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Beyond risk-based portfolios: balancing performance and risk contributions in asset allocation

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Asset allocation using a new Performance/Risk Contribution measure improves the performance of risk-based portfolios

1. Introduction

The percentage capital allocation is well known to be a bad advisor on the percentage risk allocation in multi-asset class portfolios. In a typical 60/40 US equities-bond portfolio, the equity part is often responsible for more than 90% of the total portfolio's volatility (Qian 2005). One solution is to let portfolio weights be indirectly determined by a target constraint on the percentage volatility contributions. A special case is the risk parity or equal-risk-contribution portfolio, seeking portfolios in which all components contribute equally to the portfolio's volatility (see, e.g. Qian 2005, Maillard *et al*. 2010 and Bai *et al*. 2016).

Boudt *et al*. (2012) and Roncalli (2015) generalize the approach to risk allocation based on downside risk measures of the type $\mathcal{R}_p \equiv -\mu_p + c_p\sigma_p$, with μ_p and σ_p the portfolio expected return and volatility, and c_p a multiple that may depend on the portfolio return distribution. For such downside risk measures, setting a target value on the risk contribution implies finding a balance between the expected return contribution and the portfolio volatility contribution. This objective of finding a balance between a *marginal revenue*-type measure and a *marginal cost*-type measure is intuitive from a profit-maximizing perspective, as mentioned by Lee (2011). In fact, the maximum Sharpe ratio portfolio is such that the excess return contribution of each asset is proportional to the volatility contribution of that asset, with the value of the multiplier being equal to the Sharpe ratio of the portfolio.

In this paper, we introduce a flexible framework to evaluate and optimize the balance between components' performance and risk contributions, where the performance measure (de-

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noted by $\mathcal{P}_p$) and risk measure (denoted by $\mathcal{R}_p$) can be any measure, as long as they are first-order homogeneous functions of the portfolio weights, such that they can be decomposed into performance and risk contributions using Euler's theorem.†† For the evaluation of the balance between the performance and risk contributions, we propose the Performance/Risk Contribution Concentration (PRCC) metric. This measure is designed to be minimal when, for all portfolio components, the performance and risk contributions are perfectly aligned. We show its usefulness as an *ex-post* diagnostic tool to characterize the portfolio's bets in terms of performance contributions. More precisely, we define a bet when the ratio between the portfolio component's performance contribution and its risk contribution deviates from the ratio between the aggregate portfolio performance relative to the aggregate portfolio risk. The latter is denoted as $\tau_p \equiv \mathcal{P}_p/\mathcal{R}_p$ and henceforth used as a measure of the portfolio's relative performance.

The fully invested maximum relative performance portfolio has a zero PRCC value. This stands in contrast with the potential mismatch between the component performance and risk contributions of a risk-based portfolio, which, by the definition of Lee (2011), is a portfolio for which the weights are determined without making use of a return forecast. A typical example is the equally-weighted portfolio, for which Kritzman *et al*. (2010, p. 31) criticize the absence of optimization as follows: 'If we have at least some information on the expected returns, riskiness, and diversification properties of the assets, why should we not expect optimization to improve on a naively diversified portfolio?' On the other hand, Ardia and Boudt (2015) show that risk-based portfolios can coincide with the maximum Sharpe ratio portfolio under specific conditions on the expected return. For this reason, we recommend to adjust the weights of risk-based portfolios such that their PRCC value is closer to zero.

In addition to its interpretation as a diagnostic tool, we thus propose to use the PRCC to go beyond risk-based portfolios in order to adjust the risk-based portfolio weights, such that they achieve a better balance between the performance and risk contributions at the individual component level. The adjustment is limited because of an upper-bound constraint on the mean-squared distance between the PRCC-modified weights and the original risk-based portfolio weights. This bound constraint ensures that the optimized weights can still be interpreted in relation to the traditional risk-based portfolio. We further impose that the optimized portfolio needs to have the same relative performance as the risk-based portfolio such that the PRCC is well defined. The proposed framework of considering performance and risk allocation jointly is an alternative to the traditional mean–variance optimization of Markowitz (1952).

We illustrate this framework of building PRCC-modified risk-based portfolios in the real-life asset allocation problem of finding the optimal mix across investments in developed markets' equity, emerging markets' equity, US Government bond, corporate bonds, real estate and gold over the period 1988–2015. Our out-of-sample analysis shows that, when the reference portfolio is the equally-weighted, equal-risk-contribution and maximum diversification portfolio, the PRCC is relatively high, and optimizing the PRCC under the constraint of equal relative performance and a maximum tracking error in terms of the portfolio weights, leads to a substantial increase in both the portfolio's absolute and relative performance.

The remainder of the paper is organized as follows. In section 2, we define the PRCC measure. In section 3, we introduce the PRCC-modified risk-based portfolio. In section 4, we illustrate the use of the PRCC as a diagnostic and optimization criterion in a real-life asset allocation problem. Section 5 concludes.

2. Measuring the alignment of Component Performance/Risk Contributions

2.1. General framework

We consider a portfolio invested in N assets with weight vector $\mathbf{w} \equiv (w_1, \ldots, w_N)'$. We assume to have a measure for the performance of the portfolio, denoted by $\mathcal{P}_p(\mathbf{w})$, and a measure for the portfolio risk, denoted by $\mathcal{R}_p(\mathbf{w})$. As mentioned in Caporin *et al*. (2014), it is common to evaluate the portfolio's relative performance using ratios expressing the reward per unit of risk:

$$\tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w}) \equiv \frac{\mathcal{P}_p(\mathbf{w})}{\mathcal{R}_p(\mathbf{w})}. \tag{1}$$

Under the proposed framework, we require that the performance and risk measures have the property of being first-degree homogeneous functions of the portfolio weights. This means that if the portfolio weights are multiplied by a strictly positive scalar b, then the performance and risk measures are multiplied by b (i.e. $\mathcal{P}_p(b\mathbf{w}) = b\mathcal{P}_p(\mathbf{w})$ and $\mathcal{R}_p(b\mathbf{w}) = b\mathcal{R}_p(\mathbf{w})$ for $b > 0$). Examples include the excess portfolio return, and the portfolio volatility, VaR and ES under the assumption of elliptically symmetric return distributions and modified downside risk measures under the Cornish–Fisher expansion. From Euler's homogeneous function theorem, it follows that first-degree homogeneity is a useful property for performance and risk measures, as it implies the following aggregation results:

$$\mathcal{P}_p(\mathbf{w}) = \sum_{i=1}^{N} w_i \partial_i \mathcal{P}_p(\mathbf{w}) \quad \text{and} \quad \mathcal{R}_p(\mathbf{w}) = \sum_{i=1}^{N} w_i \partial_i \mathcal{R}_p(\mathbf{w}),$$

where we denote the partial derivative $\frac{\partial}{\partial w_i}$ by ∂_i. In the literature on performance and risk budgeting, the term:

$$C_i^{\mathcal{P}}(\mathbf{w}) \equiv w_i \partial_i \mathcal{P}_p(\mathbf{w}), \tag{2}$$

is called the *component contribution* to the portfolio performance, and:

$$C_i^{\mathcal{R}}(\mathbf{w}) \equiv w_i \partial_i \mathcal{R}_p(\mathbf{w}), \tag{3}$$

is the *component risk contribution* (see, e.g. Boudt *et al*. 2008).

From the definition of $\tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w})$ as the relative performance measure in (1), it follows that the aggregate balance between the portfolio performance and risk is: $\mathcal{P}_p(\mathbf{w}) = \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w})\mathcal{R}_p(\mathbf{w})$.

††This is clearly the case for the portfolio mean (excess) return, and for the portfolio volatility, when estimated using the classical sample-based estimator. In the web appendix (Ardia *et al*. 2017), we consider also four estimators for the portfolio Value-at-Risk (VaR) and Expected Shortfall (ES) that are first-degree homogeneous functions of the portfolio weights: the parametric approaches of assuming a Gaussian or a Student-*t* distribution, the semi-parametric approach based on the Cornish–Fisher approximation, and the non-parametric technique using kernel estimators.

But how is this balanced between the portfolio performance and risk distributed across the different positions? To answer this question, we need to investigate the balance in the performance and risk contribution at the component level. To do so, let us define the *Component Performance/Risk Contribution* of asset i as:

$$\text{CPRC}_i(\mathbf{w}) \equiv C_i^{\mathcal{P}}(\mathbf{w}) - \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w}) C_i^{\mathcal{R}}(\mathbf{w}).$$

Due to the first-degree homogeneity property and Euler's theorem, we have that the sum of all component performance/risk contributions is always zero:

$$\sum_{i=1}^{N} \text{CPRC}_i(\mathbf{w}) = 0. \quad (4)$$

As an aggregate measure of the dispersion in balance between the performance and risk contributions, we propose to use the following *Performance/Risk Contribution Concentration* measure for portfolios invested in multiple assets†:

$$\text{PRCC}(\mathbf{w}) \equiv \frac{1}{2N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left\{ \left[C_i^{\mathcal{P}}(\mathbf{w}) - \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w}) C_i^{\mathcal{R}}(\mathbf{w}) \right] - \left[C_j^{\mathcal{P}}(\mathbf{w}) - \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w}) C_j^{\mathcal{R}}(\mathbf{w}) \right] \right\}^2, \quad (5)$$

where we scale with $1/(2N^2)$ because of the property that the CPRCs add up to zero in (4), which implies that many terms in the PRCC($\mathbf{w}$) cancel out. In fact, as we show in Appendix 1, it is equivalent to define the PRCC as the average-squared value of the CPRC$_i$'s:

$$\text{PRCC}(\mathbf{w}) \equiv \frac{1}{N} \sum_{i=1}^{N} \left[C_i^{\mathcal{P}}(\mathbf{w}) - \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w}) C_i^{\mathcal{R}}(\mathbf{w}) \right]^2 = \frac{1}{N} \sum_{i=1}^{N} [\text{CPRC}_i(\mathbf{w})]^2. \quad (6)$$

In practice, we use the computationally simpler expression (6) to calculate the PRCC, but for ease of interpretation, we refer to (5) as the primary definition of the PRCC. Indeed, the most natural interpretation of the PRCC is that it measures the concentration in the mismatch between performance and risk contributions of financial portfolios. The higher the PRCC is, the more concentrated the portfolio is in terms of positions where the performance contribution diverges from the risk contribution scaled by the portfolio's relative performance ratio.

The lowest PRCC value is reached by the maximum relative performance portfolio, for which the PRCC is zero. In fact, as we show in Appendix 2, for the maximum relative performance portfolio, we have that the performance and risk contributions are optimally aligned in the sense of an equality between the performance contribution and the risk contribution, scaled by the portfolio's relative performance ratio:

$$C_i^{\mathcal{P}}(\mathbf{w}^*) = \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w}^*) C_i^{\mathcal{R}}(\mathbf{w}^*), \quad (7)$$

for all $i = 1, \ldots, N$ and where $\mathbf{w}^* \equiv \text{argmax}_{\mathbf{w} \in \mathcal{C}_{\text{FI}}} \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w})$, with $\mathcal{C}_{\text{FI}} \equiv \{\mathbf{w} \in \mathbb{R}^N \mid \mathbf{w}'\boldsymbol{\iota} = 1\}$ being the set of portfolio weights satisfying the full-investment constraint. Large values of the PRCC thus indicate active bets in terms of deviating performance/risk contributions from those of the maximum relative performance portfolio.

†In the trivial case of a portfolio fully invested in a single asset, there is of course no dispersion and the PRCC is zero. Throughout the paper, we assume the portfolios to be invested in at least two assets.

2.2. The PRCC of popular risk-based portfolios

In the remaining part of the paper, we use the mean excess return as the performance measure, and volatility as the risk measure. We denote $\boldsymbol{\mu} \equiv (\mu_1, \ldots, \mu_N)'$ as the vector of expected (arithmetic) returns, and $\tilde{\boldsymbol{\mu}} \equiv (\tilde{\mu}_1, \ldots, \tilde{\mu}_N)'$ as the vector of expected excess (arithmetic) returns over the risk-free rate. We further define the $N \times N$ covariance matrix of (arithmetic) returns by $\boldsymbol{\Sigma}$. Then the portfolio's expected excess return can be written as $\tilde{\mu}_p(\mathbf{w}) \equiv \mathbf{w}'\tilde{\boldsymbol{\mu}}$, and the portfolio volatility is given by $\sigma_p(\mathbf{w}) \equiv \sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}$. Both measures are first-degree homogeneous. The portfolio Sharpe ratio is $\tau_p^{\mu,\sigma}(\mathbf{w}) \equiv \tilde{\mu}_p(\mathbf{w})/\sigma_p(\mathbf{w})$. The performance contribution of asset i is given by $C_i^{\mu}(\mathbf{w}) \equiv w_i \tilde{\mu}_i$, while the component volatility contribution of asset i is given by:

$$C_i^{\sigma}(\mathbf{w}) \equiv w_i \partial_i \sigma_p(\mathbf{w}) = w_i \frac{[\boldsymbol{\Sigma}\mathbf{w}]_i}{\sigma_p(\mathbf{w})}.$$

The PRCC for the portfolio with a Sharpe ratio target $\tau_p^{\mu,\sigma}(\mathbf{w})$ is given by:

$$\text{PRCC}(\mathbf{w}) \equiv \frac{1}{N} \sum_{i=1}^{N} \left[C_i^{\mu}(\mathbf{w}) - \tau_p^{\mu,\sigma}(\mathbf{w}) C_i^{\sigma}(\mathbf{w}) \right]^2. \quad (8)$$

The general expression of the PRCC in (8) evaluates the mismatch between the performance and risk contributions of a fully invested portfolio with weights $\mathbf{w}$. In Appendix 3, we derive specific formulas of the volatility-based PRCC for the minimum variance portfolio, the inverse volatility portfolio, the equally-weighted portfolio, the equal-risk-contribution portfolio and the maximum diversification portfolio. The resulting expressions for the PRCC are presented in table 1.

The analysis shows that, for the minimum variance and equal-risk-contribution portfolios, the percentage risk contributions have no influence on the PRCC, the value of which is a function of the percentage return contributions. For the maximum diversification portfolio, the PRCC is a function of the variability of the spread between the assets' individual Sharpe ratios and the ratio between the weighted average return and the weighted average volatility, with weights corresponding to the portfolio weights.

3. Optimizing the performance/risk allocation of a risk-based portfolio

3.1. The PRCC-modified risk-based portfolio

Up to now, we have assumed that the portfolio's relative performance and PRCC are endogenously determined by the choice of portfolio weights. In this section, we do the reverse and consider the problem of finding the weights for which the portfolio PRCC is minimized under a target value constraint on the relative performance. While return and volatility targeting are popular in practice, there is little research on targeting relative performance. We believe that the use of the PRCC to

Table 1. Simplified representation for the volatility-based PRCC measure of risk-based portfolios.

Portfolio rule	$\text{PRCC}(\mathbf{w}^*)$
Minimum variance portfolio	
$\mathbf{w}^* \equiv \text{argmin}_{\mathbf{w}\in\mathcal{C}_{\text{FI}}} \{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}\}$	$\frac{1}{N}\cdot\sum_{i=1}^{N}\left\{w_i^*\left[\tilde{\mu}_i-\tilde{\mu}_p(\mathbf{w}^*)\right]\right\}^2$
Inverse volatility portfolio	
$\mathbf{w}^* \equiv \boldsymbol{\xi}/\boldsymbol{\xi}'\boldsymbol{\iota}$	$\frac{1}{N}\cdot\frac{1}{(\boldsymbol{\xi}'\boldsymbol{\iota})^2}\cdot\sum_{i=1}^{N}\left[\frac{\tilde{\mu}_i}{\sigma_i}-\tau_p^{\mu,\sigma}(\mathbf{w}^*)\frac{[\mathbf{R}\boldsymbol{\iota}]_i}{\sqrt{\boldsymbol{\iota}'\mathbf{R}\boldsymbol{\iota}}}\right]^2$
Equally-weighted portfolio	
$\mathbf{w}^* \equiv \boldsymbol{\iota}/N$	$\frac{1}{N^3}\cdot\sum_{i=1}^{N}\left[\tilde{\mu}_i-\tau_p^{\mu,\sigma}(\mathbf{w}^*)\frac{[\boldsymbol{\Sigma}\boldsymbol{\iota}]_i}{\sqrt{\boldsymbol{\iota}'\boldsymbol{\Sigma}\boldsymbol{\iota}}}\right]^2$
Equal-risk-contribution portfolio	
$\mathbf{w}^* \equiv \text{argmin}_{\mathbf{w}\in\mathcal{C}_{\text{FI}}}\left\{\sum_{i=1}^{N}\sum_{j=1}^{N}\left[C_i^{\sigma}(\mathbf{w})-C_j^{\sigma}(\mathbf{w})\right]^2\right\}$	$\frac{1}{N}\cdot\sum_{i=1}^{N}\left[w_i^*\tilde{\mu}_i-\frac{\tilde{\boldsymbol{\mu}}'\mathbf{w}^*}{N}\right]^2$
Maximum diversification portfolio	
$\mathbf{w}^* \equiv \text{argmax}_{\mathbf{w}\in\mathcal{C}_{\text{FI}}}\left\{\frac{\mathbf{w}'\boldsymbol{\sigma}}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}}\right\}$	$\frac{1}{N}\cdot\sum_{i=1}^{N}\left[w_i^*\sigma_i\left(\frac{\tilde{\mu}_i}{\sigma_i}-\frac{\tilde{\boldsymbol{\mu}}'\mathbf{w}^*}{\boldsymbol{\sigma}'\mathbf{w}^*}\right)\right]^2$

Notes: This table presents expressions of the PRCC for five widely used risk-based portfolios. Details of the formulas can be found in Appendix 3. We use $\boldsymbol{\sigma} \equiv (\sigma_1,\ldots,\sigma_N)'$, $\boldsymbol{\xi} \equiv (1/\sigma_1,\ldots,1/\sigma_N)'$, $\boldsymbol{\iota}$ is an $N\times 1$ vector of ones, $\boldsymbol{\Sigma}$ and $\mathbf{R}$ are the $N\times N$ covariance and correlation matrices, respectively.

improve the allocation of risk-based portfolios is a relevant use case for optimizing portfolios under a target level constraint on the relative performance.

Denote the risk-based reference portfolio weights and its relative performance by $\mathbf{w}^\star$ and $\tau_p^\star$, respectively. In most cases, the PRCC computed for $\mathbf{w}^\star$ is strictly positive, indicating that some positions have a too-large contribution to risk, compared to their contribution to performance. The proposed PRCC-modified risk-based portfolio aims then at improving the risk-based weights $\mathbf{w}^\star$ by tilting them in the direction for which the performance-per-unit-of-risk contributions (i.e. $C_i^{\mathcal{P}}(\mathbf{w})/C_i^{\mathcal{R}}(\mathbf{w})$, when $w_i > 0$) are similar to the portfolio's aggregate relative performance $\tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w})$. By doing so, the weights violate less the first-order condition of the maximum relative performance portfolio. In order to preserve the interpretation of the risk-based portfolio weights and limit the impact of potential estimation error in the expected returns, we restrict the weight modifications in two ways. First, we require that the PRCC modification does not alter the relative performance of the portfolio compared with the risk-based benchmark. Second, we require the portfolios to be long-only, fully invested, and we impose an upper-bound constraint on the mean-squared value of the weight differences induced by the optimization, and refer to this as the tracking error constraint. Hence forth, as in Brandt *et al*. (2009), we use the mean-squared deviation of the weights with respect to their reference weight as the definition of the tracking error.

Altogether, this then leads to the following optimization problem, taking the risk-based portfolio weights $\mathbf{w}^\star$ as input, and transforming them into PRCC-modified risk-based portfolio weights:

$$\begin{aligned}
\underset{\mathbf{w}}{\text{minimize}} \quad & \text{PRCC}(\mathbf{w}) \\
\text{subject to} \quad & \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w}) = \tau_p^\star \\
& \frac{1}{N}\sum_{i=1}^{N}(w_i - w_i^\star)^2 \leq \zeta^2 \\
& \mathbf{w}'\boldsymbol{\iota} = 1\,,\, 0 \leq w_i < 1 \quad \forall i.
\end{aligned} \tag{9}$$

In the empirical application, we set ζ at 10%.

From a computational viewpoint, the derivation of the PRCC-modified risk-based portfolio weights is a non-linear optimization problem. When the PRCC has smooth first and second-order derivatives (with respect to $\mathbf{w}$), this problem can be easily solved using sequential quadratic programming. Note that there always exists a solution to (9), namely $\mathbf{w}^\star$, which we use as the starting value in the sequential quadratic programming.

3.2. *Effect of estimation error*

By design, the PRCC optimization in (9) leads to portfolios for which there is a higher similarity in the estimated performance-per-unit-of-risk contributions of the portfolio investments, as compared to the risk-based portfolios. The caveat is that estimation error may distort the weights compared to the theoretical solution when there is no estimation error. This is especially a concern for the estimation error in mean excess returns, which tends to be substantially higher than the estimation error in the covariance (Merton 1980). The goal of this section is to document that, as shown by Best and Grauer (1991), in case of long-only fully invested portfolios, the risk-adjusted performance of the PRCC-modified portfolio is robust to the effects of estimation error in the mean returns.

In our discussion, we distinguish between the impact on the portfolio weights and the portfolio performance. In fact, as shown in Best and Grauer (1991), we can expect that the effect on the portfolio performance is of a smaller magnitude than the effect on the portfolio weights. According to them, the reason is that 'there is so much action in the weights that, with nonnegativity constraints imposed on the problem, assets are driven out of the portfolio before large changes in the portfolio mean and variance can occur.' In case of a portfolio with a low PRCC value this result is corroborated by the property that, for the PRCC to be close to zero, it is required that changes in the weight have only a minor effect on the performance-risk allocation at the component level. Indeed, by minimizing

Table 2. Sensitivity of PRCC-modified portfolios to estimation error in mean (excess) returns.

Impact on weights								Impact on portfolio performance							
k	0.50	0.75	1.00	1.25	1.50	1.75	2.00	k	0.50	0.75	1.00	1.25	1.50	1.75	2.00
Panel A: Maximum Sharpe ratio portfolio															
$\overline{\lvert\hat{\mathbf{w}}-\mathbf{w}^\star\rvert}$	6.93	5.18	0	6.04	10.77	13.76	14.68	$\mu_p(\hat{\mathbf{w}})/\mu_p(\mathbf{w}^\star)$	1.05	1.04	1	0.97	0.95	0.92	0.90
$\#\{\hat{w}_i < 0.1\%\}$	2	2	0	1	2	3	5	$\sigma_p(\hat{\mathbf{w}})/\sigma_p(\mathbf{w}^\star)$	1.10	1.06	1	1.03	1.08	1.16	1.26
$\max\{\hat{w}_i\}$	40.07	40.07	33.26	53.53	68.95	83.33	88.11	$\tau_p(\hat{\mathbf{w}})/\tau_p(\mathbf{w}^\star)$	0.96	0.97	1	0.97	0.91	0.86	0.83
$H(\hat{\mathbf{w}})$	0.15	0.15	0.10	0.24	0.44	0.67	0.76	PRCC($\hat{\mathbf{w}}$)	<0.01	<0.01	0	0.05	0.19	0.37	0.66
Panel B: PRCC-modified minimum variance portfolio															
$\overline{\lvert\hat{\mathbf{w}}-\mathbf{w}^\star\rvert}$	7.92	6.61	0	5.26	7.40	7.77	5.44	$\mu_p(\hat{\mathbf{w}})/\mu_p(\mathbf{w}^\star)$	0.94	0.97	1	1.07	0.91	0.87	1.07
$\#\{\hat{w}_i < 0.1\%\}$	6	5	4	5	4	7	4	$\sigma_p(\hat{\mathbf{w}})/\sigma_p(\mathbf{w}^\star)$	1.10	1.03	1	1.03	1.02	1.03	0.97
$\max\{\hat{w}_i\}$	53.48	45.56	34.46	53.08	71.10	73.32	47.84	$\tau_p(\hat{\mathbf{w}})/\tau_p(\mathbf{w}^\star)$	0.90	0.95	1	1.04	0.90	0.85	1.05
$H(\hat{\mathbf{w}})$	0.30	0.25	0.13	0.27	0.48	0.53	0.22	PRCC($\hat{\mathbf{w}}$)	0.05	0.01	<0.01	0.05	0.10	0.05	0.05
Panel C: PRCC-modified inverse volatility portfolio															
$\overline{\lvert\hat{\mathbf{w}}-\mathbf{w}^\star\rvert}$	8.38	5.58	0	8.24	7.64	7.39	7.25	$\mu_p(\hat{\mathbf{w}})/\mu_p(\mathbf{w}^\star)$	0.94	1.03	1	0.93	0.95	0.96	0.96
$\#\{\hat{w}_i < 0.1\%\}$	4	3	1	4	1	1	1	$\sigma_p(\hat{\mathbf{w}})/\sigma_p(\mathbf{w}^\star)$	1.08	1.05	1	0.91	1.11	0.92	0.92
$\max\{\hat{w}_i\}$	41.89	43.46	33.24	39.86	36.58	35.16	34.45	$\tau_p(\hat{\mathbf{w}})/\tau_p(\mathbf{w}^\star)$	0.95	0.98	1	1.03	0.94	1.04	1.04
$H(\hat{\mathbf{w}})$	0.17	0.15	0.12	0.15	0.13	0.11	0.11	PRCC($\hat{\mathbf{w}}$)	0.01	0.01	<0.01	0.03	0.13	0.01	0.01
Panel D: PRCC-modified equally-weighted portfolio															
$\overline{\lvert\hat{\mathbf{w}}-\mathbf{w}^\star\rvert}$	5.17	7.27	0	5.73	7.21	8.80	10.26	$\mu_p(\hat{\mathbf{w}})/\mu_p(\mathbf{w}^\star)$	1.06	0.93	1	0.93	0.89	0.81	0.83
$\#\{\hat{w}_i < 0.1\%\}$	1	2	1	2	3	4	3	$\sigma_p(\hat{\mathbf{w}})/\sigma_p(\mathbf{w}^\star)$	1.13	0.89	1	0.91	0.84	0.75	0.74
$\max\{\hat{w}_i\}$	27.57	36.85	20.00	36.48	36.35	37.16	37.30	$\tau_p(\hat{\mathbf{w}})/\tau_p(\mathbf{w}^\star)$	0.93	1.05	1	0.97	0.94	1.10	1.13
$H(\hat{\mathbf{w}})$	0.09	0.11	0.04	0.11	0.11	0.11	0.11	PRCC($\hat{\mathbf{w}}$)	0.09	0.08	0.05	0.09	0.23	0.02	0.02
Panel E: PRCC-modified equal-risk-contribution portfolio															
$\overline{\lvert\hat{\mathbf{w}}-\mathbf{w}^\star\rvert}$	7.87	5.82	0	9.10	7.64	7.35	7.19	$\mu_p(\hat{\mathbf{w}})/\mu_p(\mathbf{w}^\star)$	0.95	1.03	1	0.93	0.95	0.96	0.96
$\#\{\hat{w}_i < 0.1\%\}$	3	3	1	3	2	1	1	$\sigma_p(\hat{\mathbf{w}})/\sigma_p(\mathbf{w}^\star)$	1.08	1.05	1	0.91	1.09	0.92	0.92
$\max\{\hat{w}_i\}$	40.21	41.59	33.18	39.42	37.15	35.73	35.00	$\tau_p(\hat{\mathbf{w}})/\tau_p(\mathbf{w}^\star)$	0.95	0.98	1	1.03	0.94	1.04	1.04
$H(\hat{\mathbf{w}})$	0.15	0.14	0.12	0.15	0.13	0.11	0.11	PRCC($\hat{\mathbf{w}}$)	0.02	0.01	<0.01	0.03	0.12	0.01	0.01
Panel F: PRCC-modified maximum diversification portfolio															
$\overline{\lvert\hat{\mathbf{w}}-\mathbf{w}^\star\rvert}$	10.56	8.26	0	8.32	8.59	8.63	8.34	$\mu_p(\hat{\mathbf{w}})/\mu_p(\mathbf{w}^\star)$	1.17	1.06	1	1.06	1.04	1.16	1.03
$\#\{\hat{w}_i < 0.1\%\}$	6	5	3	4	3	3	4	$\sigma_p(\hat{\mathbf{w}})/\sigma_p(\mathbf{w}^\star)$	1.22	1.08	1	0.97	0.96	1.26	0.97
$\max\{\hat{w}_i\}$	45.55	54.61	42.6	43.42	44.13	43.27	43.81	$\tau_p(\hat{\mathbf{w}})/\tau_p(\mathbf{w}^\star)$	0.95	0.98	1	1.04	1.05	0.92	1.05
$H(\hat{\mathbf{w}})$	0.22	0.27	0.20	0.20	0.20	0.20	0.22	PRCC($\hat{\mathbf{w}}$)	0.09	0.04	<0.01	0.02	0.01	0.35	0.01

Notes: This table investigates the sensitivity of the weights and aggregate performance of PRCC-modified portfolios to estimation error in mean (excess) returns. The true mean and covariance of the 10 mean returns are as in table 1 of Best and Grauer (1991, p. 331). We assume no estimation error, except for the mean (excess) return of the ith asset for which $\hat{\mu}_i = k\mu_i$, with $k \in \{0.50, 0.75, 1.00, 1.25, 1.50, 1.75, 2.00\}$. For each portfolio and value of k, we report the largest impact across the $i = 1, \ldots, 10$ perturbations. As in Best and Grauer (1991), we distinguish between the large impact on the portfolio weights (the left-hand), and the relatively smaller impact on portfolio performance (the right-hand). The impact on weights is characterized by the mean absolute deviation ($\overline{\lvert\hat{\mathbf{w}}-\mathbf{w}^\star\rvert}$), the number of weights smaller than 0.1% ($\#\{\hat{w}_i < 0.1\%\}$), the maximum weights ($\max\{\hat{w}_i\}$), and the normalized Herfindahl index ($H(\hat{\mathbf{w}})$). The impact on the portfolio performance is represented in ratio, including the portfolio returns ($\mu_p(\hat{\mathbf{w}})/\mu_p(\mathbf{w}^\star)$), volatility ($\sigma_p(\hat{\mathbf{w}})/\sigma_p(\mathbf{w}^\star)$), Sharpe ratio ($\tau_p(\hat{\mathbf{w}})/\tau_p(\mathbf{w}^\star)$), and PRCC (PRCC($\hat{\mathbf{w}}$)). See Section 3.2 for computational details.

the PRCC objective under the target relative performance constraint that $\tau_p(\mathbf{w}) = \tau_p^\star$, the resulting portfolio weight vector should be such that, for each position i with $w_i > 0$, the estimated performance-per-unit-of-risk contribution is close to the target relative performance, and thus:

$$\frac{\partial \mathcal{P}_p(\mathbf{w})}{\partial w_i} - \tau_p^\star \frac{\partial \mathcal{R}_p(\mathbf{w})}{\partial w_i} \approx 0. \qquad (10)$$

It follows from this property that the portfolio performance has reached a steady state in which small increments to the portfolio weight have only little effect on the portfolio relative performance. In particular, assume that the optimal portfolio weight in absence of estimation error is close to the estimated weight, then their relative performance can also be expected to be similar.

Based on the above arguments, we thus expect that, also in the presence of estimation error, the PRCC modification of the risk-based portfolio weights is still useful. Due to the various constraints, analytical results on the sensitivity of the PRCC-modified portfolios to estimation errors in the asset means are not readily available. We therefore follow Best and Grauer (1991) and use computational results to describe in table 2 the impact of estimation error on the portfolio weights and performance. We take the same set-up as they do, namely 10 assets with mean and covariance matrix as reported in table 1 of Best and Grauer (1991, p. 331).

In the reference case, we assume that there is no estimation error and thus set $\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}$ and $\hat{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma}$. We denote the corresponding optimized weight as $\mathbf{w}^\star$. We then disturb the mean estimates for each of the 10 assets separately, by considering the 6×10 scenarios of setting $\hat{\mu}_i = k\mu_i$, with $k \in \{0.5, 0.75, 1.25, 1.5, 1.75, 2\}$ for $i = 1, \ldots, 10$. For each scenario, we compute the optimized portfolio weights $\hat{\mathbf{w}}$. We summarize the effect of estimation error on the portfolio weights using the mean absolute change in portfolio weights ($\overline{\lvert\hat{\mathbf{w}}-\mathbf{w}^\star\rvert}$), the number of near zero weights ($\#\{\hat{w}_i < 0.1\%\}$), the maximum weight ($\max\{\hat{w}_i\}$), and the normalized Herfindahl index, computed as: $H(\hat{\mathbf{w}}) \equiv \frac{H'(\hat{\mathbf{w}})-1/N}{1-1/N}$, where $H'(\hat{\mathbf{w}}) \equiv \sum_{i=1}^{N} \hat{w}_i^2$.

To evaluate the impact on the portfolio performance, we use the ratio between the portfolio mean, standard deviation and Sharpe ratio of the optimized portfolio, as compared to the portfolio performance of the optimal portfolio, denoted as $\mu_p(\hat{\mathbf{w}})/\mu_p(\mathbf{w}^\star)$, $\sigma_p(\hat{\mathbf{w}})/\sigma_p(\mathbf{w}^\star)$ and $\tau_p(\hat{\mathbf{w}})/\tau_p(\mathbf{w}^\star)$, respectively. We compute all performance measures using the estimation error-free values of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ such that the obtained values summarize the true performance impact of the errors in the portfolio weights due to $\hat{\mu}_i = k\mu_i$. To save space, we summarize the 60 scenarios by reporting, for each value of k, the worst-case value of the metric across the perturbations for the 10 assets. Results are reported in table 2.

Let us first consider the impact on the portfolio weights in the left panel. We find that the average absolute error in portfolio weights is the largest for the maximum Sharpe ratio portfolio in the case of overestimation of the mean returns. Its value increases from 6.04% for $k = 1.25$ to 14.68% for $k = 2$. As noted by Best and Grauer (1991) in the case of mean–variance efficient portfolios, we find that the overestimation of the mean return of one asset can drive the weight of other assets to zero. This effect occurs also in the PRCC optimization, since assets with a zero weight have a zero contribution to the portfolio PRCC and are thus optimal in that respect. It is thus expected that for the number of near-zero positions is the highest in case of the PRCC modification of the minimum variance portfolio, since in the absence of estimation error ($k = 1$), the portfolio is already invested in only 6 out of 10 assets. In terms of maximum weight and the normalized Herfindahl index, the maximum Sharpe ratio is the most sensitive to estimation error of all portfolios considered. For $k = 2$, the maximum Sharpe ratio portfolio becomes heavily concentrated with a maximum weight of 88.11% and a normalized Herfindahl index of 0.76. The PRCC-modified portfolios aiming at balancing the performance and risk contributions have at most a normalized Herfindahl index of 0.53.

Despite the large impact on the portfolio weights of the optimized portfolios, we find in the right panel of table 2 that the overall portfolio performance is rather robust to the estimation error in the mean excess return. This is especially clear from the relative performance. For the maximum Sharpe ratio portfolio, the Sharpe ratio of the optimized portfolio is in the worst case 17% less than the Sharpe ratio of the optimal portfolio without estimation error. For the PRCC optimized portfolios, the percentage difference is at most 15%. In other words, suppose that the portfolio Sharpe ratio is 1, then, in the worst case of the 60 scenarios considered, the portfolio Sharpe ratio would be 0.85.

The main result of the numeric sensitivity analysis is thus that, for the PRCC-modified portfolios, the estimation error in the asset mean can have a rather large effect on the portfolio weights, but that the impact on portfolio risk-adjusted performance is limited.

4. Illustration in asset allocation

Risk-based portfolios are increasingly used in the construction of equity portfolios and in asset allocation. For sake of clarity in our presentation, we choose to illustrate the use of the PRCC in asset allocation because of the typically lower dimension of a realistic asset allocation portfolio compared with a realistic equity optimization problem. Our goal is to determine the weights of the portfolio invested in six asset classes: (i) developed markets equity, (ii) emerging markets equity, (iii) US Government bond, (iv) US investment grade corporate bond, (v) real estate and (vi) gold.

We start the illustration by introducing the monthly return data used for the period 1988–2015. We then use the PRCC to analyse and modify five risk-based portfolios: (1) the minimum variance portfolio, (2) the inverse volatility portfolio, (3) the equally-weighted portfolio, (4) the equal-risk-contribution portfolio, and (5) the maximum diversification portfolio. We first present our in-sample results and conclude with an extensive out-of-sample performance evaluation using rolling estimation windows of three years and monthly portfolio rebalancing. Throughout the analysis, we estimate the mean and covariance matrix of the asset returns using the standard sample mean and covariance of the monthly arithmetic (excess) returns of the six assets.† The PRCC-modified portfolios are implemented with volatility as risk measure and $\zeta = 10\%$.

4.1. Data

The sample ranges from January 1988 to August 2015. We use the end-of-month values on the total return index of the MSCI World index, the MSCI Emerging Markets index, Bloomberg US Government bond (1–10 year) index, BofA Merrill Lynch US Corp Master Total return index, All REITS Total index, and Gold Fixing price 3 p.m (London time) in the London Bullion Market. All returns computed are arithmetic returns, based on the USD value of the indices. We take the US one-month Treasury bill rate from the database of Kenneth French as the risk-free asset.‡

The summary statistics on the buy-and-hold investments in each of six assets are reported in table 3. Among the assets considered, the MSCI Emerging Markets index has the highest annualized excess return (9.77%), followed by the NAREIT index (7.55%). On the risk side, the Bloomberg US Government Bond index has the lowest level of annualized volatility (3.16%) and drawdown (3.5%). Over the period, the two bond indices have the most attractive annualized Sharpe ratio (0.78). The investment in the gold index is the least attractive in terms of average return and Sharpe ratio performance. In some cases, it may be valuable to include an investment in gold to the

†The use of rolling estimation windows reflects industry practice when the frequency of rebalancing is monthly. Alternatively, more complex estimators could be considered that use higher frequency data (see, e.g. Boudt and Zhang (2015)) or by considering a parametric approach to modelling the time-variation in the return series (see, e.g. Boudt *et al*. (2012)).

‡The MSCI World index tracks the performance of large and mid-cap equities over 23 developed market countries: https://www.msci.com/market-cap-weighted-indexes. The data of All REITS Total index is retrieved from: https://www.reit.com/investing/index-data/monthly-index-values-returns. For the US Corp Master Total return index and gold spot, the data is collected from the Federal Reserve Bank of St. Louis, while the risk-free rate data used is the one from the K. French data website, available at: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

Table 3. Descriptive statistics of asset returns.

Panel A: Buy-and-hold strategy									
Asset	$	GR	Mean	Sd	SR	Sk	Ku	MDD	mVaR
Eq-DE	7.53	7.57	5.20	14.94	0.35	−0.61	1.37	53.65	6.97
Eq-EM	16.78	10.73	9.77	23.30	0.42	−0.59	1.64	61.44	10.83
Bo-Go	4.75	5.80	2.45	3.16	0.78	−0.01	0.23	3.46	1.02
Bo-Co	7.21	7.40	4.05	5.19	0.78	−0.80	4.30	16.07	2.05
NAREIT	12.71	9.62	7.55	17.48	0.43	−0.84	8.00	67.89	7.71
Gold	2.34	3.13	1.06	15.69	0.07	0.13	1.22	47.37	6.79

Panel B: Correlation between assets					
Asset	Eq-DE	Eq-EM	Bo-Go	Bo-Co	NAREIT
Eq-EM	0.73***				
Bo-Go	−0.09	−0.15***			
Bo-Co	0.29***	0.22***	0.68***		
NAREIT	0.54***	0.45***	0.01	0.36***	
Gold	0.05	0.16***	0.10*	0.15***	0.06

Notes: This table presents the summary statistics of the monthly returns for the six assets in our universe. In Panel A, we report the cumulative terminal value of a $1 investment ($), the annualized geometric returns (GR, in per cent), the annualized excess returns (Mean, in per cent), annualized standard deviation (Sd, in per cent), annualized Sharpe ratio (SR), skewness (Sk), kurtosis (Ku), the maximum drawdown (MDD, in per cent), and the 5% modified Value-at-Risk (mVaR, in per cent). Panel B reports the correlation between the monthly asset returns. The signs ***,** and * indicate whether the Pearson correlation coefficient is significantly different from zero at the 1, 5 and 10% levels, respectively. The six asset classes considered are the MSCI World index-developed countries (Eq-DE), the MSCI Emerging markets index (Eq-EM), the US Government bond index (Bo-GO), the US corporate bond master index (Bo-CO), NAREIT, and the Gold spot index (Gold). The sample period ranges from January 1988 to August 2015 for a total of 332 monthly observations.

portfolio, because, as can be seen in Panel B of table 3, it is a good diversifier. Its correlation with the five other asset classes is below 0.2. The highest correlation is observed between the returns of the two equity indices (0.73). Finally, note that the US Government bond has negative correlations with the MSCI World index (−0.09) and the MSCI Emerging market index (−0.15) over our period, which is marked by the financial crisis and possible flights to safety.

4.2. In-sample performance and PRCC of risk-based portfolios

Given the large heterogeneity in performance of the various asset classes, it is now relevant to study how the choice of risk-based portfolio allocation affects the in-sample portfolio performance, as reported in the left part of table 4. In terms of in-sample annualized returns, we see that the equally-weighted portfolio has the highest return (around 5%), while the minimum variance portfolio offers only an average return of 2.66%. The equal-risk-contribution portfolio is the third best in terms of annualized returns (3.58%). Its volatility is 4.55%, which is in between the 3.01% volatility of the minimum variance portfolio and the 8.92% volatility of the equally-weighted portfolio. The Sharpe ratio of the maximum Sharpe ratio portfolio is twice that of the equally-weighted portfolio. In our sample, maximizing the Sharpe ratio and minimizing the variance lead to similar portfolios with a high allocation to bonds. The maximum Sharpe ratio portfolio is invested in both US Government bonds (85%) and investment grade corporate bonds (5%), while the minimum variance portfolio only invests in the government bonds (93%).

The results on portfolio performance are as expected. The main novelty in the left part of table 4 is the PRCC values, for which high values indicate concentrated bets in terms of volatility and expected return contributions that are not aligned with the portfolio's Sharpe ratio. We see that the minimum variance portfolio is close to optimal in terms of a low PRCC value, while the inverse volatility, equally-weighted, equal-risk-contribution and maximum diversification portfolios have a monthly PRCC value that is higher than 0.00075. In annualized terms, this corresponds to a PRCC value of 0.11, as obtained by multiplying the monthly PRCC value with 144 (i.e. 12^2). Table 4 shows the annualized performance and risk contributions that lead to the annualized value of the PRCC. Interestingly, we see that for the four mentioned portfolios with a large PRCC value, this is caused by the investments in the equity, NAREIT, and Gold asset classes, which cause too much volatility compared with their return contribution, while the position in bonds contributes relatively more to return than it does to risk.

In the right part of table 4, we investigate how the PRCC modification to risk-based weights changes the portfolio performance and weights. Remember that an important constraint in the PRCC modification is that the portfolio needs to have the same Sharpe ratio, as can be seen in table 4. It follows that the PRCC modification comes either at the price of a higher volatility (which is the case for the minimum variance, the equally-weighted, and the equal-risk-contribution portfolio) or a lower return (which is the case for the inverse volatility and maximum diversification portfolio). Another constraint is the 10% upper bound on the tracking error in terms of weights compared to the benchmark portfolio. We see in the 'TE' values that this constraint is binding for the inverse volatility weighted, the equal-risk-contribution and the max diversification portfolios.

Finally, it is of interest to see in table 4 that the direction of the weight changes due to the PRCC modification depends on the risk-based portfolio considered. For the equally-weighted portfolio, we see, for instance, that a better equilibrium between the performance and risk contributions is obtained by overweight-

Table 4. In-sample estimation results.

	Traditional benchmark portfolios						PRCC-modified benchmark portfolios					
	Eq-DE	Eq-EM	Bo-GO	Bo-CO	NAREIT	Gold	Eq-DE	Eq-EM	Bo-GO	Bo-CO	NAREIT	Gold
Panel A: Minimum variance portfolio												
	$\mu_p = 2.66, \sigma_p = 3.01, \tau_p^{\mu,\sigma} = 0.88$, PRCC = 0.01						$\mu_p = 2.75, \sigma_p = 3.11, \tau_p^{\mu,\sigma} = 0.88$, PRCC < 0.01, TE = 3.10					
$\mathbf{w}^*$	3.77	1.66	93.25	0	0.07	1.25	9.63	0.08	89.01	1.14	0.14	0
$\mathcal{C}_i^\mu$	0.20	0.16	2.28	0	<0.01	0.01	0.50	0.01	2.18	0.05	0.01	0
$\mathcal{C}_i^\sigma$	0.11	0.05	2.81	0	<0.01	0.04	0.57	<0.01	2.47	0.05	<0.01	0
CPRC_i	0.10	0.12	−0.20	0	<0.01	−0.02	−0.01	<0.01	−0.01	<0.01	<0.01	0
Panel B: Inverse volatility weighted portfolio												
	$\mu_p = 3.82, \sigma_p = 4.89, \tau_p^{\mu,\sigma} = 0.78$, PRCC = 0.11						$\mu_p = 3.77, \sigma_p = 4.83, \tau_p^{\mu,\sigma} = 0.78$, PRCC = 0.05, TE = 10.00					
$\mathbf{w}^*$	9.05	5.80	42.75	26.03	7.74	8.62	9.49	0	28.51	45.06	8.80	8.14
$\mathcal{C}_i^\mu$	0.47	0.57	1.05	1.05	0.58	0.09	0.49	0	0.70	1.83	0.66	0.09
$\mathcal{C}_i^\sigma$	0.94	0.90	0.58	1.01	0.90	0.57	0.86	0	0.47	1.99	1.03	0.49
CPRC_i	−0.26	−0.13	0.60	0.27	−0.12	−0.35	−0.17	0	0.33	0.27	−0.13	−0.30
Panel C: Equally-weighted portfolio												
	$\mu_p = 5.01, \sigma_p = 8.92, \tau_p^{\mu,\sigma} = 0.56$, PRCC = 0.11						$\mu_p = 5.86, \sigma_p = 10.43, \tau_p^{\mu,\sigma} = 0.56$, PRCC = 0.06, TE = 7.59					
$\mathbf{w}^*$	16.67	16.67	16.67	16.67	16.67	16.67	9.86	19.89	14.81	12.67	31.99	10.77
$\mathcal{C}_i^\mu$	0.87	1.63	0.41	0.68	1.26	0.18	0.51	1.94	0.36	0.51	2.42	0.11
$\mathcal{C}_i^\sigma$	2.01	3.27	0.04	0.41	2.11	1.09	1.14	3.79	0.01	0.30	4.71	0.48
CPRC_i	−0.27	−0.21	0.39	0.44	0.08	−0.43	−0.13	−0.19	0.36	0.35	−0.23	−0.16
Panel D: Equal-risk-contribution portfolio												
	$\mu_p = 3.58, \sigma_p = 4.55, \tau_p^{\mu,\sigma} = 0.79$, PRCC = 0.12						$\mu_p = 3.65, \sigma_p = 4.65, \tau_p^{\mu,\sigma} = 0.79$, PRCC = 0.06, TE = 10.00					
$\mathbf{w}^*$	7.84	5.22	50.36	19.65	6.93	10.01	9.39	0	35.37	38.07	8.84	8.34
$\mathcal{C}_i^\mu$	0.41	0.51	1.23	0.80	0.52	0.11	0.49	0	0.87	1.54	0.67	0.09
$\mathcal{C}_i^\sigma$	0.76	0.76	0.76	0.76	0.76	0.76	0.84	0	0.60	1.66	1.03	0.53
CPRC_i	−0.19	−0.09	0.64	0.20	−0.07	−0.49	−0.17	0	0.40	0.24	−0.14	−0.33
Panel E: Maximum diversification portfolio												
	$\mu_p = 3.14, \sigma_p = 3.96, \tau_p^{\mu,\sigma} = 0.79$, PRCC = 0.16						$\mu_p = 2.51, \sigma_p = 3.16, \tau_p^{\mu,\sigma} = 0.79$, PRCC < 0.01, TE = 10.00					
$\mathbf{w}^*$	6.01	4.89	72.05	0	6.28	10.77	0.23	0.04	92.38	4.77	0.15	2.43
$\mathcal{C}_i^\mu$	0.31	0.48	1.76	0	0.47	0.11	0.01	<0.01	2.26	0.19	0.01	0.03
$\mathcal{C}_i^\sigma$	0.50	0.63	1.27	0	0.61	0.94	<0.01	<0.01	2.89	0.18	<0.01	0.09
CPRC_i	−0.08	−0.03	0.76	0	−0.01	−0.63	0.01	<0.01	−0.03	0.05	0.01	−0.04

Notes: This table presents the results of the in-sample analysis of annualized PRCC for traditional benchmarks (left-part) and the PRCC-modified counterparts (right-part). For each strategy, the table reports the optimized weight vector $\mathbf{w}^*$, the contributions to the annualized excess portfolio return ($\mathcal{C}_i^\mu \equiv 12 w_i^* \tilde{\mu}_i$), the annualized portfolio standard deviation ($\mathcal{C}_i^\sigma \equiv \sqrt{12} w_i [\mathbf{w}^* \mathbf{\Sigma}]_i / \sigma_p(\mathbf{w}^*)$), and the component performance/risk contribution ($\mathrm{CPRC}_i \equiv \mathcal{C}_i^\mu - (\sum_{i=1}^N \mathcal{C}_i^\mu / \sum_{i=1}^N \mathcal{C}_i^\sigma) \mathcal{C}_i^\sigma$). All reported numbers are expressed in percentage points. The six asset classes considered are the MSCI World index-developed countries (Eq-DE), the MSCI Emerging markets index (Eq-EM), the US Government bond index (Bo-GO), the US corporate bond master index (Bo-CO), NAREIT, and the Gold spot index (Gold). The in-sample period ranges from January 1988 to August 2015, for a total of 332 monthly observations.

ing equities and real estate, while for the maximum diversification portfolio, the weight to bonds is substantially increased. In case of the equal-risk-contribution portfolio, the PRCC-modified portfolio invests approximately the same weights in bonds, but instead of concentrating 50% of the weight in the government bond and only 20% in the corporate bonds, the PRCC-modified equal-risk-contribution portfolio invests around 35% in both of them.

4.3. *Out-of-sample gains from optimizing the PRCC*

To assess the out-of-sample performance from optimizing the PRCC, we implement an investment strategy that rebalances the portfolios at the end of the month. At each rebalancing date, all parameters needed for the calculation of the PRCC and the portfolio optimization are estimated using the 36 most recently observed monthly returns (i.e. on a rolling-window basis). The relative performance targets are set equal to those of the risk-based portfolio rules. In terms of performance measures, we report, for all strategies: (i) the cumulative value, (ii) the annualized geometric return, (iii) the annualized excess return, (iv) the annualized volatility, (v) the Sharpe ratio, (vi) the portfolio skewness, (vii) the portfolio kurtosis, (viii) the maximum drawdown, (ix) the 5% modified VaR, and (x) the average of the tracking error of the PRCC-modified portfolios compared with their risk-based benchmark portfolio. The out-of-sample period ranges from January 1991 to August 2015 for a total of 296 monthly observations.

Results are presented in table 5. Consider first the out-of-sample performance of the benchmark risk-based portfolios in Panel A, and compare them with the maximum Sharpe ratio portfolio. We find that the risk-based portfolios are successful in achieving the proposed investment style out-of-sample. As predicted, the minimum variance portfolio has the lowest level of volatility and drawdown (2.95 and 4.57%, respectively),

Table 5. Out-of-sample performance results.

	$	GR	Mean	Sd	SR	Sk	Ku	MDD	mVaR	TE
Panel A: Risk-based portfolios										
Min variance	3.56	5.29	2.45	2.95	0.83	−0.51	1.75	4.57	1.05	
Inverse volatility weighted	5.40	7.07	4.22	4.94	0.85	−1.25	5.82	14.76	2.06	
Equally-weighted	7.01	8.22	5.59	9.12	0.61	−1.18	6.79	31.64	4.08	
Equal-risk-contribution	4.44	6.23	3.48	5.85	0.59	−3.29	29.54	24.62	2.48	
Max diversification	4.15	5.93	3.10	3.99	0.78	−0.68	2.08	7.93	1.57	
Panel B: PRCC-modified risk-based portfolios										
Min variance	3.76	5.52	2.68	3.30	0.81	−0.57	1.41	3.99	1.24	6.07
Inverse volatility weighted	5.67	7.29	4.43	5.10	0.87	−1.37	10.76	17.48	2.02	9.33
Equally-weighted	7.94	8.76	6.01	8.17	0.74	−0.83	4.65	27.95	3.44	9.98
Equal-risk-contribution	4.99	6.74	3.96	5.96	0.66	−3.28	29.07	28.28	2.51	9.52
Max diversification	4.46	6.25	3.41	4.16	0.82	−0.08	0.91	5.79	1.46	8.06
Panel C: Other benchmark portfolio										
Max Sharpe ratio	5.51	7.16	4.36	6.04	0.72	−0.36	2.63	12.72	2.35	

Notes: This table presents the out-of-sample performance results for the risk-based portfolios, their PRCC modification, and the maximum Sharpe ratio portfolio. All reported statistics are as defined in table 3. The out-of-sample period ranges from January 1991 to August 2015, for a total of 296 monthly observations.

and that the volatility of the equal-risk-contribution portfolio (5.85%) is in between the volatility of the minimum variance portfolio (2.95%) and the equally-weighted portfolio (9.12%). We further observe that, for our sample, the highest Sharpe ratio is not achieved by the maximum Sharpe ratio portfolio, but by the low-risk portfolios. This can be explained by the presence of estimation error in the expected returns, which creates a drag in the out-of-sample performance of the maximum Sharpe ratio portfolio. In terms of absolute performance, the equally-weighted portfolio has the highest end-value ($7.01 for $1 invested in 1991). It also has the highest drawdown (31.64%) of all portfolios considered.

The PRCC-modified risk-based portfolios are partially safeguarded against the estimation risk because of the balancing objective between performance and risk contributions, and the tracking error constraint on the portfolio weights. Panel B of table 5 shows that the PRCC modification leads to a substantial improvement in the performance of the equally-weighted portfolio. The annualized geometric return increases from 8.22% to 8.76%, while its annualized volatility and maximum drawdown decrease from 9.12 and 31.64% to 8.17 and 27.95%, respectively. This performance improvement effect is consistent with the expectation in Kritzman *et al*. (2010) that optimization must be able to improve performance of a naively diversified portfolio, like the equally-weighted portfolio. For the other risk-based portfolios, the effect of the PRCC modification is to increase absolute performance at the cost of an increase in risk. This trade-off effect is in line with the constraint on the equality of estimated Sharpe ratio between the traditional risk-based portfolio and the PRCC-modified portfolio. Note also in the last column ('TE') that the PRCC modification leads to the smallest weight changes for the minimum variance portfolio, while for the equally-weighted portfolio, the average tracking error is 9.98%, indicating that for almost all rebalancing dates, the upper 10% constraint is binding.

The bottom line results of the out-of-sample study is that the PRCC modification improves the performance of the equally-weighted portfolio on all performance dimensions considered. For the other risk-based portfolios, the PRCC modification improves the absolute performance, but typically also increases the risk. In the web appendix (Ardia *et al*. 2017), we find that these results are robust to tightening the tracking error constraint from $\zeta = 10\%$ to $\zeta = 5\%$, and to the use of modified Value-at-Risk rather than volatility as the risk measure.

5. Conclusion

Risk-based portfolios have the computational and practical advantages of not requiring a return forecast. We argue that this may lead to imbalances in terms of a disparity between the performance-per-unit-of-risk-contribution for the various portfolio positions. To measure this imbalance, we propose the Performance/Risk Contribution Concentration (PRCC) measure. We show how to improve the balance between the performance and risk contributions of a reference portfolio by minimizing the portfolio PRCC under the constraint of achieving the same relative performance, and that the portfolio weights must be close enough to the benchmark weights. The proposed PRCC-modified risk-based portfolio has the potential to strike a balance between investors who believe in the construction of optimized portfolios using return forecasts (see, e.g. Kritzman *et al*. 2010), and investors who emphasize the difficulty in estimating expected returns and recommend to use portfolio allocations that do not require expected return estimates (see, e.g. DeMiguel *et al*. 2009).

We analyse the usefulness of the PRCC for the asset allocation decision to invest among equities, bonds, real estate, and gold. We find that, of all portfolios considered, the inverse volatility weighted, equally-weighted, equal-risk-contribution, and maximum diversification portfolios have the highest value of PRCC. Optimizing the PRCC of risk-based portfolios is especially beneficial when considering the equally-weighted portfolio: It increases the performance and reduces the risk. For the other risk-based portfolios, we find that the PRCC modification tends to yield similar or improved values for the relative performance, by increasing total performance at the price of a higher risk.

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Appendix 1. Simplification of the PRCC formula

Let $a_i \equiv C_i^{\mathcal{P}}(\mathbf{w}) - \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w}) C_i^{\mathcal{R}}(\mathbf{w})$. The PRCC is proportional to the sum of squared differences between couples in $\{a_1, \ldots, a_N\}$. The simplification uses that $\sum_{i=1}^N a_i = 0$, implying that:

$$\left(\sum_{i=1}^N a_i\right)^2 = \sum_{i=1}^N a_i^2 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^N a_i a_j = 0,$$

and thus:

$$2\sum_{i=1}^{N-1}\sum_{j=i+1}^N a_i a_j = -\sum_{i=1}^N a_i^2. \tag{A1}$$

It follows that:

$$\begin{aligned}\sum_{i=1}^N\sum_{j=1}^N (a_i - a_j)^2 &= 2\sum_{i=1}^{N-1}\sum_{j=i+1}^N (a_i - a_j)^2 \\ &= 2(N-1)\sum_{i=1}^N a_i^2 - 4\sum_{i=1}^{N-1}\sum_{j=i+1}^N a_i a_j \\ &= 2N\sum_{i=1}^N a_i^2,\end{aligned}$$

where we use (A1) in the last equality.

Appendix 2. PRCC of the maximum relative performance portfolio

The maximum relative performance portfolio maximizes $\tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w}) \equiv \frac{\mathcal{P}_p(\mathbf{w})}{\mathcal{R}_p(\mathbf{w})}$ under a full investment constraint. The corresponding Lagrangian is:

$$\mathcal{L}(\mathbf{w}, l) \equiv \frac{\mathcal{P}_p(\mathbf{w})}{\mathcal{R}_p(\mathbf{w})} - l(\mathbf{w}'\boldsymbol{\iota} - 1),$$

with $l \in \mathbb{R}$. From the first-order conditions, the portfolio weights need to be such that:

$$\begin{aligned}&\partial_i \mathcal{L}(\mathbf{w}^*, l) \\ &= \frac{1}{\mathcal{R}_p^2(\mathbf{w}^*)}\left[\mathcal{R}_p(\mathbf{w}^*)\partial_i\mathcal{P}_p(\mathbf{w}^*) - \mathcal{P}_p(\mathbf{w}^*)\partial_i\mathcal{R}_p(\mathbf{w}^*)\right] - l \\ &= \frac{1}{\mathcal{R}_p(\mathbf{w}^*)}\left[\partial_i\mathcal{P}_p(\mathbf{w}^*) - \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w}^*)\partial_i\mathcal{R}_p(\mathbf{w}^*)\right] - l = 0.\end{aligned}$$

Multiplying both sides by w_i^*, we have $\frac{1}{\mathcal{R}_p(\mathbf{w}^*)}\left[\mathcal{C}_i^{\mathcal{P}}(\mathbf{w}^*) - \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w}^*)\mathcal{C}_i^{\mathcal{R}}(\mathbf{w}^*)\right] - l w_i^* = 0$, and thus:

$$\mathcal{C}_i^{\mathcal{P}}(\mathbf{w}^*) - \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w}^*)\mathcal{C}_i^{\mathcal{R}}(\mathbf{w}^*) = \mathcal{R}_p(\mathbf{w}^*) l w_i^*. \tag{B1}$$

Since $\sum_{i=1}^N \left[\mathcal{C}_i^{\mathcal{P}}(\mathbf{w}^*) - \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w}^*)\mathcal{C}_i^{\mathcal{R}}(\mathbf{w}^*)\right] = 0$, it follows from (B1) that $\mathcal{R}_p(\mathbf{w}^*) l \boldsymbol{\iota}'\mathbf{w}^* = 0$. Under a full investment constraint, $\boldsymbol{\iota}'\mathbf{w}^* = 1$ and $\mathcal{R}_p(\mathbf{w}^*) > 0$, therefore $l = 0$. Combining this with (B1), we obtain:

$$\mathcal{C}_i^{\mathcal{P}}(\mathbf{w}^*) - \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w}^*)\mathcal{C}_i^{\mathcal{R}}(\mathbf{w}^*) = 0,$$

for all i. The PRCC measure is thus zero for the maximum relative performance portfolio.

Appendix 3. Volatility-based PRCC for risk-based portfolios

Minimum variance portfolio

The minimum variance portfolio minimizes the portfolio variance $\sigma_p^2(\mathbf{w}) = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$ under the full investment constraint $\mathbf{w}'\boldsymbol{\iota} = 1$. The corresponding Lagrangian is:

$$\mathcal{L}(\mathbf{w}, l) \equiv \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} - l(\mathbf{w}'\boldsymbol{\iota} - 1),$$

with $l \in \mathbb{R}$. From the first-order conditions, it follows that $\boldsymbol{\Sigma}\mathbf{w}^* = \frac{1}{2}l\boldsymbol{\iota}$. Since $\sigma_p^2(\mathbf{w}^*) = (\mathbf{w}^*)'\boldsymbol{\Sigma}\mathbf{w}^* = \frac{1}{2}l\boldsymbol{\iota}'\mathbf{w}^*$ and because of the full investment constraint $\boldsymbol{\iota}'\mathbf{w}^* = 1$, it follows that $\frac{1}{2}l = \sigma_p^2(\mathbf{w}^*)$ and thus $\boldsymbol{\Sigma}\mathbf{w}^* = \sigma_p^2(\mathbf{w}^*)\boldsymbol{\iota}$. Hence the risk contribution of asset i is $C_i^\sigma \equiv w_i^* \frac{[\boldsymbol{\Sigma}\mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)} = w_i^*\sigma_p(\mathbf{w}^*)$. Using this result, we can thus rewrite PRCC in (8) as:

$$\begin{aligned}\text{PRCC}(\mathbf{w}^*) &= \frac{1}{N}\sum_{i=1}^{N}\left[C_i^\mu(\mathbf{w}^*) - \tau_p^{\mu,\sigma}(\mathbf{w}^*)w_i^*\sigma_p(\mathbf{w}^*)\right]^2 \\ &= \frac{1}{N}\sum_{i=1}^{N}\left\{w_i^*[\tilde{\mu}_i - \tilde{\mu}_p(\mathbf{w}^*)]\right\}^2,\end{aligned}$$

since $\tau_p^{\mu,\sigma}(\mathbf{w}^*) \equiv \tilde{\mu}_p(\mathbf{w}^*)/\sigma_p(\mathbf{w}^*)$.

Inverse volatility weighted portfolio

Let us define $\xi_i \equiv 1/\sigma_i$ and $\boldsymbol{\xi} \equiv (\xi_1, \ldots, \xi_N)'$. Then, the weights of the inverse volatility weighted portfolio are given by $\mathbf{w} \equiv \boldsymbol{\xi}/\boldsymbol{\xi}'\boldsymbol{\iota}$. The covariance matrix can be decomposed as $\boldsymbol{\Sigma} \equiv \mathbf{DRD}$, where $\mathbf{D}$ is a diagonal matrix containing the variances $(\sigma_1, \ldots, \sigma_N)'$ and $\mathbf{R}$ is the correlation matrix. Using $\mathbf{D}\boldsymbol{\xi} = \boldsymbol{\iota}$, the portfolio volatility is:

$$\sigma_p(\mathbf{w}^*) \equiv \sqrt{(\mathbf{w}^*)'\boldsymbol{\Sigma}\mathbf{w}^*} = \sqrt{\frac{\boldsymbol{\xi}'}{\boldsymbol{\xi}'\boldsymbol{\iota}}\mathbf{DRD}\frac{\boldsymbol{\xi}}{\boldsymbol{\xi}'\boldsymbol{\iota}}} = \frac{1}{\boldsymbol{\xi}'\boldsymbol{\iota}}\sqrt{\boldsymbol{\iota}'\mathbf{R}\boldsymbol{\iota}}.$$

Component risk contribution of asset i is then:

$$C_i^\sigma \equiv w_i^*\frac{[\boldsymbol{\Sigma}\mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)} = \frac{1}{\boldsymbol{\xi}'\boldsymbol{\iota}}\frac{1}{\sigma_i}\frac{[\mathbf{DRD}\boldsymbol{\xi}]_i}{\sqrt{\boldsymbol{\iota}'\mathbf{R}\boldsymbol{\iota}}} = \frac{1}{\boldsymbol{\xi}'\boldsymbol{\iota}}\frac{[\mathbf{R}\boldsymbol{\iota}]_i}{\sqrt{\boldsymbol{\iota}'\mathbf{R}\boldsymbol{\iota}}}.$$

Then, the PRCC of the inverse volatility weighted portfolio can be rewritten as:

$$\text{PRCC}(\mathbf{w}^*) = \frac{1}{N}\frac{1}{(\boldsymbol{\xi}'\boldsymbol{\iota})^2}\cdot\sum_{i=1}^{N}\left[\frac{\tilde{\mu}_i}{\sigma_i} - \tau_p(\mathbf{w}^*)\frac{[\mathbf{R}\boldsymbol{\iota}]_i}{\sqrt{\boldsymbol{\iota}'\mathbf{R}\boldsymbol{\iota}}}\right]^2.$$

Equally-weighted portfolio

For the equally-weighted portfolio $\mathbf{w}^* \equiv \frac{\boldsymbol{\iota}}{N}$, the portfolio risk is:

$$\sigma_p(\mathbf{w}^*) \equiv \sqrt{(\mathbf{w}^*)'\boldsymbol{\Sigma}\mathbf{w}^*} = \sqrt{\frac{1}{N}\boldsymbol{\iota}'\boldsymbol{\Sigma}\frac{1}{N}\boldsymbol{\iota}} = \frac{1}{N}\sqrt{\boldsymbol{\iota}'\boldsymbol{\Sigma}\boldsymbol{\iota}}.$$

The component risk contribution of asset i is:

$$C_i^\sigma \equiv w_i^*\frac{[\boldsymbol{\Sigma}\mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)} = \frac{1}{N}\frac{[\boldsymbol{\Sigma}\boldsymbol{\iota}]_i}{\sqrt{\boldsymbol{\iota}'\boldsymbol{\Sigma}\boldsymbol{\iota}}}.$$

Then, the PRCC of the equally-weighted portfolio can be rewritten as:

$$\begin{aligned}\text{PRCC}(\mathbf{w}^*) &= \frac{1}{N}\sum_{i=1}^{N}\left[\frac{1}{N}\tilde{\mu}_i - \tau_p^{\mu,\sigma}(\mathbf{w}^*)\frac{1}{N}\frac{[\boldsymbol{\Sigma}\boldsymbol{\iota}]_i}{\sqrt{\boldsymbol{\iota}'\boldsymbol{\Sigma}\boldsymbol{\iota}}}\right]^2 \\ &= \frac{1}{N^3}\cdot\sum_{i=1}^{N}\left[\tilde{\mu}_i - \tau_p^{\mu,\sigma}(\mathbf{w}^*)\frac{[\boldsymbol{\Sigma}\boldsymbol{\iota}]_i}{\sqrt{\boldsymbol{\iota}'\boldsymbol{\Sigma}\boldsymbol{\iota}}}\right]^2.\end{aligned}$$

Equal-risk-contribution portfolio

The equal-risk-contribution portfolio aims at equalizing the component-risk-contributions: $C_i^\sigma(\mathbf{w}^*) = C_j^\sigma(\mathbf{w}^*)\ \forall i, j$. As $\sum_{i=1}^N C_i^\sigma(\mathbf{w}^*) = \sigma_p(\mathbf{w}^*)$, it follows that $C_i^\sigma(\mathbf{w}^*) = \frac{\sigma_p(\mathbf{w}^*)}{N}$. Hence, the component performance/risk contribution of asset i is:

$$\text{CPRC}_i(\mathbf{w}^*) = C_i^\mu(\mathbf{w}^*) - \tau_p^{\mu,\sigma}(\mathbf{w}^*)\frac{\sigma_p(\mathbf{w}^*)}{N} = w_i^*\tilde{\mu}_i - \frac{\tilde{\boldsymbol{\mu}}'\mathbf{w}^*}{N}.$$

So the PRCC measure in (8) can be rewritten as:

$$\text{PRCC}(\mathbf{w}^*) = \frac{1}{N}\sum_{i=1}^{N}\left[w_i^*\tilde{\mu}_i - \frac{\tilde{\boldsymbol{\mu}}'\mathbf{w}^*}{N}\right]^2.$$

Maximum diversification portfolio

The maximum diversification portfolio maximizes the diversification ratio $\mathbf{w}'\boldsymbol{\sigma}/\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}$, where $\boldsymbol{\sigma}$ is the vector of volatilities. The corresponding Lagrangian under a full investment constraint is:

$$\mathcal{L}(\mathbf{w}, l) \equiv \frac{\mathbf{w}'\boldsymbol{\sigma}}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}} - l(\mathbf{w}'\boldsymbol{\iota} - 1),$$

with $l \in \mathbb{R}$. From the first-order conditions, it follows that:

$$\frac{\sigma_p(\mathbf{w}^*)\sigma_i - \mathbf{w}^{*\prime}\boldsymbol{\sigma}\frac{[\boldsymbol{\Sigma}\mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)}}{\sigma_p^2(\mathbf{w}^*)} = l. \tag{C1}$$

Multiplying both sides by w_i^* and taking the sum, we get:

$$\frac{\sigma_p(\mathbf{w}^*)\boldsymbol{\sigma}'\mathbf{w}^* - \boldsymbol{\sigma}'\mathbf{w}^*\sigma_p(\mathbf{w}^*)}{\sigma_p^2(\mathbf{w}^*)} = l\sum_{i=1}^{N}w_i^* = l.$$

Since the left-hand side of the equation is zero and because of the full investment constraint $\boldsymbol{\iota}'\mathbf{w}^* = 1$, it follows that $l = 0$ and thus, given (C1) we obtain:

$$\sigma_p(\mathbf{w}^*)\sigma_i - \boldsymbol{\sigma}'\mathbf{w}^*\frac{[\boldsymbol{\Sigma}\mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)} = 0.$$

Equivalently, $\frac{[\boldsymbol{\Sigma}\mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)} = \frac{\sigma_p(\mathbf{w}^*)\sigma_i}{\boldsymbol{\sigma}'\mathbf{w}^*}$. Using this result, we can rewrite PRCC in (8) as:

$$\begin{aligned}\text{PRCC}(\mathbf{w}^*) &= \frac{1}{N}\sum_{i=1}^{N}\left[w_i^*\left(\tilde{\mu}_i - \tau_p^{\mu,\sigma}(\mathbf{w}^*)\frac{[\boldsymbol{\Sigma}\mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)}\right)\right]^2 \\ &= \frac{1}{N}\sum_{i=1}^{N}\left[w_i^*\sigma_i\left(\frac{\tilde{\mu}_i}{\sigma_i} - \frac{\tilde{\boldsymbol{\mu}}'\mathbf{w}^*}{\boldsymbol{\sigma}'\mathbf{w}^*}\right)\right]^2.\end{aligned}$$

For the maximum diversification portfolio, the PRCC measure is thus zero when all assets have the same Sharpe ratio. Indeed, when σ_i is proportional to μ_i, maximizing the diversification ratio is equivalent to maximizing the portfolio's Sharpe ratio.