Université de Neuchâtel
Faculty of Science
Department of Mathematics
Research Group: Representation theory
and geometry of groups
Rue Emile-Argand 11
CH-2000 Neuchâtel

Arenberg Doctoral School of Science,
Engineering & Technology
Faculty of science
Department of Mathematics
Research group: Algebraic Topology & Group Theory
Kasteelpark Arenberg 11-box 2100
B-3001 Leuven

KATHOLIEKE UNIVERSITEIT LEUVEN

UNIVERSITÉ DE NEUCHÂTEL

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Dennis DREESEN

EQUIVARIANT AND NON-EQUIVARIANT UNIFORM
EMBEDDINGS INTO PRODUCTS AND HILBERT SPACES

March 2011

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Directors

Prof. Paul Igodt
Prof. Alain Valette

Dissertation presented in
partial fulfillment of the
requirements for the degree of
Doctor of Sciences

Members of the jury

Prof. Karel Dekimpe, referee
Prof. Cornelia Druţu, referee
Prof. Paul Igodt, co-supervisor
Prof. Paul Jolissaint, referee
Prof. Stefaan Vaes, president
Prof. Alain Valette, co-supervisor

March 9, 2011

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FACULTE DES SCIENCES
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Equivariant and non-equivariant embeddings into products and Hilbert spaces

Dennis DREESEN

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La Faculté des sciences de l'Université de Neuchâtel,
sur le rapport des membres du jury

Mme C. Drutu (Oxford, UK),
MM. A. Valette (co-directeur de thèse, Neuchâtel),
P. Igodt (co-directeur de thèse Louvain, B)
K. Dekimpe (Louvain, B), P. Jolissaint (Neuchâtel) et S. Vaes (Louvain,B)

autorise l'impression de la présente thèse.

Neuchâtel, le 22 février 2011

Le doyen :
P. Kropf

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Dutch Abstract

Een kristallografische groep is een groep die trouw, isometrisch en eigenlijk discontinu ageert op een Euclidische ruimte $\mathbb{R}^n$. Er zijn drie stellingen, de Bieberbach-stellingen, die de theorie omtrent kristallografische groepen in zekere zin "domineren". Het onderzoek, beschreven in deze verhandeling, wordt gemotiveerd vanuit het verlangen om de drie Bieberbach-stellingen te veralgemenen naar een ruimere context. In plaats van enkel acties op de Euclidische ruimte toe te laten, kunnen we bijvoorbeeld acties op producten $M \times N$ beschouwen. Hierbij onderstellen we dat M een gesloten Riemannse variëteit is en dat N een samenhangende, enkelvoudig samenhangende, nilpotente Lie-groep is die uitgerust is met een links-invariante Riemannse metriek. Om nu de eerste Bieberbach-stelling te veralgemenen, hebben we nodig dat de isometrieën van $M \times N$ splitsen, i.e. dat $\mathrm{Iso}(M \times N) = \mathrm{Iso}(M) \times \mathrm{Iso}(N)$. We besluiten daarom te onderzoeken voor welke variëteiten M en N de isometrieën van $M \times N$ splitsen. Onze bevindingen worden geformuleerd in Deel I.

In Deel II beschouwen we een andere veralgemening van de Bieberbach context. In plaats van te ageren op $\mathbb{R}^n$, een voorbeeld van een Hilbert-ruimte, laten we nu isometrische acties op eender welke Hilbert-ruimte toe. We komen zo tot de klasse van groepen met de Haagerup-eigenschap en de kwantificatie van deze eigenschap leidt tot de notie van equivariante Hilbert compressie. Dit getal bevat bepaalde interessante informatie over de groep in kwestie. Wij bestuderen het gedrag van deze invariant onder verschillende groepsconstructies.

Ten slotte eisen we in Deel III niet langer dat groepen actie voeren op een Hilbert-ruimte, maar wel dat ze er uniform in kunnen ingebed worden. Ook de eigenschap "uniform inbedbaar zijn" kan je kwantificeren en dit leidt tot de notie van (niet-equivariante) Hilbert compressie. We bestuderen hoe dit getal zich gedraagt onder groepsconstructies.

English Abstract and Keywords

A crystallographic group is a group that acts faithfully, isometrically and properly discontinuously on a Euclidean space $\mathbb{R}^n$ and the theory of crystallographic groups is in some sense governed by three main theorems, called the Bieberbach theorems. The research performed in this thesis is motivated from a desire to generalize these theorems to a more general setting. First, instead of actions on $\mathbb{R}^n$, we consider actions on products $M \times N$ where N is a simply connected, connected nilpotent Lie-group equipped with a left-invariant Riemannian metric and where M is a closed Riemannian manifold. Our proof to generalize the first Bieberbach theorem to this setting, needs that the isometries of $M \times N$ split, i.e that $\text{Iso}(M \times N) = \text{Iso}(M) \times \text{Iso}(N)$. In Part I of this thesis, we introduce a class of products on which the isometries split.

Consequently, going back to the Bierbach context, we can replace Euclidean space $\mathbb{R}^n$ by the class of all, possibly infinite-dimensional, Hilbert spaces. We here enter the world of groups with the Haagerup property. Quantifying the degree to which a group satisfies the Haagerup property leads to the notion of equivariant Hilbert space compression, and we investigate the behaviour of this number under group constructions in Part II.

Finally, dropping the condition that groups under consideration must act isometrically on a Hilbert space, we look, in part III, at mere (uniform) embeddings of groups into Hilbert spaces. Quantifying the degree to which a group embeds uniformly into a Hilbert space, leads to the notion of (ordinary) Hilbert space compression and in Part III, the behaviour of this number under group constructions is investigated.

Keywords Bass-Serre theory/ Behaviour of compression under group constructions/Bieberbach groups / Bieberbach theorems/ crystallographic groups / equivariant Hilbert space compression/ fiberwise volume non-increasing maps/ Haagerup Property/ Hilbert space compression/ L_p-compression/ splitting of isometries/ Talleli conjecture/ uniform embeddability/ property (A)

Contents

Chapter 1

Introduction

In the nineteen tens, with the formulation and proofs of the three famous Bieberbach theorems [14], [15], [50], a very important breakthrough was obtained in the research on crystallographic groups, i.e. groups acting isometrically, properly discontinuously and cocompactly on Euclidean space $\mathbb{R}^n$. The Bieberbach theorems describe the structure of crystallographic groups and their rigidity. In particular, the structure of a crystallographic group is quite constrained, as it contains a finite index subgroup which is free abelian of finite rank. Having gained a lot of insight in the crystallographic case, people tried to generalize the Bieberbach theorems to a broader setting. Instead of actions on $\mathbb{R}^n$, groups acting isometrically, properly discontinuously and cocompactly on simply connected, connected, nilpotent Lie groups equipped with left-invariant metrics were studied. It turns out that all of the three Bieberbach theorems have their respective counterpart in this context [8], [72], [43]. About 40 years later, in 2001, Dekimpe, Lee and Raymond tried to generalize the first Bieberbach theorem further, to the context of actions on simply connected, connected, *solvable* Lie groups [34]. They found that the first Bieberbach theorem does not generalize to this case.

The research performed in this thesis starts along these same lines. In joint work with Nansen Petrosyan, we consider isometric properly discontinuous and cocompact actions not on $\mathbb{R}^n$, but more generally on products $M \times N$, where M is a closed connected Riemannian manifold and where N is a simply connected, connected, nilpotent Lie group, equipped with a left-invariant metric. We will always equip such products with the natural product metric. In our attempts to generalize the first Bieberbach theorem to this setting, we come across the interesting question as for which Riemannian manifolds M and N the isometry group of $M \times N$ splits, i.e. equals the direct sum of the isometry groups of M and N. The most famous theorem on the matter is the *de Rham decomposition* for Riemannian manifolds [44], on which we will elaborate later in the Introduction. With techniques which totally differ from the classical approach by de Rham, we give our own result on splitting of isometries in Section 3.2. In our methods, the new class of *fiberwise volume non-increasing maps* will be introduced. We study and characterize this class of maps in Part I.

Another way to generalize the Bieberbach theorems consists of replacing $\mathbb{R}^n$ by an infinite-dimensional Hilbert space. Here, we encounter the class of groups with the *Haagerup property*, i.e. second countable locally compact groups which admit a metrically proper (affine) isometric action on a Hilbert space (where "metrically proper" means that every orbit goes to infinity in the Hilbert space as you go to infinity in the group). Contrary to what happens in the finite dimensional case, this defines a huge class of groups, no structure theorem whatsoever, but lots of useful properties and results, and the desire to make it quantitative by studying *equivariant compression*. Intuitively, the *equivariant (Hilbert space) compression* of a group is introduced as a number between 0 and 1 that quantifies "how fast the orbit of $0 \in \mathcal{H}$ goes to infinity" [59]. Under certain conditions, this number measures *how Haagerup* a group really is and there are also connections with amenability. It is thus interesting to try and calculate the equivariant compression of groups and one way of doing this is by checking its behaviour under group constructions [59] [100], [80]. Before this thesis, nothing was known about the behaviour of the equivariant compression under free products and HNN-extensions. This will be the main subject of study in Part II.

The third part of the thesis is related to the (ordinary) Hilbert space compression. Here, we no longer restrict ourselves to checking how fast the orbit maps under affine isometric actions on Hilbert spaces go to infinity. Instead, we will look at the large class of large-scale Lipschitz uniform embeddings from G into Hilbert spaces. The (ordinary) compression is a number that quantifies *how fast* such embeddings go to infinity, i.e. it measures to what extent a group can be quasi-isometrically embedded into a Hilbert space. Interestingly, there are also connections with other fascinating properties such as Yu's Property (A) [110], a weak form of amenability. This motivates the interest to calculate the Hilbert space compression of groups and consequently to know how it behaves under group constructions. In Part III, we investigate the behaviour of the Hilbert space compression under group constructions such as direct limits, free products, group extensions, etc. Although in Part III groups are viewed as metric spaces, and the group law plays a minor role, our results on compression are presented after our results on equivariant compression, as it reflects the chronological order of our research.

It is time to become more specific. Let us introduce the main concepts in this thesis and give an overview of the results that are proven in our work. We start with the results shown in part I.

1.1 Fiberwise volume non-increasing diffeomorphisms

While attempting to generalize the first Bieberbach theorem to the case of actions on products of Riemannian manifolds, we were led to the following question: *for which products* $M \times N$ *of Riemannian manifolds does the isometry group* $\mathrm{Iso}(M \times N)$ split*, i.e. when is* $\mathrm{Iso}(M \times N) = \mathrm{Iso}(M) \oplus \mathrm{Iso}(N)$*.* The most famous result related to this question, is de Rham's decomposition theorem for Riemannian manifolds [44]. His methods involve the study of holonomy groups. More precisely, given a point x on a connected Riemannian manifold M, choose a vector v in the tangent space T_xM. There is, in Riemannian geometry, a classical notion of parallel transport of v along a piecewise differentiable curve. We can then look at the rotations on T_xM, obtained by parallel transport along piecewise differentiable curves in M that start and end in x. These rotations form a group $\Psi(x)$, called the holonomy group of M at x. By parallel transport along a curve joining x to a point y, we obtain a $\Psi(y)$-invariant subspace of T_yM. We say that M (and T_xM) is reducible if there is a non-trivial subspace of T_xM, invariant under $\Psi(x)$. Using this splitting of the tangent bundle of M into irreducible subspaces, de Rham showed that certain reducible Riemannian manifolds split isometrically as a cartesian product of non-reducible manifolds and one Euclidean space. Here, a cartesian product of Riemannian manifolds M and N

carries the natural product metric, i.e.

$$\langle \mathbf{v_1} \oplus \mathbf{w_1}, \mathbf{v_2} \oplus \mathbf{w_2} \rangle_{(y,z)} = \langle \mathbf{v_1}, \mathbf{v_2} \rangle_y + \langle \mathbf{w_1}, \mathbf{w_2} \rangle_z,$$

for all $(y, z) \in M \times N$, $\mathbf{v_1}, \mathbf{v_2} \in T_y(M)$ and $\mathbf{w_1}, \mathbf{w_2} \in T_z(N)$.

Theorem 1.1.1 (de Rham decomposition theorem)**.** *Every simply connected, complete and reducible Riemannian manifold M is isometric to a product*

$$M_0 \times M_1 \times M_2 \times \ldots \times M_k,$$

where M_0 is a Euclidean factor and where the others are Riemannian manifolds that are of dimension ≥ 2 and irreducible. The factors M_i are uniquely determined up to their order and up to isometry. The most general isometry of a product as above is of the form

$$f(x_0, x_1, x_2, \ldots, x_k) = (g_0(x_{i_0}), g_1(x_{i_1}), g_2(x_{i_2}), \ldots, g_k(x_{i_k})),$$

where $x_i \in M_i, i = 0, 1, 2, \ldots, k$, $i_0, i_1, i_2, \ldots, i_k$ is a permutation of $\{0, 1, 2, \ldots, k\}$ with $i_0 = 0$ and g_j is an isometry of M_{i_j} onto M_i ($j = 0, 1, 2, \ldots, k$).

From this theorem, information on the splitting of isometries on products $M \times N$ can be deduced. For example, if M and N satisfy the conditions of the theorem where M is closed (i.e. compact without boundary) and N is contractible, then the above result shows that the isometry group of $M \times N$ (always equipped with the product metric) splits.

De Rham's decomposition theorem has been extended several times, quite recently by J. Eschenburg and E. Heintze in [47] (1998) and finally in a very general form by T. Foertsch and A. Lytchak [49] (2008). In all of these cases, the completeness assumption plays a crucial role.

Our methods use a totally different approach and no completeness assumption is ever required. As a starting point, we introduce the following cohomological condition on the product $M \times N$.

Definition 1.1.2 (Definition 3.1.6 in Section 3.1)**.** *Let M be an n-dimensional closed connected Riemannian manifold. We say that $M \times N$ has* minimal n-cohomology *if*

$$H^n(M \times N; \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

The terminology is due to the fact that $H^n(M \times N; \mathbb{Z}_2)$ necessarily contains a copy of $H^n(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$.

Extending the standard notion of volume on Riemannian manifolds, we discover the class of *fiberwise volume non-increasing diffeomorphisms*: a class of functions which appears interesting on its own merits.

Definition 1.1.3 (Definition 3.1.5 in Section 3.1). *Let $f : M \times N \to M \times N$ be a diffeomorphism and let $z \in N$. Equip both $M \times \{z\}$ and $f(M \times \{z\})$ with the Riemannian metric induced from $M \times N$. We say that f is* fiberwise volume non-increasing at z *if*

$$\mathrm{Vol}(f(M \times \{z\})) \leq \mathrm{Vol}(M \times \{z\}).$$

A diffeomorphism is fiberwise volume non-increasing *(fni) if it is fiberwise volume non-increasing at every point of N. We denote the set of all fiberwise volume non-increasing maps of $M \times N$ by FNI$(M \times N)$.*

This class of groups contains the isometries, and also the maps of the form (f, g) where f is a diffeomorphism of M and g is a diffeomorphism of N. Despite the name, one should not expect a simple connection with volume preserving maps, see Section 3.1.

For the class of fni-maps, we prove three main theorems. We start with Theorem 3.2.8 from Section 3.2.

Theorem 1.1.4 (Slice Theorem). *Let M be a closed connected Riemannian manifold and let N be a Riemannian manifold such that $M \times N$ has minimal n-cohomology. If $f : M \times N \to M \times N$ is fni at $z \in N$, then there exists $w \in N$ such that $f(M \times \{z\}) = M \times \{w\}$.*

Said differently, f maps M-fibers to M-fibers. From this, we deduce the following Theorem, which is Theorem 3.2.10.

Theorem 1.1.5 (Splitting Theorem). *If M is a closed connected Riemannian manifold and if N is a connected Riemannian manifold such that $M \times N$ has minimal n-cohomology, then the isometries of $M \times N$ split, i.e.* $\mathrm{Iso}(M \times N) = \mathrm{Iso}(M) \times \mathrm{Iso}(N)$.

Next, we investigate the structure of the class of fni maps under the condition of minimal n-cohomology. The Slice Theorem then implies that FNI$(M \times N)$ is a group under the operation of composition. It will turn out that FNI($M \times N$) is in some sense completely determined by the diffeomorphism groups Diffeo(M) and Diffeo(N). While describing the structure of FNI($M \times N$) in Section 3.3, we will encounter a general version of differentiability, called Fréchet differentiability. A Fréchet space can be seen as a generalization of a Banach space, where instead of one norm, the topology is given by a countable family of semi-norms. Interestingly, the classical differentiability theory on Banach spaces has been generalized to the setting of Fréchet spaces. Similarly as differentiable manifolds, which locally *look like* open subsets of $\mathbb{R}^k$, we have the parallel notion of Fréchet manifolds, which locally *look like* open subsets of Fréchet spaces. We refer the reader to Section 2.3 for the details. For now, in order to formulate our result, let us make the following observation.

Fix a point $y_0 \in M$ and consider

$$\begin{array}{rccc} \psi: & \mathrm{FNI}(M \times N) & \to & \mathrm{Diffeo}(N) \\ & (\alpha, \beta) & \mapsto & \widetilde{\beta}, \end{array}$$

where

$$\begin{array}{rccc} \widetilde{\beta}: & N & \to & N \\ & z & \mapsto & \beta(y_0, z). \end{array}$$

Note that this definition is independent of the chosen y_0 by the Slice Theorem. We prove the following result, which is Theorem 3.3.4 in Section 3.3.

Theorem 1.1.6 (Structure Theorem). *Let M be a closed connected Riemannian n-manifold and let N be a Riemannian manifold such that $M \times N$ has minimal n-cohomology. We have the following short exact sequence:*

$$1 \to K \hookrightarrow \mathit{FNI}(M \times N) \xrightarrow{\psi} \mathrm{Diffeo}(N) \to 1$$

with ψ as above and where

$$K \cong \{f : N \to \mathrm{Diffeo}(M) \mid f \textit{ is Fréchet differentiable}\}.$$

Note that this short exact sequence splits canonically, via $f \mapsto Id_M \times f$.

We conclude that the subgroup FNI($M \times N$) < Diffeo($M \times N$) is in some sense completely determined by the diffeomorphism groups of M

and N.

The last section of Part I, Section 3.4, is devoted to applications regarding properly discontinuous actions.

A first application is related to the Bieberbach theorems [14],[15], [50]. Specifically, we prove that the first Bieberbach theorem generalizes naturally to our setting.

Theorem 1.1.7. *Let M be a closed connected Riemannian manifold and let N be a simply connected, connected, nilpotent Lie group equipped with a left-invariant metric. If Γ is a group acting properly discontinuously, cocompactly and isometrically on $M \times N$, then Γ contains a finite index subgroup isomorphic to a uniform lattice of N.*

The other Bieberbach theorems do not admit a straightforward generalization. More details are given in Subsection 3.4.1.

A second application concerns a generalized version of Talelli's conjecture (Conjecture III of [101]). Talelli conjectured that torsion-free groups have periodic cohomology only if they have finite cohomological dimension. The conjecture can be reformulated in terms of smooth properly discontinuous actions of the group on products $S^n \times \mathbb{R}^k$. For more details, we refer the reader to Subsection 3.4.2.

1.2 Equivariant Hilbert space compression

Another generalization of crystallographic groups is obtained by replacing $\mathbb{R}^n$ with any infinite dimensional Hilbert space $\mathcal{H}$. This leads us to the class of groups with the Haagerup property: a huge class of groups, no structure theorem whatsoever, but lots of useful properties and results. Before jumping to a definition, let us recall a few preliminary facts and introduce some notations.

First, recall that every isometric action α of a group G on a real Hilbert space is affine [76], i.e. there is $\pi : G \to \mathcal{O}(\mathcal{H})$ and $b : G \to \mathcal{H}$ such that

$$\forall x \in G,\ \forall v \in \mathcal{H} : \alpha(x)(v) = \pi(x)(v) + b(x),$$

where $\pi : G \to \mathcal{O}(\mathcal{H})$ is an orthogonal representation of G, called the linear part of α, and where $b : G \to \mathcal{H}$ is a 1-cocycle relative to π, i.e. b

satisfies $b(xy) = \pi(x)(y) + b(x)$ for every $x, y \in G$. Note that b is the orbit map of $0 \in \mathcal{H}$.

In this thesis, we will mostly be interested in discrete groups. An important example of such a group is a finitely generated group equipped with the word length function l_S relative to a finite symmetric generating subset S. In this Introduction, we will always restrict ourselves to this case. Such a group now is called *Haagerup* if and only if it admits a 1-cocycle b, associated to some orthogonal action of G on a Hilbert space, such that

$$\forall x \in G: \ \|b(x)\| \geq \rho_-(l_S(x)),$$

where $\rho_- : \mathbb{R}^+ \to \mathbb{R}^+$ is a map satisfying $\lim_{r\to\infty} \rho_-(r) = +\infty$. Examples of such groups are the free groups F_n, the Coxeter groups, discrete subgroups of $SO(n,1)$ and $SU(n,1)$, countable subgroups of $GL_2(F)$ where F is any field, etc. [25]

Quantifying the degree to which a group satisfies the Haagerup property opens the door to fascinating new concepts and results. We study therefore *how fast* a 1-cocycle $b : G \to \mathcal{H}$ goes to infinity, i.e. how *fast* such a map ρ_- can *increase to* ∞. Clearly, ρ_- can not grow faster than every linear function. Indeed, it is easy to see that 1-cocycles on finitely generated groups (G, l_S) are large-scale Lipschitz, i.e. that there exists some $C, D > 0$ such that for every $x \in G$,

$$\|b(x)\| \leq Cl_S(x) + D \text{ (note that here, we can even take } D = 0).$$

From below, 1-cocycles are *not* always bounded by an (affine) linear map in $l_S(x)$. In fact, F_2 does not embed quasi-isometrically into a Hilbert space [17], so no 1-cocycle of F_2 admits an (affine) linear lower bound. Naturally, this invokes the following question.

What is the greatest number $r \in [0,1]$ such that a given group admits an affine isometric action on a Hilbert space such that for the associated 1-cocycle there are numbers $C, D > 0$ such that

$$\|b(x)\| \geq \frac{1}{C} l_S(x)^r - D,$$

for every $x \in G$?

Definition 1.2.1. *The equivariant (Hilbert space) compression of a finitely generated group* (G, l_S) *is the supremum of* $r \in [0,1]$ *as above. To* *be clear, the supremum is taken over all possible affine isometric actions of G in all possible Hilbert spaces. The equivariant compression of G is denoted by $\alpha^*(G)$ and it is independent of the chosen finite symmetric generating subset S.*

The equivariant compression of a group G contains some useful information on G. For example, if $\alpha^*(G) > 1/2$, then we know that G is amenable [59] and if $\alpha^*(G) > 0$, then G is Haagerup. It is thus interesting to calculate the equivariant compression of groups, and this is also quite challenging. It is for example not at all trivial to prove that $\mathbb{Z} \wr \mathbb{Z}$ has equivariant compression 2/3 [10] or that Thompson's group F has equivariant compression 1/2 [7]. To facilitate computations, it may be interesting to know how the equivariant compression behaves under group constructions. In this respect, wreath products have been very well studied [10], [80], [100]. Naor and Peres prove in [80] that the equivariant compression of the $k-$fold wreath product of $\mathbb{Z}$ with itself equals $\frac{1}{2-2^{1-k}}$. Together with 1/2 and 0, these are the only explicit values known so far for the equivariant compression of groups.

The case of direct sums has also been studied and is completely understood: the equivariant compression of a direct sum $G_1 \oplus G_2$ equals $\min(\alpha^*(G_1), \alpha^*(G_2))$ [59]. Nothing non-trivial is known about the behaviour of the equivariant compression under different group constructions.

In part II of the thesis, we want to study the case of free products and HNN-extensions. We tackle the case of free products in Section 6.1.

Theorem 1.2.2 (Theorem 6.1.12). *Let $G = G_1 *_F G_2$ be an amalgamated free product of finitely generated groups G_1 and G_2 over the finite group F. If α, α_1 and α_2 are the equivariant Hilbert space compressions of G, G_1 and G_2 respectively, then*

1. $\alpha = 1$ *if F is of index 2 in both G_1 and G_2,*
2. $\alpha = \alpha_1$ *if $F = G_2$ and $\alpha = \alpha_2$ if $F = G_1$,*
3. $\alpha = \min(\alpha_1, \alpha_2, 1/2)$ *otherwise.*

Note that the second case is stated for completeness. Distortion makes it hard to treat the general case of infinite F.

In Section 6.2, we treat the case of HNN-extensions over finite groups. Given a group H with presentation $\langle S \mid R\rangle$, recall that $G := \mathrm{HNN}(H, F, \theta)$ is the group with presentation

$$\langle S, t \mid R, t^{-1}ft = \theta(f)\ \forall f \in F\rangle.$$

Here, $\theta : F \to H$ is a group monomorphism. Again formulated in the context of finitely generated groups, we obtain the following

Theorem 1.2.3 (Theorem 6.2.8 in Section 6.2)**.** *Given a finitely generated group H, assume that F is a subgroup of H and that $\theta : F \to H$ is a group monomorphism such that the group generated by $\theta(F) \cup F$ is finite. The equivariant Hilbert space compression α of $G := HNN(H, F, \theta)$ satisfies*

1. $\alpha = 1$ *whenever* $F = H$,
2. $\alpha = \min(\alpha(H), 1/2)$ *otherwise.*

In Section 6.3, we look at a special case of quotients. Note that it is almost impossible to say something useful on the behaviour of equivariant compression under quotients. Indeed, every finitely generated group is the quotient of a free group of finite rank and these free groups have equivariant compression 1/2. Our Theorem 6.3.2 gives information about the quotient in a very specific case. We refer the reader to Section 6.3 for details.

Our last section is a bit different in nature since we will actually *calculate* the equivariant compression of the Baumslag-Solitar monsters $BS(p, q)$ with $p, q > 1$. These are the groups with presentation

$$BS(p, q) =< a, b \mid b^{-1}a^q b = a^p > .$$

For this result, we owe gratitude to Yves de Cornulier. We prove

Theorem 1.2.4 (Corollary 6.4.8 in Section 6.4)**.** *The equivariant compression of the Baumslag-Solitar monsters $BS(p, q)$ with $p, q > 1$ equals 1/2.*

If one of $p, q \in \mathbb{N}_0$ happens to be 1, then we show that $BS(p, q)$ has equivariant compression 1.

1.3 Hilbert space compression

In Part III, we study the slightly "weaker" form of compression, called (ordinary) *Hilbert space compression.* This is again a number between 0 and 1 which can be associated to a group (G, l) with l a length function, and again this number quantifies a certain property of the group. In this setting, the role of 1-cocycles will be played by a different actor: the uniform embedding of a group into a Hilbert space.

Definition 1.3.1. *A group (G, d), where d is a metric on G, is* uniformly embeddable in a Hilbert space *if there exist a Hilbert space $\mathcal{H}$, non-decreasing functions $\rho_-, \rho_+ : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to\infty} \rho_-(t) = +\infty$, and a map $f : G \to \mathcal{H}$, such that*

$$\rho_-(d(x,y)) \leq d(f(x), f(y)) \leq \rho_+(d(x,y)) \ \forall x, y \in G.$$

The map f is called a uniform embedding of G in $\mathcal{H}$. It is called large-scale Lipschitz whenever ρ_+ can be taken of the form $\rho_+ : t \mapsto Ct + D$ for some $C > 0, D \geq 0$. It is Lipschitz if we can take $D = 0$.

The definition admits a straightforward generalization to the case of *any* metric space (X, d). We, however, choose to restrict ourselves in this Introduction to finitely generated groups, equipped with the word length metric d_S relative to a finite symmetric generating subset S. In this setting, there is a standard result which shows that uniform embeddings are large-scale Lipschitz, see Proposition 2.9 in [59]. The following question then imposes itself:

What is the greatest number $r \in [0, 1]$ such that there are numbers $C, D > 0$ such that

$$\|f(x) - f(y)\| \geq \frac{1}{C} d_S(x,y)^r - D,$$

for every $x, y \in G$?

Definition 1.3.2. *The supremum of the numbers $r \in [0,1]$ as above is called the compression of f and denoted by $R(f)$. The (ordinary Hilbert space) compression of a finitely generated group (G, l_S) is the supremum of $r \in [0,1]$ such that there exists a uniform embedding of G into a Hilbert space whose compression equals r. This supremum is taken over all possible Hilbert spaces, but since finitely generated groups are* *countable, we can assume $\mathcal{H} = l^2(\mathbb{Z})$. The Hilbert space compression of G is denoted by $\alpha(G)$ and is independent of the chosen finite symmetric generating subset.*

Remark 1.3.3. *When G is a group equipped with* any *length function, then uniform embeddings are not necessarily large-scale Lipschitz. Then, in the above definition, we impose the condition that we only consider large-scale Lipschitz uniform embeddings and Definition 1.3.2 generalizes word-for-word to the notion of Hilbert space compression of a group G equipped with* any *length function l.*

There are some nice connections between compression and other famous properties. For example, if G embeds quasi-isometrically into a Hilbert space, then its compression equals 1. The free group F_2 is a counterexample for the converse statement. There are also connections with Property (A), which is a weak form of amenability. The Hilbert space compression of a group thus contains quite some information and it is useful to calculate this number explicitly. Sometimes, this is easier than calculating the equivariant Hilbert space compression, because Hilbert space compression is a quasi-isometric invariant. In general however, calculating compressions remains a complicated task and it would be helpful to understand the behaviour of the Hilbert space compression under group constructions. This will be the subject of study in Part III. Although many of the above definitions are parallel to the definitions in Section 1.2, the theorems and their proofs will turn out to be quite different.

We start the overview of our results with the case of free products, amalgamated over finite groups.

Theorem 1.3.4 (see Theorem 8.1.3 in Section 8.1)**.** *Assume that (G_1, l_S) and (G_2, l_T) are finitely generated groups. Denoting their Hilbert space compressions by α_1 and α_2 respectively, the Hilbert space compression α of the free product $G = G_1 * G_2$ satisfies*

$$\min(\alpha_1, \alpha_2, 1/2) \leq \alpha \leq \min(\alpha_1, \alpha_2).$$

One can easily show that for finite groups F, the compression of $G_1 *_F G_2$ equals the compression of $G_1 * G_2$. Similarly, for HNN-extensions

$G := \mathrm{HNN}(H, F, \theta)$ where H if finitely generated and where F is finite, it is true that $\mathrm{HNN}(H, F, \theta)$ is quasi-isometric to $H * \mathbb{Z}$ [92]. Consequently,

$$\min(\alpha(H), 1/2) \leq \alpha(G) \leq \alpha(H).$$

In Section 8.2, we introduce a different way to look at compression and proof some crucial lemmas that we will use in future proofs. In Section 8.3, these lemmas are used, among other things, to prove a result on HNN-extensions $\mathrm{HNN}(H, F, \theta)$ where F is of finite index in H.

Theorem 1.3.5 (see Corollary 8.3.10)**.** *Assume that H is a finitely generated group. Let $G = HNN(H, F, \theta)$ be an HNN-extension of H such that both F and $\theta(F)$ are of finite index in H. Equip G with the word length metric d, and denote the induced metric on H by $d_{in} = d_{|H \times H}$. Denoting the compression of (H, d_{in}) by α_1 and that of G by α, we obtain*

$$\alpha_1/3 \leq \alpha \leq \alpha_1.$$

After this result on HNN-extensions, we turn to the classical notion of group extensions. In Section 8.4, we look at extensions of the form

$$1 \to H \to \Gamma \xrightarrow{\pi} G \to 1,$$

where in this Introduction, we assume that Γ and so G are finitely generated and that H is equipped with the induced subspace metric from Γ. Denote the compression of H, with the induced metric, by α_1 and denote the compression of Γ by α. We obtain the following results:

Theorem 1.3.6 (see Corollary 8.4.6)**.** *If G is a group of polynomial growth, then*

$$\alpha_1/3 \leq \alpha \leq \alpha_1.$$

We obtain a similar result for hyperbolic groups:

Theorem 1.3.7 (see Corollary 8.4.15)**.** *If G is a word-hyperbolic group in the sense of Gromov [53], then*

$$\alpha_1/5 \leq \alpha \leq \alpha_1.$$

We refer the reader to the corresponding sections for more background on polynomial growth and hyperbolicity. In Section 8.5, we elaborate on a generalized form of hyperbolicity, namely relative hyperbolicity.

Theorem 1.3.8 (See Theorem 8.5.13)**.** *Let* Γ *be a finitely generated group, hyperbolic relative to a set of subgroups* $H_1, H_2, \ldots, H_m$, $m \in \mathbb{N}$*. Let* S *be a finite symmetric generating subset of* Γ*. Assume that the* H_i *have strictly positive Hilbert space compression when equipped with the induced metric from* Γ*. The Hilbert space compression of* $(S \bigcup \cup_{i=1}^{m} H_i)^k$ *is strictly positive for any* $k > 0$*. Precisely, if* $\alpha_1 > 0$ *is a number such that all of the* H_i *have compression* $> \alpha_1$*, then* $\alpha((S \bigcup \cup_{i=1}^{m} H_i)^k) \geq \frac{\alpha_1}{11^{k-1}}$*.*

We end the thesis by a section on limits of groups, where on the limit $G = \lim_{i\to\infty} G_i$ we consider the induced length function (See Section 8.6 for more information). Denote

$$\mathbb{Z}^{(\mathbb{Z})} = \{(f, 0) \in \mathbb{Z} \wr \mathbb{Z}\}$$

and equip it with the induced length function from $\mathbb{Z} \wr \mathbb{Z}$. We shall write $\mathbb{Z}$ and $\mathbb{Z}^{(\mathbb{Z})}$ as limits of groups of compression 1 whereas $\mathbb{Z}$ has compression 1 and $\mathbb{Z}^{(\mathbb{Z})}$ has compression at most 3/4 [7]. This shows that it is impossible to write a formula which gives the compression of the limit G purely in terms of the compressions of the G_i. This is reflected in our result, where we feel the need to include additional information on the respective ways that the G_i are embedded into their respective Hilbert spaces. We refer the reader to Theorem 8.6.1 for details. Although the result looks quite complicated, we emphasize that the formula is in fact easy to work with and that it can actually be used to obtain without great effort, compression estimates of the limit G of a directed metric system of groups, see e.g. Examples 8.6.4, 8.6.5.

Part I

Fiberwise volume non-increasing diffeomorphisms

Chapter 2

Background

We begin this thesis by recalling some basic definitions from the field of geometry. The first section is a short section devoted to differentiable manifolds and regular values of differentiable maps. The second section is related to Riemannian geometry. We recall the definition of the exponential map and the related notion of totally normal neighbourhood [46]. In the last section, we recall the concept of a Fréchet space. These are vector spaces which can be seen as generalizations of Banach spaces and a full differentiability theory has been constructed in this setting [62]. Replacing Euclidean space by Fréchet spaces, we finally introduce the reader to the geometric concept of a Fréchet manifold.

2.1 Differentiable manifolds

To establish terminology, let us give the definition of a differentiable manifold [46]. We will always deal with smooth maps, so differentiable means C^∞.

Definition 2.1.1. *A* differentiable manifold *of dimension n is a set M and a family of injective mappings $x_\alpha : U_\alpha \subset \mathbb{R}^n \to M$ of open sets U_α of $\mathbb{R}^n$ into M such that:*

1. $\cup_\alpha x_\alpha(U_\alpha) = M$,

2. *for any pair α, β with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \phi$, the sets $x_\alpha^{-1}(W)$ and $x_\beta^{-1}(W)$ are open sets in $\mathbb{R}^n$ and the mappings $x_\beta^{-1} \circ x_\alpha$ are differentiable,*

3. *the family* $\{(U_\alpha, x_\alpha)\}$ *is maximal relative to the conditions* (1) *and* (2) *above.*

The pair (U_α, x_α) *or the mapping* x_α *with* $y \in x_\alpha(U_\alpha)$ *is called a parametrization of* M *at* y*;* $x_\alpha(U_\alpha)$ *is then called a* coordinate neighbourhood *at* y*. A family* $\{(U_\alpha, x_\alpha)\}$ *satisfying* (1) *and* (2) *is called a* differentiable structure*.*

We remark that condition (3) is really just included for technical reasons. Indeed, any family $\{(U_\alpha, x_\alpha)\}$ satisfying conditions (1) and (2) can easily be completed to a maximal one: just take the union of all the parametrizations that together with any of the parametrizations x_α satisfy condition (2). A differentiable manifold comes naturally equipped with the following topology.

Definition 2.1.2. *The topology on a manifold* M *is defined as follows: given a set* $A \subset M$*, we call it* open *if and only if* $x_\alpha^{-1}(A \cap x_\alpha(U_\alpha))$ *is open in* $\mathbb{R}^n$ *for all* α*.*

Convention 2.1.3. *As is common in literature, we will assume that manifolds are Hausdorff, second countable topological spaces.*

We proceed by stating some important theorems in the realm of differentiable manifolds.

Theorem 2.1.4 (Inverse function Theorem)**.** *Let* $f : M \to N$ *be a differentiable mapping and let* $y \in M$ *be such that* $df_y : T_yM \to T_{f(y)}N$ *is an isomorphism. Then there exists an open neighbourhood* U *of* y *and* V *of* $f(y)$ *such that* $f : U \to V$ *is a diffeomorphism.*

Given a differentiable map $f : M \to N$, the subset

$$O = \{z \in N \mid f(y) = z \Longrightarrow df_y \text{ is surjective}\}$$

is called the set of *regular values* of f. Notice that, in particular, every point $z \in N$ which is not in the image of f is a regular value for f. The complement $N \backslash O$ is called the set of *critical values* of f. The following observation is easy.

Lemma 2.1.5. *Let* $f : M \to N$ *be a differentiable map between manifolds of equal dimension. If* M *is compact and if* $z \in N$ *is a regular value, then* $f^{-1}(z)$ *contains only finitely many elements. In particular, the set of critical values of* f *is closed.*

Proof. Assume that $z \in N$ is a regular value of f. Clearly, $f^{-1}(z)$ is closed, hence compact. Moreover, by the inverse function theorem, the topology on $f^{-1}(z)$ is the discrete one. We conclude that $f^{-1}(z)$ is finite. ☐

There is one last standard result about critical points which we will use. We will need the following definition.

Definition 2.1.6. *A subset A of a manifold M has measure* 0 *if* $x_{\alpha}^{-1}(A)$ *has Lebesgue measure* 0 *in* $\mathbb{R}^n$ *for every parametrization* x_α *of* M.

Theorem 2.1.7 (Sard's Theorem)**.** *Let $f : M \to N$ be a differentiable map between differentiable manifolds. The set of critical values of f has measure* 0.

2.2 Riemannian manifolds

Definition 2.2.1. *Let M be an n-dimensional differentiable manifold. A Riemannian metric on M is a correspondence which associates to each point $y \in M$ an inner product $\langle\ ,\ \rangle_y$ (that is, a symmetric, bilinear, positive definite form) on the tangent space T_yM, which varies differentiably in the following sense: assume that $x : U \subset \mathbb{R}^n \to M$ is a system of coordinates around some point of M. Denote elements of U by $q = (q_1, q_2, \ldots q_n)$ and define for every $i \in \{1, 2, \ldots, n\}$ and $q \in U$, a path x_i^q by*

$$x_i^q(t) = x(q_1, q_2, \ldots, q_i + t, q_{i+1}, \ldots, q_n).$$

If $x(q) = y$, then denote $\mathbf{X_i(y)} = (x_i^q)'(0) \in T_yM$. We demand that the maps

$$\begin{array}{rccl} g_{ij}: & x(U) & \to & \mathbb{R} \\ & y & \mapsto & \langle \mathbf{X_i(y)}, \mathbf{X_j(y)} \rangle_y, \end{array}$$

for $i, j \in \{1, 2, \ldots, n\}$, are differentiable. The g_{ij} are called the components of the metric tensor relative to x. *Often, when convenient, we will consider them as functions on U.*

There is a natural way of measuring distances on Riemannian manifolds.

Definition 2.2.2. *A piecewise differentiable curve is a continuous mapping* $c : [a, b] \to M$ *of a closed interval* $[a, b] \subset \mathbb{R}$ *into* M *satisfying the following condition: there exists a partition*

$$a = t_0 < t_1 < t_2 < \ldots < t_k = b$$

of $[a, b]$ *such that the restrictions* $c_{|[t_i,t_{i+1}]}, i = 0, 1, \ldots, k-1$, *are differentiable. The length of a piecewise differentiable curve c is defined as*

$$\int_a^b \sqrt{\langle c'(t), c'(t) \rangle_{c(t)}} dt.$$

On a connected Riemannian manifold, every two points are connected by at least one piecewise differentiable curve. The distance between two points is defined as the infimum of the lengths of all piecewise differentiable curves which connect the two points. It is a standard fact that the topology induced by this metric is the same as that in Definition 2.1.2 (see pg 123 of [89]).

A very important related concept in Riemannian geometry is that of *geodesic* and *exponential map.* To recall these concepts, we note that on a Riemannian manifold, there is a standard way of defining covariant derivatives, using the Levi-Civita connection. We refer the reader to Chapter 2 in [46] for details.

Definition 2.2.3. *A parametrized curve* $\gamma : I \to M$ *is a geodesic at* $t_0 \in I$ *if* $\frac{D}{dt}(\frac{d\gamma}{dt}) = 0$ *at the point* t_0. *If* γ *is geodesic at* t, *for all* $t \in I$, *we say that* γ *is a geodesic.*

Corollary 3.9 in Chapter 3 from [46] shows that geodesics have certain distance minimizing properties.

Proposition 2.2.4. *If a piecewise differentiable curve* $\gamma : [a, b] \to M$, *with parameter proportional to arc length, has length less or equal to the length of any other piecewise differentiable curve joining* $\gamma(a)$ *to* $\gamma(b)$ *then* γ *is a geodesic.*

It is a standard fact in literature that given a point $y \in M$, there exists a neighbourhood U of y and some $\epsilon > 0$ such that $\forall q \in U$, $\forall v \in T_qM$ with $\|v\| < \epsilon$, there exists a geodesic $\gamma_{q,v}$ such that $\gamma_{q,v}(0) = q$, $\gamma'_{q,v}(0) = v$ and $\gamma_{q,v}$ is defined on $(-2, 2)$.

Definition 2.2.5 (Exponential map)**.** *Let $y \in M$ and let U and ϵ be as above. We define the exponential map* exp *as follows:*

$$\begin{array}{rccl} \exp: & U \times \{(q,w) \mid q \in U, w \in T_qM, \|w\| < \epsilon\} & \rightarrow & M \\ & (q,w) & \mapsto & \gamma_{q,w}(1). \end{array}$$

We often denote $\exp(q,w)$ *by* $\exp_q(w)$*.*

One easily shows that for any $y \in M$, there exists some $\delta > 0$ such that $\exp_y$, restricted to the open ball $B(0,\delta)$ of radius δ around 0, is a diffeomorphism onto its image.

Definition 2.2.6. *If W is an open neighbourhood of y such that for some $\delta > 0$, $(\exp_y)_{|B(0,\delta)}$ is a diffeomorphism whose image contains W, then W is called a normal neighbourhood of y.*

There is a class of open neighbourhoods which is even stronger:

Theorem 2.2.7 (Theorem 3.7 in Chapter 3 of [46])**.** *For any $y \in M$ there exist an open neighbourhood W of y and a number $\delta > 0$ such that, for every $q \in W$, $\exp_q$ restricted to $B(0,\delta) \subset T_qM$ is a diffeomorphism onto its image and $\exp_q(B(0,\delta)) \supset W$, that is, W is a normal neighbourhood of each of its points.*

Definition 2.2.8. *A neighbourhood satisfying the conditions of Theorem 2.2.7 is called a* totally normal neighbourhood *of $y \in M$.*

When going into the proofs of the above mentioned classical results, one can derive a Lemma which is very important for our purposes (see Remark 3.8 in [46]). In order for the statement to make sense, we remind the reader that TM, the tangent bundle of M, inherits a natural differentiable structure from M.

Lemma 2.2.9. *Let y_1, y_2 be elements of a totally normal neighbourhood W. Given $y_2 \in W$, there exists a unique minimizing geodesic γ joining y_1 to y_2. Similarly, there exists a unique $v \in T_{y_1}M$ such that $\gamma'(0) = v$. The map which sends $(y_1, y_2) \in W \times W$ to $(y_1, v) \in TM$ is differentiable.*

Definition 2.2.10. *Let M be a Riemannian manifold. We say that M is geodesic if any two points $p, q \in M$ are connected by a geodesic with length equal do $d(p,q)$.*

The classical example of a Riemannian manifold which is not geodesic, is the punctured plane $\mathbb{R}^2\backslash\{(0,0)\}$. It is a standard fact that closed manifolds, i.e. compact manifolds without boundary, are geodesic. Even stronger, such manifolds are complete, meaning that exp is defined and smooth everywhere on TM.

The way diffeomorphisms identify differentiable manifolds, isometries can be used to identify Riemannian manifolds.

Definition 2.2.11. *Let M and N be Riemannian manifolds. A diffeomorphism $f : M \to N$ is called an* isometry *if:*

$$\langle \mathbf{v}, \mathbf{w} \rangle_y = \langle \mathbf{df_y}(\mathbf{v}), \mathbf{df_y}(\mathbf{w}) \rangle_{f(y)}, \text{ for all } y \in M, v, w \in T_y M.$$

The group of isometries of a Riemannian manifold M into itself will be denoted by $\mathrm{Iso}(M)$*.*

It turns out that this definition of isometry is equivalent to the metric definition of an isometry as a distance-preserving diffeomorphism [71].

Remark 2.2.12 (Product manifolds)**.** *We will be primarily interested in isometries on a product of Riemannian manifolds M and N. We shall elaborate a bit on this. Let M and N be differentiable manifolds of dimension n and m, and let $\{(U_\alpha, x_\alpha)\}$ and $\{(V_\beta, \chi_\beta)\}$ be differentiable structures on M and N respectively. For each α and β, we define the mappings*

$$\eta_{\alpha\beta}(p,q) = (x_\alpha(p), \chi_\beta(q)), \; p \in U_\alpha, \; q \in V_\beta.$$

It is a standard fact that $\{(U_\alpha \times V_\beta, \eta_{\alpha\beta})\}$ is a differentiable structure on $M \times N$ such that the projections $\pi : M \times N \to M$ and $p : M \times N \to N$ are differentiable (pg. 31, [46]).

Every curve $\psi(t)$ on $M \times N$ is of the form $(\alpha(t), \beta(t)) \in M \times N$ and

$$\begin{array}{rcl} T_{(y,z)} M \times N & = & T_y M \oplus T_z N \\ \psi'(0) & = & \alpha'(0) \oplus \beta'(0). \end{array}$$

The equality must be interpreted as follows: given a differentiable function $g : M \times N \to \mathbb{R}$, we define

$$\begin{array}{rrcl} \widetilde{g} : & M & \to & \mathbb{R} \\ & y & \mapsto & g(y, \beta(0)) \end{array}$$

and similarly

$$\begin{array}{rccl} \overline{g}: & N & \rightarrow & \mathbb{R} \\ & z & \mapsto & g(\alpha(0), z). \end{array}$$

Then $\psi'(0)(g) = \alpha'(0)(\widetilde{g}) + \beta'(0)(\overline{g})$. *Finally, the product* $M \times N$ *is again a Riemannian manifold with inner product given by*

$$\langle \mathbf{v_1} \oplus \mathbf{w_1}, \mathbf{v_2} \oplus \mathbf{w_2} \rangle_{(y,z)} = \langle \mathbf{v_1}, \mathbf{v_2} \rangle_y + \langle \mathbf{w_1}, \mathbf{w_2} \rangle_z,$$

for all $(y, z) \in M \times N$, $\mathbf{v_1}, \mathbf{v_2} \in T_y(M)$ *and* $\mathbf{w_1}, \mathbf{w_2} \in T_z(N)$.

Definition 2.2.13. *An isometry* f *on a product of manifolds is said to* split *if its* M*-component* $f_1 : M \times N \rightarrow M$ *is independent of the* N*-coordinates and its* N*-component* $f_2 : M \times N \rightarrow N$ *is independent of its* M*-coordinates. In this case, the component mappings* f_1 *and* f_2 *can be seen as isometries of* M *and* N *respectively. The statement that all isometries split is equivalent to the formula* $\mathrm{Iso}(M \times N) = \mathrm{Iso}(M) \oplus \mathrm{Iso}(N)$.

2.3 Fréchet manifolds

2.3.1 Fréchet spaces

A differentiable manifold is an object which locally *looks like* an open subset of Euclidean space $\mathbb{R}^n$. Therefore, it is possible to define what it means for a map between manifolds to be differentiable. In Section 3.3, we encounter naturally a class of intuitively "smooth" maps $f : N \rightarrow C^\infty(M, M)$, where N is a connected Riemannian manifold and where $C^\infty(M, M)$ is the set of C^∞ maps from a closed manifold M into itself. The intuitive idea of smoothness here is not covered by the classical theory of differentiable manifolds, since $C^\infty(M, M)$ is, in general, *not* a differentiable manifold. However, $C^\infty(M, M)$ does locally look like a *Fréchet space*, which can be seen as a type of vector space which generalizes the notion of Banach space, and in particular that of Euclidean space. It will turn out that $C^\infty(M, M)$ is, what is called a *Fréchet manifold*. Our definition of Fréchet space is taken from [62].

Definition 2.3.1. *A seminorm on a vector space* F *is a real-valued function* $\| \; \| : F \rightarrow \mathbb{R}^+$ *such that*

1. $\|h_1 + h_2\| \leq \|h_1\| + \|h_2\|$ *for all vectors* h_1 *and* h_2;
2. $\|ch\| = |c| \cdot \|h\|$ *for all scalars* c *and vectors* h.

A collection of seminorms $\{\| \ \|_n : n \in \mathbb{N}\}$ defines a unique topology such that a sequence or net $(h_j)_j$ converges to h if and only if $(\|h_j - h\|_n)_j \to 0$ for all $n \in \mathbb{N}$.

A *locally convex topological vector space* is a vector space that is equipped with a topology coming from a collection of seminorms. It turns out that the topology is *metrizable* if and only if it may be defined by a countable collection of seminorms $\{\| \ \|_n\}$. In this case, we may always use sequences instead of nets. Note that this topology is Hausdorff if and only if $h = 0$ whenever $\|h\|_n = 0$ for every $n \in \mathbb{N}$. We say that a sequence h_j is *Cauchy* if it is Cauchy for every norm, more precisely if $\|h_j - h_k\|_n \to 0$ as j and $k \to \infty$ for all n. The space F is called *complete* if every Cauchy sequence converges.

Definition 2.3.2. *A Fréchet space is a complete Hausdorff metrizable locally convex topological vector space.*

Let us start with an easy example.

Example 2.3.3. *Every Banach space is a Fréchet space. The collection of norms contains only one.*

We will need one specific Fréchet space in our work, see Example 2.3.5. Let us introduce it step by step.

Example 2.3.4. *Let* U *be an open relatively compact subset of* $\mathbb{R}^n$. *Denote* $K = \overline{U}$. *We say that a map* $h : K \to \mathbb{R}^m$ *is smooth on* K *if every element of* K *is contained in some open subset of* $\mathbb{R}^n$ *on which* h *can be extended to a* C^∞ *map. Consider the set*

$$\mathcal{F} := \{h : K \to \mathbb{R}^m \mid f \text{ is } C^\infty\}.$$

For every finite sequence $z = (i_1, i_2, \ldots, i_l)$ *where* $i_j \in \{1, 2, \ldots, n\}$ *for every* $j = 1, 2, \ldots, l$, *we define the norm* $\| \ \|_z$ *by*

$$\|h\|_z = \sup_{x \in K} \|\frac{\partial^l h}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_l}}(x)\|_m,$$

where $\| \ \|_m$ *is the Euclidean norm in* $\mathbb{R}^m$*. We include the degenerate case that* $l = 0$*, in which case*

$$\|h\|_{(\ )} = \sup_{x \in K} \|h(x)\|.$$

It is a standard exercise to verify that $\mathcal{F}$ *is a Fréchet space.*

Note that for every $k \in \mathbb{N}$*, the space of* C^k *maps from* K *to* $\mathbb{R}^m$ *is also a Fréchet space. Stronger even, since the family of norms introduced above becomes a finite family, we can replace it by one norm and obtain that it is a Banach space.*

We will now generalize this example to the setting of Section 3.3. Let us start by fixing some notation. We let M be a closed manifold and we take $g \in C^\infty(M, M)$, the maps from M to M which are C^∞. Consider the tangent bundle $p : TM \to M$ and denote its pullback under g by $g^*(TM)$:

$$g^*(TM) = \{(y, \epsilon) \mid y \in M, \epsilon \in TM \text{ with } p(\epsilon) = g(y)\}.$$

Given an open, relatively compact set $U \subset M$, we say that $TM_{|U}$ is trivial if it is isomorphic to a trivial bundle. For convenience, we will henceforth always assume that $\overline{U}$ is contained in the image of a coordinate chart x. Clearly, there is a local trivialization, mapping $v \in T_yM, y \in \mathrm{Im}(x)$ to $(y, b_1, b_2, \ldots, b_n) \in \mathrm{Im}(x) \times \mathbb{R}^n$ where the real numbers b_i are the coordinates of v relative to the basis of T_yM induced by x. Remark that TM is a differentiable manifold of dimension $2n$.

Furthermore, we shall say that $g^*(TM)_{|U}$ is trivial if $TM_{|U}$ is trivial and there exists a parametrization $\widetilde{x}$ and a real number $\delta > 0$ such that the image of $\widetilde{x}$ contains the δ-neighbourhood of $g(\mathrm{Im}(x))$. Again, there is a local trivialization, mapping $(y, \epsilon) \in g^*(TM)$ with $y \in \mathrm{Im}(x)$ to $(y, c_1, c_2, \ldots, c_n) \in \mathrm{Im}(x) \times \mathbb{R}^n$ where the c_i are the coordinates of ϵ relative to the basis of $T_{g(y)}M$ induced by $\widetilde{x}$.

Cover M by finitely many open sets U_α, such that each $g^*(TM)_{|U_\alpha}$ is trivial. Denote the corresponding charts, analogously to x and $\widetilde{x}$ above, by x_α and $\widetilde{x_\alpha}$. We call the finite set of triples

$$(U_\alpha, x_\alpha, \widetilde{x_\alpha})$$

a *trivializing family* for $g : M \to M$.

A map $s : M \to g^*(TM), y \mapsto (y, \cdot)$ is called a section of $g^*(TM)$. We always assume that it is smooth as a map between the differentiable manifolds M and $g^*(TM)$. By definition of trivializing family, the restriction to one of the U_α of $s : M \to g^*(TM)$, can be seen as a map $s^\alpha : U_\alpha \to U_\alpha \times \mathbb{R}^n$. The first component $U_\alpha \to U_\alpha$ is just the identity. Using the parametrization x_α, we can view the second component map $U_\alpha \to \mathbb{R}^n$ as a map from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$. We denote this map by $\overline{s^\alpha} : x_\alpha^{-1}(U_\alpha) \to \mathbb{R}^n$. Note that $\overline{s^\alpha}$ is in fact equally well defined on $x_\alpha^{-1}(\overline{U_\alpha})$. We now use Example 2.3.4 to define a family of norms on the space of smooth sections of $g^*(TM)$. For every finite sequence $z = (i_1, i_2, \ldots, i_l)$ where $i_j \in \{1, 2, \ldots, n\}$ for every $j = 1, 2, \ldots, l$ and for every α, we define the norm $\| \ \|_{z,\alpha}$ of a section s of $g^*(TM)$ by

$$\|s\|_{z,\alpha} := \|\overline{s^\alpha}\|_{z,\alpha} = \sup_{x \in x_\alpha^{-1}(\overline{U_\alpha})} \| \frac{\partial^l \overline{s^\alpha}}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_l}}(x) \|_n,$$

where $\| \ \|_n$ is the Euclidean norm in $\mathbb{R}^n$. We include the degenerate case that $l = 0$, in which case

$$\|s\|_{(\ ),\alpha} := \|\overline{s^\alpha}\|_{(\ ),\alpha} = \sup_{x \in x_\alpha^{-1}(\overline{U_\alpha})} \|\overline{s^\alpha}\|_n.$$

It is easy to verify that the space of smooth sections of $g^*(TM)$ equipped with this family of semi-norms is a Fréchet space. For reference sake, let us write this as an

Example 2.3.5. *Let M be a closed differentiable manifold and let $g : M \to M$ be a smooth map. The set of smooth sections of the pullback bundel $g^*(TM)$ equipped with the family of norms as given above, is a Fréchet space.*

The theory of differentiable maps can be extended to a context of Fréchet spaces. More concretely, let F_1 and F_2 be Fréchet spaces, U an open subset of F_1 and $f : U \subset F_1 \to F_2$ a continuous map.

Definition 2.3.6. *The derivative of f at the point $v \in U$ in the direction $h \in F_1$ is defined by*

$$Df(v)(h) = \lim_{t \to 0} \frac{f(v + th) - f(v)}{t}.$$

We say that f is differentiable at $v \in U$ in the direction h if the limit exists. We say that f is C^1 if the limit exists for every $v \in U$ and $h \in F_1$ and if $Df : U \times F_1 \to F_2$ is continuous (jointly as a function on the product).

Remark 2.3.7. *Note that this continuity-condition is weaker than the corresponding continuity-condition for C^1-maps on Banach spaces, i.e. if f is a map between Banach spaces which is Banach space differentiable, then it is Fréchet differentiable.*

In previous sections, we worked with C^∞ maps. We thus proceed by introducing the higher derivatives.

Definition 2.3.8. *Let F_1 and F_2 be Fréchet spaces and let $f : U \subset F_1 \to F_2$ be a C^1 map. We say that f is C^2 if Df is C^1, i.e. for every $v \in U, h_1, h_2 \in F_1$, we ask that*

$$D^2 f(v)(h_1, h_2) = \lim_{t \to 0} \frac{Df(v + th_2)(h_1) - Df(v)(h_1)}{t},$$

exists and that the map $D^2 f : U \times F_1 \times F_1 \to F_2$ is continuous (jointly as a function on the product). Higher derivatives are defined by induction, saying that f is C^{n+1} if $D^n f$ is C^1, i.e.

$$\lim_{t \to 0} \frac{D^n f(v + th_{n+1})(h_1, h_2, \ldots, h_n) - D^n f(v)(h_1, h_2, \ldots, h_n)}{t},$$

exists for every $v \in U$, $h_1, h_2, \ldots, h_n, h_{n+1} \in F_1$ and is continuous jointly as a function on $U \times F_1^{n+1}$.

Many of the standard analytical results on Banach spaces stay valid in the context of Fréchet spaces. We mention only the results that we will need later.

Theorem 2.3.9. *If $f : F_1 \to F_2$ and $g : F_2 \to F_3$ are C^1, then so is their composition $g \circ f$ and*

$$D(g \circ f)(v)(h) = Dg(f(v))(Df(v)(h)).$$

More general, if f and g are C^n, then so is their composition $g \circ f$.

Theorem 2.3.10. *If f is C^n, then $D^n f(v)(h_1, h_2, \ldots, h_n)$ is completely symmetric, i.e. for every $i, j \in \{1, 2, \ldots, n\}$, interchanging h_i and h_j has no effect. It is also linear seperately in $h_1, h_2, \ldots, h_n$.*

Convention 2.3.11. *In previous sections, the word* differentiable *stood for C^∞. We will honor this convention here: from now on, "differentiable" or "smooth" means C^∞.*

2.3.2 Fréchet manifolds

Definition 2.3.12. *A* Fréchet manifold *is a Hausdorff topological space $\mathcal{C}$, equipped with a family of injective mappings $x_\alpha : U_\alpha \to \mathcal{C}$, where the U_α are open subsets of Fréchet spaces F_α, such that:*

1. $\cup_\alpha x_\alpha(U_\alpha) = \mathcal{C}$,
2. *for any pair α, β with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \phi$, the sets $x_\alpha^{-1}(W)$ and $x_\beta^{-1}(W)$ are open sets and the mappings $x_\beta^{-1} \circ x_\alpha$ are Fréchet differentiable,*
3. *the family $\{(U_\alpha, x_\alpha)\}$ is maximal relative to the conditions (1) and (2).*

The pair (U_α, x_α) or the mapping x_α with $p \in x_\alpha(U_\alpha)$ is called a parametrization of $\mathcal{C}$ at p; $x_\alpha(U_\alpha)$ is then called a coordinate neighbourhood *at p. A family $\{(U_\alpha, x_\alpha)\}$ satisfying (1) and (2) is called a* Fréchet differentiable structure.

Definition 2.3.13. *The topology on a Fréchet manifold $\mathcal{C}$ is defined as follows: given a set $A \subset \mathcal{C}$, we call it* open *if and only if $x_\alpha^{-1}(A \cap x_\alpha(U_\alpha))$ is open in F_α for all α.*

Definition 2.3.14. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be Fréchet manifolds. A map $f : \mathcal{C}_1 \to \mathcal{C}_2$ is differentiable at $p \in M$ if given a parametrization $\chi : V \subset F_\chi \to \mathcal{C}_2$ at $f(p)$, there exists a parametrization $x : U \subset F_x \to M$ at p such that $f(x(U)) \subset \chi(V)$ and the mapping*

$$\chi^{-1} \circ f \circ x : U \subset F_x \to F_\chi$$

is differentiable at $x^{-1}(p)$. The map f is differentiable on an open set of $\mathcal{C}_1$ if it is differentiable at all of the points of this open set.

One can check that this definition is independent of the chosen parametrizations.

Example 2.3.15. *Every differentiable manifold is a Fréchet manifold, clearly. Moreover, the usual definition of smooth maps between differentiable manifolds completely coincides with the definition above (see also Theorem 2.3.10).*

The following example, see also [84], is key for our purposes. Precisely, we introduce the Fréchet manifold $C^\infty(M,M)$ of C^∞ self-maps of a closed n-dimensional Riemannian manifold M. In order to do so, fix $g \in C^\infty(M,M)$ and recall that the space F of smooth sections of $g^*(TM)$ is a Fréchet space (see Example 2.3.5). We show that there is an open subset $U \subset F$ which injects naturally into a subset of $C^\infty(M,M)$ which contains g.

Since every point $y \in M$ has a totally normal neighbourhood (see Definition 2.2.8) and since M is compact, there exists $\delta > 0$ such that $\exp_q$ restricted to $B(0,\delta) \subset T_qM$ is a diffeomorphism for all $q \in M$. Fix a trivializing family $(U_\alpha, x_\alpha, \widetilde{x_\alpha})$ for $g : M \to M$ and equip the space of sections of $g^*(TM)$ as a Fréchet space F (see Example 2.3.5). Given $s \in F$ and $y \in M$, denote $\overline{\pi}(s(y))$ the component of $s(y)$ inside $T_{g(y)}M$. Look at the open set $\mathcal{O} \subset F$ of smooth sections $s : M \to g^*(TM)$ in F such that $\overline{\pi}(s(y)) \in T_{g(y)}M$ has length strictly smaller than δ for every y. We define a parametrization $x : \mathcal{O} \to C^\infty(M,M)$ at $g \in C^\infty(M,M)$ as follows. Given a section $s \in \mathcal{O}$, look at the map $x(s) : M \to M$ which maps a point $y \in M$ to $exp_{g(y)}(\overline{\pi}(s(y)))$. Note that $x(0) = g$. One can verify that these parametrizations x form a Fréchet differentiable structure on $C^\infty(M,M)$. We write the conclusion as an

Example 2.3.16. *The space $C^\infty(M,M)$, where M is a closed Riemannian manifold, is a Fréchet manifold.*

One can check that the topology on $C^\infty(M,M)$ corresponds to the weak (or equivalently strong) topology on $C^\infty(M,M)$ that was introduced by Hirsch in [64]. We end this background chapter by a remark related to the diffeomorphism group of a closed Riemannian manifold M. First, a definition.

Definition 2.3.17. *Let $\mathcal{C}$ be a Fréchet manifold and Z a closed subset of $\mathcal{C}$. We say that Z is a submanifold of $\mathcal{C}$ if for every $z \in Z$, there exists* *a parametrization* $x : F \times G \to \mathcal{C}$ *of* $\mathcal{C}$ *at* z*, where* F *and* G *are Fréchet spaces and such that* $y \in Im(x)$ *lies in* Z *if and only if* $x^{-1}(y) \in F \times \{0\}$*.*

Example 2.3.18 ([62], pg. 88)**.** *Let* M *be a closed Riemannian manifold. The group of diffeomorphisms of* M *is a submanifold of* $C^\infty(M, M)$*.*

Chapter 3

Results

In this chapter, which is joint work with Nansen Petrosyan, we introduce the class of *fiberwise volume non-increasing* diffeomorphisms on a product $M \times N$ of Riemannian manifolds, where M is a closed n-manifold. This class of maps is extremely interesting if we impose the cohomological condition of *minimal n-cohomology* on the product. This condition and the class of fiberwise volume non-increasing diffeomorphisms is introduced in Section 3.1. We believe that there are three main results: the *slice, splitting* and *structure* Theorem. The slice Theorem gives a property that characterizes the class of fiberwise volume non-increasing diffeormorphisms. The splitting Theorem uses this key property to show that isometries on products with minimal n-cohomology split. The structure Theorem describes the group of fiberwise volume non-increasing diffeomorphisms on a product manifold $M \times N$ in terms of the diffeomorphism groups of M and N. We shall treat the *slice* and *splitting* Theorem in Section 3.2. The structure Theorem will be treated in Section 3.3. We end this chapter with two nice applications on properly discontinuous actions in Section 3.4.

We shall proceed under the convention that M and N are second countable Riemannian manifolds.

3.1 Fiberwise volume non-increasing diffeomorphisms and minimal n-cohomology

In this paragraph, we introduce the class of maps and the cohomological condition which are central in our work. In order to so, we note that there exists a standard way to define the volume of certain *nice* subsets of Riemannian manifolds. We extend this definition in a natural way, enabling us to define the volume of a Riemannian manifold.

Definition 3.1.1. *A subset A of a manifold M has measure* 0 *if $x^{-1}(A)$ has Lebesgue measure* 0 *in $\mathbb{R}^n$ for every parametrization x of M.*

Observe that the notion of measure 0 is invariant under diffeomorphisms.

Definition 3.1.2. *Assume that $x : U \to M$ is a parametrization. If C is an open connected set such that $\overline{C} \subset x(U)$ is compact and such that the boundary $\partial(C)$ of C has measure* 0*, then we call C a* nice open *of M.*

On Riemannian manifolds, the volume of a nice open C of M is defined by

$$\mathrm{Vol}(C) = \int\limits_{x^{-1}(C)} \sqrt{\det(g_{ij})}\, d\mu,$$

where the g_{ij} are the components of the metric tensor relative to x and where μ is the Lebesgue measure on $\mathbb{R}^n$. This definition is independent of the parametrization used.

Definition 3.1.3. *A diffeomorphism $f : M \to M$ is* volume preserving *if it preserves the volume of all nice opens of M.*

There is a standard way to find a *nice family for M*, i.e. a family of nice opens, say $(C_i)_{i\in I}$ where I is some index set, such that the C_i are pairwise disjoint and such that $M \backslash \bigcup_{i\in I} C_i$ has measure 0. Indeed, start with a countable number of nice opens $B_1, B_2, \ldots$ in M whose union is M. This is always possible since M is second countable. Consider the sets

$$B_1' := B_1,\ B_2' := B_2 \backslash \overline{B_1},\ B_3' := B_3 \backslash \overline{B_1 \cup B_2},\ \ldots,\ B_n' := B_n \backslash \overline{\cup_{i=1}^{n-1} B_i},\ \ldots$$

We can take the family $(C_i)_{i\in I}$ as the family of connected components of the sets B'_j.

Definition 3.1.4. *Let $(C_i)_{i\in I}$ be a nice family for M. We define the volume of M as*

$$\mathrm{Vol}(M) = \sum_{i\in I} \mathrm{Vol}(C_i).$$

Note that nice families are always countable because of the second countability condition on M. Also, note that the above definition does not depend on the chosen nice family and that volume preserving diffeomorphisms preserve $\mathrm{Vol}(M)$.

We come to a very important

Definition 3.1.5. *Let $f : M \times N \to M \times N$ be a diffeomorphism and let $z \in N$. Equip both $M \times \{z\}$ and $f(M \times \{z\})$ with the Riemannian metric induced from $M \times N$ and note that* $\mathrm{Vol}(M \times \{z\}) = \mathrm{Vol}(M)$. *We say that f is* fiberwise volume non-increasing at z *if*

$$\mathrm{Vol}(f(M \times \{z\})) \leq \mathrm{Vol}(M).$$

A diffeomorphism is fiberwise volume non-increasing *(fni) if it is fiberwise volume non-increasing at every point of N. We denote the set of all fiberwise volume non-increasing maps of $M \times N$ by FNI$(M \times N)$.*

It is interesting to investigate which maps exactly are fni. Clearly, isometries are fiberwise volume non-increasing, in fact they are fiberwise volume *preserving*. Maps of the form (f, g) where f is a diffeomorphism of M and g a diffeomorphism of N are also fni.

There appears to be no immediate connection between fni maps and volume preserving maps on a product $M \times N$. For example, on the cylinder $S^1 \times \mathbb{R} \subset \mathbb{R}^3$, one can consider the diffeomorphism mapping $(\cos(x), \sin(x), y) \in S^1 \times \mathbb{R}$ to $(\cos(x), \sin(x), \frac{y}{2})$. This map is clearly not volume preserving, but it is fni. Conversely, the diffeomorphism

$$\begin{array}{cccc} f: & S^1 \times \mathbb{R} & \to & S^1 \times \mathbb{R} \\ & (\cos(x), \sin(x), y) & \mapsto & (\cos(x), \sin(x), y + \sin(x)) \end{array}$$

is volume preserving, since the Jacobian of the map f has determinant one at each point of $\mathbb{R}^3$. Yet, f is not fni.

The class of fiberwise volume non-increasing diffeomorphisms on a product $M \times N$ becomes extremely interesting if we impose a certain cohomological condition on this product.

Definition 3.1.6. *Let M be an n-dimensional closed Riemannian manifold. Apart from being Riemannian, we put no conditions on N. We say that $M \times N$ has* minimal n-cohomology *if*

$$H^n(M \times N; \mathbb{Z}_2) \cong H^n(M; \mathbb{Z}_2).$$

The terminology is due to the fact that $H^n(M \times N; \mathbb{Z}_2)$ necessarily contains a copy of $H^n(M; \mathbb{Z}_2)$ because the natural projection $\pi : M \times N \to M$ is surjective and so induces an injective map π^* on the cohomology level. Note that the above definition also implies that N is connected.

Choose a point $z \in N$ and define the inclusion

$$\begin{array}{rccc} i: & M & \to & M \times N \\ & y & \mapsto & (y, z). \end{array}$$

Compactness of M implies that $H^n(M; \mathbb{Z}_2) \cong H_0(M; \mathbb{Z}_2)$ is a finite group, and so our definition of minimal n-cohomology is equivalent to the fact that

$$\pi^* : H^n(M; \mathbb{Z}_2) \to H^n(M \times N; \mathbb{Z}_2)$$

and

$$i^* : H^n(M \times N; \mathbb{Z}_2) \to H^n(M; \mathbb{Z}_2)$$

are isomorphisms.

One of the most important, yet trivial examples, is the case of products $M \times N$ where M is closed and N is contractible. Other examples include the products $S^n \times S^m$ for $m \neq n$. Indeed, for $M = S^n$ and $N = S^m$, the result follows by the following Künneth formula:

$$H^n(M \times N; \mathbb{Z}_2) \cong \oplus_{i=0}^{n} H^{n-i}(M) \otimes_{\mathbb{Z}_2} H^i(N) \cong H^n(M).$$

3.2 Slice and Splitting Theorem

3.2.1 Preliminaries

We will need two additional preliminary results for the proofs of the slice and structure Theorem. The first can be deduced from the following standard result from algebraic topology.

Theorem 3.2.1. (Poincaré-Lefschetz Duality, [19]) *Let M be a compact orientable n-manifold and let L be a closed subset of M. Denoting Čech cohomology by $\check{H}$, we have the following commutative diagram where the columns are exact and all the horizontal arrows (cap products with the orientation class) are isomorphisms:*

$$\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d] \\
\check{H}^p(M,L) \arrow[rr, "\cong"] \arrow[d] & & H_{n-p}(M\backslash L) \arrow[d] \\
\check{H}^p(M) \arrow[rr, "\cong"] \arrow[d] & & H_{n-p}(M) \arrow[d] \\
\check{H}^p(L) \arrow[rr, "\cong"] \arrow[d] & & H_{n-p}(M, M\backslash L) \arrow[d] \\
\check{H}^{p+1}(M,L) \arrow[rr, "\cong"] \arrow[d] & & H_{n-p-1}(M\backslash L) \arrow[d] \\
\vdots & & \vdots
\end{tikzcd}$$

For non-orientable M the theorem holds with $\mathbb{Z}_2$-coefficients.

One deduces the following useful

Corollary 3.2.2. *If L is a proper closed subset of a closed n-manifold M, then*

$$\mid \check{H}^n(L;\mathbb{Z}_2) \mid < \mid \check{H}^n(M;\mathbb{Z}_2) \mid .$$

Proof. The corollary follows from the fact that $\check{H}^n(L;\mathbb{Z}_2)$ is isomorphic to $H_0(M, M\backslash L;\mathbb{Z}_2)$ and this group contains less elements than $H_0(M;\mathbb{Z}_2) \cong \check{H}^n(M;\mathbb{Z}_2)$, by Theorem 3.2.1. □

Apart from this, we will need a certain algebraic fact. Denote the set of $n \times n$-matrices with $\mathbb{R}$-coefficients by $\mathcal{M}_n(\mathbb{R})$. We recall the following definition.

Definition 3.2.3. *A symmetric matrix $\mathbf{G}$ in $\mathcal{M}_n(\mathbb{R})$ is positive definite if $\mathbf{x}^T\mathbf{G}\mathbf{x} > 0$ for every non-zero vector $\mathbf{x} \in \mathbb{R}^k$. A symmetric matrix $\mathbf{H} \in \mathcal{M}_n(\mathbb{R})$ is positive semi-definite if $\mathbf{x}^T\mathbf{H}\mathbf{x} \geq 0$ for every vector $\mathbf{x} \in \mathbb{R}^k$.*

We prove the following

Lemma 3.2.4. *If* $\mathbf{G} \in \mathcal{M}_n(\mathbb{R})$ *is positive definite and* $\mathbf{H} \in \mathcal{M}_n(\mathbb{R})$ *is positive semi-definite, then* $\det(\mathbf{G}+\mathbf{H}) \geq \det(\mathbf{G})$. *The inequality is strict when* $\mathbf{H} \neq \mathbf{0}$.

Proof. We start by proving the special case where $\mathbf{H} = \mathbf{E} = (\mu, 0, 0, \ldots, 0)$ with $\mu \geq 0$. Here, the notation $(e_{11}, e_{22}, \ldots, e_{nn})$ stands for a diagonal matrix whose $(i,i)^{\text{th}}$ entry is e_{ii}.
Denote by $\widetilde{\mathbf{G}}$ the matrix obtained from $\mathbf{G}$ by removing the first row and column, i.e. $\widetilde{\mathbf{G}}_{ij} = \mathbf{G}_{(i+1)(j+1)}$ for $i, j \in \{1, 2, \ldots, n-1\}$. Expanding $\det(\mathbf{G}+\mathbf{E})$ by the first row gives

$$\det(\mathbf{G}+\mathbf{E}) = \det(\mathbf{G}) + \mu \det(\widetilde{\mathbf{G}}).$$

Since

$$(x_1, x_2, \ldots, x_{n-1})\widetilde{\mathbf{G}}(x_1, x_2, \ldots x_{n-1})^T$$

equals

$$(0, x_1, x_2, \ldots, x_{n-1})\mathbf{G}(0, x_1, x_2, \ldots, x_{n-1}),$$

for all $(x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, we have that $\widetilde{\mathbf{G}}$ is positive definite. This implies that $\det(\widetilde{\mathbf{G}}) > 0$ and thus $\det(\mathbf{G}+\mathbf{E}) \geq \det(\mathbf{G})$. Strict inequality holds if and only if $\mu > 0$. Notice that a similar proof exists when $\mathbf{H}$ equals a diagonal matrix of the form $(0, 0, \ldots, 0, \mu, 0, \ldots, 0)$.

In general, take an orthogonal matrix $\mathbf{O}$ such that $\mathbf{D} = \mathbf{OHO}^T = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Clearly, $\lambda_i \geq 0$ for all i. We have

$$\det(\mathbf{G}+\mathbf{H}) = \det(\mathbf{OGO}^T + \mathbf{D}) = \det(\mathbf{OGO}^T + \mathbf{E_1} + \mathbf{E_2} + \ldots + \mathbf{E_n}),$$

where $\mathbf{E_i}$ is the matrix that has λ_i as its $(i,i)^{th}$ entry and zeros everywhere else. By positive definiteness of $\mathbf{OGO}^T$ we have that $\mathbf{OGO}^T + \mathbf{E_1} + \mathbf{E_2} + \ldots + \mathbf{E_k}$ is positive definite for each $k \in \{1, 2, \ldots, n\}$. The proof now follows from the special case proven above. ☐

Subsequently, the following shorter proof was suggested by Alain Valette.

Proof. Assume first that $G = 1$, the identity matrix. since H is diagonalizable with non-negative eigenvalues, the result is clear. Now for the general case: write $G+H = G^{1/2}(1+H')G^{1/2}$ where $H' = G^{-1/2}HG^{-1/2}$. Then $\det(G+H) = \det(G)\det(1+H')$. As H' is positive semi-definite, we obtain the result from the first case. ☐

3.2.2 Slice Theorem

In this paragraph, we give a characteristic property for fiberwise volume non-increasing diffeomorphisms on products $M \times N$ that have minimal n-cohomology. The first auxiliary result is also interesting on its own merits.

Proposition 3.2.5. *If $M \times N$ has minimal n-cohomology, then we have that*

$$\phi := \pi \circ f \circ i : M \to M$$

is surjective for any homeomorphism $f : M \times N \to M \times N$.

Proof. Since f is a homeomorphism and since $M \times N$ has minimal n-cohomology, we know that

$$\phi^* : H^n(M; \mathbb{Z}_2) \xrightarrow{\pi^*} H^n(M \times N; \mathbb{Z}_2) \xrightarrow{f^*} H^n(M \times N; \mathbb{Z}_2) \xrightarrow{i^*} H^n(M; \mathbb{Z}_2)$$

is an isomorphism. Assume by contradiction that ϕ is not surjective. The image of ϕ is compact and thus closed. Since it misses a point, say p, it has to miss an open subset of M, say U. Take a CW-complex structure on M containing an open n-cell σ with $p \in \sigma \subset U$. Now, the forgetful map $\phi_1 : M \to M \backslash \sigma$ of ϕ induces the mapping $\phi_1^* : H^n(M\backslash\sigma; \mathbb{Z}_2) \to H^n(M; \mathbb{Z}_2)$. Let j be the inclusion mapping of $M\backslash\sigma$ into M. On the cohomology level we obtain

$$\phi_1^* \circ j^* : H^n(M; \mathbb{Z}_2) \to H^n(M; \mathbb{Z}_2),$$

and this mapping equals ϕ^*. Since ϕ^* is surjective, we conclude that ϕ_1^* must be surjective, which is a contradiction to Corollary 3.2.2 because Čech cohomology and singular cohomology are isomorphic for CW-complexes. ☐

Remark 3.2.6. *Note that the analogous result is valid for homeomorphisms $f : M_1 \times N_1 \to M_2 \times N_2$ where both products have minimal n-cohomology.*

Proposition 3.2.7. *Let $f : M \times N \to M \times N$ be a diffeomorphism of a product of Riemannian manifolds M and N. Choose $z \in N$ and equip $f(M \times \{z\})$ with the induced metric from $M \times N$. Suppose that C is a nice open in M such that the natural projection map $\pi : f(M \times \{z\}) \to M$*

restricts to a diffeomorphism Θ *onto an open set containing* $\overline{C}$*. Then,* $\mathrm{Vol}(\Theta^{-1}(C)) \geq \mathrm{Vol}(C)$*. Moreover, the equality is strict if and only if the projection* $p : M \times N \to N$ *is not constant on* $\Theta^{-1}(C)$*.*

Proof. Let $x : U \to M$ be a parametrization for M such that $\overline{C} \subset x(U)$. Let $V = x^{-1}(C)$ and consider the parametrization

$$\psi := \Theta^{-1} \circ x : V \to \Theta^{-1}(C).$$

Write $\psi = (x, \eta)$ where $x : V \to M$ is the M-component map and where $\eta : V \to N$ is the N-component map of ψ. Denote the components of the metric tensor relative to x and ψ by g_{ij} and $\widetilde{g_{ij}}$ respectively. Denote by $\mathbf{G}$ and $\widetilde{\mathbf{G}}$ the matrix whose coefficients are the maps g_{ij} and $\widetilde{g_{ij}}$ respectively. By definition we have that

$$\mathrm{Vol}(C) = \int\limits_V \sqrt{\det(\mathbf{G}_{ij})(q)}\, d\mu$$

and

$$\mathrm{Vol}(\Theta^{-1}(C)) = \int\limits_V \sqrt{\det(\widetilde{\mathbf{G}_{ij}})(q)}\, d\mu.$$

To prove that $\mathrm{Vol}(\Theta^{-1}(C)) \geq \mathrm{Vol}(C)$ it thus suffices to show that $\det(\mathbf{G}_{ij}(q)) \leq \det(\widetilde{\mathbf{G}_{ij}}(q))$ for all $q \in V$. Let us investigate the functions $\widetilde{g_{ij}}$.

For each $i \in \{1, 2, \ldots, n\}$ and $q = (q_1, q_2, \ldots, q_n) \in V$, denote the curve

$$x(q_1, q_2, \ldots, q_i + t, q_{i+1}, \ldots, q_n)$$

by $x_i^q(t)$ and

$$\psi(q_1, q_2, \ldots, q_i + t, q_{i+1}, \ldots q_n)$$

by $\psi_i^q(t) = (x_i^q(t), \eta_i^q(t)) \in M \times N$. For simplicity, we drop the upper index q in the following calculation.

$$\begin{aligned}
\widetilde{g_{ij}}(q) &= \langle \psi_i'(0), \psi_j'(0) \rangle_{\psi(q)} \\
&= \langle (x_i(t), \eta_i(t))'(0), (x_j(t), \eta_j(t))'(0) \rangle_{\psi(q)} \\
&= \langle x_i'(0), x_j'(0) \rangle_{x(q)} + \langle \eta_i'(0), \eta_j'(0) \rangle_{\eta(q)} \\
&= g_{ij}(q) + h_{ij}(q),
\end{aligned}$$

where

$$h_{ij}(q) = \langle (\eta_i^q)'(0), (\eta_j^q)'(0)\rangle_{\eta(q)}.$$

This shows that $\widetilde{g_{ij}}(q) = g_{ij}(q) + h_{ij}(q)$ for all $q \in V$. The first part of the proposition now follows from Lemma 3.2.4.

If $p \circ \Theta^{-1}$ is not constant on C, then $\mathrm{Vol}(\Theta^{-1}(C)) > \mathrm{Vol}(C)$. Indeed, in this case there exists an open set $O \subset C$ such that the linear map $D(p \circ \Theta^{-1})_y \neq 0$ for each $y \in O$. Let $W = x^{-1}(O)$. We have that for each $q \in W$ there exists $i_q \in \{1, 2, \ldots, n\}$ such that

$$D(p \circ \Theta^{-1})_{x(q)}((x_{i_q}^q)'(0)) \neq 0. \tag{3.1}$$

Since $\eta = p \circ \Theta^{-1} \circ x$, we conclude that the matrices with coefficients $h_{ij}(q)$ are non-zero. Our claim now follows from Lemma 3.2.4.

Finally, if $\mathrm{Vol}(\Theta^{-1}(C)) > \mathrm{Vol}(C)$, then $p \circ \Theta^{-1}$ is not constant. Indeed, this follows from the fact that the $h_{ij}(q)$ can not all be 0. ☐

Theorem 3.2.8 (Slice Theorem)**.** *Let M be a connected Riemannian manifold and let N be a Riemannian manifold such that $M \times N$ has minimal n-cohomology. If $f : M \times N \to M \times N$ is fni at $z \in N$, then there exists $w \in N$ such that $f(M \times \{z\}) = M \times \{w\}$.*

Proof. Assume that f is fiberwise volume non-increasing at z. We prove the theorem by showing that

$$\mathrm{Vol}(f(M \times \{z\})) > \mathrm{Vol}(M),$$

if $f(M \times \{z\})$ is not of the form $M \times \{w\}$ for some $w \in N$. For the remainder of the proof we will denote $f(M \times \{z\})$ by $f(M)$.

Let π be the natural projection map of $f(M)$ onto M. From Proposition 3.2.5, it follows that $\pi \circ f_{|M\times\{z\}}$ is surjective. Let us look at the set A of critical values of π. This set is closed by Lemma 2.1.5 and we know by Sard's theorem that it is of measure 0 in M. Take a family of nice opens $(C_i)_{i\geq 0}$ of M that are pairwise disjoint, and such that their union equals $M\backslash\widetilde{A}$ where $\widetilde{A} \supset A$ has measure 0. We can assume this family to be such that the C_i satisfy the hypotheses of proposition 3.2.7. We conclude that $\mathrm{Vol}(f(M)) \geq \mathrm{Vol}(M)$.

Assume there exists a nice open $C \subset M$ such that

1. there are open subsets $V \subset f(M)$ and $O \subset M$ with $\Theta := \pi_{|V} : V \to O$ a diffeomorphism and $\overline{C} \subset O$,

2. $\mathrm{Vol}(\Theta^{-1}(C)) > \mathrm{Vol}(C)$.

We can then look at a nice family of M containing C to conclude that $\mathrm{Vol}(f(M)) > \mathrm{Vol}(M)$, obtaining the desired contradiction. It remains thus to prove the existence of a nice open C, satisfying the two conditions above, in case $f(M)$ is not a fiber.

Denote $p : f(M) \to N$ the projection map. Let us show first that there exist $y \in f(M)$ such that both $(D\pi)_y$ is an isomorphism and the differential $(Dp)_y \neq 0$. Proceeding by contradiction, we obtain disjoint, open sets

$$\mathcal{A}_1 = \{y \in f(M) \mid Dp_y \neq 0\},$$

and

$$\mathcal{A}_2 = \{y \in f(M) \mid D\pi_y \text{ is an isomorphism}\}.$$

They are nonempty, since we have assumed that $f(M \times \{z\})$ is not of the form $M \times \{w\}$. Since f is a diffeomorphism, we have that $\mathcal{A}_1 \cup \mathcal{A}_2 = f(M)$. Since M is connected, this is a contradiction. Hence, there exists an element $y \in f(M)$ such that $(Dp)_y \neq 0$ and $(D\pi)_y$ is an isomorphism. Take a nice open $U \subset f(M)$ consisting of such points y such that $\pi_{|U}$ is a diffeomorphism onto its image. This image contains the closure of a nice open set C. Now, $\mathrm{Vol}(\Theta^{-1}(C)) > \mathrm{Vol}(C)$, as desired. □

Remark 3.2.9. *Assume for a moment that M and N are not necessarily connected. Denote the connected components of M and N by $M_1, M_2, \ldots, M_l$ and $N_1, N_2, \ldots, N_k$ respectively. Let us say that $M \times N$ satisfies* generalized minimal n-cohomology*, if*

$$H^n(M \times N; \mathbb{Z}_2) \cong \oplus_{j=1}^{k} H^n(M; \mathbb{Z}_2).$$

Intuitively, the word "minimal" makes sense: the natural projection maps $\pi_j : M \times N_j \to M$ are surjective and so each induce a monomorphism of $H^n(M; \mathbb{Z}_2)$ into $H^n(M \times N; \mathbb{Z}_2) = \oplus_{j=1}^{k} H^n(M \times N_j; \mathbb{Z}_2)$. In fact, the condition of generalized minimal n-cohomology of $M \times N$ implies that the connected components $M_i \times N_j$ have minimal n-cohomology. Now fix $z \in N_j$ and note that any diffeomorphism f of $M \times N$ restricts to a diffeomorphism from $M_i \times N_j$ onto some other connected component $M_r \times N_s$. Slightly modifying our proofs above, we conclude that f maps $M_i \times \{z\}$ to $M_r \times \{\overline{z}\}$ for some $\overline{z} \in N_s$. We conclude that a fiber $M \times \{z\}$ is mapped to a collection of l "mini-fibers" of which the N-coordinates are not necessarily equal.

3.2.3 Splitting Theorem

From the Slice Theorem, we can deduce that the isometries split on products $M \times N$ with minimal n-cohomology. Because of its significance, and the easy proof in comparison with other more general results, we state the following as a Theorem, not as a Corollary.

Theorem 3.2.10 (Splitting Theorem)**.** *If M is a closed connected Riemannian manifold and if N is a connected Riemannian manifold such that $M \times N$ has minimal n-cohomology, then the isometries of $M \times N$ split, i.e.* $\mathrm{Iso}(M \times N) = \mathrm{Iso}(M) \times \mathrm{Iso}(N)$.

Proof. Let $f = (f_1, f_2)$ be an isometry of $M \times N$. Then, f satisfies the hypotheses of Theorem 3.2.8 and therefore f_2 is independent of its M-coordinates. Notice that f_2 can thus be seen as a map from N to N.

Let $(y, z) \in M \times N$ and denote $f_1(y, z) = x$. A path γ in $\{y\} \times N$, containing (y, z), is orthogonal to every fiber $M \times \{w\}$. Since f is an isometry which maps each fiber to another fiber, we have that $f \circ \gamma$ is orthogonal to each fiber $M \times \{w\}$. It is therefore a path in $\{x\} \times N$ and connectedness of N implies that $f_1(\{y\} \times N) = \{x\}$. Since $y \in M$ is arbitrary, we conclude that f_1 does not depend on its N-coordinates. It can thus be seen as a map from M to M.

Since f is an isometry, we obtain that f_1 and f_2 are isometries of M and N respectively. $\square$

Corollary 3.2.11. *If N is contractible and M is closed, then the isometries of $M \times N$ split. Moreover, the isometries of $S^n \times S^m$ for $m \neq n$ split.*

The most famous theorem which gives information on the splitting of isometries on product manifolds is no doubt the *de Rham decomposition theorem* for Riemannian manifolds [44] (1954). This theorem has been extended several times, quite recently by J. Eschenburg and E. Heintze in [47] (1998) and finally in a very general form by T. Foertsch and A. Lytchak [49] (2008). We shall state their general result here. In order to do so, recall that the *affine rank* of a metric space is the supremum over all topological dimensions of affine spaces that admit an isometric embedding into X. Important for us is the fact that Riemannian manifolds have finite affine rank. A metric space is *geodesic* if every two points of the space are connected by a geodesic. Here, a geodesic is an isometric embedding of an interval of $\mathbb{R}$. We call a metric space *irreducible* if for every decomposition $X = Y \times Z$, one of the factors Y or Z must be a point. Here $Y \times Z$ is equipped with the metric given by

$$d((y,z),(\overline{y},\overline{z})) = \sqrt{d(y,\overline{y})^2 + d(z,\overline{z})^2}.$$

Theorem 3.2.12 (Foertsch and Lytchak, Theorem 1.1)**.** *Let X be a geodesic metric space of finite affine rank. Then X admits a unique decomposition as a direct product*

$$X = Y_0 \times Y_1 \times \ldots \times Y_n,$$

where Y_0 is a Euclidean space (possibly a point), and where the $Y_i, i = 1, 2, \ldots, n$, are irreducible metric spaces not isometric to the real line, nor to a point. Thus, if there is another direct product decomposition $X = Z_0 \times Z_1 \times \ldots \times Z_m$ of this kind then we have $m = n$ and there exists a permutation s of $\{1, \ldots, n\}$ such that for each point $x \in X$, the Y_i-fiber through x coincides with the $Z_{s(i)}$-fiber through x for all $i = 1, \ldots, n$.

Corollary 3.2.13 (Foertsch and Lytchak, Corollary 1.3)**.** *Let X be a geodesic space of finite affine rank and let $X = Y_0 \times Y_1 \times \ldots \times Y_n$ be its product decomposition as in Theorem* 1.1*. Denote by $\mathcal{P}$ the group of all permutations $s \in \sigma_n$, such that Y_i and $Y_{s(i)}$ are isometric for all $i = 1, \ldots, n$. Then there is the natural exact sequence:*

$$1 \to \operatorname{Iso}(Y_0) \times \operatorname{Iso}(Y_1) \times \operatorname{Iso}(Y_2) \times \ldots \times \operatorname{Iso}(Y_n) \xrightarrow{i} \operatorname{Iso}(X) \xrightarrow{p} \mathcal{P} \to 1.$$

We prove the following

Lemma 3.2.14. *The above Theorem implies our Splitting Theorem when N is geodesic.*

Proof. Assume that M is a closed connected Riemannian manifold such that $M \times N$ has minimal n-cohomology. We use Corollary 3.2.13 to show that the isometries of $M \times N$ split.

Since N is geodesic and M is closed, we have that $M \times N$ is geodesic. The above theorem then gives a splitting

$$M \times N = (M_1 \times M_2 \times \ldots \times M_k) \times (N_0 \times N_1 \times N_2 \times \ldots \times N_l),$$

and the isometries of $M \times N$ permute the factors of this decomposition.

We use the hypothesis of minimal n-cohomology in the following proof by contradiction. Assume by contradiction that the isometries do not split, i.e. that some isometry $f : M \times N \to M \times N$ maps a factor M_i to a factor N_j (with $j \neq 0$). Denote the dimension of M_i by n_i. By the Künneth formula and with $\mathbb{Z}_2$-coefficients, we get that $H^{n-n_i}(M)$ equals

$$\oplus_{r=0}^{n-n_i} H^r(M_1 \times M_2 \times \ldots \times M_{i-1} \times M_{i+1} \times \ldots \times M_k) \otimes_{\mathbb{Z}_2} H^{n-n_i-r}(M_i).$$

Checking the last term in this sum, we conclude that $H^{n-n_i}(M)$ contains $\mathbb{Z}_2$. Again by the Künneth formula, we get that $H^{n_i}(N)$ equals

$$\oplus_{r=0}^{n_i} H^r(N_0 \times N_1 \times N_2 \times \ldots \times N_{j-1} \times N_{j+1} \times \ldots \times N_l) \otimes_{\mathbb{Z}_2} H^{n_i-r}(N_j),$$

and the first term of this sum is $\mathbb{Z}_2$. Since M is closed and connected, it satisfies $H^n(M) \cong \mathbb{Z}_2$. Looking at the $(n-n_i)^{\text{th}}$ and the last term of

$$H^n(M \times N) = \oplus_{r=0}^{n} H^r(M) \otimes_{\mathbb{Z}_2} H^{n-r}(N),$$

we conclude that $H^n(M \times N)$ contains $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Therefore, $M \times N$ does not have minimal n-cohomology, a contradiction. ☐

The proof of Theorem 3.2.10 that we have provided is totally different from the work of Foertsch and Lytchak. Moreover, it is certainly prefered in the context of Riemannian manifolds having minimal n-cohomology, since our methods provide a quick and easy proof. Finally, we do not need the fact that N is geodesic which is central in the work of Foertsch and Lytchak.

3.3 Structure Theorem

It is at this point not very clear which maps exactly are fiberwise volume non-increasing. We know that there is apparently no direct connection with volume preserving maps (see Section 3.1) and that the class of fiberwise volume non-increasing maps contains isometries and maps of the form (f, g) where f is a diffeomorphism of M and g a diffeomorphism of N.

When M is connected, then under the condition of minimal n-cohomology, it follows from the Slice Theorem 3.2.8, that the set of fiberwise volume non-increasing maps FNI($M \times N$), equipped with the operation of composition, is a group: indeed, in this setting "fiberwise volume non-increasing" and "fiber preserving" are equivalent notions! We will prove in this section that FNI($M \times N$) is in some sense completely determined by the diffeomorphism groups of the factors M and N.

Lemma 3.3.1. *Given a point $y_0 \in M$, consider*

$$\begin{array}{rccc} \psi: & \mathit{FNI}(M \times N) & \rightarrow & \mathrm{Diffeo}(N) \\ & (\alpha, \beta) & \mapsto & \widetilde{\beta}, \end{array}$$

where

$$\begin{array}{rccc} \widetilde{\beta}: & N & \rightarrow & N \\ & z & \mapsto & \beta(y_0, z). \end{array}$$

This definition is independent of the chosen y_0. Furthermore, the map ψ is a group homomorphism whose kernel can be identified to the set K of maps

$$f: N \rightarrow \mathrm{Diffeo}(M),$$

such that

$$\overline{f}: M \times N \rightarrow M, (y, z) \mapsto f(z)(y)$$

is differentiable. Additionally, there is a short exact sequence

$$1 \rightarrow K \cong \mathit{kernel}(\psi) \hookrightarrow \mathit{FNI}(M \times N) \xrightarrow{\psi} \mathrm{Diffeo}(N) \rightarrow 1.$$

Proof. Theorem 3.2.8 implies that the definition of ψ is independent of the chosen $y_0 \in M$.

To show that ψ is a group homomorphism, let $(y, z) \in M \times N$ and $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in$ FNI($M \times N$). Then,

$$(\alpha_1, \beta_1) \circ (\alpha_2, \beta_2)(y, z) = (\alpha_1(\alpha_2(y, z), \beta_2(y, z)), \beta_1(\alpha_2(y, z), \beta_2(y, z)))$$

and thus

$$\psi((\alpha_1, \beta_1) \circ (\alpha_2, \beta_2))(z) = \beta_1(\alpha_2(y_0, z), \beta_2(y_0, z)).$$

On the other hand,

$$\psi(\alpha_1, \beta_1) \circ \psi(\alpha_2, \beta_2)(z) = \beta_1(y_0, \beta_2(y_0, z)).$$

Both expressions are equal since β_1 doesn't depend on its first argument.

Observe that ψ maps each $(\alpha, \beta) \in \mathrm{FNI}(M \times N)$ to an element of $\mathrm{Diffeo}(N)$. This follows from the fact that $(\alpha, \beta) \in \mathrm{FNI}(M \times N)$ has an inverse $(\alpha', \beta') \in \mathrm{FNI}(M \times N)$ and so $\psi(\alpha', \beta')$ is an inverse for $\psi(\alpha, \beta)$. We conclude that ψ is a well-defined group homomorphism.

Given a diffeomorphism γ of N, define

$$\begin{array}{rccc} \hat{\gamma}: & M \times N & \to & N \\ & (y, z) & \to & \gamma(z). \end{array}$$

Let $\pi : M \times N \to M$ be the natural projection onto M. Then, $(\pi, \hat{\gamma}) \in \mathrm{FNI}(M \times N)$ and $\psi(\pi, \hat{\gamma}) = \gamma$. Hence, ψ is surjective.

If f is an element of K and $p : M \times N \to N$ is the natural projection map, then $(\overline{f}, p)$ is an element of kernel(ψ): the fact that its inverse is differentiable follows from the inverse function theorem. Conversely, if $(\alpha, \beta) \in \mathrm{kernel}(\psi)$, then $\beta = p$ and $\alpha = \overline{g}$ for some $g \in K$. There is thus a bijective correspondence between K and kernel(ψ). We define the group law on K such that this bijection is a group isomorphism. ☐

It would be desirable to have an "easier" description of K. To this end, let us look at the set

$$\mathcal{D} = \{f : N \to \mathrm{Diffeo}(M)\},$$

equipped with the following group law:

$$f * g : N \to \mathrm{Diffeo}(M), z \mapsto f(z) \circ g(z) \ \forall f, g \in \mathcal{D}.$$

It is easy to see that $K < (\mathcal{D}, *)$ and that K contains those elements of $\mathcal{D}$ that satisfy a certain differentiability condition: for a given $f \in K$, the diffeomorphisms $f(z)$ should change "smoothly in z" in order for the corresponding map $\overline{f}$ to be differentiable. Recall that Diffeo(M) need not be a differentiable manifold, but that it does have the structure of a Fréchet manifold. In fact, it is a submanifold of the Fréchet manifold $C^\infty(M, M)$ of smooth self-maps of M (see Example 2.3.16 and [73], [84]). We show that

$$K = \{f \in \mathcal{D} \mid f \text{ is Fréchet } C^\infty\}.$$

Proposition 3.3.2. *A map $f : N \to C^\infty(M, M)$ is Fréchet C^∞ if and only if the corresponding map $\overline{f} : M \times N \to M, (y, z) \mapsto f(z)(y)$ is C^∞.*

Proof. **Assume first that f is Fréchet C^∞.** Then,

$$\begin{array}{rccc} j: & M \times N & \rightarrow & M \times C^\infty(M) \\ & (y,z) & \mapsto & (y, f(z)) \end{array}$$

is Fréchet C^∞. So, (Fréchet) differentiability of $\overline{f}$ would be implied by Fréchet differentiability of

$$\begin{array}{rccc} i: & M \times C^\infty(M,M) & \rightarrow & M \\ & (y,g) & \mapsto & g(y). \end{array}$$

Choose $(y,g) \in M \times C^\infty(M,M)$, fix a trivializing family for $g : M \rightarrow M$ and denote S the Fréchet space of smooth sections of the pullback bundle $g^*(TM)$ (Example 2.3.5). We denote a parametrization around (y,g) by

$$\begin{array}{rccc} x: & O \times \mathcal{O} \subset \mathbb{R}^n \times S & \rightarrow & U := U_1 \times U_2 \subset M \times C^\infty(M,M) \\ & (o,s) & \mapsto & (x_1(o), x_2(s)), \end{array}$$

where $(U_1, x_1, \widetilde{x_1})$ is inside the chosen trivializing family and where $x_2 : \mathcal{O} \rightarrow C^\infty(M,M)$ is a parametrization at g as in Example 2.3.16. We denote $\mathrm{Im}(\widetilde{x_1}) = W$.

Using the structure of $C^\infty(M,M)$ as a Fréchet manifold, we see that $i \circ x : (o,s) \mapsto \exp(\pi(s(x_1(o))))$ with π the natural projection of $g^*(TM)$ to TM. Now, Fréchet differentiability of i on $U_1 \times U_2$ is equivalent with Fréchet differentiability of

$$\begin{array}{rccc} \widetilde{i} := \widetilde{x_1}^{-1} \circ i \circ x : & O \times \mathcal{O} & \rightarrow & \widetilde{x_1}^{-1}(W) \\ & (o,s) & \rightarrow & \widetilde{x_1}^{-1}(\exp(\pi(s(x_1(o))))), \end{array}$$

on $O \times \mathcal{O}$ (where we can assume without loss of generality that $\mathcal{O}$ is small enough for $\widetilde{i}$ to be defined). Since exp is smooth on TM, it suffices to prove that

$$\begin{array}{rccc} \gamma: & O \times \mathcal{O} & \rightarrow & TM_{|\mathrm{Im}(\widetilde{x_1})} \cong \mathrm{Im}(\widetilde{x_1}) \times \mathbb{R}^n \\ & (o,s) & \mapsto & \pi \circ s(x_1(o)) \end{array}$$

is Fréchet differentiable on $O \times \mathcal{O}$. By differentiability of g, we only need to prove Fréchet differentiability for the second component map

$\gamma_2 : (o,s) \mapsto \overline{s^1}(o)$ (we use the same notations as in Example 2.3.5). It is an easy exercise to prove by induction on l that the l^{th} differential $D^l\gamma_2$

$$D^l\gamma_2 : (O \times \mathcal{O}) \times (\mathbb{R}^n \times S)^l \to \mathbb{R}^n$$

exists and that it maps $(o, s, k_1, h_1, k_2, h_2, \ldots, k_l, h_l) \in (O\times\mathcal{O})\times(\mathbb{R}^n\times S)^l$ to

$$\frac{\partial^l \overline{s^1}}{\partial k_1 \partial k_2 \ldots \partial k_l}(o) + \sum_{j=1}^{l} \frac{\partial^{l-1}\overline{h_j^1}}{\partial k_1 \partial k_2 \ldots \partial \hat{k}_j \ldots \partial k_l}(o) \in \mathbb{R}^n.$$

Continuity of the differentials of γ_2 then follows automatically and so we have proven the forward claim of the proposition.

To prove the converse, things become quite technical. First, choose $z \in N$, let χ be a parametrization of N at z and denote $f(z) = g$. Fix a trivializing family $(U_\alpha, x_\alpha, \widetilde{x_\alpha})_{\alpha\in\mathcal{A}}$ for $g : M \to M$ where $\mathcal{A}$ is some finite index set. As usual, use this family to equip S, the set of smooth sections of $g^*(TM)$, as a Fréchet space. Since M is compact, we can choose an open neighbourhood $V \subset \operatorname{Im}(\chi)$ of z such that the map

$$\begin{array}{rccl} v: & U_\alpha \times \chi^{-1}(V) & \to & g^*(TM) \\ & (y,w) & \mapsto & (y, (\exp_{g(y)})^{-1}\overline{f}(y,\chi(w))) \end{array}$$

is well-defined in the sense that $\overline{f}(y,\chi(w))$ is inside a normal neighbourhood of $g(y)$ for all y. The differentiability of $\overline{f}$ implies that of v by Lemma 2.2.9. It suffices to prove Fréchet differentiability of

$$\begin{array}{rccl} \widetilde{f}: & \chi^{-1}(V) & \to & S \\ & w & \mapsto & v(\cdot,w). \end{array}$$

Let k be the dimension of N. For $w \in \chi^{-1}(V)$, note that $v(\cdot,w)$ is indeed a section, so the notation $\overline{(v(\cdot,w))^\alpha}$ makes sense. We will henceforth look at this as a C^∞ map from $x_\alpha^{-1}(U_\alpha)\times\chi^{-1}(V)$ to $\mathbb{R}^n$. We prove now that the l^{th} differential $D^l(\widetilde{f})$ exists and maps $(w', h_1, h_2, \ldots, h_l) \in \chi^{-1}(V)\times(\mathbb{R}^k)^l$ to the section s such that $\forall\alpha \in \mathcal{A}$

$$\begin{array}{rccl} \overline{s^\alpha}: & x_\alpha^{-1}(U_\alpha) & \to & \mathbb{R}^n \\ & o' & \mapsto & \frac{\partial^l \overline{v(\cdot,w)^\alpha}(o)}{\partial h_1 \partial h_2 \ldots \partial h_l}(o', w'). \end{array}$$

The result then follows readily from compactness of $\overline{U_\alpha}$.

So, by induction, assume this hypothesis is true for some natural number l, let us prove it for $l+1$. The topology on S is induced by a countable collection of norms, and convergence of a sequence always needs to be proven for every norm. Therefore, let $j \in \mathbb{N}$ and $i_1, i_2, \ldots, i_j \in \{1, 2, \ldots, n\}$. We have to prove uniform convergence of

$$\frac{1}{t}\frac{\partial^j \overline{D^l(\widetilde{f})(w' + th_{l+1}, h_1, h_2, \ldots, h_l)^\alpha} - \partial^j \overline{D^l(\widetilde{f})(w', h_1, h_2, \ldots, h_l)^\alpha}}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_j}}$$

uniformly for $t \to 0$ over $x_\alpha^{-1}(U_\alpha)$ to

$$\frac{\partial^{l+1+j} \overline{v(\cdot, w)^\alpha}(y)}{\partial h_1 \partial h_2 \ldots \partial h_{l+1} \partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_j}}.$$

Pointwise convergence is immediate by differentiability of v. Uniform convergence follows from the lemma below. □

Lemma 3.3.3. *Given $n, k, d \in \mathbb{N}, h \in \mathbb{R}^k$, a C^1-map $v : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^d$, and a compact subset $C \subset \mathbb{R}^n$, then*

$$\lim_{t \to 0} \frac{1}{t}(v(x, w + th) - v(x, w)) = \frac{\partial v}{\partial h}(x, w)$$

uniformly over $x \in C$.

Proof. Without loss of generality we can assume that $d = 1$. Assume, by contradiction, that the convergence is not uniform over C. Then, $\exists \epsilon > 0 \ \forall N \in \mathbb{N} \ \exists t_N < \frac{1}{N} \ \exists x_N \in C$ such that

$$|\frac{1}{t_N}(v(x_N, w + t_N h) - v(x_N, w)) - \frac{\partial v}{\partial h}(x_N, w)| \geq \epsilon.$$

Consequently, $\exists \epsilon > 0 \ \forall N \in \mathbb{N} \ \exists t'_N < \frac{1}{N} \ \exists x_N \in C$ such that

$$|\frac{\partial v}{\partial h}(x_N, w + t'_N h) - \frac{\partial v}{\partial h}(x_N, w)| \geq \epsilon.$$

Since C is compact, continuity of $\frac{\partial v}{\partial h}(x, w)$ gives us a contradiction. □

Since the diffeomorphism group of M is a submanifold of $C^\infty(M, M)$, we obtain the following

Theorem 3.3.4 (Structure Theorem)**.** *Let M be a closed connected Riemannian n-manifold and let N be a Riemannian manifold such that $M \times N$ has minimal n-cohomology. We have the following short exact sequence:*

$$1 \to K \hookrightarrow \mathit{FNI}(M \times N) \xrightarrow{\psi} \mathrm{Diffeo}(N) \to 1$$

with ψ as in Lemma 3.3.1 and where $K \cong \{f : N \to \mathrm{Diffeo}(M) \mid f$ is Fréchet differentiable$\}$.

3.4 Applications to properly discontinuous actions

3.4.1 Almost-crystallographic groups

Preliminaries

A group Γ acts *properly discontinuously* on a topological space X if the set

$$\{\gamma \in \Gamma \mid \gamma K \cap K \neq \phi\}$$

is finite for any compact $K \subset X$. A *k-dimensional crystallographic group* Γ is a group acting isometrically, properly discontinuously and cocompactly on $\mathbb{R}^k$. If the action is also free, then we call Γ a *Bieberbach group.* The structure of crystallographic groups and some of its properties are described by the three famous Bieberbach theorems (see [14], [15], [50]). Let us recall what they are.

Bieberbach 3.4.1. *Let $\Gamma \subset \mathbb{R}^k \rtimes O(k) = \mathrm{Iso}(\mathbb{R}^k)$ be a k-dimensional crystallographic group. Then Γ contains a finite index subgroup $\Gamma^* = \Gamma \cap \mathbb{R}^k$ which is a uniform lattice, i.e. a discrete cocompact subgroup of $\mathbb{R}^k$.*

Bieberbach 3.4.2. *Let $\Gamma_1, \Gamma_2 \subset \mathbb{R}^k \rtimes O(k)$ be two k-dimensional crystallographic groups. If Γ_1 and Γ_2 are isomorphic, then they are conjugated by an element of $\mathit{Aff}(\mathbb{R}^k) = \mathbb{R}^k \rtimes GL(k, \mathbb{R})$.*

Bieberbach 3.4.3. *Up to isomorphism, there are only finitely many k-dimensional crystallographic groups.*

All three Bieberbach theorems have been generalized to the case of almost-crystallographic groups.

Definition 3.4.4. *An* almost-crystallographic group *is a group that acts properly discontinuously, cocompactly and isometrically on a simply connected, connected, nilpotent Lie group N that is equipped with a left-invariant metric.*

Remark 3.4.5. *For later use, remark that a finite index subgroup of an almost-crystallographic group is again almost-crystallographic.*

The left-invariant metric on N is determined by the choice of an inner product on the Lie algebra η of N. Then, $\mathrm{Iso}(N) = N \rtimes C$ where C is the group of automorphisms of N whose differential at the identity preserves the chosen inner product on η (see [109]).
In 1960, Auslander generalized the first Bieberbach theorem to almost-crystallographic groups.

Bieberbach 3.4.6 (Generalization first Bieberbach theorem, Auslander, [8])**.** *Let $\Gamma \subset N \rtimes C$ be an almost-crystallographic group. Then Γ contains a finite index subgroup $\Gamma^* = \Gamma \cap N$ which is a uniform lattice of N.*

The second Bieberbach theorem also admits a straightforward generalization.

Bieberbach 3.4.7 (Generalization second Bieberbach theorem, [72])**.** *Let $\Gamma_1 \subset N_1 \rtimes C_1$ and $\Gamma_2 \subset N_2 \rtimes C_2$ be two almost-crystallographic groups. If $\phi : \Gamma_1 \to \Gamma_2$ is an isomorphism, then one can assume $N_1 = N_2$ and there exists an element $\alpha \in \mathit{Aff}(N) = N \rtimes \mathit{Aut}(N)$ such that $\forall \gamma \in \Gamma_1 : \ \phi(\gamma) = \alpha\gamma\alpha^{-1}$.*

To obtain a generalization for the third Bieberbach theorem, consider the following reformulation of Bieberbach 3.4.3.

Bieberbach 3.4.8 (Reformulation third Bieberbach theorem)**.** *Assume that A is a finitely generated torsion-free abelian group. Then, up to isomorphism, there are only finitely many extensions of the form*

$$0 \to A \to \Gamma \to F \to 1,$$

where A is isomorphic to a maximal abelian subgroup of Γ and where F is finite.

The generalization to the almost-crystallographic case states the following.

Bieberbach 3.4.9 (Generalization third Bieberbach theorem, [43]). *Assume that A is a finitely generated torsion-free nilpotent group. Then, up to isomorphism, there are only finitely many extensions of the form*

$$1 \to A \to \Gamma \to F \to 1,$$

where A is isomorphic to a maximal nilpotent subgroup of Γ and where F is finite.

As an application to our main results, we will elaborate on generalizations of these Bieberbach theorems to actions on products $M \times N$ where M is a closed connected Riemannian manifold and where N is a simply connected, connected, nilpotent Lie group, equipped with a left-invariant metric.

Generalizing Bieberbach 3.4.6

It turns out that the first Bieberbach theorem can be generalized in our setting.

Theorem 3.4.10. *Let M be a closed connected Riemannian manifold and let N be a simply connected, connected, nilpotent Lie group equipped with a left-invariant metric. If Γ is a group acting properly discontinuously, cocompactly and isometrically on $M \times N$, then Γ contains a finite index subgroup isomorphic to a uniform lattice of N.*

Proof. Since N is contractible, we have that $M \times N$ has minimal n-cohomology. Theorem 3.2.10 thus implies that $\operatorname{Iso}(M \times N) = \operatorname{Iso}(M) \times \operatorname{Iso}(N)$. Denote

$$\psi : \operatorname{Iso}(M \times N) \to \operatorname{Iso}(N)$$

the canonical projection. Let $\overline{\Gamma} = \psi(\Gamma)$ and let Γ_1 be the kernel of $\psi_{|\Gamma}$. We obtain the following short exact sequence:

$$1 \to \Gamma_1 \to \Gamma \to \overline{\Gamma} \to 1.$$

Since Γ acts properly discontinuously and since $\Gamma_1 \subset \Gamma$ maps $M \times \{1\}$ to itself, we have that Γ_1 is finite. Clearly, $\overline{\Gamma}$ is an almost-crystallographic group. Bieberbach 3.4.6 then shows that $\overline{\Gamma}$ contains a finite index subgroup isomorphic to a uniform lattice of N. It is thus virtually-(finitely generated and nilpotent) [93]. Hence, it is poly-(cyclic or finite).

In total, we have that Γ is poly-(cyclic or finite) and therefore poly-$\mathbb{Z}$-by-finite. We obtain the following short exact sequence:

$$1 \to P\mathbb{Z} \to \Gamma \to F \to 1,$$

where $P\mathbb{Z}$ is a poly-$\mathbb{Z}$ group and F is a finite group.

It is an easy observation that poly-$\mathbb{Z}$-groups are torsion-free, since every exact sequence

$$1 \to H \to G \to \mathbb{Z} \to 0,$$

splits. Since the kernel of ψ_Γ is all torsion, we get that the restriction of ψ to the P$\mathbb{Z}-$subgroup is injective. Then, $P\mathbb{Z}$ is isomorphic to a finite index subgroup of the almost-crystallographic group $\overline{\Gamma}$. Thus, it is itself an almost-crystallographic group with a finite index subgroup isomorphic to a uniform lattice of N. We conclude that Γ contains a finite index subgroup isomorphic to a uniform lattice of N. ☐

Remark 3.4.11. *An important tool in proving Theorem 3.4.10 is the Splitting Theorem 3.2.10. Since N is locally compact with a left-invariant metric, it is geodesic. Then, Corollary 3.2.13 together with our Lemma 3.2.14 implies that* $\mathrm{Iso}(M \times N) = \mathrm{Iso}(M) \times \mathrm{Iso}(N)$ *and we obtain an alternate proof.*

The second Bieberbach theorem does *not* admit a straightforward generalization to our setting. Indeed, identify the product $S^1 \times \{0\}$ with $S^1 \subset \mathbb{C}$ and denote

$$g : S^1 \to S^1, e^{2\pi i\theta} \mapsto e^{\pi i} e^{2\pi i\theta} \ (\theta \in \mathbb{R}).$$

The group generated by g acts isometrically, cocompactly and properly discontinuously on S^1. The same is true for the group generated by the map $S^1 \to S^1, x \mapsto x^{-1}$. Both groups are isomorphic, yet not even conjugated by a diffeomorphism of S^1! Indeed, if $f \in \mathrm{Diffeo}(S^1)$, then the image of the path

$$[0,1] \to S^1, t \mapsto e^{2\pi i t}$$

under $f^{-1} \circ g \circ f$ runs counterclockwise around the circle whereas the image under $x \mapsto x^{-1}$ runs clockwise.

One could argue that a restriction to cocompact, properly discontinuous, isometric, *free* actions could do the trick. Indeed, note that two isomorphic groups acting freely, properly discontinuously and isometrically on S^1 are equal. This is an easy observation: if you see the circle as a subset of $\mathbb{C}$, then any isometry of S^1 which is not multiplication by a constant, has a fixed point. Since the action is free, the group consists only of multiplications by some constants and hence is abelian. Since the action is properly discontinuous, the group is also finite. The result then follows easily. Now, by Bieberbach 3.4.2, groups acting properly discontinuously, cocompactly and freely by isometries on $\mathbb{R}$ are conjugated by an element of $\mathrm{Aff}(\mathbb{R})$. The following example shows however that there is no similar rigidity for $S^1 \times \mathbb{R}$. More concretely, we find two isomorphic groups acting properly discontinuously, cocompactly and isometrically on $S^1 \times \mathbb{R}$ such that the induced actions on S^1 and $\mathbb{R}$ are free, but these groups cannot be conjugated by an element of $\mathrm{Diffeo}(S^1) \times \mathrm{Diffeo}(\mathbb{R})$.

Example 3.4.12. *Consider* $S^1 = \{e^{2\pi i\theta} \mid \theta \in \mathbb{R}\}$. *Choose* $\theta_1, \theta_2 \in \mathbb{R}\backslash\mathbb{Q}$ *such that* $\theta_1 \pm \theta_2 \notin \mathbb{Z}$. *Let* $\Gamma \subset \mathrm{Iso}(S^1 \times \mathbb{R})$ *be the group generated by* (α_1, α_2) *where* $\alpha_1 : S^1 \to S^1$ *is multiplication by* $e^{2\pi i\theta_1}$ *and* $\alpha_2 : \mathbb{R} \to \mathbb{R}$, $x \mapsto x + 1$. *Analogously, let* $\widetilde{\Gamma}$ *be the group generated by* (β_1, β_2) *where* $\beta_1 : S^1 \to S^1$ *is multiplication by* $e^{2\pi i\theta_2}$ *and where* $\beta_2 = \alpha_2$. *Clearly, both groups are infinite cyclic and they act isometrically, properly discontinuously and cocompactly on* $S^1 \times \mathbb{R}$. *Also, the induced actions on* S^1 *and* $\mathbb{R}$ *are free. However, let us explain why* $\langle\alpha_1\rangle$ *and* $\langle\beta_1\rangle$ *can not be conjugated by a diffeomorphism of* S^1. *Assume by contradiction that* $f^{-1} \circ \alpha_1 \circ f = \beta_1$ *for some* $f \in \mathrm{Diffeo}(S^1)$. *We can assume w.l.o.g. that* $\theta_1, \theta_2 \in [0, 2\pi[$ *and* $0 \leq \theta_1 < \theta_2$. *Look at the sequence of points* $(f^{-1} \circ \alpha_1 \circ f)^{(n)}(1)$, *where* ${}^{(n)}$ *stands for n-fold composition of a map with itself. For any* $f \in \mathrm{Diffeo}(S^1)$, *this sequence of points for n large enough passes* $1 \in S^1$ *strictly less times than the sequence of points* $\beta_1^{(n)}(1)$, *obtaining a contradiction.*

The third Bieberbach theorem does not generalize either. There are infinitely many non-isomorphic groups acting isometrically, properly discontinuously and cocompactly on $S^1 \times \{0\}$.

3.4.2 Talelli's Conjecture

We begin by recalling the definition of cohomological dimension.

Definition 3.4.13. *The* cohomological dimension *of a group* Γ *is defined by*

$$cd(\Gamma) = \sup\{n \mid H^n(\Gamma; M) \neq 0 \text{ for some } \mathbb{Z}\Gamma\text{-module } M\}.$$

There are two definitions in literature for *periodic cohomology* of a group. We use the following

Definition 3.4.14. *A group* Γ *has* periodic cohomology after k steps *if there exists an integer* $q > 0$ *such that* $H^i(\Gamma, -)$ *and* $H^{i+q}(\Gamma, -)$ *are naturally isomorphic functors for all* $i > k$.

In 2005, Talelli stated the following (Conjecture III of [101])

Conjecture 3.4.15. *(Talelli, 2005)* A torsion-free group Γ that has periodic cohomology after some steps has finite cohomological dimension.

By a result of Mislin and Talelli ([77]) we know that this conjecture holds for the large class of $LH\mathcal{F}$-groups (see [70]). Among others, this class contains all linear and all elementary amenable groups.

In 2001, Adem and Smith have proven that a countable group acts freely, properly discontinuously and smoothly on some $S^n \times \mathbb{R}^k$ if and only if it has periodic cohomology. Actually, they use the other definition of periodic cohomology which states that the isomorphisms of cohomological functors are induced by a cup product map (see [1] for more details). For the large class of $H\mathcal{F}$-groups it is known that these definitions are equivalent. It has been conjectured by Talelli that they are equivalent for all groups. The Adem-Smith Theorem suggests the following slightly weaker reformulation of the Talelli conjecture.

Conjecture 3.4.16. *(Talelli reformulated, 2005)* If Γ is a torsion-free group that acts smoothly and properly discontinuously on $S^n \times \mathbb{R}^k$, then it has finite cohomological dimension.

Now, let us replace S^n by any closed, connected Riemannian manifold M and replace $\mathbb{R}^k$ by any k-dimensional contractible Riemannian manifold N. We obtain the following generalization.

Conjecture 3.4.17 (Petrosyan, 2007). *If* Γ *is a torsion-free group acting smoothly and properly discontinuously on* $M \times N$, *then* $cd(\Gamma) \leq \dim(N)$.

In [90], Petrosyan has verified this conjecture in the case of $H\mathcal{F}$-group and when N is 1-dimensional. We prove the following

Theorem 3.4.18. *Let Γ be a torsion-free group that acts properly discontinuously on $M \times N$ where M is a closed and connected Riemannian manifold and N is a Riemannian manifold such that $M \times N$ has minimal n-cohomology. If each $\gamma \in \Gamma$ acts as a fiberwise volume non-increasing map, then Γ acts freely and properly discontinuously on N. In particular, if N is contractible, then $cd(\Gamma) \leq \dim(N)$.*

Proof. Let $y_0 \in M$ and consider the map

$$\begin{array}{rccc} \psi: & \mathrm{FNI}(M \times N) & \rightarrow & \mathrm{Diffeo}(N) \\ & (\alpha, \beta) & \mapsto & \widetilde{\beta}, \end{array}$$

where

$$\begin{array}{rccc} \widetilde{\beta}: & N & \rightarrow & N \\ & z & \mapsto & \beta(y_0, z). \end{array}$$

By Lemma 3.3.1, we have that ψ is a well-defined epimorphism. This gives us the following short exact sequence

$$1 \rightarrow \Gamma_1 \rightarrow \Gamma \rightarrow \overline{\Gamma} \rightarrow 1,$$

where $\overline{\Gamma} = \psi(\Gamma)$ and Γ_1 is the kernel of $\psi_{|\Gamma}$. Let $z \in N$ and observe that every element of Γ_1 maps $M \times \{z\}$ onto itself. Since Γ acts properly discontinuously on $M \times N$ we have that Γ_1 is finite. Since Γ is torsion-free, Γ_1 must be trivial and therefore, $\Gamma \cong \overline{\Gamma}$.

Now, $\overline{\Gamma}$ acts freely, smoothly and properly discontinuously on N. When N is contractible, we have $cd(\Gamma) \leq \dim(N)$. ☐

Part II

Equivariant Hilbert space compression

Chapter 4

Background: Affine isometric actions and the property of Haagerup

In this chapter, we will focus on affine isometric actions of groups on Hilbert spaces and we will introduce the Haagerup property for groups. We will explore this property, explain why it is useful, give examples of groups that satisfy it and provide the reader with a variety of equivalent definitions. As a main reference, we refer the reader to an excellent book on the subject, written by Pierre-Alain Cherix, Michael Cowling, Paul Jolissaint, Pierre Julg and Alain Valette [25].

Before proceeding, we draw the reader's attention to Convention 4.1.2 below.

4.1 Affine isometric actions

Let $\mathcal{B}$ denote a real (or complex) Banach space. We denote its isometry group by $\mathrm{Iso}(\mathcal{B})$ and its group of affine transformations by $\mathrm{Aff}(\mathcal{B}) = \mathcal{B} \rtimes GL(\mathcal{B})$ where $GL(\mathcal{B})$ acts on $\mathcal{B}$ in the natural way. We denote $(v, A)(w) = (v, A) \cdot w = Aw + v$ for any $(v, A) \in \mathrm{Aff}(\mathcal{B})$ and $w \in \mathcal{B}$.

An affine representation (or affine action) of G on $\mathcal{B}$ is a group homomorphism $\alpha : G \to \mathrm{Aff}(\mathcal{B})$. For $v \in \mathcal{B}$ and $x \in G$, we shall often abbreviate $\alpha(x)(v)$ by $x \cdot v$. Writing $\alpha(x) = (b(x), \pi(x))$ for every $x \in G$, the condition that α is an action implies that $\pi : G \to GL(\mathcal{B})$ is a group homomorphism and that $b : G \to \mathcal{B}$ satisfies the 1-cocycle equality

$$b(xy) = \pi(x)b(y) + b(x),$$

for every $x, y \in G$. We call π the *linear part of* α and a map $b : G \to \mathcal{B}$ satisfying the above equality is called a 1-*cocycle relative to* π. Conversely, every couple (b, π) where $\pi : G \to GL(\mathcal{B})$ is a linear representation and b a 1-cocycle relative to π gives an affine representation of G.

An affine isometric action α of a group G on a Banach space $\mathcal{B}$ is a group homomorphism $\alpha : G \to \mathrm{Aff}(\mathcal{B}) \cap \mathrm{Iso}(\mathcal{B})$. Since α is an isometric action, the linear part π must be orthogonal (or unitary). Notice that the corresponding 1-cocycle is the orbit map of $0 \in \mathcal{H}$, i.e. $\forall x \in G : \; b(x) = \alpha(x)(0)$. In particular, we have that $b(1) = 0$ and $\forall x, y \in G : \; \|b(x) - b(y)\| = \|b(x^{-1}y)\|$.

Conversely, given an orthogonal (or unitary) representation $\pi : G \to \mathcal{O}(\mathcal{B})$ and a corresponding 1-cocycle b, we get an affine isometric representation $\alpha = (b, \pi)$ of G. When studying isometric actions, it is natural to ask which isometries are affine. For the real case, the following result gives a satisfying answer.

Theorem 4.1.1 (Mazur-Ulam, [76]). *Every isometry of a real Banach space is affine.*

The same is not true for isometries on complex Hilbert space. Indeed, the action of $\mathbb{Z}_2$ on $\mathbb{C}$ by complex conjugation is isometric but not affine.

It is common to study groups through their affine isometric actions on Banach spaces (e.g. Property (T), Haagerup's property, etc.). One is then interested to know whether or not a specific group admits a 1-cocycle (always relative to some orthogonal representation of the group) that *goes to infinity.* We are interested in the properties which are typical for such groups and we wonder whether or not it is interesting to quantify *the speed at which a* 1-*cocycle goes to infinity.* To answer these questions and to make the above notions more precise, we proceed under the following convention.

Convention 4.1.2. *In the remainder of this chapter, all of the groups G under consideration will be second countable locally compact topological groups and all of the Banach spaces $\mathcal{B}$ will be Hilbert spaces $\mathcal{H}$. We also assume that representations of G on $\mathcal{B}$ are strongly continuous, i.e. that*

$$\begin{array}{ccc} G \times \mathcal{B} & \to & \mathcal{B} \\ (x, v) & \mapsto & \alpha(x)v \end{array}$$

is a continuous map. Notice that this condition is always satisfied for discrete groups.

The class of second countable locally compact groups is a very fascinating one. First of all, it is sufficiently large, containing for example all real Lie groups with finitely many connected components and all discrete countable groups. Secondly, locally compact groups have many nice properties: for example, they are equipped with a left-invariant regular Borel measure, called the Haar measure, which is unique up to multiplication by a constant. Also interesting is the fact that a second countable locally compact group can be equipped with a proper (i.e. closed metric balls are compact) left-invariant metric, inducing the topology and unique up to coarse equivalence [61]. This can be used to make the notion of *going to infinity* more precise. For now, we introduce the following

Definition 4.1.3. *Let α be an affine isometric representation of a group G on a Hilbert space $\mathcal{H}$. We say that α is (metrically) proper if for every $v \in \mathcal{H}$ and $M \in \mathbb{R}$, we can find a compact set $K_{v,M} \subset G$ such that $\|\alpha(x)v\| \geq M$ whenever $x \notin K$. This is more elegantly denoted by*

$$\lim_{x\to\infty} \|\alpha(x)v\| = \infty,$$

and we say that the orbit of v goes to infinity*. Clearly, if the orbit of some element $v \in \mathcal{H}$ goes to infinity, then the same is true for all the other orbits because*

$$\|x \cdot v - x \cdot w\| = \|v - w\|,$$

for every $x \in G$ and $w \in \mathcal{H}$. Equivalently, we can thus say that α is (metrically) proper if and only if the orbit of 0 goes to infinity, i.e. $\lim_{x\to\infty} \|b(x)\| = +\infty$.

Definition 4.1.4. *A group G is* Haagerup *if it admits a proper affine isometric action on a real or complex Hilbert space. We also say that G satisfies the Haagerup property, that it has property (H) or that it is a-T-menable.*

Note that Haagerup groups are locally compact since the sets

$$\{x \in G \mid \|b(x)\| \leq n\}$$

are compact for every natural number n.

The word a-T-menable suggests that Haagerup groups are far away from satisfying some mysterious "property (T)". It is indeed true that Haagerup groups are designed to satisfy a strong negation of the following

Definition 4.1.5. *A group G satisfies Kazhdan's property (T) if for every isometric action of G on a real or complex Hilbert space, the associated 1-cocycle b is bounded; i.e. there exists $M > 0$ such that $\forall x \in G : \ \|b(x)\| \leq M$.*

The only Haagerup groups which have property (T) are compact.

We claim that we can w.l.o.g. assume that $\mathcal{H}$ is a *complex* Hilbert space. Since every complex Hilbert space is also a real Hilbert space, every proper affine isometric action on a complex Hilbert space, *is* a proper affine isometric action on a real Hilbert space. Conversely, we also get the following

Lemma 4.1.6. *Assume that b is a 1-cocycle relative to an orthogonal action of G on a real Hilbert space $\mathcal{H}$. There exists a unitary action of G on some complex Hilbert space such that the associated 1-cocycle $\widetilde{b}$ satisfies $\|b(x)\| = \|\widetilde{b}(x)\|$ for every $x \in G$.*

Proof. Let $\alpha = (\pi, b)$ be an affine isometric action on a real Hilbert space $\mathcal{H}$. Set $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \mathbb{C}$. Given $x \in G$ and $v \otimes \lambda \in \widetilde{\mathcal{H}}$, we set $\widetilde{\alpha}(x)(v \otimes \lambda) = ((\pi(x)v) \otimes \lambda) + (b(x) \otimes 1)$. Again, $\widetilde{\alpha}$ is an affine isometric action, but this time on the *complex* Hilbert space $\widetilde{\mathcal{H}}$. Moreover, $\widetilde{b} : x \mapsto b(x) \otimes 1$ is a 1-cocycle relative to the linear part $\widetilde{\pi}$ of this action satisfying that $\|\widetilde{b}(x)\| = \|b(x)\|$ for every $x \in G$. ☐

Corollary 4.1.7. *A group admits a proper affine isometric action on a real Hilbert space if and only if it admits a proper affine isometric action on a complex Hilbert space. The definition of the Haagerup property does not depend on whether we consider real or complex Hilbert spaces.*

The reason that complex Hilbert spaces are sometimes more convenient, is that you can invoke theorems such as the spectral theorem for unitary operators. We will continue with complex Hilbert spaces, but we note that the *real analogues* of all the theorems and definitions in this chapter hold equally well.

4.2 Equivalent definitions

In literature, a variety of equivalent definitions for the Haagerup property and Kazhdan's property (T) are current. In fact, every definition for the

Haagerup property that we consider, turns out to be a strong negation for one of many definitions of property (T). At least four equivalent definitions of the Haagerup property will be important in our work. We will introduce these definitions step by step and sketch the reasons why they are equivalent. For further information, we refer the reader to [25], specifically Theorem 2.1.1, and [68].

4.2.1 Conditionally negative definite maps

We start by introducing a special type of function on a topological group.

Definition 4.2.1. *A continuous function* $\psi : G \to \mathbb{R}^+$ *is* conditionally negative definite *if* $\psi(1) = 0$, $\psi(x) = \psi(x^{-1})$ *for every* $x \in G$ *and*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\overline{a_j}\psi(x_j^{-1}x_i) \leq 0,$$

for every $n \geq 2$, $x_1, x_2, \ldots, x_n \in G$ *and* $a_1, a_2, \ldots, a_n \in \mathbb{C}$ *with* $\sum_{i=1}^{n} a_i = 0$.

Lemma 4.2.2. *Let* $\mathcal{H}$ *be a Hilbert space and* $b : G \to \mathcal{H}$ *a* 1*-cocycle relative to a unitary representation of* G*. Then*

$$\begin{array}{rccl} \psi : & G & \to & \mathbb{R}^+ \\ & x & \mapsto & \|b(x)\|^2 \end{array}$$

is a conditionally negative definite function on G*.*

Proof. Denote the associated affine isometric action by α and write $\alpha(x)(v) = x \cdot v$ for every $x \in G$, $v \in \mathcal{H}$. It is clear that $\|b(1)\|^2 = \|1 \cdot 0\|^2 = 0$ and that for every $x \in G$,

$$\|b(x)\|^2 = \|x \cdot 0 - 0\|^2 = \|0 - x^{-1} \cdot 0\|^2 = \|b(x^{-1})\|^2.$$

If for some $n \geq 2$ we have elements $x_1, x_2, \ldots, x_n$ of G and *real* numbers $a_1, a_2, \ldots, a_n$ such that $\sum_{i=1}^{n} a_i = 0$, then a simple calculation shows that

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \|b(x_j^{-1}x_i)\|^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \|b(x_i) - b(x_j)\|^2 \\ &= -2\|\sum_{i=1}^{n} a_i b(x_i)\|^2 \leq 0. \end{aligned}$$

Now, assume that the a_j are *complex* numbers and write them as $b_j + ic_j$. A simple calculation shows that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\overline{a_j}\|b(x_j^{-1}x_i)\|^2 \quad = \quad \sum_{i=1}^{n}\sum_{j=1}^{n}(b_ib_j + c_ic_j)\|b(x_j^{-1}x_i)\| \leq 0$$

where we use the fact that the imaginary parts cancel out due to symmetry ($\psi(x) = \psi(x^{-1})$). We conclude that ψ is conditionally negative definite.

□

Interestingly, the following Proposition (see Proposition 14.*iii* of [68]) shows that the converse of this Lemma is also true!

Proposition 4.2.3 (Affine GNS construction)**.** *Let $\psi : G \to \mathbb{R}^+$ be a conditionally negative definite function on a topological group G. Then there exists an affine isometric action α on a Hilbert space $\mathcal{H}$, such that the associated 1-cocycle satisfies $\psi(x) = \|b(x)\|^2$ for all $x \in G$.*

We obtain the following characterization of the Haagerup property.

Theorem 4.2.4. *A group G is Haagerup if and only if there exists a continuous function $\psi : G \to \mathbb{R}^+$ which is conditionally negative definite and proper, that is, $\lim_{g\to\infty} \psi(g) = +\infty$.*

4.2.2 Positive definite maps

Another interesting type of function on a group which is related to the Haagerup property is the positive definite function.

Definition 4.2.5. *A continuous map $\phi : G \to \mathbb{C}$ is called positive definite if $\forall n \in \mathbb{N}_0,\ \forall x_1, x_2, \ldots, x_n \in G$ and $\forall a_1, a_2, \ldots, a_n \in \mathbb{C}$:*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\overline{a_j}\phi(x_j^{-1}x_i) \geq 0.$$

Remark 4.2.6. *We emphasize the fact that by our definition, positive definite maps are continuous! In literature, this is not always the case.*

The classical example of such a map is of course given by $\phi : x \mapsto \langle\pi(x)v, v\rangle$ where v is a vector in $\mathcal{H}$ and π a unitary representation.

It follows from the definition that a positive definite map ϕ is hermitian, i.e. $\forall x \in G: \ \phi(x^{-1}) = \overline{\phi(x)}$. Indeed, taking $a_1 = a_2 = 1$ and $x_1 = x, x_2 = 1$, we see that $\phi(x^{-1}) + \phi(x) \in \mathbb{R}$. On the other hand, taking $a_1 = i, a_2 = 1$ and $x_1 = x, x_2 = 1$, we see that also $i\phi(x) - i\phi(x^{-1}) \in \mathbb{R}$. We conclude that $\phi(x^{-1}) = \overline{\phi(x)}$.

A Theorem of Schoenberg (1938), see e.g. Theorem 5.16 in [68], gives a nice connection between positive definite and negative definite functions.

Theorem 4.2.7 (Schoenberg). *Let $\psi : G \to \mathbb{R}^+$ be a function that satisfies $\psi(1) = 0$. Then ψ is a conditionally negative definite function if and only if $e^{-t\psi}$ is positive definite for every $t > 0$.*

This Theorem is the key to associate positive definite maps to the Haagerup property. Assume for a moment that G is Haagerup and let $\psi : G \to \mathbb{R}^+$ be a proper conditionally negative definite function on G. By Schoenberg's Theorem, we can associate a sequence of positive definite functions to ψ by setting $\phi_n = e^{-\frac{\psi}{n}}$ for all $n \in \mathbb{N}_0$. The functions $(\phi_n)_{n\in\mathbb{N}_0}$ satisfy certain properties. They are for example normalized, i.e. $\phi_n(1) = 1$ for every $n \in \mathbb{N}_0$. Next, they are elements of the abelian C^*-algebra $C_0(G)$, i.e. $\lim_{g\to\infty} \phi_n(g) = 0$ for every $g \in G$. And finally, the sequence $(\phi_n)_{n\in\mathbb{N}_0}$ converges to 1 uniformly over compact subsets of G. It turns out that the existence of such a sequence is equivalent to the Haagerup property.

Theorem 4.2.8. *A group G is Haagerup if and only if the abelian C^*-algebra $C_0(G)$ possesses an approximate unit of normalized, positive definite functions, that is, there exists a sequence of positive definite functions $(\phi_n)_{n\in\mathbb{N}_0}$ in $C_0(G)$ such that $\phi_n(1) = 1$ for all n and $\phi_n \to 1$ uniformly over compact subsets of G.*

For a proof, we refer the reader to Proposition 2.1 of [25].

4.2.3 C_0-representations

Starting from a family of positive definite maps, let us deduce a last equivalent definition for the Haagerup property. We start by introducing a few definitions on unitary representations.

Definition 4.2.9. *A unitary representation* $\pi : G \to \mathcal{U}(\mathcal{H})$ *is called* C_0 *if for every* $v, w \in \mathcal{H}$ *and for every constant* $\epsilon > 0$*, there exists a compact subset* $K \subset G$ *such that* $|\langle \pi(x)v, w\rangle| \leq \epsilon$ *for every* $x \notin K$*. This is denoted more elegantly by*

$$\lim_{x \to \infty} \langle \pi(x)v, w\rangle = 0,$$

for every $v, w \in \mathcal{H}$*.*

In some sense, the following definition *feels* like an opposite to Definition 4.2.9.

Definition 4.2.10. *A unitary representation* $\pi : G \to \mathcal{U}(\mathcal{H})$ *has almost invariant vectors if for every compact subset* $K \subset G$ *and for every* $\epsilon > 0$*, there exists a unit vector* $\xi_{\epsilon,K} \in \mathcal{H}$ *such that*

$$\forall x \in K : \ \|\pi(x)(\xi_{\epsilon,K}) - \xi_{\epsilon,K}\| \leq \epsilon.$$

Said differently, a unitary representation has almost invariant vectors if it weakly contains the trivial representation 1_G*.*

The following classical result is due to Gel'fand, Naimark and Segal. It associates a unitary representation to every positive definite function on G. Recall that a vector ξ in a Hilbert space $\mathcal{H}$ is *cyclic* relative to a unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ if the set $\{\pi(x)(\xi) \mid x \in G\}$ generates a dense subspace of $\mathcal{H}$.

Proposition 4.2.11 (Gel'fand-Naimark-Segal)**.** *Given a positive definite map* $\phi : G \to \mathbb{C}$ *with* $\phi(1) = 1$*, there is a standard way to associate a triple* $(\pi, \mathcal{H}, \xi)$ *to it. Here,* $\mathcal{H}$ *is a Hilbert space,* $\pi : G \to \mathcal{U}(\mathcal{H})$ *is a unitary representation and* $\xi \in \mathcal{H}$ *is a cyclic unit vector for* π *satisfying that* $\forall x \in G : \ \phi(x) = \langle \pi(x)(\xi), \xi\rangle$*. If* ϕ *is* C_0*, i.e.* $\lim_{x\to\infty} \phi(x) = 0$*, then* π *is also* C_0*. The triple* $(\pi, \mathcal{H}, \xi)$ *is called the Gel'fand-Naimark-Segal triple associated to* ϕ*.*

Proof. Let us show the result in the case of discrete countable groups. Set V the space of maps from G to $\mathbb{C}$ with finite support and define

$$\forall f_1, f_2 \in V : \ \langle f_1, f_2\rangle = \sum_{g,h \in G} f_1(g)\overline{f_2(h)}\phi(h^{-1}g).$$

In particular, we have for all $f \in V$ that

$$\langle f, f\rangle = \sum_{g,h\in G} f(g)\overline{f(h)}\phi(h^{-1}g) \geq 0,$$

because ϕ is positive definite. Using the Cauchy-Schwarz inequality for semi-inner products, one sees easily that $W := \{f \in V \mid \langle f, f\rangle = 0\}$ is a subspace of V. We set $\mathcal{H}$ equal to the Cauchy completion of V/W; this is a Hilbert space. To lighten notation, we will denote every element $f + W \in \mathcal{H}$ simply by f.

For all $x \in G$ and $f \in \mathcal{H}$, define $\pi(x) : \mathcal{H} \to \mathcal{H}$ by $\pi(x)(f) : g \mapsto f(x^{-1}g)$. Since

$$\begin{aligned}
\langle \pi(x)f_1, \pi(x)f_2\rangle &= \sum_{g,h\in G} f_1(x^{-1}g)\overline{f_2(x^{-1}h)}\phi(h^{-1}g) \\
&= \sum_{g,h\in G} f_1(g)\overline{f_2(h)}\phi(h^{-1}x^{-1}xg) \\
&= \langle f_1, f_2\rangle,
\end{aligned}$$

we conclude that $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation. It is C_0 if ϕ is C_0.

Finally, if ξ is equal to the characteristic function of $\{1\} \subset G$, then

$$\forall x \in G : \langle \pi(x)\xi, \xi\rangle = \phi(x).$$

□

Our goal is to formulate the Haagerup property in terms of unitary representations. Assume thus that G is Haagerup and take a sequence $(\phi_n)_{n\in\mathbb{N}_0}$ of C_0 positive definite maps as in Theorem 4.2.8. Denote the corresponding Gel'fand-Naimark-Segal triples by $(\pi_n, \mathcal{H}_n, \xi_n)$ and set

$$\pi = \oplus_{n\in\mathbb{N}_0}\pi_n : G \to \mathcal{U}(\oplus_{n\in\mathbb{N}_0}\mathcal{H}_n).$$

The so obtained unitary representation is C_0. Indeed, choose

$$v = \oplus_{n\in\mathbb{N}_0}v_n, w = \oplus_{n\in\mathbb{N}_0}w_n \in \mathcal{H} \text{ and } \epsilon > 0.$$

By definition,

$$\langle \pi(x)v, w\rangle = \sum_{n\in\mathbb{N}_0} \langle \pi_n(x)v_n, w_n\rangle.$$

Take N large enough such that $\sum_{n\geq N}[\|v_n\|^2 + \|w_n\|^2] \leq \epsilon/2$. Then

$$\sum_{n\geq N} \langle \pi_n(x)v_n, w_n\rangle \leq \epsilon/2.$$

Take a compact set $K \subset G$ such that $\langle \pi_n(x)v_n, w_n\rangle < \frac{\epsilon}{2N}$ for every $x \notin K$ and $n \leq N$. This is possible since the ϕ_n and thus the π_n are C_0. We conclude that for every $x \notin K$, the inner product $\langle \pi(x)v, w\rangle < \epsilon$, concluding the proof that π is C_0.

We claim that π also contains almost invariant vectors. Indeed, fix a compact set $K \subset G$ and choose $\epsilon > 0$. Next, take N large enough such that $\forall x \in K : \ |\phi_N(x) - 1| \leq \frac{\epsilon^2}{2}$. We claim that $\|\pi(x)(\xi_N) - \xi_N\| \leq \epsilon$ for every $x \in K$. Indeed, if $x \in K$, then

$$\begin{aligned} \|\pi(x)(\xi_N) - \xi_N\|^2 &= \|\pi_N(x)(\xi_N) - \xi_N\|^2 \\ &= |2 - \phi_N(x) - \overline{\phi_N(x)}| \\ &\leq |1 - \phi_N(x)| + |1 - \overline{\phi_N(x)}| \leq \epsilon^2. \end{aligned}$$

We conclude that $\|\pi(x)(\xi_N) - \xi_N\| \leq \epsilon$. This implies that a group with property (H) admits a C_0 unitary representation which has almost invariant vectors. It turns out that the converse is also true.

Theorem 4.2.12 (see Proposition 2.1.1 in [25]). *A group is Haagerup if and only if it admits a C_0 unitary representation $G \to \mathcal{U}(\mathcal{H})$ which has almost invariant vectors.*

Summarizing, we obtain the following equivalent definitions for the Haagerup property.

Theorem 4.2.13. *A second countable, locally compact group G satisfies the Haagerup property (we also say it is a-T-menable or has Property (H)) if and only if any of the following equivalent conditions is satisfied:*

1. *there exists a continuous function $\psi : G \to \mathbb{R}^+$ which is conditionally negative definite and proper, that is,* $\lim_{g\to\infty} \psi(g) = \infty$*,*

2. *the abelian C^*-algebra $C_0(G)$ possesses an approximate unit of normalized, positive definite functions, that is, there exists a sequence of positive definite functions $(\phi_n)_{n\in\mathbb{N}_0}$ in $C_0(G)$ such that $\phi_n(1) = 1$ for all n and $\phi_n \to 1$ uniformly over compact subsets of G,*

3. *it admits a C_0 unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ which has almost invariant vectors,*

4. *it admits an affine isometric and metrically proper action on a Hilbert space.*

4.3 Examples, permanence properties and applications

The class of groups satisfying the Haagerup property has been extensively studied, e.g. [102], [25], [30], [66],... The reason for the interest in this class of groups is at least four-fold. Firstly, it is true that the Haagerup property knows translations and applications in various fields of mathematics such as representation theory, harmonic analysis, operator K-theory,... Secondly, groups with the Haagerup property satisfy nice conjectures such as the strong Baum-Connes conjecture and the Novikov conjecture [63], [104]. Thirdly, since amenable groups are Haagerup, the latter can be seen as a weak form of amenability and so it is interesting to compare this property with other weak forms of amenability. Finally, last but not least, it must be noted that the Haagerup property is satisfied for a rather large class of groups, thus rendering the class of Haagerup groups one which is hard to overlook. We shall start this section by giving examples and by stating basic permanence properties. We will end this chapter by elaborating on the usefulness of the Haagerup property.

4.3.1 Examples

We give some classes of groups that admit proper affine isometric actions on Hilbert spaces without giving too many details as to why the Haagerup property is satisfied. Such details will be mentioned later in the context of equivariant Hilbert space compression, see Chapter 5.

1. Starting with a trivial example: compact groups have the Haagerup property.

2. The Lie groups $SO(n,1)$ and $SU(n,1)$ which are the isometry groups of the n-dimensional real and complex hyperbolic space respectively, satisfy the Haagerup property [106], [107].

3. The free group $F_2 =< a, b >$ on 2 generators has property (H). Indeed, take the finite symmetric generating subset $S = \{a, b, a^{-1}, b^{-1}\}$ of F_2 and look at the corresponding Cayley-graph T. Denote $V = F_2$ the collection of vertices and E the collection of oriented edges of T. Define

$$\begin{array}{rccc} b: & G & \to & l^2(E) \\ & x & \mapsto & \chi_x, \end{array}$$

where

$$\chi_x : e \mapsto \begin{cases} 1 & \text{if } e \text{ lies on the geodesic path in } T \text{ from } 1 \text{ to } x \\ -1 & \text{if } e \text{ lies on the geodesic path in } T \text{ from } x \text{ to } 1 \\ 0 & \text{else.} \end{cases}$$

Given some $x \in F_2$ and some oriented edge $e = [g, h]$, we set $x \cdot e = [xg, xh]$ and we define a unitary action π of F_2 on $l^2(E)$ by setting $\pi(x)(f) : \overline{e} \mapsto f(x^{-1} \cdot \overline{e})$ for every $\overline{e} \in E$ and $x \in F_2$. One verifies that b is a 1-cocycle relative to π. Moreover, denoting l the word length function on F_2 relative to S, one can verify easily that $||b(x)|| = \sqrt{2l(x)}$. This implies that $\lim_{x\to\infty} ||b(x)|| = +\infty$.

It is a simple observation that closed subgroups of Haagerup groups are Haagerup. We conclude that free groups of any countable rank have property (H).

4. More generally, let T be a simplicial tree [1]. We consider the natural distance on T, i.e. the distance $d(x, y)$ between two vertices x and y is the number of (non-oriented) edges in a path without backtracking from x to y. Fixing any *base*-vertex x_0 on the tree, it is known that the map

$$\begin{array}{rccc} \psi: & \mathrm{Aut}(T) & \to & \mathbb{R}^+ \\ & g & \mapsto & d(gx_0, x_0) \end{array}$$

is conditionally negative definite [108], [2] (see also [68], Proposition 6.2). Said differently, a group acting properly on a tree is Haagerup.

5. Similarly, if a group acts properly on an $\mathbb{R}$-tree, then it is Haagerup since the function $g \mapsto d(g \cdot x_0, x_0)$ is again conditionally negative definite (pg. 73, [68]).

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[1] i.e. a tree whose edges have length 1.

6. A Coxeter group G is a discrete finitely presented group with presentation
$$\langle r_1, r_2, \ldots, r_n \mid (r_i r_j)^{m_{ij}} = 1 \rangle,$$
where $m_{ii} = 1$ and $m_{ij} \geq 2$ if $i \neq j$. We allow the case that $m_{ij} = +\infty$, meaning that no relation of the form $(r_i r_j)^{m_{ij}}$ is imposed. On a Coxeter group, we consider the word length metric $\psi = l : G \to \mathbb{R}^+$ relative to $\{r_1, r_2, \ldots, r_n\}$. In [18], it is shown that l is a conditionally negative definite function on G, i.e. Coxeter groups satisfy the Haagerup property.

7. Amenable groups have the Haagerup property. Indeed, let G be a locally compact group with Haar measure μ. Let $\lambda : G \to \mathcal{U}(L^2(G, \mu))$ be the regular representation, i.e.
$$\forall x, y \in G,\ f \in L^2(G, \mu) :\ \lambda(x)(f) : y \mapsto f(x^{-1}y).$$
Given $f, g \in C_c(G)$, then $x \mapsto \langle \lambda(x)(f), g \rangle$ has compact support. Since the set $C_c(G)$ of continuous functions with compact support is dense in $L^2(G, \mu)$, this implies that λ is C_0. The result then follows from the standard fact that λ has almost invariant vectors if and only if G is amenable [88].

   The converse is of course false, e.g. F_2 is Haagerup but not amenable.

8. The Baumslag-Solitar monsters $B(p, q)$ with $p, q \geq 1$ are discrete finitely presented groups with presentation given by
$$B(p, q) =< a, b \mid b^{-1} a^q b = a^p > .$$
They satisfy the Haagerup property [51]. For more details and an alternate proof, we refer the reader to Section 6.4.

In order to find groups not satisfying property (H), we could look at the class of groups with property (T), e.g. $SL_n(\mathbb{R})$ for $n \geq 3$. In fact, there is some weaker relative version for property (T) which often prevents a group from being Haagerup.

Definition 4.3.1. *Let G be a compactly generated group and let $H < G$ be a closed subgroup. We say that the pair (G, H) has* relative property (T) *if every conditionally negative definite map on G is bounded on H.*

If G has the Haagerup property and (G, H) has relative property (T), then H must be compact. For connected Lie groups, the converse is also true, i.e. if (G, H) can have relative property (T) only if H is compact, then G is Haagerup (Theorem 4.0.1, [25]). In general, there exist examples of discrete groups which are not Haagerup but do not have property (T) relative to an infinite subgroup: they have property (T) relative to an infinite *subset* [30].

In [68], page 18 and 94, it is shown that $(\mathbb{R}^2 \rtimes SL_2(\mathbb{R}), \mathbb{R}^2)$ and $(\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z}), \mathbb{Z}^2)$ have relative property (T) so that $\mathbb{R}^2 \rtimes SL_2(\mathbb{R})$ and $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ are not Haagerup.

4.3.2 Permanence properties

The permanence properties for the class of Haagerup groups have been well studied and are of major importance in our work. We summarize the main results here.

1. **closed subgroups.** A closed subgroup of a Haagerup group is (clearly) Haagerup.

2. **direct limits.** Assume that G is the increasing union of a sequence $(G_n)_{n \in \mathbb{N}_0}$ of open subgroups. If all G_n have the Haagerup property, then so does G (Proposition 6.1.1 in [25]).

3. **amalgamated free products over finite groups.** If G and H are *discrete* groups with the Haagerup property, containing a common finite subgroup A, then the amalgamated free product $G *_A H$ is also Haagerup (Proposition 6.2.3 of [25]). In particular, $SL_2(\mathbb{Z}) = \mathbb{Z}_6 *_{\mathbb{Z}_2} \mathbb{Z}_4$ is Haagerup.

4. **HNN-extensions over finite groups.** Let H be a discrete group, A a finite subgroup of H and $\theta : A \to H$ a monomorphism. If H has the Haagerup property, then the Higman-Neumann-Neumann extension $\mathrm{HNN}(A, H, \theta)$ has the Haagerup property (Proposition 6.2.7 of [25]).

5. **semi-direct products.** The class of Haagerup groups is *not* closed under semi-direct products. Indeed, we have already mentioned that $\mathbb{R}^2 \rtimes SL_2(\mathbb{R})$ is not Haagerup, yet $\mathbb{R}^2$ and $SL_2(\mathbb{R})$ are! However, as

the following example shows, there exist special types of semi-direct products which preserve the Haagerup property.

6. **wreath products.** The class of Haagerup groups is closed under wreath products, i.e. if G and H are Haagerup then so is $H \wr G$ [31]. Recall that $H \wr G = W \rtimes G$ where $W = H^{(G)}$, the space of functions from G to H with finite support, and where G acts on W as follows: $\forall g, x \in G, f \in W : \; g \cdot f : x \mapsto f(g^{-1}x)$.

7. **quotients.** The class of Haagerup groups is not closed under quotients, since every countable group is a quotient of a free group. However, there is a type of special quotient in which the Haagerup property is preserved.

8. **Some special kind of quotient for which the Haagerup property is preserved.** Let Γ_0 be a discrete group and let $\rho : \Gamma_0 \to \mathrm{Aut}(A)$ be an action by automorphisms on a discrete abelian group A. Assume that there is $a \in A$ such that its stabilizer $\{\gamma \in \Gamma_0 \mid \rho(\gamma)(a) = a\}$ in Γ_0 is finite. Let Γ be a discrete group together with a surjective homomorphism $p : \Gamma \twoheadrightarrow \Gamma_0$ and consider the action of Γ on A given by $\widetilde{\rho} = \rho \circ p : \Gamma \to \mathrm{Aut}(A)$. If $A \rtimes_{\widetilde{\rho}} \Gamma$ is Haagerup, then so is Γ_0 (See Theorem 3.1 of [26]).

4.3.3 Applications

We end this chapter by a quick word regarding the usefulness of the Haagerup property.

First of all, we note that the Haagerup property can be seen as some strong form of *non-rigidity*. Indeed, let Γ be a discrete group with the Haagerup property. Fix a proper conditionally negative definite map on Γ and perturb this map by a bounded function to obtain a conditionally negative definite ψ satisfying $\psi(x) = 0$ if and only if $x = 0$ (see Lemma 6.1.9, which is Lemma 6.2.1 in [25]). For $t > 0$, the maps $e^{-t\psi}$ are C_0 positive definite maps and the corresponding Gel'fand-Naimark-Segal triples give C_0 unitary representations π_t of G. Consequently, there exists a one-parameter family of C_0 unitary representations $(\pi_t)_{t>0}$ that interpolates between the trivial representation at $t = 0$ and the regular representation at $t = \infty$.

Another interesting fact is that Haagerup groups satisfy the Baum-Connes and the Novikov conjecture. We refer the reader to [63], [104], [25] (pg.9) for details.

Finally, since group actions form an important bridge between the land of groups and the land of functional analysis, it is not a big surprise that the Haagerup property, and property (T), can be reformulated in terms of von Neumann algebras [66], [67], [4], [27], [26],... Again, one can see the Haagerup property as a strong negation of property T and one can define notions for von Neumann algebras which correspond to the relative property (T). We will not go into this much further, restricting ourselves only to the following basic definitions.

Definition 4.3.2. *A C^*-algebra is a Banach algebra $\mathcal{B}$ over $\mathbb{C}$ together with a map $* : \mathcal{B} \to \mathcal{B}$ such that $\forall f, g \in \mathcal{B}$ and $\forall z \in \mathbb{C}$:*

$$(f+g)^* = f^* + g^*,$$

$$(zf)^* = \overline{z} f^*,$$

$$f^{**} = f,$$

$$(fg)^* = g^* f^*.$$

Moreover, the Banach norm on $\mathcal{B}$ must satisfy the additional condition that $\|f^* f\| = \|f\|^2$ for all $f \in \mathcal{B}$.

The most famous C^*-algebra associated to a Hilbert space $\mathcal{H}$ is probably $\mathcal{B}(\mathcal{H})$, the space of bounded operators on $\mathcal{H}$, equipped with the operator norm. Here, the $*$-map sends every $T \in \mathcal{B}(\mathcal{H})$ to its hermitian conjugate.

Before we proceed, we mention that the weak operator topology on $\mathcal{B}(\mathcal{H})$ is the smallest topology such that the functional $T \mapsto \langle Tv, w \rangle$ is continuous for every $v, w \in \mathcal{H}$. We come to the following Definition.

Definition 4.3.3. *A von Neumann algebra A is a weakly closed *-algebra of bounded operators (on a Hilbert space) containing the identity.*

There is a natural way to associate a von Neumann algebra to a locally compact group.

Definition 4.3.4. *Let G be a locally compact group with Haar measure μ and denote $\lambda : G \to \mathcal{U}(L^2(G,\mu))$ its left regular representation, i.e. $\forall x, y \in G,\ \forall f \in L^2(G) : \lambda(x)(f) : y \mapsto f(x^{-1}y)$. We define the group von Neumann algebra as the von Neumann algebra $W^*(G)$ generated by $\lambda(G) \subset \mathcal{B}(L^2(G))$.*

For the class of finite von Neumann algebras, there is a notion of Haagerup's property. We refer the reader to [67] for details on the following definition.

Definition 4.3.5 (Definition 2.1 in [67])**.** *Let A be a finite von Neumann algebra and let τ be a finite, faithful, normal, normalized trace on A. We say that A has the* Haagerup property *if there exists a sequence $(\Phi_n)_{n\geq 1}$ of completely positive, normal maps from A to itself such that:*

1. *$\tau \circ \Phi_n \leq \tau$ and Φ_n is L^2-compact for every n;*
2. *for every $x \in A$, $\|\Phi_n(x) - x\|_{2,\tau} \to 0$ as $n \to \infty$.*

The following theorem is due to Choda [27] (see also Proposition 4.16 of [4]).

Theorem 4.3.6. *For a countable group G, the following conditions are equivalent.*

1. *G is Haagerup in the (usual) sense of Definition 4.1.4,*
2. *The associated von Neumann algebra $W^*(G)$ is Haagerup in the sense of Definition 4.3.5.*

Chapter 5

Equivariant Hilbert space compression

Equivariant Hilbert space compression arose as a desire to quantify the Haagerup property for topological groups. While the Haagerup property depends completely on the topology, the quantification of this property requires the choice of a length function on the group. We start this chapter by recalling elementary notions related to length functions on groups. We shall then give a precise definition of *equivariant Hilbert space compression* in Section 5.2. In Section 5.3, we give an overview of examples of groups together with their equivariant compression. Additional properties and an introduction to our main research question are formulated in Section 5.4. For definiteness, we state the following

Convention 5.0.7. *From here on and throughout Part II, all of our groups are topological groups and we only consider strongly continuous group actions* $\alpha : G \times \mathcal{H} \to \mathcal{H}$ *on Hilbert spaces* $\mathcal{H}$*, i.e. the map*

$$\begin{array}{ccc} G \times \mathcal{H} & \to & \mathcal{H} \\ (x, v) & \mapsto & \alpha(x)v \end{array}$$

is continuous. Hence, all of our 1*-cocycles, conditionally negative definite maps,... are assumed continuous. On the group, we will each time also specify a length function* l*. We emphasize that this length function does not need to induce the given topology on the group.*

5.1 Length functions on groups

Definition 5.1.1. *A length function* l *on a topological group* G *is a (not necessarily continuous) function* $l : G \to \mathbb{R}^+$ *such that*

1. *$\forall x \in G : l(x) = 0 \Leftrightarrow x = 1$,*
2. *$\forall x \in G :\ l(x) = l(x^{-1})$ and*
3. *$\forall x, y \in G :\ l(xy) \leq l(x) + l(y)$.*

A length function on G induces a left-invariant metric on G by setting $d(x, y) = l(x^{-1}y)$, $\forall x, y \in G$. Conversely, if d is a left-invariant metric on G, then $x \mapsto d(x, 1)$ is a length function on G.

Definition 5.1.2. *We say that (G, l), or l, is uniformly discrete if there exists $\epsilon > 0$ such that $d(x, y) = l(x^{-1}y) > \epsilon$ for every two distinct elements $x, y \in G$. We say that l is proper, whenever*

$$\forall M \in \mathbb{R}^+ :\ \{x \in G \mid l(x) \leq M\} \text{ is compact.}$$

Note that groups equipped with proper length functions are σ-compact.

There exist two important equivalence relations on the set of distances or length functions on a group: coarse equivalence and quasi-isometric equivalence.

Definition 5.1.3. *Two length functions l_1, l_2 on a group G are coarsely equivalent if the identity $Id : (G, l_1) \to (G, l_2)$ is a coarse equivalence, i.e. if the following two conditions are satisfied:*

- *for every $R > 0$ there exists $S > 0$ such that the l_1-ball of radius R and center 1 is contained in the l_2-ball of radius S and center 1,*
- *for every $R > 0$ there exists $S > 0$ such that the l_2-ball of radius R and center 1 is contained in the l_1-ball or radius S and center 1.*

Famous properties which are invariant under the coarse equivalence class of a length function l are uniform embeddability into a Hilbert space and property (A), see Section 7.1 for details.

Definition 5.1.4. *Let $f : X \to Y$ be a map between 2 metric spaces (X, d) and (Y, ρ). We say that it is a quasi-isometric embedding if there are constants $C, D > 0$ such that*

$$\forall x, x' \in X :\ \frac{1}{C} d(x, x') - D \leq \rho(f(x), f(x')) \leq C d(x, x') + D.$$

It is a quasi-isometry if there is also a map $g : Y \to X$ which is a quasi-isometric embedding such that $d(g \circ f(x), x)$ is uniformly bounded over X and $\rho(f \circ g(y), y)$ is uniformly bounded over Y.

Definition 5.1.5. *Two length functions l_1, l_2 on a group G are quasi-isometric whenever the identity $Id : (G, l_1) \to (G, l_2)$ is a quasi-isometry.*

Every coarse invariant is clearly a quasi-isometric invariant. Other properties of groups which are invariant under quasi-isometry are polynomial growth, amenability, etc.

There is an interesting class of groups that admit a (proper) length function in a natural way.

Definition 5.1.6. *A second countable locally compact group G is* compactly generated *if there exists a compact subset $S \subset G$ such that $G = \cup_{n\in\mathbb{N}} S^n$. We will without loss of generality assume that S contains an open neighbourhood of $1 \in G$ and that it is symmetric, i.e. that $x \in S \Leftrightarrow x^{-1} \in S$.*

We define the length $|x|_S$ of $x \in G$ as the smallest natural number n such that $x \in S^n$. The so obtained length function is called the *word length function relative to S*. Among all symmetric compact generating sets $S \subset G$ containing an open neighbourhood of the identity, the quasi-isometric class of $|\cdot|_S$ does not depend on the choice of S. Note that $|\cdot|_S$ in general does not induce the topology on G.

Example 5.1.7. *All connected locally compact groups are compactly generated. Indeed, if V is a compact neighbourhood of the identity, then the subgroup $\langle V \rangle$ generated by V is open and closed, hence $G = \langle V \rangle$ by connectedness.*

Example 5.1.8. *Another example is given by the class of finitely generated groups. If S is a finite symmetric generating subset for G which contains the identity, then $|x|_S$ is the least number of (non-oriented) edges that a path between 1 and x on the Cayley graph of (G, S) must contain. The word length function associated to another finite generating subset S' will be quasi-isometric to $|\cdot|_S$.*

In the case of compactly generated groups, one can define the Haagerup property solely in terms of the word length function $|\cdot|_S$. Indeed, G has the Haagerup property if and only if there exists a 1-cocycle b on G such that for all $M > 0$ there exists R such that $\|b(x)\| \geq M$ whenever $x \notin \overline{B(1, R)}$.

5.2 Definition

Let G be a compactly generated group and let S be a compact generating subset (which is symmetric and contains an open neighbourhood of $1 \in G$). Assume that the word length function associated to S is not bounded, said differently there is no natural number n such that $G = S^n$.

It is an easy observation that all 1-cocycles b, relative to unitary actions of G on Hilbert spaces $\mathcal{H}$, are Lipschitz. Indeed, if we set $M = \max_{s \in S} \|b(s)\|$, then for $x = s_1 s_2 \dots s_n$ with $s_1, s_2, \dots, s_n \in S$, we get

$$\|b(s_1 s_2 \dots s_n)\| \leq \sum_{i=1}^{n} \|b(s_i)\| \leq Mn.$$

This implies that b is Lipschitz since

$$\|b(x) - b(y)\| = \|b(x^{-1}y)\| \leq M|x^{-1}y|_S = M d_S(x, y).$$

The fact that $\|b(x)\|$ admits a linear upper bound raises a natural question: *What can we say about a lower bound for* $\|b(x)\|$*?*

Not every 1-cocycle needs to admit a linear lower bound. Even stronger, a result by Bourgain [17] shows that the free group F_2 on 2 generators does not embed quasi-isometrically into a Hilbert space, so it admits no 1-cocycle b with linear lower bound at all. This invokes the following question: *How close to linear can the lower bound of* $\|b(x)\|$ *be?*

Definition 5.2.1. *Let G be a topological (not necessarily compactly generated) group. Let b be a (always continuous) 1-cocycle on G, relative to some (always strongly continuous) unitary representation of G on a Hilbert space. Assume that l is a length function on G. We do not require that the topology on G is induced by l. The supremum of $r \in [0, 1]$ such that there are numbers $C, D > 0$ satisfying*

$$\forall x \in G: \; (1/C) l(x)^r - D \leq \|b(x)\| \leq C l(x) + D,$$

is called the compression of b (relative to l). We denote this by $R(b)$.

Remark 5.2.2. *The compression of a 1-cocycle b is only defined if b is large-scale Lipschitz relative to l, i.e. if there exist $C, D > 0$ such that*

$$\|b(x)\| \leq C l(x) + D,$$

for every $x \in G$.

The above definition is a statement on the 1-cocycle itself. Being interested in the underlying group, the following natural question imposes itself: *How fast does the fastest increasing* 1*-cocycle go to infinity?* This leads to the definition of equivariant Hilbert space compression.

Definition 5.2.3. *Let G be a topological group and l a length function on G. The equivariant (Hilbert space) compression $\alpha^*(G)$ of (G,l) is the supremum of $R(b)$ where b is a (always continuous)* 1*-cocycle on G, large-scale Lipschitz relative to l.*

Remark 5.2.4. *Let us mention explicitly that the supremum in Definition 5.2.3 is taken over* 1*-cocycles, relative to all unitary representations of G on all possible Hilbert spaces. Moreover, continuity is taken with respect to the original topology on the group, not the topology induced by l.*

Another way to interpret compression is by looking at the class of G-equivariant maps. A map $f : G \to \mathcal{H}$ is called G-equivariant if there exists an affine isometric action α of G on $\mathcal{H}$ such that $f(xy) = \alpha(x)f(y)$ for every $x, y \in G$. Clearly, such maps are continuous and they *grow at the same speed* as their corresponding 1-cocycles. Indeed, let b be the 1-cocycle associated to α and denote the linear part of α by π, then

$$\forall x \in G: \ \|f(x)\| = \|\alpha(x)f(0)\| = \|\pi(x)f(0) + b(x)\|.$$

If we set $\|f(0)\| = M$, then from the fact that π is unitary, we get that

$$\|b(x)\| - M \leq \|f(x)\| \leq \|b(x)\| + M.$$

Said differently, instead of restricting ourselves to 1-cocycles in Definition 5.2.3, we could have also defined equivariant compression by looking at the class of *all* G-equivariant maps.

The above interpretation gives rise to a notion of equivariant compression for a topological space (X,τ) relative to a metric d which is not necessarily compatible with the topology τ. Assume that β is a (always strongly continuous) isometric action of some group G on (X,τ). A continuous map $f : (X,\tau) \to \mathcal{H}$ is called G-equivariant relative to β, if there exists a Hilbert space $\mathcal{H}$ and an affine isometric action α of G on $\mathcal{H}$ such that $f(\beta(g)x) = \alpha(g)f(x)$ for every $g \in G,\ x \in X$.

Definition 5.2.5. *Let X be a topological space and let d be a metric on X, not necessarily compatible with the topology. Assume that $\beta : G \to \mathrm{Iso}(X)$ is an isometric action. The equivariant compression of X, relative to β, is the supremum of $r \in [0,1]$ such that there exist constants $C > 1, D \geq 0$ and a map $f : X \to \mathcal{H}$, $G-$equivariant relative to β, such that*

$$(1/C)d(x,y)^r - D \leq \|f(x) - f(y)\| \leq Cd(x,y) + D,$$

for all $x, y \in X$.

We conclude this section by a few remarks related to the quasi-isometric invariance of the equivariant Hilbert space compression.

Lemma 5.2.6. *The equivariant Hilbert space compression of (G,l) only depends on the quasi-isometric class of l.*

Proof. Let l_1 and l_2 be quasi-isometric length functions on a group G. Find constants $\overline{C}, \overline{D} > 1$ such that

$$\forall x \in G: \ \frac{1}{\overline{C}} l_2(x) - \overline{D} \leq l_1(x) \leq \overline{C} l_2(x) + \overline{D}. \tag{5.1}$$

Now, assume that there exists a 1-cocycle b on G and numbers $C, D > 1$, $r \in [0,1]$ such that

$$\forall x \in G: \ \frac{1}{C} l_1(x)^r - D \leq \|b(x)\| \leq C l_1(x) + D. \tag{5.2}$$

Combining (5.1) and (5.2), we obtain that

$$\forall x \in G: \ \frac{1}{C}(\frac{1}{\overline{C}} l_2(x) - \overline{D})^r - D \leq \|b(x)\| \leq C(\overline{C} l_2(x) + \overline{D}) + D.$$

The upper bound is already affine. Regarding the lower bound, notice that if $l_2(x) \geq 2\overline{CD}$, then

$$\frac{1}{\overline{C}} l_2(x) - \overline{D} \geq \frac{1}{\overline{C}} l_2(x) - \frac{1}{2\overline{C}} l_2(x) = \frac{1}{2\overline{C}} l_2(x).$$

Consequently,

$$\|b(x)\| \geq \frac{1}{C}(\frac{1}{2\overline{C}})^r l_2(x)^r - D.$$

If $l_2(x) \leq 2\overline{C}\overline{D}$, then

$$\|b(x)\| \geq 0 \geq \frac{1}{C}(\frac{1}{2\overline{C}})^r l_2(x)^r - \frac{\overline{D}^r}{C}.$$

Denoting $\widetilde{D} = \max(\frac{\overline{D}^r}{C}, C\overline{D} + D)$ and $\widetilde{C} = \max(C(2\overline{C})^r, C\overline{C})$, we get that

$$\forall x \in G: \ \frac{1}{\widetilde{C}} l_2(x)^r - \widetilde{D} \leq \|b(x)\| \leq \widetilde{C} l_2(x) + \widetilde{D}.$$

□

Corollary 5.2.7. *The equivariant compression of a compactly generated group is independent of the choice of the word length function* $|\cdot|_S$ *where* S *is a compact symmetric generating subset containing a neighbourhood of the identity.*

Corollary 5.2.8. *If* (H, l_H) *is a quasi-isometrically embedded subgroup of* (G, l_G) *then the equivariant compression of* (H, l_H) *is greater than the equivariant compression of* (G, l_G)*.*

Corollary 5.2.9. *If* G *is equipped with a uniformly discrete length function* l*, then multiplication of* l *by a constant provides another quasi-isometric uniformly discrete length function. In this case, we may thus always assume that*

$$\inf_{x \in G \setminus \{1\}} l(x) \geq 1.$$

Remark 5.2.10. *There is no reason why the value of the equivariant compression should in general be a quasi-isometric invariant, i.e. if*

$$f : (G, l_G) \to (H, l_H)$$

is a quasi-isometry, then we can not simply conclude that the equivariant compressions of (G, l_G) *and* (H, l_H) *are equal. However, we* can *conclude that the equivariant compressions are equal if both* f *and its "quasi-isometric inverse* g*" are continuous group homomorphisms. The idea is that every* 1*-cocycle* b *on* H *gives a* 1*-cocycle* $\widetilde{b}$ *on* G *by* $\widetilde{b}(x) = b(f(x))$ *with the same compression. Similarly, one can associate to every* 1*-cocycle on* G *a* 1*-cocycle on* H *with the same compression.*

5.3 Examples

In this section, we list examples of groups with known equivariant Hilbert space compression. To make the reader more familiar with the subject, we shall in some cases elaborate on how the equivariant compression can be calculated. As a guideline, we use the list of examples of the Haagerup property in Section 4.3, see also [25].

1. All 1-cocycles on compact groups are bounded and equipping these groups with the word length function, we see that they have equivariant compression 1.

2. Polycyclic groups and connected amenable Lie groups have equivariant compression 1 when equipped with the word length metric relative to a compact generating subset [103].

3. Richard Thompson's group F has equivariant compression 1/2, see [7]. This is in fact a special case of a result on finitely generated diagram groups G. Concretely, if we denote the diagram metric on G by d, then the proof of Theorem 1.13 in [7] shows that $\psi : x \mapsto d(x,1)$ is a conditionally negative definite map on G. So, if the diagram metric is bi-Lipschitz equivalent to the word length metric, then the equivariant Hilbert space compression of G is at least 1/2.

4. The Baumslag-Solitar monsters $BS(p,q)$ with $p,q > 1$ have equivariant compression 1/2. This is calculated in Section 6.4.

5. The group $SO(1,n)$ of isometries of real hyperbolic n-space has equivariant compression 1/2. The result can be deduced from the work of Robertson, see Corollary 2.5 in [95], on which we would like to elaborate here.

   Denote real hyperbolic n-space by X and recall that X is the set
   $$\{x \in \mathbb{R}^{n+1} : \ x_0 > 0, -x_0^2 + x_1^2 + x_2^2 + \ldots + x_n^2 = -1\}.$$
   Denote $O(1,n)$ the group of linear automorphisms of $\mathbb{R}^{n+1}$ which preserve the quadratic form $-x_0^2 + x_1^2 + x_2^2 + \ldots + x_n^2$. The identity component $SO(1,n)$ of $O(1,n)$ acts transitively and by isometries

on X. The stabilizer of $(1,0,0,\dots,0)$ is $SO(n)$, yielding $X = SO(1,n)/SO(n)$.

The (hyperbolic) distance between two elements is defined as follows. Consider the Minkowski inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^{n+1}$, i.e. $\langle x,y\rangle = -x_0y_0 + \sum_{i=1}^n x_iy_i$. Now, given two points x, y in X, we define the distance between two points x, y of X as $d(x,y) = \operatorname{Arccosh}(-\langle\widetilde{x},\widetilde{y}\rangle)$.

Let H_0 be the half space of X, consisting of points with last coordinate strictly positive. Let Ω be the set of all half spaces of X, i.e. the $SO(1,n)$-translates of H_0. The stabilizer of H_0 is $SO(1,n-1)$, so Ω is isomorphic to $SO(1,n)/SO(1,n-1)$. The groups $SO(1,n)$ and $SO(1,n-1)$ are unimodular, i.e. the right Haar measure is a left Haar measure ([13], Proposition C.4.11). It follows that there is a nonzero positive $SO(1,n)$-invariant measure μ_Ω on the quotient ([79], Chapter 3, p.140, Corollary 4).

Given $x \in X$, denote the set of all half-spaces that contain x by Σ_x. Corollary 2.5 in [95] states that

Corollary 5.3.1. *There is a constant $k > 0$ such that $d(x,y) = k\mu_\Omega(\Sigma_x\Delta\Sigma_y)$ for every $x, y \in X$.*

Now, fix $x_0 \in X$ and define

$$\begin{array}{rccl} f: & X & \to & L^2(\Omega,\mu_\Omega) \\ & x & \mapsto & \chi_x - \chi_{x_0}, \end{array}$$

where χ_x is the characteristic function of Σ_x. Let

$$\lambda : SO(1,n) \to \mathcal{O}(L^2(\Omega,\mu_\Omega))$$

be the orthogonal representation induced by the regular representation and set

$$\begin{array}{rccl} b: & SO(1,n) & \to & L^2(\Omega,\mu_\Omega) \\ & g & \mapsto & \lambda(g)(\chi_{x_0}) - \chi_{x_0} = \chi_{gx_0} - \chi_{x_0}. \end{array}$$

One verifies that f is $SO(1,n)$-equivariant relative to the natural $SO(1,n)$-action on X and the affine isometric action with linear part λ and 1-cocycle b on $L^2(\Omega,\mu_\Omega)$. Moreover, since

$$\|f(x)\| = \|\chi_x - \chi_{x_0}\| = \sqrt{\mu_\Omega(\Sigma_x\Delta\Sigma_{x_0})} = \sqrt{(1/k)d(x,x_0)},$$

we conclude that the equivariant compression of X, relative to the action by its isometry-group, is at least $1/2$.

De Cornulier, Tessera and Valette have proved in [32] that the equivariant compression of non-amenable, locally compact compactly generated groups equipped with the word length function, is smaller or equal to $1/2$. Since the $1-$cocycle b has equivariant compression equal to $1/2$ and since $SO(1,n)$ is not amenable, we conclude that $\alpha^*(SO(1,n)) = 1/2$.

Regarding the complex case, we note that the equivariant compression of $SU(1,n)$ is again $1/2$. It is smaller or equal to $1/2$ since $SU(1,n)$ is not amenable. On the other hand, let us denote the hyperbolic distance on the complex hyperbolic plane by d and let x_0 be any fixed point in the complex hyperbolic plane. A result by Faraut and Harzallah [48] shows that $g \mapsto d(gx_0, x_0)$ is a conditionally negative definite map on G and so the equivariant compression of $SU(1,n)$ is at least $1/2$. We conclude that the equivariant compression of $SU(1,n)$ is $1/2$.

6. The free group on any finite number of generators has compression at most $1/2$ since it is not amenable. It follows from the 1-cocycle on page 70, that the equivariant compression is in fact equal to $1/2$.

7. Let G be a Coxeter group with finite generating subset S. Bozejko, Januszkiewick and Spatzier showed in [18] that the word length function relative to S is a conditionally negative definite function on G. Consequently, the equivariant Hilbert space compression of Coxeter groups is at least $1/2$.

8. In the case of amenable groups, we are not able to give a general statement regarding the value of the equivariant Hilbert space compression. There are amenable groups whose equivariant compression is 1, e.g. all polycyclic groups. On the other side of the spectrum, Tim Austin [9] proved the existence of a finitely generated amenable group whose equivariant compression, relative to the word length metric, equals 0. We can say that at this time, only a countable number of explicit values for the equivariant Hilbert space compression are known (see also Section 5.4).

5.4 Properties and research interests

In this paragraph, we restrict ourselves to compactly generated groups equipped with the word length metric. Recall that by our definition such groups are locally compact and second countable, see Definition 5.1.6. In this context, a group G is Haagerup if and only if it admits a 1-cocycle b satisfying that

$$\forall M \in \mathbb{R}^+,\ \exists R > 0 :\ x \notin \overline{B(1,R)} \Longrightarrow \|b(x)\| \geq M.$$

In this section, we would like to state some of the main results related to compression and formulate our main research question.

Intuitively, equivariant compression comes from the desire to quantify the Haagerup property. If the equivariant compression of G is 1, then not only does G admit a metrically proper 1-cocycle (i.e. is Haagerup), it admits one which grows really fast. In fact, for every $r \in [0,1[$ there is a 1-cocycle b such that $\|b(x)\|$ is bounded from below by an affine function in $|x|^r$! Conversely, if the equivariant compression is a strictly positive number which is close to 0, then again G is Haagerup, all be it in a *weaker* sense: 1-cocycles converge to infinity, but slower.

It is natural to ask why quantifying the degree of being Haagerup may be useful. To this end, recall first that amenable groups are Haagerup, but that the converse is not necessarily true: F_2 is Haagerup but not amenable. Using equivariant compression, we are able to formulate a special kind of converse. Indeed, an interesting result due to Guentner and Kaminker [59] which is later generalized by de Cornulier, Tessera and Valette (see [32], Theorem 4.1) states that $\alpha^*(G) > 1/2$ implies amenability of G. Said differently, a compactly generated group which is Haagerup in *a strong sense* is amenable! One of the reasons why this result is useful was illustrated by an attempt of Arzhantseva, Guba and Sapir in [7] to investigate the amenability of Thompson's groups F. They calculated the equivariant compression of F but unfortunately discovered that it is exactly 1/2.

A natural question to ask is whether a f.g. group is Haagerup if and only it has a strictly positive equivariant Hilbert space compression. Clearly, if the equivariant compression of G is strictly positive, then G is Haagerup. The converse was disproved by Tim Austin [9] who found a finitely generated amenable group with equivariant Hilbert space compression equal to 0.

The reader should now be convinced that it is interesting to calculate the equivariant compression of a group. It would thus be desirable to know how equivariant compression behaves under group constructions. This is what we will investigate in Part II. Given two or more groups $G_1, G_2, \ldots$, one can construct many new groups by taking direct sums $G_1 \oplus G_2$, wreath products $G_1 \wr G_2$, amalgamated free products $G_1 *_C G_2$, HNN-extensions $HNN(H, F, \theta)$, directed limits, group extensions, quotients,…. Oddly enough, the behaviour of the equivariant Hilbert space compression has been examined in only two of the above situations.

- The case of direct sums was treated in [59]: starting from 1-cocycles $b_1 : G_1 \to \mathcal{H}_1$ and $b_2 : G_2 \to \mathcal{H}_2$, with compressions $R(b_1)$ and $R(b_2)$ respectively, Guentner and Kaminker constructed a new cocycle $b : G_1 \oplus G_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$ by setting $b(g_1 \oplus g_2) = b(g_1) \oplus b(g_2)$. Here, $G = G_1 \oplus G_2$ is equipped with the straightforward length function and topology. One verifies easily that the compression of b equals $\min(R(b_1), R(b_2))$. Consequently, $\alpha^*(G) \geq \min(\alpha^*(G_1), \alpha^*(G_2))$. Since G_1 and G_2 are also metric subspaces of G, we conclude that $\alpha^*(G) = \min(\alpha^*(G_1), \alpha^*(G_2))$,

- The case of wreath products is complicated and has been examined for example in [80], [100], [7], [103]. It is impossible to give the equivariant compression of $G_1 \wr G_2$ solely in terms of the equivariant compression of G_1 and G_2. Indeed, the equivariant compression of $\mathbb{Z} \wr \mathbb{Z}$, two groups of compression 1, is equal to 2/3 (see [10]), whereas the equivariant compression of $F \wr \mathbb{Z}$ with F finite, equals 1 (see [103]).

  In this context, there is a very nice result by Naor and Peres in [80].

  Theorem 5.4.1. *Let G be a finitely generated group equipped with the word lenght metric relative to a finite symmetric generating subset. Then*

  $$\alpha^*(G) \geq \frac{1}{2} \Longrightarrow \alpha^*(G \wr \mathbb{Z}) \geq \frac{2\alpha^*(G)}{2\alpha^*(G)+1},$$

  and

  $$\alpha^*(G) \leq \frac{1}{2} \Longrightarrow \alpha^*(G \wr \mathbb{Z}) = \alpha^*(G).$$

> Denoting the k-fold wreath product of $\mathbb{Z}$ with itself by $Z_{(k)}$, i.e. $\mathbb{Z}_{(1)} = \mathbb{Z}, \mathbb{Z}_{(2)} = \mathbb{Z}\wr\mathbb{Z}, \mathbb{Z}_{(3)} = \mathbb{Z}_{(2)}\wr\mathbb{Z}, \ldots$, they found that $\alpha^*(\mathbb{Z}_{(k)}) = \frac{1}{2-2^{1-k}}$. Astonishingly, together with 0 and 1/2, these are the only known explicit values for the equivariant compression of a finitely generated group.

In Part II of this thesis, we study the behaviour of the equivariant Hilbert space compression under free products, amalgamated over finite groups, and of HNN-extensions $\mathrm{HNN}(G, F, \theta)$ where F is finite. Our results give exact values, i.e. we do not obtain upper and lower bounds but equalities. We refer the reader to Sections 6.1 and 6.2 for details. After that, we will treat the behaviour of the equivariant Hilbert space compression under a special type of quotient. We refer the reader to Section 6.3 for details. We end this chapter by calculating explicitly the equivariant compression of the Baumslag-Solitar monsters $BS(p, q)$ with $p, q \geq 1$.

Chapter 6

Results

In this Chapter, the reader may find our results related to equivariant Hilbert space compression. In Section 6.1, we calculate the equivariant Hilbert space compression of free products under finite amalgamation. In Section 6.2, we have a similar result, this time for HNN-extensions. Section 6.3 is related to the equivariant compression of a special type of quotient of a group. In Section 6.4 finally, we calculate the equivariant Hilbert space compression of the Baumslag-Solitar monsters. For this, we owe gratitude to Y. de Cornulier. Notice that Section 6.4 is quite different in nature from the other sections, since here we do not investigate the behaviour of the equivariant Hilbert space compression under a group construction.

Throughout this chapter, we shall only work with discrete groups.

6.1 Amalgamated free products

Convention 6.1.1. *In this section, we always equip our groups with uniformly discrete length functions.*

6.1.1 Amalgamated free products and the associated Bass-Serre tree

Definition 6.1.2. *Assume that* G_1 *and* G_2 *are groups. A* word *in* G_1 *and* G_2 *is a tuple of the form*

$$(s_1, s_2, \ldots, s_n),$$

where each s_i *is an element of* $G_1 \sqcup G_2$*. A word may be reduced by*

- *removing s_i if s_i is the identity element of G_1 or G_2,*
- *replacing two neighbouring elements s_i, s_{i+1} which belong to the same group, by one element which is their product in that group.*

The free product $G_1 * G_2$ *is the group whose elements are the reduced words in G_1 and G_2, under the operation of concatenation followed by reduction.*

We are interested in the more general notion of amalgamated free products.

Definition 6.1.3. *Let G_1, G_2 and F be groups and let $i_1 : F \hookrightarrow G_1$ and $i_2 : F \hookrightarrow G_2$ be group monomorphisms. Set*

$$D = \{i_1(f)i_2(f)^{-1} \mid f \in F\},$$

and denote the normal closure of D in $G_1 * G_2$ by $\mathcal{N}(D)$. The amalgamated free product $G = G_1 *_F G_2$ *is defined as $\frac{G_1 * G_2}{\mathcal{N}(D)}$. Although the notation $G_1 *_F G_2$ does not contain a reference to the maps i_1 and i_2, they must always be specified.*

Note that dividing out by D corresponds to identifying $i_1(F) = i_2(F)$. If $\pi : G_1 * G_2 \twoheadrightarrow G$ is the canonical surjection, then we will lighten notation by denoting $\pi(i_1(F)) = \pi(i_2(F))$ again by F. Sometimes, we will also denote $i_1(F)$ or $i_2(F)$ shortly by F. From the context it will always be clear what we mean. When we need to be very precise, we will distinguish further between elements $x \in G$ and reduced words in $G_1 * G_2$ whose image under π is x. Such words are called *words representing x*.

A very important fact about an amalgamated free product $G = G_1 *_F G_2$ is that you can naturally associate a tree to it. This tree is called the Bass-Serre tree associated to G and is denoted by T. Recall that a *tree* consists of a set V of vertices and a set E of edges. Every two vertices in a tree are connected by a unique path without backtracking. Whenever two vertices $v, v' \in V$ are connected by an edge, then we denote this edge by $[v, v']$, or equivalently by $[v', v]$, i.e. edges do not carry an orientation. The Bass-Serre tree $T = (V, E)$ is defined as follows. The vertices are the left cosets of G_1 and G_2 in G, i.e. $V = \frac{G}{G_1} \sqcup \frac{G}{G_2}$. The edges are the left cosets of F in G, i.e. $E = \frac{G}{F}$. For every $x \in G$, the edge xF connects the vertices xG_1 and xG_2. The key observation that

T is a tree follows from [99]. All of the used results regarding Bass-Serre theory have quite elementary proofs once you know that T is a tree. This is why we decide not to give many detailed proofs here. We refer the reader to [98] and [99] for more information.

Proposition 6.1.4. *Consider an amalgamated free product $G = G_1 *_F G_2$. Let R and S be sets containing exactly one representative for each left coset of $i_1(F)$ in G_1 and $i_2(F)$ in G_2 respectively. Assume $1_{G_1} \in R$ and $1_{G_2} \in S$ and denote elements of R, S and F by α_i, β_j and f respectively. Every element $x \in G$ has a unique normal form:*

$$x = \alpha_1\beta_1\alpha_2\beta_2 \dots \alpha_k\beta_k f, \tag{6.1}$$

where $k \geq 1$ is some natural number depending on x and where none of the α_i and β_j, except maybe for α_1 or β_k, are equal to 1.

If, in addition, we can write

$$x = \gamma_1\delta_1\gamma_2\delta_2 \dots \gamma_k\delta_k,$$

where k is as in (6.1) and where the γ_i and δ_i belong to G_1 and G_2 respectively, then

$$\gamma_i \in F\alpha_i F \text{ and } \delta_i \in F\beta_i F,$$

for all $i \in \{1, 2, \dots, k\}$.

A nice connection between the equivariant compression of an amalgamated free product $G = G_1 *_F G_2$ and the equivariant compression of its factors (G_1, l_1) and (G_2, l_2) can only be expected if the length function on G is somehow related to the length functions on G_1 and G_2.

Construction 6.1.5. *Given uniformly discrete length functions l_1 on G_1 and l_2 on G_2, we will construct a length function on $G_1 *_F G_2$. We note to this end that each element $x \in G$ can be represented by words $(s_1, s_2, s_3, \dots, s_n)$ in G_1 and G_2, where n is a natural number, $x =_G \prod_{i=1}^n s_i$ and where the s_i belong alternately to G_1 and G_2. We define the length of a word as the sum $l_1(s_1)+l_2(s_2)+l_1(s_3)+\dots+l_{1,2}(s_n)$ where $l_{1,2}$ is l_1 or l_2 as appropriate. We define the length of g as the infimum of the lengths of all words representing x. It is easy to check that this defines a uniformly discrete length function on G. From now on, we will tacitly assume that amalgamated free products are equipped with the length function from this construction.*

Remark 6.1.6. *When G_1 and G_2 are finitely generated groups equipped with the word length metric relative to finite symmetric generating subsets S_1 and S_2 respectively, then the above construction equips $G_1 *_F G_2$ with the word length metric relative to $S_1 \cup S_2$.*

Remark 6.1.7. *Multiplying l_1 by $c_1 \in \mathbb{R}_0^+$ and l_2 by $c_2 \in \mathbb{R}_0^+$, we obtain length functions $c_1 l_1$ on G_1 and $c_2 l_2$ on G_2. Note that applying Construction 6.1.5 to $c_1 l_1$ and $c_2 l_2$ gives a length function on G which is quasi-isometric to the length function on G induced by l_1 and l_2. In particular, if we want to calculate the equivariant compression of an amalgamated free product, then we can assume without loss of generality that*

$$\inf_{x \in G_i \setminus \{1\}} l_i(x) \geq 1,$$

for every $i = 1, 2$.

We were not able to find the following Lemma in literature, so we will provide a proof.

Lemma 6.1.8. *If F is finite, then the inclusion maps $i_1 : (G_1, l_1) \hookrightarrow (G_1 *_F G_2, l)$ and $i_2 : (G_2, l_2) \hookrightarrow (G_1 *_F G_2, l)$ are quasi-isometric embeddings.*

Proof. We assume w.l.o.g. that l_1 and l_2 are as in Remark 6.1.7. We consider the case $(G_1, l_1) \hookrightarrow (G_1 *_F G_2, l)$, the other case is proven analogously.

By definition, $l_1 \geq l$. Conversely, set $M = \max\{l_i(f) \mid f \in F, i = 1, 2\}$ and take an element $x \in G_1$. Choose a word in G_1 and G_2 representing x, say $(s_1, s_2, s_3, \ldots, s_n)$, where the s_i belong alternately to G_1 and G_2 and are not equal to the identity element of G_1 or G_2, except possibly for s_1. Denote

$$k = l_1(s_1) + l_2(s_2) + l_1(s_3) + \ldots + l_{1,2}(s_n),$$

where $l_{1,2}$ is l_1 or l_2 as appropriate. We prove that $l_1(x) \leq 2(M+1)k$, which then implies $l_1 \leq 2(M+1)l$. Note that we are ready if $n = 1$ or if $x = 1$, so in the sequel, assume that this is not the case.

On the Bass-Serre tree T, we can consider the path γ

$$G_1 \to s_1 G_2 \to s_1 s_2 G_1 \to \ldots \to s_1 s_2 \ldots s_n G_j,$$

where j is either 1 or 2. If $j = 1$, do nothing, if $j = 2$, then consider the word $(s_1, s_2, \ldots, s_n, 1_{G_2})$ instead of $(s_1, s_2, \ldots, s_n)$ and set $n := n + 1$. Either way, note that k remains unchanged and so we can always assume that the word $(s_1, s_2, \ldots, s_n)$ induces a closed path on the Bass-Serre tree T.

Assume first that $n > 2$ and set $s_0 = 1_{G_2}$. Since γ is a closed loop in the tree T, there is some backtracking somewhere, i.e. there is some $0 \leq j < n - 1$ such that one of the following conditions is satisfied

1. $$s_0 s_1 s_2 \ldots s_j G_1 = s_0 s_1 s_2 \ldots s_j s_{j+1} s_{j+2} G_1,$$

2. $$s_0 s_1 s_2 \ldots s_j G_2 = s_0 s_1 s_2 \ldots s_j s_{j+1} s_{j+2} G_2.$$

Take j as small as possible. In case (1), we have $s_{j+1} \in G_1, s_{j+2} \in G_2$ and so $s_{j+2} \in G_1 \cap G_2 = F$. In the second case, we have $s_{j+1} \in G_2$ and we obtain analogously that $s_{j+2} \in F$. Consequently, we can merge s_{j+2} with its neighbours and again remove $s_0 = 1_{G_2}$ to obtain a word $(t_1, t_2, \ldots, t_m)$ with $m < n$, where the t_i belong alternately to G_1 and G_2 and such that

$$l_1(t_1) + l_2(t_2) + l_1(t_3) + \ldots + l_{1,2}(t_m) \leq k + M.$$

Continuing this procedure, we obtain eventually the word (a_1, a_2) consisting of two letter a_1 and a_2 and

$$l_1(a_1) + l_2(a_2) \leq k + (n - 2)M.$$

Since $a_1 a_2 \in G_1$ and $a_1 \in G_1$, we conclude that $a_2 \in F$. Merging with a_1, we get the word (x) such that

$$l_1(x) \leq l_1(a_1) + l_1(a_2) \leq l_1(a_1) + l_2(a_2) + M \leq k + (n - 1)M.$$

It must be noted that $n - 2 \leq k$ since

$$k = l_1(s_1) + l_2(s_2) + l_1(s_3) + \ldots + l_{1,2}(s_n)$$

and at least $n - 2$ of these terms are greater or equal to 1. Consequently, we get

$$l_1(x) \leq (M + 1)(k + 1) \leq 2(M + 1)k,$$

as desired.

□

6.1.2 A bit about 1-cocycles

In Chapter 4, we have introduced the notion of a 1-cocycle associated to a unitary/ orthogonal representation of a group on a Hilbert space. We have already proven some general properties regarding 1-cocycles, but we will need some very specific results. As a first Lemma, we prove a version of Lemma 6.2.1 in [25] whose original formulation we are unsure of. Here, the finiteness of F plays a crucial role.

Lemma 6.1.9 (A version of Lemma 6.2.1 in [25]). *Let F be a finite subgroup of a discrete group G. If ψ is a a conditionally negative definite map on G then there is a conditionally negative definite map ψ' on G such that*

1. *ψ' is F-bi-invariant, i.e.*
$$\forall x \in G,\ \forall f, f' \in F:\ \psi'(fxf') = \psi'(x);$$
2. *$\psi'(f) = 0\ \forall f \in F$, and $\psi'(x) \geq 1$ for all $x \in G \setminus F$;*
3. *If b and b' are the 1-cocycles associated to ψ and ψ' via Proposition 4.2.3, then $\|b\| - \|b'\|$ is bounded.*

Proof. Proposition 4.2.3 associates an affine isometric action to ψ. Denote this action by α and its linear part by π. Note first that the vector $\xi := \frac{1}{|F|}\sum_{f\in F} b(f)$ stays fixed under F: for every $f' \in F$, we have

$$\begin{aligned}
\alpha(f')(\xi) &= \pi(f')\xi + b(f') \\
&= \frac{1}{|F|}\sum_{f\in F}(\pi(f')b(f) + b(f')) \\
&= \frac{1}{|F|}\sum_{f\in F} b(f'f) \\
&= \frac{1}{|F|}\sum_{f\in F} b(f) \\
&= \xi.
\end{aligned}$$

Next, look at the collection of left cosets of F in G and denote the characteristic function of F in G/F by δ_F. We can then define maps

$$\begin{aligned}
\lambda(x):\ l^2(G/F) &\to l^2(G/F) \\
\gamma(\cdot) &\mapsto \gamma(x^{-1}\cdot):\ yF \mapsto \gamma(x^{-1}yF),
\end{aligned}$$

for every $x \in G$. We introduce the candidate for ψ' by setting

$$\psi'(x) = \|\alpha(x)\xi - \xi\|^2 + \frac{1}{2}\|\lambda_{G/F}(x)(\delta_F) - \delta_F\|^2, \tag{6.2}$$

for every $x \in G$. This map is conditionally negative definite as a sum of conditionally negative definite maps. Indeed, the first term is conditionally negative definite as $x \mapsto \alpha(x)\xi - \xi$ is a 1-cocycle relative to π. The second term is conditionally negative definite as the norm squared of a 1-coboundary. It is easy to show that ψ' satisfies the necessary conditions $(1), (2), (3)$ from the Lemma. We only elaborate on (3). Since

$$\|\alpha(x)\xi - \xi\| = \|\pi(x)\xi - \xi + b(x)\|,$$

we get that

$$\|b(x)\| - 2\|\xi\| \leq \|\alpha(x)\xi - \xi\| \leq \|b(x)\| + 2\|\xi\|.$$

Therefore, the distance between $\|b(x)\|$ and $\|\alpha(x)\xi - \xi\| + \frac{1}{2}\|\lambda_{G/F}(x)(\delta_F) - \delta_F\|$ is bounded. Since

$$\forall M \geq 0 : \ |\sqrt{M^2+1} - M| \leq 1,$$

and $\frac{1}{2}\|\lambda_{G/F}(x)(\delta_F) - \delta_F\|^2$ is always bounded by 1, we conclude that the distance between $\|b(x)\|$ and

$$\|b'(x)\| = \sqrt{\|\alpha(x)\xi - \xi\|^2 + \frac{1}{2}\|\lambda_{G/F}(x)(\delta_F) - \delta_F\|^2},$$

is also bounded. ☐

Corollary 6.1.10. *If the equivariant Hilbert space compression of* (G, l) *is strictly greater than some number* $\epsilon \geq 0$ *and if* $F < G$ *is a finite subgroup, then there exists a conditionally negative definite map* $\psi : G \to \mathbb{R}^+$ *such that*

1. ψ *is* F*-bi-invariant;*
2. $\psi(f) = 0, \ \forall f \in F$ *and* $\psi(x) \geq 1$ *for all* $x \in G \setminus F$
3. *the* 1*-cocycle* b *associated to* ψ *satisfies* $(1/C)\ l(x)^\epsilon \leq \|b(x)\| \leq C\ l(x), \ \forall x \in G \backslash F$.

Proof. From Lemma 6.1.9 it follows quite easily that there is a conditionally negative definite map on G satisfying conditions (1) and (2) and such that the 1-cocycle associated to ψ by Proposition 4.2.3 satisfies

$$(1/C)\ l(x)^\epsilon - D \leq \|b(x)\| \leq Cl(x) + D,$$

for some $C > 1,\ D \geq 0$. It remains to get rid of the term D.

By uniform discreteness of l, we can replace the upper bound $Cl(x)+D$ by $(C + \frac{D}{\inf\{l(x)|x\neq 1\}})l(x)$, obtaining an upperbound of the desired form. Regarding the lower bound: if $l(x) \geq M := (2CD)^{1/\epsilon}$, then $\frac{1}{2C}l(x)^\epsilon \geq D$, so that

$$\|b(x)\| \geq \frac{1}{C}\ l(x)^\epsilon - D \geq \frac{1}{2C}l(x)^\epsilon.$$

If necessary, then enlarge C, but keep M fixed, such that $\frac{1}{2C}M^\epsilon \leq 1$. Now if $x \in G\backslash F$ with $l(x) \leq M$, then

$$\|b(x)\| \geq 1 \geq \frac{1}{2C}l(x)^\epsilon.$$

We conclude that for every $x \in G\backslash F$,

$$\|b(x)\| \geq \frac{1}{2C}l(x)^\epsilon,$$

obtaining a lower bound of the desired form. □

Lemma 6.1.11. *Denote G any topological group equipped with a length function l and let F be a finite normal subgroup of G. If we define the length of an element $\overline{x}$ of G/F as the minimum of $l(y)$ where $y \in xF$, then the equivariant Hilbert space compressions of G and G/F are equal.*

Proof. Given a 1-cocycle $b : G \to \mathcal{H}$ which is large-scale Lipschitz, we get a conditionally negative definite map $\psi : x \mapsto \|b(x)\|^2$. By Lemma 6.1.9, there exists a conditionally negative definite function, ψ', such that

1. ψ' is F-bi-invariant;
2. $\psi'(f) = 0\ \forall f \in F$, and $\psi'(x) \geq 1$ for all $x \in G \setminus F$.

Moreover, the associated 1-cocycle b' is at bounded distance from b and thus has the same compression. Conditions (1) and (2) imply that the map ψ' is in fact a conditionally negative definite function on G/F. This way, we obtain a 1-cocycle $\bar{b}$ on the quotient G/F. It is easy to show that it is large-scale Lipschitz and that it has the same compression as b', and thus as b.

To prove the reverse implication in the lemma, we start with a large-scale Lipschitz $1-$cocycle $\bar{b} : G/F \to \mathcal{H}$ and look at the associated conditionally negative definite function $\psi'(\overline{x}) = \|\bar{b}(\overline{x})\|^2$. Define $\psi : G \to \mathbb{R}^+$ by setting $\psi(x) = \psi'(\overline{x})$. This map is clearly conditionally negative definite and so Proposition 4.2.3 associates a $1-$cocycle b to it. Reasoning as in the proof of Lemma 5.2.6, it is clear that the compressions of b and $\bar{b}$ are equal and that b is large-scale Lipschitz.

□

6.1.3 Proof and formulation of the main result

We are ready to prove the following result.

Theorem 6.1.12. *Let G_1 and G_2 be groups equipped with uniformly discrete length functions l_1 and l_2 respectively. Let $G = G_1 *_F G_2$ be an amalgamated free product where F is finite and equip G with a (uniformly discrete) length function l as in Construction 6.1.5. If α, α_1 and α_2 are the equivariant Hilbert space compressions of G, G_1 and G_2 respectively, then*

1. $\alpha = 1$ *if F is of index 2 in both G_1 and G_2,*
2. $\alpha = \alpha_1$ *if $F = G_2$ and $\alpha = \alpha_2$ if $F = G_1$,*
3. $\alpha = \min(\alpha_1, \alpha_2, 1/2)$ *otherwise.*

Proof. First of all, let us use Remark 6.1.7 to assume that l_1 and l_2 satisfy

$$\inf_{x \in G_1 \setminus \{1\}} l_1(x) \geq 1 \text{ and } \inf_{x \in G_2 \setminus \{1\}} l_2(x) \geq 1.$$

Regarding (1), the assumptions imply that F is a normal subgroup of both G_1 and G_2 and so it is a normal subgroup of G with quotient $G_1/F * G_2/F = \mathbb{Z}_2 * \mathbb{Z}_2$. Lemma 6.1.11 implies that the equivariant compression of G is equal to that of $\mathbb{Z}_2 * \mathbb{Z}_2$, which is clearly 1 since $\mathbb{Z}_2 * \mathbb{Z}_2$ is the infinite dihedral group.

Regarding (2), the two cases are proven analogously. Let us consider the case $F = G_2$. In this case, G and G_1 are equal as groups and the result follows easily from Lemma 6.1.8.

To prove (3)**, let us first find an upper bound for** α**.** Note that F is of index at least 3 in one of the factors G_1, G_2 and that it is not equal to any of G_1 and G_2. We assume that F is of index at least 3 in G_2, the other case is analogous. Take $x \in G_1 \backslash F$ and $y_1, y_2 \in G_2 \backslash F$ such that $y_1 F \neq y_2 F$. Lemma 2.28 in [78] shows that $(xy_1)^2$ and $(xy_2)^2$ generate the free group F_2. Denote the set which contains these generators and their inverses by $\widetilde{S}$. We claim that F_2, equipped with the word length metric relative to $\widetilde{S}$, actually embeds quasi-isometrically into G.

Indeed, note first that for all $z \in F_2 < G$, we have that $l(z) \leq \max_{s \in \widetilde{S}}(l(s)) l_{\widetilde{S}}(z)$. Conversely, take any reduced word, say $(s_1, s_2, \ldots, s_n)$ in the elements of $\widetilde{S}$. On the Bass-Serre tree, it is easy to see that for every $i = 1, 2, \ldots, n-1$, the vertex $s_1 s_2 \ldots s_{i+1} G_j$, where $j = 1$ iff s_{i+1} ends on y_1 or y_2, is further away from G_1 then $s_1 s_2 \ldots s_i G_{\widetilde{j}}$, where again $\widetilde{j} = 1$ iff s_i ends on y_1 or y_2. This implies that for any element $z \in F_2 < G$, we have $l(z) \geq l_{\widetilde{S}}(z)$. We conclude that l is quasi-isometric to $l_{\widetilde{S}}$ on $F_2 < G$.

Since the free group has equivariant compression $1/2$, we conclude that the equivariant compression of G is bounded from above by $1/2$. From Lemma 6.1.8, we know that the inclusion maps of the factors G_1 and G_2 into G are quasi-isometric embeddings, so we conclude that $\alpha \leq \min(\alpha_1, \alpha_2, 1/2)$.

Conversely, we look for a lower bound. Assume $0 \leq \epsilon < \min(\alpha_1, \alpha_2, 1/2)$. By Corollary 6.1.10, there exist $C > 0$ and conditionally negative definite functions $\psi_i : G_i \to \mathbb{R}^+$ for $i \in \{1, 2\}$, such that

1. ψ_i is F-bi-invariant;
2. $\psi_i(f) = 0,\ \forall f \in F$ and $\psi_i(x) \geq 1$ for all $x \in G_i \setminus F$
3. the 1-cocycle b_i associated to ψ_i satisfies $(1/C)\ l_i(g)^\epsilon \leq \|b_i(g)\| \leq C\ l_i(g),\ \forall g \in G_i \backslash F$.

Let R and S be sets that contain exactly one representative for each left coset of F in G_1 and G_2 respectively. Assume $1_{G_1} \in R, 1_{G_2} \in S$

and denote elements of R, S and F by α_i, β_j and f respectively. By Proposition 6.1.4, every element $x \in G$ has a unique normal form:

$$x = \alpha_1 \beta_1 \alpha_2 \beta_2 \dots \alpha_k \beta_k f, \tag{6.3}$$

such that none of the α_i and β_j, except maybe for α_1 or β_k, are equal to 1. In [25] (see the proof of Proposition 6.2.3), it is shown that the map $\psi : G \to \mathbb{R}^+$, defined by

$$\psi(x) = \sum_{i=1}^{k} \psi_1(\alpha_i) + \sum_{j=1}^{k} \psi_2(\beta_j),$$

is a conditionally negative definite function on G. Application of Proposition 4.2.3 gives an affine isometric action of G on a Hilbert space $\mathcal{H}$ with $1-$cocycle b satisfying $\|b(x)\|^2 = \psi(x)$. Let us check that b is Lipschitz (as l_1, l_2 are not necessarily given as word length, this is not automatic). Choose $x \in G$, and write $x = \alpha_1 \beta_1 \alpha_2 \beta_2 \dots \alpha_k \beta_k f$ in normal form as above. We obtain that

$$\begin{aligned}
\|b(x)\|^2 &= \psi(x) \\
&= \sum_{i=1}^{k} \psi_1(\alpha_i) + \sum_{j=1}^{k} \psi_2(\beta_j) \\
&= \sum_{i=1}^{k} \|b_1(\alpha_i)\|^2 + \sum_{j=1}^{k} \|b_2(\beta_j)\|^2
\end{aligned}$$

Since $\forall a, b \in \mathbb{R}^+, \sqrt{a^2 + b^2} \leq a + b$, we get

$$\|b(x)\| \leq C[\sum_{i=1}^{k} l_1(\alpha_i) + \sum_{j=1}^{k} l_2(\beta_j)]. \tag{6.4}$$

Denote $l^{SB}(x)$ the *shortest blocklength* of x, meaning that it is the minimum of the lengths of all words representing x which are of the form $(\gamma_1, \delta_1, \gamma_2, \dots, \gamma_k, \delta_k)$ where k is as in Equation (6.3) and where the γ_i and δ_i belong to G_1 and G_2 respectively. Take such a word representing x. It follows from Proposition 6.1.4 that $\alpha_i \in F \gamma_i F$ and $\beta_i \in F \delta_i F$ for all $i \in \{1, 2, \dots, k\}$. Using Equation (6.4), we obtain

$$\|b(x)\| \leq C[l^{SB}(x) + 4Mk],$$

where $M = \max\{l_i(f) \mid f \in F, i = 1, 2\}$. Now, γ_1 and δ_k may be zero, but the remaining $2k-2$ letters all have l_1 or l_2-length greater than 1. Consequently, $l^{SB} \geq 2k-2$. This implies

$$\|b(x)\| \leq C(l^{SB}(x) + 2Ml^{SB}(x) + 4M) = (C + 2MC)l^{SB}(x) + 4MC.$$

From the proof of Lemma 6.1.8, we deduce that $\forall g \in G_1 : l_1(g) \leq 2(M+1)l(g)$ and similarly $\forall g \in G_2 : l_2(g) \leq 2(M+1)l(g)$. We claim that $l^{SB} \leq 2(M+1)l$. Indeed, take a word $(s_1, s_2, \ldots, s_m)$ representing x, where the s_i belong alternately to G_1 and G_2. This word gives a path in the Bass-Serre tree T, namely

$$G_1 \to s_1G_2 \to s_1s_2G_1 \to \ldots.$$

Assume that the last time that this path passes through G_1 is at the i^{th} vertex, i.e. $G_1 = s_1s_2s_3 \ldots s_{i-1}G_1$. Then the sum

$$\begin{aligned} l_1(s_1) + l_2(s_2) + \ldots + l_2(s_{i-1}) + l_1(s_i) &\geq l(s_1s_2 \ldots s_{i-1}s_i) \\ &\geq \frac{1}{2(M+1)} l_1(s_1s_2 \ldots s_{i-1}s_i), \end{aligned}$$

since $s_1s_2 \ldots s_i \in G_1$. The next vertex is $s_1s_2 \ldots s_iG_2$, and again you can look at the last time that your path passes through this vertex. Following the same reasoning, we obtain finally that

$$l_1(s_1) + l_2(s_2) + \ldots + l_{1,2}(s_n) \geq \frac{1}{2(M+1)} l^{SB}(x),$$

so that $l^{SB} \leq 2(M+1)l$. We conclude that

$$\forall x \in G : \ \|b(x)\| \leq 2(C + 2MC)(M+1)l(x) + 4MC,$$

and so b is large-scale Lipschitz.

Conversely, we have

$$\|b(x)\|^2 \geq \sum_{i=1}^{l} (1/C^2) l_1(\alpha_i)^{2\epsilon} + \sum_{j=1}^{l} (1/C^2) l_2(\beta_j)^{2\epsilon}.$$

Since $\forall a, b \in \mathbb{R}^+, \epsilon \in [0, 1/2[$: $a^{2\epsilon} + b^{2\epsilon} \geq (a+b)^{2\epsilon}$, we obtain

$$\|b(x)\|^2 \geq (1/C^2)(\sum_{i=1}^{l} l_1(\alpha_i) + \sum_{j=1}^{l} l_2(\beta_j))^{2\epsilon}.$$

Setting $\overline{M} = \sup_{f\in F}\{l(f)\}$, we obtain

$$\|b(x)\| \geq (1/C)(l(x) - \min(l(x), \overline{M}))^{\epsilon}.$$

Reasoning as in the proof of Lemma 5.2.6, we conclude that

$$\|b(x)\| \geq (1/C')l(x)^{\epsilon} - D', \tag{6.5}$$

for some $C' > 0, D' \geq 0$. It follows that the equivariant compression of G is greater or equal than $\min(\alpha_1, \alpha_2, 1/2)$, which was already an upper bound for α. We conclude that $\alpha = \min(\alpha_1, \alpha_2, 1/2)$. □

6.2 HNN-extensions

6.2.1 HNN-extensions and the associated Bass-Serre tree

Definition 6.2.1. *Let H be a group with presentation $\langle S \mid R\rangle$, let F be a subgroup of H and $\theta : F \to H$ a group monomorphism. The HNN-extension of H over F and relative to θ is denoted by $HNN(H, F, \theta)$ and has presentation*

$$HNN(H, F, \theta) = \langle S, t \mid R, t^{-1}ft = \theta(f) \text{ for every } f \in F\rangle.$$

The generator t is called the stable letter of the HNN-extension.

We look at some trivial cases.

- If $F = \{1\}$, then θ must be the inclusion map $F \hookrightarrow H$ and $HNN(H, \{1\}, \theta) = H * \mathbb{Z}$.
- If $F = H$ and θ is the identity map, then $HNN(H, F, \theta) = H \oplus \mathbb{Z}$.
- The Baumslag-Solitar group $BS(p, q)$ with $p, q \in \mathbb{Z}$ is the group with presentation

  $$BS(p, q) =< a, t \mid t^{-1}a^q t = a^p > .$$

  Note that $BS(p, q) = \text{HNN}(\mathbb{Z}, q\mathbb{Z}, \theta)$, where $\theta : q\mathbb{Z} \to \mathbb{Z}$ maps q to p.

As in the case for free products with amalgamation, one can naturally associate a tree $T=(V,E)$ to an HNN-extension $\mathrm{HNN}(H,F,\theta)$. This tree is called the Bass-Serre tree and is defined as follows. The set V of vertices is the set of left cosets of H in G, i.e. $V=G/H$. The set E of (non-oriented) edges is given by the left cosets of F in G, i.e. $E=G/F$. Given $x\in G$, the edge xF connects xH and xtH. Note that if $xH=yH$, then $xtH=ytH$ only if $yF=xF$, so a vertex xH has a different neighbour for every left coset of F in xH. Similarly, $xt^{-1}H=yt^{-1}H$ only if $x\theta(F)=y\theta(F)$, so we have an additional extra neighbour for every left coset of $\theta(F)$ in xH. If, in H, F is of finite index i and $\theta(F)$ is of finite index j, then every vertex xH has $i+j$ neighbours. Details on the key-insight that T is a tree can be found in [98], [99]. The following result is an easy consequence of this.

Proposition 6.2.2. *Let R and S be sets that contain exactly one representative for each left coset of F in H and $\theta(F)$ in H respectively, such that $1\in R$ and $1\in S$. We denote elements of $R,S,R\sqcup S$ and F by $\alpha_i,\beta_i,\gamma_i$ and f respectively. Every element $x\in G=HNN(H,F,\theta)$ can be uniquely written in a normal form*

$$x=\gamma_1t^{i_1}\gamma_2t^{i_2}\cdot\ldots\cdot\gamma_kt^{i_k}\alpha_{k+1}f, \tag{6.6}$$

where $k\in\mathbb{N}$, where $i_j=1$ whenever $\gamma_j\in R$ and $i_j=-1$ whenever $\gamma_j\in S$ and where no two subwords of the form $\gamma_1t^{i_1}\gamma_2t^{i_2}\cdot\ldots\cdot\gamma_lt^{i_l}$ with $l\leq k$ belong to the same left coset of H in G.

Moreover, assume that $x=h_1t^{i_1}h_2t^{i_2}\cdot\ldots\cdot h_kt^{i_k}h_{k+1}$, where the h_j are elements of H and where k and the i_j are equal to those in Equation (6.6). If A is the group generated by $F\cup\theta(F)$, then $h_j\in A\gamma_jA$, $\forall j\in\{1,2,\ldots,k\}$ and $h_{k+1}\in A\alpha_{k+1}fA$.

We note that you can only expect a nice relation between the equivariant compression of a group (H,l_H) and the equivariant compression of an HNN-extension $(\mathrm{HNN}(H,F,\theta),l)$ if the lenght functions l and l_H are somehow related.

Construction 6.2.3. *Assume that H is a group equipped with a (not necessarily uniformly discrete) length function l_H. There is a natural way to equip an HNN-extension $G=HNN(H,F,\theta)$ with a length function l. Indeed, each element $x\in G$ can be represented by a collection of words*

$$(a_1,t^{i_1},a_2,t^{i_2},\ldots,a_{n-1},t^{i_{n-1}},a_n)$$

in H and $\mathbb{Z}$ where n runs over the natural numbers, where

$$x = a_1 t^{i_1} a_2 t^{i_2} \cdot \ldots \cdot a_n,$$

where the a_i belong to H and where $i_1, i_2, \ldots i_{n-1} \in \{1, -1\}$. We define the length of such a word as the sum $\sum_{j=1}^{n-1} (l_H(a_j) + \mid i_j \mid) + l_H(a_n)$. We define the length of x as the infimum of the lengths of all words representing x. It is easy to see that this defines a length function on G.

Remark 6.2.4. *When H is a finitely generated group equipped with the word length metric relative to a finite symmetric generating subset S, then the above construction equips HNN(H, F, θ) with the word length metric relative to $S \cup \{t\}$.*

Construction 6.2.3 implies a crucial lemma.

Lemma 6.2.5. *Let $G = HNN(H, F, \theta)$ be an HNN-extension of a group (H, l_H) over a* finite *group $F < H$. Let l be the length function on G constructed as in Construction 6.2.3. Then the inclusion $i : (H, l_H) \hookrightarrow (G, l)$ is a quasi-isometric embedding.*

Proof. To begin, note that clearly $l_{|H} \leq l_H$, where $l_{|H}$ is l restricted to $H < G$.

Conversely, set $M = \max\{l_H(a) \mid a \in F \cup \theta(F)\}$ and take an element $x \in H \backslash \{1\}$. Choose a word in H and $\mathbb{Z}$ representing $x \neq 1$, say $(s_1, t^{i_1}, s_2, t^{i_2}, \ldots, t_{n-1}^{i_{n-1}}, s_n)$, with $i_j \in \{1, -1\}$ and the $s_i \in H$ and denote its length by

$$k = l_H(s_1) + l_H(s_2) + l_H(s_3) + \ldots + l_H(s_n) + (n-1).$$

We prove that $l_H(x) \leq (M+1)k$. Since this holds for *any* word representing x, this then implies $l_H \leq (M+1)l$.

The proof is trivial for $n = 1$. Assume thus $n > 1$ and note that the path

$$s_1 H \to s_1 t^{i_1} s_2 H \to s_1 t^{i_1} s_2 t^{i_2} s_3 H \to \ldots \to xH,$$

on the Bass-Serre tree T of G is a closed loop. Since the $i_j \neq 0$, our word contains at least two letters of the form t^c with $c \in \{-1, 1\}$ so that necessarily $n > 2$. Moreover, since T is a tree, there must be backtracking, i.e. there exists $j \in \{1, \ldots n-2\}$ such that

$$s_1 t^{i_1} s_2 t^{i_2} \ldots s_{j-1} H = s_1 t^{i_1} s_2 t^{i_2} \ldots s_{j-1} t^{i_{j-1}} s_j t^{i_j} H.$$

Consequently, $t^{i_{j-1}}s_jt^{i_j} \in H$ and so $t^{i_{j-1}}s_jt^{i_j} = a \in F \cup \theta(F)$. Merging $(t^{i_{j-1}}, s_j, t^{i_j})$ to (a) in the word

$$(s_1, t^{i_1}, s_2, t^{i_2}, \ldots, t_{n-1}^{i_{n-1}}, s_n),$$

and then merging a with its neighbours, we obtain the word

$$(b_1, t^{c_1}, b_2, t^{c_2}, \ldots, b_m) = (s_1, t^{i_1}, s_2, t^{i_2}, \ldots, s_{j-1}as_{j+1}, t^{i_{j+1}}, \ldots, s_n),$$

which contains $2m - 1 = 2n - 5$ letters and such that

$$\sum_{j=1}^{m-1} (l_H(b_j) + 1) + l_H(b_m) \leq k + M.$$

Applying the above reasoning over and over, we finally obtain the word (x) and

$$l_H(x) \leq k + (n-1)M \leq (M+1)k,$$

where we use the fact that $n - 1 \leq k$. □

6.2.2 Proof and formulation of the main result

We will need a few lemmas before proceeding to the proof of our equivariant compression formula for HNN-extensions. A first lemma and its proof are taken from [25], Lemma 6.2.2. The proof of the second lemma is trickier. We have tried to adapt ideas of Proposition 6.2.3 in [25].

Lemma 6.2.6 (Lemma 6.2.2 in [25]). *Let G be a discrete group acting (on the left) on a set Y; let H be a group, and let $c : Y \times G \to H$ be a map verifying the cocycle relation*

$$c(y, g_1g_2) = c(y, g_1)c(g_1^{-1}y, g_2) \tag{6.7}$$

for all $y \in Y$ and $g_1, g_2 \in G$. Let ψ be a conditionally negative definite function on H, vanishing on a subset A of H. Assume that for every $g \in G$, the set $\{y \in Y : c(y, g) \notin A\}$ is finite; then the function $\widetilde{\psi}$ on G may be defined by

$$\widetilde{\psi}(g) = \sum_{y \in Y} \psi(c(y, g)),$$

and $\widetilde{\psi}$ is conditionally negative definite on G.

Lemma 6.2.7. *Let $G = HNN(H, F, \theta)$ be an HNN-extension where the group A generated by $F \cup \theta(F)$ is finite. Assume that ψ is a conditionally negative definite map on H which is A-bi-invariant. Let R and S be sets that contain exactly one representative for each left coset of F in H and $\theta(F)$ in H respectively. Assume $1 \in R$ and $1 \in S$. We denote elements of $R, S, R \sqcup S, F$ and A by $\alpha_i, \beta_i, \gamma_i, f$ and a respectively. Given $x \in G$, recall from Proposition 6.2.2 that we can uniquely write it as*

$$x = \gamma_1 t^{i_1} \gamma_2 t^{i_2} \cdot \ldots \cdot \gamma_k t^{i_k} \alpha_{k+1} f, \tag{6.8}$$

where $i_j = 1$ whenever $\gamma_j \in R, i_j = -1$ whenever $\gamma_j \in S$ and no two subwords of the form $\gamma_1 t^{i_1} \gamma_2 t^{i_2} \cdot \ldots \cdot \gamma_l t^{i_l}$ with $l \leq k$ belong to the same left coset of H in G. The map

$$\widetilde{\psi} : x \mapsto \sum_{i=1}^{k} \psi(\gamma_i) + \psi(\alpha_{k+1}),$$

where x is written as in (6.8), is a conditionally negative definite function on G.

Proof. To prove this, we remark that G/H, the collection of left cosets of H in G, can be identified with the elements whose normal form as in (6.8) is of the form

$$\gamma_1 t^{i_1} \gamma_2 t^{i_2} \cdot \ldots \cdot \gamma_k t^{i_k}.$$

This provides a section $\sigma : G/H \to G$ for the canonical projection map $\pi : G \to G/H$. Define $c : G/H \times G \to H$ by setting

$$c(y, x) = \sigma(y)^{-1} x \sigma(x^{-1} y),$$

where $x^{-1}y$ stands for $\pi(x^{-1}\sigma(y))$. It is easy to check that c satisfies Equation (6.7). We will apply Lemma 6.2.6 on c to prove that $\widetilde{\psi}$ is conditionally negative definite. Therefore, choose any elements $x \in G$ and $y \in G/H$. Assume first that x, when written as in (6.8), does not start with the word $\sigma(y)$. We write $\sigma(y) = y_0 y_1$ and $x = y_0 x_1 \alpha_1 f$ where y_0 is the subword common to $\sigma(y)$ and x, and where y_1 ends with some non-zero power of t. Then

$$x^{-1}\sigma(y) = f^{-1} \alpha_1^{-1} x_1^{-1} y_1.$$

Using the uniqueness described at the bottom of Proposition 6.2.2, where in our case $h_{k+1} = 1$, we can write $f^{-1}\alpha_1^{-1}x_1^{-1}y_1$ as in Equation (6.8), obtaining

$$x^{-1}\sigma(y) = y_1'a,$$

where y_1' ends with some non-zero power of t and where $a \in A$. This implies that

$$\sigma(x^{-1}y) = y_1'$$

and so that $c(y,x) = a^{-1} \in A$. This already shows that for any $x \in G$, the set $\{y \in G/H \mid c(y,x) \notin A\}$ contains only a finite number of elements.

Assume next that x begins with the word $\sigma(y)$ and write

$$x = \sigma(y)\gamma_l t^{i_l}\gamma_{l+1}t^{i_{l+1}} \cdot \ldots \cdot \gamma_k t^{i_k}\alpha_{k+1}f$$

as in (6.8). Then

$$\sigma(y)^{-1}x = \gamma_l t^{i_l}\gamma_{l+1}t^{i_{l+1}} \cdot \ldots \cdot \gamma_k t^{i_k}\alpha_{k+1}f,$$

as in (6.8). We can now apply the uniqueness at the bottom of Proposition 6.2.2 to obtain

$$\begin{aligned} x^{-1}\sigma(y) &= (f^{-1}\alpha_{k+1}^{-1}t^{-i_k}\gamma_k^{-1} \cdot \ldots \cdot t^{-i_{l+1}}\gamma_{l+1}^{-1}t^{-i_l})\gamma_l^{-1} \\ &= (\gamma_{k+1}'t^{-i_k}\gamma_k' \cdot \ldots \cdot t^{-i_{l+1}}\gamma_{l+1}'t^{-i_l}a')\gamma_l^{-1}, \end{aligned}$$

where the $\gamma_j' \in R \sqcup S$ and $a' \in A$. This gives

$$\begin{aligned} \sigma(x^{-1}y) &= \gamma_{k+1}'t^{-i_k}\gamma_k' \cdot \ldots \cdot t^{-i_{l+1}}\gamma_{l+1}'t^{-i_l} \\ &= f^{-1}\alpha_{k+1}^{-1}t^{-i_k}\gamma_k^{-1} \cdot \ldots \cdot t^{-i_{l+1}}\gamma_{l+1}^{-1}t^{-i_l}a'^{-1}. \end{aligned}$$

Therefore $c(y,x) = \gamma_l a'^{-1}$ and $\psi(c(y,x)) = \psi(\gamma_l a'^{-1}) = \psi(\gamma_l)$. By Lemma 6.2.6, we conclude that $\widetilde{\psi}$ is conditionally negative definite. □

We have come to our main Theorem.

Theorem 6.2.8. *Let (H, l_H) be a discrete group equipped with a length function and denote its equivariant Hilbert space compression by α_1. Assume that F is a subgroup of H and that $\theta : F \to H$ is a group monomorphism such that the group generated by $\theta(F) \cup F$ is finite (e.g. if F is normal in H). Denote $G :=$ HNN(H, F, θ) and equip it with a length function l as in Construction 6.2.3. Then, the equivariant Hilbert space compression α of $G :=$ HNN(H, F, θ) satisfies*

1. $\alpha = 1$ *whenever* $F = H$,
2. $\alpha = \min(\alpha_1, 1/2)$ *otherwise.*

Proof. **The first claim** follows trivially from Lemma 6.1.11, but we have added it for completeness. Let us focus on the second claim.

We first look for an upper bound on α. Since $F \neq H$, we can choose $b \in H \backslash F$ and $c \in H \backslash \theta(F)$. Denote $\overline{S}$ the set which contains $t^{-1}b, tc$ and their inverses $(t^{-1}b)^{-1}, (tc)^{-1}$. We claim that $\overline{S}$ generates a copy of F_2, the free subgroup on 2 generators. Moreover, we show that this copy, equipped with the word lenght metric relative to $\overline{S}$, embeds quasi-isometrically inside G. In order to prove this, choose some element $x \in \langle \overline{S} \rangle$ and write it as a product of elements of $\overline{S}$, say $x = s_1 s_2 s_3 \dots s_n$, where no two neighbours are each other's inverse. Denote $1_H = s_0$ and look at the corresponding path

$$1_H H \to s_1 H \to s_1 s_2 H \to \dots \to xH,$$

in the Bass-Serre tree. We claim that this is a path without backtracking. Indeed, assume by contradiction that for some $i \leq n-2$, we have $s_0 s_1 s_2 \dots s_i H = s_0 s_1 s_2 \dots s_{i+2} H$, then $s_{i+1} s_{i+2} \in H$. There are only finitely many possibilities and one can check that the only case that $s_{i+1} s_{i+2} \in H$ is when $s_{i+2} = s_{i+1}^{-1}$. Now, this implies first that $\langle \overline{S} \rangle$ is indeed the free group on 2 generators. Secondly, it shows that $|x|_{\overline{S}}$ equals the length of the geodesic path in T which connects H to xH. In particular, we have that $|x|_{\overline{S}} \leq l(x)$, for every $x \in G$. Since for every $x \in F_2 = \langle \overline{S} \rangle$, $l(x) \leq |x|_{\overline{S}} \max_{s \in \overline{S}} l(s)$, we see that indeed F_2 embeds quasi-isometrically inside G. From this, we deduce that $\alpha \leq 1/2$. Together with Lemma 6.2.5, we conclude that $\alpha \leq \min(1/2, \alpha_1)$. We now proceed by showing that $\min(1/2, \alpha_1)$ is also a lower bound for α.

Conversely, if the compression of H is zero, then everything is trivial, so assume that α_1 is non-zero and choose $0 \leq \epsilon < \min(\alpha_1, 1/2)$. Set A the group generated by $F \cup \theta(F)$. By Corollary 6.1.10, we can take an A-bi-invariant conditionally negative definite map ψ on H such that the associated $1-$cocycle b on H satisfies

$$\forall h \in H \backslash A: \ (1/C)\ l_H(h)^{\epsilon} \leq \|b(h)\| \leq C\ l_H(h) + D,$$

for some $C, D \geq 1$. We can not assume $D = 0$ here, because we did not assume that l is uniformly discrete. Note that the *upper* bound holds for *every* $h \in H$, since $\|b(a)\| = 0$ for $a \in A$. Now introduce the exact same notations as in the formulation of Lemma 6.2.7. Given $x \in G$, we can write

$$x = \gamma_1 t^{i_1} \gamma_2 t^{i_2} \cdot \ldots \cdot \gamma_k t^{i_k} \alpha_{k+1} f, \tag{6.9}$$

as in Equation (6.8) and the map

$$\widetilde{\psi} : x \mapsto \sum_{i=1}^{k} \psi(\gamma_i) + \psi(\alpha_{k+1}),$$

is conditionally negative definite.

Next, consider the Bass-Serre tree T associated to the HNN-extension G. We define the tree length $l_T(xH)$ of a vertex xH in T as the number of edges on a path from H to xH without backtracking. The map

$$\begin{array}{rccl} \psi' : & G & \to & \mathbb{R}^+ \\ & x & \mapsto & l_T(xH) \end{array}$$

is conditionally negative definite on G (see Proposition 2 in §6.a of [68] or Example 4 in Section 4.3). We will from now on denote $l_T(xH)$ more briefly by $l_T(x)$.

Define a new conditionally negative definite map $\overline{\psi}$ by $\overline{\psi} = \widetilde{\psi} + \psi'$. Proposition 4.2.3 associates to $\overline{\psi}$ a $1-$cocycle $\overline{b}$ relative to some affine isometric action of G on a Hilbert space $\mathcal{H}$. It suffices to prove that $\overline{b}$ is large-scale Lipschitz and that its compression is at least ϵ. We give the proof in the form of two seperate lemmas.

Lemma 6.2.9. *The* 1*-cocycle* $\overline{b}$ *is Lipschitz (as* l_1, l_2 *are not necessarily given as word length, this is not automatic).*

Proof. Choose any $x \in G$ and write

$$x = \gamma_1 t^{i_1} \gamma_2 t^{i_2} \cdot \ldots \cdot \gamma_k t^{i_k} \alpha_{k+1} f$$

as in equation (6.6). Take $M = \max\{l_H(a) \mid a \in A\}$. If $k = l_T(x) = 0$, then

$$\|\overline{b}(x)\| = \sqrt{\psi(\alpha_1)} \leq C l_H(\alpha_1) + D \leq C l_H(x) + CM + D.$$

Using Lemma 6.2.5, we see that there exist constants $\widetilde{C}$ and $\widetilde{D}$ such that for every x with $l_T(x) = 0$, we have

$$\|\bar{b}(x)\| \leq \widetilde{C}l(x) + \widetilde{D}.$$

So from now on, let us assume w.l.o.g. that $k = l_T(x) > 0$. We have that

$$\begin{aligned} \|\bar{b}(x)\|^2 &= \overline{\psi}(x) \\ &= (\sum_{i=1}^{k} \psi(\gamma_i)) + \psi(\alpha_{k+1}) + \psi'(x) \\ &= (\sum_{i=1}^{k} \|b(\gamma_i)\|^2) + \|b(\alpha_{k+1})\|^2 + \psi'(x) \end{aligned}$$

Consequently,

$$\|\bar{b}(x)\| \leq (\sum_{i=1}^{k} Cl_H(\gamma_i) + D) + Cl_H(\alpha_{k+1}) + D + \psi'(x).$$

If $k = l_T(x) \geq 1$, we get

$$\|\bar{b}(x)\| \leq C[(\sum_{i=1}^{k} l_H(\gamma_i)) + l_H(\alpha_{k+1}) + 3l_T(x)D],$$

and so

$$\|\bar{b}(x)\| \leq 3CD[\sum_{i=1}^{k} l_H(\gamma_i) + l_H(\alpha_{k+1}) + l_T(x)]. \tag{6.10}$$

Denote $l^{SB}(x)$ the *shortest blocklength* of x, meaning that it is the length of x looking only at the representatives of x in $H * \mathbb{Z}$ of the form $h_1 t^{i_1} h_2 t^{i_2} \cdot \ldots \cdot h_k t^{i_k} h_{k+1}$, where the h_i are in H and where k and the i_j are as in Equation (6.6). It follows from Proposition 6.2.2 that $\gamma_j \in Ah_jA$, $\forall j \in \{1, 2, \ldots, k\}$ and $\alpha_{k+1} f \in Ah_{k+1}A$. Together with Equation (6.10) and the fact that $l^{SB}(x) \geq l_T(x)$, we get that

$$\begin{aligned} \|\bar{b}(x)\| &\leq 3CD[l^{SB}(x) + 2M(k+1)] \\ &\leq 3CD(l^{SB}(x) + 2M + 2Ml^{SB}(x)) \\ &\leq 3CD(1 + 4M)l^{SB}(x). \end{aligned}$$

By Lemma 6.2.5, we see that $\forall h \in H : l_H(h) \leq (M+1)l(h)$. We claim that this implies that $l^{SB} \leq (M+1)l$. Indeed, fix $x \in G\backslash\{1\}$ and take a word $(s_1, t^{i_1}, s_2, t^{i_2}, \ldots, t^{i_{n-1}}, s_n)$, with the $s_i \in H$, which represents $x \in G$. Now, let σ be the geodesic path from H to xH and let j be maximal such that $s_1 t^{i_1} s_2 t^{i_2} \ldots s_j H$ equals the first vertex, i.e. H, of σ. By Lemma 6.2.5, we get that $l_H(s_1 t^{i_1} s_2 t^{i_2} \ldots s_j) \leq (M+1) l(s_1 t^{i_1} s_2 t^{i_2} \ldots s_j)$. Next, take j_1 maximal such that

$$s_1 t^{i_1} s_2 t^{i_2} \ldots s_{j_1} H$$

equals the second vertex of σ. We obtain that $l_H(s_{j+1} t^{i_{j+1}} \ldots s_{j_1}) \leq (M+1) l(s_{j+1} t^{i_{j+1}} \ldots s_{j_1})$. Continuing in this fashion, it is easy to conclude that $l^{SB}(x) \leq (M+1)l(x)$.

We conclude that

$$\|\overline{b}(x)\| \leq 3CD(M+1)(1+4M)\; l(x), \; \forall x \in G,$$

so that $\overline{b}$ is large-scale Lipschitz. □

Lemma 6.2.10. *The 1-cocycle $\overline{b}$ has compression at least ϵ.*

Proof. To show that $\overline{b}$ has compression at least ϵ, we fix $x \in G\backslash\{1\}$, write it as in Equation (6.6) and denote $l_1(x) := k + \sum_{i=1}^{k} l_H(\gamma_i) + l_H(\alpha_{k+1} f)$. We consider two cases. First, assume that $k = l_T(x) \geq \frac{1}{2M+1} l_1(x)$. Then

$$\|\overline{b}(x)\| \geq \sqrt{l_T(x)} \geq \frac{1}{\sqrt{2M+1}} l_1(x)^{1/2} \geq \frac{1}{\sqrt{2M+1}} l(x)^{1/2}. \qquad (6.11)$$

If $k < \frac{1}{2M+1} l_1(x)$, then $\sum_{i=1}^{k+1} l_H(\gamma_i) > \frac{2M}{2M+1} l_1(x)$, where we denote $\alpha_{k+1} f = \gamma_{k+1}$. Denote V the set of γ_i where $i = 1, 2, \ldots k+1$ that do not belong to A. Clearly, $|V| \leq k+1$. We get

$$\begin{aligned} \sum_{\gamma_i \in V} l_H(\gamma_i) &\geq \sum_{i=1}^{k+1} l_H(\gamma_i) - M(k+1) \\ &\geq \frac{2M}{2M+1} l_1(x) - M\left(\frac{1}{2M+1} l_1(x) + 1\right) \\ &= \frac{M}{2M+1} l_1(x) - M. \end{aligned}$$

Now, if we take x such that $\frac{M}{4M+2}l(x) \geq M$, then since $l_1(x) \geq l(x)$, we get

$$\begin{aligned}\sum_{\gamma_i \in V} l_H(\gamma_i) &\geq \frac{M}{2M+1}l(x) - M \\ &\geq \frac{M}{4M+2}l(x).\end{aligned}$$

Since on $H\backslash A$ we have that $\|b(\gamma_i)\| \geq (1/C)l_H(\gamma_i)^\epsilon$, we get

$$\begin{aligned}\|\bar{b}(x)\|^2 &\geq \sum_{\gamma_i \in V} \|b(\gamma_i)\|^2 \\ &\geq (1/C)^2 \sum_{\gamma_i \in V} l_H(\gamma_i)^{2\epsilon} \\ &\geq (1/C)^2 (\sum_{\gamma_i \in V} l_H(\gamma_i))^{2\epsilon} \\ &\geq (1/C)^2 (\frac{M}{4M+2})^{2\epsilon} l(x)^{2\epsilon}.\end{aligned}$$

Together with Equation (6.11), we conclude that there is $\overline{C} \geq 1$ such that for every x with $l(x)$ large enough:

$$\|\bar{b}(x)\| \geq (1/\overline{C})l(x)^\epsilon.$$

Consequently, there is $C' \geq 1$ and $D' \geq 0$ such that for every $x \in G$:

$$\|\bar{b}(x)\| \geq (1/C')l(x)^\epsilon - D'.$$

□

Since ϵ was any number between 0 and $\min(\alpha_1, 1/2)$, we conclude that the equivariant compression of G is at least $\min(1/2, \alpha_1)$ and we are ready.

□

6.3 A special type of quotient

We denote by A some countable abelian group and by Γ_0 a group acting on A by automorphisms. Given $a \in A$ and $\gamma \in \Gamma_0$, we denote by $\gamma \cdot a$

the element of A obtained by letting γ act on a. Let Γ be a group that surjects onto Γ_0, i.e. there exists an epimorphism $p : \Gamma \twoheadrightarrow \Gamma_0$. This induces a Γ-action on A and we denote the associated semi-direct product by $G := A \rtimes \Gamma$. We shall assume that this is a finitely generated group. Take a finite symmetric generating subset S of G and consider the word length metric relative to S. Clearly, S projects in a natural way onto a finite generating subset of Γ_0, relative to which we consider the word length metric on Γ_0. On A, we consider the induced subspace-metric from G.

Theorem 3.1 of [26] states the following.

Theorem 6.3.1. *If G is Haagerup, then at least one of the following conditions is satisfied:*

1. *Γ_0 is Haagerup,*
2. *$\forall a \in A$: the Γ_0-orbit $(\gamma \cdot a)_{\gamma \in \Gamma_0}$ is bounded.*

Using ideas from the proof of Theorem 6.3.1, we find the following bound on the equivariant compression of Γ_0.

Theorem 6.3.2. *Assume that for some $l > 0$, there exists $a \in A, M \in \mathbb{N}$ such that*

$$\forall \gamma \in \Gamma_0 \textit{ with } |\gamma|_{\Gamma_0} \geq M : |\gamma \cdot a|_A \geq |\gamma|_{\Gamma_0}^l.$$

Denoting the equivariant Hilbert space compression of G by α and that of Γ_0 by β, we obtain

$$\beta \geq \alpha l/2.$$

Note that this gives a relation between the equivariant compression of G and the equivariant compression of the quotient Γ_0 of G. We will need some preliminary definitions, notations and results before we can give the proof of Theorem 6.3.2.

6.3.1 Preliminaries

Given a locally compact topological space P, recall that the Borel σ-algebra $\mathcal{B}$ is the smallest σ-algebra on P which contains the topology of P. The elements of $\mathcal{B}$ are called *Borel sets* of P. A *Borel measure* μ on P is any measure for which the Borel sets are μ-measurable and such that $\mu(K) < \infty$ for every compact $K \subset P$. It is called *inner regular* if

$$\mu(B) = \sup\{\mu(K) \mid K \subset B \text{ is a compact subset of } P\},$$

for every $B \in \mathcal{B}$. It is called *outer regular* if

$$\mu(B) = \inf\{\mu(U) \mid U \supset B \text{ is an open set of } P\},$$

for every $B \in \mathcal{B}$. It is called *regular* if it is inner and outer regular.

A *complex valued* measure on a measurable space $(P, \mathcal{P})$ is a map $\widetilde{\mu} : \mathcal{P} \to \mathbb{C}$ such that for every sequence $(E_n)_{n\in\mathbb{N}}$ of disjoint measurable subsets in P,

$$\widetilde{\mu}(\cup_{n\geq 0} E_n) = \sum_{n\geq 0} \widetilde{\mu}(E_n).$$

We note that this definition implies that the sum on the right hand side converges in $\mathbb{C}$. To a complex measure, one can associate a finite (positive) measure $|\widetilde{\mu}|$ on $\mathcal{P}$ by setting $|\widetilde{\mu}|(E)$ equal to the supremum of

$$\sum_{n\geq 0} |\widetilde{\mu}(E_n)|$$

over all countable partition $(E_n)_{n\geq 0}$ of E consisting of measurable sets. It is a standard result that the set of all complex measures on $(P, \mathcal{P})$, equipped with the evident pointwise summation and scalar multiplication, is a complex Banach space if we equip it with the norm

$$\|\widetilde{\mu}\| = |\widetilde{\mu}|(P).$$

We will mostly deal with positive probability measures μ, ν on $(P, \mathcal{P})$. In this case, one can prove that $\|\mu - \nu\| = 2\sup_{B\in\mathcal{P}} |\mu(B) - \nu(B)|$. This justifies the following definition.

Definition 6.3.3. *Given probability measures μ, ν on a measure space $(P, \mathcal{P})$, we define their total variational distance as*

$$d(\mu, \nu) = \|\mu - \nu\| = 2 \sup_{B\in\mathcal{P}} |\mu(B) - \nu(B)|.$$

We will use the following classical result.

Theorem 6.3.4 (Radon-Nikodym Theorem)**.** *Assume that* $(X, \mathcal{B}, \mu)$ *is a* σ*-finite measurable space (where as usual* μ *is a positive measure). If a complex measure* ν *has the same sets of measure* 0 *as* μ *then there exists* $h \in L^1(X, \mu)$ *such that*

$$\nu(E) = \int_\Omega h\chi_E d\mu,$$

for all $E \in \mathcal{B}$*. Here* χ_E *is the characteristic function of* E *and* h *is called the Radon-Nikodym derivative. We denote* h *by* $\frac{d\nu}{d\mu}$*.*

The following classical result in functional analysis can be found in [94], pg.389.

Theorem 6.3.5. *Let* $\widetilde{\pi} : A \to \mathcal{U}(\mathcal{H})$ *be a unitary representation of a locally compact abelian group and let* $\xi \in \mathcal{H}$ *be a vector of length* 1*. There exists a regular Borel probability measure* μ_ξ *on* $\hat{A} = \{\chi : A \to S^1 \mid \chi$ *is a group homomorphism* $\}$ *such that*

$$\forall a \in A : \ \langle \widetilde{\pi}(a)\xi, \xi\rangle = \int_{\hat{A}} \chi(a) d\mu_\xi(\chi). \tag{6.12}$$

Here, $\hat{A}$ *is equipped with the topology of uniform convergence on compact sets.*

Moreover, the measures μ_ξ *are completely determined by Equation (6.12): for every* $a \in A$*, define* $f_a : \hat{A} \to S^1, \chi \mapsto \chi(a)$*. There exists a unique positive linear functional* L *on the space* $C(\hat{A})$ *of bounded continuous maps on* $\hat{A}$ *such that*

$$\forall a \in A : \ L(f_a) = \langle \widetilde{\pi}(a)\xi, \xi\rangle.$$

In the setting of Theorem 6.3.2, there is an action of a group Γ on the abelian group A. Notice that this action induces an action of Γ on $\hat{A}$ by defining $\gamma \cdot \chi : a \mapsto \chi(\gamma^{-1} \cdot a)$. Given a Borel measure μ on $\hat{A}$, we can then define a measure $\gamma_*\mu$ as follows. For any Borel set B of $\hat{A}$, we set $\gamma_*\mu(B) = \mu(\gamma^{-1} \cdot B) = \mu(\{\gamma^{-1} \cdot \chi \mid \chi \in B\})$. Using the above, one easily shows the following

Lemma 6.3.6. *Let $\pi : G = A \rtimes \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of G and denote $\widetilde{\pi} = \pi_{|A} : A \to \mathcal{U}(\mathcal{H})$. Choose $a \in A$, $\gamma \in \Gamma$ and let $\xi \in \mathcal{H}$ be a unit vector. Using the same notations as above, we have that $\mu_{\pi(\gamma)\xi} = \gamma_* \mu_\xi$.*

The following is also standard, see for example Proposition 7 in [22].

Lemma 6.3.7. *Assume that $\widetilde{\pi} : A \to \mathcal{U}(\mathcal{H})$ is a unitary representation of a discrete countable abelian group. Given two unit vectors $\xi, \theta \in \mathcal{H}$, denote their associated Borel measures on $\hat{A}$ by μ_ξ and μ_θ respectively. Then*

$$\|\mu_\xi - \mu_\theta\| \leq 4\|\xi - \theta\|.$$

6.3.2 Proof of main result

We proceed with the proof of Theorem 6.3.2. Since the full proof contains many calculations, inequalities, ... we decide to start with an *outline*.

Outline 6.3.8. *Assume that $\overline{b} : G \to \mathcal{H}$ is a $1-$cocycle, relative to a unitary action of G on $\mathcal{H}$, such that $\lim_{g\to\infty} \|\overline{b}(g)\| = \infty$. Then $\overline{\psi} : g \to \|\overline{b}(g)\|^2$ is a conditionally negative definite map going to infinity, i.e. $\lim_{g\to\infty} \overline{\psi}(g) = \infty$. Setting $\overline{\phi}_n = e^{\frac{-\overline{\psi}}{n}}$, we obtain a family $(\overline{\phi}_n)_{n\in\mathbb{N}_0}$ of positive definite maps such that $\overline{\phi}_n(e) = 1$, $\lim_{g\to\infty} \overline{\phi}_n(g) = 0$ and $\overline{\phi}_n \overset{n\to\infty}{\to} 1$ uniformly on compact subsets of G. Denote the associated Gel'fand-Naimark-Segal triples by $(\pi_n, \mathcal{H}_n, \xi_n)$ and set $\pi = \oplus_{n\geq 1}\pi_n$. Using the restrictions to A of the unitary representations π_n and applying Theorem 6.3.5 to the vectors ξ_n, we are going to construct a family $(\sigma_n)_{n\in\mathbb{N}_0}$ of unitary representations of Γ_0 on Hilbert spaces $\mathcal{H}_n$. For every $n \in \mathbb{N}_0$, we fix a certain vector $\eta_n \in \mathcal{H}_n$ and we obtain a nice upper and lower bound for $\|\sigma_n(\gamma)\eta_n - \eta_n\|$ (see Equations (6.14) and (6.15) below). Next, we look at the positive definite map*

$$\begin{array}{rccl} \phi_n : & \Gamma_0 & \to & \mathbb{C} \\ & \gamma & \mapsto & \langle \sigma_n(\gamma)\eta_n, \eta_n\rangle. \end{array}.$$

Using Equations (6.14) and (6.15), we construct a conditionally negative definite map on Γ_0. We show that the associated 1-cocycle will have compression at least $\alpha l/2$, provided that $\overline{b}$ has compression at least α.

Proof of Theorem 6.3.2. If $\alpha = 0$, then there is nothing to prove. Assume thus that $\alpha > 0$ and fix any number $0 < \delta < \alpha$. By definition of equivariant Hilbert space compression, there is a $1-$cocycle $\overline{b} : G \to \mathcal{H}$, relative to a unitary representation $\overline{\pi} : G \to \mathcal{U}(\mathcal{H})$, that has compression $> \delta$. Setting $\overline{\psi}(g) = \|\overline{b}(g)\|^2$, we get a conditionally negative definite map such that

$$\forall g \in G : \ \frac{1}{C}|g|^{2\delta} \leq \overline{\psi}(g) \leq C|g|^2,$$

for some $C \geq 1$ (the fact that we can omit the additive constant D in the upper and lower bound above follows from Lemma 6.1.9, applied for $F = \{1\}$). For all $n \in \mathbb{N}_0$, we define $\overline{\phi}_n := e^{\frac{-1}{n}\overline{\psi}} : G \to \mathbb{R}^+$. By Schoenberg's Theorem (pg. 66, [68]), the maps $(\overline{\phi}_n)_{n\in\mathbb{N}_0}$ are a family of positive definite maps. Moreover, they satisfy that $\overline{\phi}_n(e) = 1$, $\lim_{g\to\infty}\overline{\phi}_n(g) = 0$ and $\overline{\phi}_n \overset{n\to\infty}{\to} 1$ uniformly on compact subsets of G.

For each $n \in \mathbb{N}_0$, let $(\pi_n, \mathcal{H}_n, \xi_n)$ be the Gel'fand-Naimark-Segal triple associated to $\overline{\phi}_n$ as in Proposition 4.2.11. Set

$$\pi = \oplus_n \pi_n.$$

For every $n \in \mathbb{N}_0$, take a probability measure μ_n on $\hat{A}$ such that

$$\forall a' \in A : \ \overline{\phi}_n(a') = \langle \pi(a')\xi_n, \xi_n \rangle = \int_{\hat{A}} \chi(a') d\mu_n(\chi).$$

Let $\gamma_1, \gamma_2, \ldots$ be an enumeration of the elements of Γ_0 and define probability measures ν_n on $\hat{A}$ by

$$\nu_n = (1 - \frac{1}{2^n})\mu_n + \sum_{i\geq 1} \frac{1}{2^{n+i}}\gamma_{i*}\mu_n.$$

Clearly,

$$\|\nu_n - \mu_n\| \leq \frac{1}{2^{n-1}}, \tag{6.13}$$

so μ_n and ν_n are close to each other when n is large. The reason that we introduce the approximations ν_n of μ_n is that they are Γ_0-quasi-invariant, i.e. ν_n has the same sets of measure 0 as $\gamma_*\nu_n$ for every $\gamma \in \Gamma_0$. This means the Radon-Nikodym derivatives $\frac{d\gamma_*\nu_n}{d\nu_n} : \hat{A} \to \mathbb{R}^+$ exist and for every $n \in \mathbb{N}$ we can define

$$\sigma_n : \Gamma_0 \to \mathcal{U}(L^2(\hat{A}, \nu_n))$$

by setting

$$\sigma_n(\gamma)(f) : \chi \mapsto (\frac{d\gamma_*\nu_n}{d\nu_n}(\chi))^{1/2} f(\gamma^{-1}\cdot\chi).$$

The $(\sigma_n)_{n\geq 0}$ are unitary representations of Γ_0:

$$\begin{aligned}
\langle \sigma_n(\gamma)(f), \sigma_n(\gamma)(g)\rangle &= \int_{\hat{A}} \frac{d\gamma_*\nu_n}{d\nu_n}(\chi) f(\gamma^{-1}\cdot\chi)\overline{g(\gamma^{-1}\cdot\chi)} d\nu_n \\
&= \int_{\hat{A}} f(\gamma^{-1}\cdot\chi)\overline{g(\gamma^{-1}\cdot\chi)} d\gamma_*\nu_n \\
&= \int_{\hat{A}} f(\chi)\overline{g(\chi)} d\nu_n \\
&= \langle f, g\rangle.
\end{aligned}$$

For all $n \in \mathbb{N}_0$, set $\eta_n = 1 \in L^2(\hat{A}, d\nu_n)$ and define

$$\phi_n : \Gamma_0 \to \mathbb{R}, \gamma \mapsto \langle \sigma_n(\gamma)\eta_n, \eta_n\rangle.$$

Clearly, all of these maps are positive definite and $\forall \gamma \in \Gamma_0 :\ 0 \leq \phi_n(\gamma) \leq 1$ (CS-inequality). Choose $2 < B < e = \exp(1)$ and write

$$\begin{array}{rccl}
\psi : & \Gamma_0 & \to & \mathbb{R}^+ \\
 & \gamma & \mapsto & \sum_{k=1}^{\infty} B^k (1 - \phi_{\lfloor e^{2k}\rfloor}(\gamma)),
\end{array}$$

where $\lfloor e^{2k}\rfloor$ denotes the integer part of e^{2k}. We will prove later that ψ is well-defined, but for now let us just assume it. It is then easy to see that ψ is a real valued conditionally negative definite map on Γ_0. The rest of the proof consists of showing that there exists $M \in \mathbb{R}^+$ such that $\psi(\gamma) \geq |\gamma|_{\Gamma_0}^{\alpha l}$ if $|\gamma|_{\Gamma_0} \geq M$.

The first step to investigate $\psi(\gamma)$ is to investigate the terms $|1-\phi_n(\gamma)|$. Since

$$\begin{aligned}
1 - \phi_n(\gamma) &= \frac{1}{2}(2 - 2\langle \sigma_n(\gamma)\eta_n, \eta_n\rangle) \\
&= \frac{1}{2}\|\sigma_n(\gamma)\eta_n - \eta_n\|^2,
\end{aligned}$$

we investigate the terms $\|\sigma_n(\gamma)\eta_n - \eta_n\|^2$. Let us start with an upper bound.

Lemma 6.3.9. *Using the same notations as above, we get that* $\forall n \in \mathbb{N}_0$ *and* $\gamma \in \Gamma_0$*:*

$$\|\sigma_n(\gamma)\eta_n - \eta_n\|^2 \leq \frac{1}{2^{n-2}} + 4\|\pi(\widetilde{\gamma})\xi_n - \xi_n\|, \tag{6.14}$$

where $\widetilde{\gamma} \in \Gamma \subset G$ *is any element such that* $p(\widetilde{\gamma}) = \gamma$*.*

Proof. We calculate

$$\begin{aligned}
\|\sigma_n(\gamma)\eta_n - \eta_n\|^2 &= \int_{\hat{A}} |(\frac{d\gamma_*\nu_n}{d\nu_n})^{1/2}(\chi) - 1|^2 d\nu_n \\
&\leq \int_{\hat{A}} |(\frac{d\gamma_*\nu_n}{d\nu_n})^{1/2}(\chi) - 1| \cdot |(\frac{d\gamma_*\nu_n}{d\nu_n})^{1/2}(\chi) + 1| d\nu_n \\
&= \int_{\hat{A}} |\frac{d\gamma_*\nu_n}{d\nu_n}(\chi) - 1| d\nu_n \\
&= \|\gamma_*\nu_n - \nu_n\| \\
&= \|\gamma_*\nu_n - \gamma_*\mu_n + \gamma_*\mu_n - \mu_n + \mu_n - \nu_n\| \\
&\leq 2\|\nu_n - \mu_n\| + \|\gamma_*\mu_n - \mu_n\|.
\end{aligned}$$

Using Lemma 6.3.6 and Equation (6.13), we continue

$$\begin{aligned}
\|\sigma_n(\gamma)\eta_n - \eta_n\|^2 &\leq \frac{1}{2^{n-2}} + \|\mu_{\pi(\widetilde{\gamma})\xi_n} - \mu_n\| \text{ where } p(\widetilde{\gamma}) = \gamma \\
&\leq \frac{1}{2^{n-2}} + 4\|\pi(\widetilde{\gamma})\xi_n - \xi_n\| \text{ (see Lemma 6.3.7).}
\end{aligned}$$

This ends the proof of Lemma 6.3.9. □

Next, we look for a lower bound on $\|\sigma_n(\gamma)\eta_n - \eta_n\|^2$.

Lemma 6.3.10. *Using the same notations as above, we get that* $\forall n \in \mathbb{N}_0,\ \gamma \in \Gamma_0$ *and* $a \in A$:

$$\|\sigma_n(\gamma)(\eta_n) - \eta_n\| \geq \frac{1}{4}[\|\pi(\gamma \cdot a)(\xi_n) - \xi_n\|^2 - \|\pi(a)(\xi_n) - \xi_n\|^2 - \frac{1}{2^{n-3}}]. \tag{6.15}$$

Proof. Using the Cauchy-Schwarz inequality in the third step, we note first that

$$\begin{aligned}
\|\gamma_*\nu_n - \nu_n\| &= \int_{\hat{A}} |\frac{d\gamma_*\nu_n}{d\nu_n} - 1| d\nu_n \\
&= \int_{\hat{A}} |(\frac{d\gamma_*\nu_n}{d\nu_n}(\chi))^{1/2} - 1|\ |(\frac{d\gamma_*\nu_n}{d\nu_n}(\chi))^{1/2} + 1| d\nu_n \\
&\leq \sqrt{\int_{\hat{A}} |(\frac{d\gamma_*\nu_n}{d\nu_n}(\chi))^{1/2} - 1|^2 d\nu_n} \sqrt{\int_{\hat{A}} |(\frac{d\gamma_*\nu_n}{d\nu_n}(\chi))^{1/2} + 1|^2 d\nu_n} \\
&\leq 2\sqrt{\int_{\hat{A}} |(\frac{d\gamma_*\nu_n}{d\nu_n}(\chi))^{1/2} - 1|^2 d\nu_n},
\end{aligned}$$

where we use the fact that $\forall a, b \in \mathbb{R} :\ (a+b)^2 \leq 2(a^2+b^2)$. This implies that

$$\|\gamma_*\nu_n - \nu_n\| \leq 2\|\sigma_n(\gamma)(\eta_n) - \eta_n\|. \tag{6.16}$$

Notice further that

$$\begin{aligned}
|\langle \pi(\gamma \cdot a)\xi_n, \xi_n\rangle - \langle \pi(a)\xi_n, \xi_n\rangle| &= |\int_{\hat{A}} \chi(a) d\gamma_*\mu_n - \int_{\hat{A}} \chi(a) d\mu_n| \\
&\leq \|\gamma_*\mu_n - \mu_n\| \\
&\leq 2\|\nu_n - \mu_n\| + \|\gamma_*\nu_n - \nu_n\| \\
&\leq \frac{1}{2^{n-2}} + 2\|\sigma_n(\gamma)\eta_n - \eta_n\|,
\end{aligned}$$

where we used Equations (6.13) and (6.16). Consequently,

$$\begin{aligned}
\|\pi(\gamma \cdot a)\xi_n - \xi_n\|^2 &= 2 - 2\langle \pi(\gamma \cdot a)\xi_n, \xi_n\rangle \\
&\leq 2 + 2[\frac{1}{2^{n-2}} + 2\|\sigma_n(\gamma)\eta_n - \eta_n\| - \langle \pi(a)\xi_n, \xi_n\rangle] \\
&= \frac{1}{2^{n-3}} + 4\|\sigma_n(\gamma)\eta_n - \eta_n\| + \|\pi(a)\xi_n - \xi_n\|^2,
\end{aligned}$$

which concludes the proof of Lemma 6.3.10. □

Recall the definition

$$\begin{aligned}
\psi :\ \Gamma_0 &\rightarrow \mathbb{R}^+ \\
\gamma &\mapsto \textstyle\sum_{k=1}^{\infty} B^k (1 - \phi_{\lfloor e^{2k} \rfloor}(\gamma)),
\end{aligned}$$

where $2 < B < e$ is any real number. Let us check that ψ is well defined, i.e. we look for an upper bound on $\psi(\gamma)$.

Lemma 6.3.11. *The map ψ is well-defined.*

Proof. Denoting $n_k = \lfloor e^{2k} \rfloor$ for short, we use Equation (6.14) to show that for every $\gamma \in \Gamma_0$,

$$\begin{aligned} B^k(1-\phi_{\lfloor e^{2k}\rfloor}(\gamma)) &= \frac{1}{2}[B^k\|\sigma_{n_k}(\gamma)(\eta_{n_k}) - \eta_{n_k}\|^2] \\ &\leq \frac{1}{2}[B^k(\frac{1}{2^{n_k-2}} + 4\|\pi(\widetilde{\gamma})(\xi_{n_k}) - \xi_{n_k}\|)] \\ &= \frac{(\frac{B}{4})^k}{2^{n_k-1-2k}} + 2B^k\sqrt{2-2\overline{\phi}_{n_k}(\widetilde{\gamma}))} \\ &\leq \frac{1}{2^{n_k-1-2k}} + 2B^k\sqrt{2-2e^{\frac{-1}{n_k}\overline{\psi}(\widetilde{\gamma})}}. \end{aligned}$$

It is clear that

$$\sum_{k=1}^{\infty} \frac{1}{2^{n_k-1-2k}}$$

converges in $\mathbb{R}$, so it remains to show that

$$\sum_{k=1}^{\infty} 2B^k\sqrt{2-2e^{\frac{-1}{n_k}\overline{\psi}(\widetilde{\gamma})}} < +\infty.$$

Let us take $\epsilon > 1$ such that $B\epsilon < e$. Then for any constant $c \in \mathbb{R}^+$, we have that

$$\lim_{k\to\infty} 2(B\epsilon)^k\sqrt{2-2e^{\frac{-1}{n_k}c}} = 0,$$

such that for k large enough $2B^k\sqrt{2-2e^{\frac{-1}{n_k}\overline{\psi}(\widetilde{\gamma})}} \leq \frac{1}{\epsilon^k}$. This concludes the proof that ψ is well-defined. □

Next, in order to calculate the compression of the 1-cocycle associated to ψ, let us find a lower bound on $\psi(\gamma)$.

Lemma 6.3.12. *Denote $n_k = \lfloor e^{2k} \rfloor$ for short. There exists some $D \geq 0$ such that*

$$\psi(\gamma) \geq \frac{1}{8}\sum_{k=1}^{\infty} B^k(1-e^{\frac{-1}{n_k C}|\gamma|_{\Gamma_0}^{2\delta l}})^2 - D \tag{6.17}$$

for every $\gamma \in \Gamma_0$ with $|\gamma|_{\Gamma_0}$ sufficiently large.

Proof. Reasoning as in the proof that ψ is well-defined, one can verify that all of the infinite sums in the calculation below converge in $\mathbb{R}$. Using Equation (6.15), we obtain $\forall \gamma \in \Gamma_0$ and for $a \in A$ satisfying the conditions of Theorem 6.3.2 that

$$\begin{aligned}
\psi(\gamma) &= \frac{1}{2}\sum_{k=1}^{\infty} B^k \|\sigma_{n_k}(\gamma)\eta_{n_k} - \eta_{n_k}\|^2 \\
&\geq \frac{1}{32}\sum_{k=1}^{\infty} B^k (\|\pi(\gamma\cdot a)(\xi_{n_k}) - \xi_{n_k}\|^2 - \\
&\quad \|\pi(a)(\xi_{n_k}) - \xi_{n_k}\|^2 - \frac{1}{2^{n_k-3}})^2 \\
&= \frac{1}{32}\sum_{k=1}^{\infty} B^k \|\pi(\gamma\cdot a)(\xi_{n_k}) - \xi_{n_k}\|^4 + \\
&\quad \frac{1}{32}\sum_{k=1}^{\infty} B^k (\|\pi(a)(\xi_{n_k}) - \xi_{n_k}\|^2 + \frac{1}{2^{n_k-3}})^2 - \\
&\quad \frac{1}{16}\sum_{k=1}^{\infty} B^k \|\pi(\gamma\cdot a)(\xi_{n_k}) - \xi_{n_k}\|^2 (\|\pi(a)(\xi_{n_k}) - \xi_{n_k}\|^2 + \frac{1}{2^{n_k-3}}) \\
&\geq \frac{1}{32}\sum_{k=1}^{\infty} B^k \|\pi(\gamma\cdot a)(\xi_{n_k}) - \xi_{n_k}\|^4 - \\
&\quad \frac{1}{4}\sum_{k=1}^{\infty} B^k (\|\pi(a)(\xi_{n_k}) - \xi_{n_k}\|^2 + \frac{1}{2^{n_k-3}}),
\end{aligned}$$

where the last step follows from the fact that $\|\pi(\gamma\cdot a)(\xi_{n_k}) - \xi_{n_k}\|^2 \leq 4$. Denote $\frac{1}{4}\sum_{k=1}^{\infty} B^k (\|\pi(a)(\xi_{n_k}) - \xi_{n_k}\|^2 + \frac{1}{2^{n_k-3}})$ by D and take $\gamma \in \Gamma_0$ with length *sufficiently large* such that $|\gamma\cdot a|_A \geq |\gamma|_{\Gamma_0}^{l}$. Then,

$$\begin{aligned}
\psi(\gamma) &\geq \frac{1}{32}\sum_{k=1}^{\infty} B^k (2 - 2\overline{\phi}_{n_k}(\gamma\cdot a))^2 - D \\
&\geq \frac{1}{32}\sum_{k=1}^{\infty} B^k (2 - 2e^{\frac{-1}{n_k C}|\gamma\cdot a|_A^{2\delta}})^2 - D \\
&\geq \frac{1}{8}\sum_{k=1}^{\infty} B^k (1 - e^{\frac{-1}{n_k C}|\gamma|_{\Gamma_0}^{2\delta l}})^2 - D.
\end{aligned}$$

□

Finally, let us show that the 1-cocycle associated to ψ has compression at least $\alpha l/2$. For this, assume that $|\gamma|_{\Gamma_0}$ is large enough for Equation 6.17 to hold and also larger than $C^{\frac{1}{2\delta l}}$. Then we can fix $k_0 \in \mathbb{N}$ such that

$$(Ce^{2k_0})^{\frac{1}{2\delta l}} \leq |\gamma|_{\Gamma_0} < (Ce^{2(k_0+1)})^{\frac{1}{2\delta l}}.$$

Consequently,

$$\begin{aligned}
\psi(\gamma) &\geq \frac{1}{8}\sum_{k=1}^{k_0-1} B^k\Big(1 - e^{\frac{-1}{\lfloor e^{2k}\rfloor}e^{2k_0}}\Big)^2 - D \\
&\geq \frac{1}{8}\sum_{k=1}^{k_0-1} B^k\big(1 - e^{-e^{2(k_0-k)}}\big)^2 - D \\
&\geq \frac{1}{8}\sum_{k=1}^{k_0-1} B^k\big(1 - e^{-e^{2}}\big)^2 - D \\
&\geq \frac{1}{8}\sum_{k=1}^{k_0-1} B^k\frac{1}{B} - D \\
&\geq \sum_{k=1}^{k_0-1} B^{k-4} - D \\
&\geq B^{k_0-5} - D.
\end{aligned}$$

We conclude that there exists some $M > 0$ such that $\forall\gamma \in \Gamma_0$ with $|\gamma|_{\Gamma_0} \geq M$ we have

$$(Ce^{2k_0})^{\frac{1}{2\delta l}} \leq |\gamma|_{\Gamma_0} < (Ce^{2(k_0+1)})^{\frac{1}{2\delta l}} \Longrightarrow \psi(\gamma) \geq B^{k_0-5} - D.$$

Take $\omega = \frac{2\ln(B)\delta l}{(1+p)(2+p)}$, where $p > 0$ is any (small) real number. We claim that the compression of ψ is greater than ω. Indeed, by our choice of ω, we get that $\frac{\ln(B)}{1+p} = (2+p)\frac{\omega}{2\delta l}$. Multiplying both sides by $k_0 - 5$, we obtain

$$\frac{\ln(B)}{1+p}(k_0-5) = (2+p)\frac{\omega}{2\delta l}(k_0-5).$$

Now, take $\gamma \in \Gamma_0$ with $|\gamma|_{\Gamma_0}$ large enough in order that the associated k_0 satisfies

$$(k_0-5)(1+p)(2+p) \geq 2(k_0+1) + \ln(C).$$

Then,

$$\ln(B)(k_0-5) \geq (2(k_0+1)+\ln(C))\frac{\omega}{2\delta l}.$$

Applying the exponential map on both sides, we get

$$B^{k_0-5} \geq (Ce^{2(k_0+1)})^{\frac{\omega}{2\delta l}},$$

and so

$$\psi(\gamma) > |\gamma|^{\omega} - D.$$

We conclude that the equivariant Hilbert space compression of Γ_0 is at least $\omega/2 = \frac{\ln(B)\delta l}{(1+p)(2+p)}$. Taking the limit for $(p,B,\delta) \rightarrow (0,e,\alpha)$, we conclude that the equivariant compression of Γ_0 is greater or equal to $\alpha l/2$. ☐

6.4 The equivariant compression of Baumslag-Solitar monsters

6.4.1 Introduction

In this section, we investigate the equivariant Hilbert space compression of the Baumslag-Solitar monsters $G = BS(p,q)$ with $p,q \geq 1$. We show that G, when equipped with the word length metric relative to a finite symmetric generating subset, can be quasi-isometrically embedded into the product $T \times H$, where T is the Bass-Serre tree associated to G and H is the hyperbolic plane. From this, we will be able to calculate the equivariant Hilbert space compression. We are thankful to Yves de Cornulier for suggesting the result (see Corollary 6.4.8 for the statement) and the strategy of the proof.

Recall that the Baumslag-Solitar group $G = BS(p,q)$ with $p,q \geq 1$ is the group with presentation

$$G = BS(p,q) =< a,b \mid b^{-1}a^q b = a^p > .$$

Notice that the roles of p and q can be interchanged:

$$\begin{aligned} < a,b \mid b^{-1}a^q b = a^p > &= < a,b \mid a^q = ba^p b^{-1} > \\ &= < a,b^{-1} \mid a^q = (b^{-1})^{-1}a^p b^{-1} > \\ &= < a,b \mid b^{-1}a^p b = a^q > . \end{aligned}$$

Therefore, we assume w.l.o.g. that $p \leq q$. The Baumslag-Solitar groups were originally introduced to find some easy examples of non-Hopfian groups [28]. Let me recall that a group is Hopfian if every epimorphism from the group onto itself is an isomorphism. One can proof that $BS(3, 2)$ is non-Hopfian.

6.4.2 Notations and definitions

Set $S := \{a, a^{-1}, b, b^{-1}\}$ and note that it is a finite symmetric generating subset of G. From now on we shall always equip G with the word length l relative to S.

One can easily verify that $G = \mathrm{HNN}(\mathbb{Z}, q\mathbb{Z}, \theta)$, where

$$\begin{array}{rccc} \theta: & q\mathbb{Z} & \to & \mathbb{Z} \\ & qz & \mapsto & pz. \end{array}$$

Let T be the associated Bass-Serre tree, defined as in Section 6.2. Since $q\mathbb{Z}$ is of index q in $\mathbb{Z}$ and $p\mathbb{Z}$ is of index p, we see that every vertex has exactly $p + q$ neighbours. On T, the tree distance d_T between two vertices $x\mathbb{Z}, x'\mathbb{Z}$ is defined as the number of edges contained in a path from $x\mathbb{Z}$ to $x'\mathbb{Z}$ without backtracking. The multiplication action of G on T, i.e. $x \cdot (y\mathbb{Z}) = (xy)\mathbb{Z}$ for every $x, y \in G$, gives an isometric action of G on T. Note that this action can naturally be extended to an isometric action on T, when seen as a metric simplicial 1-complex.

The Baumslag-Solitar group $BS(p, q) = G$ also acts on the hyperbolic plane. To see this, look at the Poincaré upper half plane model of H. For all $(c, d) \in H$, let $a \in G$ act by $a \cdot (c, d) = (c + 1, d)$ and $b \in G$ by $b \cdot (c, d) = (q/p)(c, d) = ((q/p)c, (q/p)d)$. This action is well-defined, since

$$b^{-1}a^q b \cdot (c, d) = b^{-1} \cdot ((q/p)c + q, (q/p)d) = (c + p, d) = a^p \cdot (c, d).$$

Recalling that $d_H((c, d), (c', d')) = \operatorname{arccosh}(1 + \frac{(c-c')^2+(d-d')^2}{2dd'})$, we see that the so obtained action of G on H is isometric.

6.4.3 Calculation of the equivariant compression

Definition 6.4.1. *Denote the vertex $\mathbb{Z}$ on T by v_0 and define*

$$\begin{array}{rccl} f: & G & \to & T \times H \\ & x & \mapsto & x \cdot (v_0, (0, 1)) := (x \cdot v_0, x \cdot (0, 1)). \end{array}$$

Equipping $T \times H$ with the maximum metric, it is our goal to show that f is a quasi-isometric embedding of G into $T \times H$.

Since the action of G is isometric, it is clear that for

$$C := \max(d(a \cdot (v_0, (0,1)), (v_0, (0,1)), d(b \cdot (v_0, (0,1)), (v_0, (0,1))),$$

we have

$$d_{T\times H}(f(x), f(y)) \leq C d_G(x, y), \tag{6.18}$$

for all $x, y \in G$.

Let us prove the converse, i.e. that there exists $C > 1$ such that

$$(1/C) d_G(x, y) \leq d_{T\times H}(f(x), f(y)),\ \forall x, y \in G.$$

To this end, define $\pi : G \to \mathbb{Z}$ the unique group homomorphism mapping a to 0 and b to 1. On the vertices of T, we define $\gamma(x \cdot v_0) := \pi(x)$. This is well defined by Britton's Lemma (pg. 181 of [75]). Identifying each edge isometrically to $[0, 1]$, we extend γ as a polynomial map of degree 1 on each edge.

On H, we set $\gamma(x \cdot (0, 1)) = \pi(x)$. Clearly, this is well-defined, since if $x \cdot (0, 1) = y \cdot (0, 1)$, then their second coordinates are equal. The result follows by noting that the second coordinate of a point in H only changes under the action of b, not of a, i.e. the second coordinate of $x \cdot (0, 1)$ only depends on $\pi(x)$ and in fact equals $(q/p)^{\pi(x)}$. We extend γ to the entire hyperbolic plane by setting $\gamma((c, d)) =^{q/p} \log(d)$. Notice that γ is constant on the horizontal lines $d =$cte. There is a problem with our definitions if we consider the special case $p = q$. This is why in many of the following lemmas, we shall include the condition that $p \neq q$.

Definition 6.4.2. *Assume $p \neq q$. We set*

$$\mathcal{W} = \{(t, (c, d)) \in T \times H \mid \gamma(t) = \gamma((c, d))\} \subset T \times H.$$

Notice that $\mathcal{W}$ contains the image $Im(f)$ of f. We emphasize the fact that $t \in T$ is not necessarily a vertex *of T.*

Lemma 6.4.3. *Assume that $p \neq q$. There is a constant $\overline{C} > 0$ such that every element of $\mathcal{W}$ is at distance less than $\overline{C}$ from the image of f.*

Proof. Denote $V = [0,1[\times[p/q, q/p] \subset H$, where we use the Poincaré upper half plane model for H. Set $\overline{C} := \max(1/2, \operatorname{diam}(V))$, where $\operatorname{diam}(V)$ is the diameter of $V \subset H$ in the hyperbolic distance.

Choose $(t,(c,d)) \in \mathcal{W}$ and take $x \in G$ such that $x \cdot v_0 \in T$ is at distance $\leq 1/2$ from t. Take the point of the form $(n,1)$ with $n \in \mathbb{Z}$ such that $x^{-1} \cdot (c,d) \in V_n := [n, n+1[\times[p/q, q/p] \subset H$. Clearly, $a^n \cdot (0,1) = (n,1) \in V_n$ and so

$$d_H(xa^n \cdot (0,1), (c,d)) = d_H(a^n \cdot (0,1), x^{-1} \cdot (c,d)) \leq \operatorname{diam}(V_n) = \operatorname{diam}(V).$$

Moreover, on T, we also get that

$$d_T(xa^n \cdot v_0, t) = d_T(x \cdot v_0, t) \leq 1/2.$$

This implies $d_{T\times H}(f(xa^n), (t,(c,d))) \leq \overline{C}$ and so we have proven the lemma. □

To continue, the idea is as follows. First, we find constants $\mathcal{C}, \mathcal{D} > 0$ satisfying the following property: for every $n \in \mathbb{N}$ and for every two elements $f(x), f(y)$ in the image of f at distance smaller than n from each other, there is a discrete path in $\mathcal{W} \subset T \times H$ starting in $f(x)$ and ending in $f(y)$ of at most $\mathcal{C}n$ elements such that two consecutive elements in the path lie at distance smaller than $\mathcal{D}$ from each other. Next, we show that for every constant $B > 0$, there exists a constant $\overline{B}$ such that whenever elements $f(x), f(y)$ in the image of f are at distance $< B$, then $d_G(x,y) < \overline{B}$. Using these facts and Lemma 6.4.3, we find in Proposition 6.4.7 the existence of some $C > 0$ such that

$$\forall x, y \in G: \ \frac{1}{C} d_G(x,y) \leq d_{T\times H}(f(x), f(y)).$$

Lemma 6.4.4. *Assume that $p \neq q$. There are constants $\mathcal{C}, \mathcal{D} > 0$ satisfying the following property: for every $n \in \mathbb{N}$ and for every two elements $f(x), f(y)$ in the image of f at distance less than n from each other, there is a discrete path in $\mathcal{W} \subset T \times H$ starting in $f(x)$ and ending in $f(y)$ of at most $\mathcal{C}n$ elements such that two consecutive elements in the path lie at distance smaller than $\mathcal{D}$ from each other.*

Proof. It is not difficult to see that the G-action on $T \times H$ restricts to $\mathcal{W}$ and so without loss of generality, we can assume that $x = 1$. Now, fix

$n \in \mathbb{N}$ and choose $y \in G$. We denote $f(y) = (v, (c, d))$ and assume that $d_{T\times H}((v_0, (0, 1)), (v, (c, d))) \leq n$.

Choose a path $v_0, v_1, v_2, \dots v_n = v$ in T such that $d(v_i, v_{i+1}) \leq 1$ for $i = 0, 1, \dots, n-1$. Keeping c fixed, find numbers d_i to obtain a discrete path $(v_i, (c, d_i))_{i=0,1,2,\dots n}$ in $\mathcal{W}$. Here, it is easy to see that $d_0 = 1$ and one calculates that the distance between two consecutive elements (c, d_i) and (c, d_{i+1}) is bounded by $\operatorname{arccosh}(1/2(q/p + p/q))$. We thus have a discrete path of n elements of $\mathcal{W}$, connecting $(v_0, (c, 1))$ to $(v, (c, d))$, where the distance between two consecutive elements is bounded by a constant $K := \max(1, \operatorname{arccosh}(1/2(q/p + p/q)))$.

It now remains to find a discrete path in $\mathcal{W}$ connecting $(v_0, (0, 1))$ and $(v_0, (c, 1))$. To this end, note first that

$$d_H((0, 1), (c, 1)) = d_{T\times H}((v_0, (0, 1)), (v_0, (c, 1)),$$

which by the triangle inequality is smaller than

$$d_{T\times H}((v_0, (0, 1)), (v, (c, d))) + d_{T\times H}((v, (c, d)), (v_0, (c, 1))),$$

hence $d_H((0, 1), (c, 1)) \leq n + nK$. Take a discrete path

$$(0, 1) = (a_0, b_0), (a_1, b_1), \dots, (a_m, b_m) = (c, 1),$$

in H where m is the smallest integer greater than

$$\frac{n + nK}{\operatorname{arccosh}(1/2(q/p + p/q))}$$

and where two consecutive elements are at most distance $\operatorname{arccosh}(1/2(q/p+ p/q))$ from each other. Inside the path through the vertices

$$\dots, t^{-2}\mathbb{Z}, t^{-1}\mathbb{Z}, \mathbb{Z}, t\mathbb{Z}, t^2\mathbb{Z}, \dots,$$

choose a discrete path $t_0 = v_0, t_1, t_2, \dots, t_m = v_0$ in T, not necessarily consisting out of vertices of T, such that the points $(t_j, (a_j, b_j))_{j=1,2,\dots,m}$ belong to $\mathcal{W}$ and the distance between two consecutive t_j is at most 1. To see that this can be done, note first that for every $a \in \mathbb{R}$, the distance

$$d_H((a, b_i), (a, b_{i+1})) \leq d_H((a_i, b_i), (a_{i+1}, b_{i+1})) \leq \operatorname{arccosh}(1/2(q/p+p/q)).$$

This implies that $b_{i+1} \in [(p/q)b_i, (q/p)b_i]$. Consequently, $|\gamma((a_{i+1}, b_{i+1})) - \gamma((a_i, b_i))|$ is at most 1. This implies that we can find some t_{i+1} at distance less than 1 from t_i which satisfies the desired condition.

We have obtained a discrete path of

$$\frac{n + nK}{\operatorname{arccosh}(1/2(q/p + p/q))}$$

elements of $\mathcal{W}$ connecting $(v_0, (0, 1))$ and $(v_0, (c, 1))$, such that the distance between two consecutive elements is bounded by $\max(1, \operatorname{arccosh}(1/2(q/p + p/q)))$.

Concatenating the path $(t_j, (a_j, b_j))_{j=0,1,2,\ldots,m}$ with $(v_i, (c, d_i))_{i=1,2,\ldots,n}$, we obtain a discrete path in $\mathcal{W}$, connecting $(v_0, (0, 1))$ to $(v, (c, d))$ which consists out of

$$n + \frac{n + nK}{\operatorname{arccosh}(1/2(q/p + p/q))}$$

elements such that two consecutive elements are at most a distance $\max(1, \operatorname{arccosh}(1/2(q/p + p/q)))$ from each other.

The proof now ends by setting

$$\mathcal{C} = 1 + \frac{1 + K}{\operatorname{arccosh}(1/2(q/p + p/q))}$$

and by setting

$$\mathcal{D} = \max(1, \operatorname{arccosh}(1/2(q/p + p/q))).$$

□

Lemma 6.4.5. *Assume that $p \neq q$. For every constant $B > 0$ there exists a constant $\overline{B}$ such that whenever elements $f(x), f(y)$ are at distance $< B$, then $d_G(x, y) < \overline{B}$.*

The proof of this claim also follows from Theorem 1 in [51]. We decide to give an alternate proof because it will take care of the case $p = q$ (see Remark 6.4.6).

Proof of Lemma 6.4.5. Without loss of generality, we can assume that $x = 1$ and $B \in \mathbb{N}$. Assume next that $f(y)$ is at distance $< B$ from $f(1) = (v_0, (0, 1))$. Since $d_T(v_0, y \cdot v_0) < B$ and using Proposition 6.2.2, we can write $y = a^{r_1} b^{i_1} a^{r_2} b^{i_2} \cdot \ldots \cdot a^{r_{B-1}} b^{i_{B-1}} a^{r_B}$ where $r_B \in \mathbb{Z}$, where

$i_1, i_2, \dots i_{B-1} \in \{-1, 1\}$ and $r_1, r_2, \dots, r_{B-1} \in \{0, 1, \dots, q\}$. Notice that the collection of elements

$$\mathcal{C} := \{a^{r_1}b^{i_1}a^{r_2}b^{i_2} \cdot \ldots \cdot a^{r_{B-1}}b^{i_{B-1}} \mid r_j \in \{0, 1, \dots, q\}, i_j \in \{-1, 1\}\},$$

is finite. Define $\widetilde{B} = B + \max\{d_H(\overline{y} \cdot (0, 1), (0, 1)) \mid \overline{y} \in \mathcal{C}\}$ and take B' such that $\operatorname{arccosh}(1 + \frac{B'^2}{2}) = \widetilde{B}$. Clearly, if $|r_B| > B'$, then

$$d_H(y \cdot (0, 1), (yr_B^{-1}) \cdot (0, 1)) = d_H((r_B, 1), (0, 1)) = \operatorname{arccosh}(1 + \frac{r_B^2}{2}) > \widetilde{B}. \tag{6.19}$$

This implies that $y \cdot (0, 1)$ is not in the B neighbourhood of $(0, 1)$: indeed, $d_H(y \cdot (0, 1), (0, 1)) < B$ together with the triangle inequality would give a contradiction to Equation (6.19). Now, the fact that $y \cdot (0, 1)$ is not in the B neighbourhood of $(0, 1)$, shows that

$$d_{T\times H}((y \cdot v_0, y \cdot (0, 1)), (v_0, (0, 1))) > B,$$

a contradiction. We conclude that $|r_B| \leq B'$ and that

$$l(y) \leq (B - 1)(q + 1) + B'.$$

We conclude the proof by setting $\overline{B} = (B - 1)(q + 1) + B'$. □

Remark 6.4.6. *One can prove the same result when $p = q$. In this case, we again obtain the same finite set $\mathcal{C}$. Next, because $d_H(y\cdot(0, 1), (0, 1)) < B$ and since b acts trivially on H, one sees easily that necessarily $|r_B| \leq (\sum_{i=1}^{B-1}|r_i|) + \frac{B}{d(a\cdot(0,1),(0,1))}$. We thus obtain*

$$l(y) \leq (B - 1)(q + 1) + (B - 1)q + \frac{B}{\operatorname{arccosh}(3/2)}.$$

This implies

$$l(y) \leq B[2q + 1 + \frac{1}{\operatorname{arccosh}(3/2)}].$$

Together with Equation (6.18), this implies that f is a quasi-isometric embedding when $p = q$.

Proposition 6.4.7. *The map $f : BS(p, q) \to T \times H$ where $p, q \geq 1$ is a bi-Lipschitz embedding.*

Proof. The case that $p = q$ follows from Remark 6.4.6. Else, denote by $\overline{C}$ the constant from Lemma 6.4.3 and denote by $\mathcal{C}$ and $\mathcal{D}$ the constants from Lemma 6.4.4.

Choose two elements $x, y \in G$ and denote $d(f(x), f(y)) = n$. Using Lemmas 6.4.4 and 6.4.3, find a discrete path $v_0 = f(x), v_1 = f(x_1), v_2 = f(x_2), \ldots, v_m = f(y)$ in the image of f where $m = \mathcal{C}n$ and such that the distance between two consecutive elements in the path is bounded by $\overline{D} := \mathcal{D} + 2\overline{C}$. Applying Lemma 6.4.5 to $\overline{D}$, we obtain a constant $\overline{D}'$, such that the distances between two consecutive elements $x_i, x_{i+1} \in G$ is bounded by $\overline{D}'$. By the triangle inequality, we obtain that

$$d_G(x, y) \leq \sum_{i=1}^{m} d_G(x_{i-1}, x_i) \leq m\overline{D}' = \mathcal{C}\overline{D}'n.$$

Using Equation 6.18 and the above, we obtain the existence of a constant $C > 0$ satisfying

$$\forall x, y \in G: \ (1/C)d_G(x, y) \leq d_{T\times H}(f(x), f(y)) \leq C d_G(x, y).$$

□

Corollary 6.4.8. *The equivariant compression of the Baumslag-Solitar monsters* $BS(p, q)$ *with* $p, q > 1$ *equals* $1/2$.

Proof. From the proof of the upperbound of Theorem 6.2.8, it follows that $BS(p, q)$ contains a subgroup isomorphic to F_2 whenever $p, q > 1$. This shows that the equivariant compression of G is at most $1/2$.

Next, we claim that the equivariant compression of $BS(p, q)$ is also greater than $1/2$. Indeed, f is an equivariant quasi-isometric embedding of G into $T \times H$ relative to the multiplication action of G on itself and an isometric action on $T \times H$. It thus suffices to prove that the equivariant compression of $T \times H$ relative to this action is at least $1/2$. Now, the equivariant compression of T relative to this action is at least $1/2$, see Proposition 2 in §6.a of [68]. The same is true for the equivariant compression of the hyperbolic plane relative to this action (Example 5 in Section 5.3). The equivariant compression of a product being the minimum of the equivariant compression of the factors (Proposition 4.1 in [59]), we conclude that the equivariant compression of $T \times H$ relative to the G-action, and thus the equivariant compression of G, is greater than $1/2$.

Combining the above bounds, we obtain that the equivariant compression of G equals $1/2$. $\square$

Remark 6.4.9. *For completeness, let us look at the case* $G = BS(1, q)$ *with* $q \geq 1$*. It is standard that* G *is solvable, hence amenable and so by [32] its equivariant compression equals its ordinary Hilbert space compression (see Definition 7.2.4). The proof above shows that* B *quasi-isometrically embeds into the product* $T \times H$ *which has Hilbert space compression* 1 *(see Examples* 2 *and* 3 *in Subsection 7.2.2). From this, we get that the Hilbert space compression of* G *equals* 1*.*

Part III

Hilbert space compression

Chapter 7

Hilbert space compression

In this Chapter, we start with a section providing background on groups which are *uniformly embeddable in a Hilbert spaces* and groups with *Property* (A). We give basic definitions, describe permanence properties for both classes of groups and provide examples. In Section 7.2, we introduce the *Hilbert space compression of a group.* This is a number between 0 en 1 which can be associated to a group G and which in some sense quantifies *how well* G uniformly embeds in a Hilbert space. We provide basic definitions, discuss some examples and formulate our main research question.

7.1 Uniform embeddability of groups

In [54], Gromov introduced the notion of uniform embeddability of a finitely generated group into a Hilbert space and suggested that such a group would satisfy the Novikov Conjecture [55]. Six years later, Yu came up with a formal proof of this claim [110]. He also proved that such uniformly embeddable groups satisfy the coarse Baum-Connes Conjecture (see also [96]). Let us give a definition of this interesting property.

Definition 7.1.1. *A metric space* (X, d) *is* uniformly embeddable in a Hilbert space*, if there exist a Hilbert space* $\mathcal{H}$*, non-decreasing functions* $\rho_-, \rho_+ : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\lim_{t\to\infty} \rho_-(t) = +\infty$*, and a map* $f : X \to \mathcal{H}$*, such that*

$$\rho_-(d(x,y)) \leq d(f(x), f(y)) \leq \rho_+(d(x,y)) \ \forall x, y \in X.$$

The map f is called a uniform embedding of X in $\mathcal{H}$. It is called large-scale Lipschitz whenever ρ_+ can be taken of the form $\rho_+ : t \mapsto Ct + D$ for some $C > 0, D \geq 0$. It is Lipschitz if we can take $D = 0$.

Remark 7.1.2. *Clearly, when X is countable, we can assume that $\mathcal{H} = l^2(\mathbb{Z})$ in Definition 7.1.1.*

The following equivalent formulation for uniform embeddability of (X, d) is quite common in literature. It has the disadvantage of being less intuitive, but the advantage of simplifying the proofs of several important results [39]. In most of our proofs, this is the definition we will use.

Definition 7.1.3. *Let (X, d) be a metric space. Then X is uniformly embeddable in a Hilbert space if and only if for every $R > 0$ and $\epsilon > 0$ there exists a Hilbert space valued map $\xi : X \to \mathcal{H}, x \to \xi_x$ such that $\|\xi_x\| = 1$ for all $x \in X$ and such that*

1. $\|\xi_x - \xi_{x'}\| \leq \epsilon$ *whenever* $d(x, x') \leq R$,
2. $\lim_{S\to\infty} \sup\{|\langle \xi_x, \xi_y\rangle|, d(x, x') \geq S, x, x' \in X\} = 0$.

Remark 7.1.4. *From analyzing why the above two definitions for uniform embeddability are equivalent (Proposition 8.2.1), one deduces that the second condition can equally be replaced by the weaker condition*

$$\lim_{S\to\infty} \inf\{\|\xi_x - \xi_{x'}\|, d(x, x') \geq S, x, x' \in X\} = \sqrt{2}.$$

Intuitively, a space is uniformly embeddable if for any $\epsilon > 0$ *as small as you want* and for every $R > 0$ *as big as you want*, there is a collection of unit vectors $(\xi_x)_{x\in X}$ in some Hilbert space such that ξ_x and ξ_y are ϵ-close to each other if $d(x, x')$ is not too big, i.e. $d(x, x') \leq R$. Yet simultaneously, if $d(x, x')$ becomes very large, then the vectors should be sufficiently far away from each other. More precisely, they should tend to be orthogonal to each other.

There is a property for metric spaces which is closely related to uniform embeddability.

Definition 7.1.5 ([105])**.** *Let (X, d) be a metric space. Then X has* Property (A) *if and only if for every $R > 0$ and $\epsilon > 0$ there exists an $S > 0$ and a map $\xi : X \to l^2(X)$ such that for all $x, x' \in X$ we have $\|\xi_x\| = 1$ and*

1. *$\|\xi_x - \xi_{x'}\| \leq \epsilon$ whenever $d(x, x') \leq R$,*
2. *the support of ξ_x lies inside $B_S(x)$ for all $x \in X$.*

Clearly, Property (A) implies uniform embeddability.

Remark 7.1.6. *It must be noted that Definition 7.1.5, which is due to Tu, is slightly different from Yu's original definition of property (A) [110]. The above definition is always implied yet only equivalent to Yu's original definition if X is a discrete space with* bounded geometry*, i.e. if for every $R > 0$, there is a uniform bound on the number of elements in the balls of radius R in X [105]. This is for example the case if G is a discrete countable group equipped with a proper length function.*

For the remainder of this section, let us restrict our attention to discrete countable groups equipped with length functions. A result of Tu [105] shows that every such group admits a proper length function that is unique up to coarse equivalence. Clearly, uniform embeddability and also property (A) of a group only depend on the coarse equivalence class of the chosen length function. This implies the following definition.

Definition 7.1.7. *A discrete countable group G is uniformly embeddable in a Hilbert space (has property (A)) if and only if (G, l) is uniformly embeddable in a Hilbert space (resp. has Property (A)) where l is* any *proper length function on G.*

If we don't explicitly specify a length function l on G, then it will be this definition for uniformly embeddable group (and Property (A)) that we shall use. Interestingly, with this definition, property (A) is equivalent to exactness, i.e. the reduced C^*-algebra of G is exact ([87], see also [3]).

The class of discrete countable groups which are uniformly embeddable or have property (A) is quite large. For example, finitely generated groups with the Haagerup property are uniformly embeddable into a Hilbert space, clearly. The converse is not true: $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ is known not to be Haagerup, but from the permanence properties for the class of uniformly embeddable groups (see Theorem 7.1.8), it is easy to check that $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ is uniformly embeddable. In fact, it even satisfies property (A). Other groups with property (A) are hyperbolic groups and one-relator groups, i.e. groups with presentations of the form $\langle X \mid R\rangle$ where X is countable and R is a single word over X [57].

It turns out that finding discrete countable groups without property (A) is very hard. In this respect, we mention that the only finitely generated groups known not to have property (A) are Gromov's groups which contain expanders in their Cayley graphs [56]. They are not uniformly embeddable and hence do not satisfy property (A). There exist also groups of which it is not known whether or not they satisfy property (A), e.g. Thompson's group F [81].

As mentioned above, property (A) implies uniform embeddability. The converse is not true in the setting of *metric spaces* [81], but in the case of *discrete countable groups*, no counterexamples are known! The close connection between Property (A) and uniform embeddability for discrete countable groups also reflects in the fact that both classes of groups have about the same permanence properties. The following Theorem is due to Guentner and Dadarlat [39].

Theorem 7.1.8. *The class of discrete countable groups that are uniformly embeddable in a Hilbert space (or that have property* (A)*) is closed under subgroups, direct sums, direct limits, amalgamated free products, HNN-extensions and extensions by exact groups.*

The case of subgroups and direct sums is clear and has simply been mentioned for completeness. We do not know if the class of uniformly embeddable groups is closed under extension by any uniformly embeddable group. In fact, even the case of a central extension of $\mathbb{Z}$ by a uniformly embeddable group remains open [39].

7.2 Hilbert space compression

7.2.1 Definition

In this section, we introduce the *Hilbert space compression of a metric space* (X, d). We start with the definition of a quasi-geodesic metric space.

Definition 7.2.1. *A metric space* (X, d) *is* quasi-geodesic *if there exist* $\delta, \lambda > 0$ *such that for all* $x, y \in X$*, there exists a sequence* $x = x_0, x_1, \ldots, x_n = y \in X$ *for some* $n \in \mathbb{N}$ *such that*

$$\sum_{i=1}^{n} d_X(x_{i-1}, x_i) \leq \lambda d_X(x, y),$$

and

$$d_X(x_{i-1}, x_i) \leq \delta \text{ for all } 0 \leq i \leq n-1.$$

Note that $\lambda \geq 1$ *by the triangle inequality.*

Example 7.2.2. *Compactly generated groups equipped with the word length metric relative to a compact symmetric generating subset are quasi-geodesic metric spaces.*

The following example shows that not every uniform embedding $f : X \to \mathcal{H}$ must be large-scale Lipschitz [59].

Example 7.2.3. *Let* $X = \{n^2 \mid n \in \mathbb{N}\} \subset \mathbb{R}$ *with the induced metric. This space is not quasi-geodesic. Define* $f : X \to \mathbb{R}$ *by* $f(x) = x^2$. *Choose* $m > n \in \mathbb{N}$. *Since*

$$2n^2 < ((n+1)^2 - n^2)^2 \leq (m^2 - n^2)^2,$$

we see that

$$f(m^2) - f(n^2) = (m^2 + n^2)(m^2 - n^2) = (2n^2 + (m^2 - n^2))(m^2 - n^2)$$

is bounded by a function in $m^2 - n^2$. *One then easily checks that* f *is a uniform embedding. It is however not large-scale Lipschitz.*

The fact that f in the above example is not large-scale Lipschitz is due to the fact that X is not quasi-geodesic. If (X, d) *is* quasi-geodesic, then there is a standard result which shows that any uniform embedding $f : X \to \mathcal{H}$ must be large-scale Lipschitz [59].

Let (X, d) be a quasi-geodesic metric space. Let $f : X \to \mathcal{H}$ be a uniform embedding, i.e. there exist $C \geq 1$, $D > 0$ and $\rho_- : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to\infty} \rho_-(t) = \infty$ and such that

$$\forall x, y \in X : \rho_-(d(x,y)) \leq d(f(x), f(y)) \leq C d(x,y) + D.$$

If ρ_- is also a linear (affine) map, i.e. $\exists C \geq 1, D \geq 0$ such that

$$\forall t : \ \rho_-(t) \geq \frac{1}{C} t - D,$$

then f is called a quasi-isometric embedding (Definition 5.1.4). Not every metric space admits a quasi-isometric embedding into a Hilbert space; even in the case of compactly generated groups this is not true, e.g. F_2 [17]. It is thus natural to ask *how close to quasi-isometric* a quasi-geodesic metric space can be embedded in a Hilbert space. More precisely, we ask for which $r \in [0,1]$ there exists some uniform embedding $f : (X,d) \to \mathcal{H}$ into a Hilbert space and numbers $C \geq 1,\ D \geq 0$ such that

$$\forall x, y \in X : \ d(f(x), f(y)) \geq \frac{1}{C} d(x,y)^r - D.$$

The supremum of such r is called *the Hilbert space compression of* (X,d). To be precise, we write the following

Definition 7.2.4. *Let (X,d) be a (not necessarily quasi-geodesic) metric space. Assume that $f : (X,d) \to \mathcal{H}$ is a large-scale Lipschitz uniform embedding in a Hilbert space $\mathcal{H}$. The supremum of $r \in [0,1]$ such that there exist $C \geq 1,\ D \geq 0$ such that*

$$\forall x, y \in X : \ \frac{1}{C} d(x,y)^r - D \leq d(f(x), f(y)) \leq C d(x,y) + D,$$

is called the compression of f and is denoted by $R(f)$. The Hilbert space compression $\alpha(X)$ is defined as the supremum of $R(f)$ over all large-scale Lipschitz uniform embeddings of X into Hilbert spaces.

Intuitively speaking, the Hilbert space compression of a metric space is a number between 0 and 1 which quantifies *how well* it embeds quasi-isometrically in a Hilbert space. The latter is a very important property which has some nice consequences. In [32] for example, they deduce that a compactly generated, locally compact group G, equipped with the word length relative to a compact symmetric generating subset containing an open neighbourhood of the identity, which embeds quasi-isometrically into a Hilbert space, is amenable and unimodular.

Until now, and in most of the sequel, we will only consider uniform embeddings into Hilbert spaces, i.e. into L^2-spaces. By straightforwardly modifying the definitions, one can choose however any $p \geq 1$ and replace the class of L^2-spaces by the class of L^p-spaces. More concretely, we obtain the following

Definition 7.2.5. *Let (X,d) be a (not necessarily quasi-geodesic) metric space. Assume that $f : (X,d) \to L^p(Y,\mu)$ is a large-scale Lipschitz* *uniform embedding of X into some L^p-space. The supremum of $r \in [0,1]$ such that there exist $C \geq 1$, $D \geq 0$ such that*

$$\forall x, y \in X: \ \frac{1}{C} d(x,y)^r - D \leq d(f(x), f(y)) \leq C d(x,y) + D,$$

is called the compression of f and is denoted by $R(f)$. The L^p-compression $\alpha_p(X)$ is defined as the supremum of $R(f)$ over all large-scale Lipschitz uniform embeddings of X into L^p-spaces.

The following Proposition is easily verified. A proof is similar to that of Lemma 5.2.6.

Proposition 7.2.6. *If $f : (X, d_X) \to (Y, d_Y)$ is a quasi-isometry, then $\alpha_p(X) = \alpha_p(Y)$ for every $p \geq 1$, i.e. L^p-compression is a quasi-isometric invariant.*

7.2.2 Basic properties, examples and research question

In this thesis, our metric spaces will always be groups, equipped with some length function l. Each time in this section, if we don't specify the given length function l on G, then we will assume that G is a finitely generated group, equipped with the word length metric relative to a finite symmetric generating subset.

It is natural to ask why it would be interesting to calculate the Hilbert space compression, i.e. why is it interesting to quantify how well a group embeds quasi-isometrically into Hilbert space? A first motivation could be a statement by Guentner and Kaminker in [59]. They show that a finitely generated group G satisfies property (A) if $\alpha(G) > 1/2$. Unfortunately, this did not solve the question as to whether Thompson's group F has property (A) or not: $\alpha(F) = 1/2$ [7].

A second reason of importance is the relation between Hilbert space compression and *equivariant* Hilbert space compression. It is sometimes easier to calculate the Hilbert space compression of a group because this is a quasi-isometric invariant. One can then try to deduce information on the equivariant compression. From the comments below Remark 5.2.4 for example, it is clear that $\alpha^*(G) \leq \alpha(G)$, i.e. the equivariant compression of a group is smaller than its (ordinary) Hilbert space compression. It was remarked by Gromov, see Proposition 4.4 in [32], that equality holds for every amenable (locally compact) compactly generated group, equipped with word length. In general unfortunately, equality does not hold. The free group F_2 on two generators has Hilbert space compression 1 [59], yet its equivariant compression is 1/2.

We proceed by discussing some further properties and some examples of groups with known Hilbert space compression.

1. Finite groups have Hilbert space compression 1.

2. The free groups F_n have Hilbert space compression equal to 1. This can be deduced from the following result by Tessera, see Theorem 7.3 in [103], see also Proposition 4.2 in [59] and Theorem 2.6 in [21].

   Theorem 7.2.7. *Let T be a simplicial tree, i.e. every edge has length 1. For every increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying, for some $1 \leq p < \infty$*

   $$\int_1^\infty (\frac{f(t)}{t})^p \frac{dt}{t} < \infty,$$

   there exists a uniform embedding F of T into $l^p(T)$ and constants $C, M > 0$ such that

   $$\|F(x) - F(y)\| \geq Cf(d(x,y)),$$

   for every $x, y \in T$ with $d(x,y) \geq M$.

3. Clearly, hyperbolic groups containing an F_2 do not embed quasi-isometrically into a Hilbert space. However, their Hilbert space compression is always equal to 1 (see Theorem 3.1 in [21]). Indeed, hyperbolic groups embed quasi-isometrically into a real hyperbolic space $\mathbb{H}^n$ for some n [16]. Hyperbolic space embeds quasi-isometrically into a finite product of locally finite trees equipped with the l^1-metric [45]. This product is quasi-isometric to the same product with the l^2-metric, and so we can conclude that the compression of hyperbolic groups equals 1. In particular, hyperbolic space also has Hilbert space compression equal to 1.

4. For a similar reason, the Baumslag-Solitar groups $BS(p,q)$ with $p, q \geq 1$ have Hilbert space compression 1 (See Proposition 6.4.7).

5. Let G be a right-angled Coxeter group, i.e. a finitely generated group whose generators are of order 2 and such that any two different generators either commute or are unrelated. Such groups embed in a finite product of locally finite trees [45] and so have Hilbert space compression 1.

6. Polycyclic groups and connected lie groups, equipped with the word length relative to a compact symmetric generating subset, have Hilbert space compression equal to 1 [103].

7. The first group known not to have compression 1 was Richard Thompson's group F. It has compression 1/2 [7].

8. For compactly generated (locally compact) amenable groups, the Hilbert space compression is equal to the equivariant Hilbert space compression [32]. We can thus refer the reader to Section 5.3 for examples of amenable groups and their compression. There exist finitely generated amenable groups with compression 1, but also with compression 0 [9].

9. In [6], Arzhantseva, Druţu and Sapir construct finitely generated groups with arbitrary prescribed Hilbert space compression $\alpha \in [0, 1]$. This contrasts with the case of *equivariant* Hilbert space compression in which case the only known values so far are $0, 1/2$ and $\frac{1}{2-2^{1-k}}$ with $k \in \mathbb{N}_0$.

10. Regarding the behaviour of Hilbert space compression under group constructions, we mention the easy observation that every subgroup $H < G$, equipped with the induced metric from G, satisfies $\alpha(H) \geq \alpha(G)$.

11. In [74], Li proves that the wreath product of two groups with positive compression is again positive.

12. The only other group construction that has been studied is that of taking direct sums. In this respect, we mention the following result, which is Theorem 4.1 of [59].

    If $f_1 : G_1 \to \mathcal{H}_1$ and $f_2 : G_2 \to \mathcal{H}_2$ are uniform embeddings, then $f = (f_1, f_2) : G_1 \oplus G_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$ is a uniform embedding such that $R(f) = \min(R(f_1), R(f_2))$. Consequently, we obtain $\alpha(G_1 \oplus G_2) = \min(\alpha(G_1), \alpha(G_2))$.

The fact that only so little is known about the behaviour of the Hilbert space compression under group constructions, has led us to the following

Research Question: What is the behaviour of the Hilbert space compression under group constructions? Specifically, if $G_1, G_2, G_3, \ldots$ is a collection of groups with known Hilbert space compression, what is the compression of the amalgamated free product $G_1 *_{G_2} G_3$? What is the compression of an HNN-extensions $\mathrm{HNN}(G_1, G_2, \theta)$ with $G_2 < G_1$? What is the compression of $\lim_{i\to\infty} G_i$, or of a group extension of G_1 by G_2, ...

We will investigate the above situations in Chapter 8.

Chapter 8

Results

8.1 Free products and HNN-extensions over finite groups

Convention 8.1.1. *In this section, we only deal with uniformly discrete metric spaces, i.e. spaces X such that* $\inf\{d_X(x,y) \mid x,y \in X,\ x \neq y\} > 0$.

It is our goal to attack the same problems as in Sections 6.1 and 6.2, but this time in the non-equivariant case. In the important case that our groups are finitely generated, and equipped with the word length metric, then free products amalgamated over finite groups and HNN-extensions over finite groups are quasi-isometrically speaking *non-amalgamated free products* [92]. It thus suffices to study this situation. We ease into things, starting with the following general lemma.

Lemma 8.1.2. *Let $\mathcal{H}$ be a Hilbert space and let X be a uniformly discrete metric space. If $f : X \to \mathcal{H}$ is a map such that*

$$(1/C)\ d(x,y)^{\epsilon} - D \leq d(f(x), f(y)) \leq C\ d(x,y) + D$$

for some $\epsilon > 0, C \geq 1, D \geq 0$ and $\forall x, y \in X$, then there exist $\tilde{f} : X \to \mathcal{H} \oplus l^2(X)$ and a real number $\overline{C} \geq 1$ such that

$$(1/\overline{C})\ d(x,y)^{\epsilon} \leq d(f(x), f(y)) \leq \overline{C}\ d(x,y)\ \forall x, y \in X. \tag{8.1}$$

Proof. Denote $B = \inf\{d(x,y) \mid x,y \in X\ x \neq y\}$. Define $\tilde{f} : X \to \mathcal{H} \oplus l^2(X), x \mapsto f(x) \oplus \delta_x$ where δ_x is the Dirac function at x. Then for every two distinct elements x, y of X, we have that

$$\|\tilde{f}(x) - \tilde{f}(y)\|^2 = \|f(x) - f(y)\|^2 + 2.$$

Therefore

$$\|\tilde{f}(x) - \tilde{f}(y)\| \leq Cd(x,y) + D + \sqrt{2} \leq (C + \frac{D+\sqrt{2}}{B})d(x,y),$$

and so we obtain an upper bound like the one in equation (8.1) by setting $\overline{C} = C + \frac{D+\sqrt{2}}{B}$. With respect to the lower bound, we obtain that

$$\|\tilde{f}(x) - \tilde{f}(y)\| \geq \frac{1}{\sqrt{2}}((1/C)d(x,y)^\epsilon - D) + 1.$$

When $d(x,y)^\epsilon \geq 2CD$, then

$$\|\tilde{f}(x) - \tilde{f}(y)\| \ \geq \frac{1}{\sqrt{2}}(\frac{1}{2C}d(x,y)^\epsilon + \frac{1}{2C}d(x,y)^\epsilon - D) \geq (1/\overline{C})d(x,y)^\epsilon,$$

if we take $\overline{C} \geq 2\sqrt{2}C$. When $d(x,y)^\epsilon \leq 2CD$, then

$$\|\tilde{f}(x) - \tilde{f}(y)\| \geq 1 \geq \frac{1}{2CD}d(x,y)^\epsilon.$$

Finally, putting $\overline{C} := \max(2\sqrt{2}C, 2CD, C + \frac{D+\sqrt{2}}{B})$, we obtain

$$(1/\overline{C})\ d(x,y)^\epsilon \leq \|\tilde{f}(x) - \tilde{f}(y)\| \leq \overline{C}\ d(x,y) \ \ \forall x,y \in X.$$

□

Predicting the Hilbert space compression of $G_1 * G_2$ in terms of those of G_1 and G_2 is only possible if the length function on $G_1 * G_2$ is somehow related to the length function on G_1 and that on G_2. Let us start by fixing two finitely generated groups G_1 and G_2. We denote the chosen finite symmetric generating subset, the word length function, the word length distance and the Hilbert space compression of G_1 by S_1, l_1, d_1 and α_1 respectively. We use similar notations (S_2, l_2, d_2 and α_2) for G_2. We denote the generating subset $S_1 \cup S_2$ on $G := G_1 * G_2$ by S and equip G with the word length metric d_G relative to S. Let us introduce some standard notations regarding free products of groups (see for example [24]). Two non-trivial elements x, y of $G_1 * G_2$ can always be written in reduced form as

$$\begin{aligned} x &= a_1 a_2 \dots a_m \\ y &= b_1 b_2 \dots b_n, \end{aligned} \tag{8.2}$$

where m, n are natural numbers and where the a_i, b_j are elements of $G_1 \setminus \{1\} \sqcup G_2 \setminus \{1\}$ such that no two consecutive elements a_i, a_{i+1} or b_j, b_{j+1} both belong to G_1 or both belong to G_2. If i_0 is the highest index such that $a_1, a_2, \dots, a_{i_0-1}$ and a_{i_0} are equal to $b_1, b_2, \dots, b_{i_0-1}$ and b_{i_0} respectively, then $h := a_1 a_2 \dots a_{i_0}$, where an empty product equals 1, is called the common part of x and y. This way, we write

$$\begin{aligned} x &= h\ g_x\ x_1\ x_2 \dots x_n \\ y &= h\ g_y\ y_1\ y_2 \dots y_m, \end{aligned} \tag{8.3}$$

where h is the common part of x and y. If $g_x \in G_1 \setminus \{1\}$, and $g_y \in G_2 \setminus \{1\}$ (which can happen only when $h = 1$), then redefine $y_{m+1} := y_m, y_m := y_{m-1}, \dots y_2 := y_1, y_1 := g_y, g_y := 1_{G_1}$, in order that g_x, g_y both belong to G_1. Similar remarks hold when $g_x \in G_2 \setminus \{1\}$ and $g_y \in G_1 \setminus \{1\}$. With the convention that empty sums are zero, we obtain

$$d_G(x, y) = \sum_{i=1}^{n} l_{1,2}(x_i) + d_{1,2}(g_x, g_y) + \sum_{j=1}^{m} l_{1,2}(y_j) \tag{8.4}$$

where $l_{1,2}$ stands for l_1 or l_2 as appropriate and similarly for $d_{1,2}$.

This natural construction for a distance on the free product can easily be generalized. Given two metric spaces $(X_1, d_1), (X_2, d_2)$ and points $\tilde{x}_1 \in X_1, \tilde{x}_2 \in X_2$, let us define the free product $X_1 * X_2$ of $(X_1, d_1, \tilde{x}_1)$ and $(X_2, d_2, \tilde{x}_2)$. As a set, this metric space is equal to the collection of all words whose letters are alternately elements from $X_1 \setminus \{\tilde{x}_1\}$ and $X_2 \setminus \{\tilde{x}_2\}$. We also include the word $\tilde{x}_1$ which we identify with $\tilde{x}_2$ (the idea being that $\tilde{x}_1$ and $\tilde{x}_2$ play the role of "the identity elements" of X_1 and X_2). Given $x, y \in X \setminus \{\tilde{x}_1\}$, write them similarly as in Equation (8.3):

$$\begin{aligned} x &= h g_x x_1 x_2 \dots x_n \\ y &= h g_y y_1 y_2 \dots y_m. \end{aligned}$$

Again, if $g_x \in X_1 \backslash \{\tilde{x}_1\}$ and $g_y \in X_2 \backslash \{\tilde{x}_2\}$, then redefine $y_{m+1} := y_m, y_m := y_{m-1}, \ldots, y_2 := y_1, y_1 := \tilde{x}_1$ in order that g_x and g_y both belong to X_1. The case where $g_x \in X_2 \backslash \{\tilde{x}_1\}$ and $g_y \in X_1 \backslash \{\tilde{x}_2\}$ is similar. Using d_1 and d_2, we define the distance between two elements $x, y \in X$, written as above by

$$\sum_{i=1}^{n} d_{1,2}(x_i, \tilde{x}_{1,2}) + d_{1,2}(g_x, g_y) + \sum_{j=1}^{m} d_{1,2}(y_j, \tilde{x}_{1,2}),$$

where $d_{1,2}$ stands for d_1 or d_2 and $\tilde{x}_{1,2}$ stands for $\tilde{x}_1$ or $\tilde{x}_2$ as appropriate. We prove the following

Theorem 8.1.3. *Assume that* $(X_1, d_1, \tilde{x}_1)$ *and* $(X_2, d_2, \tilde{x}_2)$ *are uniformly discrete metric spaces as above. Denoting their Hilbert space compressions by* α_1 *and* α_2 *respectively, the Hilbert space compression* α *of the free product* $X = X_1 * X_2$*, equipped with the natural metric as just defined, satisfies*

$$\min(\alpha_1, \alpha_2, 1/2) \leq \alpha \leq \min(\alpha_1, \alpha_2).$$

Proof. It is not hard to find the desired upper bound for α since X_1 and X_2 are metric subspaces of X and so $\alpha \leq \min(\alpha_1, \alpha_2)$.

We are ready if this minimum is 0. If not, then choose a number $0 \leq \epsilon < \min(\alpha_1, \alpha_2, 1/2)$ and let us take for each $i \in \{1, 2\}$ a map $f_i : X_i \to \mathcal{H}_i$ which maps $\tilde{x}_i$ to 0 and such that

$$\exists C \geq 1,\ \forall x, y \in X_i : (1/C) d(x, y)^{\epsilon} \leq d(f_i(x), f_i(y)) \leq C d(x, y). \quad (8.5)$$

Use inclusion to view the f_i as maps to $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$. Borrowing, but generalizing, an idea of Chen, Dadarlat, Guentner and Yu [24], we construct an embedding of $X_1 * X_2$ into $\mathcal{H}$.

Denote by W_i $(i = 1, 2)$ the set that contains $\tilde{x}_i$ and those elements of X whose expression as a word begins with an element of $X_i \backslash \{\tilde{x}_i\}$. Notice that $W_1 \cap W_2 = \{\tilde{x}_1\}$. Define a Hilbert space $\overline{\mathcal{H}}$ by

$$\overline{\mathcal{H}} = (\bigoplus_{W_1} \mathcal{H}) \oplus (\bigoplus_{W_2} \mathcal{H}).$$

Consider a map $f : X \to \overline{\mathcal{H}}$ defined as follows: set $f(\tilde{x}_1) = 0$. Next, choose any element $x \in X \setminus \{\tilde{x}_1\}$ and write it as a word $x = x_1\, x_2 \ldots x_n$ where the x_i are alternately elements of $X_1 \setminus \{\tilde{x}_1\}$ and $X_2 \setminus \{\tilde{x}_2\}$ (or

$X_2 \setminus \{\tilde{x}_2\}$ and $X_1 \setminus \{\tilde{x}_1\}$). If $x_1 \in X_1$, then we define $f(x) = f(x)_1 \oplus f(x)_2 \in \overline{\mathcal{H}}$ by setting $f(x)_2 = 0$ and

$$f(x)_1 = \bigoplus_{h \in W_1} (f(x)_1)_h,$$

where $(f(x)_1)_h$ equals

$$\begin{cases} f_1(x_{2k+1}), & \exists k \geq 0 \text{ such that } h = x_1\, x_2 \ldots x_{2k} \\ f_2(x_{2k}), & \exists k \geq 1 \text{ such that } h = x_1\, x_2 \ldots x_{2k-1} \\ 0 & \text{otherwise} \end{cases},$$

using the convention that an empty product is $\tilde{x}_1 = \tilde{x}_2$. In particular, $(f(x)_1)_{\tilde{x}_1} = f_1(x_1), (f(x)_1)_{x_1} = f_2(x_2), (f(x)_1)_{x_1 x_2} = f_1(x_3), \ldots$. A similar formula is used when the reduced word expression of x begins with an element of X_2. Let us show that

$$(1/C)\ d(x,y)^{\epsilon} \leq d(f(x), f(y)) \leq C\ d(x,y) \text{ for all } x, y \in X.$$

Choose $x, y \in X$ and write $x = hg_x x_1 x_2 \ldots x_n$ and $y = hg_y y_1 y_2 \ldots y_m$, where we use the same notations as in (8.3). With the convention that empty sums are 0, we obtain:

$$\|f(x) - f(y)\|^2 = \sum_{i=1}^{n} \|f_{1,2}(x_i)\|^2 + \|f_{1,2}(g_x) - f_{1,2}(g_y)\|^2 + \sum_{j=1}^{m} \|f_{1,2}(y_j)\|^2.$$

By Equation (8.5), we obtain the upper bound

$$\|f(x) - f(y)\|^2 \leq \sum_{i=1}^{n} C^2 d_{1,2}(x_i, \tilde{x}_{1,2})^2 + C^2 d_{1,2}(g_x, g_y)^2 + \sum_{j=1}^{m} C^2 d_{1,2}(y_j, \tilde{x}_{1,2})^2.$$

Using the fact that $\sqrt{a^2 + b^2} \leq a + b$ for all $a, b \geq 0$, we obtain

$$d(f(x), f(y)) \leq C d(x,y).$$

On the other hand, we have that

$$\begin{aligned} d(f(x), f(y)) \quad \geq \quad & [\sum_{i=1}^{n} (1/C^2) d_{1,2}(x_i, \tilde{x}_{1,2})^{2\epsilon} + (1/C^2) d_{1,2}(g_x, g_y)^{2\epsilon} \ + \\ & \sum_{j=1}^{m} (1/C^2)(d_{1,2}(y_j, \tilde{x}_{1,2}))^{2\epsilon}]^{1/2}. \end{aligned}$$

Since $a^{2\epsilon} + b^{2\epsilon} \geq (a+b)^{2\epsilon}$ for all $a, b \geq 0$ and $\epsilon \leq 1/2$, we get that

$$d(f(x), f(y)) \geq (1/C)\sqrt{d(x,y)^{2\epsilon}} = (1/C)d(x,y)^{\epsilon},$$

which concludes the proof of the Theorem. □

The proof of Theorem 8.1.3 can easily be adapted to provide information about the L^p-compression of a free product of groups. Explicitly, we obtain

Corollary 8.1.4. *Assume that* $(X_1, d_1, \tilde{x}_1)$ *and* $(X_2, d_2, \tilde{x}_2)$ *are uniformly discrete metric space. If we denote their* L^p*-compressions by* α_1 *and* α_2 *respectively, then the* L^p*-compression* α *of the free product* $X = X_1 * X_2$*, when equipped with the same metric as in Theorem 8.1.3, satisfies*

$$\min(\alpha_1, \alpha_2, 1/p) \leq \alpha \leq \min(\alpha_1, \alpha_2).$$

It is particularly interesting to notice that for $p = 1$, we obtain that the L^1-compression of the free product $G_1 * G_2$ equals the minimum of the L^1-compressions of G_1 and G_2.

Moreover, the same result holds when we replace the class of L^1-spaces, with a class $\mathcal{C}$ of Banach spaces which is stable under l^1-direct sum.

Let us end this section by a remark regarding the L^p-compression ($p \geq 1$) of $G_1 *_F G_2$ and $\mathrm{HNN}(H, F, \theta)$ where F is a finite group and where H, G_1 and G_2 are finitely generated groups. Our claim is that the L^p-compressions of $G_1 *_F G_2$ and $\mathrm{HNN}(H, F, \theta)$ are equal to that of $G_1 * G_2$ and $G_1 * \mathbb{Z}$ respectively. We distinguish two cases. First, assume that G_1, G_2 and H are finite (or more generally are hyperbolic). It follows from [69] that $G_1 *_F G_2, G_1 * G_2, H * \mathbb{Z}$ and $\mathrm{HNN}(H, F, \theta)$ are also hyperbolic and thus all have L^p-compression 1 [103]. Secondly, assume that at least one of G_1 and G_2 is infinite. Then our claim regarding amalgamated products follows from Theorem 0.2 in [92], where it is shown that $G_1 *_F G_2$ is quasi-isometric to $G_1 * G_2$. Our claim regarding HNN-extensions follows from the same result, where it is proven that $\mathrm{HNN}(H, F, \theta)$ and $H * \mathbb{Z}$ are quasi-isometric for infinite H.

In order to prove a result on HNN-extensions $\mathrm{HNN}(H, F, \theta)$ where F is of finite index in H, we introduce first a new way to interpret Hilbert space compression.

8.2 Hilbert space compression from a different viewpoint

In the sequel, we introduce an interpretation of compression which is different from what is common in literature. To introduce this different interpretation, let us analyze the proof that Definitions 7.1.1 and 7.1.3 of uniform embeddability are equivalent. To this end, we give a slightly modified version of Proposition 2.1 in [39].

Proposition 8.2.1. *Let (X, d) be a metric space. Then X is uniformly embeddable in a Hilbert space in the sense of Definition 7.1.1 if and only if for every $R > 0$ and $\epsilon > 0$ there exists a Hilbert space valued map $\xi : X \to \mathcal{H}, x \to \xi_x$ such that $\|\xi_x\| = 1$ for all $x \in X$ and such that*

1. $\sup\{\|\xi_x - \xi_y\| : d(x, y) \leq R, x, y \in X\} \leq \epsilon$,
2. $\lim_{S\to\infty} \inf\{\|\xi_x - \xi_y\| : d(x, y) \geq S, x, y \in X\} = \sqrt{2}$.

It follows easily from the proof that the second condition may be replaced by the following condition which is stronger:

$$\lim_{S\to\infty} \sup\{|\langle \xi_x, \xi_y\rangle| : d(x, y) \geq S, x, y \in X\} = 0.$$

Proof. Assume that X is uniformly embeddable and let $F : X \to \mathcal{H}$ be a uniform embedding of X in a real Hilbert space $\mathcal{H}$. Let ρ_- and ρ_+ be functions such that

$$\rho_-(d(x, y)) \leq \|F(x) - F(y)\| \leq \rho_+(d(x, y)).$$

Denote

$$\mathrm{Exp}(\mathcal{H}) = \mathbb{R} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus \cdots$$

and define $\mathrm{Exp} : \mathcal{H} \to \mathrm{Exp}(\mathcal{H})$ by

$$\mathrm{Exp}(\zeta) = 1 \oplus \zeta \oplus (\frac{1}{\sqrt{2!}}\zeta \otimes \zeta) \oplus (\frac{1}{\sqrt{3!}}\zeta \otimes \zeta \otimes \zeta) \oplus \cdots .$$

Note that $\langle \mathrm{Exp}(\zeta), \mathrm{Exp}(\zeta')\rangle = e^{\langle \zeta,\zeta'\rangle}$, for all $\zeta, \zeta' \in \mathcal{H}$. For $t > 0$ define

$$\xi_x = e^{-t\|F(x)\|^2}\mathrm{Exp}(\sqrt{2t}F(x)).$$

It is easily verified that $\langle \xi_x, \xi_y \rangle = e^{-t\|F(x)-F(y)\|^2}$. Consequently, for all $x, y \in X$ we have $\|\xi_x\| = 1$, and

$$e^{-t\rho_+(d(x,y))^2} \leq \langle \xi_x, \xi_y \rangle \leq e^{-t\rho_-(d(x,y))^2}. \tag{8.6}$$

Putting $t = \frac{-\ln(1-\epsilon^2/2)}{\rho_+(R)^2}$, it is easy to verify conditions 1 and 2 above.

Conversely, choose $p > 0$ and assume that X satisfies the conditions in the statement. There exist a sequence of maps $\eta_n : X \to \mathcal{H}_n$ and a sequence of numbers $S_0 = 0 < S_1 < S_2 < \ldots$, increasing to infinity, such that for every $n \geq 1$ and every $x, y \in X$,

1. $\|\eta_n(x)\| = 1$
2. $\|\eta_n(x) - \eta_n(y)\| \leq \frac{1}{n^{1/2+p}}$, provided $d(x, y) \leq \sqrt{n}$,
3. $\|\eta_n(x) - \eta_n(y)\| \geq 1$, provided $d(x, y) \geq S_n$.

Choose a base point $x_0 \in X$ and define $F : X \to \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ by

$$F(x) = \frac{1}{2}((\eta_1(x) - \eta_1(x_0)) \oplus (\eta_2(x) - \eta_2(x_0)) \oplus \cdots).$$

It is not hard to verify that F is well defined and

$$\rho_-(d(x, y)) \leq \|F(x) - F(y)\| \leq d(x, y) + D, \text{ for all } x, y \in X,$$

where $D > 0$ is some constant, $\rho_- = \frac{1}{2}\sum_{n=1}^{\infty} \sqrt{n-1}\chi_{[S_{n-1},S_n)}$, and the $\chi_{[S_{n-1},S_n)}$ are the characteristic functions of the sets $[S_{n-1}, S_n)$.

Indeed, let $x, y \in X$. If n is such that $\sqrt{n-1} \leq d(x, y) < \sqrt{n}$, we have

$$\begin{aligned}\|F(x) - F(y)\|^2 &= \frac{1}{4}\sum_{i\leq n-1} \|\eta_i(x) - \eta_i(y)\|^2 + \frac{1}{4}\sum_{i\geq n} \|\eta_i(x) - \eta_i(y)\|^2 \\ &\leq (n-1) + \frac{1}{4}\sum_{i\geq n} \frac{1}{i^{1+2p}} \leq d(x, y)^2 + \overline{D}\end{aligned}$$

where $\overline{D} = \frac{1}{4}\sum_{i\geq n} \frac{1}{i^{1+2p}} < \infty$.

Similarly, if n is such that $S_{n-1} \leq d(x, y) < S_n$, we have

$$\|F(x) - F(y)\|^2 \geq \frac{1}{4}\sum_{i\leq n-1} \|\eta_i(x) - \eta_i(y)\|^2 \geq \frac{n-1}{4} = \rho_-(d(x, y))^2.$$

$\square$

In Definition 7.1.1, compression hides in the function ρ_-, describing how fast this map can go to infinity. Similarly, compression should also hide somewhere in Definition 7.1.3 of uniform embeddability. To see where, we introduce the following

Lemma 8.2.2. *Assume that X is a metric space with compression $\alpha > 0$. Fix numbers $a, b, r \in \mathbb{R}^+$ and set $R_n = n^r, \epsilon_n = \frac{1}{an^b}$ for all $n \in \mathbb{N}_0$. For any $0 < p < \alpha$ and every n larger than some number M_p, we can find a collection of unit vectors $(\xi_n^x)_{x \in X}$ in some Hilbert space $\mathcal{H}_p$ satisfying*

1. *$\|\xi_n^x - \xi_n^y\| \leq \epsilon_n$ provided $d(x,y) \leq R_n$,*
2. *$\|\xi_n^x - \xi_n^y\| \geq 1$ provided $d(x,y) \geq S_n := n^{\frac{r+b+p}{\alpha-p}}$.*

The second condition can be replaced by

$$|\langle \xi_n^x, \xi_n^y \rangle| \leq \frac{1}{2} \text{ provided } d(x,y) \geq S_n := n^{\frac{r+b+p}{\alpha-p}}.$$

Moreover, $\lim_{S \to \infty} \inf\{\|\xi_n^x - \xi_n^y\| \mid d(x,y) \geq S\} = \sqrt{2}$, for every $n \in \mathbb{N}_0$.

Proof. Choose $0 < p < \alpha$ and let $F : X \to \mathcal{H}_p$ be a uniform embedding of X into a Hilbert space satisfying

$$\forall x, y \in X: \ \frac{1}{C} d(x,y)^{\alpha-p} - D \leq d(F(x), F(y)) \leq \widetilde{C} d(x,y) + \widetilde{D},$$

for some $C, \widetilde{C} > 0$ and $D, \widetilde{D} \geq 0$. Denote $\rho_-(d(x,y)) := \frac{1}{C} d(x,y)^{\alpha-p} - D$ and $\rho_+(d(x,y)) := \widetilde{C} d(x,y) + \widetilde{D}$. In the proof of Proposition 8.2.1, set $t_n = \frac{-\ln(1-\frac{\epsilon_n^2}{2})}{\rho_+(R_n)^2}$ and obtain vectors $(\xi_n^x)_{x \in X}$ such that

$$e^{-t_n \rho_+(d(x,y))^2} \leq \langle \xi_n^x, \xi_n^y \rangle \leq e^{-t_n \rho_-(d(x,y))^2}.$$

It is easy to verify that the vectors $(\xi_n^x)_{x \in X}$ satisfy condition (1) of this Corollary. Regarding the second condition, note that

$$\begin{aligned} \|\xi_n^x - \xi_n^y\|^2 &= 2 - 2\langle \xi_n^x, \xi_n^y \rangle \\ &\geq 2 - 2e^{-t_n \rho_-(d(x,y))^2} \\ &= 2 - 2e^{\frac{\ln(1-\frac{\epsilon_n^2}{2})}{(\widetilde{C}R_n+\widetilde{D})^2}((1/C)\ d(x,y)^{\alpha-p}-D)^2}. \end{aligned}$$

Consequently we have $\|\xi_n^x - \xi_n^y\| \geq 1$ whenever

$$2 - 2e^{\frac{\ln(1-\frac{\epsilon_n^2}{2})}{(\widetilde{C}R_n+\widetilde{D})^2}((1/C)\ d(x,y)^{\alpha-p}-D)^2} \geq 1,$$

i.e. whenever

$$(1-\frac{\epsilon_n^2}{2})^{\frac{((1/C)d(x,y)^{\alpha-p}-D)^2}{(\widetilde{C}R_n+\widetilde{D})^2}} \leq \frac{1}{2}.$$

This is true if

$$\frac{(1/C)d(x,y)^{\alpha-p}-D}{\widetilde{C}R_n+\widetilde{D}} \geq \sqrt{\frac{-\ln(2)}{\ln(1-\frac{\epsilon_n^2}{2})}},$$

if and only if

$$d(x,y) \geq [C(\sqrt{\frac{-\ln(2)}{\ln(1-\frac{1}{2a^2n^{2b}})}}(\widetilde{C}n^r+\widetilde{D})+D)]^{\frac{1}{\alpha-p}}.$$

Since $\frac{-\ln(2)}{\ln(1-\frac{1}{2a^2n^{2b}})} \leq \ln(2)2a^2n^{2b} \leq 2a^2n^{2b}$, it suffices to take

$$d(x,y) \geq [C\sqrt{2}an^b(\widetilde{C}n^r+\widetilde{D})+CD]^{\frac{1}{\alpha-p}} := A_n.$$

If n is large enough, then $A_n \leq n^{\frac{r+b+p}{\alpha-p}}$, so we obtain $\|\xi_n^x - \xi_n^y\| \geq 1$ provided $d(x,y) \geq n^{\frac{r+b+p}{\alpha-p}}$. ☐

In the particular case that $r = 1/2 = b$ and the compression of (X,d) is $\alpha > 0$, then $S_n = n^{\frac{1+p}{\alpha-p}}$, where $p > 0$ is arbitrarily small. Said differently:

> The compression of X hides in the speed at which the sequence S_n converges to infinity. If the compression is high, then the sequence $(S_n)_{n\in\mathbb{N}_0}$ increases slow, if the compression is low, then the sequence converges fast.

The converse is also true.

Lemma 8.2.3. *Let (X,d) be a metric space. If for some $a,p > 0$ and for every $n \in \mathbb{N}_0$ greater than some natural number M, you can find a collection of unit vectors $(\xi_n^x)_{x\in X}$ such that*

1. *$\|\xi_n^x - \xi_n^y\| \leq \frac{1}{n^{1/2+p}}$ provided $d(x,y) \leq \sqrt{n}$*
2. *$|\langle \xi_n^x, \xi_n^y \rangle| \leq \frac{1}{2}$ provided $d(x,y) \geq n^a$,*

then the compression of X is greater than $\frac{1}{2a}$.

Proof. Following the proof of Proposition 8.2.1, the above families of vectors give an explicit large-scale Lipschitz uniform embedding F of (X,d) into a Hilbert space satisfying

$$\forall x,y \in X: \ \|F(x)-F(y)\| \geq \sum_{n=1}^{\infty} \frac{1}{2}\sqrt{n-1}\chi_{[S_{n-1},S_n)}(d(x,y)),$$

where $S_n \leq n^a$ if n is sufficiently large. The compression of F is at least β if

$$S_n^{\beta} \leq \sqrt{n-1},$$

for all n sufficiently large. We conclude that the compression of (X,d) is at least $\frac{1}{2a}$. ☐

In many cases, one is interested in the Hilbert space compression of compactly generated groups equipped with the word length distance relative to a compact symmetric generating subset. Such spaces are quasi-geodesic metric spaces, hence any uniform embedding is large-scale Lipschitz. This means that the proof of Proposition 8.2.1 may be adapted to obtain stronger compression information. Concretely, in the second part of the proof of Proposition 8.2.1, we use the condition

$$\|\eta_n(x)-\eta_n(y)\| \leq \frac{1}{n^{1/2+p}}, \text{ provided } d(x,y) \leq \sqrt{n}, \tag{8.7}$$

to prove that F is large-scale Lipschitz. In the quasi-geodesic case, it suffices to relax condition (8.7) to

$$\|\eta_n(x)-\eta_n(y)\| \leq \frac{1}{n^{1/2+p}}, \text{ provided } d(x,y) \leq \ln(n).$$

The second condition

$$\|\eta_n(x)-\eta_n(y)\| \geq 1, \text{ provided } d(x,y) \geq S_n,$$

then holds for smaller S_n and we obtain that the function

$$\rho_- = \frac{1}{2}\sum_{n=1}^{\infty} \sqrt{n-1}\chi_{[S_{n-1},S_n)}$$

becomes larger. Concretely,

Lemma 8.2.4. *Let* (X, d) *be a quasi-geodesic metric space. If for some* $a, p > 0$ *and for every* $n \in \mathbb{N}_0$ *greater than some natural number* M*, you can find a collection of unit vectors* $(\xi_n^x)_{x \in X}$ *such that*

1. $\|\xi_n^x - \xi_n^y\| \leq \frac{1}{n^{1/2+p}}$ *provided* $d(x, y) \leq \ln(n)$
2. $|\langle \xi_n^x, \xi_n^y \rangle| \leq \frac{1}{2}$ *provided* $d(x, y) \geq n^a$,

then the compression of X *is greater than* $\frac{1}{2a}$.

8.3 HNN-extensions over a finite index subgroup

Let us start by introducing some notations and definitions as in [39].

Recall that a *tree* (V, E) consists of a set V of vertices, a set E of edges and two endpoint maps $E \to V$, associating to each edge its endpoints. Every two vertices in a tree are connected by a unique path without backtracking. Whenever two vertices $v, v' \in V$ are connected by an edge, then we denote this edge by $[v, v']$, or equivalently by $[v', v]$, i.e. edges do not carry an orientation.

A *tree of metric spaces* consists of families $(X_v)_{v \in V}$ and $(X_e)_{e \in E}$ of metric spaces where (V, E) is a tree. Moreover, there exist maps $\sigma_{e,v} : X_e \to X_v$ whenever v is an endpoint of e. The maps $\sigma_{e,v}$ are called *structural maps*, the spaces X_v are called *vertex spaces* and the spaces X_e are called *edge spaces.*

Given an HNN-extension $G := \mathrm{HNN}(H, F, \theta)$ of any group H over any group F, we can use Bass-Serre theory to associate a tree T to it as follows. As the set V of vertices we take G/H, the collection of left cosets of H in G. As the set E of edges we take G/F, the left cosets of F in G. Given $x \in G$, the edge xF connects xH and xtH.

Notice that the vertices and edges of the above tree are actually subsets of G, so we can equip them as metric subspaces of G. We can define structural maps $\sigma_{xF,xH} : xF \hookrightarrow xH$ by inclusion and $\sigma_{xF,xtH} : xF \to xtH$ by $xf \mapsto xft = xt\theta(f)$. This way, we obtain a tree of metric spaces which is called *the tree of metric spaces associated to the HNN-extension* $G = HNN(H, F, \theta)$. We emphasize again that the cosets xF and xH are equipped with the induced metric from G.

The Hilbert space compression of a group depends on the chosen length function on this group. On $G = \mathrm{HNN}(H, F, \theta)$, we choose the same length function as in Section 6.2, see Construction 6.2.3.

Remark 8.3.1. *There is a connection between this distance on G and the distance on the underlying Bass-Serre tree. Precisely, given $x, y \in G = HNN(H, F, \theta)$, remark that the distance $d(x, y)$ in G equals*

$$d_T(xH, yH) + \inf\{d(x_0, x_1) + d(x_2, x_3) + \ldots + d(x_{p-1}, x_p)\},$$

where d_T is the distance on the underlying tree T (i.e. the number of edges in the shortest path connecting xH and yH) and where the infimum is taken over all sequences $x_0, x_1, \ldots, x_p$, where $p = 2d_T(xH, yH) + 1$ and

- $x = x_0, y = x_p$
- $x_{2k} = x_{2k-1}t$ *or* $x_{2k} = x_{2k-1}t^{-1}$ *for* $k = 1, \ldots, d_T(xH, yH)$,
- x_{2k}, x_{2k+1} *lie in the same coset of H for $k = 0, 1, \ldots, d_T(xH, yH)$.*

In this section, we prove a connection between the Hilbert space compression of $H < G := HNN(H, F, \theta)$ with the induced metric from G and the Hilbert space compression of G. In parts of our proof, one can recognize careful adaptations of a proof of Guentner and Dadarlat (Theorem 5.3 in [39]).

Theorem 8.3.2. *Consider $G := HNN(H, F, \theta)$ where both F and $\theta(F)$ are finite index subgroups of H. Equip H with a length function l_1 and G with the metric d from Construction 6.2.3. Next, equip H with the induced metric d_{in} from G. Then,*

$$\alpha_1/6 \leq \alpha \leq \alpha_1,$$

where α_1 and α denote the Hilbert space compressions of (H, d_{in}) and (G, d) respectively.

From now on, we work under the hypotheses of Theorem 8.3.2. Let us fix some notations. For a given vertex $v \in V$, we denote by $\alpha(v) \in V$ the unique vertex such that $[v, \alpha(v)]$ points towards the infinite geodesic $H, tH, t^2H, \ldots$. Here, just for this once, $[v, \alpha(v)]$ was considered as an *oriented* edge. Given vertices $v, v' \in V$, we denote by (k, l) the unique pair of integers such that $\alpha^k(v) = \alpha^l(v')$ and $d_T(v, v') = k + l$. Here, α^k is k-fold composition of α with itself and α^0 is the identity map. Write $Y_v = \sigma_{[v,\alpha(v)],v}(X_{[v,\alpha(v)]}) \subset X_v$ and remark that it is a left coset of F or $\theta(F)$. Set $f_v = \sigma_{[v,\alpha(v)],\alpha(v)} \circ \sigma^{-1}_{[v,\alpha(v)],v} : Y_v \to X_{\alpha(v)}$. Finally, let $Z > 0$ be a real number such that every right coset of F and $\theta(F)$ in (H, d_{in}) contains a representative whose length is strictly smaller than Z.

Definition 8.3.3. *Given $x_0 \in G$, an s-*chain starting in x_0 *is a sequence $\mathbf{x} = (x_0, x_1, \ldots, x_{s-1})$ with $x_i \in X_{\alpha^i(v)}$ such that for each $0 \le i \le s-2$ there exists $\overline{x_i} \in Y_{\alpha^i(v)}$ such that $d(x_i, \overline{x_i}) < d(x_i, Y_{\alpha^i(v)}) + 1$ and $x_{i+1} = f_{\alpha^i(v)}(\overline{x_i})$.*

Lemma 8.3.4. *Assume that R is a strictly positive real number, let $x_0 \in X_v$ and $x'_0 \in X_{v'}$ with $d(x_0, x'_0) \le R$ and let k, l and Z be as just described. Then, any chains $(x_0, x_1, \ldots, x_k)$, $(x'_0, x'_1, \ldots, x'_l)$ are such that*

$$\max\{(\sup_{0\le i\le k-1} d(x_i, x_{i+1})), (\sup_{0\le j\le l-1} d(x'_j, x'_{j+1})), d(x_k, x'_l)\} \le (Z+3)R.$$

Proof. The result is clear for $R < 1$. For $R \ge 1$, fix $i \in \{0, 1, \ldots, k-1\}$. Write $v_i = \alpha^i(v)$, denote $e = [v_i, \alpha(v_i)]$ and take $a \in G$ such that $X_e = aF$. This implies either that $X_{v_i} = aH$ and $X_{\alpha(v_i)} = atH$ or that $X_{v_i} = atH$ and $X_{\alpha(v_i)} = aH$. We only prove the second case, leaving the first case as an exercise to the reader.

The elements of Y_{v_i} are of the form $aft = at\theta(f)$ where $f \in F$. Writing $x_i = ath$ for some $h \in H$, take b a representative of $\theta(F)h$ whose length is smaller than Z. Then, $x_i b^{-1} \in Y_{v_i}$, so $d(x_i, x_{i+1}) < d(x_i, Y_{v_i}) + 2 \le d(x_i, x_i b^{-1}) + 2 < Z + 2 < (Z+3)R$.

Analogously, one proves that $d(x'_j, x'_{j+1}) < Z + 2 < (Z+3)R$ for $j \in \{0, 1, \ldots, l-1\}$. For the case $i = k$, we use the triangle inequality to get that

$$d(x_k, x'_l) \le (Z+2)(k+l) + R \le (Z+3)R.$$

□

Notation 8.3.5. *Given $R > 0$ and $\epsilon > 0$, choose and fix $s, n \in \mathbb{N}_0$ such that*

$$\sqrt{2/s} \le \frac{\epsilon}{2(R+1)}, \; n \ge (Z+3)R. \tag{8.8}$$

Next, using Proposition 8.2.1, find a Hilbert space $\overline{\mathcal{H}}$, an $S > 0$ and unit vectors $\{\tilde{\xi}_x \mid x \in H\} \subset \overline{\mathcal{H}}$ satisfying the conditions

$$\sup\{\|\tilde{\xi}_y - \tilde{\xi}_{y'}\| : d(y, y') \leq n + 2s(Z + 2)\} \leq \frac{\epsilon}{2(R + 1)}, \tag{8.9}$$

$$\sup\{|\langle\tilde{\xi}_y, \tilde{\xi}_{y'}\rangle| : d(y, y') \geq S\} \leq \frac{1}{2}. \tag{8.10}$$

For each $v \in V$, denote $\mathcal{H}_v := \overline{\mathcal{H}}$. Since G is the disjoint union of the vertex spaces X_v, we can take unit vectors $\{\xi_x \mid x \in G\} \subset \mathcal{H} := \oplus_{v\in V}\mathcal{H}_v$ such that $\xi_x \in \mathcal{H}_v$ whenever $x \in X_v$ and such that

$$\sup\{\|\xi_y - \xi_{y'}\| : d(y, y') \leq n + 2s(Z + 2), y, y' \in X_v, v \in V\} \leq \frac{\epsilon}{2(R + 1)}, \tag{8.11}$$

$$\sup\{|\langle\xi_y, \xi_{y'}\rangle| : d(y, y') \geq S, y, y' \in X_v, v \in V\} \leq \frac{1}{2}. \tag{8.12}$$

Finally, for every s-chain $\mathbf{x} = (x_0, x_1, \ldots, x_{s-1})$, define the unit vector $\eta^{\mathbf{x}} \in \mathcal{H}$ by

$$\eta^{\mathbf{x}} = \sqrt{1/s}\sum_{i=0}^{s-1} \xi_{x_i}. \tag{8.13}$$

Our initial goal is to prove Proposition 8.3.9, namely that the vectors $\eta^{\mathbf{x}}$ satisfy properties similar to those of Proposition 8.2.1.

Lemma 8.3.6. *Let $\mathbf{x} = (x_0, x_1, \ldots, x_{s-1})$ and $\mathbf{x}' = (x'_0, x'_1, \ldots, x'_{s-1})$ be s-chains starting in X_v. If $d(x_0, x'_0) \leq n$, then $\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\| \leq \frac{\epsilon}{2(R+1)}$.*

Proof. We write

$$\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\| = \|\sqrt{1/s}\sum_{i=0}^{s-1}(\xi_{x_i} - \xi_{x'_i})\|. \tag{8.14}$$

Since by the triangle inequality $d(x_i, x'_i) \leq n + 2i(Z + 2) \leq n + 2s(Z + 2)$, we can bound (8.14) by

$$\sup_{0\leq i\leq s-1} \|\xi_{x_i} - \xi_{x'_i}\| \leq \sup_{\Delta} \|\xi_y - \xi_{y'}\| \leq \frac{\epsilon}{2(R + 1)},$$

where $\Delta = \{(y, y') \mid d(y, y') \leq n + 2s(Z + 2), y, y' \in X_v, v \in V\}$. □

Lemma 8.3.7. *Let* $\mathbf{x} = (x_1, \dots, x_s)$ *and* $\mathbf{x}' = (x_0, x'_1, x'_2, \dots, x'_{s-1})$ *be* s*-chains with* $x_0 \in X_v$ *and* $x_1 \in X_{\alpha(v)}$*. If* (x_0, x_1) *is a* 2*-chain and* $d(x_0, x_1) \leq n$*, then* $\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\| \leq \frac{\epsilon}{R+1}$*.*

Proof. Denote $\overline{x}_0 = f_v^{-1}(x_1)$ and set $\overline{\mathbf{x}} = (\overline{x}_0, x_1, \dots, x_{s-1})$. Then

$$\eta^{\mathbf{x}} = \eta^{(x_1, x_2, \dots, x_s)} = \sqrt{1/s}(\sum_{i=1}^{s-1} \xi_{x_i} + \xi_{x_s}),$$

$$\eta^{\overline{\mathbf{x}}} = \eta^{(\overline{x}_0, x_1, \dots, x_{s-1})} = \sqrt{1/s}(\xi_{\overline{x}_0} + \sum_{i=1}^{s-1} \xi_{x_i}).$$

Therefore,

$$\|\eta^{\overline{\mathbf{x}}} - \eta^{\mathbf{x}}\| = \sqrt{1/s}\|\xi_{\overline{x}_0} - \xi_{x_s}\| = \sqrt{2/s} \leq \frac{\epsilon}{2(R+1)},$$

where the final inequality comes from the choice of s in Equation (8.8). Since $d(x_0, \overline{x}_0) = d(x_0, x_1) - 1 \leq n$, we can apply Lemma 8.3.6 to the chains $\mathbf{x}'$ and $\overline{\mathbf{x}}$ to conclude that

$$\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\| \leq \|\eta^{\mathbf{x}} - \eta^{\overline{\mathbf{x}}}\| + \|\eta^{\overline{\mathbf{x}}} - \eta^{\mathbf{x}'}\| \leq \frac{\epsilon}{2(R+1)} + \frac{\epsilon}{2(R+1)} = \frac{\epsilon}{R+1}.$$

□

Lemma 8.3.8. *For any* 2 s*-chains* $\mathbf{x} = (x_0, x_1, \dots, x_{s-1})$ *and* $\mathbf{x}' = (x'_0, x'_1, \dots, x'_{s-1})$*, we have that* $|\langle \eta^{\mathbf{x}}, \eta^{\mathbf{x}'} \rangle|$ *is smaller than*

$$\sup\{\langle \xi_y, \xi_{y'} \rangle \mid: y, y' \in X_v, v \in T, d(y, y') \geq d(x_0, x'_0) - 2s(Z+2)\}.$$

Proof. Assume that $x_0 \in X_{\tilde{v}}$ and $x'_0 \in X_{v'}$. As before, denote (k, l) the unique pair of natural numbers such that $d_T(\tilde{v}, v') = k + l$ and $\alpha^k(\tilde{v}) = \alpha^l(v')$. By symmetry, we will assume that $k \geq l$. Further, we will assume that $k < s$, because $k \geq s$ implies that $\langle \eta^{\mathbf{x}}, \eta^{\mathbf{x}'} \rangle = 0$. We obtain by definition that

$$\langle \eta^{\mathbf{x}}, \eta^{\mathbf{x}'} \rangle = (1/s) \sum_{i=0}^{s-k-1} \langle \xi_{x_{k+i}}, \xi_{x'_{l+i}} \rangle.$$

Notice that $d(x_{k+i}, x'_{l+i}) \geq d(x_0, x'_0) - (k+i)(Z+2) - (l+i)(Z+2) \geq d(x_0, x'_0) - 2s(Z+2)$, so

$$|\langle \eta^{\mathbf{x}}, \eta^{\mathbf{x}'} \rangle| \leq \sup_{\Omega} |\langle \xi_y, \xi'_y \rangle|,$$

where $\Omega = \{(y, y') : y, y' \in X_v, v \in V, d(y, y') \geq d(x_0, x'_0) - 2s(Z+2)\}$.

□

Proposition 8.3.9. *Given $R > 0$ and $\epsilon > 0$, let s, n and $(\xi_x)_{x \in G}$ be constructed as in Notation 8.3.5. For each $x_0 \in G$, choose and fix an s-chain $\mathbf{x} = (x_0, x_1, \ldots, x_{s-1})$ and consider the corresponding vector $\eta^{\mathbf{x}} = \eta^{(x_0, x_1, \ldots x_{s-1})}$. Then*

$$\sup\{\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\| : d(x_0, x'_0) \leq R\} \leq \epsilon, \tag{8.15}$$

and

$$|\langle \eta^{\mathbf{x}}, \eta^{\mathbf{x}'} \rangle| \leq \sup_{\Omega} \{|\langle \xi_y, \xi_{y'} \rangle|\}, \tag{8.16}$$

where $\Omega = \{(y, y') : y, y' \in X_v, v \in V, d(y, y') \geq d(x_0, x'_0) - 2s(Z+2)\}$.

Proof. Condition (8.16) was proven in Lemma 8.3.8. To prove (8.15), let us choose $x_0, x'_0 \in G$ such that $d(x_0, x'_0) \leq R$. Choose any two s-chains $\mathbf{x}$ and $\mathbf{x}'$ starting at x_0 and x'_0, respectively. We want to prove that $\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\| \leq \epsilon$.

Therefore, let k and l be as before. Take first any chains $(x_0, x_1, \ldots, x_k)$ and $(x'_0, x'_1, \ldots, x'_l)$. By Lemma 8.3.4, we have that

$$\max\{(\sup_{0 \leq i \leq k-1} d(x_i, x_{i+1})), (\sup_{0 \leq j \leq l-1} d(x'_j, x'_{j+1})), d(x_k, x'_l)\}$$

is smaller than $(Z+3)R \leq n$ (see Equation (8.8)). Consequently, we can apply Lemma 8.3.6 and Lemma 8.3.7 repeatedly to s-chains $\mathbf{x}(i)$ and $\mathbf{x}'(j)$ whose initial elements are $x_i, i = 0 \ldots k$ and $x'_j, j = 0 \ldots l$, respectively. We obtain

$$\begin{aligned}
\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\| &\leq \sum_{i=0}^{k-1} \|\eta^{\mathbf{x}(i)} - \eta^{\mathbf{x}(i+1)}\| + \|\eta^{\mathbf{x}(k)} - \eta^{\mathbf{x}'(l)}\| + \\
&\quad \sum_{j=0}^{l-1} \|\eta^{\mathbf{x}'(j)} - \eta^{\mathbf{x}'(j+1)}\| \\
&\leq \frac{(k+l+1)\epsilon}{R+1} \leq \epsilon.
\end{aligned}$$

□

We are now ready to conclude the proof of Theorem 8.3.2.

Proof of Theorem 8.3.2. Clearly $\alpha \leq \alpha_1$ since (H, d_{in}) embeds isometrically in (G, d).

Conversely, assume that $\alpha_1 > 0$ and fix any real number $0 < p < \alpha_1$. For each $m \in \mathbb{N}_0$, define $\epsilon_m = \frac{1}{m^{1/2+p}}$, $R_m = \sqrt{m}$ and define $n_m = m^{\frac{1}{2}+p}$ and $s_m = m^{2+5p}$. Clearly then

$$n_m \geq (Z+3)R_m, \ \sqrt{2/s_m} \leq \frac{\epsilon_m}{2(R_m+1)}, \tag{8.17}$$

whenever m is larger than some natural number r_p. Next, use Lemma 8.2.2 to find a collection of unit vectors $\{\xi_y \mid y \in H\}$ in some Hilbert space $\overline{\mathcal{H}}$ such that for m larger than some $\widetilde{r_p} \geq r_p$, we have

1. $\|\xi_y - \xi_{y'}\| \leq \frac{1}{m^{1+2p}}$ whenever $d(y, y') \leq m^{2+6p}$
2. $|\langle \xi_y, \xi_{y'} \rangle| \leq 1/2$ whenever $d(y, y') \geq S_m := m^{\frac{3+9p}{\alpha_1 - p}}$.

Since for m large enough, $\frac{1}{m^{1+2p}}$ is smaller than $\frac{\epsilon_m}{2(R_m+1)}$ and m^{2+6p} is greater than $n_m + 2s_m(Z+2)$, we see that for m large enough, i.e. larger than some $\delta(p) \geq \widetilde{r_p}$, Equations (8.8), (8.9) and (8.10) are satisfied.

For every $x_0, x_0' \in G$ and for every $m \in \mathbb{N}_0$ larger than $\delta(p)$, Proposition 8.3.9 gives vectors $(\eta_m^{\mathbf{x}})$, $(\eta_m^{\mathbf{x}'}) \in \mathcal{H} := \oplus_{v \in V} \overline{\mathcal{H}}$ where $\mathbf{x}$ and $\mathbf{x}'$ are s_m-chains starting in x_0 and x_0' respectively. Moreover,

$$\sup\{\|\eta_m^{\mathbf{x}} - \eta_m^{\mathbf{x}'}\| : d(x_0, x_0') \leq R_m\} \leq \epsilon_m$$

and

$$|\langle \eta_m^{\mathbf{x}}, \eta_m^{\mathbf{x}'} \rangle| \leq 1/2 \text{ whenever } d(x_0, x_0') \geq S_m + 2s_m(Z+2).$$

Denote $S_m' = S_m + 2s_m(Z+2)$. For m large enough, this is smaller than $m^{\frac{3+10p}{\alpha_1 - p}}$. From Lemma 8.2.3, we get that the compression of G is at least

$$\frac{\alpha_1 - p}{6 + 20p}.$$

Recalling that $p > 0$ can be taken arbitrarily small, we can let p go to 0 and obtain that the compression of G is at least $\alpha_1/6$. □

We obtain stronger bounds when G is quasi-geodesic. Using Lemma 8.2.4, we can deduce the following Corollary.

Corollary 8.3.10. *Assume that G is also a quasi-geodesic space. In the above proof, it then suffices to take $R_m = \ln(m)$. Passing through the proof exactly as above, we obtain the improved bound*

$$\alpha_1/3 \leq \alpha \leq \alpha_1.$$

The above Corollary is especially interesting when H and thus G are finitely generated and equipped with the word length metric relative to finite symmetric generating subsets.

Remark 8.3.11. *The demand that F and $\theta(F)$ should be of finite index in H can be replaced by the following statement:*

> *There exists $A \geq 0$ such that every right coset of F and $\theta(F)$ in (H, d_{in}) has a representative with length smaller than A.*

8.4 Group extensions

8.4.1 Extensions of uniformly embeddable groups

In this section, we investigate the behaviour of the Hilbert space compression under group extensions. We will always consider short exact sequences of the form

$$1 \to H \to \Gamma \xrightarrow{\pi} G \to 1.$$

If we want to predict the Hilbert space compression of Γ in terms of that of H and G, then clearly, we need to equip H and G with length functions that are somehow related to the length $l_\Gamma := l$ on Γ. To begin, we equip H with the induced length function l_H from Γ. We equip G with the length function l_G defined by

$$l_G(\pi(a)) = \inf\{l(b) \mid b \in \Gamma \text{ and } \pi(b) = \pi(a)\}, \tag{8.18}$$

for every $a \in \Gamma$. This length function is always well-defined if we are in one of the following two cases. The first case is when l_Γ is uniformly discrete, the second case is when all balls of finite radius are finite. In the sequel, we will assume that l_Γ is uniformly discrete, but all the proofs and results hold equally well for the second case.

Remark 8.4.1. *We start with an easy observation, namely that the compression of* Γ *is equal to the compression of* H *whenever* G *is finite: indeed, choose a set* R *of representatives for the right cosets of* H *in* Γ*. Then, given a uniform embedding* $f : H \to \mathcal{H}$*, and any element* $x = sc \in \Gamma$ *where* $c \in R$*, we define* $\overline{f} : \Gamma \to \mathcal{H}, sc \mapsto f(s)$*. This is a uniform embedding of* Γ *whose compression is equal to that of* f*. Similarly, it is easy to verify that the compression of* G *equals that of* Γ *whenever* H *is finite.*

In Theorem 4.1 of [39], Guentner and Dadarlat prove in the case of discrete countable groups, thus equipped with proper length functions by Definition 7.1.7, that uniform embeddability of H and the fact that G has property (A) in a short exact sequence as above implies uniform embeddability of Γ. For countable G, their methods generalize easily for *any uniformly discrete* length function on any discrete group Γ.

We give a short outline of their proof because the details will be important to us. Fixing $R > 0$ and $\epsilon > 0$, then by the fact that G has property (A) (see Definition 7.1.5), we can find a number $S^G > 0$ and a map $g : G \to l^2(G)$ such that $\|g(x)\| = 1$ for all $x \in G$ and such that

1. $|1 - \langle g(x), g(y)\rangle| \leq \frac{\epsilon}{2}$ provided $d_G(x, y) \leq R$;
2. the support of $g(x)$ lies in $B_{S^G}(x)$, for all $x \in G$.

Similarly, from the fact that H uniformly embeds into a Hilbert space, there exists a number $S^H > 0$ and a map $h : H \to \mathcal{H}$ such that $\|h(s)\| = 1$ for all $s \in H$ and such that

1. $|1 - \langle h(s), h(t)\rangle| \leq \frac{\epsilon}{2}$ provided $d_H(s, t) \leq 2S^G + R$;
2. $|\langle h(s), h(t)\rangle| \leq 1/2$ if $d_H(s, t) \geq S^H$.

Next, they define a map $f : \Gamma \to l^2(G, \mathcal{H})$ by

$$f(a)(x) = g(\pi(a), x)h(\sigma(x)^{-1}a\sigma(\pi(a)^{-1}x)), \ \forall a \in \Gamma, \ x \in G,$$

where $\sigma : G \to \Gamma$ is a set-theoretic section, i.e. $\pi \circ \sigma = Id_G$, such that $l_\Gamma(\sigma(x)) = l_G(x)$; (in general, such $\sigma(x)$ should not exist if balls of finite radius can be infinite, but then we can choose $r > 0$ very small, take σ such that $l_\Gamma(\sigma(x)) \leq l_G(x) + r$ for every $x \in G$ and basically continue their proof with this σ). Clearly, the $f(a) \in l^2(G, \mathcal{H})$ are vectors of norm 1. Moreover, it is shown that

1. $|1-\langle f(a), f(b)\rangle| \leq \epsilon$ whenever $d_\Gamma(a,b) \leq R$.
2. $|\langle f(a), f(b)\rangle| \leq \frac{1}{2}$ whenever $d_\Gamma(a,b) \geq 2S^G + S^H$.

In this Section, we will check the behaviour of compression under extensions as above, while putting conditions on G. First, we investigate the case that G is a group of polynomial growth. Afterwards, we consider the case that G is a finitely generated word-hyperbolic group. In both cases, the compression of G equals 1 and we will try to give bounds on the compression of Γ in terms of the compression of H. Our global idea is to quantify somehow *how well* the group G satisfies property (A). This is similar to compression, seen as a quantification of uniform embeddability. We will then use this quantification and the extension result mentioned above to find bounds on the Hilbert space compression of Γ.

8.4.2 Extensions by a group of polynomial growth

Let us start by the definition of a metric space with polynomial growth.

Definition 8.4.2. *A metric space X has polynomial growth if there exists a polynomial P such that $|\overline{B(x,R)}| \leq P(R)$ for every $x \in X$ and every $R \geq 0$. Here $\overline{B(x,R)}$ is the closed ball with radius R and center x.*

In Lemma 6.6 of [105], Tu proves that groups of polynomial growth have property (A). We use this as a starting point to quantify *how well* groups of polynomial growth satisfy property (A).

Lemma 8.4.3. *Let G be a group, equipped with a uniformly discrete length function, that has polynomial growth. Let $p \in]0,1[$ be any real number. There exists $n_0 \in \mathbb{N}$ such that for every natural number $n \geq n_0$, there exists a collection of unit vectors $(g_n(x))_{x\in G}$ in $l^2(G)$ such that $\|g_n(x)\|_2 = 1,\ \forall x \in G$ and*

1. *$|1-\langle g_n(x), g_n(y)\rangle| \leq \frac{1}{4n^{1+2p}}$ provided $d_G(x,y) \leq \sqrt{n}$,*
2. *$\mathrm{supp}(g_n(x)) \subset B(x, S^G_n)$ for all $x \in G$ where $S^G_n = n^{3/2+5p}$.*

Proof. For each $x \in G$ and $r \in \mathbb{R}$, denote by $B(x,r) \subset G$ the ball of radius r and center x. Denote the characteristic function of $B(x,r)$ by χ^r_x. We shall denote $B(1,r)$ simply by B_r and χ^r_1 by χ_r. For $n \in \mathbb{N}_0$, denote $R_n = \sqrt{n}$ and, with the convention that $\forall a \in \mathbb{R} : a/0 = \infty$, let $\overline{m_n}$ be the infimum of all real numbers r such that

$$\frac{|B_{r+R_n}|}{|B_{r-R_n}|} \leq 1 + \frac{1}{2n^{1+2p}}.$$

Clearly, such $\overline{m_n}$ exists, since if it didn't exist, then $\forall i \in \mathbb{N}_0$,

$$|B_{2iR_n+R_n}| \geq |B_{2(i-1)R_n+R_n}|(1 + \frac{1}{2n^{1+2p}}) \geq \ldots \geq |B_{R_n}|(1 + \frac{1}{2n^{1+2p}})^i,$$

obtaining a contradiction since the left hand side depends polynomially on i whereas the right hand side depends exponentially on i.

We claim that there exists $\overline{n} \in \mathbb{N}_0$ such that $\forall n \geq \overline{n} : \overline{m_n} \leq 2n^{3/2+4p}$. Assume therefore, that such $\overline{n}$ does not exist. Then there exists a strictly monotone increasing sequence $(n_i)_{i\in\mathbb{N}}$ such that

$$\forall i : \frac{|B_{2n_i^{3/2+4p}+R_{n_i}}|}{|B_{2n_i^{3/2+4p}-R_{n_i}}|} > 1 + \frac{1}{2n_i^{1+2p}}.$$

Denoting the integer part of a real number a by $[a]$ and assuming for the last inequality below that $\forall i : n_i^p \geq 2$, we obtain that

$$\begin{aligned} |B_{2n_i^{3/2+4p}+R_{n_i}}| &> (1 + \frac{1}{2n_i^{1+2p}})|B_{2n_i^{3/2+4p}-2R_{n_i}+R_{n_i}}| \\ &> (1 + \frac{1}{2n_i^{1+2p}})^2|B_{2n_i^{3/2+4p}-4R_{n_i}+R_{n_i}}| \\ &> \ldots \\ &> (1 + \frac{1}{2n_i^{1+2p}})^{[n_i^{1+4p}]}|B_{R_{n_i}}| \\ &> (1 + \frac{1}{2n_i^{1+2p}})^{n_i^{1+3p}}|B_{R_{n_i}}|. \end{aligned}$$

Since $\lim_{i\to\infty}(1 + \frac{1}{2n_i^{1+2p}})^{n_i^{1+2p}} = \exp(1/2)$, it is clear that the right hand side depends exponentially on n_i, whereas the left hand side depends polynomially on n_i. We obtain a contradiction.

Denote $m_n < 2n^{3/2+4p}n^{p/2}$ such that

$$\frac{|B_{m_n+R_n}|}{|B_{m_n-R_n}|} \leq 1 + \frac{1}{2n^{1+2p}}.$$

Consider now the functions $\chi_x^{m_n}$. They are elements of $l^1(G)$ such that $d_G(x,y) \leq \sqrt{n} = R_n$ implies

$$\begin{aligned}\frac{\|\chi_x^{m_n} - \chi_y^{m_n}\|_1}{\|\chi_x^{m_n}\|_1} &\leq \frac{|B(x, m_n + R_n)| - |B(x, m_n - R_n)|}{|B(x, m_n - R_n)|} \\ &= \frac{|B_{m_n+R_n}|}{|B_{m_n-R_n}|} - 1 \leq \frac{1}{2n^{1+2p}}.\end{aligned}$$

Moreover, the support of $\chi_x^{m_n}$ lies inside

$$\overline{B(x, m_n)} \subset B(x, n^{3/2+5p}),$$

whenever n is larger than some natural number $\overline{n_1}$. To conclude, take $n \geq n_0 := \max(\overline{n}, \overline{n_1})$ and define $g_n(x) = \sqrt{\frac{\chi_x^{m_n}}{\|\chi_{m_n}\|_1}}$. Clearly, these are elements of norm 1 in $l^2(G)$ that satisfy condition (2) of this lemma. To show that they also satisfy condition (1), take x, y such that $d_G(x,y) \leq R_n$. Then

$$\begin{aligned}\|g_n(x) - g_n(y)\|_2^2 &= \sum_{z\in X} |g_n(x)(z) - g_n(y)(z)|^2 \\ &\leq \sum_{z\in X} (|g_n(x)(z) - g_n(y)(z)| \cdot |g_n(x)(z) + g_n(y)(z)|) \\ &= \sum_{z\in X} |g_n(x)(z)^2 - g_n(y)(z)^2| \\ &= \frac{\|\chi_x^{m_n} - \chi_y^{m_n}\|_1}{\|\chi_{m_n}\|_1} \leq \frac{1}{2n^{1+2p}}.\end{aligned}$$

Therefore $|1 - \langle g_n(x), g_n(y)\rangle| \leq \frac{1}{4n^{1+2p}}$ as desired. □

Carefully modifying the proof above, we obtain the following corollary for $R_n = \ln(n)$ and $m_n = 2n^{1+4p}$.

Corollary 8.4.4. *Let G be a group, equipped with a uniformly discrete length function, that has polynomial growth. Let $p \in]0,1[$ be any real number. There exists $n_0 \in \mathbb{N}$ such that for every natural number $n \geq n_0$, there exists a collection of unit vectors $(g_n(x))_{x\in G}$ in $l^2(G)$ such that $\|g_n(x)\|_2 = 1$, $\forall x \in G$ and*

1. $|1 - \langle g_n(x), g_n(y)\rangle| \leq \frac{1}{4n^{1+2p}}$ *provided* $d_G(x,y) \leq \ln(n)$,

2. $\operatorname{supp}(g_n(x)) \subset B(x, S_n^G)$ *for all* $x \in G$ *where* $S_n^G = n^{1+5p}$.

Theorem 8.4.5. *Assume that* Γ *is a group, equipped with a uniformly discrete length function* $l = l_\Gamma$, *that fits in a short exact sequence*

$$1 \to H \to \Gamma \xrightarrow{\pi} G \to 1.$$

If G *with the induced metric from* Γ *(Equation (8.18)) has polynomial growth and if* H *with the induced metric from* Γ *has Hilbert space compression* α_1, *then*

$$\alpha_1/4 \leq \alpha(\Gamma) \leq \alpha_1.$$

Proof. Denote the Hilbert space compression of H by $\alpha_1 > 0$ and choose $0 < p < \alpha_1$. Take $S_n^G = n^{3/2+5p}$ as in Lemma 8.4.3. For n sufficiently large, Lemma 8.4.3 provides maps $g_n : G \to l^2(G)$ such that $\|g_n(x)\|_2 = 1,\ \forall x \in G$ and such that

- $|1 - \langle g_n(x), g_n(y)\rangle| \leq \frac{1}{4n^{1+2p}}$ provided $d_G(x,y) \leq \sqrt{n}$,
- $\operatorname{supp}(g_n(x)) \subset B(x, S_n^G)$ for all $x \in G$ where $S_n^G = n^{3/2+5p}$.

If n is large enough, then $n^{3/2+6p} \geq 2S_n^G + \sqrt{n} = 2n^{3/2+5p} + \sqrt{n}$. For n *sufficiently large*, Lemma 8.2.2 applied for $r = 3/2 + 6p, a = \sqrt{2}, b = 1/2 + p$ gives a Hilbert space $\mathcal{H}$ and maps $h_n : H \to \mathcal{H}$ such that $\|h_n(s)\| = 1\ \forall s \in H$ and

- $|1 - \langle h_n(s), h_n(\tilde{s})\rangle| \leq \frac{1}{4n^{1+2p}}$ provided $d_H(s,\tilde{s}) \leq 2S_n^G + \sqrt{n}$
- $|\langle h_n(s), h_n(\tilde{s})\rangle| \leq \frac{1}{2}$ whenever $d(s,\tilde{s}) \geq S_n^H := n^{\frac{2+8p}{\alpha_1 - p}}$.

In the proof of Theorem 4.1 in [39], Guentner and Kaminker fix n and use the maps g_n and h_n to construct a map $f_n : \Gamma \to l^2(G, \mathcal{H})$ such that $\|f_n(a)\| = 1,\ \forall a \in \Gamma$ and

- $|1 - \langle f_n(a), f_n(b)\rangle| \leq \frac{1}{2n^{1+2p}}$ if $d(a,b) \leq \sqrt{n}$,
- $|\langle f_n(a), f_n(b)\rangle| \leq \frac{1}{2}$ if $d(a,b) \geq 2S_n^G + S_n^H$.

Denoting $\overline{S_n} = n^p S_n^H$, and because $S_n^H \geq S_n^G$, we obtain for n larger than some $n_1 \in \mathbb{N}_0$ that

- $\|f_n(a) - f_n(b)\| \leq \frac{1}{n^{1/2+p}}$ if $d(a,b) \leq \sqrt{n}$,
- $\|f_n(a) - f_n(b)\| \geq 1$ if $d(a,b) \geq \overline{S_n}$.

Applying Lemma 8.2.3 and letting p go to 0, we conclude that the compression of Γ is greater or equal to $\frac{\alpha_1}{4}$. □

We get an improved bound in the case that Γ is quasi-geodesic (Definition 7.2.1). In particular, the following corollary is valid when Γ is a finitely generated group equipped with the word length metric relative to a finite symmetric generating subset.

Corollary 8.4.6. *Let* (Γ, l) *be a group, equipped with a uniformly discrete length function* $l = l_\Gamma$*, which is quasi-geodesic as a metric space. Assume that* Γ *fits in a short exact sequence*

$$1 \to H \to \Gamma \xrightarrow{\pi} G \to 1.$$

If G *with the induced metric from* Γ *(Equation (8.18)) has polynomial growth and if* H *with the induced metric from* Γ *has Hilbert space compression* α_1*, then*

$$\alpha_1/3 \leq \alpha(\Gamma) \leq \alpha_1.$$

Proof. In the proof of Theorem 8.4.5, take $S_n^G = n^{1+5p}$ as in Corollary 8.4.4. For n sufficiently large, Corollary 8.4.4 provides maps $g_n : G \to l^2(G)$ such that $\|g_n(x)\|_2 = 1,\ \forall x \in G$ and such that

- $|1 - \langle g_n(x), g_n(y)\rangle| \leq \frac{1}{4n^{1+2p}}$ provided $d_G(x,y) \leq \ln(n)$,
- $\operatorname{supp}(g_n(x)) \subset B(x, S_n^G)$ for all $x \in G$ where $S_n^G = n^{1+5p}$.

If n is large enough, then $n^{1+6p} \geq 2S_n^G + \ln(n)$. For n *sufficiently large*, Lemma 8.2.2 applied for $r = 1 + 6p, a = \sqrt{2}, b = 1/2 + p$ gives a Hilbert space $\mathcal{H}$ and maps $h_n : H \to \mathcal{H}$ such that $\|h_n(s)\| = 1\ \forall s \in H$ and

- $|1 - \langle h_n(s), h_n(\tilde{s})\rangle| \leq \frac{1}{4n^{1+2p}}$ provided $d_H(s,\tilde{s}) \leq 2S_n^G + \ln(n)$
- $|\langle h_n(s), h_n(\tilde{s})\rangle| \leq \frac{1}{2}$ whenever $d(s,\tilde{s}) \geq S_n^H := n^{\frac{(3/2)+8p}{\alpha_1 - p}}$.

For n sufficiently large and denoting $\overline{S_n} = n^p S_n^H$, we obtain maps $f_n : \Gamma \to l^2(G, \mathcal{H})$ such that $\|f_n(a)\| = 1,\ \forall a \in \Gamma$ and

- $\|f_n(a) - f_n(b)\| \leq \frac{1}{n^{1/2+p}}$ if $d(a,b) \leq \ln(n)$,
- $\|f_n(a) - f_n(b)\| \geq 1$ if $d(a,b) \geq \overline{S_n}$.

By Lemma 8.2.4, we get that the compression of Γ is at least $\frac{\alpha_1}{3}$. $\square$

8.4.3 Extensions by word-hyperbolic groups

Background on word-hyperbolic groups

In this Section, we investigate the behaviour of the Hilbert space compression under extensions by a word-hyperbolic group. Such groups G are finitely generated and equipped with the word length metric relative to a finite symmetric generating subset. Let us start with some background on word-hyperbolic groups, see [53] and [65].

Definition 8.4.7. *If A is a subset of some metric space (X, d) and $r \in \mathbb{R}^+$, then we denote*

$$B_r(A) = \{x \in X \mid d(x, A) \leq r\},$$

and we call it the r-neighbourhood *of A. The* Hausdorff distance *between subsets A and B of X is defined as*

$$\inf\{r > 0 \mid A \subset B_r(B) \text{ and } B \subset B_r(A)\},$$

if this expression makes sense. If not, then we define it by $+\infty$.

Definition 8.4.8. *Let G be a finitely generated group and fix a finite symmetric generating subset S. Denote its Cayley graph relative to S, viewed as a simplicial complex where edges have length 1, by C. Given $x, y \in G$, a geodesic between x and y is an isometry $g : [0, n] \to C$, for some $n \in \mathbb{N}$, such that $g(0) = x$ and $g(n) = y$. A geodesic triangle in C is a subset $\Delta(x, y, z)$ with $x, y, z \in G$ which is the union $[x, y] \cup [y, z] \cup [z, x]$ of three geodesic "sides", where the notation $[x, y]$ refers to the image of any geodesic connecting x and y. A group G is called word-hyperbolic if there exists some real number $\delta > 0$ such that for every geodesic triangle in C, every one of its sides is at Hausdorff distance $\leq \delta$ from the union of the other two sides.*

Remark 8.4.9. *The term* word-hyperbolic *group is derived from the fact that for $\delta = 2$, any geodesic triangle in the hyperbolic plane is contained in the δ-neighbourhood of the union of the other two sides. The same is of course not true for Euclidean space $\mathbb{E}^2$.*

It must be noted that Definition 8.4.8 is independent of the chosen finite symmetric generating subset. Indeed, word-hyperbolicity is a quasi-isometric invariant (see e.g. Theorem 12 on page 88 in [53]). Let us give some examples.

All finite groups are word-hyperbolic, clearly. The same is true for all virtually cyclic groups. The most classical examples of word-hyperbolic groups are the free groups F_n, on n generators. The corresponding Cayley graph is then a tree and so any side of a geodesic triangle is contained in the union of the two other sides. It is not difficult to find groups which are not word-hyperbolic, e.g. any group which has Hilbert space compression different from 1 is not word-hyperbolic. The group $\mathbb{Z} \oplus \mathbb{Z}$ is easily seen not to be word-hyperbolic.

The class of word-hyperbolic groups satisfies some nice properties, definitely in the field of combinatorial group theory. Specifically, they are all finitely presented and have solvable word and conjugacy problem. They also have solvable isomorphism problem, i.e. there is an algorithm that takes as input two presentations of word-hyperbolic groups, and which decides whether these groups are isomorphic or not [41]. Also, they satisfy some strong form of automaticity, i.e. they are strongly geodesically automatic. Finally, there is a combinatorial characterization of hyperbolic groups which is quite fascinating and which allows one to generalize the notion of hyperbolic group to the notion of *relatively hyperbolic group*, see Section 8.5 and [86].

In our work, we will briefly make use of the *Gromov boundary* of a word-hyperbolic group.

Definition 8.4.10. *Let G be a word-hyperbolic group. Fix a finite symmetric generating subset S of G and denote the Cayley-graph of G, relative to S, by C. A (geodesic) ray is an isometry $g : \mathbb{R}^+ \to C$. Two rays g_1, g_2 are called* equivalent *if the images of g_1 and g_2, viewed as subsets of C, lie at bounded Hausdorff distance from each other. The* Gromov boundary *of G is the set of equivalence classes of geodesic rays of C.*

Main result

It is our interest to investigate the behaviour of the Hilbert space compression under extensions by (infinite) word-hyperbolic groups. Our strategy is similar as in the case of groups with polynomial growth: we will *quantify* Property (A) for word-hyperbolic groups. To this end, we prove Lemma 8.4.11, based on Tu's proof (i.e. Proposition 8.1 in [105]) that word-hyperbolic groups have Property (A).

Lemma 8.4.11. *Let G be an infinite finitely generated word-hyperbolic group, equipped with the word length relative to a finite symmetric generating subset. Let $p \in]0,1[$ be any real number. There exists $n_0 \in \mathbb{N}$ such that for every natural number $n \geq n_0$, there exists a collection of unit vectors $(g_n(x))_{x\in G}$ in $l^2(G)$ such that $\|g_n(x)\| = 1$, $\forall x \in G$ and*

- $|1 - \langle g_n(x), g_n(y)\rangle| \leq \frac{1}{4n^{1+2p}}$ *provided* $d_G(x,y) \leq \sqrt{n}$,
- *supp*$(g_n(x)) \subset B(x, S_n^G)$ *for all* $x \in G$ *where* $S_n^G = n^{3+6p}$.

Proof. Choose $a \in \partial G$, the Gromov boundary of G. For all $x \in G$, let $[[x, a[[$ be the set of infinite geodesics that belong to a and such that $g(0) = x$. For every $x \in G$ and $k, n \in \mathbb{N}_0$, we define elements of $l^1(G)$ as follows:

$$F(x,k,n) = \text{ characteristic function of } \bigcup_{\substack{d(x,y)<k \\ g\in[[y,a[[}} g([n,2n]),$$

$$H(x,n) = \frac{1}{n^{3/2}} \sum_{k<\sqrt{n}} F(x,k,n).$$

It follows easily from these definitions that $\|F(x,k,n)\|_1 \geq n$ for every $k, n \in \mathbb{N}_0$. Consequently, we have that $\|H(x,n)\|_1 \geq 1$. Also, it is clear that the support of $H(x,n)$ is inside $B(x, \sqrt{n} + 2n)$. In the proof of Proposition 8.1 in [105], Tu shows that there is a constant $C > 0$ such that for every $R > 0$:

$$\|H(x,n) - H(y,n)\|_{l^1(G)} \leq \frac{2C(R+1)}{n^{1/2}}, \text{ whenever } d_G(x,y) \leq R. \quad (8.19)$$

Let $m_n = n \cdot n^{2+5p} = n^{3+5p}$ play the role of n. Then we have constants $C, D > 0$ such that the $H(x, m_n)_{x\in G, n\in\mathbb{N}_0}$ satisfy the following conditions for all n greater than some natural number n_0:

1. $\|H(x,m_n)\|_1 \geq 1$
2. $\|H(x,m_n)-H(y,m_n)\|_1 \leq \frac{C\sqrt{n}+D}{n^{(2+5p)1/2}\sqrt{n}} \leq \frac{1}{2n^{1+2p}}$ provided $d_G(x,y) \leq \sqrt{n}$
3. $\mathrm{supp}(H(x,m_n)) \subset B(x,n^{3+6p})$,

where in (3), we use the fact that for n sufficiently large: $2m_n + \sqrt{m_n} \leq n^{3+6p}$. For all $x \in G, n \geq n_0$, set $g_n(x) = \sqrt{\frac{H(x,m_n)}{\|H(1,m_n)\|_1}}$ to obtain a collection of elements of $l^2(G)$. Calculating as in the end of the proof of Lemma 8.4.3, we obtain

$$\begin{aligned} \|g_n(x)-g_n(y)\|_2^2 &\leq \frac{\|H(x,m_n)-H(y,m_n)\|_1}{\|H(1,m_n)\|_1} \\ &\leq \|H(x,m_n)-H(y,m_n)\|_1. \end{aligned}$$

The $g_n(x)$ with $n \geq n_0$ satisfy the conditions of this Lemma. □

Remark 8.4.12. *If we take $m_n = n^{2+5p}$, then the proof shows that there exists $n_0 \in \mathbb{N}$ such that for every natural number $n \geq n_0$, there exists a collection of unit vectors $(g_n(x))_{x\in G}$ in $l^2(G)$ such that $\|g_n(x)\| = 1,\ \forall x \in G$ and*

- $|1-\langle g_n(x), g_n(y)\rangle| \leq \frac{1}{4n^{1+2p}}$ *provided* $d_G(x,y) \leq \ln(n)$,
- *supp*$(g_n(x)) \subset B(x,S_n^G)$ *for all* $x \in G$ *where* $S_n^G = n^{2+6p}$.

Remark 8.4.13. *If we equip G with a metric d_G which is quasi-isometric to the word length function d, then we obtain similar results: simply replace m_n by $m_n n^{2p}$ in the proof of Lemma 8.4.11 to obtain vectors satisfying*

- $|1-\langle g_n(x), g_n(y)\rangle| \leq \frac{1}{4n^{1+2p}}$ *provided* $d(x,y) \leq \sqrt{n}n^p$,
- *supp*$(g_n(x)) \subset B_d(x,S_n^G)$ *for all* $x \in G$ *where* $S_n^G = n^{3+8p}$.

Next, by quasi-isometry of the length functions, we have obtained for every n sufficiently large a collection of unit vectors $g_n(x) \in l^2(G)$ such that

- $|1-\langle g_n(x), g_n(y)\rangle| \leq \frac{1}{4n^{1+2p}}$ *provided* $d_G(x,y) \leq \sqrt{n}$,

- *supp*$(g_n(x)) \subset B_{d_G}(x, S_n^G)$ *for all* $x \in G$ *where* $S_n^G = n^{3+9p}$.

Theorem 8.4.14. *Let* Γ *be a discrete group, equipped with a uniformly discrete length function* $l = l_\Gamma$. *Assume that it fits in a short exact sequence*

$$1 \to H \to \Gamma \xrightarrow{\pi} G \to 1,$$

where G *is a finitely generated word-hyperbolic group. Assume that the metric on* G *is quasi-isometric to the induced metric from* Γ *(Equation (8.18)). If* H*, with the induced metric from* Γ*, has Hilbert space compression* α_1*, then the Hilbert space compression of* Γ *is at least* $\frac{\alpha_1}{7}$.

Proof. Recall from Remark 8.4.13 that, for n large enough, we have proven the existence of maps $g_n : G \to l^2(G)$ such that $\|g_n(x)\|_2 = 1,\ \forall x \in G$ and such that

- $|1 - \langle g_n(x), g_n(y)\rangle| \leq \frac{1}{4n^{1+2p}}$ provided $d_G(x, y) \leq \sqrt{n}$,
- $\text{supp}(g_n(x)) \subset B(x, S_n^G)$ for all $x \in G$ where $S_n^G = n^{3+9p}$.

Use Lemma 8.2.2 for $r = 3 + 10p, a = \sqrt{2}, b = (1/2) + p$, to find a Hilbert space $\mathcal{H}$ and unit vectors $(h_n(s))_{s\in H} \in \mathcal{H}$ for every n large enough such that

- $|1 - \langle h_n(s), h_n(\tilde{s})\rangle| \leq \frac{1}{4n^{1+2p}}$ provided $d_H(s, \tilde{s}) \leq 2S_n^G + \sqrt{n}$
- $|\langle h_n(s), h_n(\tilde{s})\rangle| \leq \frac{1}{2}$ whenever $d(s, \tilde{s}) \geq S_n^H := n^{\frac{(7/2)+12p}{\alpha_1 - p}}$.

Similarly as in the proof of Theorem 8.4.5, we obtain for every $n \in \mathbb{N}$ larger than some $n_1 \in \mathbb{N}$, a map $f_n : \Gamma \to l^2(G, \mathcal{H})$ such that $\|f_n(a)\| = 1,\ \forall a \in \Gamma$ and

- $\|f_n(a) - f_n(b)\| \leq \frac{1}{n^{1/2+p}}$ if $d(a, b) \leq \sqrt{n}$,
- $\|f_n(a) - f_n(b)\| \geq 1$ if $d(a, b) \geq \overline{S_n} := 2S_n^G + S_n^H$.

Clearly, $2S_n^G + S_n^H \leq n^p S_n^H \leq n^{\frac{(7/2)+13p}{\alpha_1 - p}}$ for n sufficiently large. Using Lemma 8.2.3 and letting p go to 0, we conclude that the compression of Γ as at least $\frac{\alpha_1}{7}$. ☐

As always, something special happens in the case that Γ is quasi-geodesic. More concretely, we obtain the following result for finitely generated groups.

Corollary 8.4.15. *Assume that* Γ *is a finitely generated group, equipped with the word length function* $l = l_\Gamma$ *relative to a finite symmetric generating subset* S *and that it fits in a short exact sequence*

$$1 \to H \to \Gamma \xrightarrow{\pi} G \to 1.$$

Equip G *with the word length function* l_G *relative to* $\pi(S)$*. If* G *is a finitely generated word-hyperbolic group in the sense of Gromov [53] and if* H*, with the induced metric from* Γ*, has Hilbert space compression* α_1*, then the Hilbert space compression of* Γ *is at least* $\alpha_1/5$*.*

Proof. Use Remark 8.4.12 to obtain $S_n^G = n^{2+6p}$ and thus $r = 2 + 7p$ in the proof of Theorem 8.4.14. ☐

Alain Valette pointed out that a stronger result is valid in the following special case.

Theorem 8.4.16. *Let* A *and* G *be finitely generated groups, each equipped with the word length metric relative to a finite symmetric generating subset. Assume that* A *is abelian, that* G *is word-hyperbolic and that*

$$0 \to A \to \Gamma \xrightarrow{\pi} G \to 1,$$

is a central extension. The compression of Γ*, equipped with the word length metric relative to a finite symmetric generating subset, equals* 1*.*

Proof. Denote the second bounded cohomology group of G, defined using bounded cocycles, by $H_b^2(G, A)$. By [83], the comparison map

$$H_b^2(G, A) \to H^2(G, A),$$

is onto for G word-hyperbolic, i.e. every 2-cocycle has a bounded representative.

Now, let $s : G \to \Gamma$ be a (set-theoretic) section, i.e. $\pi \circ s = Id_G$ and define

$$c(x, y) = s(xy)^{-1} s(x) s(y) \ \forall x, y \in G.$$

By the above, we can assume that c is bounded and so Gersten's result [52],[82] implies that Γ is quasi-isometric to $G \times A$. Consequently, the compression of Γ equals the minimum of the compressions of G and A [59], which is 1.

□

8.5 Remarks on relatively hyperbolic groups

After the results in Section 8.4 regarding extensions by hyperbolic groups, we now turn to a generalized notion of hyperbolicity. More precisely, we study the class of *relatively hyperbolic groups*. Let us start with an interesting combinatorial characterization of hyperbolic groups, due to Gromov, which lies at the basis of Osin's definition [86].

8.5.1 Isoperimetric inequality for word-hyperbolic groups.

Let Γ be a finitely presented group $\Gamma = \langle S \mid \mathcal{R}\rangle$, where S and $\mathcal{R}$ are finite. Consider the free product $F = F(S)$ with basis S, equipped with the word length metric $|\cdot|_S$ relative to S. The kernel of the natural projection $F \twoheadrightarrow \Gamma$ is the normal closure of $\mathcal{R}$, so every word w in F which represents the identity $1 \in \Gamma$ can be written as

$$w =_F \prod_{i=1}^{n} f_i^{-1} r_i^{\pm 1} f_i, \tag{8.20}$$

where $n \in \mathbb{N}$ and such that $r_i \in \mathcal{R}$ and $f_i \in F$ for every $i \leq n$. The smallest number n such that w can be written in the form (8.20) is called the *area of* w. The Dehn function $\mathrm{Dehn}(n)$ of the presentation $\langle S \mid \mathcal{R}\rangle$ is defined as follows:

$$\begin{array}{rccl} \mathrm{Dehn}: & \mathbb{N} & \rightarrow & \mathbb{N} \cup \{\infty\} \\ & n & \mapsto & \max\{\mathrm{area}(w) \mid w \in F \text{ represents } 1 \text{ in } \Gamma, |w|_S \leq n\}. \end{array}$$

Definition 8.5.1. *A finite presentation $\langle S \mid \mathcal{R}\rangle$ has linear isoperimetric inequality if there is a linear function f such that Dehn$(n) \leq f(n)$, for all $n \in \mathbb{N}$. Similarly, if we can take f to be quadratic, exponential,..., then $\langle S \mid \mathcal{R}\rangle$ has quadratic, exponential, ... isoperimetric inequality.*

Given a finitely presented group $\Gamma = \langle S \mid \mathcal{R} \rangle$, one can easily verify that the above definition is independent of the chosen generator system S. The property of having linear (quadractic, exponential, etc.) isoperimetric inequality does not depend on the chosen finite presentation of G.

Example 8.5.2 ([38]). *A finitely generated nilpotent group of nilpotency class c has isoperimetric inequality of degree at most $c+1$.*

The following result is due to Gromov [60].

Theorem 8.5.3 (Theorem 43 in [53]). *For a finitely presented group Γ, the following two conditions are equivalent:*

1. *Γ is word-hyperbolic*
2. *Γ has a linear isoperimetric inequality.*

Interestingly, there are no groups with isoperimetric inequality of degree d with $d \in]1, 2[$, but the set of real numbers $d \in [2, +\infty)$ such that there exists a finitely presented group with isoperimetric inequality of degree d forms a dense subset of $[2, +\infty)$ [20].

8.5.2 Relatively hyperbolic groups

Let Γ be a group and $(H_i)_{i\in I}$ a collection of subgroups of Γ. Assume that there exists a finite set S such that Γ is generated by $S \bigcup \cup_{i\in I} H_i$. Consider the free product $F = F(S) * (*_{i\in I} H_i)$, where $F(S)$ is the free group with basis S. If the kernel of the natural projection $F \twoheadrightarrow \Gamma$ is the normal closure of a finite set $\mathcal{R}$, then we say that Γ is finitely presented relative to $(H_i)_{i\in I}$ and we denote

$$\Gamma = \langle S, (H_i)_{i\in I} \mid \mathcal{R} \rangle.$$

Set

$$\mathcal{H} = \sqcup_{i\in I} (H_i \backslash \{1\}) \subset F.$$

Given a word w in the alphabet $S \cup \mathcal{H}$ representing $1 \in \Gamma$, there exists an expression

$$w =_F \prod_{i=1}^{k} f_i^{-1} r_i^{\pm 1} f_i \tag{8.21}$$

with the equality in the group F, where $r_i \in \mathcal{R}$ and $f_i \in F$ for $i = 1, 2, \ldots, k$. The smallest possible number k in a representation of the form (8.21) is called the *relative area* of w and is denoted by $area^{rel}(w)$.

Definition 8.5.4 ([86]). *A group Γ is hyperbolic relative to a collection of subgroups $(H_i)_{i\in I}$ if it is finitely presented relative to $(H_i)_{i\in I}$ and if there is a constant $L > 0$ such that for any word w in $S \cup \mathcal{H}$ representing the identity in Γ, we have $Area^{rel}(w) \leq L|w|$, where $|w|$ is the word length of $w \in F$ relative to $S \cup \mathcal{H}$.*

Example 8.5.5. *Clearly, all hyperbolic groups are hyperbolic relative to the trivial subgroup $\{1\}$. Another example is given by any $C'(1/6)$-small cancelation quotient of the free product of groups $X_1, X_2, \ldots, X_k$ (see [75]) relative to the natural images of the subgroups X_i in G. [86]*

The above definition does not require the group G and the subgroups H_i to be finite, as well as the collection $(H_i)_{i\in I}$ to be finite. The following result is Theorem 1.1 in [85].

Proposition 8.5.6. *Let Γ be a group, hyperbolic relative to a collection of subgroups $(H_i)_{i\in I}$. If Γ is generated by a finite set in the ordinary (non relative) sense and hyperbolic relative to a collection of subgroups $(H_i)_{i\in I}$, then I is finite and the groups H_i are finitely generated.*

There is a notion of *weak hyperbolicity relative to a collection of subgroups.* A group Γ is weakly hyperbolic relative to a collection of subgroups $(H_i)_{i\in I}$ if the Cayley graph of Γ with respect to the generating subset $S \cup \bigcup_{i\in I} H_i$ is hyperbolic, where S is a finite generating set of Γ modulo $(H_i)_{i\in I}$. Although relative hyperbolicity implies weak hyperbolicity, the converse is not true: Szczepański noted that the group $\mathbb{Z} \times \mathbb{Z}$ is weakly hyperbolic but not hyperbolic relative to the subgroup $\{(m, m) \mid m \in \mathbb{Z}\}$ [97].

8.5.3 Preliminaries

Let us give some definitions and formulate some results that we will need.

Definition 8.5.7. *A collection of subsets $(U_i)_{i\in I}$ of a metric space (X, d) is called s-separated whenever $d(U_i, U_j) > s$ for every 2 distinct elements $i, j \in I$.*

The following lemma is the quantification of part of a proof by Osin (Lemma 13 in [85]). There is not much difficulty in quantifying his proof, but it would be difficult for the interested reader to check everything himself. This is why we refer the reader to Subsection 8.5.5 for details. The details and the definitions introduced there, will not be used anywhere else in the text.

Lemma 8.5.8. *Suppose that* Γ *is a group which is generated by a finite set* S *and which is hyperbolic relative to a collection of subgroups* $H_1, H_2, \ldots, H_m$*. Denote the word length on* Γ *by* $|\cdot|_S$*. Choose* $i \in \{1, 2, \ldots, m\}$*,* $k \in \mathbb{N}_0$ *and let* $p > 0$ *be a real number. Denote* $B(k) = (S \bigcup \cup_{i=1}^m H_i)^{k-1} \subset \Gamma$ *and let* $R(k) \subset B(k)$ *denote a set containing exactly one representative for each left coset of* H_i *in* $B(k)H_i$ *and such that for all* $a \in R(k) : |a|_{S\cup\mathcal{H}}$ *is minimal. For* $s \in \mathbb{R}^+$*, denote* $T_s = \{x \in \Gamma \mid |x|_S \leq s^{2+p}\}$*. Then the family*

$$(aH_i \backslash B(k)T_s)_{a\in R(k)}$$

is s*-separated whenever* s *is larger than some number* s_p *which depends on* p *(but not on* k*).*

We introduce the following notation.

Notation 8.5.9. *Let* X *be a metric space and let* $U \subset X$*. We denote*

$$U_r = \{x \in X \mid \ \exists y \in U : d(x, y) < r\}.$$

Lemma 8.5.10. *Let* X *be a metric space and* $U \subset X$*. Choose* $r > 0$*. Assume that there exists a collection of unit vectors* $(\xi_u)_{u\in U}$ *in some Hilbert space* $\mathcal{H}$ *and numbers* $R, \epsilon, S > 0$ *such that*

- $\|\xi_u - \xi_v\| \leq \epsilon$ *provided* $d(u, v) \leq R$*,*
- $\|\xi_u - \xi_v\| \geq 1$ *provided* $d(u, v) \geq S$*,*

then this collection of unit vectors can be extended to a family $(\xi_u)_{u\in U_r} \subset \mathcal{H}$ *satisfying*

- $\|\xi_u - \xi_v\| \leq \epsilon$ *provided* $d(u, v) \leq R - 2r$*,*
- $\|\xi_u - \xi_v\| \geq 1$ *provided* $d(u, v) \geq S + 2r$*.*

Proof. For any $u \in U_r \backslash U$, choose a point $\overline{u} \in U$ at distance smaller than r from u and set $\xi_u := \xi_{\overline{u}}$. One easily verifies that the so defined vectors satisfy the conditions of this Lemma. ☐

The following definitions and results come from an article by Dadarlat and Guentner (see [40]). Let $\mathcal{U}$ be a cover of a metric space X by subsets of X. If $L > 0$ is a number such that any subset $B \subset X$ with diameter less than L is contained in some $U \in \mathcal{U}$, then L is called a Lebesque number for $\mathcal{U}$. If every $x \in X$ is contained in at most k sets $U \in \mathcal{U}$, then $\mathcal{U}$ is said to have multiplicity at most k.

A partition of unity on X is a family of maps $(\phi_i)_{i\in I}$ with $\phi_i : X \to [0, 1]$ and such that $\sum_{i\in I} \phi_i(x) = 1$ for all $x \in X$. Given $x \in X$, we do not require that the set $\{i \in I \mid \phi_i(x) \neq 0\}$ is finite. Finally, we say that $(\phi_i)_{i\in I}$ is subordinated to a cover $\mathcal{U} = (U_i)_{i\in I}$ if each ϕ_i vanishes outside U_i. The following proposition is Proposition 4.1 from [40].

Proposition 8.5.11. *Let $\mathcal{U}$ be a cover of a metric space X with multiplicity at most $k + 1 \geq 1$, and Lebesque number $L > 0$. There is a partition of unity $(\phi_U)_{U\in\mathcal{U}}$ subordinated to $\mathcal{U}$ satisfying*

$$\sum_{U\in\mathcal{U}} |\phi_U(x) - \phi_U(y)| \leq \frac{(2k + 2)(2k + 3)}{L}\, d(x, y),\ \forall x, y \in X.$$

The following Theorem follows directly from the proof of Theorem 3.2 of [40]. The idea is to prove uniform embeddability of a metric space (X, d) by using covers of X consisting of uniformly embeddable subsets.

Theorem 8.5.12. *Let X be a metric space and let $R > 0$ and $\epsilon > 0$ be real numbers. Assume that there exist a cover $\mathcal{U} = (U_i)_{i\in I}$ of X and a partition of unity $(\phi_i)_{i\in I}$ subordinated to $\mathcal{U}$ such that*

$$\forall x, y \in X : d(x, y) \leq R \Rightarrow \sum_{i\in I} |\phi_i(x) - \phi_i(y)| \leq \frac{\epsilon^2}{4}.$$

Assume further that there exists $S > 0$ and that for all $i \in I$ there is a family of unit vectors $(\xi^i_x)_{x\in(U_i)_R}$ in a Hilbert space $\mathcal{H}_i$ satisfying

- $\|\xi^i_x - \xi^i_y\| \leq \epsilon/2$ *provided* $d(x, y) \leq R$,
- $\|\xi^i_x - \xi^i_y\| \geq 1$ *provided* $d(x, y) \geq S$.

Then there exist unit vectors $(\xi_x)_{x\in X}$ satisfying

- $\|\xi_x - \xi_y\| \leq \epsilon$ *provided* $d(x, y) \leq R$,
- $\|\xi_x - \xi_y\| \geq 1$ *provided* $d(x, y) \geq S$.

8.5.4 Main result

Using methods of Dadarlat and Guentner [40], we prove the following result.

Theorem 8.5.13. *Let* Γ *be a finitely generated group, hyperbolic relative to a set of subgroups* $H_1, H_2, \ldots, H_m$, $m \in \mathbb{N}$. *Equip* Γ *with the word length metric relative to a finite symmetric generating subset* S *and assume that the* H_i *have strictly positive Hilbert space compression when equipped with the induced metric from* Γ. *The Hilbert space compression of* $(S \bigcup \cup_{i=1}^m H_i)^k$ *is strictly positive for any* $k > 0$. *Precisely, if* $\alpha_1 > 0$ *is a number such that all of the* H_i *have compression* $> \alpha_1$, *then* $\alpha((S \bigcup \cup_{i=1}^m H_i)^k) \geq \frac{\alpha_1}{11^k}$.

Proof. For each $k \in \mathbb{N}_0$, we denote $B(k) = (S \bigcup \cup_{i=1}^m H_i)^{k-1}$, where we set $(S \bigcup \cup_{i=1}^m H_i)^0 = \{1\}$. The result is proven by induction on $k \in \mathbb{N}_0$. Clearly, the result holds for $k = 1$.

Assume that $k > 1$. From now on, when we talk about vectors, we always mean elements of a *Hilbert* space. We denote the Hilbert space compression of $B(k-1)$ by $\alpha > 0$. Let $\delta > 0$ be the minimum of the Hilbert space compressions of the groups H_i ($i \in \{1, 2, \ldots m\}$). Choose $0 < p < \min(\delta, \alpha)$.

If n is large enough, then Lemma 8.2.2 gives unit vectors $(\xi_n^x)_{x \in B(k-1)}$ and a number S_n such that

1. $\|\xi_n^x - \xi_n^y\| \leq \frac{1}{4n^{1/2+p}}$ provided $d(x,y) \leq n^{5+29p}$
2. $\|\xi_n^x - \xi_n^y\| \geq 1$ provided $d(x,y) \geq S_n/n^p := n^{\frac{11/2+31p}{\alpha - p}}$.

If n is large enough, then $n^{5+29p} - 2n^{5+28p} = (n^p - 2)n^{5+28p} \geq n^{5+28p} \geq n^{3/2+5p}$ and $(S_n/n^p) + 2n^{5+28p} \leq 3S_n/n^p \leq S_n$. For every n large enough, Lemma 8.5.10 now provides a collection of unit vectors, again denoted ξ_n^x, but this time for every $x \in B(k-1)_{n^{5+28p}}$ such that

1. $\|\xi_n^x - \xi_n^y\| \leq \frac{1}{4n^{1/2+p}}$ provided $d(x,y) \leq n^{3/2+5p}$
2. $\|\xi_n^x - \xi_n^y\| \geq 1$ provided $d(x,y) \geq S_n$.

The same holds for the groups $(H_i)_{i=1,2,\ldots,m}$. Specifically, for every $i \in \{1, 2, \ldots, m\}$ and when n is large enough, Lemma 8.2.2 gives unit vectors $(\chi_n^x)_{x \in H_i}$ such that

1. $\|\chi_n^x - \chi_n^y\| \leq \frac{1}{4n^{1/2+p}}$ provided $d(x,y) \leq n^{5/2+10p}$
2. $\|\chi_n^x - \chi_n^y\| \geq 1$ provided $d(x,y) \geq n^{\frac{3+12p}{\delta-p}}$.

If n is large enough, then $n^{5/2+10p} - 2n^{5/2+9p} \geq n^{5/2+9p} \geq n^{3/2+5p}$ and $n^{\frac{3+12p}{\delta-p}} + 2n^{5/2+9p} \leq 3n^{\frac{3+12p}{\delta-p}} \leq n^{\frac{3+13p}{\delta-p}}$. Consequently, Lemma 8.5.10 provides a collection of unit vectors, again denoted χ_n^x, but this time for every $x \in (H_i)_{n^{5/2+9p}}$ such that

1. $\|\chi_n^x - \chi_n^y\| \leq \frac{1}{4n^{1/2+p}}$ provided $d(x,y) \leq n^{3/2+5p}$
2. $\|\chi_n^x - \chi_n^y\| \geq 1$ provided $d(x,y) \geq S_n' := n^{\frac{3+13p}{\delta-p}}$.

It is clear that

$$B(k) = (\bigcup_{i=1}^{m} B(k-1)H_i) \cup (\bigcup_{z\in S} B(k-1)z).$$

The sets $(B(k-1)z)_{z\in S}$ are easily analysed: they have the same compression as $B(k-1)$. It is more difficult to say something about the compression of the $B(k-1)H_i$. For any $i \in \{1,2,\ldots,m\}$, we now look for unit vectors on $B(k-1)H_i$ satisfying certain conditions. More precisely, unit vectors $(\eta_x^n)_{x\in B(k-1)H_i}$, for n large enough, such that

1. $\|\eta_n^x - \eta_n^y\| \leq \frac{1}{2n^{1/2+p}}$ provided $d(x,y) \leq n^{3/2+5p}$
2. $\|\eta_n^x - \eta_n^y\| \geq 1$ provided $d(x,y) \geq \widetilde{S_n}$,

for certain numbers $\widetilde{S_n}$. As a first step, we find explicit information on the corresponding numbers $\widetilde{S_n}$. As a second step, we prove the existence of unit vectors $(\zeta_n^x)_{x\in B(k)}$ for n large enough such that

1. $\|\zeta_n^x - \zeta_n^y\| \leq \frac{1}{n^{1/2+p}}$ provided $d(x,y) \leq \sqrt{n}$
2. $\|\zeta_n^x - \zeta_n^y\| \geq 1$ provided $d(x,y) \geq \widetilde{S_n}n^p$.

As a third step, we shall extract information from this regarding the compression of $B(k)$, showing the result.

Step 1: Let us fix $i \in \{1,2,\ldots,m\}$ and write $B(k-1)H_i = \sqcup_{g\in R(k-1)} gH_i$, where $R(k-1) \subset B(k-1)$ is a set containing exactly one representative for each left coset of H_i in $B(k-1)H_i$ and such that for all $g \in R(k-1)$: $|g|_{S\cup\mathcal{H}}$ is minimal. We denote the word length function on Γ, relative to S, by l and for all $s \in \mathbb{R}^+$, we define $T_s = \{x \in \Gamma \mid l(x) \leq s^{2+p}\}$. Take $s_n = n^{5/2+8p}$ and denote

$$\mathcal{U}_n = (B(k-1)T_{s_n})_{\frac{s_n}{2}} = (B(k-1))_{s_n^{2+p}+\frac{s_n}{2}}$$

and

$$\mathcal{U}^i_{n,a} = (aH_i \backslash B(k-1)T_{s_n})_{\frac{s_n}{2}} \text{ where } a \in R(k-1).$$

By Lemma 8.5.8, we see that the family $(aH_i\backslash B(k-1)T_{s_n})_{a\in R(k-1)}$ is s_n-separated whenever n is *large enough*. Consequently, the cover $\mathcal{V}^i_n := \{\mathcal{U}_n \cap B(k-1)H_i, \mathcal{U}^i_{n,a} \cap B(k-1)H_i \mid a \in R(k-1)\}$ of $B(k-1)H_i$ is such that every element $x \in B(k-1)H_i$ belongs to at most 2 elements of $\mathcal{V}^i_n$. Moreover, every subset of $B(k-1)H_i$ of diameter smaller than $s_n/2$ is contained in some element of $\mathcal{V}^i_n$. Applying Proposition 8.5.11 for $L = s_n/2$ and $k = 1$, we get a partition of unity $(\phi_U)_{U\in\mathcal{V}^i_n}$ on $B(k-1)H_i$ subordinated to $\mathcal{V}^i_n$, such that for n large enough

$$\sum_{U\in\mathcal{V}^i_n} |\phi_U(x) - \phi_U(y)| \leq \frac{4\cdot 5\cdot 2}{s_n}\, d(x,y) \leq \frac{1}{16n^{5/2+7p}}\, d(x,y) \leq \frac{1}{16n^{1+2p}},$$

whenever $d_S(x,y) \leq n^{3/2+5p}$.

Notice that $(\mathcal{U}_n)_{n^{3/2+5p}} \subset (B(k-1))_{n^{5+28p}}$ and that $(\mathcal{U}^i_{n,a})_{n^{3/2+5p}} \subset (aH_i)_{n^{5/2+9p}}$ for n large enough. From our conditions on the vectors ξ^x_n and χ^x_n and from Theorem 8.5.12, we obtain for every $i \in \{1,2,\dots,m\}$ and for every n large enough a collection of unit vectors $(\eta^i_{n,x})_{x\in B(k-1)H_i}$ satisfying

1. $\|\eta^i_{n,x} - \eta^i_{n,y}\| \leq \frac{1}{2n^{1/2+p}}$ provided $d(x,y) \leq n^{3/2+5p}$
2. $\|\eta^i_{n,x} - \eta^i_{n,y}\| \geq 1$ provided that $d(x,y) \geq \widetilde{S_n} := \max(S_n, S'_n) = \max(n^{\frac{3+13p}{\delta-p}}, n^{\frac{11/2+31p}{\alpha-p}+p})$.

Now, for $k \geq 3$, we have that $\alpha \leq \delta$ since $B(k-1)$ contains the groups H_i for every $i \in \{1,2,\dots,m\}$. Therefore, we get that

$$\widetilde{S_n} = \begin{cases} S_n = n^{\frac{11/2+31p}{\alpha-p}+p} & \text{if } k \geq 3; \\ \max(n^{\frac{3+13p}{\delta-p}}, n^{\frac{11/2+31p}{1-p}+p}) & \text{if } k = 2. \end{cases} \tag{8.22}$$

Step 2: Denote $|S| = \overline{m}$. The sets $(B(k-1)H_i)_{i\in\{1,2,\dots,m\}}$ together with the $(B(k-1)z)_{z\in S}$ cover $B(k)$. Denote this cover by $\mathcal{W} = \{\mathcal{W}_i \mid i = 1,2,\dots,m+\overline{m}\}$, where $\mathcal{W}_i = B(k-1)H_i$ for $i = 1,\dots,m$ and where the $(\mathcal{W}_i)_{i=m+1,\dots,m+\overline{m}}$ are the $(B(k-1)z)_{z\in S}$.

If n is large enough, then $n^{3/2+5p} - 2n^{3/2+4p} \geq n^{3/2+4p} \geq \sqrt{n}$ and $\widetilde{S_n} + 2n^{3/2+4p} \leq \widetilde{S_n}n^p$. For $i \in \{1,2,\dots,m\}$ and for n large enough, Lemma 8.5.10 provides a collection of unit vectors, again denoted $(\eta^i_{n,x})$ but this time for $x \in (\mathcal{W}_i)_{n^{3/2+4p}} \cap B(k)$ satisfying

1. $\|\eta^i_{n,x} - \eta^i_{n,y}\| \leq \frac{1}{2n^{1/2+p}}$ provided $d(x,y) \leq \sqrt{n}$
2. $\|\eta^i_{n,x} - \eta^i_{n,y}\| \geq 1$ provided $d(x,y) \geq \widetilde{S_n}n^p$.

Regarding the $(\mathcal{W}_i)_{i=m+1,m+2,\dots,m+\overline{m}}$ (i.e. the $(B(k-1)z)_{z\in S}$), note that Lemma 8.2.2 shows that for n large enough, there are unit vectors $(\gamma^x_n)_{x\in B(k-1)z}$ such that

1. $\|\gamma^x_n - \gamma^y_n\| \leq \frac{1}{2n^{1/2+p}}$ provided $d(x,y) \leq n^{3/2+5p}$
2. $\|\gamma^x_n - \gamma^y_n\| \geq 1$ provided $d(x,y) \geq n^{\frac{2+7p}{\alpha-p}}$.

If n is large enough, then $n^{3/2+5p} - 2n^{3/2+4p} \geq n^{3/2+4p} \geq \sqrt{n}$ and $n^{\frac{2+7p}{\alpha-p}} + 2n^{3/2+4p} \leq n^{\frac{2+7p}{\alpha-p}+p} \leq \widetilde{S_n} \leq \widetilde{S_n}n^p$. For each $i = m+1,\dots,m+\overline{m}$ and for each n large enough, Lemma 8.5.10 then provides unit vectors, denoted $(\eta^i_{n,x})$ but this time for $x \in (\mathcal{W}_i)_{n^{3/2+4p}} \cap B(k)$, such that

1. $\|\eta^i_{n,x} - \eta^i_{n,y}\| \leq \frac{1}{2n^{1/2+p}}$ provided $d(x,y) \leq \sqrt{n}$
2. $\|\eta^i_{n,x} - \eta^i_{n,y}\| \geq 1$ provided $d(x,y) \geq \widetilde{S_n}n^p$.

In $B(k)$, take the $n^{3/2+3p}$ neighbourhoods of the $(\mathcal{W}_i)_{i=1,2,\dots,m+\overline{m}}$, obtaining a new cover of $B(k)$ which we denote by $(\mathcal{T}^i_n)_{i=1,2,\dots,m+\overline{m}}$. This cover clearly has Lebesque number at least $n^{3/2+3p}$ and multiplicity at most $m+\overline{m}$. Applying Proposition 8.5.11, we obtain a partition of unity $(\phi_{\mathcal{T}^i_n})_{i=1,2,\dots,m+\overline{m}}$ relative to this cover such that

$$\sum_{i=1}^{m+\overline{m}} |\phi_{\mathcal{T}^i_n}(x) - \phi_{\mathcal{T}^i_n}(y)| \leq \frac{(2m+2\overline{m})(2m+2\overline{m}+1)}{n^{3/2+3p}} d(x,y) \leq \frac{1}{4n^{1+2p}},$$

whenever $d_X(x,y) \leq \sqrt{n}$ and n is large enough.

Denote $\mathcal{R}_n^i = (\mathcal{T}_n^i)_{\sqrt{n}} \cap B(k) \subset (\mathcal{W}_i)_{n^{3/2+3p}+\sqrt{n}} \subset (\mathcal{W}_i)_{n^{3/2+4p}}$. Then the conditions on the vectors $\eta_{n,x}^i$ here above, together with Theorem 8.5.12 provide a family of unit vectors $(\zeta_n^x)_{x\in B(k)}$ for each n large enough such that

1. $\|\zeta_n^x - \zeta_n^y\| \leq \frac{1}{n^{1/2+p}}$ provided $d(x,y) \leq \sqrt{n}$
2. $\|\zeta_n^x - \zeta_n^y\| \geq 1$ provided $d(x,y) \geq \widetilde{S_n} n^p$.

Step 3: Let us use the vectors ζ_n^x to find a lower bound on the compression of the $B(k)$. For $k = 2$, we get that $\widetilde{S_n} = \max(n^{\frac{3+13p}{\delta-p}}, n^{\frac{11/2+31p}{1-p}+p})$. From Lemma 8.2.3, where we let p go to 0, we conclude that the Hilbert space compression of $B(2)$ is at least $\min(\delta/6, 1/11)$ and so it is greater than $\delta/11$.

Using the value $\delta/11$ for the compression of $B(2)$, we get vectors $(\zeta_n^x)_{x\in B(3)}$ and numbers $\widetilde{S_n} n^p$ equal to $n^{\frac{11/2+31p}{\frac{\delta}{11}-p}+2p}$ (see Equation (8.22)). Again using Lemma 8.2.3 and letting p go to 0, one finds for $k = 3$ that the compression of $B(3)$ is at least $\frac{\delta}{11\cdot 11}$. Continuing in this manner, we find that the compression of $B(k)$ is at least $\frac{\delta}{11^{k-1}}$ and hence strictly greater than 0. ☐

8.5.5 Quantifying a result by Osin

For the readers convenience, we explain how one comes to the following result, which is Lemma 8.5.8. The main idea is to quantify the proof of Lemma 13 in [85].

Lemma 8.5.14. *Suppose that* Γ *is a group which is generated (in the ordinary, non relative sense) by a finite set* S *and which is hyperbolic relative to a collection of subgroups* $H_1, H_2, \ldots, H_m$. *Choose* $i \in \{1, 2, \ldots, m\}$, $k \in \mathbb{N}_0$ *and let* $p > 0$ *be a real number. Denote* $B(k) = (S \bigcup \cup_{i=1}^m H_i)^{k-1} \subset \Gamma$ *and let* $R(k) \subset B(k)$ *denote a set containing exactly one representative for each left coset of* H_i *in* $B(k)H_i$ *and such that for all* $a \in R(k)$: $|a|_{S\cup\mathcal{H}}$ *is minimal. For* $s \in \mathbb{R}^+$, *denote* $T_s = \{x \in \Gamma \mid |x|_S \leq s^{2+p}\}$. *Then the family*

$$(aH_i \backslash B(k)T_s)_{a\in R(k)}$$

is s*-separated whenever* s *is larger than some number* s_p *depending on* p *(but not on* k*).*

We will need to introduce some terminology [86]. Let Γ be a group which is generated by a finite set S and which is hyperbolic relative to a collection of subgroups $H_1, H_2, \ldots, H_m$. Recall that Γ is a quotient of the free group $F(S) * (*_{i\in I}\overline{H_i})$, where the $\overline{H_i}$ are isomorphic copies of the H_i and where $F(S)$ is the free group with basis S. Let us denote by $\mathcal{H}$ the disjoint union

$$\mathcal{H} = \bigsqcup_{i=1}^{m} (\overline{H_i} \backslash \{1\}).$$

We denote $(S \cup \mathcal{H})^*$ the free monoid generated by $S \cup \mathcal{H}$.

Given a group Γ generated by a symmetric set S, the Cayley graph $C(\Gamma, S)$ of Γ with respect to S is an oriented labelled 1-complex with the vertex set $V(C(\Gamma, S)) = \Gamma$ and the edge set $E(C(\Gamma, S)) = \Gamma \times S$. An edge $e = (x, z)$ goes from the vertex x to the vertex xz and has label $\mathrm{Lab}(e) = z$. As usual, we denote the origin and the terminus of the edge e, i.e., the vertices x and xz, by e_- and e_+ respectively. Given a combinatorial path $p = e_1e_2\ldots e_k$ in the Cayley graph $C(\Gamma, S)$, where $e_1, e_2, \ldots, e_k \in E(C(\Gamma, S))$, we denote by $\mathrm{Lab}(p)$ its label. By definition,

$$\mathrm{Lab}(p) \equiv \mathrm{Lab}(e_1)\mathrm{Lab}(e_2)\ldots\mathrm{Lab}(e_n),$$

where "$\equiv$" denotes letter for letter equality. We often denote $\phi(p) = \mathrm{Lab}(p)$, $p_- = (e_1)_-$ and $p_+ = (e_n)_+$.

To each element $x \in \Gamma$, we can associate its word length relative to S, denoted $|x|_S$, but also its *relative length with respect to the collection of subgroups* $H_1, H_2, \ldots, H_m$. We denote this length, i.e. the length of a shortest word in $(S \cup \mathcal{H})^*$ representing $x \in \Gamma$, by $|x|_{S\cup\mathcal{H}}$.

Definition 8.5.15 (H_i-subwords)**.** *Given a word $w \in (S \cup \mathcal{H})^*$, we say that a subword v of w is an H_i-subword if v consists of letters from $\overline{H_i}$. An H_i-subword of w is called an H_i-syllable if it is maximal, i.e., it is not contained in a bigger H_i-subword of w.*

Definition 8.5.16 (H_i-components)**.** *Let q be a path in the Cayley graph of Γ with respect to $S \cup \mathcal{H}$. A subpath p of q is called an H_i-subpath, if the label of p is an H_i-subword of the word $\phi(q)$. A component (or more precisely an H_i-component) of q is an H_i-subpath p such that the label of p is an H_i-syllable of the word $\phi(q)$.*

Definition 8.5.17 (Connected components)**.** *Two H_i-components p_1, p_2 of a path q in $C(\Gamma, S \cup \mathcal{H})$ are called* connected *if there exists a path $c \in C(\Gamma, S \cup \mathcal{H})$ that connects some vertex of p_1 to some vertex of p_2 and $\phi(c)$ is a word consisting of letters from H_i. In algebraic terms this means that these two vertices belong to the same coset xH_i.*

Definition 8.5.18 (Isolated components)**.** *An H_i-component p of a path q is called* isolated *if no (distinct) H_i-component is connected to p.*

Lemma 3.1 in [86] shows that the finite generating subset S of Γ can always be chosen such that the following condition is satisfied:

> There is a constant $M > 0$ such that for any cycle q in $C(\Gamma, S \cup \mathcal{H})$, i.e. a path starting and ending in the same point, for any $i = 1, 2, \ldots, m$ and for any set of isolated H_i components $p_1, p_2, \ldots, p_k$ of q, we have
>
> $$\sum_{i=1}^{k} d_S((p_i)_-, (p_i)_+) \leq Ml(q),$$
>
> where $l(q)$ is the number of letters in $\phi(q) \in (S \cup \mathcal{H})^*$.

We will always choose S satisfying this condition.

Definition 8.5.19. *Let Y be a metric space such that there is $\delta > 0$ such that every side of every geodesic triangle is contained in the union of the closed δ-neighbourhoods of the other two sides. Then, we will say that Y is δ-hyperbolic.*

In our context, Y will be the Cayley-graph of Γ with respect to $S \cup \mathcal{H}$ and equipped with the word length metric relative to $S \cup \mathcal{H}$ (see the remarks below Proposition 8.5.6).

Lemma 8.5.20 (Corollary 3.7 in [86])**.** *Let Y be a δ-hyperbolic space and let p and q be geodesic paths in Y such that $d(p_-, q_-) \leq s$ and $d(p_+, q_+) \leq s$, then p and q belong to the closed $(s + 2\delta)$-neighborhood of each other.*

We denote $\overline{S}(\delta, s) = s + 2\delta$ for every $s \in \mathbb{R}$.

Definition 8.5.21. *Let p be a path in $C(\Gamma, S \cup \mathcal{H})$, v a vertex of a component k of p. If $v \neq k_-$ and $v \neq k_+$, we say that v is an inner vertex of k. A vertex u of p is called* non-phase*, if u is an inner vertex of some component of p. All other vertices are called phase.*

Definition 8.5.22. *Two paths p, q in $C(\Gamma, S \cup \mathcal{H})$ are called s-similar whenever $d_S(p_-, q_-) \leq s$ and $d_S(p_+, q_+) \leq s$.*

The following Proposition has a long and technical proof. From Formula 3.16 on page 48 in [86], we can write it as follows.

Proposition 8.5.23. *Take δ such that $C(\Gamma, S \cup \mathcal{H})$ is a δ-hyperbolic space. For any $s \geq 0$, the following is true. Assume that p and q are two s-similar geodesic paths in $C(\Gamma, S \cup \mathcal{H})$ (geodesic always with respect to the relative metric $d_{S \cup \mathcal{H}}$). Then for any phase vertex u of p, there exists a phase vertex v of q such that*

$$dist_S(u, v) \leq 10(S')^2 M.$$

Here $S'(\delta, s) = \overline{S}(\delta, \overline{S}(\delta, s)) + 1/2$, i.e. $S' = s + 4\delta + 1/2$.

Notation 8.5.24. *For every $s \in \mathbb{R}$, we denote $\epsilon(\delta, s) = 10(S')^2 M$ where $S' = s + 4\delta + 1/2$.*

Lemma 8.5.25 (Lemma 3.21 of [86])**.** *For every $s \geq 0$, there exists $C(s)$ satisfying the following conditions. Let p and q be a pair of s-similar geodesics in $C(\Gamma, S \cup \mathcal{H})$. Then for any $i = 1, 2, \ldots, m$ and any H_i-component a of p satisfying the condition $d_S(a_-, a_+) > C(s)$, there exists an H_i-component b of q such that b is connected to a.*

Notation 8.5.26. *The proof of the above Lemma shows that $C := C_s$ can be taken as*

$$M(2 + 4\epsilon(\delta, s)).$$

Lemma 8.5.27 (Lemma 12 in [85])**.** *Let $p_1 = q_1 e_1, p_2 = q_2 e_2$ be two s-similar geodesics in $C(\Gamma, S \cup \mathcal{H})$. Suppose that for a certain $i \in \{1, 2, \ldots, m\}$, e_1 and e_2 are H_i-components satisfying the inequality*

$$d_S((e_i)_-, (e_i)_+) > \max\{C_s, 2M(s+1)\}.$$

Then e_1 and e_2 are connected.

Finally, we follow the proof from Lemma 13 in [85] to prove the desired Lemma 8.5.14.

Proof of Lemma 8.5.14. Given $s > 0$, set

$$\overline{T}_s = \{x \in \Gamma \mid |x|_S \leq \max\{C(s), 2M(s+1)\}\}.$$

We define

$$Y_s = B(k)\overline{T}_s.$$

Suppose that $x \in g_1H_i \backslash Y_s, y \in g_2H_i \backslash Y_s$ for different $g_1, g_2 \in R(k)$. Then, $x = g_1h_1, y = g_2h_2$ for some $h_1, h_2 \in H_i \backslash \overline{T}_s$. Let us show that $d_S(x, y) > s$, so assume by contradiction that $d_S(x, y) \leq s$. Let $A_i, i = 1, 2$, denote a shortest word in $S \cup \mathcal{H}$ representing g_i. Let also $p_i, i = 1, 2$, denote the path in $C(\Gamma, S \cup \mathcal{H})$ such that $(p_i)_- = 1$ and $\mathrm{Lab}(p_i) = A_ih_i$. Clearly, p_i is geodesic in $C(\Gamma, S \cup \mathcal{H})$. Indeed, otherwise we would have

$$|g_ih_i|_{S\cup\mathcal{H}} = d_{S\cup\mathcal{H}}((p_i)_-, (p_i)_+) < l(p_i) = \|A_i\| + 1 = |g_i|_{S\cup\mathcal{H}} + 1 \leq k$$

and hence $g_ih_i \in B(k) \subset Y_s$, which contradicts our assumption. Note also that

$$d_S((p_1)_+, (p_2)_+) = d_S(x, y) \leq s.$$

As $h_i \notin T_s$, we have $|h_i|_S > \max\{C(s), 2M(s+1)\}$ for $i = 1, 2$. By Lemma 8.5.27, the H_i-components of p_1 and p_2 labelled h_1 and h_2 respectively are connected. This means that $g_1H_i = g_2H_i$, a contradiction.

The proof now follows from the observation that for s larger than some number $r(p)$ depending on p, we have

$$C(s) = M(2 + 4[10M(s + 4\delta + 1/2)^2]) \leq s^{2+p},$$

hence

$$s^{2+p} \geq \max(C(s), 2M(s+1)),$$

and so T_s in the formulation of Lemma 8.5.14 contains $\overline{T}_s$. Since the

$$(aH_i \backslash B(k)\overline{T}_s)_{a\in R(k)}$$

are s-separated, the same is thus true for the

$$(aH_i \backslash B(k)T_s)_{a\in R(k)}.$$

□

8.6 The limit of a directed system of groups

Let $G_1 \overset{j_1}{\to} G_2 \overset{j_2}{\to} G_3 \overset{j_3}{\to} \ldots$ be a directed system of groups G_i, equipped with length functions l_i, such that the maps $G_i \to G_{i+1}$ are isometric group homomorphisms. Denote G the direct limit of this system. By definition, G is the disjoint union of all the G_i, divided by the following equivalence relation:

$$x \in G_k, y \in G_l \text{ are equivalent iff } x = y \text{ or } y = j_{l-1} \circ j_{l-2} \circ \ldots \circ j_k(x),$$

where we assume without loss of generality that $k \leq l$. We define the induced length function l on G by $l := \lim_{i\to\infty} l_i$, i.e.

$$\forall x \in G: \ l(x) := l_i(x),$$

where i is large enough such that $x \in G_i$. We proceed under the assumption that balls of finite radius in G contain finitely many elements. In this section, we ask how the Hilbert space compression of G, denoted by $\alpha(G)$, is related to the Hilbert space compressions of the G_i.

To begin, notice that every G_i can be seen as a metric subspace of G and so $\alpha(G) \leq \inf_{i\in\mathbb{N}} \alpha(G_i)$. Clearly, this bound is sharp, since as a family of subgroups we can take $G_i = G$ ($\forall i \in \mathbb{N}$). It proves more challenging to find a good lower bound for $\alpha(G)$. First, note that the same bound as above, i.e. $\inf_{i\in\mathbb{N}} \alpha(G_i)$, is not always a lower bound. As an example, equip the group

$$\mathbb{Z}^{(\mathbb{Z})} = \{f : \mathbb{Z} \to \mathbb{Z} \text{ with finite support}\} = \{(f, a) \in \mathbb{Z} \wr \mathbb{Z} \mid a = 0\}$$

with the induced word length metric from $\mathbb{Z} \wr \mathbb{Z}$. This group is obtained as the direct limit of the family of subgroups $G_n := \mathbb{Z}^{2n+1} = \{f : [-n, n] \to \mathbb{Z}\}$, where each $\mathbb{Z}^{2n+1}$ is equipped with the subspace metric from $\mathbb{Z}^{(\mathbb{Z})}$. Since this metric is quasi-isometric to the standard word length metric on $\mathbb{Z}^{2n+1}$, we obtain $\mathbb{Z}^{(\mathbb{Z})}$ as a limit of groups with compression 1. However, it follows from the proof of Theorem 3.9 in [7] that $\mathbb{Z}^{(\mathbb{Z})}$ has compression less than $\frac{3}{4}$.

The fact that $\mathbb{Z}$ and $\mathbb{Z}^{(\mathbb{Z})}$ have different compressions but are both limits of groups of compression 1 implies that there can not simply be a formula giving the compression of $G = \lim_{i\to\infty} G_i$ purely in terms of the compressions of the G_i. This is reflected in our result, where we feel the need to include more information on how the G_i are embedded in their respective Hilbert spaces.

We propose the following

Theorem 8.6.1. *Let G be the direct limit of a sequence $(G_i)_{i\in\mathbb{N}}$ of groups, equipped with the induced length function. Assume that balls of finite radius in G contain only finitely many elements. If $\inf_{i\in\mathbb{N}}(\alpha(G_i)) = 0$, then $\alpha(G) = 0$.*
Else, choose $0 < \alpha_1 < \inf_{i\in\mathbb{N}} \alpha(G_i)$ and choose for every $i \in \mathbb{N}$, a Hilbert space $\mathcal{H}_i$, constants $C_i > 0, \widetilde{C_i}, D_i, \widetilde{D_i} \geq 0$, but $\widetilde{C_i}, \widetilde{D_i}$ not both 0, and a map

$$f_i : G_i \to \mathcal{H}_i$$

satisfying

$$(1/C_i)\ d(x,y)^{\alpha_1} - D_i \leq d(f_i(x), f_i(y)) \leq \widetilde{C_i}\ d(x,y) + \widetilde{D_i}\ \forall x, y \in G_i.$$

Denote $g : \mathbb{N} \to \mathbb{N}$ such that for all $x \in G$ we have $x \in G_{g(n)}$ whenever $l(x) \leq \sqrt{n}$. Then,

$$\alpha(G) \geq \liminf_{n\to\infty} \frac{(\alpha_1/2)\ln(n-1)}{\ln(C_{g(n)}\sqrt{2\ln(2)n}(\widetilde{C_{g(n)}}\ \sqrt{n} + \widetilde{D_{g(n)}}) + C_{g(n)}D_{g(n)})}.$$

Proof. Choose $n \in \mathbb{N}_0, p > 0$ and denote $R_n = \sqrt{n}, \epsilon_n = \frac{1}{n^{1/2+p}}$. Next, take $g(n) \in \mathbb{N}$ such that $x \in G_{g(n)}$ whenever $l_G(x) \leq R = \sqrt{n}$. Set $t = \frac{-\ln(1-\epsilon_n^2/2)}{(\widetilde{C_{g(n)}}\ R_n + \widetilde{D_{g(n)}})^2}$ and take vectors $(\xi_x)_{x\in G_{g(n)}}$ as in the proof of proposition 8.2.1, i.e. such that for all $x, y \in G_{g(n)}$:

$$e^{-t(\widetilde{C_{g(n)}}\ d_{g(n)}(x,y)+\widetilde{D_{g(n)}})^2} \leq \langle \xi_x, \xi_y \rangle \leq e^{-t((1/C_{g(n)})\ d_{g(n)}(x,y)^{\alpha_1} - D_{g(n)})^2}.$$

From the lower bound on $\langle \xi_x, \xi_y \rangle$, one derives

$$\|\xi_x - \xi_y\| \leq \epsilon_n \text{ whenever } d_{g(n)}(x,y) \leq R_n. \tag{8.23}$$

Calculating as in Lemma 8.2.2, we derive that $\|\xi_x - \xi_y\| \geq 1$ whenever

$$\frac{(1/C_{g(n)})d_{g(n)}(x,y)^{\alpha_1} - D_{g(n)}}{\widetilde{C_{g(n)}}R_n + \widetilde{D_{g(n)}}} \geq \sqrt{\frac{-\ln(2)}{\ln(1-\frac{\epsilon_n^2}{2})}},$$

if and only if

$$d_{g(n)}(x,y) \geq S_n := [C_{g(n)}(\sqrt{\frac{-\ln(2)}{\ln(1-\frac{1}{2n^{1+2p}})}}(\widetilde{C_{g(n)}}\sqrt{n}+\widetilde{D_{g(n)}})+D_{g(n)})]^{\frac{1}{\alpha_1}}. \tag{8.24}$$

In the proof of Proposition 3.1 of [39], Dadarlat and Guentner explain how the family $(\xi_x)_{x\in G_{g(n)}}$ can be extended to a family of unit vectors $(\hat{\xi}_x)_{x\in G}$ in a larger Hilbert space, but still satisfying the same inequalities as (8.23) and (8.24). More precisely, we obtain unit vectors $(\hat{\xi}_x)_{x\in G}$ in a Hilbert space satisfying

1. $\|\xi_x - \xi_y\| \leq \frac{1}{n^{1/2+p}}$ whenever $d(x,y) \leq \sqrt{n}$;
2. $\|\xi_x - \xi_y\| \geq 1$ whenever $d(x,y) \geq S_n$.

From the proof of proposition 8.2.1, we derive the existence of a large-scale uniform embedding of G into a Hilbert space whose compression function ρ_-, is greater than $\frac{1}{2}\sum_{n=1}^{\infty}\sqrt{n-1}\chi_{[S_{n-1},S_n)}(t)$. Choose some $\beta \in [0,1]$, and define $\gamma : \mathbb{R}^+ \to \mathbb{R}^+, t \mapsto t^{\beta}$. If γ eventually lies under some multiple of ρ_-, then the compression of G is greater than β. There exists $T, \overline{C} \in \mathbb{R}^+$ such that $\gamma(t) \leq \overline{C}\rho_-(t),\ \forall t \geq T$ if

$$\limsup_{n\to\infty} \frac{S_n^{\beta}}{\sqrt{n-1}} < \infty.$$

This is true if there is $M, N \in \mathbb{N}$ such that $\forall n \geq N$, the exponent that we have to give to S_n to obtain S_n^{β} is smaller or equal to the exponent we have to give to S_n to obtain $\sqrt{(n-1)M^2}$. This is true if

$$\beta < \liminf_{n\to\infty}\{\frac{(\alpha_1/2)\ln(n-1)}{\ln(C_{g(n)}(\sqrt{\frac{-\ln(2)}{\ln(1-\frac{1}{2n^{2p+1}})}}(\widetilde{C_{g(n)}}\sqrt{n}+\widetilde{D_{g(n)}})+D_{g(n)}))}\}.$$

Recalling that $\lim_{n\to\infty}[(\frac{-\ln(2)}{\ln(1-\frac{1}{2n^{2p+1}})})/(2\ln(2)n^{2p+1})] = 1$, that compression is by definition a supremum and that we can let p go to 0 as it was any arbitrary positive number, we get the desired lower bound

$$\liminf_{n\to\infty} \frac{(\alpha_1/2)\ln(n-1)}{\ln(C_{g(n)}\sqrt{2\ln(2)n}(\widetilde{C_{g(n)}}\sqrt{n}+\widetilde{D_{g(n)}})+C_{g(n)}D_{g(n)})},$$

for the Hilbert space compression of G. □

If G happens to be a quasi-geodesic space, then, we can reason similarly as in Corollary 8.4.6, to improve our result. Using the same notations as in Theorem 8.6.1 and assuming that G is a quasi-geodesic space, we obtain the following.

Corollary 8.6.2. *Denote $g : \mathbb{N} \to \mathbb{N}$ a function such that for all $x \in G$ we have $x \in G_{g(n)}$ whenever $l(x) \leq \ln(n)$. Then,*

$$\alpha(G) \geq \liminf_{n \to \infty} \frac{(\alpha_1/2)\ln(n-1)}{\ln(C_{g(n)}\sqrt{2\ln(2)n}(\widetilde{C_{g(n)}}\ \ln(n) + \widetilde{D_{g(n)}}) + C_{g(n)}D_{g(n)})}.$$

Remark 8.6.3. *All of the above easily generalizes to directed systems of groups $(G_i)_{i \in I}$ where I is* any *directed set.*

We end this Section with a few examples.

Example 8.6.4. *Assume that G is an infinite direct sum of finite groups $G = F_0 \oplus F_1 \oplus F_2 \oplus \ldots$ where $F_0 = \{1\}$. We can equip G with a proper length function by setting $l(g) = \min\{n \in \mathbb{N} \mid g \in \oplus_{i=0}^{n} F_i\}$. Clearly, then G is the direct limit of the spaces $G_n = \oplus_{i=0}^{n} F_i$. Consider the 0-map $f_n : G_n \to \mathbb{R}$. Recalling the fact that finite groups have Hilbert space compression equal to 1, we see that this is a uniform embedding as in Theorem 8.6.1, where $\widetilde{D_n} = 1, \widetilde{C_n} = 0, C_n = 1, D_n = n$. We can apply Theorem 8.6.1, obtaining $\alpha(G) = 1$.*

Example 8.6.5. *Let G and H be finitely generated groups and equip $\Gamma = G \wr H$ with the word length metric relative to a finite symmetric generating subset. Theorem 8.6.1 can sometimes be used to estimate the compressions of spaces $G^{(H)} := \{f : H \to G \mid f$ has finite support $\}$, equipped with the induced length function from $G \wr H$. Concretely, let us show that $\alpha(\mathbb{Z}^{(\mathbb{Z})}) \geq 2/5$.*

Set $\Gamma = \mathbb{Z}^{(\mathbb{Z})}$ and $\Gamma_n := \{f : [-n, n] \to \mathbb{Z}\}$. Identifying Γ_n with $\mathbb{Z}^{2n+1}$, we see that the inclusion map $f_n : \Gamma_n \hookrightarrow \mathbb{R}^{2n+1}$ is a uniform embedding in the Hilbert space $\mathbb{R}^{2n+1}$. Let us calculate the numbers $C_n, D_n, \widetilde{C_n}, \widetilde{D_n}$ from Theorem 8.6.1. We denote the classical word length metric on $\mathbb{Z}^{2n+1} = \Gamma_n$ by d_S and the induced metric from $\mathbb{Z} \wr \mathbb{Z}$ by $d_\wr$. Choosing

$x=(x_i)_{i=-n,\dots,n}, y=(y_i)_{i=-n,\dots,n}\in\Gamma_n$, we get

$$\begin{aligned} d(f_n(x),f_n(y)) &= \sqrt{\sum_{i=-n}^{n}|x_i-y_i|^2} \\ &\le \sum_{i=-n}^{n}|x_i-y_i| \\ &= d_S(x,y)\le d_{\wr}(x,y), \end{aligned}$$

so we can take $\widetilde{C_n}=1,\widetilde{D_n}=0$. Next, one calculates

$$\begin{aligned} d(f_n(x),f_n(y)) &= \sqrt{\sum_{i=-n}^{n}|x_i-y_i|^2} \\ &\ge \frac{1}{\sqrt{2n+1}}\sum_{i=-n}^{n}|x_i-y_i| \\ &= \frac{1}{\sqrt{2n+1}}d_S(x,y) \\ &\ge \frac{1}{\sqrt{2n+1}}(d_{\wr}(x,y)-4n) \\ &\ge \frac{1}{\sqrt{2n+1}}d_{\wr}(x,y)-4\sqrt{n}, \end{aligned}$$

so we can take $C_n=\frac{1}{\sqrt{2n+1}}$ and $D_n=4\sqrt{n}$.

Clearly, if $d_\Gamma(a,b)\le\sqrt{n}$, then $a,b\in\Gamma_{g(n)}$, where $g(n)=[\sqrt{n}]$ is the smallest integer greater than $\sqrt{n}$. Using $\widetilde{C_{g(n)}}=1,\widetilde{D_{g(n)}}=0,C_{g(n)}=\sqrt{2[\sqrt{n}]+1},D_{g(n)}=4\sqrt{[\sqrt{n}]}$, in combination with Theorem 8.6.1, we get that $\alpha(\mathbb{Z}^{(\mathbb{Z})})\ge 2/5$.

Appendix A

Nederlandstalige samenvatting

Een kristallografische groep is een groep die trouw, isometrisch en kristallografisch (i.e. cocompact en eigenlijk discontinu) actie voert op een Euclidische ruimte $\mathbb{R}^n$. De theorie rond kristallografische groepen wordt in zekere zin gedomineerd door drie belangrijke resultaten: de Bieberbach-stellingen [14], [15], [50]. Deze zijn genoemd naar de Duitse wiskundige Ludwig Bieberbach, die ook twee van de drie stellingen zelf bewees. Het eerste resultaat verwijst naar de structuur van kristallografische groepen en toont aan dat ze virtueel $\mathbb{Z}^n$ zijn. De tweede stelling zegt dat twee kristallografische groepen enkel isomorf zijn als ze dezelfde dimensie, zeg n, hebben en toegevoegd zijn door een affiene transformatie van $\mathbb{R}^n$. De derde Bieberbach-stelling ten slotte geeft weer dat er op isomorfisme na slechts eindig veel kristallografische groepen per gegeven dimensie bestaan. Het onderzoek dat in deze doctoraatsverhandeling beschreven wordt, werd gemotiveerd vanuit het verlangen om de Bieberbach-resultaten te veralgemenen naar een ruimere context.

A.1 Producten waarop isometrieën splitsen

Reeds enige tijd geleden slaagde men erin om de Bieberbach-stellingen te veralgemenen naar de context van *bijna-kristallografische groepen* [8], [72], [43]. Dit zijn groepen die opnieuw trouw, isometrisch en kristallografisch actie voeren, maar in plaats van te eisen dat ze op zulke wijze ageren op een Euclidische ruimte, mogen ze nu ageren op eender welke samenhangende, enkelvoudig samenhangende, nilpotente Lie-groep, uitgerust met een links-invariante Riemannse metriek.

Als eerste onderzoeksproject, trachtten we de Bieberbach-stellingen te veralgemenen naar trouwe, isometrische, kristallografische acties op productruimten $M \times N$ waarbij M een gesloten (i.e. compact en zonder rand) Riemannse variëteit is en waar N een Lie-groep is zoals hierboven beschreven. Het was reeds snel duidelijk dat de tweede en derde Bieberbach-stelling geen natuurlijke veralgemening naar deze context toelaten. Echter, we bewezen dat de eerste Bieberbach-stelling als volgt veralgemeent:

Theorem A.1.1 (Dreesen, Petrosyan)**.** *Zij M een gesloten Riemannse variëteit en N een samenhangende, enkelvoudig samenhangende, nilpotente Lie-groep, uitgerust met een links-invariante Riemannse metriek. Op het product $M \times N$ beschouwen we de natuurlijke product-metriek. Indien Γ een groep is die trouw, isometrisch en kristallografisch actie voert op $M \times N$, dan bevat Γ een deelgroep van eindige index die isomorf is met een discrete cocompacte deelgroep van N.*

Het cruciaal argument in het bewijs is het feit dat, onder de assumpties van bovenstaande stelling, de isometrieën van $M \times N$ *splitsen*, i.e. $\mathrm{Iso}(M \times N) = \mathrm{Iso}(M) \times \mathrm{Iso}(N)$, waarbij de notatie $\mathrm{Iso}(M \times N)$ verwijst naar de groep der isometrieën van $M \times N$. Deze observatie leidde tot de volgende vraag: *Gegeven een gesloten Riemannse variëteit M, voor welke Riemannse variëteiten N splitsen de isometrieën van $M \times N$?* De zoektocht naar een antwoord op deze vraag leidde tot het invoeren van volgende definities.

Definition A.1.2. *Zij M en N Riemannse variëteiten en stel dat M gesloten is en van dimensie n. We zeggen dat het product $M \times N$* minimale n-cohomologie *heeft indien $H^n(M \times N; \mathbb{Z}_2) = H^n(M; \mathbb{Z}_2)$.*

De volgende definitie maakt gebruik van het feit dat er een natuurlijke wijze bestaat om het *volume* van een Riemannse variëteit M te definiëren.

Definition A.1.3. *Zij $f : M \times N \to M \times N$ een diffeomorfisme. We zeggen dat f* het volume van een M-vezel niet vergroot *(afgekort f is VVNG), indien voor elke $z \in N$:*

$$\mathrm{Vol}(f(M \times \{z\})) \leq \mathrm{Vol}(M).$$

Hierbij beschouwen we de Riemannse deelruimtemetriek op $f(M \times \{z\})$, geïnduceerd door $M \times N$.

We bewezen vervolgens volgende stellingen.

Theorem A.1.4. *Indien* $M \times N$ *minimale* n*-cohomologie heeft, en als* $f : M \times N \to M \times N$*, VVNG is, dan beeldt* f *elke* M*-vezel* $M \times \{z\} \subset M \times N$ *af op een* M*-vezel* $M \times \{w\} \subset M \times N$*.*

Theorem A.1.5. *Indien* $M \times N$ *minimale* n*-cohomologie heeft, dan splitsen de isometrieën van* $M \times N$*.*

Theorem A.1.6. *Indien* $M \times N$ *minimale* n*-cohomologie heeft, dan is de verzameling* $VVNG(M \times N)$ *van VVNG-afbeeldingen een groep en deze past in een korte exacte rij*

$$1 \to K \to VVNG(M \times N) \to \mathrm{Diffeo}(N) \to 1,$$

waar Diffeo(N) *de groep van diffeomorfismen is van* N *en waarbij*

$$K = \{f : N \to \mathrm{Diffeo}(M) \mid f \textit{ is Fréchet } C^\infty\}.$$

Deze exacte rij splits bovendien.

De bovenstaande theorie werd ten slotte toegepast om een speciaal geval van een conjectuur van Talelli aan te tonen (Conjectuur III in [101]).

A.2 Equivariante Hilbert compressie

In het eerste deel van deze verhandeling veralgemeenden we de kristallografische context door niet langer op $\mathbb{R}^n$ te ageren, maar wel op bepaalde productruimten $M \times N$. Een tweede manier om de context van kristallografische groepen te veralgemenen is door $\mathbb{R}^n$, i.e. de n-dimensionale Hilbert ruimte, te vervangen door eender welke, mogelijks oneindig dimensionale, Hilbert ruimte. Opnieuw beperken we ons tot het geval waarbij de *orbieten onder deze actie naar oneindig gaan.* We komen zo tot volgende definitie.

Definition A.2.1 ([25])**.** *Een aftelbare discrete groep* Γ *heeft de* Haagerup *eigenschap indien hij isometrisch actie voert op een Hilbert ruimte* $\mathcal{H}$*, maar wel zodanig dat de orbiet* $b : \Gamma \to \mathcal{H}, x \mapsto b(x) := x \cdot 0$ *van* $0 \in \mathcal{H}$ *naar oneindig gaat, i.e. voor elk natuurlijk getal* M*, moet er een eindig deel* $F \subset \Gamma$ *bestaan zodat* $\|b(x)\| \geq M$ *indien* $x \notin F$*.*

De Haagerup eigenschap werd grondig bestudeerd en de theorie rond groepen met de Haagerup eigenschap bevat vele interessante resultaten: groepen met de Haagerup eigenschap voldoen bvb. aan de conjectuur van Baum-Connes. Bovendien kent de Haagerup eigenschap vertalingen naar en toepassingen in andere wiskundigen contexten: bvb. de context van C^*-algebra's [25].

Rond 2004, ontdekten Guentner en Kaminker dat het kwantificeren van de Haagerup eigenschap tot een reeks interessante problemen en resultaten leidde. Concreet genomen, trachtten ze te kwantificeren hoe snel nu precies banen, onder isometrische acties van een gegeven groep, naar oneindig konden gaan. Om deze kwantificatie door te voeren, hadden ze meer nodig dan gewoon een topologie op de groep Γ. We zullen vanaf nu dus stilzwijgend onderstellen dat groepen uitgerust zijn met vooraf gekozen lengtefuncties. We bekomen volgende definitie.

Definition A.2.2 ([59]). *Zij $f : \Gamma \to \mathcal{H}$ een afbeelding van Γ naar een Hilbert ruimte $\mathcal{H}$. De compressie $R(f)$ van f is het supremum van de getallen $r \in [0,1]$ waarvoor er getallen $C, D > 0$ bestaan zodat*

$$\forall x, y \in \Gamma : \ \frac{1}{C} d(x,y)^r - D \leq \|f(x) - f(y)\| \leq C d(x,y) + D.$$

Een afbeelding $f : \Gamma \to \mathcal{H}$ die aan bovenstaande rechtse ongelijkheid voldoet, noemen we *veralgemeend Lipschitz*. We beperken ons in al het volgende enkel tot afbeeldingen die *veralgemeend Lipschitz* zijn.

Definition A.2.3 ([59]). *De equivariante compressie van Γ is het supremum van $R(f)$, genomen over alle mogelijke afbeeldingen $f : \Gamma \to \mathcal{H}$ waarbij $\mathcal{H}$ een Hilbert ruimte is en f Γ-equivariant is t.o.v. een isometrische actie van Γ op $\mathcal{H}$ en de linkse vermenigvuldigingsactie van Γ op zichzelf.*

Een voorbeeld van een interessant compressie-resultaat is het volgende:

Theorem A.2.4 ([59]). *Indien de equivariante compressie van een eindig voortgebrachte groep, uitgerust met woordlengte, strikt groter is dan $1/2$, dan is de groep amenabel.*

Dit resultaat geeft een soort omgekeerde van de stelling dat alle amenabele groepen Haagerup zijn.

In deze thesis onderzochten we hoe de equivariante compressie zich gedraagt onder groepsconstructies. We behandelden o.a. vrije producten en HNN-extensies, beiden genomen over eindige groepen. We bekeken ook het geval van quotiënten van groepen en berekenden ten slotte de equivariante compressie van de Baumslag-Solitar groepen $BS(m,n)$ met $m,n \geq 1$. We zullen hier niet alle resultaten formuleren en om niet alles nodeloos ingewikkeld te maken, zullen we steeds onderstellen dat groepen eindig voortgebracht zijn en uitgerust met woordlengte t.o.v. een eindig symmetrisch voortbrengend deel.

Theorem A.2.5. *Zij* (Γ_1, l_1) *en* (Γ_2, l_2) *eindig voortgebrachte groepen uitgerust met woordlengte* l_1 *en* l_2 *en noteer hun equivariante compressies respectievelijk met* α_1 *en* α_2*. De equivariante compressie van* $\Gamma_1 *_F \Gamma_2$*, met* F *eindig, is dan gelijk aan*

1. *1, indien* F *van index* 2 *in* Γ_1 *en* Γ_2 *is,*
2. α_1*, indien* $F = \Gamma_2$ *en* α_2 *indien* $F = \Gamma_1$*,*
3. $\min(\alpha_1, \alpha_2, 1/2)$*, in alle andere gevallen.*

Voor HNN-extensies bekomen we het volgende resultaat.

Theorem A.2.6. *Zij* (H, l_1) *een eindig voortgebrachte groep met woordlengte* l_1 *en equivariante compressie* α*. Dan bedraagt de equivariante compressie van een HNN-extensie* $HNN(H, F, \theta)$ *van* H *over een eindige groep* F

1. 1*, indien* $H = F$*,*
2. $\min(\alpha, 1/2)$*, in de andere gevallen.*

A.3 (Niet-equivariante) Hilbert compressie

In het derde deel van deze verhandeling, keken we wat er gebeurde indien we de eis van G-equivariantie lieten vallen.

Definition A.3.1 ([59])**.** *De Hilbert compressie van een groep* (Γ, l) *is het supremum van* $R(f)$ *genomen over alle Hilbert ruimten en alle (veralgemeend Lipschitz) afbeeldingen* $f : \Gamma \to \mathcal{H}$*.*

Ook in dit geval werden er interessante stellingen bewezen.

Theorem A.3.2 ([59]). *Zij* Γ *een eindig voortgebrachte groep met woordlengte. Indien de compressie van* Γ *strikt groter is dan* $1/2$*, dan voldoet de groep aan eigenschap* (A)*, als geïntroduceerd door Yu [110].*

Een groep met voldoende grote compressie voldoet dus aan een zwakke vorm van amenabiliteit.

Wij onderzochten in Deel 3 van deze verhandeling het gedrag van compressie onder groepsconstructies. Verschillende constructies werden bestudeerd, waaronder vrije producten geamalgameerd over eindige groepen, HNN-extensies over eindige groepen en deelgroepen van eindige index, extensies van groepen door hyperbolische groepen of groepen met polynomiale groei, limieten van groepen,...

Om de lezer een idee te geven van een typisch resultaat, formuleren we hier nu het resultaat i.v.m. groepsextensies. Om alles simpel te houden, onderstellen we dat Γ steeds eindig voortgebracht is en uitgerust met woordlengte t.o.v. een eindig symmetrisch voortbrengend deel.

Theorem A.3.3. *Onderstel dat*

$$1 \to H \to \Gamma \to G \to 1,$$

kort exact is. Op H *beschouwen we de deelruimte metriek van* Γ *en op* G *beschouwen we de woordlengte. Noteer de compressie van* H *door* α *en noteer de compressie van* Γ *met* β*. Indien* G *polynomiale groei heeft, dan geldt*

$$\alpha/3 \leq \beta \leq \alpha.$$

Indien G *hyperbolisch is, dan geldt*

$$\alpha/5 \leq \beta \leq \alpha.$$

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