here, we show that there exist positive constants $\tilde{c}_1$, $\tilde{c}_2$ and $\tilde{\Lambda}$ depending only on n, p, κ and ε such that

$(i)_{\tilde{\mathcal{S}}}$ $\|d\tilde{\mathcal{S}}(c)\|^2 \leq \tilde{c}_1\|\delta c\|^2$, for any $c \in \delta^*\mathcal{C}^{p+1}(\mathcal{U})$,

$(ii)_{\tilde{\mathcal{S}}}$ $\|\tilde{\mathcal{S}}c\|^2 \geq \tilde{c}_2\|c\|^2$, for any $c \in \delta^*\mathcal{C}^{p+1}(\mathcal{U})$ satisfying $\|\delta c\|^2 \leq \tilde{\Lambda}\|c\|^2$.

The inequality $(i)_{\tilde{\mathcal{S}}}$ is a direct consequence of Lemma III.5.4 and Lemma III.5.5. Indeed, as $dc = 0$ we have

$$d\tilde{\mathcal{S}}(c) = dW(c) = (p+1)W(\delta c)$$

and Lemma III.5.5 leads to $(i)_{\tilde{\mathcal{S}}}$.

The second inequality is less obvious and it can be shown adding a point to the first induction in the construction of $\mathcal{S}$ in Section III.3.2. The idea is to construct a p-form $u^{(0)}$ linking $\mathcal{S}(c)$ and $\tilde{\mathcal{S}}(c)$ playing the same role as $v^{(p)}$ in the construction of $\mathcal{D}$ (see Section III.3.1). Then the control on the norm of $\mathcal{S}(c)$ (see $(ii)_{\mathcal{S}}$) and a control on the norm of $u^{(0)}$ will imply the desired inequality.

Proof of $(ii)_{\tilde{\mathcal{S}}}$: in the "first step of induction" (of Section III.3.2), add

$(c')_1$ there exists a positive constant c_3' depending only on n, p, κ and ε such that if

$$u^{(p)} = (-1)^{p+2}\frac{1}{p+1}W(c_{0,p})$$

then $\|u^{(p)}\|^2 \leq c_3'\|\delta c\|^2$ and

$$dW(c) = (-1)^{p+2}(p+1)\left(du^{(p)} + \frac{(-1)^{(p+2)(p+1)}}{p+1}W(c_{1,p})\right).$$

Indeed, by Lemma III.5.5 and $(a')_1$, $\|u^{(p)}\|^2 \leq cst\|c_{0,p}\|^2 \leq c_3'\|\delta c\|^2$. Moreover, by Lemma III.5.4 and $(b')_1$

$$\begin{aligned} du^{(p)} &= \frac{(-1)^{p+2}}{p+1}\left((p+1)W(c_{0,p+1}) + (-1)^pW(c_{1,p})\right) \\ &= \frac{(-1)^{p+2}}{p+1}\left(dW(c) - (-1)^{p+1}W(c_{1,p})\right). \end{aligned}$$

The induction hypothesis gets

$(c')_q$ there exists a positive constant c_3' depending only on n, p, κ and ε such that if

$$u^{(p+1-q)} = u^{(p+1-(q-1))} + \frac{(-1)^{p+2}(-1)^{p+1}\dots(-1)^{p+2-(q-1)}}{(p+1)p(p-1)\dots(p+2-q)}W(c_{q-1,p+1-q})$$

then $\|u^{(p+1-q)}\|^2 \leq c_3'\|\delta c\|^2$ and

$$\frac{(-1)^{p+2}}{p+1}dW(c) = du^{(p+1-q)} + \frac{(-1)^{p+2}\dots(-1)^{p+2-q}}{(p+1)\dots(p+2-q)}W(c_{q,p+1-q}).$$

Then, the proof goes as follows. Let us consider

$$u^{(p-q)} = u^{(p+1-q)} + \frac{(-1)^{p+2}(-1)^{p+1}\dots(-1)^{p+2-q}}{(p+1)p(p-1)\dots(p+2-(q+1))}W(c_{q,p-q}).$$

Then, by $(c')_q$, by Lemma III.5.5 and by $(a')_{q+1}$, we obtain $\|u^{(p-q)}\|^2 \leq c'_3\|\delta c\|^2$. Moreover, by $(c')_q$ and Lemma III.5.4 we have

$$\begin{aligned}
\frac{(-1)^{p+2}}{p+1}dW(c) &= du^{(p-q)} - \frac{(-1)^{p+2}\dots(-1)^{p+2-q}}{(p+1)\dots(p+2-(q+1))}dW(c_{q,p-q}) \\
&\quad + \frac{(-1)^{p+2}\dots(-1)^{p+2-q}}{(p+1)\dots(p+2-q)}W(c_{q,p+1-q}) \\
&= du^{(p-q)} - \frac{(-1)^{p+2}\dots(-1)^{p+2-q}}{(p+1)\dots(p+2-q)}W(c_{q,p+1-q}) \\
&\quad + \frac{(-1)^{p+2}\dots(-1)^{p+2-q}}{(p+1)\dots(p+2-(q+1))}(-1)^{p+1-q}W(c_{q+1,p-q}) \\
&\quad + \frac{(-1)^{p+2}\dots(-1)^{p+2-q}}{(p+1)\dots(p+2-q)}W(c_{q,p+1-q})
\end{aligned}$$

and the claim follows.
At the end of the induction (for $q = p+1$), we obtain a p-form $u^{(0)}$ such that $\|u^{(0)}\|^2 \leq c'_3\|\delta c\|^2$ and

$$\begin{aligned}
dW(c) &= (-1)^p(p+1)\left(du^{(0)} + k(p)W(c_{p+1,0})\right) \\
&= (-1)^p(p+1)\left(du^{(0)} + k(p)W(r(d\mathcal{S}(c)))\right) \\
&= (-1)^p(p+1)\left(du^{(0)} + k(p)d(\mathcal{S}(c))\right)
\end{aligned}$$

where $k(p)$ is a constant depending only on p. Moreover, as $\mathcal{S}(c)$ is a coexact p-form, this implies

$$coex(W(c)) = (-1)^p(p+1)\left(coex(u^{(0)}) + k(p)\mathcal{S}(c)\right)$$

so that

$$\begin{aligned}
\|coex(W(c))\| &\geq (p+1)|k(p)|\cdot\|\mathcal{S}(c)\| - (p+1)\|u^{(0)}\| \\
&\geq (p+1)|k(p)|\cdot\|\mathcal{S}(c)\| - (p+1)(c'_3)^{\frac{1}{2}}\|\delta c\|.
\end{aligned}$$

But, by $(ii)_{\mathcal{S}}$, if $\|\delta c\|^2 \leq \Lambda'\|c\|^2$ then $\|\mathcal{S}(c)\| \geq (c'_2)^{\frac{1}{2}}\|c\|$. Therefore,

$$\|coex(W(c))\| \geq (p+1)|k(p)|(c'_2)^{\frac{1}{2}}\left(\|c\| - \sqrt{\frac{c'_3}{c'_2k(p)^2}}\|\delta c\|\right)$$

Finally, if $\|\delta c\|^2 \leq \tilde{\Lambda}\|c\|^2$, with $\tilde{\Lambda} = \min\left\{\Lambda', \frac{k(p)^2 c_2'}{4c_3'}\right\}$, then

$$\|coex(W(c))\| \geq \frac{1}{2}(p+1)|k(p)|(c_2')^{\frac{1}{2}}\|c\|$$

which is the desired inequality in $(ii)_{\tilde{S}}$. □

III.4.3 Another proof of "McGowan lemma"

In [50], the author gives a lower bound for the N^{th} eigenvalue of Δ on exact 2-forms on a compact Riemannian manifold M (see Lemma 2.3 in [50]) where N depends on an open cover of M. In particular, if the open cover is contractible then $N-1$ is the number of non-empty intersections of triples of open sets in the open cover. The lower bound depends then essentially on lower bounds for the smallest positive eigenvalue of Δ on exact forms on the open sets of the cover, on the intersection of pairs of such open sets and on the intersection of triples of such open sets. The proof of McGowan relies also on the double complexe of Čech - de Rham and can be compared to the induction done in Section III.3.1 to construct the discretizing operator $\mathcal{D}$. So it is not so surprising that we obtain the following generalization of the lemma. The main difference is that in our technique, if the discretization is of sufficiently small mesh then Lemma III.2.10 gives the lower bound for the spectrum on the intersections. But, then N can get quite large as it is comparable to the number of open sets in the open cover. Let us now state and prove the result.

Lemma III.4.5 *Let $n \geq 1$, $\kappa \geq 0$ and $r_0 > 0$. Then there exists a positive constant $\lambda(n,\kappa,r_0)$ depending only on n, κ and r_0 such that for any $(M,g) \in \mathbb{M}(n,\kappa,r_0)$ we have*

$$\lambda_{N,p}^{d^*}(M) \geq \lambda(n,\kappa,r_0)$$

where $N \leq c(n,p,\kappa,r_0)Vol(M)$ and $c(n,p,\kappa,r_0)$ is a positive constant.

Proof: let ρ_0 be given by Lemma III.2.9 and let X a $\frac{\rho_0}{4}$-discretization of M. Then the discretizing operator

$$\mathcal{D} : d^*\Lambda^{p+1}(M) \to \delta^*\mathcal{C}^{p+1}(\mathcal{U})$$

constructed in Section III.3.1 satisfies $(i)_{\mathcal{D}}$ and $(ii)_{\mathcal{D}}$. Let then

$$N = dim\left(\delta^*\mathcal{C}^{p+1}(\mathcal{U})\right) + 1.$$

Consider moreover $\phi_1, \ldots, \phi_N$ the N first eigenforms in $d^*\Lambda^{p+1}(M)$. By definition of N, there exist $a_1, \ldots, a_N$ such that $\sum_{i=1}^{N} a_i\mathcal{D}(\phi_i) = 0$ and

$\sum_{i=1}^{N} a_i\phi_i \neq 0$. In particular, by $(ii)_{\mathcal{D}}$, we get

$$\left\| d\left(\sum_{i=1}^{N} a_i\phi_i\right)\right\|^2 \geq \Lambda \left\|\sum_{i=1}^{N} a_i\phi_i\right\|^2$$

and thanks to the variational characterization of the spectrum

$$\lambda_{N,p}^{d^*}(M) = \max\left\{\frac{\|d\phi\|^2}{\|\phi\|^2} : \phi \in \langle\phi_1,\ldots,\phi_N\rangle \setminus \{0\}\right\} \geq \Lambda.$$

Note that by Remark III.2.8, we have $N \leq |\mathcal{S}_p(\mathcal{U}_X)| \leq c_2 \frac{\nu^p}{(p+1)!} Vol(M)$ where c_2 and ν depend only on n, p, κ and r_0. □

III.5 Appendix

III.5.1 Whitney forms

Let (M^n, g) be a compact connected n-dimensional Riemannian manifold without boundary. Let $\mathcal{U}$ be a finite contractible open cover of M. Let $\{\varphi_j\}$ be a partition of unity subordinated to $\mathcal{U}$. Let ν a bound on the cardinality of the sets $\{j : U_j \cap U_I \neq \emptyset\}$, $I \in S_q(\mathcal{U})$, $q \geq 0$.

Definition III.5.1 *For any $I = \{i_0,\ldots,i_q\} \in S_q(\mathcal{U})$, we define the* ***Whitney form*** *$W_I \in \Lambda^q(M)$ by*

$$W_I = \sum_{j=0}^{q} (-1)^j \varphi_{i_j} d\varphi_{i_0} \wedge \ldots \wedge d\varphi_{i_{j-1}} \wedge d\varphi_{i_{j+1}} \wedge \ldots \wedge d\varphi_{i_q}$$

Remark III.5.2 *Note that W_I has support in U_I. Moreover, we have $dW_I = (q+1)d\varphi_{i_0} \wedge \ldots \wedge d\varphi_{i_q}$, for $I = \{i_0,\ldots,i_q\}$. In the sequel, we will write $d\varphi_I = d\varphi_{i_0} \wedge \ldots \wedge d\varphi_{i_q}$.*

We can extend the definition of Whitney forms to q-cochains as follows.

Definition III.5.3 *Let $W : \mathcal{C}^q(\mathcal{U}, \Lambda^p) \to \Lambda^{p+q}(M)$ the application defined by*

$$W(c) = \sum_{I \in S_q(\mathcal{U})} W_I \wedge c(I).$$

The application W restricted to Čech cochains is the Whitney map introduced by Whitney (see [60]) (up to a constant). The following lemma generalizes the well-known fact that the Whitney map commutes with the exterior differential and the coboundary.

Lemma III.5.4 *For any* $c \in \mathcal{C}^q(\mathcal{U}, \Lambda^p)$, *we have*

$$dW(c) = (q+1)W(\delta c) + (-1)^q W(dc).$$

Proof: we have

$$\begin{aligned} dW(c) &= \sum_{I \in S_q(\mathcal{U})} d(W_I \wedge c(I)) \\ &= \sum_{I \in S_q(\mathcal{U})} dW_I \wedge c(I) + (-1)^q \sum_{I \in S_q(\mathcal{U})} W_I \wedge dc(I) \\ &= (q+1) \sum_{I \in S_q(\mathcal{U})} d\varphi_I \wedge c(I) + (-1)^q W(dc). \end{aligned}$$

Let us now compute $W(\delta c)$. We have

$$W(\delta c) = \sum_{J \in S_{q+1}(\mathcal{U})} W_J \wedge \left(\sum_{j \in J} \epsilon(j, J \setminus j) c(J \setminus j) \right)$$

where $\epsilon(j, J \setminus j)$ is ± 1 according to the signature of the permutation ordering the set $\{j\} \cup (J \setminus j)$ in J. If we let $I = J \setminus j$, we can write

$$W(\delta c) = \sum_{I \in S_q(\mathcal{U})} \sum_{j : U_j \cap U_I \neq \emptyset} W_{\{j,I\}} \wedge c(I)$$

so that it suffices to show that

$$\sum_{j : U_j \cap U_I \neq \emptyset} W_{\{j,I\}} = d\varphi_I \qquad (63)$$

to conclude the proof. Let us rewrite the expression as follows

$$\sum_{j : U_j \cap U_I \neq \emptyset} W_{\{j,I\}} = \sum_{j : U_j \cap U_I \neq \emptyset} \varphi_j d\varphi_I - d\varphi_j \wedge W_I. \qquad (64)$$

But as $\{\varphi_j\}$ is a partition of unity $\sum_{j : U_j \cap U_I \neq \emptyset} \varphi_j = 1$ and $\sum_{j : U_j \cap U_I \neq \emptyset} d\varphi_j = 0$, hence (64) implies (63). □

Lemma III.5.5 *There exists a positive constant* k *depending only on* n, ν *and on* $\|d\varphi_j\|_\infty$ *such that for any Čech cochain* c, $\|W(c)\|^2 \leq k \|c\|^2$.

Proof: it follows from the definition of W and from a direct calculation. □

III.5.2 About the convexity of balls

Proof of Lemma III.2.9: the main idea to prove this lemma is to smooth g to obtain a more regular metric $\tilde{g}$ and then compare $\tilde{g}$ to a Euclidean metric $\tilde{e}$. We do not compare directly g with a Euclidean metric as we need to control the difference between the different connections involved. So let $(M,g) \in \mathbb{M}(n,\kappa,r_0)$. It follows from a result of Abresch (see [22], Theorem 1.12) that there exists a Riemannian metric $\tilde{g}$ on M such that

(a) $e^{-\frac{1}{4}}g \leq \tilde{g} \leq e^{\frac{1}{4}}g$

(b) $|\nabla^g - \nabla^{\tilde{g}}| \leq \frac{1}{4}$

(c) $|K_{\tilde{g}}| \leq \tilde{\kappa}(n,\kappa)$ and $|\nabla^{\tilde{g}} R_{\tilde{g}}| \leq k(n,\kappa)$

where $\tilde{\kappa}$ and k depend only on n and κ. In particular, (a) implies that, the length of the curves, the distances and the volumes are comparable within a ratio depending only on n. Moreover, if B denotes a ball for g and $\tilde{B}$ a ball for $\tilde{g}$, we get $B(p,e^{-\frac{1}{2}}r) \subseteq \tilde{B}(p,r) \subseteq B(p,e^{\frac{1}{2}}r)$. First, we show that there exists $\tilde{r}_0 > 0$ depending only on n, κ, r_0 such that

$$inj(M,\tilde{g}) \geq \tilde{r}_0. \tag{65}$$

This is a direct consequence of a theorem of Klingenberg and a theorem of Cheeger. Indeed, by Klingenberg's Theorem (see for instance [7], Theorem 89, or [44] and [42]) and as we have bounded sectional curvature, the injectivity radius satisfies

$$inj(M,\tilde{g}) \geq \min\left\{\frac{\pi}{\sqrt{\tilde{\kappa}}}, \frac{1}{2}\tilde{l}(\tilde{\gamma})\right\} \tag{66}$$

where $\tilde{l}(\tilde{\gamma})$ is the length with respect to $\tilde{g}$ of the shorter smooth geodesic (w.r.t. $\tilde{g}$) loop. Moreover, we show there exists $L > 0$ depending only on n, κ and r_0 such that $\tilde{l}(\tilde{\gamma}) \geq L$ as follows. First, if $\frac{1}{2}\tilde{l}(\tilde{\gamma}) \geq e^{-\frac{1}{2}}r_0$, $L = 2e^{-\frac{1}{2}}r_0$ is suitable. Then suppose $\frac{1}{2}\tilde{l}(\tilde{\gamma}) < e^{-\frac{1}{2}}r_0$. By construction of $\tilde{g}$ we get then $\frac{1}{2}l(\tilde{\gamma}) < r_0$ i.e. $\tilde{\gamma}$ is contained in $B(\tilde{\gamma}(0), r_0) = B$. Again by construction of $\tilde{g}$, $\widetilde{Vol}(B) \geq c(n)Vol(B)$ and as $(M,g) \in \mathbb{M}(n,\kappa,r_0)$, there exists $c(n,\kappa,r_0) > 0$ such that $Vol(B) \geq c(n,\kappa,r_0)$ so that $\widetilde{Vol}(B)$ is bounded below by a constant V depending only on n, κ and r_0. Moreover, $\widetilde{diam}(B) \leq 2e^{\frac{1}{2}}r_0 = d$. So that we can apply Theorem 2.1. of Cheeger in [19] that ensures the existence of a positive constant L depending only on d, V and $\tilde{\kappa}$ and therefore only on n, κ and r_0 such that $\tilde{l}(\tilde{\gamma}) \geq L$. Together with (66), this implies (65).

A suitable candidate to be the diffeomorphism cited in the claim is the exponential map with respect to the metric $\tilde{g}$. Let then

$$\varphi = \widetilde{\exp}_p : B(0, \tilde{r}_0) \to \tilde{B}(p, \tilde{r}_0)$$

and $\tilde{e}$ the Euclidean metric on $\tilde{B}(p, \tilde{r}_0)$ induced by φ^{-1} and the normal coordinates. As soon as $e^{\frac{1}{2}} r \leq \tilde{r}_0$, we have $B(p,r) \subseteq \tilde{B}(p, \tilde{r}_0)$ and then $\varphi^{-1}(B(p,r))$ is well-defined. We are going to show now that there exists a positive constant $0 < \rho_0(n, \kappa, r_0) \leq e^{-\frac{1}{2}} \tilde{r}_0$ such that for any $B(q, \rho) \subseteq B(p, \rho_0) \subseteq \tilde{B}(p, \tilde{r}_0)$ we have

$$\varphi^{-1}(B(q, \rho)) \text{ is a Euclidean convex.} \tag{67}$$

This is equivalent to showing that the application

$$f : (B(q, \rho), \tilde{e}) \to \mathbb{R} \ , \ x \mapsto \frac{1}{2} d(q, x)^2 \tag{68}$$

is convex (w.r.t. $\tilde{e}$), in other words that the Hessian of f with respect to $\tilde{e}$ is non-negative i.e. $D^2_{\tilde{e}} f(U,U) \geq 0$ on $B(q, \rho)$, for ρ and ρ_0 well-chosen. Let us recall the following definition of the Hessian

$$D^2 f(U,V) = U \cdot df(V) - df(\nabla_U V)$$

where ∇ is the Levi-Civita connection. Using this definition of the Hessian for $\tilde{e}$ and g, we get

$$\begin{aligned} D^2_{\tilde{e}} f(U,U) &= D^2_g f(U,U) + df(\nabla^g_U U - \nabla^{\tilde{e}}_U U) \\ &= D^2_g f(U,U) + df(\nabla^g_U U - \nabla^{\tilde{g}}_U U) + df(\nabla^{\tilde{g}}_U U - \nabla^{\tilde{e}}_U U). \end{aligned} \tag{69}$$

Proposition 6.4.6. of Buser and Karcher in [15] says that

$$D^2_g f(U,U) \geq \rho \frac{s'_\kappa(\rho)}{s_\kappa(\rho)} g(U,U)$$

where $s_\kappa(\rho) = \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}\rho)$. So that $\frac{s'_\kappa(\rho)}{s_\kappa(\rho)} = \sqrt{\kappa} \cot(\sqrt{\kappa}\rho)$ and hence there exists $\rho_1(\kappa) > 0$ such that for any $0 < \rho < \rho_1$, $\frac{s'_\kappa(\rho)}{s_\kappa(\rho)} \geq 1$. Therefore, on $B(q, \rho)$ with $\rho \leq \rho_1$ we have

$$D^2_g f(U,U) \geq \rho g(U,U) \tag{70}$$

and this shows also that for such ρ's, $B(q, \rho)$ is convex (w.r.t. g). Also as a consequence of Proposition 6.4.6. of [15], we get

$$g(\nabla^g f, \nabla^g f) \leq \rho^2 \tag{71}$$

where $\nabla^g f$ is the gradient of f with respect to g.
Moreover, by construction of $\tilde{g}$ and by (b) in the result of Abresch, we have

$$|\nabla^g_U U - \nabla^{\tilde{g}}_U U|_g \leq \frac{1}{4} g(U,U). \tag{72}$$

By construction of $\tilde{e}$ and as the $\nabla^{\tilde{g}} R_{\tilde{g}}$ is uniformly bounded, Corollary 1 of Kaul in [43] asserts the existence of an application $h \geq 0$ such that

$$|\nabla^{\tilde{g}}_U U - \nabla^{\tilde{e}}_U U|_{\tilde{g}}(y) \leq h(\tilde{d}(p,y))\tilde{g}(U,U)$$

with $h(0) = 0$ and h depends only on bounds on $K_{\tilde{g}}$ and $\nabla^{\tilde{g}} R_{\tilde{g}}$. Hence, there exists $R(n,\kappa,r_0) > 0$ such that for any $r \leq R$, $h(r) \leq \frac{1}{4} e^{-\frac{3}{4}}$. So that we obtain on $\tilde{B}(p,r)$ with $r \leq R$

$$|\nabla^{\tilde{g}}_U U - \nabla^{\tilde{e}}_U U|_g \leq e^{\frac{1}{2}} |\nabla^{\tilde{g}}_U U - \nabla^{\tilde{e}}_U U|_{\tilde{g}} \leq \frac{1}{4} e^{-\frac{1}{4}} \tilde{g}(U,U) \leq \frac{1}{4} g(U,U). \tag{73}$$

Finally, introduce (70), (71), (72) and (73) in (69) and let us define $\rho_0 = \min\{e^{-\frac{1}{2}}\tilde{r}_0, \rho_1, e^{-\frac{1}{2}}R\}$ to obtain the following. We have $B(p,\rho_0) \subseteq \tilde{B}(p,\tilde{r}_0)$, $B(p,\rho_0) \subseteq \tilde{B}(p,R)$ and for any $B(q,\rho) \subseteq B(p,\rho_0)$, $\rho \leq \rho_1$ holds. Hence on $B(p,\rho_0)$ and for any $B(q,\rho) \subseteq B(p,\rho_0)$ we have

$$D^2_{\tilde{e}} f(U,U) \geq \rho g(U,U) - \frac{1}{4}\rho g(U,U) - \frac{1}{4}\rho g(U,U) = \frac{1}{2}\rho g(U,U) \geq 0 \tag{74}$$

i.e. f is convex. To conclude the proof, we remark that

$$B(q,\rho) \subseteq B(p, d(p,q)+\rho) \subseteq \tilde{B}(p, e^{\frac{1}{2}}(d(p,q)+\rho)) \subseteq \tilde{B}(p,\tilde{r}_0)$$

so that φ^{-1} restricted to $\tilde{B}(p, e^{\frac{1}{2}}(d(p,q)+\rho))$ is a quasi-isometry with constants of quasi-isometry depending only on n, κ and $d(p,q)+\rho$. More precisely, $(\tilde{B}(p, e^{\frac{1}{2}}(d(p,q)+\rho)), \tilde{e})$ is quasi-isometric to $(\tilde{B}(p, e^{\frac{1}{2}}(d(p,q)+\rho)), \tilde{g})$ with constants of quasi-isometry depending only on $d(p,q)+\rho$ and $\tilde{\kappa}(n,\kappa)$ and by construction of $\tilde{g}$ we can deduce that $(B(q,\rho), g)$ is quasi-isometric to $(B(q,\rho), \tilde{e})$ with constants of quasi-isometry depending only on n, κ and $d(p,q)+\rho$. This ends the proof of the lemma. $\square$

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