Topological entropy of positive contactomorphisms

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Lucas Dahinden

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Accepted on the recommendation of the jury :

Felix Schlenk	Université de Neuchâtel, CH	Director of thesis
Bruno Colbois	Université de Neuchâtel, CH	Committee Member
Urs Frauenfelder	Universität Augsburg, DE	Committee Member

Institut de Mathématiques de l'Université de Neuchâtel,
Rue Emile Argand 11, 2000 Neuchâtel, Switzerland

Faculté des Sciences
Secrétariat-décanat de Faculté
Rue Emile-Argand 11
2000 Neuchâtel – Suisse
Tél : + 41 (0)32 718 21 00
E-mail : secretariat.sciences@unine.ch

IMPRIMATUR POUR THESE DE DOCTORAT

La Faculté des sciences de l'Université de Neuchâtel autorise l'impression de la présente thèse soutenue par

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- Prof. Felix Schlenk, directeur de thèse, Université de Neuchâtel, Suisse
- Prof. Bruno Colbois, Université de Neuchâtel, Suisse
- Prof. Urs Frauenfelder, Universität Augsburg, Allemagne

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Abstract

In this thesis we study volume growth properties of positive contactomorphisms, using Rabinowitz–Floer homology. We prove that positive dimensional growth of Rabinowitz–Floer homology implies that every positive contactomorphism has positive topological entropy. We found two instances where Rabinowitz–Floer homology has positive dimensional growth. The first instances are unit cosphere bundles of energy hyperbolic manifolds. Examples are manifolds with exponentially growing fundamental group, or spaces such that the singular homology of the Loop space has exponential dimensional growth. The second instances are the boundaries of Liouville domains such that for a certain Lagrangian the wrapped Floer homology has exponential dimensional growth. Alves and Meiwes recently constructed a large class of spaces with this property, including exotic contact spheres of dimension $\geqslant 7$.

The classical Bott–Samelson theorem states that, if on a Riemannian manifold there is a point such that every geodesic from that point returns, then this manifold must be homotopy equivalent to a quotient of a sphere, and if furthermore the first return time is equal for all geodesics, then the manifold must be homotopy equivalent to a sphere or $\mathbb{R}P^2$. This theorem was fully generalized to Reeb flows on the unit cotangent bundle by Frauenfelder–Labrousse–Schlenk, and partially to positive Legendrian isotopies. We prove a full generalization of the Bott–Samelson theorem to positive Legendrian isotopies, situating the theorem properly in contact topology. The proof revolves around the slow growth of Rabinowitz–Floer homology.

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Keywords: Symplectic geometry, Contact topology, Topological entropy, Floer homology.

Résumé

Dans cette thèse nous étudions des propriétés de croissance de volume de contactomorphismes positives en utilisant l'homologie de Rabinowitz–Floer. Nous montrons que la croissance positive de la dimension de l'homologie de Rabinowitz–Floer implique que tout contactomorphisme positive a d'entropie topologique positive. Nous trouvons deux cas où l'homologie de Rabinowitz–Floer a de croissance de dimension positive. La première est le fibré de cosphères d'unité de variétés qui sont énergiquement hyperboliques. Par exemple ce sont des variétés avec un groupe fondamental expenentiellement croissant, ou des espaces dont l'espace de lacets a d'homologie singulaire de dimension exponentiellement croissante. Les deuxièmes exemples sont des bords de domaines de Liouville tel que pour un certain Lagrangien l'homologie enroulée de Floer a de croissance dimensionnelle exponentielle. Alves et Meiwes ont récemment construit une grande classe d'espaces avec cette propriété qui inclut des sphères de contact exotiques de dimension $\geqslant 7$.

Le théorème de Bott–Samelson classique dit que, si sur une variété Riemannienne il y a un point tel que chaque géodésique départante de ce point retourne, alors cette variété est homotopiquement équivalent au quotient d'une sphère, et si de plus le temps du premier retour est égal pour chaque géodésique, alors la variété est homotopiquement équivalente à une sphère ou $\mathbb{R}P^2$. Ce théorème était généralisé complètement pour des flots de Reeb sur le fibré de cosphères d'unité par Frauenfelder–Labrousse–Schlenk, et partiellement pour des isotopies Legendriens positivs. Nous montrons la généralisation complète du théorème pour des isotopies Legendriens positivs, ce qui place le théorème dans la topologie de contact. La preuve inclut la croissance lente de l'homologie Rabinowitz–Floer.

2010 Mathematics Subject Classification: Primary 53D35; Secondary 37B40, 53D40, 57R17.
Mots-clés: Géométrie symplectique, Topologie de contacte, Entropie topologique, Homologie de Floer.

Contents

Chapter 1

Introduction

This is my PhD-thesis. It contains the results that I developed during my doctorate in Neuchâtel in the years 2014–2018 under the guidance of my advisor Felix Schlenk. The results are also contained in the articles [14, 15, 16]. The first two articles are already published in the journals "Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg" and "Israel Journal of Mathematics". In all three articles I study the chaotic behaviour of certain dynamical systems and the main tool is Rabinowitz–Floer homology in all articles. In this introduction I will briefly explain what kind of dynamical system I study and what I mean by chaotic behaviour. After the preliminaries I focus on each of the three articles individually and state the main results therein. The articles themselves can be found in the later chapters.

Before you, dear reader, start the lecture of this thesis, be warned that the three articles are printed exactly as they are published. There is no nice exposition for people outside my field and there are not more details for further reading. If you are not a researcher with a background in Floer homology, or even a non-mathematician, then the explanations in my thesis are not enough to learn the theory and you will be lost after a few (possibly 0) pages. I apologize to these people and hope that this manuscript at least serves as a nice decoration in a book shelf.

1.1 Preliminaries

A contact form on an odd dimensional manifold M^{2n+1} is a 1-form α such that $\alpha \wedge d\alpha^n$ nowhere vanishes. The kernel of such a contact form $\xi = \ker \alpha$ is called a (cooriented) contact structure (where the coorientation comes from the side of ξ where α is positive). Two contact forms α_1, α_2 generating the same contact structure are always related by $\alpha_1 = f\alpha_2$ for some nowhere vanishing function $f : M \to \mathbb{R}$ (if the function is positive, α_1 and α_2 induce the same coorientation). A manifold together with a cooriented contact structure (M, ξ) is called cooriented contact manifold and a manifold together with a contact form (M, α) is called exact contact manifold. In this thesis we only consider coorientable contact structures.

An exact contact manifold (M, α) comes with the Reeb vector field R_α defined by

$$\begin{cases} \alpha(R_\alpha) = 1, \\ d\alpha(R_\alpha, \cdot) \equiv 0. \end{cases}$$

The flow induced by R_α is denoted by φ_α^t and is called Reeb flow of α. It is a path of contactomorphisms, that is for every t the map φ_α^t is a diffeorphism and $D\varphi_\alpha^t(\xi) = \xi$.

More generally, given an exact contact manifold (M,α) and a time dependent function $h^t : M \times \mathbb{R} \to \mathbb{R}$, called contact Hamiltonian[1], we define the contact Hamiltonian vector field X_{h^t} by

$$\begin{cases} \alpha(X_{h^t}) = h^t, \\ d\alpha(X_{h^t},\cdot) = dh^t(R_\alpha)\alpha - dh^t. \end{cases}$$

The flow induced by X_{h^t} is denoted by $\varphi^t_{h^t}$ and is called contact Hamiltonian flow of h^t. If $h^t \equiv 1$, then $\varphi^1_{h^t} = \varphi^t_\alpha$ is the Reeb flow of α. A contact Hamiltonian flow is a path of contactomorphisms. Every path of contactomorphisms that starts at the identity is the contact Hamiltonian flow of some Hamiltonian h^t. This is in stark contrast to symplectic geometry where Hamiltonian flows are in general rare amongst paths of symplectomorphisms.

In a cooriented contact manifold there is a natural notion of moving positively or negatively transversely to the contact structure. We say that a path $\gamma : I \to (M,\xi)$ is positive (or positively transverse) if $\alpha(\dot\gamma) > 0$. By extension to maps, a one parameter family of maps $f : X_x \times I_t \to M$ is called positive if $\alpha(\frac{d}{dt}f(x,t)) > 0\ \forall (x,t) \in X \times I$. We are interested in two instances:

- if $X = L^n$ is a closed manifold and $f(\cdot,t) = j_t$ is a Legendrian embedding for all t, then $L_t = j_t(L)$ is a positive path of Legendrian submanifolds,
- if $X = M$ and $f(\cdot,t) = \varphi^t$ is a contactomorphism for all t, then φ^t is a positive path of contactomorphisms.

Because of the Legendrian homotopy extension theorem [25, Theorem 2.6.2], one can always extend positive paths of Legendrian submanifolds to positive paths of contactomorphisms. One can equivalently define positivity for paths of contactomorphisms through contact Hamiltonians by saying that φ^t is positive if the contact Hamiltonian generating φ^t is positive. We call a contactomorphism φ positive if there exists a positive path of contactomorphisms starting at id and ending at φ.

Given a map, we are interested in its chaotic behavior. In this thesis, by chaos we a priori mean volume growth which we define as follows

Definition 1.1.1 (Growth)**.** For a function $f : X \to \mathbb{R}^+$, where $X = \mathbb{N}$ or $X = \mathbb{R}$, we define the exponential and polynomial growth of f as

$$\Gamma_e(f) = \limsup_{a\to\infty} \frac{1}{a}\log(f(a)),$$
$$\Gamma_p(f) = \limsup_{a\to\infty} \frac{1}{\log a}\log(f(a)).$$

If $\Gamma_e(f) > 0$ we say that f grows exponentially. If $\Gamma_p(f) < \infty$, we say that f grows polynomially.

Definition 1.1.2 (Volume growth)**.** Given a manifold X and a smooth map $\varphi : X \to X$. Choose a Riemannian metric g on X, which gives rise to a volume form Vol_g. Given a compact submanifold S of any codimension, define the volume growth of S under iterations of φ

$$\Gamma_e^{vol}(S;\varphi) = \Gamma_e(\mathrm{Vol}(\varphi^n(S)),$$
$$\Gamma_p^{vol}(S;\varphi) = \Gamma_p(\mathrm{Vol}(\varphi^n(S)).$$

[1] In this thesis, contact Hamiltonians are always denoted by a small h^t, whereas symplectic Hamiltonians are denoted by capital H^t.

We then define the volume growth of the dynamical system as the supremum over all compact submanifolds S.

$$\Gamma_e^{vol}(\varphi) = \sup_S \Gamma_e^{vol}(S;\varphi),$$
$$\Gamma_p^{vol}(\varphi) = \sup_S \Gamma_p^{vol}(S;\varphi).$$

If X is a compact manifold, then this notion is independent of the choice of g.

By the combined theorems of Yomdin [37] and Newhouse [38], exponential volume growth is equal to topological entropy for differentiable maps in closed manifolds. The fact that topological entropy is a much more fundamental property in chaos theory (it is meaningful for many non-differentiable maps on many non-manifolds) is responsible for the fact that the theorems in this thesis are statements about topological entropy, but the proofs establish volume growth.

1.2 Lower complexity bounds for positive contactomorphisms

In [14] we study a special class of contact manifolds: cosphere bundle S^*Q of closed manifolds Q. They are a special instance of exactly fillable contact manifolds, where the filling is given by the codisk bundles D^*Q. The Liouville one-form, or tautological one-form, is can be defined because of the fact that the tangent bundle of the cotangent bundle $T_{q,\alpha}T^*Q$ at (q,α) naturally splits into $T_qQ \oplus TT_q^*Q$. The tautological one-form λ is then defined as $\lambda_{q,\alpha}(\overline{q},\overline{\alpha}) = \alpha(\overline{q})$. In dual coordinates (q,p), the Liouville form can be written as $\lambda = pdq$. Since $d\lambda = dp \wedge dq$ is a symplectic form and the Liouville vector field $p\partial_p$ is transverse to the cosphere bundle S^*Q, the codisk bundle is indeed a Liouville domain.

To state the theorem we need a few definitions: $\Omega_0 Q(q)$ is the connected component of the based loop space that contains the constant path. The growth of homology of $\Omega_0 Q(q)$ is the growth of $\sum_{k=0}^{n} \dim H_k(\Omega_0 Q(q))$. The growth of a group with a finite generator set is the growth of balls in the wordlength metric. It is independent of the generator set. In this setup we prove the following theorem, which is a generalization of its analogon for autonomous Reeb flows from [28] to positive contactomorphisms:

Theorem. *Let φ be a positive contactomorphism on the spherization S^*Q of the closed manifold Q and let q be any point in Q.*

1. *If the fundamental group grows exponentially or if the fundamental group is finite and the homology of $\Omega Q_0(q)$ grows exponentially, then*

$$\Gamma_e^{\mathrm{vol}}(\varphi) \;\geqslant\; \Gamma_e^{\mathrm{vol}}(\varphi; S_q^*Q) \;>\; 0.$$

2. *If the fundamental group and the homology of $\Omega Q_0(q)$ both grow polynomially, then*

$$\Gamma_p^{\mathrm{vol}}(\varphi) \;\geqslant\; \Gamma_p^{\mathrm{vol}}(\varphi; S_q^*Q) \;\geqslant\; \Gamma_p(\pi_1(Q)) + \Gamma_p(\Omega Q_0(q)) - 1.$$

The novelty in the proof of this theorem is the proof that dimensional growth of time dependent Rabinowitz–Floer homology filtered by action is invariant under continuation homomorphisms. From there, the proof proceeds as follows.

From the conditions of the theorem we can estimate the growth of Morse homology of the energy functional filtered by index. By a theorem of Gromov, growth of the Morse homology of the energy functional filtered by action is the same as filtered by index. By the Abbondandolo–Schwarz isomorphism we relate Rabinowitz–Floer homology with Morse homology of the energy functional with respect to some Riemannian metric. Then we deform the geodesic Hamiltonian to the Hamiltonian generating φ, which preserves polynomial growth and positivity of exponential growth. From the growth of Rabinowitz–Floer homology we deduce volume growth which shows the theorem.

This is the first instance of contact manifolds for which positive topological entropy for all positive contactomorphisms was established.

1.3 Positive contactomorphisms in exactly fillable contact manifolds

In [16] we study exactly fillable contact manifolds. We prove the following theorem:

Theorem. *Let W be a Liouville domain and L be an asymptotically conical exact Lagrangian with connected spherical boundary $\Lambda = \partial L$ such that $\lambda|_L = 0$, such that (λ, L) is regular and such that $\omega|_{\pi_2(W;L)} = 0$. Suppose that $\Gamma^{\mathrm{symp}}(W, L) > 0$ and let φ be a positive contactomorphism of $M = \partial W$. Then the topological entropy of φ is positive.*

Here, regularity is the assumption that the functional defining wrapped Floer homology is Morse. The final step of the proof is as in the paper [14] to deduce volume growth from dimensional growth of time dependent Rabinowitz–Floer homology filtered by action. In [16] the growth comes from growth of wrapped Floer homology filtered by action. To transport the growth from one homology to the other, we prove along the way the following proposition that might be of independent interest:

Proposition. *Under the Assumption of the theorem above, for all $a, b \notin \mathcal{S}$ with $0 < a < b$ we have*

$$\mathrm{WH}^{(a,b)}(W, L) \cong \mathrm{RFH}^{(a,b)}(W, L),$$

where $\mathcal{S}$ is the set of lengths of Reeb chords from L to L, WH *is wrapped Floer homology and* RFH *is autonomous Rabinowitz–Floer homology. These isomorphisms commute with morphisms induced by inclusion of filtered chain complexes.*

The main result is of large interest because Alwes and Meiwes [6] recently proved growth of wrapped Floer homology for many examples of Liouville domains different from codisk bundles. Notably they found examples of

- Liouville domains with boundary diffeomorphic to S^{2n+1} for $2n + 1 \geqslant 7$,
- Liouville domains with boundary diffeomorphic to $S^3 \times S^2$,
- Liouville domains whose boundary is diffeomorphic to the boundary of a plumbing tree whose vertices are cotangent bundles,

for which the wrapped Floer homology has positive growth. Note that the first two examples yield dynamically exotic contact structures on the respective boundaries. These examples are found through the a product structure on wrapped Floer homology whose growth is stable under geometric operations such as handle attachments on the Liouville domain. Alves and Meiwes used the growth of wrapped Floer homology to show that every Reeb flow has positive topological entropy, using only wrapped Floer homology.

Note that on the one hand there is no corresponding product structure on Rabinowitz–Fleor homology and that on the other hand wrapped Floer homology is not suited to study time-dependent Reeb dynamics. Therefore, the combined power of the two theories is necessary to prove that on the examples found by Alves and Meiwes every positive contactomorphism has positive topological entropy.

1.4 A Bott–Samelson theorem for positive Legendrian isotopies

In [15] (chronologically the second paper) we put ourselves in a situation where, by geometric assumptions, we conclude that the Rabinowitz–Floer homology has extremely slow growth (indeed linear). Before we state the result, we first cite the following classical theorem from Riemannian geometry:

Theorem (Bott–Samelson)**.** *Let Q be a closed connected manifold of dimension $\geqslant 2$. Suppose there exists a Riemannian metric, a point p and a time T such that $\exp^T(p, S^1) = p$, where S^1 is the unit circle in T_pQ. Then the fundamental group of Q is finite and the integral cohomology ring of the universal cover of Q is generated by one element.*

If furthermore there is no $v \in S^1$ and $t < T$ such that $\exp^t(p, v) = p$, i.e. T is the first return time, then Q is simply connected or homotopy equivalent to $\mathbb{R}P^n$.

In the paper [20] this theorem is generalized to a corresponding statement for Reeb flows on the cosphere bundle T^*Q. The same paper also contains the following partial generalization to positive Legendrian isotopies, but the second part of the theorem is missing:

Theorem. *[20, Theorem 2.13] Let Q be a closed connected manifold of dimension $\geqslant 2$. Suppose there exists a positive Legendrian isotopy L_t in the spherization S^*Q that connects the fiber over a point with itself, i.e. $L_0 = L_1 = S^*_qQ$. Then the fundamental group of Q is finite and the integral cohomology ring of the universal cover of Q is generated by one element.*

In our paper we prove that the missing part indeed holds:

Theorem. *Under the assumptions of Theorem [20, Theorem 2.13], if furthermore $L_t \cap L_0 = \varnothing$ for $0 < t < 1$, then Q is simply connected or homotopy equivalent to $\mathbb{R}P^n$.*

We show the theorem by carefully extending the positive Legendrian isotopy to a positive path of contactomorphisms. This enables us to construct a Morse–Bott Rabinowitz–Floer homology. From there the proof works as in the autonomous case.

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Chapter 2

Lower complexity bounds for positive contactomorphisms

2.1 Introduction and result

Before stating the main theorem, we define positive contactomorphisms, the volume growth of maps and the growth of loop spaces. All our objects are in the smooth category and all manifolds have dimension $\geqslant 2$.

The spherization S^*Q of a manifold Q is the space of positive line elements in the cotangent bundle T^*Q. The tautological one-form λ on T^*Q does not restrict to S^*Q, but its kernel does. This endows S^*Q with a co-oriented contact structure ξ. Choose a contact form α for ξ. We call a smooth path of contactomorphisms $\varphi^t : [0,1] \to \mathrm{Cont}(S^*Q)$ starting at the identity *positive* if its generating vector field $X^t(\varphi^t(x)) := \frac{d}{dt}\varphi^t(x)$ is positively transverse to the contact structure: $\alpha(X^t) > 0$. A contactomorphism φ is called *positive* if there is a positive path φ^t of contactomorphisms with $\varphi^0 = id$, $\varphi^1 = \varphi$. This notion is independent of the choice of contact form.

Let $\varphi : M \to M$ be a smooth diffeomorphism of a manifold M. Let $S \subset M$ be a compact submanifold and fix a Riemannian metric g on M. We denote by $\gamma_{\mathrm{vol,pol}}(\varphi; S)$ and $\gamma_{\mathrm{vol,exp}}(\varphi; S)$ the polynomial and exponential volume growth of S under iterations of φ, where the volume is induced by g:

$$\begin{aligned} \gamma_{\mathrm{vol,pol}}(\varphi; S) &= \liminf_{m\to\infty} \frac{1}{\log m} \log \mathrm{Vol}(\varphi^m(S)), \\ \gamma_{\mathrm{vol,exp}}(\varphi; S) &= \liminf_{m\to\infty} \frac{1}{m} \log \mathrm{Vol}(\varphi^m(S)). \end{aligned}$$

The polynomial and exponential volume growth of a map is the supremum of polynomial and exponential volume growths over all compact submanifolds:

$$\begin{aligned} \gamma_{\mathrm{vol,pol}}(\varphi) &= \sup_S \gamma_{\mathrm{vol,pol}}(\varphi; S), \\ \gamma_{\mathrm{vol,exp}}(\varphi) &= \sup_S \gamma_{\mathrm{vol,exp}}(\varphi; S). \end{aligned}$$

These numbers are clearly independent of the choice of Riemannian metric.

Let Q be a connected manifold and consider the connected component $\Omega Q_0(q)$ of contractible loops of the space of loops based at q. Denote by $\mathbb{P}$ the set of primes and zero. For p prime let $\mathbb{F}_p$ be the field $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{F}_0 = \mathbb{Q}$. Define the homological polynomial and exponential growth of $\Omega Q_0(q)$ by

$$\begin{aligned}\gamma_{\text{pol}}(\Omega Q_0(q)) &:= \sup_{p\in\mathbb{P}} \liminf_{m\to\infty} \frac{1}{\log m} \log \sum_{k=0}^{m} \dim(H_k(\Omega Q_0(q);\mathbb{F}_p)),\\ \gamma_{\text{exp}}(\Omega Q_0(q)) &:= \sup_{p\in\mathbb{P}} \liminf_{m\to\infty} \frac{1}{m} \log \sum_{k=0}^{m} \dim(H_k(\Omega Q_0(q);\mathbb{F}_p)).\end{aligned}$$

Note that $\Omega Q_0(q)$ is homotopy equivalent to any connected component of the space $\Omega Q(q,q')$ of paths in Q from q to q'. Thus if we had used in the above definition a connected component of $\Omega Q(q,q')$ instead of $\Omega Q_0(q)$, we would have got the same number.

Denote by $\gamma_{\text{pol}}(\pi_1(Q))$ and $\gamma_{\text{exp}}(\pi_1(Q))$ the polynomial and exponential growth of the fundamental group of Q for some (and thus every) set of generators. With this notation we can state the main result of this paper.

Theorem 1. *Let φ be a positive contactomorphism on the spherization S^*Q of the closed manifold Q and let q be any point in Q.*

1. *If $\gamma_{\text{exp}}(\pi_1(Q)) > 0$ or if $\pi_1(Q)$ is finite and $\gamma_{\text{exp}}(\Omega Q_0(q)) > 0$, then*
$$\gamma_{\text{vol,exp}}(\varphi) \;\geqslant\; \gamma_{\text{vol,exp}}(\varphi; S_q^*Q) \;>\; 0.$$
2. *If $\gamma_{\text{pol}}(\pi_1(Q))$ and $\gamma_{\text{pol}}(\Omega Q_0(q))$ are finite, then*
$$\gamma_{\text{vol,pol}}(\varphi) \;\geqslant\; \gamma_{\text{vol,pol}}(\varphi; S_q^*Q) \;\geqslant\; \gamma_{\text{pol}}(\pi_1(Q)) + \gamma_{\text{pol}}(\Omega Q_0(q)) - 1.$$

Yomdin and Newhouse [37, 38] related the exponential volume growth to the topological entropy. They showed that $\gamma_{\text{vol,exp}}(\varphi) = h_{\text{top}}(\varphi)$. This results in the following reformulation of Theorem 1 (1).

Corollary 2. *If $\gamma_{\text{exp}}(\pi_1(Q)) > 0$ or if $\pi_1(Q)$ is finite and $\gamma_{\text{exp}}(\Omega Q_0(q)) > 0$, then for every positive contactomorphism φ on the spherization S^*Q,*

$$h_{top}(\varphi) > 0.$$

The first versions of this theorem were proved by Dinaburg, Gromov, Paternain and Petean for geodesic flows, using Morse theory [17, 26, 31, 32, 33], see also [24]. Frauenfelder–Schlenk [23] generalized the theorem to certain Hamiltonian flows on T^*M, using Lagrangian Floer homology. A further generalization to Reeb flows was found by Macarini–Schlenk [28] (exponential) and Frauenfelder–Labrousse–Schlenk [20] (polynomial), also using Lagrangian Floer homology. In this paper we extend these results to positive contactomorphisms. These maps can be realized as time-dependent Reeb flows.

Remark 2.1.1. By the same arguments, part (1) of Theorem 1 holds for the larger class of energy hyperbolic manifolds. We refer to [28] for the definition of this class and for more examples.

Remark 2.1.2. There are several approaches to dealing with the time-dependence of the Reeb flow. One is to absorb the time-dependence in an additional space factor, such as T^*S^1. Another approach is to cook up an action functional for our problem, and to deform it to the action functional for a Reeb flow. While we did not succeed with the geometric approach, the second approach worked out well.

Remark 2.1.3. Positive entropy for all Reeb flows on many contact 3-manifolds different from spherizations has recently been established by Alves in [1, 2, 3].

The identity map is a non-negative contactomorphism that can be uniformly approximated by positive contactomorphisms by slowing down a fixed positive contact isotopy. The class of positive contactomorphisms thus seems to be the largest natural class of contact geometric maps for which one has positive topological entropy on S^*Q under the topological condition on Q given in Theorem 1. Indeed, without the positivity assumption various scenarii are possible:

Example 2.1.4.

1. There are closed manifolds Q with $\pi_1(Q)$ of exponential growth whose spherization S^*Q carries a non-negative contactomorphism φ such that $h_{\text{top}}(\varphi) = 0$ and such that φ is generated by an autonomous contact isotopy that is positive outside a submanifold of positive codimension.
2. There are closed manifolds Q with $\pi_1(Q)$ of exponential growth whose spherization S^*Q carries a non-negative contactomorphism φ such that $h_{\text{top}}(\varphi) > 0$ and such that φ is generated by an autonomous contact isotopy that restricts to the identity on a subset of S^*Q with nonempty interior.

Two specific examples are given in Section 2.5.

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2.2 Recollections

In this section we first represent the spherization S^*Q as a hypersurface in T^*Q and specify the choice of positive path of contactomorphism. Then we cite a theorem that relates the volume growth of a flow on a spherization to the volume growth of an extension of the flow to the sublevel. Finally we state some facts about the version of Rabinowitz–Floer homology that will be used in the proof of Theorem 1. The precise definition of this Rabinowitz–Floer homology is given in Section 2.4.

The spherization of a manifold can be naturally represented as a fiberwise starshaped hypersurface $\Sigma \subset T^*Q$ in the cotangent bundle with contact structure $\xi = \ker \lambda|_\Sigma$, where λ is the Liouville one-form. The map that sends a positive line element to its intersection with Σ is a contactormorphism. The radial dilation of a fiberwise starshaped hypersurface by a positive function is a contactomorphism to its image. Every contact form of (S^*Q, ξ) is realized as $\lambda|_\Sigma$ for some hypersurface. The symplectization $(\Sigma \times \mathbb{R}_{>0}, d(r\alpha))$ naturally embeds into $T^*Q \backslash Q$. A contact isotopy φ^t admits a lift to a Hamiltonian isotopy of $\Sigma \times \mathbb{R}_{>0}$, generated by a time-dependent 1-homogenous Hamiltonian H^t.

Fix $(\Sigma \subset T^*Q, \lambda|_\Sigma)$ representing S^*Q. For every positive contactomorphism φ the path $\{\varphi^t\}_{t\in[0,1]}$ can be chosen to be the Reeb flow for t near 0 and 1, as explained in the second part of the proof of [20, Proposition 6.2]. This means that the contact Hamiltonian h^t generating φ^t is constant $\equiv c_0$ for t near 0 and 1. Thus h^t permits smooth periodic or constant extensions. In this paper we always extend φ^t such that h^t is constant c_0 for $t \leqslant 0$ and periodic for $t \geqslant 0$, see Figure 2.1. The reason for this will become clear in the proof of Theorem 5.

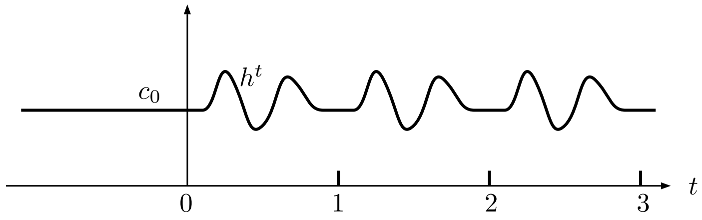

Figure 2.1: The function h^t, extended to $\mathbb{R}$.

Fix a Riemannian metric on Q and consider the induced metrics on TQ and T^*Q. Denote by μ_k the induced volume form on k-dimensional submanifolds of T^*Q and denote by $\mathrm{Vol}(\cdot)$ the integral of μ_k on $\cdot$. Using this metric, represent S^*Q as the 1-cosphere-bundle in T^*Q. Let φ be a positive contactomorphism on S^*Q and choose a path of positive contactomorphisms φ^t with $\varphi^1 = \varphi$. For each t we extend φ^t to $T^*Q\backslash Q$ by

$$\varphi^t(q, sp) \quad = \quad s\varphi^{st}(q, p) \tag{2.2.1}$$

for $s > 0$. Note that the q-coordinate of $\varphi^t(q, sp)$ is the q-coordinate of $\varphi^{st}(q, p)$. Also note that the extension (2.2.1) is not the Hamiltonian lift mentioned above. The following theorem relates the volume growth of a sphere S^*_qQ with the volume growth of its punctured sublevel disk $\dot{D}^*_qQ$ under a general twisted periodic flow. The proof can be found in the proof of Proposition 4.3 in [20], where the statement is proven for the slow growth of Reeb flows.

Theorem 3. *Let $\varphi^t : S^*Q \to S^*Q$ be a smooth family of diffeomorphisms with $\varphi^0 = \mathrm{id}$ whose generating vector field is 1-periodic. Extend φ^t to $T^*Q\backslash Q$ by (2.2.1). Then*

$$\begin{aligned} \gamma_{\mathrm{vol,exp}}(\varphi^1; \dot{D}^*_qQ) &\leqslant \gamma_{\mathrm{vol,exp}}(\varphi^1; S^*_qQ), \\ \gamma_{\mathrm{vol,pol}}(\varphi^1; \dot{D}^*_qQ) - 1 &\leqslant \gamma_{\mathrm{vol,pol}}(\varphi^1; S^*_qQ). \end{aligned}$$

Remark 2.2.1. The proof in [20] extends the flow on a hypersurface to its sublevel by extending the contact Hamiltonian to a homogeneous symplectic Hamiltonian. If one extends the flow directly as in (2.2.1) without using Hamiltonians and on a 1-cosphere bundle for an arbitrary Riemannian metric, the proof still goes through.

Albers and Frauenfelder [5] built a version of Rabinowitz–Floer homology for the space $\Omega^1_{q,q'}$ of $W^{1,2}$ paths from T^*_qQ to $T^*_{q'}Q$. Given a positive path of contactomorphisms φ^t we construct a certain modification H^t of a 1-homogenous Hamiltonian in T^*Q corresponding to φ^t, for details see Section 2.4. Define the functional $\mathcal{A}(\varphi^t; q, q') : \Omega^1_{q,q'} \times \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{A}(\varphi^t; q, q')(x, \eta) \quad = \quad \int_0^1 x^*\lambda - \eta \int_0^1 H^{\eta t}(x(t))\, dt. \tag{2.2.2}$$

A pair $(x, \eta) \in \Omega^1_{q,q'} \times \mathbb{R}$ is a critical point of the functional (2.2.2) if and only if it satisfies the equations

$$\begin{cases} \dot{x}(t) &= \eta X_{H^{\eta t}}(x(t)), \\ H^\eta(x(1)) &= 0. \end{cases}$$

The first equation implies that x is an orbit of X_{H^t}, but with time scaled by η. The second equation implies that the orbit ends on $(H^\eta)^{-1}(0)$. For $H^t = H$ autonomous $H^{-1}(0)$ is a hypersurface for which η plays the role of a Lagrange multiplier. For time-dependent H^t, however, there is no such surface and $H^{\eta t}(x(t))$ might be very large for $t < 1$. The chain complex $\mathrm{RFC}^T = \mathrm{RFC}^T(\varphi^t; q, q')$ of the filtered Rabinowitz–Floer homology is generated by the critical points of $\mathcal{A}(\varphi^t; q, q')$ with action value $\leqslant T \in \mathbb{R} \cup \{\infty\}$, for more details see Section 2.4. The boundary operator ∂^T is defined by counting solutions of a negative gradient flow with respect to a suitable L^2-metric. For $T \leqslant T'$ denote by $\iota^{T,T'} : \mathrm{RFC}^T \to \mathrm{RFC}^{T'}$ the inclusion. We denote by $\mathrm{RFC}^T_+ = \mathrm{RFC}^T / \iota^{0,T}(\mathrm{RFC}^0)$ the positive part of RFC^T and set $\mathrm{RFH}^T_+ = H(\mathrm{RFC}^T_+, \partial^T_+)$, where $\partial^T_+ : \mathrm{RFC}^T_+ \to \mathrm{RFC}^T_+$ is the induced boundary operator. For $T \leqslant T'$ let $\iota^{T,T'}_+ : \mathrm{RFC}^T_+ \to \mathrm{RFC}^{T'}_+$ be the homomorphism induced by inclusions.

The next four theorems describe the properties of RFH^T_+ used in this paper.

Theorem 4. *The functional $\mathcal{A}(\varphi^t; q, q')$ is Morse for generic points $q' \in Q$. In this case the Rabinowitz–Floer homology $\mathrm{RFH}^T_+(\varphi^t; q, q')$ is well-defined for all T.*

This theorem follows from standard theory, see [5, Sections 6 and 7]. Denote by Q_{gen} the set of q' for which $\mathcal{A}(\varphi^t; q, q')$ is Morse and that are different from q. Then Q_{gen} has full measure in Q. The following theorem is the key ingredient of our proof.

Theorem 5. *Let φ_i, $i = 0, 1$, be two positive contactomorphisms of S^*Q, let φ^t_i be corresponding positive paths of contactomorphisms and let Q^i_{gen} be the corresponding sets from Theorem 4. Then for every $q' \in Q^0_{\mathrm{gen}} \cap Q^1_{\mathrm{gen}}$ the exponential growth of $\dim(\iota^{T,\infty}_+)_* \mathrm{RFH}^T_+(\varphi^t_0; q, q')$ is positive if and only if the exponential growth of $\dim(\iota^{T,\infty}_+)_* \mathrm{RFH}^T_+(\varphi^t_1; q, q')$ is positive, and the polynomial growths coincide.*

Thus the positivity of the growth of the homology b n nbm is preserved by a deformation of the flow φ^t. Theorem 5 is stated in [5, Section 7]. We give a proof in Section 2.4.

Theorem 6. *Assume that φ^t_g is a geodesic flow. Then for $q' \in Q_{\mathrm{gen}}$ the Rabinowitz–Floer homology is isomorphic to the Morse homology of the energy functional $\mathcal{E}(x) = \int_0^1 g(\dot{x}, \dot{x})\, dt$ on the space $\Omega Q(q, q')$ of paths in Q from q to q':*

$$\mathrm{RFH}^T_+(\varphi^t_g; T^*_q Q, T^*_{q'} Q) \quad \cong \quad \mathrm{HM}^T(\mathcal{E}; \Omega Q(q, q')).$$

This isomorphism commutes with $(\iota^{T,\infty}_+)_*$.

This theorem is contained in Merry's work [29, Theorem 3.16]. Note that for autonomous flows (in particular geodesic flows) the action functional is the more classical Rabinowitz–Floer action functional, which Merry used in his work:

$$\mathcal{A}(x, \eta) = \int_0^1 x^* \lambda - \eta \int_0^1 H^{\eta t} = \int_0^1 x^* \lambda - \eta \int_0^1 H.$$

We finally need a link between the sublevel growth of the homology of $\mathcal{E}$ and the growth of the homology of the based loop space.

Theorem 7. *Let q' be non-conjugate to q.*
If $\gamma_{\exp}(\pi_1(Q)) > 0$ or if $\pi_1(Q)$ is finite and $\gamma_{\exp}(\Omega Q_0(q)) > 0$, then

$$\liminf_{T \to \infty} \frac{1}{T} \log \dim(\iota^{T,\infty}_+)_* \mathrm{HM}^T(\mathcal{E}; \Omega Q(q, q')) \quad > \quad 0.$$

If both $\gamma_{\mathrm{pol}}(\Omega Q_0(q))$ and $\gamma_{\mathrm{pol}}(\pi_1(Q))$ are finite, then

$$\liminf_{T \to \infty} \frac{1}{\log T} \log \dim(\iota^{T,\infty}_+)_* \mathrm{HM}^T(\mathcal{E}; \Omega Q(q, q')) \quad \geqslant \quad \gamma_{\mathrm{pol}}(\pi_1(Q)) + \gamma_{\mathrm{pol}}(\Omega Q_0(q)).$$

This is a result for geodesic flows taken from [26], [30] and [31] in the exponential case and from [21] in the polynomial case.

2.3 Proof of Theorem 1

Fix a Riemannian metric g on Q, consider the induced Riemannian metric on T^*Q and represent S^*Q as the 1-cosphere-bundle in T^*Q with respect to this metric as in Section 2.2. Given the positive contactomorphism $\varphi : S^*Q \to S^*Q$, choose a positive path of contactomorphisms φ^t with $\varphi^0 = id$, $\varphi^1 = \varphi$, extended in time as in Section 2.2. Fix $q \in Q$. The exponential (polynomial) volume growth of φ is not less than the exponential (polynomial) volume growth of the cosphere S^*_qQ under φ:

$$\gamma_{\mathrm{vol,exp}}(\varphi) \geqslant \gamma_{\mathrm{vol,exp}}(\varphi; S^*_qQ), \quad \gamma_{\mathrm{vol,pol}}(\varphi) \geqslant \gamma_{\mathrm{vol,pol}}(\varphi; S^*_qQ).$$

Extend φ^t to $T^*Q\backslash Q$ by (2.2.1). By Theorem 3,

$$\gamma_{\mathrm{vol,exp}}(\varphi; S^*_qQ) \geqslant \gamma_{\mathrm{vol,exp}}(\varphi^1; \dot{D}^*_qQ), \quad \gamma_{\mathrm{vol,pol}}(\varphi; S^*_qQ) \geqslant \gamma_{\mathrm{vol,pol}}(\varphi^1; \dot{D}^*_qQ) - 1.$$

The projection of $\varphi^T(\dot{D}^*_qQ)$ to the base manifold Q has an open and dense set of regular values $Q_{\mathrm{gen}}(T)$. The set $Q^0_{\mathrm{gen}} := \bigcap_{T\in\mathbb{N}} Q_{\mathrm{gen}}(T)\backslash\{q\}$ is comeager and thus has full measure. For $q' \in Q^0_{\mathrm{gen}}$ the disk $\varphi^T(\dot{D}^*_qQ)$ intersects $\dot{D}^*_{q'}Q$ transversally for all T and consequentially the number $\#(\varphi^T(\dot{D}^*_qQ) \cap \dot{D}^*_{q'}Q)$ is well defined. The volume of $\varphi^T(\dot{D}^*_qQ)$ for $T > 0$ is not less than the volume of its projection (counted with multiplicity) to Q^0_{gen},

$$\mathrm{Vol}\left(\varphi^T(\dot{D}^*_qQ)\right) \geqslant \int_{Q^0_{\mathrm{gen}}} \#\left(\varphi^T(\dot{D}^*_qQ) \cap \dot{D}^*_{q'}Q\right) dq'.$$

By the homogeneity of the flow, the elements of $\varphi^T(\dot{D}^*_qQ) \cap \dot{D}^*_{q'}Q$ correspond to orbits of φ^t from S^*_qQ to $S^*_{q'}Q$ that arrive at the latest at time T.

To count these orbits, define the Rabinowitz–Floer action functional $\mathcal{A}$ as in (2.2.2). It is Morse for $q' \in Q^1_{\mathrm{gen}}$, see Theorem 4. The intersection $Q^0_{\mathrm{gen}} \cap Q^1_{\mathrm{gen}}$ is also comeager. Since $q \neq q'$, the critical orbits (x, η) of $\mathcal{A}$ with $0 < \mathcal{A}(x, \eta) \leqslant T$ are in bijection with the orbits of φ^t from S^*_qQ to $S^*_{q'}Q$ that arrive at time $\leqslant T$. On the other hand, these critical orbits generate the Rabinowitz–Floer chain complex $\mathrm{RFC}^T_+(\varphi^t; T^*_qQ, T^*_{q'}Q)$. Thus

$$\begin{aligned}\mathrm{Vol}\left(\varphi^T(\dot{D}^*_qQ)\right) &\geqslant \int_{Q^0_{\mathrm{gen}}\cap Q^1_{\mathrm{gen}}} \dim \mathrm{RFC}^T_+(\varphi^t; T^*_qQ, T^*_{q'}Q)\, dq' \\ &\geqslant \int_{Q^0_{\mathrm{gen}}\cap Q^1_{\mathrm{gen}}} \dim(\iota^{T,\infty}_+)_* \,\mathrm{RFH}^T_+(\varphi^t; T^*_qQ, T^*_{q'}Q)\, dq'.\end{aligned}$$

Denote by Q^2_{gen} the set of points $q' \in Q$ that are not conjugate to q with respect to the geodesic flow φ^t_g of g. By Theorem 5 for $q' \in Q^0_{\mathrm{gen}} \cap Q^1_{\mathrm{gen}} \cap Q^2_{\mathrm{gen}}$ the exponential growth of

$$\dim(\iota^{T,\infty}_+)_* \,\mathrm{RFH}^T_+(\varphi^t; T^*_qQ, T^*_{q'}Q)$$

is positive if and only if the exponential growth of

$$\dim(\iota^{T,\infty}_+)_* \,\mathrm{RFH}^T_+(\varphi^t_g; T^*_qQ, T^*_{q'}Q)$$

is positive, and the polynomial growths coincide. The group $(\iota_+^{T,\infty})_* \operatorname{RFH}_+^T(\varphi_g^t; T_q^*Q, T_{q'}^*Q)$ is isomorphic to the group $(\iota^{T,\infty})_* \operatorname{HM}^T(\mathcal{E}; \Omega Q(q,q'))$ by Theorem 6. If $\gamma_{\exp}(\pi_1(Q)) > 0$ or if $\pi_1(Q)$ is finite and $\gamma_{\exp}(\Omega Q_0(q)) > 0$, then

$$\liminf_{T\to\infty} \frac{1}{T} \log \dim (\iota^{T,\infty})_* \operatorname{HM}^T(\mathcal{E}; \Omega Q(q,q')) > 0$$

by Theorem 7. Altogether, $\gamma_{\mathrm{vol,exp}}(\varphi^t; S_q^*Q) > 0$. If both $\gamma_{\mathrm{pol}}(\Omega Q_0(q))$ and $\gamma_{\mathrm{pol}}(\pi_1(Q))$ are finite, then

$$\liminf_{T\to\infty} \frac{1}{\log T} \log \dim (\iota^{T,\infty})_* \operatorname{HM}^T(\mathcal{E}; \Omega Q(q,q')) \geqslant \gamma_{\mathrm{pol}}(\pi_1(Q)) + \gamma_{\mathrm{pol}}(\Omega Q_0(q))$$

by Theorem 7. Altogether, $\gamma_{\mathrm{vol,pol}}(\varphi^t; S_q^*Q) \geqslant \gamma_{\mathrm{pol}}(\Omega Q_0(q)) + \gamma_{\mathrm{pol}}(\pi_1) - 1$.

2.4 Proof of Theorem 5

The Rabinowitz–Floer homology we used in Section 2.3 was constructed by Albers–Frauenfelder in [5]. For the proof of Theorem 5, they do not give details. In this paper we use a sandwiching argument which allows us to concentrate on monotone deformations, bypassing the problems that arise for more general deformations. We first introduce the action functional properly and then prove Theorem 5.

To define the action functional (2.2.2), we want to associate to a positive contactomorphism $\varphi : S^*Q \to S^*Q$ a Hamiltonian on T^*Q. First we choose a positive path $\{\varphi^t\}_{t\in[0,1]}$ with $\varphi^0 = id$ and $\varphi^1 = \varphi$. We represent the spherization as $(\Sigma \subset T^*Q, \lambda|_\Sigma)$ as in Section 2.2 and generate $\{\varphi^t\}_{t\in[0,1]}$ by the contact Hamiltonian $h^t : \Sigma \times [0,1] \to \mathbb{R}$. Since the path is positive, $h^t > 0$. As explained in Section 2.2 we can choose $h^t = c_0$ for some fixed constant c_0 in a neighbourhood of 0 and 1. We extend h^t on $\Sigma \times \mathbb{R}$ constantly for $t \leqslant 0$ and 1-periodically for $t \geqslant 0$.

Embed the symplectization of Σ in T^*Q and extend the coordinate r by $r = 0$ on $T^*Q \backslash (\Sigma \times \mathbb{R}_{>0})$. Fix $\kappa, R \geqslant 1$,

$$m \;\leqslant\; \min\{h^t(\theta) \mid \theta \in S^*Q,\, t \in S^1\}, \quad M \;\geqslant\; \max\{h^t(\theta) \mid \theta \in S^*Q,\, t \in S^1\}$$

and choose smooth functions $\beta : \mathbb{R}_{\geqslant 0} \to [0,1]$ and $h_m^M : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{>0}$ with

$$\beta(r) \;=\; \begin{cases} 0 & \text{if} \quad r \leqslant 1 \quad \text{or} \quad r \geqslant R\kappa + 1, \\ 1 & \text{if} \quad 2 \leqslant r \leqslant R\kappa, \end{cases}$$

$$h_m^M(r) \;=\; \begin{cases} m & \text{if} \quad r \leqslant 2, \\ M & \text{if} \quad R\kappa \leqslant r. \end{cases}$$

Then we define the Hamiltonian

$$H_{\kappa,R}^t(\theta, r) \;=\; r\Big(\beta(r)\, h^t(\theta) + (1 - \beta(r))\, h_m^M(r)\Big) - \kappa.$$

Apart from the shift by $-\kappa$, the Hamiltonian $H_{\kappa,R}^t$ is the 1-homogenous extension of h^t for $2 \leqslant r \leqslant R\kappa$ and the Hamiltonian of a Reeb flow for $r \leqslant 1$ and $r \geqslant R\kappa + 1$. The Rabinowitz–Floer functional $\mathcal{A}_{\kappa,R} := \mathcal{A}_{\kappa,R}(\varphi^t; q, q') : \Omega_{q,q'}^1 \times \mathbb{R} \to \mathbb{R}$ is given by

$$\mathcal{A}_{\kappa,R}(\varphi^t; q, q')(x, \eta) \;=\; \frac{1}{\kappa}\left(\int_0^1 x^*\lambda - \eta \int_0^1 H_{\kappa,R}^{\eta t}(x(t))\, dt\right).$$

A pair $(x, \eta) \in \Omega^1_{q,q'} \times \mathbb{R}$ is a critical point of $\mathcal{A}_{\kappa,R}$ if and only if it satisfies the equations

$$\begin{cases} \dot{x}(t) & = \eta X_{H^{\eta t}_{\kappa,R}}(x(t)), \\ H^{\eta}_{\kappa,R}(x(1)) & = 0. \end{cases}$$

The factor $\frac{1}{\kappa}$ does not change the critical points, just the critical values. The functional $\mathcal{A}_{\kappa,R}$ depends on the choice of κ, R, but the following lemma shows that for large enough κ, R the critical points with action in a fixed range are independent of the choice. This justifies that we suppressed κ and R in the main text.

Lemma 2.4.1. *Given $a < b$, there are constants κ_0, R_0 such that for $\kappa \geqslant \kappa_0$ and $R \geqslant R_0$ the following holds. If (x, η) is a critical point with $a \leqslant \mathcal{A}_{\kappa,R}(x, \eta) \leqslant b$, then the radial component of x stays in $[2, R\kappa]$ for $t \in [0, 1]$ and $\mathcal{A}_{\kappa,R}(x, \eta) = \eta$.*

Proof. A detailed proof is given in [5, Proposition 4.3]. □

We define the L^2-metric g^κ on $\Omega^1_{q,q'} \times \mathbb{R}$ by choosing an almost complex structure J compatible with ω and setting

$$g^\kappa((\hat{x}, \hat{\eta}), (\hat{x}', \hat{\eta}')) = \frac{1}{\kappa} \int_0^1 \omega(\hat{x}, J\hat{x}')\, dt + \frac{\hat{\eta}\hat{\eta}'}{\kappa}.$$

With this scalar product the gradient of $\mathcal{A}_{\kappa,R}(\varphi^t; q, q')$ has the form

$$\nabla \mathcal{A}_{\kappa,R}(\varphi^t; q, q')(x, \eta) = \begin{pmatrix} \dot{x} - \eta X_{H^{\eta t}}(x(t)) \\ \int_0^1 H^{\eta t}(x(t)) + \eta t \dot{H}^{\eta t}(x(t))\, dt \end{pmatrix}.$$

Assume that the functional $\mathcal{A}_{\kappa,R}(\varphi^t; q, q')$ is Morse. The chain complex $\mathrm{RFC}^b = \mathrm{RFC}^b(\varphi^t; q, q')$ of the filtered Rabinowitz–Floer homology is generated by the critical points of $\mathcal{A}_{\kappa,R}(\varphi^t; q, q')$ with action $\leqslant b \in \mathbb{R}$. The boundary operator ∂^b is defined by counting solutions of the negative gradient flow. For $a \leqslant b$ denote by $\iota^{a,b} : \mathrm{RFC}^a \to \mathrm{RFC}^b$ the inclusion. Choose κ_0, R_0 so large that Lemma 2.4.1 holds for critical points with action in $[a, b]$. Denote $\mathrm{RFC}^b_a = \mathrm{RFC}^b / \iota^{a,b}(\mathrm{RFC}^a)$ and set $\mathrm{RFH}^b_a = H(\mathrm{RFC}^b_a, \partial^b_a)$, where $\partial^b_a : \mathrm{RFC}^b_a \to \mathrm{RFC}^b_a$ is the induced boundary operator. These groups are independent of $\kappa \geqslant \kappa_0, R \geqslant R_0$. For such κ, R and for $a' \leqslant a \leqslant b \leqslant b'$ let $\iota^{b,b'}_{a,a'} : \mathrm{RFC}^b_a \to \mathrm{RFC}^{b'}_{a'}$ be the homomorphism induced by inclusions. Denote by $\mathrm{RFC}^b_{-\infty} = \lim_{a \to -\infty} \mathrm{RFC}^b_a$ the inverse limit and for $a \in \mathbb{R} \cup \{-\infty\}$ by $\mathrm{RFC}^\infty_a = \lim_{b \to \infty} \mathrm{RFC}^b_a$ the direct limit, while adjusting κ, R. For $\kappa \geqslant \kappa_0, R \geqslant R_0$, $\mathrm{RFC}^b_a = \mathrm{RFC}^b_{-\infty} / \iota^{a,b}_{-\infty,-\infty}(\mathrm{RFC}^a_{-\infty})$. For better readability we omit the subscript $-\infty$ and denote by $\mathrm{RFC}^T_+ = \mathrm{RFC}^T_0$ the positive part of the chain complex, by $\iota^{T,T'}_+ = \iota^{T,T'}_{0,0}$ the inclusion and by $\mathrm{RFH}^T_+ = \mathrm{RFH}^T_0$ the positive part of the homology.

Consider now a family φ^t_s of paths of contactomorphisms induced by a family of Hamiltonians h^t_s such that $\partial_s h^t_s = 0$ for $s \notin [0, 1]$. Suppose that for the associated family of functionals $\mathcal{A}_s(\varphi^t_s) := \mathcal{A}_{\kappa,R}(\varphi^t_s; q, q')$ the constants from Lemma 2.4.1 are chosen uniformly large enough. We set $\mathcal{A}_- = \mathcal{A}_s$ for $s \leqslant 0$ and $\mathcal{A}_+ = \mathcal{A}_s$ for $s \geqslant 1$. The continuation homomorphism $\Phi : \mathrm{RFC}^\infty(\mathcal{A}_-) \to \mathrm{RFC}^\infty(\mathcal{A}_+)$ is defined in the standard way by counting solutions $(x(s), \eta(s))$ of the equation

$$\partial_s(x(s), \eta(s)) = -\nabla \mathcal{A}_s(x(s), \eta(s)), \tag{2.4.1}$$

such that $\lim_{s \to \pm\infty}(x_s, \eta_s) = (x_\pm, \eta_\pm)$ exist and are critical points of $\mathcal{A}_\pm$. Then Φ induces an isomorphism $\mathrm{RFH}^\infty(\mathcal{A}_-) \to \mathrm{RFH}^\infty(\mathcal{A}_+)$, because η is bounded along deformations, cf. [10, Corollary 3.4].

Proof of Theorem 5. First we consider two positive Hamiltonians h_0^t, h_1^t such that $h_0^t \leqslant h_1^t$. We show that the action is non-increasing along solutions of (2.4.1).

For the deformation from h_0^t to h_1^t define a function $\chi : \mathbb{R} \to [0,1]$ such that $\chi' \geqslant 0$ and

$$\chi(s) \quad = \quad \begin{cases} 0 & \text{if} \quad s \leqslant 0, \\ 1 & \text{if} \quad s \geqslant 1, \end{cases}$$

and set $h_s^t := h_0^t + \chi(s)(h_1^t - h_0^t)$. Denote by H_s^t, φ_s^t and $\mathcal{A}_s$ the associated Hamiltonians, paths of contactomorphisms and functionals. The deformation satisfies

$$\tfrac{d}{ds} H_s^t \quad = \quad \chi'(s)(H_1^t - H_0^t) \quad = \quad \chi'(s)\, r\, \beta(r)(h_1^t - h_0^t) \quad \geqslant \quad 0.$$

For $(x, \eta) \in \Omega^1_{q,q'} \times \mathbb{R}$,

$$\frac{\partial}{\partial s} \mathcal{A}_s(x, \eta) \quad = \quad \int_0^1 -\frac{\eta}{\kappa} \chi'(s)(H_1^{\eta t} - H_0^{\eta t})\, dt.$$

Now consider a solution $(u(s), \eta(s))$ of (2.4.1). Denote $E = \int_{-\infty}^{\infty} \|\partial_s(x(s), \eta(s))\|^2 \, ds$ and $\mathcal{A}_\pm = \mathcal{A}_\pm(u_\pm, \eta_\pm)$. We calculate

$$\begin{aligned} \mathcal{A}_+ \quad &= \quad \mathcal{A}_- + \int_{-\infty}^{\infty} \frac{d}{ds} \mathcal{A}_s(x(s), \eta(s))\, ds \\ &= \quad \mathcal{A}_- + \int_{-\infty}^{\infty} \Big(\frac{\partial}{\partial s} \mathcal{A}_s\Big)(x(s), \eta(s)) + \big\langle \nabla \mathcal{A}_s\big(x(s), \eta(s)\big), \partial_s\big(x(s), \eta(s)\big)\big\rangle\, ds \\ &= \quad \mathcal{A}_- - E + \int_{-\infty}^{\infty} \int_0^1 -\frac{\eta(s)}{\kappa} \chi'(s)(H_1^{\eta t} - H_0^{\eta t})\, dt\, ds \end{aligned}$$

If $\eta(s) \geqslant 0$, then $-\frac{\eta(s)}{\kappa}\chi'(s)(H_1^{\eta t} - H_0^{\eta t}) \leqslant 0$. If $\eta(s) \leqslant 0$, then $h_0^{\eta t} = h_1^{\eta t} = c_0$ and thus $-\frac{\eta(s)}{\kappa}\chi'(s)(H_1^{\eta t} - H_0^{\eta t}) = 0$. It follows that $\mathcal{A}_+ \leqslant \mathcal{A}_-$.

We have just shown that Φ restricts to $\Phi|_{\mathrm{RFC}^T(\varphi_0^t)} : \mathrm{RFC}^T(\varphi_0^t) \to \mathrm{RFC}^T(\varphi_1^t)$. Furthermore $\varphi_0^t = \varphi_1^t$ for $t \leqslant 0$. Thus $\mathcal{A}(\varphi_0^t)$ and $\mathcal{A}(\varphi_1^t)$ have the same critical points with nonpositive action, and constant critical points (x, η) with $\eta \leqslant 0$ are solutions of (2.4.1). Together with the fact that the action is non-increasing along solutions of (2.4.1) we get that

$$\Phi|_{\mathrm{RFC}^0(\varphi_0^t)} \; : \; \mathrm{RFC}^0(\varphi_0^t) \;\to\; \mathrm{RFC}^0(\varphi_1^t)$$

is a lower diagonal isomorphism. Thus for the homomorphism Φ_* induced in the quotient we have

$$\begin{aligned} \Phi_*(\mathrm{RFC}_+^T(\varphi_0^t)) \quad &= \quad \Phi(\mathrm{RFC}^T(\varphi_0^t))/\Phi(\mathrm{RFC}^0(\varphi_0^t)) \\ &= \quad \Phi(\mathrm{RFC}^T(\varphi_0^t))/\iota^{0,T}\, \mathrm{RFC}^0(\varphi_1^t) \\ &\subseteq \quad \mathrm{RFC}_+^T(\varphi_1^t). \end{aligned}$$

Since Φ induces an isomorphism in RFH^∞, abbreviating $\iota = \iota_+^{T,\infty}$, we conclude that

$$\dim \iota_*\, \mathrm{RFH}_+^T(\varphi_0^t) \quad \leqslant \quad \dim \iota_*\, \mathrm{RFH}_+^T(\varphi_1^t). \tag{2.4.2}$$

Now choose $c, C > 0$ such that $c \leqslant h^t \leqslant C$. Denote by $\varphi_c^t, \varphi_{h^t}^t$ and φ_C^t the induced flows. The constants c, C are not equal to c_0 for t near 0 or 1, so we need to modify them to fit our setup. From the proof of [20, Proposition 6.2] it becomes clear that there are functions $h_c^t, h_C^t : S^*Q \times [0,1] \to \mathbb{R}$ with $h_c^t = h_C^t = c_0$ for t near 0 and 1, that satisfy $h_c^t \leqslant h^t \leqslant h_C^t$, and such that the flows $\varphi_{h_c^t}^t$ and $\varphi_{h_C^t}^t$ induced by h_c^t and h_C^t are time-reparametrizations of the geodesic flows φ_c^t and φ_C^t that satisfy $\varphi_{h_c^t}^1 = \varphi_c^1$ and $\varphi_{h_C^t}^1 = \varphi_C^1$. Extend h_c^t and h_C^t as in Section 2.2, see Figure 2.2.

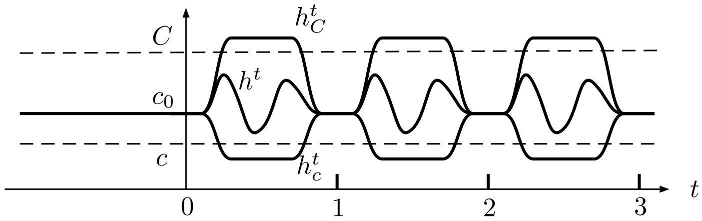

Figure 2.2: The functions h_c^t, h^t and h_C^t, extended to $\mathbb{R}$.

We apply (2.4.2) twice, first to a monotone deformation from h_c^t to h^t and then to a monotone deformation from h^t to h_C^t. By construction of h_c^t and h_C^t it is clear that there exists a function $\tau : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ such that $\varphi_{h_c^t}^{\tau(t)} = \varphi_{h_C^t}^t$ and such that $t \leqslant \tau(t) \leqslant 2\frac{C}{c}t$. Thus,

$$\iota_* \operatorname{RFH}_+^{\tau(T)}(\varphi_{h_c^t}^t) \quad \cong \quad \iota_* \operatorname{RFH}_+^T(\varphi_{h_C^t}^t).$$

With (2.4.2), this results in

$$\dim \iota_* \operatorname{RFH}_+^T(\varphi_{h_c^t}^t) \leqslant \dim \iota_* \operatorname{RFH}_+^T(\varphi^t) \leqslant \dim \iota_* \operatorname{RFH}_+^T(\varphi_{h_C^t}^t) = \dim \iota_* \operatorname{RFH}_+^{\tau(T)}(\varphi_{h_c^t}^t).$$

We conclude that for every positive path of contactomorphisms the Rabinowitz–Floer homology grows exponentially if and only if it is grows exponentially for a Reeb flow, and it has the same polynomial growth as for a Reeb flow. □

2.5 Complementary examples

In this section we provide two examples that confirm Example 2.1.4.

The first example is described in detail in [28, Section 7]. Consider the semidirect product $G = \mathbb{R}^2 \ltimes \mathbb{R}$ with multiplication

$$(x, y, z) \bullet (x', y', z') = (x + e^z x', y + e^{-z} y', z + z').$$

Choose a cocompact lattice $\Gamma \subseteq G$ of exponential growth. Then the fundamental group Γ of the closed manifold $Q = \Gamma \backslash G$ has exponential growth. The functions

$$(M_x, M_y, M_z) = (e^z p_x, e^{-z} p_y, p_z)$$

on T^*G are left-invariant and thus descend to T^*Q. Consider the Hamiltonian

$$H = \frac{1}{2}\left((M_x + 1)^2 + M_y^2 + M_z^2\right)$$

that describes an exact magnetic flow on Q. For every energy value $k > 0$ the hypersurface $\Sigma_k = H^{-1}(k)$ is contactomorphic to the spherization S^*Q. The zero section is enclosed by Σ_k if and only if $k > \frac{1}{2}$, and the zero section is contained in $\Sigma_{\frac{1}{2}}$. For every $k > \frac{1}{2}$ the flow φ_k^t induced by H on Σ_k is a positive contact isotopy and thus has positive topological entropy. On the other hand Macarini–Schlenk showed that for $k \leqslant \frac{1}{2}$ the flow φ_k^t has zero topological entropy. In particular, $\varphi_{\frac{1}{2}}^t$ is the smooth limit of positive contact isotopies, and thus a non-negative contact isotopy. It fails to be positive only on the zero section, which is a codimension 2 subset of $\Sigma_{\frac{1}{2}}$.

The second example was pointed out to me by Marcelo Alves. Let Σ_k be the closed orientable surface of genus $k \geqslant 2$. Choose a Riemannian metric g on Σ_k such that a closed disc $D \subseteq \Sigma_k$ is isometric to a round sphere S^2 deprived of an open disc that is strictly contained in a hemisphere. Equip the spherization $S^*\Sigma_k$ with the contact form whose Reeb flow is the geodesic flow φ_g^t. Let $U \subseteq S^*\Sigma_k$ be the set of points whose φ_g^t flow lines intersect fibers over $\Sigma_k \backslash D$. By construction, U^c is closed with non-empty interior. Further, the geodesic flow φ_g on U^c is periodic (the flow lines project to great circles on S^2). Thus U^c is a closed invariant set on which φ_g^t has zero topological entropy, and $\overline{U}$ is also a closed invariant set. From the maximum formula for topological entropy on decompositions into closed invariant sets [27, Proposition 3.1.7(2)] we conclude that φ_g has positive topological entropy on $\overline{U}$. Now consider the contact Hamiltonian flow φ^t induced by a Hamiltonian that is constant 1 on $\overline{U}$ and 0 outside a small neighbourhood $\widetilde{U}$ of $\overline{U}$. Since φ^t coincides with φ_g^t on $\overline{U}$, it has positive topological entropy, but restricts to the identity on $\widetilde{U}^c$, which is a set with nonempty interior. Note that for all $\varepsilon > 0$ we can choose g such that $\mu_g(\widetilde{U}^c) \geqslant (1 - \varepsilon)\mu_g(S^*(\Sigma_k))$, where μ_g is the measure induced by g.

Chapter 3

Growth of wrapped Floer homology and positive contactomorphisms

3.1 Introduction and results

An important problem in the study of dynamical systems is to understand the complexity of the mappings in question. A good numerical measure for complexity is the topological entropy. Consider a compact manifold M and a class of diffeomorphisms $\mathcal{D}$ of M. A much studied question is whether a generic map from $\mathcal{D}$ has positive topological entropy. A very different question is whether *every* map from $\mathcal{D}$ has positive topological entropy. This latter question is only interesting under further assumptions on M and $\mathcal{D}$. Here, we assume that M is a compact manifold endowed with a contact structure ξ, namely a completely non-integrable distribution of hyperplanes in the tangent bundle TM. If we also assume that ξ is co-orientable, namely that there exists a 1-form α on M with $\xi = \ker \alpha$, then associated with every choice of such an α there is a natural flow generated by the vector field R_α implicitly defined by the two equations

$$d\alpha(R_\alpha, \cdot) = 0, \qquad \alpha(R_\alpha) = 1.$$

Such flows are called Reeb flows of α. They arise as the restriction of many classical Hamiltonian systems to fixed energy levels. In particular, geodesic flows are Reeb flows.

The first result on positive topological entropy of all Reeb flows on a class of contact manifolds was obtained by Macarini–Schlenk in [28], who generalized previous results by Dinaburg, Paternain–Petean and Gromov on geodesic flows: Every Reeb flow on the cosphere bundle over a closed manifold Q has positive topological entropy, provided that the topology of Q is "sufficiently complicated" (for instance, if the fundamental group or the homology of the based loop space of Q has exponential growth).

Reeb flows form a quite special class of mappings on a contact manifold. In [14] the above result was generalized to a much larger class of diffeomorphisms on the same manifolds, namely to time-dependent Reeb flows. Let (M, ξ) be a co-oriented contact manifold. A smooth path φ_t on M is a time-dependent Reeb flow if it is generated by a time-dependent vector field R_{α_t}, where each α_t is a contact form for ξ. There is a more topological perspective on such flows: They are exactly the positive contact isotopies on (M, ξ), namely the isotopies φ_t with $\varphi_0 = \mathrm{id}$

that are everywhere positively transverse to ξ:

$$\alpha\left(\tfrac{d}{dt}\varphi_t(x)\right) > 0$$

for all t and all $x \in M$, for one and hence any contact form α for ξ.

Definition 3.1.1. A *positive contactomorphism* on (M, ξ) is the end point φ_1 of a positive contact isotopy on φ_t, $t \in [0, 1]$.

The first results on positive topological entropy of all Reeb flows on contact manifolds different from cosphere bundles were given by Alves [1, 2, 3] in dimension three. More recently, Alves and Meiwes [6] constructed many examples of higher dimensional contact manifolds for which every Reeb flow has positive topological entropy. In particular, they found on every sphere of dimension at least seven a contact structure with this property. They asked the natural question whether their results extend from Reeb flows to positive contactomorphisms. The present paper answers this question in the affirmative.

We will work in the following geometric setting that is further explained in Section 3.2.

Assumption 3.1.2. The pair (W, L) consists of a Liouville domain (W, ω, λ) with compact contact boundary $(M, \xi = \ker \lambda|_M)$ and an asymptotically conical exact Lagrangian L with connected Legendrian boundary $\Lambda = \partial L$ such that $\lambda|_L = 0$, such that $[\omega]|_{\pi_2(W,L)} = 0$, and such that (λ, L) is regular. Here, regular means that $\bigcup_{t \neq 0} \varphi^t_{\lambda|_M}(\Lambda)$ and Λ intersect transversely, where $\varphi^t_{\lambda|_M}$ is the Reeb flow of $\lambda|_M$.

Under this assumption we can define a $\mathbb{Z}_2$-vector space $\mathrm{WH}(W, L)$, which we call wrapped Floer homology, see Section 3.2.1 for the definition. This is a filtered homology, thus for every a there is a vector space $\mathrm{WH}^a(W, L)$ and a morphism $\iota_a : \mathrm{WH}^a(W, L) \to \mathrm{WH}(W, L)$. The vector spaces $\mathrm{WH}^a(W, L)$ are finite dimensional. The following notion is taken from [6].

Definition 3.1.3 (Symplectic growth)**.** For a function $f : X \to \mathbb{R}$, where $X = \mathbb{N}$ or $X = \mathbb{R}$, we define the exponential growth of f as

$$\Gamma(f(a)) = \limsup_{a \to \infty} \frac{1}{a} \log(f(a)).$$

If $\Gamma(f) > 0$, we say that f grows exponentially. We define the symplectic growth of the pair (W, L) as the growth in dimension of the filtered wrapped Floer homology

$$\Gamma^{\mathrm{symp}}(W, L) = \Gamma(\dim \mathrm{WH}^a(W, L)).$$

All but finitely many of the generators of the chain complexes underlying wrapped Floer homology correspond to Reeb chords from Λ to itself, and the filtration corresponds to the length of these Reeb chords. With length we mean time of arrival.

Alves and Meiwes showed that if $\Lambda = \partial L$ is a sphere, then positivity of $\Gamma^{\mathrm{symp}}(W, L)$ implies that every Reeb flow on $M = \partial W$ has positive topological entropy. Our main result is that this theorem extends to positive contactomorphisms.

Theorem 8. *Under Assumption 3.1.2 assume that* $\Gamma^{\mathrm{symp}}(W, L) > 0$. *Then the topological entropy of every positive contactomorphism of* (M, ξ) *is positive.*

Since a generic fiber of a cosphere bundle over a closed manifold satisfies Assumption 3.1.2, this result also generalizes the works [28] mentioned earlier.

Theorem 8 implies in particular that in the examples constructed by Alves and Meiwes every positive contactomorphism has positive topological entropy.

Corollary 9. *Let M be the sphere S^{2n+1} of dimension $2n+1 \geqslant 7$, or $S^3 \times S^2$, or the boundary of a plumbing tree whose vertices are unit codisc bundles over manifolds of dimension $\geqslant 4$. Then M admits a contact structure ξ such that every positive contactomorphism of ξ has positive topological entropy.*

Method of proof

Alves and Meiwes prove their theorem using wrapped Floer homology WH, which is a Lagrangian (or open string) version of symplectic homology. WH has the advantage that it admits product structures, notably a Pontrjagin product, and is functorial under various geometric operations, which Alves and Meiwes ingeniously combine to find examples such that $\mathrm{WH}(W, L)$ has exponential growth. Then they construct a $\mathrm{WH}(W, L)$-module structure on the wrapped Floer homology $\mathrm{WH}(W, L \to L')$, whose generators are Reeb chords from L to nearby Lagrangians L' to find positive volume growth and thus positive topological entropy.

In our time-dependent case we did not succeed to prove Theorem 8 by working with WH alone. Instead, we also work with time-dependent Lagrangian Rabinowitz–Floer homology (abbreviated by TH), which is Lagrangian Rabinowitz–Floer homology based on a time-dependent Hamiltonian and whose generators correspond to time-dependent Reeb chords from L to itself, see Section 3.3.3 for the definition. We use TH because so far there seems to be no wrapped Floer homology that encodes time-dependent Reeb dynamics in a transparent way. The problem with WH is that the main tool for understanding the homology are radial Hamiltonians, whose radial coordinate explicitly corresponds to the slope and thus to the length of Reeb chord. But for time-dependent Hamiltonians this correspondence breaks down since time-dependent Hamiltonians are not constant along their chords.

In TH, however, the information of the length of a chord and its radial position are decoupled, thus the loss of radial control does not affect the understanding of the dynamics. On the other hand, it seems hard to set up a Pontrjagin product structure on TH or a $\mathrm{TH}(W, L)$-module structure on the homology $\mathrm{TH}(W, L \to L')$ whose generators correspond to time-dependent Reeb chords from L to L'. Our solution is to combine the advantages of the two theories: We show that the growth of WH, which was obtained through algebraic structures by Alves and Meiwes, implies growth of TH, which then can be used to count the chords of a time-dependent Hamiltonian.

The transition to counting chords between different Lagrangians L, L' is then performed inside the action functional for TH, capitalizing on the fact that in TH we can encode geometric information directly in the Hamiltonian. Thus, we do not need a module structure to deduce volume growth from growth of TH.

To relate WH to TH, we use various intermediate homologies, closely following [12]. As a first step we use V-shaped wrapped Floer homology $\check{\mathrm{W}}\mathrm{H}$ (in the language of [13] the wrapped Floer homology of the trivial Lagrangian cobordism). An alternative approach would be to follow [12] and to elaborate a long exact sequence connecting wrapped Floer homology, wrapped Floer cohomology and the Morse cohomology of Lagrangians and Legendrians. However, since we are only interested in the asymptotic behavior of the homology, it is enough to relate the positive parts of the homologies, which is shorter since we do not need to consider cohomology.

Propositions along the way

The following is a list of our results that combine to the proof of Theorem 8. The individual results might be of independent interest.

Proposition 3.1.4. *Under Assumption 3.1.2, for all $a, b \notin \mathcal{S}$ with $0 < a < b$ we have*

$$\mathrm{WH}^{(a,b)}(W, L) \cong \check{\mathrm{w}}\mathrm{H}^{(a,b)}(W, L),$$

where $\mathcal{S}$ is the set of lengths of Reeb chords from L to L. These isomorphisms commute with morphisms induced by inclusion of filtered chain complexes.

The V-shaped wrapped Floer homology $\check{\mathrm{w}}\mathrm{H}$ can be identified with the standard Rabinowitz–Floer homology AH of (W, L) (Here the A stands for autonomous) through Rabinowitz–Floer homologies with perturbed Lagrange multiplier. This identification is analogous to the long exact sequence connecting symplectic homology and closed string Rabinowitz–Floer homology discovered in [12]:

Proposition 3.1.5. *Under Assumption 3.1.2, for all $a, b \notin \mathcal{S}$ with $-\infty < a < b < \infty$ we have*

$$\check{\mathrm{w}}\mathrm{H}^{(a,b)}(W, L) \cong \mathrm{AH}^{(a,b)}(W, L).$$

These isomorphisms commute with morphisms induced by inclusion of filtered chain complexes.

When combined, these two theorems imply that the positive part of WH coincides with the positive part of AH in a way that preserves the filtration.

The dimension of the positive part of AH is a lower bound to the number of Reeb chords from Λ to Λ. We now deform the action functional of AH to the one of time-dependent Rabinowitz–Floer homology TH in order to count time-dependent Reeb chords from Λ to Λ. We show that monotone deformations do not decrease the growth of the dimension of filtered homology groups, and by a sandwiching argument we then show :

Proposition 3.1.6 (Preservation of positivity of growth)**.** *Let (W, L) and h^t be as in Assumption 3.3.4 below, which is analogous to Assumption 3.1.2 in the new setup. If the exponential growth of $\dim \mathrm{TH}^{(0,T)}(h^t)$ is positive, then the exponential growth of $\dim \mathrm{TH}^{(0,T)}(\widetilde{h}^t)$ is also positive for every other $\widetilde{h}^t$ that satisfies Assumption 3.3.4.*

Quantitatively, if $c \leqslant h^t \leqslant C$, then the exponential growth γ of $\dim \mathrm{TH}^{(0,T)}(h^t)$ satisfies $c\Gamma^{\mathrm{symp}}(W, L) \leqslant \gamma \leqslant C\Gamma^{\mathrm{symp}}(W, L)$.

Finally, to find positive topological entropy, we need to count the chords of our positive path of contactomorphisms between *different* Legendrians. The following proposition is established again by deforming the functional.

Proposition 3.1.7. *Let (W, L) and h^t be as in Assumption 3.3.4. Suppose that $\Gamma^{\mathrm{symp}}(W, L) > 0$. Let Λ' be a Legendrian that is isotopic through Legendrians to $\Lambda = \partial L$. Then the number of φ^t-chords from Λ to Λ' of length $\leqslant T$ grows exponentially.*

Quantitatively, let ψ be a contactomorphism that takes Λ to Λ' so that $(\psi^{-1})^*\alpha = f\alpha$. Then the exponential growth of the number of φ^t-chords from Λ to Λ' of length $\leqslant T$ is at least $\geqslant \min f \cdot \min h^t \cdot \Gamma^{\mathrm{symp}}(W, L)$.

Organization of the paper

In Section 3.2 we describe the general geometric setup and the Floer homology of an action functional in a generality that suffices for this paper. In Subsections 3.2.1 and 3.2.2 we describe wrapped Floer homology and V-shaped wrapped Floer homology and show that their positive parts coincide (Proposition 3.1.4).

In Section 3.3 we discuss the results concerning Rabinowitz–Floer homology. In Subsection 3.3.1 we define the standard autonomous Rabinowitz–Floer homology AH. In Subsection 3.3.2 we show that AH is isomorphic to V-shaped wrapped Floer homology $\check{\text{w}}$H(Proposition 3.1.5) by introducing perturbed Rabinowitz–Floer homology. In Subsection 3.3.3 we introduce time-dependent Rabinowitz–Floer homology TH and show how the homological growth changes under the change of the dynamics (Propositions 3.1.6 and 3.1.7).

In Section 3.4 we puzzle together all these results to prove Theorem 8.

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3.2 Wrapped Floer Homology

In this section we explain the theorems concerning wrapped Floer homologies. We begin with an exposition of the geometric setup, where we explain the terms in Assumption 3.1.2 and justify these assumptions. Then we outline the construction of Lagrangian Floer homology in a general setting.

In the two following subsections we present wrapped Floer homology and V-shaped wrapped Floer homology and show that their positive parts coincide. In all versions we use $\mathbb{Z}_2$-coefficients and no grading. The wrapped Floer homology we present here was introduced in [7] and coincides with the version in [6]. For the entire section we follow [12], where analogous results for symplectic homology were established.

Liouville domains

A Liouville domain (W, ω, λ) is a compact manifold W with boundary $\partial W = M$ endowed with an exact symplectic form $\omega = d\lambda$ and a choice of primitive λ such that the so-called Liouville vector field Y defined by $\iota_Y \omega = \lambda$ is transverse to the boundary, pointing outwards. Then $(\partial W = M, \xi = \ker \alpha)$, where $\alpha = \lambda|_M$, is a contact manifold. Let $\widehat{W} = W \cup_M M \times [1, \infty)_r$ be the completion of W. The symplectization $(\widehat{M} = M \times \mathbb{R}^{>0}, d(r\alpha))$ embeds into $\widehat{W}$ such that $M \times \{1\} = M$, such that $\lambda = r\alpha$ and such that the Liouville vector field coincides with $r\partial_r$ on $\widehat{M}$.

Example 3.2.1. A starshaped domain $(D, dy \wedge dx, \frac{1}{2}(y\,dx - x\,dy))$ in $\mathbb{R}^{2n}$ is a Liouville domain with completion $\mathbb{R}^{2n}$ and Liouville vector field $x\partial_x + y\partial_y$.

Similarly, the sublevel $(D^*Q, dp \wedge dq, p\,dq)$ of a fiberwise starshaped hypersurface of T^*Q is a Liouville domain with completion T^*Q and Liouville vector field $p\,\partial_p$.

Asymptotically conical exact Lagrangians

Let $L \subset W$ be a Lagrangian submanifold with Legendrian boundary $\partial L = \Lambda \subset M$. We say that L is conical in a set $U \subset W$ if the Liouville vector field is tangent to $L \cap U$. We assume that

- L is exact, i.e. $\lambda|L = df$ for some function $f : L \to \mathbb{R}$,
- L is asymptotically conical, i.e. L is conical in $M \times [1 - \varepsilon, 1]$ for $\varepsilon > 0$ small enough.

An exact asymptotically conical Lagrangian satisfies $L \cap (M \times [1-\varepsilon, 1]) = \Lambda \times [1-\varepsilon, 1]$ for $\varepsilon > 0$ small enough. Since Λ is Legendrian and λ vanishes along ∂_r, $\lambda|_L$ vanishes in the region where L coincides with $\Lambda \times [1-\varepsilon, 1]$, and hence f is locally constant in this region. Thus one can extend an asymptotically conical exact Lagrangian L to an exact Lagrangian $\widehat{L} = L \cup_\Lambda (\Lambda \times [1, \infty))$ in $\widehat{M}$ by extending f locally constantly. We will also refer to $\widehat{L}$ as asymptotically conical.

Later, we are mainly interested in the case where Λ is a sphere, so we will assume throughout that Λ is connected. We can modify the Liouville domain such that $\lambda|_L = 0$ as follows. For connected Λ and with $\lambda|_L = df$, we can change f by a constant such that $f \equiv 0$ on a collar neighborhood of $\widehat{W}\backslash W$. Extend f to a function F on $\widehat{W}$ with support inside W minus a collar neighborhood of the boundary, and then add $-dF$ to λ. With respect to this new λ the Lagrangian is still exact with $\lambda|_L \equiv 0$. This changes λ in the interior of W, but not on the boundary $\partial W = M$, and thus also the Reeb flow on M, in which we are ultimately interested, is unchanged.

Example 3.2.2 (Continuation of Example 3.2.1)**.** Any Lagrangian plane through 0 is an (asymptotically) conical Lagrangian in $(\mathbb{R}^{2n}, dy \wedge dx, \frac{1}{2}(ydx - xdy))$.

A cotangent fiber T_q^*Q is an (asymptotically) conical Lagrangian in T^*Q.

Remark 3.2.3. If L_1, L_2 are two asymptotically conical exact Lagrangians that intersect, then we can in general not change λ such that Assumption 3.1.2 holds for both Lagrangians simultaneously. As a result there is an additional term $+[f_1(x(0)) - f_2(x(1))]$ in the action functional $\mathcal{A}_H$ defined below. Therefore the intersection of the Lagrangians, that later on will correspond to constant orbits, will not have zero action and thus one cannot separate the constant orbits from the others by action. The subsequent results should also be valid for a pair of Lagrangians, modulo finite dimensional terms stemming from the impossibility of separating different kinds of orbits. These terms do not influence the asymptotic behavior of the homology. In this paper, however, we take a different approach and consider pairs of Lagrangians only in the proof of Proposition 3.1.7, where we use a trick to detect the chords between L_1 and L_2 in the space of paths from L_1 to L_1.

Path space, Reeb chords and regularity

For an asymptotically conical exact Lagrangian L in W we denote by $\mathcal{P}(L)$ the space of smooth paths $x : [0, 1] \to \widehat{W}$ from $\widehat{L}$ to $\widehat{L}$ (i.e. $x(0), x(1) \in \widehat{L}$). Denote by R_α the Reeb vector field of α on M and by φ_α^t its flow. A *Reeb chord of length* T from Λ to Λ is a path $\gamma : [0, 1] \to M$ such that $\dot{\gamma} = TR_\alpha$, where by length we mean the time it takes the Reeb flow to run through the cord. We call a Reeb chord of length T *transverse* if the subspaces $T_{\gamma(1)}(\varphi_\alpha^T(\Lambda))$ and $T_{\gamma(1)}\Lambda$ of $T_{\gamma(1)}M$ intersect only in the origin. Note that the constant maps $t \mapsto x \in \Lambda$, which are Reeb chords of length 0, are never transverse. The spectrum of (M, α, Λ) is the set $\mathcal{S}(M, \alpha, \Lambda)$ ($\mathcal{S}$ for short) of lengths of Reeb chords from Λ to Λ, including negative lengths for "backward" Reeb flows. This set is nowhere dense in $\mathbb{R}$.

Given a contact manifold (M, ξ), the pair (α, Λ) consisting of a contact form α for ξ and a Legendrian submanifold Λ is called *regular* if all nonconstant Reeb chords of α from Λ to Λ are transverse. Given a Liouville domain (W, ω, λ), the pair (λ, L) consisting of the Liouville form and an asymptotically conical exact Lagrangian is called *regular* if $(\lambda|_{M=\partial W}, \Lambda = L \cap M)$ is regular.

Discussion of Assumption 3.1.2

Examples 3.2.1, 3.2.2 are natural examples of Liouville domains with asymptotically conical exact Lagrangians with spherical boundary. We have also seen that for (W, L) with ∂L connected, we can modify the Liouville form λ such that $\lambda|_L = 0$ without changing the Reeb dynamics of $\lambda|_{\partial W}$ on ∂W.

If $\pi_1(L) = 0$, then the assumption $[\omega]|_{\pi_2(W,L)} = 0$ holds automatically since any disk D with boundary on L can be completed by a disk in L to a sphere S such that $\int_D \omega = \int_S \omega = \int_{\partial S} \lambda = 0$ since L is Lagrangian and ω is exact. If $\pi_1(L) \neq 0$, then this assumption is nontrivial. We make this assumption to prevent bubbling of holomorphic disks.

The strong assumption in 3.1.2 is that (λ, L) is regular. For given L and generic λ this is the case, so we can force regularity by perturbing the dynamics. If we choose not to perturb λ, then we have to face the fact that there are forms λ that are not regular for any Lagrangian L. For example, the unit codisc bundle over the round sphere has periodic Reeb flow on its boundary, and thus any Legendrian gets mapped to itself after a full period, resulting in high degeneracy. Other degenerate examples are exact fillings of exactly fillable prequantization bundles, e.g. the subenergy level of the harmonic oscillator on $\mathbb{R}^{2n}$ (the energy level S^{2n-1} is a prequantization bundle over $\mathbb{C}P^{n-1}$). Note that to apply Theorem 8, it suffices to find positive symplectic growth for *one* regular pair (λ, L). Thus, as long as one is able to guarantee positive symplectic growth, one is free to perturb λ. This is the case for the examples in [6], where positive symplectic growth is guaranteed algebraically for a regular pair (λ, L) that Alves and Meiwes construct from any regular pair (λ', L'), where λ' restricts on the boundary to the standard contact structure and the constructed λ restricts on the boundary to a dynamically exotic contact structure.

Action functionals

For a Hamiltonian $H : \widehat{W} \to \mathbb{R}$, the Hamiltonian vector field X_H is defined by $\iota_{X_H}\omega = \omega(\cdot, X_H) = dH$. Its flow is denoted by φ_H^t. We define the action functional $\mathcal{A}_H : \mathcal{P}(L) \to \mathbb{R}$ by

$$\mathcal{A}_H(x) = \int_0^1 x^*\lambda - \int_0^1 H(x(t))\, dt. \tag{3.2.1}$$

The critical points of $\mathcal{A}_H$ are Hamiltonian chords with $x(0), x(1) \in \widehat{L}$. We denote the set of critical points by $\operatorname{Crit} \mathcal{A}_H$. A Hamiltonian H is called *regular* if $\widehat{L}$ and $\varphi_H^1(\widehat{L})$ intersect transversely (i.e. all critical points of $\mathcal{A}_H$ are non-degenerate). We call the Hamiltonian *Morse–Bott regular* if $\widehat{L}$ and $\varphi_H^1(\widehat{L})$ intersect in closed manifolds such that $T(\widehat{L} \cap \varphi_H^1(\widehat{L})) = T\widehat{L} \cap T\varphi_H^1(\widehat{L})$. Note that regular implies Morse–Bott regular.

We will later specify the Hamiltonians we use, by imposing in particular a certain behavior at infinity. We assume throughout and without mentioning that all our Hamiltonians are regular, except if we consider Morse–Bott situations. Since the only non-regular behavior will happen at the set of constant orbits, and since we are ultimately interested in the asymptotic behavior of the homology, the Morse–Bott situation is of marginal interest and not elaborated here. For an exposition, see for example [11].

Floer strips

An almost complex structure J on $\widehat{W}$ compatible with ω is called *conical* at a point in $\widehat{M}$ if it commutes with translations in the r-coordinate, preserves ξ and sends the Reeb vector field to the Liouville vector field $JR_\alpha = r\partial_r$. Further, we call J *asymptotically conical* if J is conical on $M \times [r, \infty)$ for some $r > 0$. Using an asymptotically conical almost complex structure J, we can define the L^2-metric on $\mathcal{P}(L)$ by

$$\langle \xi_1, \xi_2 \rangle = \int_0^1 \omega(\xi_1, J\xi_2)\, dt.$$

We interpret negative gradient flow lines $x_s(t)$ of $\mathcal{A}_H$ as Floer strips $u : \mathbb{R} \times [0,1] \to \widehat{W}$,

$$\begin{cases} \partial_s u + J(\partial_t u - X_H) = 0, \\ u(\cdot, i) \in \widehat{L},\ i = 0, 1. \end{cases} \tag{3.2.2}$$

We switch between the notations $x_s(t)$ and $u(s,t)$ according to whether we wish to see this object as a negative gradient flow line or as a perturbed holomorphic curve.

Given two critical points x_+ and x_-, we define the moduli space of parametrized Floer strips

$$\widetilde{\mathcal{M}}(x_-, x_+, H, J) = \{x_s \text{ Floer strip}, \lim_{s\to\pm\infty} x_s = x_\pm \text{ uniformly in } t\}.$$

In the sequel we suppress H and J in the notation. Denote by $\widetilde{\mathcal{M}}^k(x_-, x_+)$ the subset of $\widetilde{\mathcal{M}}(x_-, x_+)$ on which the operator obtained by linearizing Floer's equation (3.2.2) has Fredholm index k. There is an $\mathbb{R}$-action on $\widetilde{\mathcal{M}}(x_-, x_+)$ coming from translations on the domain in the s-variable. Denote the quotient by this action by

$$\mathcal{M}^k(x_-, x_+) = \widetilde{\mathcal{M}}^{k+1}(x_-, x_+)/\mathbb{R}.$$

The *energy* of $u \in \widetilde{\mathcal{M}}(x_-, x_+)$ is given by

$$E(u) := \int_{-\infty}^{\infty} \langle \nabla \mathcal{A}_H(x_s), \nabla \mathcal{A}_H(x_s) \rangle \, ds = \mathcal{A}_H(x_-) - \mathcal{A}_H(x_+),$$

a quantity that is invariant under translation of the domain and thus descends to the quotient. Since $E(u)$ is non-negative, $\mathcal{M}(x_-, x_+)$ is empty if $\mathcal{A}_H(x_-) < \mathcal{A}_H(x_+)$.

From now on we assume that $\mathcal{M}^k(x_-, x_+)$ is a k-dimensional manifold that is compact modulo breaking. Compactness modulo breaking follows from L^∞- bounds on u and its derivatives by bubbling analysis, and the manifold property follows if one can show that the set $\mathcal{M}(x_-, x_+)$ is cut out transversally from the space of all smooth strips from x_- to x_+ with boundary on $\widehat{L}$. For a regular Hamiltonian with appropriate asymptotic behavior these two properties are satisfied for a generic asymptotically conical almost complex structure. For all the Floer homologies in this section these are classical facts.

For Morse–Bott regular Hamiltonians we consider moduli spaces of flow lines with cascades, where compactness modulo breaking and transversality hold for an additional generic choice of Riemannian metric on the critical manifolds.

Floer chain complex and homology

To define a homology, for $a \in \mathbb{R}\backslash\mathcal{S}$ assume that the number of critical points of $\mathcal{A}_H$ with action less than a is finite. Then we consider as chain group the free $\mathbb{Z}_2$-vector space

$$\mathrm{FC}^a(H, J, L) = \bigoplus_{x \in \mathrm{Crit}\,\mathcal{A}_H,\ \mathcal{A}_H(x) < a} \mathbb{Z}_2 \cdot x.$$

We abbreviate $\mathrm{FC}^a := \mathrm{FC}^a(H, J, L)$ and $\mathrm{FC} := \mathrm{FC}^\infty := \bigcup \mathrm{FC}^a$. We equip FC with a boundary operator $\partial : \mathrm{FC} \to \mathrm{FC}$ by counting isolated Floer strips mod 2,

$$\partial x = \sum_{y \in \mathrm{Crit}\,\mathcal{A}_H} \#_{\mathbb{Z}_2} \mathcal{M}^0(x, y) \cdot y.$$

There are only finitely many nonzero summands since Floer strips decrease in action and FC^a is finite. Every summand is well-defined since $\mathcal{M}^0(x, y)$ is a compact 0-manifold and thus finite.

The operator ∂ is therefore well defined. The property $\partial^2 = 0$ holds since broken flow lines from x to y via intermediate critical points form exactly the boundary $\partial\mathcal{M}^1(x, y)$, as is seen by gluing and thus come in pairs. Hence, (FC, ∂) forms a chain complex, and we can define its homology $\mathrm{FH} = \ker\partial / \operatorname{im}\partial$.

Since Floer strips decrease in action, the boundary operator descends to a boundary operator ∂^a on FC^a. Further, we can define the chain complex with action window (a, b), $a, b \notin \mathcal{S}$, as the quotient $\mathrm{FC}^{(a,b)} = \mathrm{FC}^b / \mathrm{FC}^a$. This yields $\mathbb{R}$-filtered Floer homology groups

$$\mathrm{FH}^a = \ker\partial^a / \operatorname{im}\partial^a,$$
$$\mathrm{FH}^{(a,b)} = \ker\partial^{(a,b)} / \operatorname{im}\partial^{(a,b)}.$$

As for all $\mathbb{R}$-filtered homologies we have for $a < b < c$ long exact sequences

$$\ldots \to \mathrm{FH}^{(a,b)} \to \mathrm{FH}^{(a,c)} \to \mathrm{FH}^{(b,c)} \to \mathrm{FH}^{(a,b)} \to \ldots \tag{3.2.3}$$

where the first two arrows are induced by inclusion of chain complexes.

In the following we investigate two classes of admissible Hamiltonians that will in a direct limit result in different versions of wrapped Floer homology. The first is the standard WH, the second the V-shaped $\check{\mathrm{W}}\mathrm{H}$.

3.2.1 Wrapped Floer homology

We begin with the classical wrapped Floer homology, as defined in [7].

Admissible Hamiltonians

We say that a Hamiltonian $H : \widehat{W} \to \mathbb{R}$ is WH-*admissible with slope* $\mu > 0$ if

$$\begin{cases} H < 0 \text{ on } W, \\ \exists b < -\mu\colon H(x, r) = h(r) = \mu r + b \text{ on } M \times [1, \infty). \end{cases}$$

Denote by $\acute{\mathcal{H}}$ the set of WH-admissible regular Hamiltonians. For $H \in \acute{\mathcal{H}}$, the slope μ is not in $\mathcal{S}$, and so its orbits $x \in \operatorname{Crit}\mathcal{A}_H$ have image in W.

If H only depends on r in $\widehat{W}$ and is constant < 0 for $r < 1 - \delta$, then $X_H = (\frac{d}{dr}H)R_\alpha$. Thus, the elements of $\operatorname{Crit}\mathcal{A}_H$ have constant r-coordinate and correspond to the Reeb chords in M from Λ to Λ of period $\frac{d}{dr}H$ (they run backwards if $\frac{d}{dr}H < 0$). Of course, such a Hamiltonian is not regular at points where $\frac{d}{dr}H = 0$. This can be mended by adding a C^2-small Morse function supported in the region where $\frac{d}{dr}H$ is smaller than $\min\mathcal{S}$, which perturbs all the constant orbits and leaves the interesting Reeb orbits unchanged.

Continuation morphisms and wrapped Floer homology

A monotone increasing homotopy H_s from H_0 to H_1 through admissible Hamiltonians induces a chain map $\mathrm{FC}(H_0) \to \mathrm{FC}(H_1)$ that decreases in action and thus restricts to a chain map $\mathrm{FC}^{(a,b)}(H_0) \to \mathrm{FC}^{(a,b)}(H_1)$ for $a, b \in \mathbb{R} \cup \{\pm\infty\} \backslash \mathcal{S}$. The morphism

$$\Phi_{H_s} : \mathrm{FH}^{(a,b)}(H_0) \to \mathrm{FH}^{(a,b)}(H_1)$$

induced in homology is called *continuation morphism.* It is independent of the monotone homotopy H_s, and if H_s does not depend on s, then $\Phi_{H_s} = id$.

The set $\acute{\mathcal{H}}$ admits a partial order where $H_0 \leqslant H_1$ if the order is satisfied pointwise. With this partial order, $\acute{\mathcal{H}}$ becomes a directed set. The set of homologies $\{\mathrm{FH}^{(a,b)}(H)\}$ thus forms a direct system indexed by $\acute{\mathcal{H}}$. We define the wrapped Floer homology as the direct limit of this system,

$$\mathrm{WH}^{(a,b)}(W) = \varinjlim \mathrm{FH}^{(a,b)}.$$

For $a = -\infty$ we abbreviate $\mathrm{WH}^b(W, L) := \mathrm{WH}^{(-\infty,b)}(W, L)$. Since long exact sequences are preserved by direct limits, the sequence (3.2.3) holds for WH.

The generators of WH fall into two different classes: forward Reeb orbits on M, and "short orbits" on W. We are mainly interested in Reeb orbits of $(M, \lambda|_M)$. They are singled out by action, as the following lemma shows.

Lemma 3.2.4. *In the geometric situation (3.1.2) and for positive $a \notin \mathcal{S}$ the positive part of the homology $\mathrm{WH}^a_+(W, L) := \mathrm{WH}^{(\varepsilon,a)}(W, L)$, where $0 < \varepsilon < \min \mathcal{S}_{>0}$, is generated by the Reeb chords from Λ to Λ of length $< a$, and their action is given by their length.*

Proof. We start with Hamiltonians $H = H_{\mu,\varepsilon'}$ as in Figure 3.1 defined for $\mu \notin \mathcal{S}$, $0 < 4\varepsilon' < \varepsilon$ by

- H only depends on r,
- $H \equiv -\varepsilon'$ for $r < 1$,
- $H = \mu(r-1) - 2\varepsilon'$ for $r \geqslant 1 + \varepsilon'$,
- H is convex.

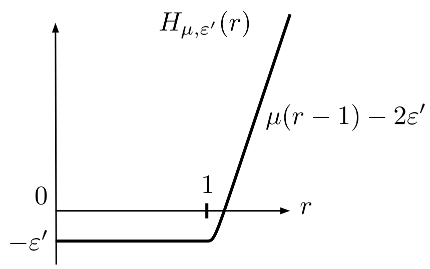

Figure 3.1: The function $H_{\mu,\varepsilon'}$

Then we perturb H in the region where $|\frac{d}{dr}H(r)| < \varepsilon'$ to a function still denoted by H such that L and $\varphi^1_H(L)$ intersect transversely and such that in the perturbation region the new H satisfies $\|H\|_{C^0} \leqslant \varepsilon'$ and $\|\lambda(X_H)\|_{C^0} < \varepsilon'$, making it a regular admissible Hamiltonian in $\acute{\mathcal{H}}$. Note that $\{H_{\mu,\varepsilon'}\}$ is cofinal in $\acute{\mathcal{H}}$ for $\mu \to \infty$, $\varepsilon' \to 0$. Then critical points x of $\mathcal{A}_H$ correspond either to Reeb chords from Λ to Λ with $H(x(t))$ and $(\frac{d}{dr}H)(x(t))$ constant in t and have action

$$\begin{aligned}
\mathcal{A}_H(x) &= \int_0^1 x^*\lambda - \int_0^1 H(x(t))\,dt \\
&= \int_0^1 \lambda\left(\frac{d}{dr}H \cdot R_{\lambda|_M}(x(t))\right)\,dt - H \\
&= \frac{d}{dr}H - H,
\end{aligned}$$

or they are ε'-short X_H-chords, namely $\int_0^1 |\lambda(\dot{x}(t))|\,dt < \varepsilon'$, from L to L, such that

$$\begin{aligned} |\mathcal{A}_H(x)| &\leqslant \int_0^1 |\lambda(\dot{x}(t))| + |H(x(t))|\,dt \leqslant 2\varepsilon' \\ &\leqslant \frac{1}{2}\varepsilon. \end{aligned}$$

In the first case the term $H(x)$ tends to zero in the direct limit, so Reeb chords have limit action $\frac{d}{dr}H = \mathrm{length}(x) > \varepsilon$, and in the second case the action of the critical points lies outside the action window. □

In analogy to the positive part we define the non-positive part $\mathrm{WH}^0(W,L) := \mathrm{WH}^\varepsilon(W,L)$ for $0 < \varepsilon < \min \mathcal{S}_{>0}$. For all $a \notin \mathcal{S}$ the long exact sequence (3.2.3) becomes

$$\ldots \to \mathrm{WH}^0(W,L) \to \mathrm{WH}^a(W,L) \to \mathrm{WH}^a_+(W,L) \to \mathrm{WH}^0(W,L) \to \ldots.$$

One can perform the perturbation of the family of functions $\{H_{\mu,\varepsilon'}\}$ in the proof above such that finite set $L \cap \varphi^1_H(L)$ in the perturbation region is constant, which implies that $\mathrm{WH}^a(W,L)$ and $\mathrm{WH}^a_+(W,L)$ are isomorphic up to an error of finite dimension independent of a. Thus, WH grows exponentially if and only if WH_+ grows exponentially. In [13] it is mentioned that $\mathrm{WH}^0(W,L)$ corresponds to the Morse-cohomology of L.

Note that even though WH is defined as a direct limit, for finite action windows (a,b) the homology $\mathrm{WH}^{(a,b)}$ is already attained by a Hamiltonian $H \in \acute{\mathcal{H}}$ that is C^2-small for $r < 1$, at $r = 1$ sharply increases and has asymptotic slope $\mu > b$.

3.2.2 V-shaped wrapped Floer homology

We construct V-shaped wrapped Floer homology by using a different class of Hamiltonians. A Hamiltonian is called $\check{\mathrm{w}}\mathrm{H}$-admissible if

$$\begin{cases} H < 0 \text{ on } M \times \{1\}, \\ \exists b < -\mu \colon H(x,r) = h(r) = \mu r + b \text{ on } M \times [1,\infty). \end{cases}$$

Denote the set of $\check{\mathrm{w}}\mathrm{H}$-admissible regular Hamiltonians by $\check{\mathcal{H}}$. Again, using continuation homomorphisms we can define for $a, b \notin \mathcal{S}$ the direct limit homology

$$\check{\mathrm{w}}\mathrm{H}^{(a,b)}(W,L) = \varinjlim \mathrm{FH}^{(a,b)}(W,L).$$

In the language of [13] this is the homology of the trivial Liouville cobordism with Lagrangian $([0,1]\times M, [0,1]\times\Lambda)$ with filling (W,L). This homology is different from wrapped Floer homology. The paper [12] suggests that there is a long exact sequence splitting $\check{\mathrm{w}}\mathrm{H}$ into wrapped Floer homology and wrapped Floer cohomology with interesting behavior in the "0-part" $\check{\mathrm{w}}\mathrm{H}^{(-\varepsilon,\varepsilon)}(W,L)$. Since we are only interested in the positive part of the homology, Proposition 3.1.4 is sufficient for our purposes.

Proof of Proposition 3.1.4. We proceed by deforming the Hamiltonians. We consider a cofinal family of Hamiltonians in $\acute{\mathcal{H}}$ and show that each such Hamiltonian can be deformed to a Hamiltonian in $\check{\mathcal{H}}$ such that the set of deformed Hamiltonians forms a cofinal family. The cofinal family in $\acute{\mathcal{H}}$ is the family $\{H_{\mu,\varepsilon'}\}$ from the proof of Lemma 3.2.4. Recall that critical points of $H_{\mu,\varepsilon'}$ are either ε'-short trajectories with action $\leqslant 2\varepsilon' \leqslant \frac{1}{2}\varepsilon$ or Reeb trajectories with length $> \varepsilon$ and action $> \varepsilon - \varepsilon' > \frac{1}{2}\varepsilon$. Thus in the chain complex $\mathrm{FC}^{(\frac{1}{2}\varepsilon,a)}(H_{\mu,\varepsilon'})$ the trajectories of the first type are quotiented out.

To define the cofinal family in $\check{\mathcal{H}}$ choose $\delta > 0$ such that L is conical for $r \in [1-\delta, 1]$ and $0 < \varepsilon < \min|\mathcal{S}|$ such that $2\varepsilon < \delta$. Start with $G = G_{\mu,\nu,\varepsilon'}$ as depicted in Figure 3.2 for $\nu \leqslant 0 < \mu$ with $\mu, \nu \notin \mathcal{S}, 0 < 4\varepsilon' < \varepsilon$ with the following properties:

- G depends only on r,
- $G(r) \equiv -\frac{1}{2}\delta\nu$ for $r < 1 - \delta$,
- $G(r) = \nu(r-1) - 2\varepsilon'$ for $r \in [1 - \frac{1}{2}\delta, 1 - \varepsilon']$,
- $G(1) = -\varepsilon'$ and $G'(0) = 0$,
- $G(r) = \mu(r-1) - 2\varepsilon'$ for $r \geqslant 1 + \varepsilon'$,
- G is convex for $r \in [1 - \varepsilon', 1 + \varepsilon']$ and concave for $r \in [1 - \delta, 1 - \frac{1}{2}\delta]$.

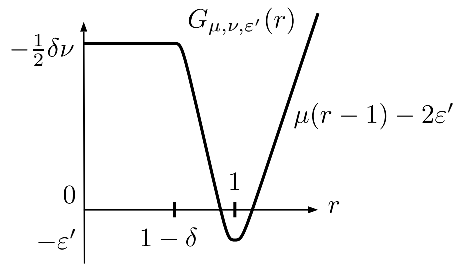

Figure 3.2: The function $G_{\mu,\nu,\varepsilon'}$

Then we perturb G in the region where $|\frac{d}{dr}G(r)| < \varepsilon'$ to a function still denoted by G such that L and $\varphi^1_G(L)$ intersect transversely and such that in the perturbation region the new G satisfies $\|G\|_{C^0} \leqslant \varepsilon'$ and $\|\lambda(X_G)\|_{C^0} \leqslant \varepsilon'$, making it a regular admissible Hamiltonian in $\acute{\mathcal{H}}$. Note that for $\nu = 0$ we have $G_{0,\mu,\varepsilon'} = H_{\mu,\varepsilon'}$. Critical points of $\mathcal{A}_G$ come in several types distinguished by their location. We can compute their action as in the proof of Lemma 3.2.4:

(I) $r \leqslant 1 - \delta$: $\frac{1}{4}\varepsilon$-short trajectories with $G \sim -\frac{1}{2}\delta\nu > 0$ and with action $< \frac{1}{4}\varepsilon - (-\frac{1}{2}\delta\nu) < \frac{1}{2}\varepsilon$,

(II) $1 - \delta < r < 1 - \frac{1}{2}\delta$: backward Reeb trajectories with length in $(\nu, -\varepsilon)$, with $G \sim -\frac{1}{2}\delta\nu > 0$ and action $< -\varepsilon - (-\frac{1}{2}\delta\nu) < 0$,

(III) $1 - \varepsilon' < r < 1$: backwards Reeb trajectories with length in $(\nu, -\varepsilon)$, with $G \sim -\varepsilon'$ and action $< -\varepsilon - (-\varepsilon') < 0$,

(IV) $r \approx 1$: ε'-short trajectories with $G \sim -\varepsilon'$ and with action $< \varepsilon' - (-\varepsilon') < \frac{1}{2}\varepsilon$,

(V) $r > 1$: Reeb trajectories with length in (ε, μ), $G \sim -\varepsilon'$ and action $> \varepsilon - (-\varepsilon') > \frac{1}{2}\varepsilon$.

Thus in the chain complex $\mathrm{FC}^{(\frac{1}{2}\varepsilon,a)}(G)$ all critical points other than of type (V) get quotiented out. It is clear from this description that we can monotonously deform $H_{\mu,\varepsilon'} = G_{0,\mu,\varepsilon'}$ to $G_{\nu,\mu,\varepsilon'}$ through Hamiltonians of type G by lowering the parameter ν. Since for $r \geqslant 1$ this deformation does not change the function, we conclude that the continuation homomorphism $\mathrm{FH}^{(\frac{1}{2}\varepsilon)}(H_{\mu,\varepsilon'}) \to \mathrm{FH}^{(\frac{1}{2},a)}(G_{\nu,\mu,\varepsilon'})$ is lower diagonal and thus an isomorphism. Taking the limit

$(\varepsilon', \nu, \mu) \to (-\infty, \infty, 0)$ for $\mathrm{FH}^{(\frac{1}{2}\varepsilon, a)}(G_{\nu,\mu,\varepsilon'})$ is thus the same as taking the limit $(\mu, \varepsilon') \to (0, \infty)$ for $\mathrm{FH}^{(\frac{1}{2}\varepsilon, a)}(H_{\mu,\varepsilon'})$.

To show that the isomorphisms commute with morphisms induced by inclusion of filtered chain complexes note that for parameters (ν, μ, ε') we have that the two chain complexes are not only isomorphic, but identical, $\mathrm{FC}^I(G_{(\nu,\mu,\varepsilon')}) = \mathrm{FC}^I(H_{(\mu,\varepsilon')})$, for any interval $I = (a, b)$ with $\frac{1}{2}\varepsilon < a < b \leqslant \infty$. This means that at the chain level inclusions trivially commute with the identity, thus morphisms induced by inclusion commute with isomorphisms induced by identity. Taking the direct limit preserves commutative diagrams and we are done.

For finite action windows one can even take a shortcut in the above argument since one can find parameters (ν, μ, ε') sufficiently close to $(-\infty, \infty, 0)$ such that $\mathrm{FH}^I(G_{\nu,\mu,\varepsilon'}) \sim \check{\mathrm{w}}\mathrm{H}^I(W, L)$ and $\mathrm{FH}^I(H_{\mu,\varepsilon'}) \sim \mathrm{WH}^I(W, L)$ for both $I = (a, b)$ and $I = (a', b')$ and the proof finishes before taking direct limits. □

As an alternative we can choose not to perturb G around $r = 1$. If $G'(0) = 0$ and $G''(0) > 0$, the critical manifold of type (IV) consists of constant orbits, can be identified with $\{0\} \times \Lambda$ and is Morse–Bott. This way it becomes transparent that the 0-part $\check{\mathrm{w}}\mathrm{H}^0(W, L)$ of V-shaped wrapped Floer homology can be identified with the Morse cohomology of Λ.

3.3 Lagrangian Rabinowitz–Floer Homology

We introduce three types of Lagrangian Rabinowitz–Floer homology. We start with an exposition of Lagrangian Rabinowitz–Floer homology with autonomous Hamiltonian (AH). This is the standard Lagrangian Rabinowitz–Floer homology. That the Hamiltonian is autonomous means that the critical orbits of the functional are contained in a fixed energy surface which leads to a much lighter analysis than for time-dependent Hamiltonians. In our construction of AH we will work with just one fixed Hamiltonian function H. While many different choices of autonomous Hamiltonians would result in isometric homologies AH, where filtered versions have the same dimension growth, we do not elaborate on this, since this independence will later on automatically follow in the setting of TH, where an even larger class of (time-dependent) Hamiltonians is used.

To show that AH is isomorphic to $\check{\mathrm{w}}\mathrm{H}$, we introduce Lagrangian Rabinowitz–Floer homology with perturbed Lagrange multiplier (PH), following [12]. Since the proof of Proposition 3.1.5 can be found in the paper [12] up to small changes, we only sketch the construction of PH.

Finally, we introduce Lagrangian Rabinowitz–Floer homology for time-dependent Hamiltonians (TH). We will study invariance of growth of TH under monotone changes of the Hamiltonian and derive uniform growth properties by a sandwich argument. Further, we show how to encode changes of the target Legendrian in the functional and how to derive uniform growth properties for time-dependent Reeb chords from Λ to Λ', where Λ' is a Legendrian isotopic to Λ.

3.3.1 Autonomous Lagrangian Rabinowitz–Floer homology (AH)

The action functional

Let $H : \widehat{W} \to \mathbb{R}$ be a smooth function on $\widehat{W}$ such that 0 is a regular value (later H is specifically chosen). We define the action functional $\mathfrak{a}_H : \mathcal{P}(L) \times \mathbb{R} \to \mathbb{R}$ by

$$\mathfrak{a}_H(x, \eta) = \int_0^1 x^*\lambda - \eta \int_0^1 H(x(t))\, dt. \tag{3.3.1}$$

A pair (x,η) is a critical point of $\mathfrak{a}_H$ if and only if it satisfies the equations

$$\begin{cases} \dot{x}(t) &= \eta X_H(x(t)), \\ H\circ x &\equiv 0. \end{cases}$$

The first equation implies that x is a Hamiltonian orbit from L to L with period η (flowing backwards if $\eta<0$). The second equation implies that the image of x is contained in the hypersurface defined by $H=0$.

If H depends only on r, $H(r)=0$ only for $r=1$ and $H'(1)=1$, then $\operatorname{Crit}\mathfrak{a}_H$ is the set of Reeb orbits from Λ to Λ with period η (running backwards if $\eta<0$), and $\mathfrak{a}_H(x,\eta)=\eta$ at critical points. Note that for $\eta=0$ the critical points are constant orbits that form the critical manifold $\Lambda=L\cap H^{-1}(0)$. Thus $\mathfrak{a}$ is never Morse. If (W,L) is regular, all critical points with $\eta\neq 0$ are regular and the critical manifold at $\eta=0$ is Morse–Bott. Since we only have one nontrivial critical manifold, we do not focus on the Morse–Bott situation. We choose a Morse function on Λ and abusing notation we denote by $\operatorname{Crit}\mathfrak{a}_H$ the union of the isolated critical points of $\mathfrak{a}_H$ with the critical points of the Morse function. The action of a critical point of the Morse function is, by definition, $0=\mathfrak{a}_H(\Lambda)$.

Choice of Hamiltonian

We fix a smooth Hamiltonian such that

- H depends only on r,
- $H\equiv -\frac{2}{3}\delta$ for $r<1-\delta$,
- $H=r-1$ for $r>1$,
- H is convex.

Then the critical points of $\mathfrak{a}_H$ are as described above. The choices of constants are for compatibility with the other homologies.

Moduli spaces of Floer strips

We choose again an asymptotically conical almost complex structure J on $\widehat{W}$. It induces the metric on $\mathcal{P}(L)\times\mathbb{R}$

$$\langle(\hat{x}_1,\hat{\eta}_1),(\hat{x}_2,\hat{\eta}_2)\rangle=\int_0^1\omega(\hat{x}_1,J\hat{x}_2)\,dt+\hat{\eta}_1\hat{\eta}_2.$$

For this inner product the L^2-gradient equation of $\mathfrak{a}_H$ is the Rabinowitz–Floer equation

$$\begin{cases}\partial_s x+J(x)[\partial_t x-\eta X_H(x(s,t))]=0,\\ \partial_s\eta+\int_0^1 H(x(s,t))\,dt=0.\end{cases}$$

In addition we choose a Riemannian metric g on the only nontrivial critical manifold $\Lambda\times\{\eta=0\}$. For two critical points (x_1,η_1) and (x_2,η_2) of $\mathfrak{a}_H$ we consider the moduli space $\widetilde{\mathcal{M}}((x_1,\eta_1),(x_2,\eta_2),H,J)$ of gradient flow lines with cascades from (x_1,η_1) to (x_2,η_2). We denote the subset where the linearization of the Floer equation has Fredholm index k by $\widetilde{\mathcal{M}}^k((x_1,\eta_1),(x_2,\eta_2),H,J)$. On this space there is a natural action by s-translation of the domain (if there are multiple cascades, then there is one such action per cascade). We take the quotient by this action to obtain the reduced moduli space of gradient flow lines $\mathcal{M}^{k-1}((x_1,\eta_1),(x_2,\eta_2),H,J)$ (if there are multiple cascades, then the dimensional shift is by the number of cascades).

For regular (W,L) and generic J and g we have transversality and compactness modulo breaking for these moduli-spaces.

Chain complex

We define the chain groups $\mathrm{AC}^a(H)$ as free $\mathbb{Z}_2$-module generated over $\operatorname{Crit}\mathfrak{a}_H$,

$$\mathrm{AC}^a(H) = \sum_{\mathfrak{a}_H(x,\eta)<a} \mathbb{Z}_2 \cdot (x,\eta).$$

The differential is defined by counting modulo $\mathbb{Z}_2$ isolated Rabinowitz–Floer-strips:

$$\partial(x,\eta) = \sum_{(x',\eta')\in \operatorname{Crit}\mathfrak{a}} \#_{\mathbb{Z}_2}\mathcal{M}^0((x,\eta),(x',\eta'),H,J).$$

By a gluing argument we can identify ∂^2 with counting broken cascades in $\partial\mathcal{M}^1$, which is zero modulo 2. Thus we can define $\mathrm{AH}^{(a,b)}(W,L)$ as the filtered homology of this chain complex.

3.3.2 The relation between AH and V-shaped wrapped Floer homology

The goal of this subsection is to prove Proposition 3.1.5. It follows almost exactly as its analogue for the closed string case in [12]. The difference is in the analysis when we show L^∞-bounds for x and its derivatives in order to prove compactness modulo breaking of moduli spaces of Floer strips $(x(s,t),\eta(s))$. For the L^∞-bound on x we invoke the maximum principle which is possible because Floer strips satisfy Neumann conditions at their boundary. For the L^∞-bound on the first derivatives of x we perform a bubbling analysis, where we use the hypothesis that $[\omega]$ vanishes on $\pi_2(W,L)$ to exclude bubbling of disks at the boundary.

The remaining argumentation remains completely unchanged. We sketch it here, for details see [12].

Perturbed Lagrangian Rabinowitz–Floer homology

As our main tool for the proof we introduce perturbed Lagrangian Rabinowitz–Floer homology (PH), which is defined like AH but for the functional $\mathfrak{pa} : \mathcal{P}(L) \times \mathbb{R} \to \mathbb{R}$,

$$\mathfrak{pa}_{H,\alpha,\beta}(x,\eta) = \int_0^1 x^*\lambda - \alpha(\eta)\int_0^1 H(x(t))\,dt + \beta(\eta),$$

depending on the smooth function $H : \widehat{W} \to \mathbb{R}$ (with properties specified later on) and $\alpha, \beta : \mathbb{R} \to \mathbb{R}$. In this subsection we always assume $H = H(r)$ to depend only on r.

Characterization of the critical points

A pair (x,η) is a critical point of $\mathfrak{pa}_{H,\alpha,\beta}$ iff

$$\begin{cases} \dot{x}(t) = \alpha(\eta)X_H(x(t)), \\ \dot{\alpha}(\eta)\int_0^1 H(x(t))\,dt = \dot{\beta}(\eta). \end{cases} \tag{3.3.2}$$

On the Morse–Bott component of the set of critical points, we consider critical points of a Morse function on this manifold. Note that, since H only depends on r, $X_H = H'(r)R_\lambda$ and thus all critical points of $\mathfrak{pa}$ correspond to Reeb chords of length $T = \alpha(\eta)H'$. Passing from the set of

Reeb chords of length T to the corresponding set of critical points amounts to finding numbers r, η such that

$$\begin{cases} T = \alpha(\eta) H'(r), \\ \dot{\alpha}(\eta) H(r) = \dot{\beta}(\eta). \end{cases} \tag{3.3.3}$$

The action at a critical point equals $\mathfrak{pa}_{H,\alpha,\beta}(x,\eta) = \alpha(\eta)(H'(r) - H(r)) + \beta(\eta)$.

Floer equations

Given an asymptotically conical almost complex structure J, the negative L^2-gradient equation is equivalent to the Floer equations for $(u(s,t), \eta(s))$:

$$\begin{cases} \partial_s u + J(u)[\partial_t u - \alpha(\eta) X_H(u)] = 0, \\ \partial_s \eta - \dot{\alpha}(\eta) \int_0^1 H(u)\, dt + \dot{\beta}(\eta) = 0. \end{cases} \tag{3.3.4}$$

In the case of nontrivial critical manifolds we use an additional Riemannian metric to define negative gradient equations with cascades.

Since we use PH to interpolate between AH and w̌H, it is important to observe that the two homologies are indeed special cases of PH.

Special case 1: Rabinowitz–Floer homology

Note that for H as in AH and for $\alpha(\eta) = \eta, \beta(\eta) = 0$ we have $\mathfrak{pa}_{H,\eta,0} = \mathfrak{a}_H$. Thus we see directly that

$$\mathrm{PH}^{(a,b)}(H, \eta, 0) \cong \mathrm{AH}^{(a,b)}(W, L).$$

Special case 2: Wrapped Floer homology

Recall that even if w̌H is defined as a limit, for finite action windows (a, b) the limit is attained for $H \in \check{\mathcal{H}}$ that is C^2-small for $r < 1$, then steeply increases and has slope at infinity $\mu > b$, and then $\mathrm{FH}^{(a,b)}(\mathcal{A}_H) = \check{\mathrm{w}}\mathrm{H}^{(a,b)}$. For (H, α, β) with such a function H, with $\alpha(\eta) = 1$ and β a Morse function with only one critical point in 0 the critical equation (3.3.2) splits: (x, η) is a critical point iff x is a critical point of $\mathcal{A}_H$ and η is 0. The Floer equation (3.3.4) also splits: $(x(s,t), \eta(s))$ is a Floer trajectory if $x(s,t)$ is a Floer trajectory of $\mathcal{A}_H$ and $\eta(s)$ is a Morse gradient trajectory. Since the Morse complex of η consists of just one point, we have directly

$$\mathrm{PH}^{(a,b)}(H, 1, \beta) \equiv \check{\mathrm{w}}\mathrm{H}^{(a,b)}(W, L).$$

Invariance property

To prove Proposition 3.1.5 we thus need a way to show that different PH are isomorphic. Of course $\mathrm{PH}^{(a,b)}(H, \alpha, \beta)$ depends on the functions (H, α, β), but the following proposition shows that it is invariant under homotopies for which the spectrum does not cross the boundaries of the action window.

Proposition 3.3.1. *Let $(H_s, \alpha_s, \beta_s), s \in [0,1]$ be a homotopy that is supported in $(0,1)$, such that for all values of s the resulting homology is well defined and such that for no value of s the boundaries of the action window (a,b) lie in the spectrum $\mathcal{S}_s$ of $\mathfrak{pa}_{H_s,\alpha_s,\beta_s}$. Then*

$$\mathrm{PH}^{(a,b)}(H_0, \alpha_0, \beta_0) \cong \mathrm{PH}^{(a,b)}(H_1, \alpha_1, \beta_1).$$

This result follows by the usual continuation technique.

The steps connecting the special cases

One can connect the two special cases above by the five steps from [12] such that no step has an action crossing and thus Proposition 3.3.1 is applicable. We outline the steps (including a preparatory step), discuss what they do to the functional and give details where they differ from [12]. Remember that we start with $(H, \eta, 0)$, where $\mathfrak{pa}_{(H,\eta,0)} = \mathfrak{a}_H$ and want to end with $(H, 1, \beta)$, where $\mathrm{PH}^{(a,b)}(H, 1, \beta) = \check{\mathrm{w}}\mathrm{H}^{(a,b)}(W, L)$. We suppose that a, b are not in the spectrum $\mathcal{S}$. Since the set $\mathcal{S}$ is nowhere dense, there is an $\varepsilon > 0$ such that a, b are ε-far from $\mathcal{S}$.

We deform in the following steps, cf. Figure 3.3:

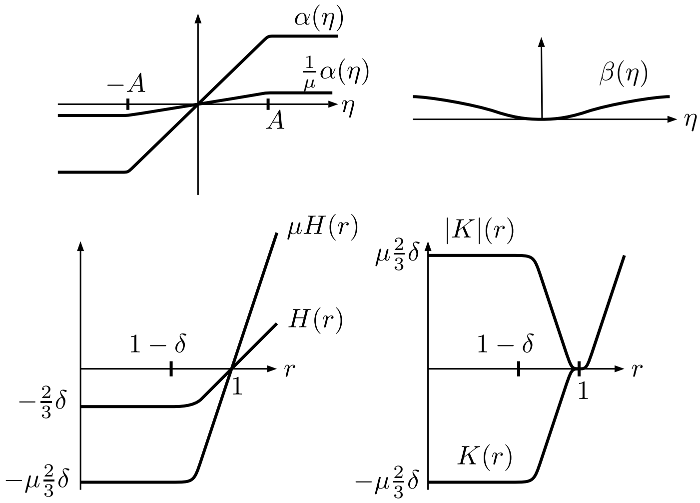

Figure 3.3: The functions appearing along the deformation from $(H, \eta, 0)$ to $(|K|, 1, \beta)$

1. Replace $(H, \eta, 0)$ by $(H, \alpha(\eta), 0)$ where $\alpha = id$ for $\eta \in (-A, A)$ and $\alpha = \pm A$ for $\pm\eta > A$ (up to a smoothing) for A so large that no Floer strip $(x_s(t), \eta_s)$ with asymptotics in the chosen action window exceeds $|\eta_s| > A$, and such that $-\frac{1}{2}\delta A < a$.
2. Replace $(H, \alpha, 0)$ by $(\mu H, \frac{1}{\mu}\alpha, 0)$ with $\mu = A$.
3. Deform $(\mu H, \frac{1}{\mu}\alpha, 0)$ to $(\mu H, \frac{1}{\mu}\alpha, \beta)$ for β a C^2-small Morse function with unique minimum $\beta(0) = 0$.
4. Replace $(\mu H, \frac{1}{\mu}\alpha, \beta)$ by $(K, \frac{1}{\mu}\alpha, \beta)$ where K is μH with a small terrace point at $r = 1$.
5. Homotope $\mathfrak{pa}_{(K,\frac{1}{\mu}\alpha,\beta)}$ to $\mathfrak{pa}_{(|K|,1,\beta)}$.
6. Perturb $(|K|, 1, \beta)$ to $\check{H} \in \check{\mathcal{H}}$.

Discussion of the steps

For each situation and the intermediate deformations we have to check that the triples (H, α, β) define a Morse–Bott action functional in the desired action window. To this end we have to show that the moduli spaces of Floer strips are compact modulo breaking which requires establishing L^∞-bounds on x, η and the derivatives of x along Floer-strips.

First we outline the proof that there is no action crossing, which coincides with seeing what effect the deformation of the functional has to the set of critical points and the action spectrum. For the complete discussion we refer the reader to [12, Section 6].

Step 1 reduces the support of $\dot{\alpha}$ to a compact interval. This step is a replacement, not a deformation. The Kazdan–Warner type inequality in [12] suggests that we could also achieve this step through deformations, but it is then much harder to establish bounds (since $\dot{\alpha}$ is not compactly supported). The isomorphism can be shown directly: For a Reeb chord the equations (3.3.3) have two kinds of solutions, the ones that coincide for $(H, \eta, 0)$ and $(H, \alpha, 0)$ and others that lie outside the action window in question. By the choice of A, also the Floer strips coincide and thus the chain complexes are the same, for a detailed discussion see [12, Section 5.4]. The reason for the condition $-\frac{1}{2}\delta A < a$ is that it guarantees that critical points of $\mathcal{A}_H$ at $r < 1 - \frac{1}{2}\delta$ have action outside the action window. These critical points will also be visible as critical points of $\mathfrak{pa}$ from Step 3 on, but not within the action window.

Step 2 gives H the asymptotic slope $\mu = A$, and in compensation flattens α so that it is constant ± 1 for $|\eta|$ large. This leaves the action functional unchanged, hence induces trivially an isomorphism by the identity at the chain level.

Step 3 puts β into its intended form. One can choose the isotopy $(\mu H, \frac{1}{\mu}\alpha, \beta_s)$ such that β_s is always a C^2-small Morse function with one unique minimum $\beta_s(0) = 0$ for $s > 0$. The first equation of (3.3.3) suggests that $\alpha(\eta)$ and thus η is not changed by this and the second equation, that locally looks like $(r-1) = \dot{\beta}(\eta)$, has solutions for r close to 1 since β_s is small. Thus the change moves nonconstant critical points of the functional slightly away from $\{r = 1\}$, for $\eta > 0$ to $r > 1$ and for $\eta < 0$ to $r < 1$, which changes the action a little. Also, some new critical points appear around $r \sim 1 - \delta$ for $\dot{\alpha}$ small enough to satisfy the second equation and thus $\eta \sim -A$. There $\mathfrak{pa} \sim -(\mu H' - (-\mu\delta))) < -\mu\delta < a$ and thus the critical points lie outside the action window.

Step 4 deforms H to K by introducing a terrace point in a neighborhood of $r = 1$ that contains only constant (H, α, β)-critical points. The deformation to the terrace translates the existing nonconstant critical points in the r-direction by a small amount and leaves their action unchanged. It introduces new critical points at r close to 1. These have action $\mathfrak{pa} = \alpha(H_s' - H_s) + \beta \approx \alpha H_s' = T$ and are therefore at all times close to $\mathcal{S}$ and therefore far from the boundary of the action window. This is a preparatory step such that the mirroring in Step 5 is smooth. This perturbation is small.

Step 5 directly homotopes the functional rather than the triple $(K, \frac{1}{\mu}\alpha, \beta)$. The homotopy is made such that

$$\mathfrak{pa}_s(x, \eta) = \int_0^1 \lambda(\dot{x}) - (1-s)\frac{1}{\mu}\alpha K - s|K|\, dt + \beta.$$

In total this lifts up the part left of the terrace point, converting the terrace to a minimum and introducing the desired V-shape for the Hamiltonian. This movement is compensated by changing η to 1. This perturbation is large. During the deformation critical points can be characterized as Reeb chords from L to L of period T together with numbers r, η such that

$$\begin{cases} T = (1-s)\frac{1}{\mu}\alpha(\eta)K'(r) + s|K|'(r), \\ (1-s)\frac{1}{\mu}\dot{\alpha}(\eta)K(r) = \dot{\beta}(\eta). \end{cases} \tag{3.3.5}$$

The action at a critical point equals

$$\mathfrak{pa}_{H,\alpha,\beta}(x, \eta) = (1-s)\frac{1}{\mu}\alpha(\eta)(K'(r) - K(r)) + s\frac{1}{\mu}\alpha(\eta)(|K|'(r) - |K|(r)) + \beta(\eta).$$

In the following we use that a, b are far from $\mathcal{S}$. It is sufficient to show that $r \sim 1$, because then $K(r) \sim |K|(r) \sim 0$ and thus from the first equation (3.3.5) we have $\mathfrak{pa}_s(x, \eta) \sim T$. But then all critical actions are close to $\mathcal{S}$ and therefore far from a, b. We distinguish several cases.

[$\eta = 0$]: Then the second equation implies that $K = 0$, thus $r = 1$. Furthermore we see that $K' = 0$ and by the first equation $T = 0$ and we have a constant orbit.

[$\eta > 0$]: Then $\dot{\beta}(\eta) > 0$ and thus by the second equation $r > 1$, which implies $K = |K|$. The following are subcases.

[$\eta > 0$, s is close to 1]: Then $(1-s)\frac{1}{\mu}\alpha(\eta) + s \sim 1$ and the first equation tells us that $T \sim K'$, but since the asymptotic slope μ of K is far from $\mathcal{S}$, this implies $r \sim 1$.

[$\eta > 0$, s is far from 1, $\alpha(\eta) = \eta$]: The second equation $(1-s)\frac{1}{\mu}K(r) = \dot{\beta}(\eta)$ tells us that $K(r) \sim 0$, hence $r \sim 1$.

[$\eta > 0$, s is far from 1, $\alpha(\eta) \neq \eta$]: Then $\alpha(\eta) \sim A = \mu$. Then the first equation tells us that $T \sim K'$, but since the asymptotic slope μ of K is far from $\mathcal{S}$, this implies $r \sim 1$. This finishes the case $\eta > 0$.

[$\eta < 0$]: Then $\dot{\beta}(\eta) < 0$ and thus by the second equation $r < 1$, which implies $-K = |K|$. The following are subcases.

[$\eta < 0$, s is close to 1]: Then $(1-s)\frac{1}{\mu}\alpha - s \sim -1$ and by the first equation $T \sim -K' = |K|'$. Since μ is far from $\mathcal{S}$ this implies that either $r \sim 1$ or that $r < 1 - \frac{1}{2}\delta$ (where H is bent into the constant $-\frac{2}{3}\delta$). In the first case we are done. The second case implies that $|K(r)| \sim \frac{1}{2}\delta\mu$ and with $(1-s)\frac{1}{\mu}\alpha - s \sim -1$ we get $\mathfrak{pa} \sim -K' + K \leqslant K \leqslant -\frac{1}{2}\delta\mu$. By our choice of μ this is well out of the action window.

[$\eta < 0$, s is far from 1, $\alpha(\eta) = \eta$]: Then the second equation tells as that $K(r) \sim 0$, hence $r \sim 1$.

[$\eta < 0$, s is far from 1, $\alpha(\eta) \neq \eta$]: Then $\alpha(\eta) \sim -A = -\mu$ and thus $(1-s)\frac{1}{\mu}\alpha + s \sim -1$. Then the first equation tells us that $T \sim -K' = |K|'$, and thus r is either close to 1 or $\leqslant 1 - \frac{1}{2}\delta$. But then $\mathfrak{pa}_s \sim -K' + K$ and we conclude as in the case when s is close to 1.

In conclusion in all cases the action of all critical points lies either close to $\mathcal{S}$ or outside $[a, b]$ and therefore far from a, b. All the above estimates are quantified rigorously in [12, Section 6, Step 4].

Step 6 can already be performed in the framework of Section 3.2. It replaces $|K|$ by $|K| - \varepsilon$ and perturbs $|K|$ on $\{r \leqslant \frac{\delta}{2}\}$ and also at the minimum at 1 such that the functional is really Morse and not just Morse–Bott. The resulting Hamiltonian is in $\check{\mathcal{H}}$ and has the form for which $\mathrm{FH}^{(a,b)}$ coincides with $\check{\mathrm{w}}\mathrm{H}^{(a,b)}$.

Compactness of moduli spaces

To get a well defined homology, the moduli spaces of solutions of the Floer equations with specified asymptotics must be compact modulo breaking. This follows from C^∞_{loc} compactness of the spaces $\widetilde{\mathcal{M}}((x, \eta), (x', \eta'), H, J)$. For this it is enough to show L^∞-bounds for x and η and L^∞-bounds for the first derivatives of x. Using the Floer equation (3.3.4) one then also has an L^∞-bound for η'. Then bootstrapping (3.3.4) yields L^∞-bounds on higher derivatives.

The bound for the first derivatives of x follows as always by a bubbling analysis since $[\omega]$ vanishes on $\pi_2(W)$ (because ω is exact) and on $\pi_2(W, L)$ (by hypothesis).

L^∞-bound on η

For the situation before Step 1 this is classic. After Step 1 the bound of before still holds by our choice of A.

In Step 2 we have $\beta = 0$ and for $|\eta| \geqslant A$ we have $\dot{\alpha}(\eta) = 0$, so the equation for η becomes $\partial_s \eta(s) = 0$ for $|\eta| > A$ and thus $|\eta|$ cannot exceed A.

In Steps 3 to 6 the equation for η is $\partial_s \eta(s) = -\dot{\beta}(\eta(s))$ for $|\eta| > A$. Since $\operatorname{sign} \dot{\beta}(\eta) = \operatorname{sign} \eta$, it is clear that $|\eta|$ is bounded by A.

L^∞-bound on x

The proof follows the usual pattern. Since the energy of the Floer strip (x_s, η_s) is bounded, the gradient $\nabla \mathfrak{pa}(x_s, \eta_s)$ can be large only for a finite time. One shows separately that a small gradient implies a bound on $r \circ u(s,t)$ and that in the finite time where the gradient is large, $r \circ u(s,t)$ cannot grow too much. Note that there are positive constants A, A', B, C such that at each moment of the whole process we have the bounds

$$\begin{cases} H = h(r) = Ar + A' \text{ for all } r \geqslant 2, \\ \|\dot{\alpha}\|_{L^\infty} \leqslant B < \infty, \\ \|\dot{\beta}\|_{L^\infty} \leqslant C < \infty. \end{cases} \tag{3.3.6}$$

We also set $D = \min |H|$.

The following fundamental property says that at values of s where $\nabla \mathfrak{pa}(x_s, \eta_s)$ is small, the radius r stays bounded. This allows us to restrict our attention to the region where $\nabla \mathfrak{pa}$ is large, but this region must be compact since the energy is finite.

$$\forall \varepsilon > 0 \; \exists S \text{ such that } \|\nabla \mathfrak{pa}_{H,\alpha,\beta}(x,\eta)\| \leqslant \varepsilon \Rightarrow \max_{t \in [0,1]} r \circ x(t) \leqslant S. \tag{3.3.7}$$

This property holds during all the Steps 2–6, as is shown in [12, Lemma 4.7]. The proof holds verbatim in our situation.

Now we can analyze the radial coordinate $r : M \times \mathbb{R} \to \mathbb{R}; (x,r) \to r$ along a local solution $(x_s(t), \eta_s) = (u(s,t), \eta(s))$ of the Floer equation. The crucial observation is the following estimate for the Laplacian.

Lemma 3.3.2. *If $H = h(r)$ depends only on r, a local solution $(u(s,t), \eta(s))$ of (3.3.4) satisfies at image points in $M \times \mathbb{R} \subset W$ the bounds*

$$\begin{aligned} \Delta(r \circ u) &= \|\partial_s u\|^2 - \partial_s(h'(r)\alpha(\eta))(r \circ u), \\ \Delta(\log r \circ u) &\geqslant -\partial_s(h'(r)\alpha(\eta)). \end{aligned}$$

This is Lemma 4.1 in [12], and the proof is not affected by the change to the open string situation.

If (H, α, β) satisfies furthermore (3.3.6), then we obtain for $r \circ u \geqslant 2$ the bound

$$\Delta(\log r \circ u) \geqslant -A^2 B^2 D - ABC \tag{3.3.8}$$

as in [12, Lemma 4.2].

To invoke the maximum principle, we will need that at the sets $\{t = 0\}$ and $\{t = 1\}$ (which get mapped to L), the function $r \circ u$ satisfies the Neumann condition. At $t = 0$ we compute

$$\begin{aligned} \partial_t(r \circ u(s,t))|_{t=0} &= \langle \nabla r, \partial_t u(s,0) \rangle \\ &= \langle \nabla r, J[\partial_s u(s,0) - J\alpha(\eta) X_H] \rangle \\ &= \omega(\nabla r, \partial_s u(s,0) - J\alpha(\eta) X_H) = 0. \end{aligned} \tag{3.3.9}$$

The last equality holds since JX_H and ∇r are parallel, and since $\nabla r, \partial_s u(s,0) \in T_{x(s,0)} L$ which is Lagrangian. For $t = 1$ the computation is identical. It also holds for the function $\log r \circ u$.

Lemma 3.3.3 (L^∞-bound on x_s)**.** *Suppose that (H, α, β) satisfies conditions (3.3.6) and (3.3.7). Also suppose that $A \notin \mathcal{S}$. Consider a negative gradient flow line $(x_s(t), \eta_s) = (u(s,t), \eta(s))$ with is asymptotic to the critical points (x_1, η_1) and (x_2, η_2). Let $\mathcal{E} := \mathcal{E}(u, \eta) := \mathfrak{pa}(x_1, \eta_1) - \mathfrak{pa}(x_2, \eta_2)$ be the energy of this flow line. Then for all ε and the corresponding S from (3.3.7) and for all (s,t) we have the following estimate*

$$\log r \circ u(s,t) \leqslant \max(\log 2, \log S) + \frac{(A^2B^2D + ABC)\mathcal{E}(u,\eta)^2}{2\varepsilon^4}.$$

Proof. We follow [12, Proposition 4.3]. Fix s_0. Let $[s_-, s_+]$ be the maximal compact interval containing s_0 such that

$$\forall \sigma \in [s_-, s_+] : \|\nabla \mathfrak{pa}(x_\sigma, \eta(\sigma))\| \geqslant \varepsilon.$$

It is possibly empty, but it exists since

$$\mathcal{E} = \int_{\mathbb{R}} \|\nabla \mathfrak{pa}(x_s, \eta_s)\|^2 \, ds = \mathfrak{pa}(x_1, \eta_1) - \mathfrak{pa}(x_2, \eta_2)$$

is finite and $s_+ - s_- \leqslant \frac{\mathcal{E}}{\varepsilon^2}$. Because of (3.3.7) we have $r \circ x_{s_\pm}(t) < S \ \forall t \in [0,1]$.

On the strip $[s_-, s_+] \times [0,1]$ the function

$$\chi = \log r \circ u + \frac{1}{2}(A^2B^2D + ABC)(s - s_0)^2$$

is subharmonic at places where $r \circ u \geqslant 2$ because of (3.3.8). Define the set

$$\Omega = \{(s,t) \in [s_-, s_+] \times [0,1] \mid \log r \circ u \geqslant \max(\log 2, \log S)\}.$$

At the boundary $\partial\Omega$ either $\log r \circ u \leqslant \max(\log 2, \log S)$ or $\log r \circ u$ satisfies the Neumann condition (3.3.9). By the maximum principle we conclude that χ has no maximum in the interior of Ω nor at a boundary point where the Neumann condition holds. We conclude that for $(s,t) \in \Omega$,

$$\begin{aligned}
\log r \circ u(s,t) &\leqslant \chi(s,t) \leqslant \max_{(s,t)\in\Omega} \chi(s,t) \\
&\leqslant \max(\log 2, \log S) + \frac{1}{2}(A^2B^2D + ABC)(\max\{s_0 - s_-, s_+ - s_0\})^2 \\
&\leqslant \max(\log 2, \log S) + \frac{1}{2}(A^2B^2D + ABC)(s_+ - s_-)^2 \\
&\leqslant \max(\log 2, \log S) + \frac{1}{2}(A^2B^2D + ABC)\left(\frac{\mathcal{E}}{\varepsilon^2}\right)^2,
\end{aligned}$$

which is the desired bound. □

3.3.3 Time-dependent Lagrangian Rabinowitz–Floer homology (TH)

In contrast to the homologies we encountered so far, the aim of TH is not to capture the dynamics of the Reeb flow of $\lambda|_M$, but rather to set up a tool that allows us to study positive contactomorphisms. This will result in a homology TH which is an invariant of a Hamiltonian H on $\widehat{W}$. We will find a uniform way to construct such a Hamiltonian from a positive contact Hamiltonian h^t such that TH becomes an invariant of h^t.

The action functional

Let $H^t : W \times [0,1] \to \mathbb{R}$ be a time-dependent smooth function on W. We define the action functional $\mathfrak{ta}_{H^t} : \mathcal{P}(L) \times \mathbb{R} \to \mathbb{R}$ by

$$\mathfrak{ta}_{H^t}(x,\eta) \quad = \quad \frac{1}{\kappa}\left(\int_0^1 x^*\lambda - \eta\int_0^1 H^{\eta t}(x(t))\,dt\right), \qquad (3.3.10)$$

where κ is some positive constant discussed later. A pair (x,η) is a critical point of $\mathfrak{ta}_{H^t}$ if and only if it satisfies the equations

$$\begin{cases} \dot{x}(t) &= \eta X_{H^t}(x(t)), \\ H^\eta(x(1)) &= 0, \end{cases}$$

where X_{H^t} is the Hamiltonian vector field generated by H^t. The first equation implies that x is an orbit of X_{H^t}, but with time scaled by η. The second equation, which one deduces by partial integration, implies that the orbit ends on $(H^\eta)^{-1}(0)$. If $H^t = H$ is autonomous, then (up to the constant κ) the functional $\mathfrak{ta}_H$ is as for autonomous Rabinowitz–Floer homology AH. Then $H^{-1}(0)$ is a hypersurface for which η plays the role of a Lagrange multiplier, and $H(x(t)) = 0$ for all t. For time-dependent H^t, however, there is no such hypersurface, and $H^{\eta t}(x(t))$ might be very large or very small for $t < 1$.

Construction of the Hamiltonians

For the study of positive contactomorphisms φ it is crucial to carefully construct Hamiltonians H^t on W in such a way that critical points of $\mathfrak{ta}_{H^t}$ encode dynamical information on φ, such that the resulting TH is well defined and such that for monotone deformations of H^t the continuation morphisms are monotone with respect to the action.

We start with the object we actually want to study: a positive contactomorphism φ of (M,α). By definition there is a positive path of contactomorphisms φ^t such that $\varphi^0 = id$, $\varphi^1 = \varphi$. By the second part of the proof of [20, Proposition 6.2], φ^t can be deformed with fixed endpoints to $\tilde{\varphi}^t$ such that $\tilde{\varphi}^t$ is the Reeb flow of α for t near 0 and 1. This means that the contact Hamiltonian h^t on M generating φ^t is constant $\equiv 1$ for t near 0 and 1. This deformation can be performed such that order is preserved: $\tilde{h}_1^t \leqslant \tilde{h}_2^t$ whenever $h_1^t \leqslant h_2^t$. From now on we assume that this deformation is already performed if not stated differently. Then the concatenation of positive contactomorphisms is a positive contactomorphism. More specifically, h^t permits smooth periodic or constant extensions to $t \in \mathbb{R}$. We choose to extend $h^t \equiv 1$ for $t \leqslant 0$, which later will guarantee monotonicity of continuation morphisms. For positive time we choose $h = t$ for every integer step $t \in [k, k+1]$ individually such that h^t and its derivatives have uniform bounds. In the actual applications the choice will be that h^t is periodic for $t \geqslant 0$, see Figure 3.4. Later on we will use the interval $[0,1]$ to encode the information that we count orbits from our base Legendrian to another Legendrian, and then extend h^t to a periodic function for $t \geqslant 1$. We can now update Assumption 3.1.2 to the following analogon for the new situation:

Assumption 3.3.4. The pair (W,L) consists of a Liouville domain (W,ω,λ) with contact boundary $(M, \xi = \ker\lambda|_M)$ and an asymptotically conical exact Lagrangian L with connected Legendrian boundary $\Lambda = \partial L$ such that $\lambda|_L = 0$, and such that $[\omega]|_{\pi_2(W,L)} = 0$. The contact Hamiltonian $h^t : M \times \mathbb{R} \to \mathbb{R}$ satisfies for some positive constants c, C, c'

- $h^t \equiv 1$ for $t \leqslant 0$,
- $0 < c \leqslant h^t \leqslant C$ and $|\frac{d}{dt}h^t| \leqslant c'$,

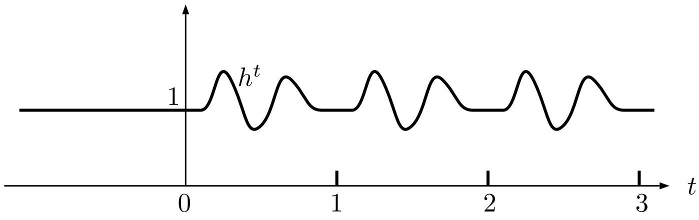

Figure 3.4: The function $h^t(x)$ at a given point $x \in M$

- $\bigcup_{t\neq 0} \varphi^t_{h^t}\Lambda$ and Λ intersect transversely.

The third assumption implies that the action functional $\mathfrak{ta}_{H^t}$ for h^t defined by 3.3.11 is Morse away from $\eta = 0$.

To work in the Liouville domain we construct from the contact Hamiltonian h^t on M a Hamiltonian H^t on W in a uniform way, depending on two large enough parameters κ and K, which we choose for every finite action window individually. The flow lines of X_{H^t} are lifts of φ^t-flow lines if we set on $M \times \mathbb{R}^{>0}$

$$H^t := rh^t - \kappa. \tag{3.3.11}$$

For such an H^t the critical points of $\mathfrak{ta}_{H^t}$ end in $\{r = \kappa/h^\eta\}$. Changing κ does not change critical points in an essential way (provided they do not run into $r = 0$), but translates them in the r-direction. In order to have a smooth Hamiltonian on $\widehat{W}$ and to get compactness of moduli spaces later on, we deform H^t to depend only on r but not on t for $r \leqslant 1$ and $r \geqslant \kappa K$, see Figure 3.5. This is made precise by choosing uniformly for all action windows a constant $M \geqslant \max\{h^t(x) \mid x \in M\}$, a convex smooth function $\rho : \mathbb{R}^{\geqslant 0} \to \mathbb{R}$, that will play the role of the radius smoothed out over W, and a smooth function $\beta : \mathbb{R}^{\geqslant 0} \to [0, 1]$, that will serve as a transition parameter, such that

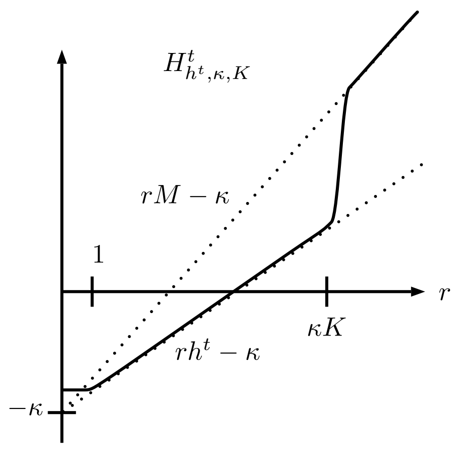

Figure 3.5: The function $H^t_{h^t,\kappa,K}$ at a fixed time t on a line $\mathbb{R}^{\geqslant 0} \times \{x\}$ in dependence of r.

$$\rho(r) = \begin{cases} 1 - \frac{2}{3}\delta & \text{if} \quad r \leqslant 1-\delta, \\ r & \text{if} \quad r \geqslant 1, \end{cases}$$

$$\beta(r) = \begin{cases} 0 & \text{if} \quad r \leqslant 1-\delta \quad \text{or} \quad r \geqslant \kappa K + 1, \\ 1 & \text{if} \quad 1 \leqslant r \leqslant \kappa K. \end{cases}$$

Then we define the Hamiltonian

$$H^t_{h^t,\kappa,K}(x,r) = \rho(r)\Big(\beta(r)\, h^t(x) + (1-\beta(r))\, M\Big) - \kappa.$$

The factor $\frac{1}{\kappa}$ in Definition (3.3.10) does not influence the critical points, but only their action values. In fact, the following lemma shows that for κ, K large enough, the critical points (up to translation in the r-direction) and their actions do not depend on the choice of the constants.

Lemma 3.3.5. *Let h^t satisfy (3.3.4). Given $a < b$, there are constants κ_0, K_0 such that for $\kappa \geqslant \kappa_0$ and $K \geqslant K_0$ the following holds. If (x,η) is a critical point with $a \leqslant \mathfrak{ta}_{h^t,\kappa,K}(x,\eta) \leqslant b$, then the radial component of x stays in $[1, K\kappa]$ for $t \in [0,1]$ and $\mathfrak{ta}_{h^t,\kappa,K}(x,\eta) = \eta$.*

Proof. A detailed proof and in particular explicit upper bounds for the constants κ_0, K_0 are given in [5, Proposition 4.3] in the setup of cotangent bundles. It applies verbatim in the present setting. □

In the following we abbreviate $\mathfrak{ta}_{h^t,\kappa,K} = \mathfrak{ta}$.

Remark 3.3.6. For $h^t \equiv 1$ one can choose for all action windows $K = \kappa = 1$. The function H^t does not depend on M and coincides with H in the definition of $\mathfrak{a}$ in (3.3.1). Thus the functional $\mathfrak{ta}$ coincides with $\mathfrak{a}$, and AH is a special case of TH.

The differential

The differential is constructed exactly as in the autonomous case: Choose an asymptotically conical almost complex structure J to define an L^2-metric with respect to which one considers negative gradient flow lines that correspond to solutions of a perturbed Cauchy–Riemann equation. We get transversality of moduli spaces by perturbing J. The desired L^∞-bounds on the flow lines follow as before since for $r \geqslant K\kappa + 1$ the Hamiltonian H^t is autonomous and linear in r. Thus, the choice to deform H^t to an autonomous Hamiltonian for large r guarantees compactness. The drawback is that the resulting homology counts the orbits of h^t only in the chosen action window.

Action windows and definition of homology

As Lemma 3.3.5 shows, the definition of TH must be done for finite action windows first, and then extended. To do this we choose κ_0, K_0 so large that Lemma 3.3.5 holds for critical points with action in $[a,b]$. We first generate the chain complex $\mathrm{TC}^b = \mathrm{TC}^b(h^t,\kappa,K)$ by the critical points of $\mathfrak{ta}$ with action $\leqslant b \in \mathbb{R}$ and then define $\mathrm{TC}^{(a,b)} = \mathrm{TC}^b / \mathrm{TC}^a$ and $\mathrm{TH}^{(a,b)} = \mathrm{TH}^{(a,b)}(h^t,\kappa,K)$ its homology. These groups are independent of $\kappa \geqslant \kappa_0, K \geqslant K_0$, which is why we denote them by $\mathrm{TH}^{(a,b)}(h^t)$ for brevity.

For $a \leqslant a', b \leqslant b'$ there are (for κ, K large enough) homomorphisms induced by inclusion of generators $\mathrm{TC}^{(a,b)} \to \mathrm{TC}^{(a',b')}$. We define $\mathrm{TC}^{(-\infty,b)}$ as the inverse limit, $\mathrm{TC}^{(a,\infty)}$ as the direct limit and $\mathrm{TC} = \mathrm{TC}^{(-\infty,\infty)}$ as direct inverse limit (in this order, to preserve exactness of long exact sequences), while adjusting κ, K. It still holds that $\mathrm{TC}_a^b = \mathrm{TC}^{(-\infty,b)}/\mathrm{TC}^{(-\infty,a)}$. We denote by $\mathrm{TC}_+^T = \mathrm{TC}^{(0,T)}$ the positive part of the chain complex and by ι the homomorphisms $\mathrm{TH}_+^T \to \mathrm{TH}_+^\infty$ induced by inclusion.

Invariance properties

Consider now a family of Hamiltonians h_s^t such that $\partial_s h_s^t$ is supported in $s \in [0,1]$. Suppose that for the associated family of functionals $\mathfrak{ta}_s := \mathfrak{ta}_{h^t,\kappa,K}$ the constants κ, K are chosen uniformly large enough for $[a,b]$. We set $\mathfrak{ta}_- = \mathfrak{ta}_s$ for $s \leqslant 0$ and $\mathfrak{ta}_+ = \mathfrak{ta}_s$ for $s \geqslant 1$. The continuation homomorphism $\Phi : \mathrm{TC}(\mathfrak{ta}_-) \to \mathrm{TC}(\mathfrak{ta}_+)$ is defined as in the definition of the differential by counting the 0-dimensional components of the moduli space of curves (x_s, η_s) that satisfy the equation

$$\partial_s(x_s,\eta_s) \quad = \quad -\nabla \mathfrak{ta}_s(x_s,\eta_s), \tag{3.3.12}$$

such that $\lim_{s\to\pm\infty}(x_s,\eta_s) = (x_\pm,\eta_\pm)$ for critical points $(x_\pm,\eta_\pm)$ of $\mathfrak{ta}_\pm$. Then Φ induces an isomorphism $\mathrm{TH}(\mathfrak{ta}_-) \to \mathrm{TH}(\mathfrak{ta}_+)$, because η is bounded along deformations, and actually does not depend on the homotopy h_s but only on the endpoints $h_\pm$.

Unfortunately, this isomorphism a priori loses information on the filtration of the homology. To keep this information, we restrict our attention to monotone deformations, i.e. $\partial_s h_s^t(x) \geqslant 0 \ \forall\ s,t,x$. The following proposition says that such monotone deformations are compatible with the action filtration. In the proof it becomes clear why we extend h^t to be constant for $t < 0$.

Proposition 3.3.7 (Monotonicity)**.** *Let (W,L) be a Liouville domain with asymptotically conical exact Lagrangian with $\lambda|_L = 0$ such that $[\omega]|_{\pi_2(W,L)} = 0$. Let $h_-^t \leqslant h_+^t$ be two time-dependent positive contact Hamiltonians that satisfy (3.3.4). Then the continuation homomorphism*

$$\Phi : \mathrm{TH}(h_-^t) \to \mathrm{TH}(h_+^t)$$

restricts for every a to

$$\Phi|_{\mathrm{TC}^a(h_-^t)} : \mathrm{TH}^a(h_-^t) \to \mathrm{TH}^a(h_+^t).$$

Proof. It suffices to show that the action is non-increasing along solutions of (3.3.12).

For the deformation from h_-^t to h_+^t, define a monotone smooth function $\chi : \mathbb{R} \to [0,1]$ such that

$$\chi(s) \quad = \quad \begin{cases} 0 & \text{if} \quad s \leqslant 0, \\ 1 & \text{if} \quad s \geqslant 1, \end{cases}$$

and set $h_s^t := h_-^t + \chi(s)(h_+^t - h_-^t)$. Denote by H_s^t and $\mathfrak{ta}_s$ the associated Hamiltonians and functionals. The deformation satisfies

$$\tfrac{d}{ds} H_s^t \quad = \quad \chi'(s)(H_+^t - H_-^t) \quad = \quad \chi'(s)\, r\, \beta(r)(h_+^t - h_-^t) \quad \geqslant \quad 0.$$

For every (x,η) we have,

$$\frac{\partial}{\partial s}\mathfrak{ta}_s(x,\eta) \quad = \quad \int_0^1 -\frac{\eta}{\kappa}\chi'(s)(H_+^{\eta t} - H_-^{\eta t})\, dt.$$

Now consider a solution $(u(s),\eta(s))$ of (3.3.12). Set $E=\int_{-\infty}^{\infty}\|\partial_s(x(s),\eta(s))\|^2\,ds$ and $\mathfrak{ta}_\pm=\mathfrak{ta}_\pm(u_\pm,\eta_\pm)$. We calculate

$$\begin{aligned}\mathfrak{ta}_+ &= \mathfrak{ta}_- + \int_{-\infty}^{\infty}\frac{d}{ds}\mathfrak{ta}_s(x(s),\eta(s))\,ds\\ &= \mathfrak{ta}_- + \int_{-\infty}^{\infty}\Big(\frac{\partial}{\partial s}\mathfrak{ta}_s\Big)(x(s),\eta(s)) + \big\langle\nabla\mathfrak{ta}_s\big(x(s),\eta(s)\big),\partial_s\big(x(s),\eta(s)\big)\big\rangle\,ds\\ &= \mathfrak{ta}_- - E + \int_{-\infty}^{\infty}\int_0^1 -\frac{\eta(s)}{\kappa}\chi'(s)(H_+^{\eta t}-H_-^{\eta t})\,dt\,ds.\end{aligned}$$

If $\eta(s)\geqslant 0$, then $-\frac{\eta(s)}{\kappa}\chi'(s)(H_+^{\eta t}-H_-^{\eta t})\leqslant 0$. If $\eta(s)\leqslant 0$, then $h_+^{\eta t}=h_-^{\eta t}=1$ and thus $-\frac{\eta(s)}{\kappa}\chi'(s)(H_+^{\eta t}-H_-^{\eta t})=0$. It follows that $\mathfrak{ta}_+\leqslant\mathfrak{ta}_-$. □

We use this proposition to show that along deformations, although the exponential growth of TH might change, its positivity is preserved. We accomplish this by comparing with a Reeb flow.

Proof of Proposition 3.1.6. Suppose for now that $h_0^t\leqslant h_1^t$. Proposition 3.3.7 shows that then the deformation morphism Φ restricts to $\Phi|_{\mathrm{TC}^T(\varphi_0^t)}:\mathrm{TC}^T(h_0^t)\to\mathrm{TC}^T(h_1^t)$. Furthermore $h_0^t=h_1^t$ for $t\leqslant 0$, thus $\mathfrak{ta}(h_0^t)$ and $\mathfrak{ta}(h_1^t)$ have the same critical points with non-positive action, and constant critical points (x,η) with $\eta\leqslant 0$ are solutions of (3.3.12). Since action is non-increasing along solutions of (3.3.12) we get that

$$\Phi|_{\mathrm{TC}^0(h_0^t)}\;:\;\mathrm{TC}^0(h_0^t)\;\to\;\mathrm{TC}^0(h_1^t)$$

is a lower diagonal isomorphism. For the homomorphism Φ_* induced in the quotient we thus have

$$\begin{aligned}\Phi_*(\mathrm{TC}_+^T(h_0^t)) &= \Phi(\mathrm{TC}^T(h_0^t))/\Phi(\mathrm{TC}^0(h_0^t))\\ &= \Phi(\mathrm{TC}^T(h_0^t))/\iota^{0,T}\,\mathrm{TC}^0(h_1^t)\\ &\subseteq \mathrm{TC}_+^T(h_1^t).\end{aligned}$$

Since Φ induces an isomorphism in TH^∞, abbreviating $\iota=\iota_+^{T,\infty}$, we conclude that

$$\dim\iota_*\,\mathrm{TH}_+^T(h_0^t)\;\leqslant\;\dim\iota_*\,\mathrm{TH}_+^T(h_1^t). \tag{3.3.13}$$

Now choose $c,C>0$ such that $c\leqslant h^t\leqslant C$. Denote by $\varphi_c^t,\varphi_{h^t}^t,\varphi_C^t$ the induced flows. The constants c,C are not equal to 1 for t near 0 or 1, so we need to modify them to fit our setup. From the proof of [20, Proposition 6.2] it is clear that there are functions $h_c^t,h_C^t:S^*Q\times[0,1]\to\mathbb{R}$ with $h_c^t=h_C^t=1$ for t near 0 and 1, that satisfy $h_c^t\leqslant h^t\leqslant h_C^t$, and such that the flows $\varphi_{h_c^t}^t$ and $\varphi_{h_C^t}^t$ induced by h_c^t and h_C^t are time-reparametrizations of the geodesic flows φ_c^t and φ_C^t that satisfy $\varphi_{h_c^t}^1=\varphi_c^1$ and $\varphi_{h_C^t}^1=\varphi_C^1$. Extend h_c^t and h_C^t constantly for $t<0$ and periodically for $t>0$, see Figure 3.6.

We apply (3.3.13) twice, first to a monotone deformation from h_c^t to h^t and then to a monotone deformation from h^t to h_C^t. By construction of h_c^t and h_C^t it is clear that there exists a function $\tau:\mathbb{R}^{\geqslant 0}\to\mathbb{R}^{\geqslant 0}$ such that $\varphi_{h_c^t}^{\tau(t)}=\varphi_{h_C^t}^t$ and such that $t\leqslant\tau(t)\leqslant 2\frac{C}{c}t$. Thus,

$$\iota_*\,\mathrm{TH}_+^{\tau(T)}(h_c^t)\;\cong\;\iota_*\,\mathrm{TH}_+^T(h_C^t).$$

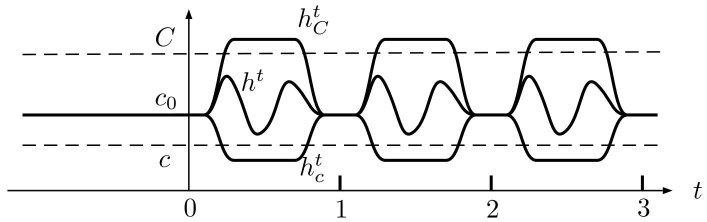

Figure 3.6: The sandwiching of h^t by h_c^t and h_C^t.

With (3.3.13), this results in

$$\dim \iota_* \operatorname{TH}_+^T(h_c^t) \leqslant \dim \iota_* \operatorname{TH}_+^T(h^t) \leqslant \dim \iota_* \operatorname{TH}_+^T(h_C^t) = \dim \iota_* \operatorname{TH}_+^{\tau(T)}(h_c^t).$$

We conclude that for every positive path of contactomorphisms the growth of Rabinowitz–Floer homology is positive if and only if it is positive for a Reeb flow.

From Section 3.3.2 we know that $\Gamma^{\mathrm{symp}}(W, L)$ is the growth of $\mathrm{AH}(W, L)$, thus also of $\mathrm{TH}(1)$. Since the flow of $\min h^t$ is an asymptotically linear reparametrization of the Reeb flow (Since we require that around integer values the Hamiltonian is constant 1, we need to reparametrize in a nontrivial way.), the quantitative bound follows. □

Finally we show that from the homology growth from L to L we can deduce information about the growth of time-dependent Reeb chords from Λ to a Legendrian Λ' that is isotopic to Λ through Legendrians.

Proof of Proposition 3.1.7. The idea of the proof is a rearrangement of information: If $\psi(\Lambda) = \Lambda'$, then the set of φ^t-chords from Λ to Λ' is in bijection with the set of $\varphi^t \circ \psi^{-1}$-chords from $\psi^{-1}\Lambda$ to Λ, see Figure 3.7.

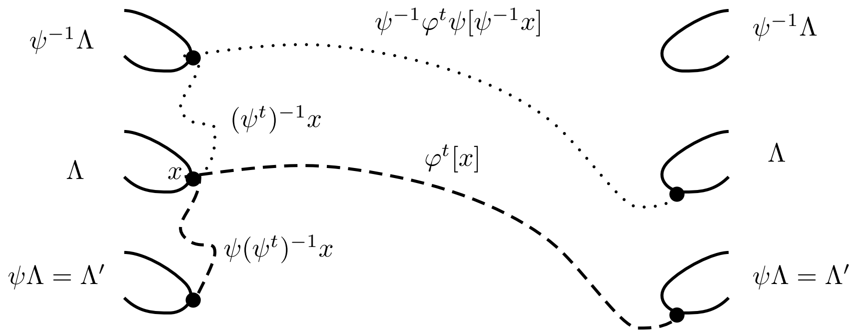

Figure 3.7: A sketch of the geometric situation. A point x is taken along the dotted line from Λ to $\psi^{-1}\Lambda$ by $(\psi^t)^{-1}$ and then to Λ by the flow of φ^t, conjugated by ψ^{-1}. Applying ψ to everything yields the dashed line which, ignoring the first part from Λ' to Λ, is a φ^t-chord from Λ to Λ'.

Let φ^t denote the flow of h^t. By the isotopy extension theorem we can extend the isotopy of Legendrians to a path of contactomorphisms $\{\psi^t\}_t$ with $\psi^0 = \mathrm{id}$ and $\psi^1 = \psi$. Let g^t be the contact Hamiltonian that generates $(\psi^t)^{-1}$. Let $\tilde{h}^t$ denote the Hamiltonian that generates $\tilde{\varphi}^t = (\psi)^{-1} \circ \varphi^t \circ (\psi)$. We choose k so large that the Hamiltonian $\tilde{g}$ of $\tilde{\varphi}^{kt} \circ (\psi^t)$ is always larger than $\max\{\tilde{h}^t\}$. Then we deform $\tilde{h}^t$ to $\overline{h}^t$ such that

$$\overline{h}^t = \begin{cases} 1 & \text{if } t < 0, \\ \tilde{g} & \text{if } t \in [0,1], \\ \tilde{h}^t & \text{if } t > 1. \end{cases}$$

By our choice of k this deformation is positive, thus induces a monotone morphism in TH. The time-1 flow of $\overline{h}$ is $\tilde{\varphi}^k \circ \psi$. By Proposition 3.1.6, also $\mathrm{TH}(\overline{h}^t; L)$ grows exponentially. The critical points of $\mathfrak{pa}_{\overline{h}^t}$ of period $T+1 > 1$ correspond to orbits from Λ to Λ that first travel to $\psi^{-1}\Lambda$, then follow $\tilde{\varphi}^t$ for time $T+k$ to end at Λ. By application of ψ this is equivalent to orbits that start at $\psi(\Lambda) = \Lambda'$, travel to Λ and then follow φ^t for time $T+k$ to end at Λ'. Since $\psi(\Lambda) = \Lambda'$, every φ^t-orbit from Λ to Λ' has a "starting tail" from Λ to Λ'.

From this we conclude that the number of φ^t-orbits from Λ to Λ' with length between k and T grows at least as fast as the dimension of the chain complex $\mathrm{TC}^{(1,T)}(\overline{h}^t; L)$, and thus at least as fast as the dimension of $\mathrm{TH}^{(1,T)}(\overline{h}^t; L)$ which has by Proposition 3.1.6 at least exponential growth $\min\{\overline{h}^t\}\Gamma^{\mathrm{symp}}(W,L)$. But $\min\{\overline{h}^t\} = \min\{\tilde{h}^t\}$ by construction. If we denote $(\psi^{-1})^*\alpha = f\alpha$, then $\tilde{h}^t(\psi^{-1}\varphi^t\psi x) = f(\varphi^t\psi x)h(\psi x)$. Thus, $\min\{\tilde{h}^t\} \geqslant \min f \min h^t$ and the quantitative assertion follows.

□

3.4 Proof of Theorem 8

The main result follows by composition of the results of the previous sections. The lower bounds on volume growth using a method introduced in [3].

Proof of Theorem 8. Let $\varphi^t, t \in \mathbb{R}$ be a positive path of contactomorphisms with $\varphi^0 = id, \varphi^1 = \varphi$ such that its underlying contact Hamiltonian h^t is constant 1 for $t \leqslant 0$ and 1-periodic for $t > 0$. We will show that the exponential growth of $\mathrm{Vol}(\varphi^t\Lambda)$ is positive, where Vol is taken with respect to some well chosen Riemannian metric. By a theorem of Yomdin [37] this volume growth provides a lower bound on the topological entropy.

From Propositions 3.1.4 and 3.1.5 we deduce the following chain of isomorphisms

$$\mathrm{WH}^a_+(W,L) \cong \check{\mathrm{w}}\mathrm{H}^a_+(W,L) \cong \mathrm{AH}^{(0,a)}(W,L),$$

for all $a > 0$ such that $a \notin \mathcal{S}$. Thus, the exponential growth of $\dim \mathrm{AH}^{(0,a)}(W,L)$ is $\Gamma^{\mathrm{symp}}(W,L)$. Proposition 3.1.6 shows that the exponential growth of $\dim \mathrm{TH}^{(0,a)}(h^t)$ is at least $c\Gamma^{\mathrm{symp}}(W,L)$, where $c = \min h^t > 0$.

Since Λ is a Legendrian sphere, there is a tubular neighborhood $\mathcal{N} = B^n \times \Lambda$ of Λ in M that is a product of a ball and the Legendrian spheres Λ. By isotopy extension each of the fibers Λ' is the image of Λ by a contactomorphism ψ of M isotopic to the identity such that $(\psi^{-1})^*\alpha = f\alpha$. After choosing a smaller neighborhood, one can assume that $\min f \geqslant 1 - \varepsilon$ for a uniform $\varepsilon > 0$. By Proposition 3.1.7 we see that for all fibers Λ' the number of φ^t-chords from Λ to Λ' has growth at least $\gamma := (1-\varepsilon)c\Gamma^{\mathrm{symp}}(W,L)$.

Now we choose our Riemannian metric g such that g orthogonally splits on $\mathcal{N} = B^n \times \Lambda$. We show that $\mathrm{Vol}_n(\bigcup_{t\in[0,T]}\varphi^t\Lambda)$ has growth at least γ since it cuts through $\mathcal{N}$ many times. In the following we regard φ^t as a map $\varphi(x,t) : (\Lambda, \mathbb{R}) \to \bigcup_{t\in\mathbb{R}}\varphi^t\Lambda = \varphi(\Lambda \times \mathbb{R})$.

Let $\pi : \mathcal{N} \to B^n$ be the projection to the fiber. Then by Sard's theorem there is a subset $B' \subset B^n$ of full measure such that the map

$$P : \varphi^{-1}(\pi^{-1}(B')) \subset \Lambda \times \mathbb{R} \to B';\ (x,t) \to \pi \circ \varphi(x,t)$$

has only regular values. At these points $P^{-1}(b)\cap\Lambda\times[0,T]$ is finite for every T, so we can consider its number of elements $n_b(T)$. Note that P being regular implies that the functional in the proof of Proposition 3.1.6 is Morse for all action windows $(1, T+1), T>0$, so the corresponding TH is well defined for these action windows. Since $n_b(T)$ counts the number of φ^t-trajectories from Λ to $b\times\Lambda\subset\mathcal{N}$, $n_b(T)$ has growth at least γ. Since

$$\mathrm{Vol}_n(\varphi(\Lambda\times[0,T])\cap\mathcal{N}\geqslant\int_{B'}n_b(T)\,d\,\mathrm{Vol}_n,$$

and since the integrand on the right hand side uniformly has growth at least γ, we conclude that $\mathrm{Vol}_n(\bigcup_{t\in[0,T]}\varphi^t\Lambda)\geqslant\mathrm{Vol}_n(\bigcup_{t\in[0,T]}\varphi^t\Lambda)\cap\mathcal{N}$ has growth at least γ.

Now to relate the growth of $\mathrm{Vol}_n(\bigcup_{t\in[0,T]}\varphi^t\Lambda)$ to the growth of $\mathrm{Vol}_n(\varphi^t\Lambda)$, we note that $|X_{h^t}|_g\leqslant C$ is bounded in length from above by compactness of M. Thus we can estimate

$$\mathrm{Vol}_n(\varphi(\Lambda\times[0,T])\leqslant C\int_0^T\mathrm{Vol}_{n-1}(\varphi^t\Lambda)\,dt.$$

Since the left hand side has growth at least γ, the same holds for the right hand side. This is only possible if the integrand also has growth at least γ. We conclude that the volume growth of $\varphi^t\Lambda$, thus the volume growth of φ on M, and thus the topological entropy of φ are at least $c\Gamma^{\mathrm{symp}}(W,L)>0$, since $\varepsilon>0$ was arbitrary. □

Remark 3.4.1. Note that for the definition of symplectic growth we required that (λ, L) is regular. In contrast we do not require regularity for φ^t. This is because we only consider Hamiltonians as in the proof Proposition 3.1.7, which are perturbed in the interval $[0,1]$.

Chapter 4

The Bott–Samelson theorem for positive Legendrian isotopies

4.1 Introduction and result

The spherization S^*Q of a manifold Q is the space of positive line elements in the cotangent bundle T^*Q. The tautological one-form λ on T^*Q does not pass to the quotient, but its kernel does. This endows S^*Q with a cooriented contact structure ξ.

Let $j_t : L \to S^*Q$ be a smooth family of embeddings such that $j_t(L)$ is a Legendrian submanifold of S^*Q for all t. Then $L_t = j_t(L)$ is called a Legendrian isotopy. If $\alpha(\frac{d}{dt}j_t(x)) > 0$ for one and hence any coorientation preserving contact form α for ξ and all $x \in L$, then L_t is called positive. Frauenfelder–Labrousse–Schlenk proved the following Theorem.

Theorem 10. *[20, Theorem 2.13] Let Q be a closed connected manifold of dimension $\geqslant 2$. Suppose there exists a positive Legendrian isotopy L_t in the spherization S^*Q that connects the fiber over a point with itself, i.e. $L_0 = L_1 = S^*_qQ$. Then the fundamental group of Q is finite and the integral cohomology ring of the universal cover of Q is generated by one element.*

We note that by a deep result in algebraic topology, a manifold with integral cohomology ring generated by one element is homotopy equivalent to S^n, $\mathbb{R}P^n$ or $\mathbb{C}P^n$ or has the integral cohomology ring of $\mathbb{H}P^n$ or the Cayley plane, see [8] and the references therein.

In this paper we prove the following addition to Theorem 10, which was conjectured in [20].

Theorem 11. *Under the assumptions of Theorem 10, if furthermore $L_t \cap L_0 = \varnothing$ for $0 < t < 1$, then Q is simply connected or homotopy equivalent to $\mathbb{R}P^n$.*

The union of these two theorems is the complete generalization of the classical Bott–Samelson theorem from geodesic flows to positive Legendrian isotopies.

The first versions of the Bott–Samelson theorem were for geodesic flows and used Morse theory of the energy functional on the based loop space, see [9], [35] and [8]. Frauenfelder, Labrousse and Schlenk [20] proved versions of Theorem 10 and 11 for autonomous Reeb flows, using Rabinowitz–Floer homology. They also proved Theorem 10 using Rabinowitz–Floer homology for positive Legendrian isotopies as stated above. The puzzle piece missing in [20] to generalize Theorem 11 from autonomous Reeb flows to positive Legendrian isotopies is the fact that the action functional in the construction is Morse–Bott. We provide this in Lemma 4.3.2, and thus complete the proof in [20]. The key ingredient is the choice of Hamiltonian, which is elaborated in Lemma 4.2.1. We cannot avoid the Hamiltonian to be time-dependent, but we can control the time-dependence along the Legendrian isotopy. At critical points, the resulting action functional then behaves like in the autonomous case. This paper is heavily based on [20], which also contains an extensive introduction to the topic.

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4.2 Recollections

The Rabinowitz–Floer homology we use depends on a time-dependent Reeb flow, not on a Legendrian isotopy. We first explain how we choose such a flow that restricts to a given Legendrian isotopy. Then we briefly present the version of Rabinowitz–Floer homology we use and discuss its properties. We only sketch the proofs, since they are contained in or are analguous to proofs in [5, 11, 12, 20]. For a general exposition of Morse–Bott homology we refer the reader to the Appendix of [22].

The choice of flow

Let $j_t : L \hookrightarrow M$, $t \in [0,1]$, be a positive Legendrian isotopy in a cooriented exact contact manifold (M, α). We denote $L_t = j_t(L)$. By the Legendrian isotopy extension theorem, see for example [25, Theorem 2.6.2], there exists a positive contact isotopy ψ^t of M such that $\psi^t(L_0) = L_t$. If furthermore $L_0 = L_1$, then there exists a positive and twisted periodic (that is $\varphi^t = \varphi^{t-k} \circ \varphi^k$ for all $t \in \mathbb{R}, k \in \mathbb{Z}$) contact isotopy φ^t such that $\varphi^k(L_0) = L_0$ for all $k \in \mathbb{N}$, see [20, Proposition 6.2]. This isotopy is generated by a contact Hamiltonian h^t that is a convex combination of the contact Hamiltonian of ψ^t and the one of the Reeb flow ψ^t_R (namely $h \equiv 1$), such that for t near 0 or 1, φ^t coincides with ψ^t_R. Note that in general $\varphi^t(L_0) \neq \psi^t(L_0)$ for $t \notin \mathbb{N}$.

Lemma 4.2.1. *Given a periodic Legendrian isotopy L_t that is the restriction of the Reeb flow generated by the contact Hamiltonian $h \equiv 1$ for t near 0 and 1 (as given by [20, Proposition 6.2]), then the corresponding twisted periodic positive contact isotopy φ^t can be chosen such that the time-dependent contact Hamiltonian h^t that generates φ^t satisfies $\dot{h}^t = 0$ along L_t.*

Proof. The construction of h^t is performed as in [25, Theorem 2.6.2]. We emphasize for a function h^t and a path $\gamma(t)$ the distinction between $(\frac{d}{dt}h^t)(\gamma(t))$ and $\frac{d}{dt}(h^t(\gamma(t)))$ by using the notation $\dot{h}^t := \frac{d}{dt}h^t$.

Recall that a contact Hamiltonian h^t and a contact vector field X_t determine each other through the equations $h^t = \alpha(X_t)$ and $\iota_{X_t} d\alpha = dh^t(R_\alpha)\alpha - dh^t$. We define the 1-jet of h^t along L_t as follows.

$$h^t(j_t(x)) = \alpha\left(\frac{d}{dt}j_t(x)\right) \quad \forall x \in L, \tag{4.2.1}$$

$$dh^t(v) = -\iota_{\frac{d}{dt}j_t(x)} d\alpha(v) \quad \forall v \in \xi|_{L_t}, \tag{4.2.2}$$

$$dh^t\left(\frac{d}{dt}j_t(x)\right) = \frac{d}{dt}\left(h^t(j_t(x))\right) \quad \forall x \in L. \tag{4.2.3}$$

Any Hamiltonian h^t that satisfies the first two equations generates a vector field X_t such that $X_t(j_t(x)) = \frac{d}{dt}j_t(x)$ for all $x \in L$. Equation (4.2.2) holds for all $v \in TL_t$ since $TL_t \subseteq \xi|_{L_t}$.

Equation (4.2.3) does not contradict (4.2.2) since $\frac{d}{dt}j_t(x)$ is positively transverse to ξ for all $x \in L$. (Here we differ from [25] where the choice in (4.2.3) is $dh^t(R_\alpha) = 0$.) Since $\frac{d}{dt}\left(h^t(j_t(x))\right) = \dot{h}^t(j_t(x)) + dh^t(\frac{d}{dt}j_t(x))$ for all $x \in L$, equation (4.2.3) implies $\dot{h}^t = 0$ along L_t. We extend h^t to a neighbourhood of L_t by identifying a neighbourhood of L_t with the normal bundle $NL_t \to L_t$ and choosing h_t linear on each fiber.

Finally we extend h^t to a positive function that is constant 1 outside a neighbourhood of L_t. Since the Legendrian isotopy is the restriction of the Reeb flow generated by $h \equiv 1$ for t near 0 and 1, the function h^t thus constructed satisfies $h^t \equiv 1$ for t near 0 and 1, and admits a 1-periodic extension. □

The spherization (S^*Q, ξ) of a manifold Q is represented by any fiberwise starshaped hypersurface $\Sigma \subset T^*Q$ in the cotangent bundle with contact structure $\ker \lambda|_\Sigma$. The map that sends a positive line element to its intersection with Σ is a contactormorphism. The radial dilation of a fiberwise starshaped hypersurface by a positive function is a contactomorphism onto its image. Every cooriented contact form of (S^*Q, ξ) is realized as $\lambda|_\Sigma$ for some fiberwise starshaped hypersurface Σ. We choose a Riemannian metric g on Q and represent S^*Q henceforth as the unit cosphere bundle with respect to this metric. With $\alpha = \lambda|_\Sigma$, the symplectization $(\Sigma \times \mathbb{R}_{>0}, d(r\alpha))$ is naturally symplectomorphic to $T^*Q \backslash Q$. A contact isotopy φ^t_Σ of Σ admits a lift to a Hamiltonian isotopy φ^t of $\Sigma \times \mathbb{R}_{>0}$, defined by $\varphi^t(x, r) = (\varphi^t_\Sigma(x), \frac{r}{\rho_t(x)})$ where $\rho_t(x)$ is defined by $(\varphi^t_\Sigma)^*(\alpha)|_x = \rho_t(x)\alpha|_x$, see [5, Proposition 2.3]. If φ^t_Σ is generated by the contact Hamiltonian h^t, then φ^t is generated by the Hamiltonian $H^t = rh^t$.

The functional

Let h^t be a positive, periodic contact Hamiltonian on $(\Sigma, \ker \lambda)$. Following [5] we choose a lift of the contact isotopy φ^t of Σ generated by h^t to the symplectization $(\Sigma \times \mathbb{R}_{>0}, d(r\alpha))$, depending on parameters $\kappa \geqslant 2, R \geqslant 2$ and constants c, C such that uniformly $0 < c < h^t < C$. We define $\widetilde{H}^t = rh^t - \kappa$. The Hamiltonian H^t is a deformation of $\widetilde{H}^t$ such that $H^t = cr - \kappa$ for $r \leqslant 1$, $H^t = \widetilde{H}^t$ for $2 \leqslant r \leqslant \kappa R - 1$ and $H^t = Cr - \kappa$ for $r > \kappa R$. This has the effect that H^t induces reparametrized g-geodesic flows for $r \in (0,1] \cup [\kappa R, \infty)$, and a lift of the h^t-contact flow for $r \in [2, \kappa R - 1]$.

Denote by $\Omega_{T^*_qQ}T^*Q$ the set of $W^{1,2}$ paths $x : [0,1] \to T^*Q$ such that $x(0), x(1) \in T^*_qQ$. Define the functional $\mathcal{A} : \Omega_{T^*_qQ}T^*Q \times \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{A}(x, \eta) \quad = \quad \frac{1}{\kappa}\left(\int_0^1 \left[\lambda(\dot{x}) - \eta H^{\eta t}(x(t))\right]\, dt\right).$$

This functional depends of course on h^t, but also on the parameters κ, R and the constants c, C. A pair (x, η) is a critical point of $\mathcal{A}$ if and only if $\dot{x} = \eta X_{H^{\eta t}}$ and $\int_0^1 H^{\eta t}(x(t)) + \eta t \dot{H}^{\eta t}(x(t))\, dt = 0$. This is equivalent to

$$\begin{cases} \dot{x} = \eta X_{H^{\eta t}}, \\ H^\eta(x(1)) = 0, \end{cases} \tag{4.2.4}$$

as one sees by using that $\eta X_{H^{\eta t}}$-chords satisfy $\frac{d}{dt}H^{\eta t}(x(t)) = \eta \dot{H}^{\eta t}(x(t))$ and by integration by parts. Note that the factor $\frac{1}{\kappa}$ does not change the critical point equations (4.2.4), but only the critical values. In fact, Lemma 4.2.3 below shows that this factor normalizes the action for critical points in such a way that the action spectrum is independent of κ.

Remark 4.2.2. For autonomous Hamiltonians $H^t = H$ the second equation of (4.2.4) becomes $H(x(t)) = 0\ \forall\ t \in [0,1]$. Thus the critical points are flow lines on the hypersurface $H^{-1}(0)$.

This hypersurface is not well-defined for time-dependent Reeb flows and $H^{\eta t}(x(t))$ might be very large for $t \neq 1$. We deal with this defect through the parameters κ, R. Intuitively speaking, the parameters create safe space (κ towards the zero section, and R towards infinity), where critical orbits are free to roam. This is made precise in the next lemma from [5, Proposition 4.3, Corollary 4.4].

Lemma 4.2.3. *For all $a < b$ there exist constants $\kappa_0 \geqslant 2, R_0 \geqslant 2$ such that for $\kappa \geqslant \kappa_0$ and $R \geqslant R_0$, all critical points (x, η) of $\mathcal{A}$ with action between a and b satisfy $2 \leqslant |x(t)|_g \leqslant \kappa R - 1$ for all t and $\mathcal{A}(x, \eta) = \eta$. As a consequence, the critical point equation (4.2.4) and the action values are independent of the choice of $\kappa \geqslant \kappa_0, R \geqslant R_0, c, C$.*

The chain group

Assume from now on that the functional $\mathcal{A}$ is Morse–Bott for critical points with action between a and b. Choose in addition a Morse function f on $\operatorname{Crit} \mathcal{A}$. Then for $b \in \mathbb{R}$ we define the filtered Rabinowitz–Floer chain group $\mathrm{RFC}^b(\mathcal{A})$ as the $\mathbb{Z}_2$-vector space generated by the critical points of f on $\operatorname{Crit} \mathcal{A}$ with action $\leqslant b$.

The index

The index of a critical point $c = (x, \eta)$ of f on $\operatorname{Crit} \mathcal{A}$ is defined as follows. Let TT_q^*Q be the vertical Lagrangian distribution. Denote by $\mu_{RS}(x, \eta)$ the Robbin–Salamon index of the path $d(\varphi^{\eta t})^{-1}(TT^*_{x(t)}Q)$ with respect to $TT^*_{x(0)}Q$, and by μ_M the Morse index of f on $\operatorname{Crit} \mathcal{A}$, see [34]. Then the index of c is defined as

$$\mu(c) = \mu_{RS}(x, \eta) - \frac{n-1}{2} + \frac{1}{2}\mu_M(c),$$

where the shift by $-\frac{n-1}{2}$ is introduced such that the index μ agrees with the Morse index for geodesic Hamiltonians. Denote by $\mathrm{RFC}_*^{>0}(\mathcal{A})$ the chain groups graded by the index μ.

The differential

For the differential, we choose an ω-compatible almost complex structure $J = J_{t,\eta}$ on T^*Q that satisfies the following properties for $r \in [0, 1] \cup [\kappa R, \infty)$, following [12, Chapter 3]:

- J is independent of t, η,
- J maps $r\partial_r$ to $X_{\frac{1}{2}r^2}$ and preserves $\ker \lambda|_{\{r=const\}}$,
- J is invariant under the Liouville flow $(y, r) \mapsto (y, e^t r),\ t \in \mathbb{R}$.

Define the L^2-metric

$$\langle (v_1, \eta_1), (v_2, \eta_2) \rangle_J = \frac{1}{\kappa} \int_0^1 \omega(v_1, Jv_2)\, dt + \frac{\eta_1 \eta_2}{\kappa}$$

on $\Omega_{T_q^*Q} T^*Q \times \mathbb{R}$. Further, choose a Morse–Smale metric m on $\operatorname{Crit} \mathcal{A}$. The differential of degree -1 is defined by the $\mathbb{Z}_2$-count of finite energy negative gradient flow lines with cascades. A flow line with cascades starts at a critical point of f at time $-\infty$, then runs until a finite time as negative m-gradient flow line on $\operatorname{Crit} \mathcal{A}$, then runs as negative $\langle \cdot, \cdot \rangle_J$-flow line from one component of $\operatorname{Crit} \mathcal{A}$ to an other (from time $-\infty$ to $+\infty$), then runs for a finite time along a negative m-gradient flow line, ..., and after finitely many such changes (cascades) ends in a critical point of f at time $+\infty$. To show that this differential is well defined and $d^2 = 0$, one has to show that for $\mathcal{A}(c^+), \mathcal{A}(c^-) \in [a,b]$ the moduli space of finite energy negative gradient flow lines with cascades from c^+ to c^- is compact modulo breaking. This follows from standard arguments as soon as one has established L^∞ bounds on the Floer strips underlying the $\langle\cdot,\cdot\rangle_J$-parts of the flow lines, on the derivatives of the Floer strips, and on η. The L^∞ bounds on the Floer strips follow from a maximum principle since our Hamiltonian is convex for $r \notin [1, \kappa R]$. The L^∞ bounds on the derivatives follow from the exactness of $\omega = d\lambda$ that prevents bubbling. The following lemma shows that for almost critical points, η is bounded by the action.

Lemma 4.2.4 (Fundamental Lemma)**.** *There exists $\varepsilon > 0$ such that*

$$\|\nabla \mathcal{A}(x,\eta)\| < \varepsilon \Rightarrow |\eta| \leqslant \frac{1}{\varepsilon}(\mathcal{A}(x,\eta)+1).$$

This is a version with Lagrangian boundary conditions of [5, Lemma 4.5] and is proved using a by now standard scheme, see [11, Proposition 3.1]. The L^∞ bound on η is then obtained as in [11, Corollary 3.3].

The Homology

We define $\mathrm{RFC}^b_{a,*}(\mathcal{A})$ as the quotient chain complex $\mathrm{RFC}^b_* / \mathrm{RFC}^a_*$. By Lemma 4.2.3, for $\kappa \geqslant \kappa_0$, and $R \geqslant R_0$ the generators and actions of this chain complex do not depend on the choice of κ, R, c, C, and by standard continuation arguments the resulting homology is independent up to canonical isomorphisms of a generic choice of g, J, m. Finally, define $\mathrm{RFC}^{>0}_*(\mathcal{A})$ as the inverse direct limit $\lim_{b\nearrow\infty} \lim_{a\searrow 0} \mathrm{RFC}^b_{a,*}(\mathcal{A})$ under the homomorphisms induced by inclusion and denote the resulting homology by $\mathrm{RFH}^{>0}_*(\mathcal{A})$.

Invariance

For any other twisted periodic and positive contact isotopy $\widetilde{\varphi}^t$ such that the corresponding functional $\widetilde{\mathcal{A}}$ is Morse–Bott, we have

$$\mathrm{RFH}^{>0}_*(\mathcal{A}) \cong \mathrm{RFH}^{>0}_*(\widetilde{\mathcal{A}}).$$

This can be shown like the invariance (29) in the proof of [20, Lemma 5.4] with the additional explanation after [20, Lemma 5.5], by considering the path of Hamiltonians $H^t_s = (1-\beta(s))H^t + \beta(s)\widetilde{H}^t$, where $\beta(s)$ is a smooth monotone function with $\beta(s) = 0$ for $s \leqslant 0$ and $\beta(s) = 1$ for $s \geqslant 1$, generating a path of functionals $\mathcal{A}_s$ that connects $\mathcal{A}$ and $\widetilde{\mathcal{A}}$, where the constants κ, R, c, C are chosen uniformly in s. Note that $\partial_s H^t_s$ is compactly supported, thus a continuation homomorphism can be defined. Also note that there exists an $\varepsilon > 0$ such that for all $s \in \mathbb{R}$ the action spectrum of $\mathcal{A}_{H^t_s}$ and the interval $(0,\varepsilon]$ are disjoint. Using this, we can exclude that critical values cross 0 during the continuation. The isomorphism follows then by standard arguments.

In particular for $h^t \equiv 1$ the corresponding functional $\mathcal{A}_g$ is the functional of the g-geodesic flow. Denote by $\mathrm{HM}^{>0}_*(\mathcal{E})$ the $\mathbb{Z}_2$-Morse homology relative the constant loop of the energy functional $\mathcal{E}(x) = \int_0^1 \frac{1}{2} g(\dot{x},\dot{x})\, dt$ on the space of based loops in Q. The following result is a special case of Merry's theorem [29, Theorem 3.16].

$$\mathrm{RFH}^{>0}_*(\mathcal{A}_g) \cong \mathrm{HM}^{>0}_*(\mathcal{E}).$$

Since $\mathrm{HM}^{>0}_*(\mathcal{E})$ is isomorphic to the homology $H_*(\Omega_q, q; \mathbb{Z}_2)$ relative the constant loop, we obtain

Lemma 4.2.5. $\mathrm{RFH}^{>0}_*(\mathcal{A}_g) \cong H_*(\Omega_q, q; \mathbb{Z}_2)$.

4.3 Proof of Theorem 11

Recall that Theorem 10 is shown in [20]. In this section we prove Theorem 11, using the results that are already established in Theorem 10. The main step is to show that in this situation the action functional is Morse–Bott. Theorem 11 then follows exactly as in [20].

Remark 4.3.1. Let (x, η) be a critical point of $\mathcal{A}$ for κ, R, c, C as in Lemma 4.2.3 and h^t as in Lemma 4.2.1. Then along x we have $H^t = \widetilde{H}^t = rh^t - \kappa$, and hence $\eta \dot{H}^{\eta t}(x(t)) = 0$. Since $\frac{d}{dt} H^{\eta t}(x(t)) = \eta \dot{H}^{\eta t}(x(t))$ we thus have $H^{\eta t}(x(t)) = 0$ for all t and $\mathcal{A}(x, \eta) = \eta$. In this sense the choice made in Lemma 4.2.1 is designed such that the functional that arises from the situation of Theorem 11 behaves at critical points as in the autonomous case.

Lemma 4.3.2. *In the situation of Theorem 11 and for h^t chosen as in Lemma 4.2.1, the action functional $\mathcal{A}$ defined above is Morse–Bott at the critical sets with positive action, the components of the critical manifold being diffeomorphic to $S_q^* Q \times \{k\}, k \in \mathbb{N}$.*

Proof. A diffeomorphism from the critical manifolds to $S_q^* Q \times \{k\}$ is given by mapping critical points $(x, k) \in \Omega_q Q \times \mathbb{R}$ to $(x(1), k) \in T_q^* Q \times \{k\}$. Since h^t is constant 1 for t near $k \in \mathbb{N}$ and by the equations (4.2.4), the image of this map is $\{(q, p) \in T^* Q \mid |p|_g = 1\} \times \{k\} \cong S_q^* Q \times \{k\}$.

The functional $\mathcal{A}$ is Morse–Bott if the kernel of the Hessian $\mathcal{HA}$ is exactly the tangent space of the critical manifold. The inclusion $T\,\mathrm{Crit}\,\mathcal{A} \subseteq \ker \mathcal{HA}$ is obvious, we will show the converse. A tangent vector to $x \in \Omega_q Q$ is a section $\hat{x}$ of the pullback bundle $x^* TT^* Q$. Assume that $(\hat{x}, \hat{\eta}) \in \ker \mathcal{H}(\mathcal{A})$. Since $\hat{x} \in T\Omega_q Q$, the endpoints of $\hat{x}$ are in the vertical subbundle, $\hat{x}(i) \in T_{x(i)} T_q^* Q \subseteq \ker \lambda$, for $i = 0, 1$.

We will compute $\mathcal{HA}((\hat{x}, \hat{\eta}), (\check{x}, \check{\eta}))$ where $(\check{x}, \check{\eta})$ is another vector based at (x, η). Assume for the moment that x lies in a single Darboux chart and that in local coordinates we have $x = (q, p)$ and $\hat{x} = (\hat{q}, \hat{p})$. As a preparation we compute

$$\begin{aligned} d\left(\int_0^1 \lambda(\dot{x})\, dt\right)(\hat{x}) &= \int_0^1 \frac{d}{d\epsilon}(p + \epsilon\hat{p})(\dot{q} + \epsilon\dot{\hat{q}}) \mid_{\epsilon=0}\ dt \\ &= \int_0^1 \hat{p}\dot{q} + p\dot{\hat{q}}\, dt \\ &= \int_0^1 \hat{p}\dot{q} - \dot{p}\hat{q}\, dt + p\hat{q} \mid_0^1 \\ &= \int_0^1 \omega(\hat{x}, \dot{x})\, dt, \end{aligned}$$

where the last equality holds because the endpoints of $\hat{x}$ lie in the vertical subbundle and thus $\hat{q}(i) = 0$ for $i = 0, 1$. If x does not lie in a single chart, the same follows after finitely many coordinate changes. Similarly we compute

$$d\left(\int_0^1 \omega(\hat{x}, \dot{x})\, dt\right)(\check{x}) = \int_0^1 \omega(\hat{x}, \dot{\check{x}})\, dt = \int_0^1 \omega(\check{x}, \dot{\hat{x}})\, dt + \omega(\hat{x}, \check{x})|_0^1 = \int_0^1 \omega(\check{x}, \dot{\hat{x}})\, dt,$$

where the last equality holds because the endpoints of $\hat{x}$ and $\check{x}$ lie in the vertical subbundle which is a Lagrangian. Using these preparations we can now compute

$$\begin{aligned}
\mathcal{A}(x,\eta) &= \int_0^1 \lambda(\dot{x}) - \eta H^{\eta t}(x(t))\,dt,\\
d\mathcal{A}(\hat{x},\hat{\eta}) &= \int_0^1 \omega(\hat{x},\dot{x}) - \eta dH^{\eta t}(\hat{x}) - \hat{\eta}\left(H^{\eta t}(x(t)) + \eta t\dot{H}^{\eta t}(x(t))\right)\,dt,\\
\mathcal{H}\mathcal{A}((\hat{x},\hat{\eta}),(\check{x},\check{\eta})) &= \int_0^1 \omega(\check{x},\dot{\hat{x}}) - \hat{\eta}\left(dH^{\eta t}(\check{x}) + \eta t d\dot{H}^{\eta t}(\check{x})\right)\\
&\quad - \check{\eta}\left(dH^{\eta t}(\hat{x}) + \eta t\, d\dot{H}^{\eta t}(\hat{x}) + \hat{\eta}\big(2t\dot{H}^{\eta t} + \eta t^2\ddot{H}^{\eta t}\big)\right)\,dt.
\end{aligned}$$

Thus $(\hat{x},\hat{\eta})$ lies in $\ker \mathcal{H}\mathcal{A}$ if and only if the following equations are satisfied.

$$\dot{\hat{x}} = \hat{\eta}\big(X_{H^{\eta t}} + \eta t X_{\dot{H}^{\eta t}}\big) \qquad \forall t, \tag{4.3.1}$$

$$0 = \int_0^1 dH^{\eta t}(\hat{x}) + \eta t d\dot{H}^{\eta t}(\hat{x}) + \hat{\eta}(2t\dot{H}^{\eta t} + \eta t^2\ddot{H}^{\eta t})\,dt. \tag{4.3.2}$$

We translate these equations to the fixed vector space $T_{x(0)}T^*Q$ by pulling back along $\varphi^{\eta t}$: Define

$$v(t) = D\varphi^{-1}\hat{x}(t),$$

where we abbreviated $\varphi^{\eta t}$ to φ for better readability. Since φ is a symplectomorphism, $D\varphi^{-1}X_{H^{\eta t}} = X_{\varphi^* H^{\eta t}}$. Thus equation (4.3.1) becomes

$$\dot{v} = \hat{\eta}(X_{\varphi^* H^{\eta t}} + \eta t X_{\varphi^* \dot{H}^{\eta t}}) \qquad \forall t, \tag{4.3.3}$$

Integrating equation (4.3.3), we obtain

$$v(1) = v(0) + \hat{\eta}\int_0^1 X_{\varphi^* H^{\eta t}} + \eta t X_{\varphi^* \dot{H}^{\eta t}}\,dt.$$

Since $H^{\eta t}$ is (after addition of the constant κ) 1-homogeneous in the fibers near $\operatorname{Crit}\mathcal{A}$, the flow $\varphi^{\eta t}$ commutes with dilations by a factor close to 1. Thus also $\varphi^* H^{\eta t}$ is (after addition of κ) 1-homogeneous, thus $\varphi^*\dot{H}^{\eta t}$ is 1-homogeneous and so near $\operatorname{Crit}\mathcal{A}$, $X_{\varphi^*\dot{H}^{\eta t}}$ is a lift of the contact Hamiltonian vector field $X_{\varphi^*\dot{h}^{\eta t}}$ on the spherization S^*Q. For h^t chosen as in Lemma 4.2.1, we have $\varphi^*\dot{h}^{\eta t} = 0$. Thus $X_{\varphi^*\dot{h}^{\eta t}}$ lies in the contact structure, $X_{\varphi^*\dot{h}^{\eta t}} \in \ker\lambda|_{S^*Q}$, and thus $X_{\varphi^*\dot{H}^{\eta t}} \in \ker\lambda|_{S^*Q} \oplus \langle\partial_r\rangle = \ker\lambda$. By the geometric setup of the theorem, $D\varphi^{-1}T_{x(1)}T_q^*Q = T_{x(0)}T_q^*Q$, so with $\hat{x}(i)$ also the endpoints of v lie in the vertical subbundle, $v(i) \in T_{x(0)}T_q^*Q \subseteq \ker\lambda$. Thus we conclude that $\hat{\eta}\int_0^1 X_{\varphi^* H^{\eta t}}dt \in \ker\lambda$. But $X_{\varphi^* H^{\eta t}} \pitchfork_+ \ker\lambda$ for all t since $h^{\eta t}$ is positive, and thus $\int_0^1 X_{\varphi^* H^{\eta t}}dt \pitchfork_+ \ker\lambda$. We conclude that $\hat{\eta} = 0$ and with (4.3.3) that v is constant.

Recall that our task is to show that $(\hat{x},\hat{\eta}) = (\hat{x},0) \in T\operatorname{Crit}\mathcal{A}$, and recall from (4.2.4) that $\operatorname{Crit}\mathcal{A} = \{x \mid \dot{x}(t) = \eta X_{H^{\eta t}}\} \times \mathbb{N} \cap \{x \mid H^{\eta}(x(1)) = 0\} \times \mathbb{N}$. We first define the path $(x_s,\eta_s) \in \{x \mid \dot{x}(t) = \eta X_{H^{\eta t}}\} \times \mathbb{N}$ by $x_s(t) = \varphi^{\eta t}(x(0) + sv), \eta_s \equiv \eta$. Then $\frac{d}{ds}(x_s,\eta_s)\,|_{s=0} = (\hat{x},0)$. Thus,

$$(\hat{x},0) \in T(\{x \mid \dot{x}(t) = \eta X_{H^{\eta t}}\} \times \mathbb{N}).$$

Since $\dot{x}_s = \eta X_{H^{\eta t}}(x_s)$ for all s, $\frac{d}{dt} H^{\eta t}(x_s(t)) = \eta \dot{H}^{\eta t}(x_s(t))$ and thus also $\frac{d}{dt} dH^{\eta t}_{x_0}(\hat{x}) = \eta d\dot{H}^{\eta t}_{x_0}(\hat{x})$. Together with $\hat{\eta} = 0$, equation (4.3.2) becomes

$$\begin{aligned} 0 &= \int_0^1 dH^{\eta t}(\hat{x}) + t\frac{d}{dt} dH^{\eta t}(\hat{x})\, dt \\ &\overset{\int by\, parts}{=} \int_0^1 dH^{\eta t}(\hat{x}) - dH^{\eta t}(\hat{x})\, dt + 1 \cdot dH^{\eta 1}(\hat{x}(1)) - 0 \cdot dH^{\eta 0}(\hat{x}(0)) \\ &= dH^{\eta}(\hat{x}(1)). \end{aligned}$$

Thus,

$$(\hat{x}, 0) \in T(\{x \mid \dot{x}(t) = \eta X_{H^{\eta t}}\} \times \mathbb{N}) \cap T(\{x \mid H^{\eta}(x(1)) = 0\} \times \mathbb{N}) = T\,\mathrm{Crit}\,\mathcal{A},$$

as claimed. □

Before we can continue we need two observations about the index. Since the components of $\mathrm{Crit}\,\mathcal{A}$ are spheres $S^*_q Q \times \{k\}$, the Morse function f on $\mathrm{Crit}\,\mathcal{A}$ can be chosen with exactly two critical points c_k^-, c_k^+ per component, with Morse index 0 and $d-1$.

Lemma 4.3.3. *The Robbin–Salamon index of* $(x(t), k) \in \mathrm{Crit}\,\mathcal{A}$ *depends only on* k *and is equal to* $k\mu_0$ *for some constant* $\mu_0 \geqslant 1$.

Proof. The proof goes exactly as in [20, Section 5.2] and uses Rabinowitz–Floer homology over $\mathbb{Z}$ coefficients, which is developed in [20] to prove Theorem 10. We repeat the argument without developing the theory over $\mathbb{Z}$ coefficients and refer the interested reader to [20]. Note that the change of coefficients changes neither the critical point equation nor the index.

The subset of $\mathrm{Crit}\,\mathcal{A}$ with $\eta = k$ is a sphere and thus connected. Let $(x_0, k), (x_1, k)$ be two critical points of $\mathcal{A}$ and (x_s, k) be a path in $\mathrm{Crit}\,\mathcal{A}$ connecting them. Identify the vector spaces $T_{x_s(0)}T^*Q$ in such a way that $TT^*_{x_s(0)}Q$ is constant. Then $d(\varphi^{kt})^{-1}(TT^*_{x_s(t)}Q)$ is a homotopy with parameter s with constant endpoints of paths with parameter t of Lagrangian subspaces. Thus the two paths $d(\varphi^{kt})^{-1}(TT^*_{x_0(t)}Q)$ and $d(\varphi^{kt})^{-1}(TT^*_{x_1(t)}Q)$ are stratum homotopic in the sense of [34] and thus $\mu_{\mathrm{RS}}(x_0, k) = \mu_{\mathrm{RS}}(x_1, k)$. We conclude that the Robbin–Salamon index only depends on k. Since every φ^{kt} flow line is the k-fold concatenation of φ^t flow lines, $\mu_{\mathrm{RS}}(\varphi^{kt}x(0), k) = k\mu_{\mathrm{RS}}(\varphi^t x(0), 1) =: k\mu_0$ by the concatenation property of the Robbin–Salamon index.

Assume that $\mu_0 \leqslant 0$. Since the signatures of $c_i^{\pm}$ are $\pm\frac{d-1}{2}$ (in particular bounded), there exists a k_0 such that $\mu(c) < k_0\ \forall c \in \mathrm{Crit}\,\mathcal{A}$. Thus for $k \geqslant k_0$ we have by deformation of $\mathcal{A}$ to a geodesic functional $\mathcal{A}_g$, and by the $\mathbb{Z}$-version of Lemma 4.2.5 (also contained in [29]),

$$0 = \mathrm{RFH}_k^{>0}(\mathcal{A}; \mathbb{Z}) \cong \mathrm{RFH}_k^{>0}(\mathcal{A}_g; \mathbb{Z}) \cong H_k(\Omega_q Q, q; \mathbb{Z}) \cong \widetilde{H}_k(\Omega_q Q; \mathbb{Z}),$$

and thus also $H_k(\Omega_q \widetilde{Q}; \mathbb{Z}) \cong 0$. Thus for all $k \geqslant k_0 + 1$ and $\mathbb{F} = \mathbb{Z}_p$ for any prime number p or $\mathbb{F} = \mathbb{Q}$ we have $H_*(\Omega_q \widetilde{Q}, q; \mathbb{F}) \cong 0$. By [36, Proposition 10] this implies $H_k(\widetilde{Q}, \mathbb{F}) \cong 0$ for all $k \geqslant 1$ and for all $\mathbb{F} = \mathbb{Z}_p$ or $\mathbb{Q}$ and thus $\widetilde{Q}$ is contractible. Since $\dim Q \geqslant 2$ and Q is closed, we must have $|\pi_1(Q)| = \infty$ which contradicts Theorem 10. □

With this our main theorem follows exactly as in [20]. We repeat the proof for the convenience of the reader.

Proof of Theorem 11. The chain group $\mathrm{RFC}_*^{>0}(\mathcal{A})$ of the Rabinowitz–Floer homology is generated by the critical points $c_k^\pm$, $k \geqslant 1$, where

$$\mu(c_k^-) = k\mu_0 - (d-1), \quad \mu(c_k^+) = k\mu_0.$$

By Lemma 4.3.3, $\mu_0 \geqslant 1$. Hence there is one critical point of index zero if μ_0 is a divisor of $(d-1)$ and no critical point of index zero otherwise. Hence after a deformation to the functional $\mathcal{A}_g$ of a geodesic flow, with Lemma 4.2.5 and the reduced long exact $\mathbb{Z}_2$-homology sequence of the pair $(\Omega_q Q, q)$ we find that

$$\mathrm{RFH}_0^{>0}(\mathcal{A}) \cong \mathrm{RFH}_0^{>0}(\mathcal{A}_g) \cong H_0(\Omega_q Q, q; \mathbb{Z}_2) \cong \widetilde{H}_0(\Omega_q Q; \mathbb{Z}_2)$$

is 0 or $\mathbb{Z}_2$, thus $\pi_1(Q)$ is 0 or $\mathbb{Z}_2$. In the first case we are done, so assume the second case. By Theorem 10, Q is a closed manifold such that $H^*(\widetilde{Q}; \mathbb{Z}_2)$ is generated by one element. Then by [21, Corollary 3.8], Q is either homotopy equivalent to $\mathbb{R}P^d$, or $\widetilde{Q}$ is homotopy equivalent to $\mathbb{C}P^{2n+1}$. In the former case we are done, so assume the latter.

We denote $\dim(\mathbb{C}P^{2n+1}) = 2(2n+1) = d$. Assume first that $\mu_0 \geqslant 2$, then $\mu(c_1^-) = \mu_0 - d + 1 \leqslant \mu(c) - 2$ for all other critical points c. This means that c_1^- is the lowest index generator of $\mathrm{RFH}_*^{>0}(\mathcal{A})$. The lowest index non-vanishing group is $\mathrm{RFH}_0^{>0}(\mathcal{A}) \cong \widetilde{H}_0(\Omega_q Q; \mathbb{Z}_2) \cong \mathbb{Z}_2$ and thus $\mu(c_1^-) = 0$ and $\mu_0 = d - 1$. Recall that

$$H_*(\mathbb{C}P^{2n+1}; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } * = kd \text{ or } kd+1 \text{ for } k \in \mathbb{N}_0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\Omega_q Q$ is homotopy equivalent to the disjoint union of two copies of $\Omega_q \mathbb{C}P^{2n+1}$,

$$H_*(\Omega_q Q; \mathbb{Z}_2) \cong H_*(\Omega_q \mathbb{C}P^{2n+1}) \oplus H_*(\Omega_q \mathbb{C}P^{2n+1}).$$

In particular $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong H_{2d}(\Omega_q Q; \mathbb{Z}_2) \cong \widetilde{H}_{2d}(\Omega_q Q; \mathbb{Z}_2) \cong \mathrm{RFH}_{2d}^{>0}(\mathcal{A})$, which is only possible if we have two generators of index $2d$. The c_k^+ have pairwise different indices and so do the c_k^-, thus there must be k^+ and k^- such that $\mu(c_{k^+}^+) = k^+\mu_0 - d + 1 = \mu(c_{k^-}^-) = k^-\mu_0 = 2d$. But since $\mu_0 = d - 1$ and $d \geqslant 6$, this is impossible.

The remaining case is $\mu_0 = 1$. Since $d \geqslant 6$, the critical points with negative index are exactly

$$\mu(c_1^-) = -d + 2, \ \mu(c_2^-) = -d + 3, \ \ldots, \ \mu(c_{d-2}^-) = -1.$$

If the chord underlying c_1^- were contractible in $\Omega_{T_q^*Q} T^*Q$, then all chords underlying $c_k^\pm$ would be contractible since they all are concatenations of chords homotopic to the chord underlying c_1^-. This contradicts the fact that they also generate the $\mathbb{Z}_2$-homology of the connected component of noncontractible chords in $\Omega_q Q$. Thus the chord underlying c_1^- must be noncontractible. Since $\pi_1(Q) = \mathbb{Z}_2$ and since the chord underlying c_2^- is the concatenation of two chords homotopic to the chord underlying c_1^-, the chord underlying c_2^- is contractible and in particular not homotopic to the chord underlying c_1^-. The boundary operator is defined by flow lines with cascades with underlying Floer strips and paths in $\operatorname{Crit} \mathcal{A}$, thus every chord underlying a critical point is homotopic to the chords underlying the summands of its boundary. Thus c_1^- cannot contribute to the boundary of c_2^-. Since all other critical points have higher index, we conclude that c_1^- is not a boundary. Since c_1^- is in the lowest degree chain group, it is closed and hence represents a non-trivial homology class. Thus $\mathrm{RFH}_{-d+2}^{>0}(\mathcal{A}) \cong \widetilde{H}_{-d+2}(\Omega_q Q; \mathbb{Z}_2)$ does not vanish, which is impossible since $-d + 2 < 0$. □

Chapter 5

Synopsis

In this chapter we briefly explain how the theorems of the previous chapters are connected.

We investigate two phenomena that are classically theorems of Riemannian dynamics and that are shown with Morse homology: first, that the geodesic flow on an energy hyperbolic manifold has positive topological entropy, and second, the Bott–Samelson theorem. Both theorems admit the generalization that "geodesic flow" can be replaced by "Reeb flow on the cosphere bundle", see [12, 28], which reveals that the theorems are actually phenomena of Reeb dynamics and thus are better situated in the world of contact geometry. In the papers [14, 15] we show that the two results further generalize to positive contactomorphisms, which reveals that the theorems are really contact topological and not geometrical. (The Bott–Samelson theorem even generalizes to positive Legendrian isotopies, but this is by the Legendrian isotopy extension theorem an immediate equivalence.) The proof of both theorems involves the construction of Rabinowitz–Floer homology for positive paths of contactomorphisms, the study of its invariance properties under deformation of the dynamical law and the Abbondandolo–Schwarz isomorphism between the Rabinowitz–Floer homology of a geodesic action functional and Morse homology of the energy functional.

The paper [16] shows that in an exactly fillable contact manifold positive growth of wrapped Floer homology implies that every positive contactomorphism has positive topological entropy. This can be seen as a continuation of [14], where we study positive contactomorphisms in cosphere bundles, which are special cases of exactly fillable contact manifolds. In both cases the main step of the proof is to show that Rabinowitz–Floer homology has positive exponential growth, from where one can show that topological entropy is positive. In [14] the growth comes from the Morse homology of the loop space on the base manifold, filtered by Morse index which corresponds to Conley–Zehnder index for Rabinowitz–Floer homology. In the more general setting of [16] neither do we have an isomorphism to Morse homology, nor can we define an index. Instead, we show that the positive part of wrapped Floer homology filtered by action is isomorphic to the positive part of Rabinowitz–Floer homology filtered by action. This is of large interest because Alves and Meiwes [6] recently discovered a large family of examples of exactly fillable contact manifolds for which wrapped Floer homology has positive exponential growth.

One might ask if there is an alternative proof showing that every positive contactomorphism of the examples of Alves and Meiwes has positive topological entropy that relies either only on wrapped Floer homology or only Rabinowitz–Floer homology. Such a proof is thinkable, supposing that one can mimick the construction of Alves and Meiwes for Rabinowitz–Floer homology, or that one can count chords of positive paths of contactomorphisms with wrapped Floer homology. Unfortunately both generalizations are not possible with the theories as they are today, and that therefore the combined strength of the two theories is necessary. On the one hand Alves and Meiwes managed to prove their results because wrapped Floer homology admits a Pontrjagin product structure that is stable under geometric operations, which is not (yet) established in Rabinowitz–Floer homology. On the other hand there is no wrapped Floer homology (known to us) that encodes time-dependent Reeb dynamics in a transparent way. The problem (or brilliant feature, depending on viewpoint) with wrapped Floer homology is that the main tool for understanding the homology are radial Hamiltonians, for which the radial coordinate explicitly corresponds to the slope and thus to the length of the chord. But for time-dependent Hamiltonians this correspondence breaks down since time-dependent Hamiltonians are not constant along their chords.

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