

Semantics for Analytic Containment

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Abstract. In 1977, R. B. Angell presented a logic for *analytic containment*, a notion of “relevant” implication stronger than Anderson and Belnap’s entailment. In this paper I provide for the first time the logic of *first degree* analytic containment, as presented in [2] and [3], with a semantical characterization—leaving higher degree systems for future investigations. The semantical framework I introduce for this purpose involves a special sort of truth–predicates, which apply to pairs of collections of formulas instead of individual formulas, and which behave in some respects like Gentzen’s sequents. This semantics captures very general properties of the truth–functional connectives, and for that reason it may be used to model a vast range of logics. I briefly illustrate the point with classical consequence and Anderson and Belnap’s “tautological entailments”.

Keywords: analytic containment, relevant implication, semantics.

1. Analytic Containment

Analytic containment, as R. B. Angell understands the notion, is that relation which holds between two sentences, when the meaning of the first contains the meaning of the second: p analytically contains q iff the meaning of p contains the meaning of q .¹

In [2] and [3], Angell devised his system AC as a logic of analytic containment.

Angell actually introduced two systems under the label ‘AC’. Both are formulated in a classical propositional language with negation (\neg) and conjunction (\wedge) as primitive truth–functional connectives, augmented by the binary operator \leftrightarrow representing mutual analytic containment (sameness of meaning). The formation rules of the 1977 system are as expected: the atoms are formulas, and if A and B are formulas, then $\neg A$, $A \wedge B$ and $A \leftrightarrow B$ are formulas. The formulas of the 1989 system are only those of the form $A \leftrightarrow B$, where A and B contain only truth–functional compounds of atoms. Analytic containment (\rightarrow) is in both systems defined by $A \rightarrow B =_{df} A \leftrightarrow (A \wedge B)$. The theorems of the 1989 system all have the form $A \leftrightarrow B$, while the 1977 system also includes classical logic, but no specific axioms or rules for formu-

¹ [3], p. 119.

las with nested \leftrightarrow . The latter system thus essentially embodies properties of first degree analytic containment.

In this paper I shall focus on the 1989 system, and on the first degree fragment of the 1977 system (which was actually Angell's center of interest in [2]), and the aim is basically to provide them with semantical characterizations. I shall leave the task of dealing with systems involving formulas of arbitrary degrees for future work.

It is perhaps worth saying a word or two about what analytic containment is. Angell's informal definition of analytic containment is not very perspicuous. For instance, in one sense, the meaning of a sentence is contained in the meaning of its negation. But it is clear that Angell does not want to say that for any sentence p , $\neg p$ analytically contains p . ("Proof": his systems would represent the principle as a theorem, but they do not.)

As far as I can see, analytic containment amounts to this: sentence p analytically contains sentence q when one can "extract" the information given by q from the information given by p only by looking at the meaning of the components of p and q —or again, more objectively, when in virtue of the meaning of the constituents of p and q , the information given by q is carried by, or is part of, the information given by p .

It may be thought that in the above explication reference to meanings is useless. For, one might argue, information containment is always a question of meaning: for any sentences p and q , if the information given by p contains the information given by q , then this is true in virtue of what p and q mean.

However, reference to meanings allows one to formulate a certain distinction between two notions of analytic containment which, I think, is worth formulating. One of these notions is the one which has been defined above (keep on calling it 'analytic containment'), the other one is stronger and we shall call it 'logical containment'. Logical containment is related to analytic containment as logical truth to analytic truth. An analytic truth (e.g. 'Sam is a man if Sam is a bachelor') is a sentence true in virtue of the meaning of its components; a logical truth (e.g. 'Sam is a man if Sam is a man and Sam likes chocolate') is a sentence true in virtue of the meaning of its *logical* components. Similarly, sentence q (e.g. 'Sam is a man') is analytically contained in sentence p (e.g. 'Sam is a bachelor') when one can extract the information given by q from the information given by p by looking at the meaning of the components of p and q ; and sentence q (e.g. 'Sam is a man') is logically contained in sentence p (e.g. 'Sam is a man and Sam likes chocolate') when one can extract the information given by q from the information given by p only by looking at the meaning of the *logical* components of p and q .

Although the two notions are distinct, in each of Angell's systems \rightarrow may be interpreted indifferently as analytic containment or as logical containment.

2. The System AC

My formulation of Angell's 1989 system is somewhat different than his own, but the differences are minor. In the sequel, 'system AC' will denote my version of the system.

We start with a classical propositional language L with primitive operators \neg and \vee (\wedge is defined in the usual way in terms of \neg and \vee) augmented with the binary operator \rightarrow representing analytic containment. The formulas of the system are all truth-functional compounds of atoms, plus every formula of type $A \rightarrow B$, where A and B contain no occurrence of \rightarrow . In what follows, I shall use A, B, \dots for formulas containing no arrows, and I shall call them 'TFFs'.

System AC is then defined by the following axiom schemata and rules (here and below, any occurrence of \rightarrow has to be understood as having the widest possible scope):

$$A \rightarrow \neg\neg A \quad (\text{AC1a})$$

$$\neg\neg A \rightarrow A \quad (\text{AC1b})$$

$$A \rightarrow A \wedge A \quad (\text{AC2})$$

$$A \wedge B \rightarrow A \quad (\text{AC3})$$

$$A \vee B \rightarrow B \vee A \quad (\text{AC4})$$

$$A \vee (B \vee C) \rightarrow (A \vee B) \vee C \quad (\text{AC5a})$$

$$(A \vee B) \vee C \rightarrow A \vee (B \vee C) \quad (\text{AC5b})$$

$$A \vee (B \wedge C) \rightarrow (A \vee B) \wedge (A \vee C) \quad (\text{AC6a})$$

$$(A \vee B) \wedge (A \vee C) \rightarrow A \vee (B \wedge C) \quad (\text{AC6b})$$

$$A \rightarrow B, B \rightarrow A / \neg A \rightarrow \neg B \quad (\text{AC7})$$

$$A \rightarrow B / A \vee C \rightarrow B \vee C \quad (\text{AC8})$$

$$A \rightarrow B, B \rightarrow C / A \rightarrow C \quad (\text{AC9})$$

We shall use ' $A \vdash B$ ' to mean that $A \rightarrow B$ is a theorem of AC, and ' $A \dashv\vdash B$ ' to mean that both $A \vdash B$ and $B \vdash A$. One can then prove:

$$A \vdash A \quad (\text{T1})$$

$$A \dashv\vdash A \wedge A \quad (\text{T2})$$

$$A \dashv\vdash A \vee A \quad (\text{T3})$$

$$A \vee B \dashv\vdash B \vee A \quad (\text{T4})$$

$$A \wedge B \dashv\vdash B \wedge A \quad (\text{T5})$$

$$A \wedge (B \wedge C) \dashv\vdash (A \wedge B) \wedge C \quad (\text{T6})$$

$$A \wedge (B \vee C) \dashv\vdash (A \wedge B) \vee (A \wedge C) \quad (\text{T7})$$

$$A \wedge B \vdash A \vee B \quad (\text{T8})$$

$$A \wedge (B \vee C) \vdash A \vee B \quad (\text{T9})$$

$$A \vdash B \wedge C \text{ iff } A \vdash B \text{ and } A \vdash C \quad (\text{T10})$$

$$\text{If } A \vdash B \text{ and } C \vdash D, \text{ then } A \wedge C \vdash B \wedge D \quad (\text{T11})$$

$$\text{If } A \vdash B \text{ and } C \vdash D, \text{ then } A \vee C \vdash B \vee D \quad (\text{T12})$$

$$\text{If } A \vdash B \text{ and } A \vdash C \vee D, \text{ then } A \vdash B \vee C \quad (\text{T13})$$

$$\text{If } A \dashv\vdash B, \text{ then } \neg A \dashv\vdash \neg B \quad (\text{T14})$$

$$\text{If } A \dashv\vdash B, \text{ then } A \wedge C \dashv\vdash B \wedge C \quad (\text{T15})$$

$$\text{If } A \dashv\vdash B, \text{ then } A \vee C \dashv\vdash B \vee C \quad (\text{T16})$$

$$\text{If } A \dashv\vdash B, \text{ then } C \dashv\vdash C[A/B], \text{ where } C[A/B] \text{ is the result of} \\ \text{replacing some occurrences of } B \text{ in } C \text{ by } A \quad (\text{T17})$$

$$C \dashv\vdash C[\neg\neg A/A] \text{ (same notation as above)} \quad (\text{T18})$$

For a comparison, Angell's original system in [3] is formulated as follows (with mutual analytic containment \leftrightarrow taken as primitive, and $A \rightarrow B$ defined as $A \leftrightarrow (A \wedge B)$):

$$A \leftrightarrow \neg\neg A$$

$$A \leftrightarrow A \wedge A$$

$$A \wedge B \leftrightarrow B \wedge A$$

$$A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C$$

$$A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$$

$$A \leftrightarrow B, X / X[A/B], \text{ where } X[A/B] \text{ is the result of replacing some} \\ \text{occurrences of } B \text{ in } X \text{ by } A$$

It is easy to see that theoremhood in system AC formalizes a notion of consequence which is strictly stronger than classical consequence: if $A \vdash B$, then B is a classical consequence of A , but the converse does not hold. In fact, as one can check from the list of axioms, $A \vdash B$ only if all atoms in B appear in A . Analytic containment is a form of "relevant" implication: in particular, neither of $A \rightarrow B \vee \neg B$ and $A \wedge \neg A \rightarrow B$ is derivable.

One important feature of AC is that $A \rightarrow A \vee B$ is not derivable. This is a crucial difference with Anderson and Belnap’s system E_{fde} for “tautological entailment”.² Actually, E_{fde} is just AC plus axiom $A \rightarrow A \vee B$.³

3. Semantics for AC

One natural idea to model analytic containment is to follow a standard way of modelling e.g. classical consequence: introduce a certain kind of models which assign certain semantic properties to the atoms of the language, then recursively define a notion of truth in a model for all TFFs, in such a way that $A \vdash B$ iff B is true in every model where A is true. This is the way I am going to proceed.

Such a semantical treatment of analytic containment meets some difficulties, mainly because of the behavior of disjunction. Given the fact that $A \rightarrow A \vee B$ is not derivable in AC, the truth of A in a model m should not yield the truth in m of $A \vee B$ for any arbitrary B . This means that, in the present context, we must do without the familiar sounding truth–clause for disjunction ‘ $A \vee B$ is true in m iff A is true in m or B is true in m ’.

The semantics I present in this section is very unusual. In this semantics, instead of assigning semantic properties to *individual* atoms (like e.g. having or lacking this or that truth–value), the models assign semantic properties to *collections* of atoms—more precisely, to pairs of such collections. Relative to any model, a truth predicate which applies to pairs of collections of formulas is defined, as well as a truth predicate which applies to individual formulas—the second being defined in terms of the first.

More concretely, we define a *valuation* on language L as a set of *pairs* of finite subsets of the set At of all atoms of L . For any given valuation v , we define a truth predicate which applies to arbitrary pairs of finite sets of TFFs, \vDash_v . For the sake of readability, I shall write $\Gamma \vDash_v \Delta$ instead of $\vDash_v (\langle \Gamma, \Delta \rangle)$, and I shall use a standard convention used in sequent calculi of using commas instead of union signs, and of omitting the set–theoretical brackets around TFFs. ‘ $\Gamma \vDash_v \Delta$ ’ should be read ‘according to v , the disjunction of all the negated members of Γ and of all the members of Δ is true’. As the reader will see, the analogy with sequents goes beyond the above notational convention.

The definition of \vDash_v is as follows:

1. If Γ and Δ are sets of atoms, then $\Gamma \vDash_v \Delta$ iff $\langle \Gamma, \Delta \rangle \in v$;

² See [1], ch. III, §15. The theorems of E_{fde} are all the formulas of type $A \rightarrow B$ (where A and B are TFFs) which are theorems of system E for entailment.

³ See [2].

2. (a) $\Gamma \models_v \Delta, \neg A$ iff $\Gamma, A \models_v \Delta$;
 (b) $\Gamma, \neg A \models_v \Delta$ iff $\Gamma \models_v \Delta, A$;
3. (a) $\Gamma \models_v \Delta, A \vee B$ iff $\Gamma \models_v \Delta, A, B$;
 (b) $\Gamma, A \vee B \models_v \Delta$ iff both $\Gamma, A \models_v \Delta$ and $\Gamma, B \models_v \Delta$.

The clause for conjunction is then:

- $\Gamma \models_v \Delta, A \wedge B$ iff both $\Gamma \models_v \Delta, A$ and $\Gamma \models_v \Delta, B$;
 $\Gamma, A \wedge B \models_v \Delta$ iff $\Gamma, A, B \models_v \Delta$.

Finally, we shall say that TFF A is *true in valuation* v —for short, $\models_v A$ —iff $\emptyset \models_v A$. Note that no TFF is true in every valuation. Also note that truth in a valuation has a special behavior with respect to disjunction (but not conjunction): from the truth of A one cannot in general infer the truth of $A \vee B$, and conversely from the truth of $A \vee B$ one cannot infer that A is true or B is true.⁴ Truth in a valuation behaves in this respect like membership to a (real) belief set: you may believe that Sam is either at home or at the stadium without having a belief as to exactly where he is; and you may believe that Sam is at home without believing that he is at home or the number of pens in my office is 7.

Arbitrary valuations, though, are not appropriate for modelling analytic containment in the way suggested at the beginning of this section. For let p, q and r be atoms of L . By (T9) $p \wedge (q \vee r) \vdash p \vee q$. Now let v be the valuation whose sole members are $\langle \emptyset, \{p\} \rangle$ and $\langle \emptyset, \{q, r\} \rangle$. Then $\models_v p \wedge (q \vee r)$ but $\not\models_v p \vee q$. We need valuations of a certain kind, which I shall call *AC-valuations*.

An *AC-valuation* is a valuation which satisfies the following condition:

- for all sets of atoms $\Gamma, \Gamma', \Gamma'', \Delta, \Delta', \Delta''$, if $\langle \Gamma, \Delta \rangle \in v$ and $\langle \Gamma' \cup \Gamma'', \Delta' \cup \Delta'' \rangle \in v$, then $\langle \Gamma \cup \Gamma', \Delta \cup \Delta' \rangle \in v$.

Let us say that A *semantically contains* B —for short $A \Vdash B$ —iff B is true in every AC-valuation where A is true. The next two sections are devoted to showing that $A \vdash B$ iff $A \Vdash B$ —in other words, that AC is both sound and complete with respect to the semantics of AC-valuations.

4. Soundness

The soundness of AC with respect to the previous semantics—i.e. the fact that $A \vdash B$ entails $A \Vdash B$ —will be proved indirectly.

⁴ Of course, taking ‘ A is false in v ’ to mean ‘ $\neg A$ is true in v ’, the dual claim about conjunction holds as well: falsity in a valuation has a special behavior with respect to conjunction (but not disjunction).

Where v is a valuation, let us say that $A \Vdash_v B$ iff for every finite set of TFFs Γ , the following two conditions hold:

- If $\Gamma \vDash_v A$, then $\Gamma \vDash_v B$;
- $A \vDash_v \Gamma$ iff $A, B \vDash_v \Gamma$.

We shall first prove that for every AC-valuation v , if $A \vdash B$ then $A \Vdash_v B$. Then given that for every valuation v , if $A \Vdash_v B$, then $\vDash_v A$ implies $\vDash_v B$, this will establish the soundness of AC with respect to the proposed semantics.⁵

We shall need the following important proposition, which says that the defining condition of AC-valuations described in the previous section transmits from atoms to arbitrary TFFs:

PROPOSITION 4.1. *Let v be an AC-valuation. Then for all sets of TFFs $\Gamma, \Gamma', \Gamma'', \Delta, \Delta', \Delta''$, if $\Gamma \vDash_v \Delta$ and $\Gamma', \Gamma'' \vDash_v \Delta', \Delta''$, then $\Gamma, \Gamma' \vDash_v \Delta, \Delta'$.*

PROOF. Let the *degree of a TFF A* be the natural number $d(A)$ recursively defined by the following 3 clauses: (i) if A is an atom, then $d(A) = 0$; (ii) $d(\neg A) = 1 + d(A)$; (iii) $d(A \vee B) = 1 + \max\{d(A), d(B)\}$. The *degree of a set Γ of TFFs* is then defined as $\max\{d(A) : A \in \Gamma\}$. One may prove the proposition by induction on the degree of $\Gamma \cup \Gamma' \cup \Gamma'' \cup \Delta \cup \Delta' \cup \Delta''$ in the following way. Let v be an AC-valuation. Then by definition, proposition 1 holds when $\Gamma \cup \Gamma' \cup \Gamma'' \cup \Delta \cup \Delta' \cup \Delta''$ is of degree 0. Suppose now that it holds whenever $\Gamma \cup \Gamma' \cup \Gamma'' \cup \Delta \cup \Delta' \cup \Delta''$ is of degree $\leq n$, and let $\Gamma, \Gamma', \Gamma'', \Delta, \Delta'$ and Δ'' be such that their union is of degree $n + 1$. Then each set Ψ among the above 6 may be decomposed into three components: Ψ_1 , the set of all TFFs in Ψ of degree $\leq n$; Ψ_2 , the set of all TFFs in Ψ of the form $\neg A$ of degree $n + 1$; and Ψ_3 , the set of all TFFs in Ψ of the form $A \vee B$ of degree $n + 1$.

Before proceeding further, some definitions are in order. In what follows, where Ψ is any set of TFFs of form $\neg A$, ' $\bar{\Psi}$ ' will denote the set of all TFFs A such that $\neg A \in \Psi$; and where Ψ is any set of TFFs of form $A \vee B$, ' $\hat{\Psi}$ ' will denote the set of all TFFs A such that for some TFF B , $A \vee B \in \Psi$ or $B \vee A \in \Psi$. In case Ψ is empty, both ' $\bar{\Psi}$ ' and ' $\hat{\Psi}$ ' will denote the empty set. Finally, let Ψ be any set of cardinal $n \geq 2$, containing only TFFs of form $A \vee B$. Assume a numbering of all the disjunctions of L , and say that Ψ is $\{A_1 \vee B_1, \dots, A_n \vee B_n\}$, where the indices respect the numbering. Then let Ψ' be $\{A_1, B_1\} \times \dots \times \{A_n, B_n\}$. Where ξ is any n -tuple $\langle a_1, \dots, a_n \rangle$, let $s(\xi)$

⁵ On the use of \Vdash , see REMARK I on page 98.

be $\{a_1, \dots, a_n\}$. Then we let $\vec{\Psi}$ be $\{s(\xi) : \xi \in \Psi'\}$. (It should be clear that the choice of the original numbering is immaterial, i.e. if $\vec{\Psi}_1$ is obtained from a given numbering and $\vec{\Psi}_2$ from another, then $\vec{\Psi}_1 = \vec{\Psi}_2$.) In case Ψ contains only one element $A \vee B$, we let $\vec{\Psi}$ be $\{\{A\}, \{B\}\}$. In case Ψ is empty, we put $\vec{\Psi} = \{\emptyset\}$.

Then suppose that

1. $\Gamma \vDash_v \Delta$, and
2. $\Gamma', \Gamma'' \vDash_v \Delta', \Delta''$.

Then:

1. $\Gamma_1, \Gamma_2, \Gamma_3 \vDash_v \Delta_1, \Delta_2, \Delta_3$, and
2. $\Gamma'_1, \Gamma'_2, \Gamma'_3, \Gamma''_1, \Gamma''_2, \Gamma''_3 \vDash_v \Delta'_1, \Delta'_2, \Delta'_3, \Delta''_1, \Delta''_2, \Delta''_3$.

So:

1. $\forall x \in \vec{\Gamma}_3 \quad \Gamma_1, x, \overline{\Delta}_2 \vDash_v \Delta_1, \widehat{\Delta}_3, \overline{\Gamma}_2$, and
2. $\forall y \in \vec{\Gamma}'_3 \quad \forall z \in \vec{\Gamma}''_3 \quad \Gamma'_1, y, \overline{\Delta}'_2, \Gamma''_1, z, \overline{\Delta}''_2 \vDash_v \Delta'_1, \widehat{\Delta}'_3, \overline{\Gamma}'_2, \Delta''_1, \widehat{\Delta}''_3, \overline{\Gamma}''_2$.

Then by induction hypothesis, for all $x \in \vec{\Gamma}_3$, $y \in \vec{\Gamma}'_3$ and $z \in \vec{\Gamma}''_3$:

- $\Gamma_1, x, \overline{\Delta}_2, \Gamma'_1, y, \overline{\Delta}'_2 \vDash_v \Delta_1, \widehat{\Delta}_3, \overline{\Gamma}_2, \Delta'_1, \widehat{\Delta}'_3, \overline{\Gamma}'_2$,

i.e.:

- $\Gamma_1, \Gamma_2, x, \Gamma'_1, \Gamma'_2, y \vDash_v \Delta_1, \Delta_2, \Delta_3, \Delta'_1, \Delta'_2, \Delta'_3$.

But this implies that: $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma'_1, \Gamma'_2, \Gamma'_3 \vDash_v \Delta_1, \Delta_2, \Delta_3, \Delta'_1, \Delta'_2, \Delta'_3$, i.e. that $\Gamma, \Gamma' \vDash_v \Delta, \Delta'$. ■

PROPOSITION 4.2. *For every AC-valuation v and all TFFs A and B , if $A \vdash B$, then $A \Vdash_v B$.*

PROOF. It suffices to prove that for every AC-valuation v , (i) every axiom of AC $A \rightarrow B$ is such that $A \Vdash_v B$, and (ii) that every rule of inference of AC $A_1 \rightarrow B_1, A_2 \rightarrow B_2, \dots / A \rightarrow B$ is such that if $A_1 \Vdash_v B_1, A_2 \Vdash_v B_2, \dots$, then $A \Vdash_v B$. Point (i) is obvious for (AC1a)–(AC5b). For the remaining cases, let v be an AC-valuation.

1. (for (AC6a) and (AC6b)) For any arbitrary finite set of TFFs Γ , and TFFs A, B and C , we have:

- (a) $\Gamma \vDash_v A \vee (B \wedge C)$ iff $\Gamma \vDash_v A, B$ and $\Gamma \vDash_v A, C$;
- (b) $\Gamma \vDash_v (A \vee B) \wedge (A \vee C)$ iff $\Gamma \vDash_v A, B$ and $\Gamma \vDash_v A, C$;
- (c) $A \vee (B \wedge C) \vDash_v \Gamma$ iff $A \vDash_v \Gamma$ and $B, C \vDash_v \Gamma$;
- (d) $(A \vee B) \wedge (A \vee C) \vDash_v \Gamma$ iff $A \vDash_v \Gamma, B, C \vDash_v \Gamma, A, B \vDash_v \Gamma$ and $A, C \vDash_v \Gamma$;
- (e) $A \vee (B \wedge C), (A \vee B) \wedge (A \vee C) \vDash_v \Gamma$ iff $A \vDash_v \Gamma, B, C \vDash_v \Gamma, A, B \vDash_v \Gamma, A, C \vDash_v \Gamma$ and $A, B, C \vDash_v \Gamma$.

By (a) and (b), $\Gamma \vDash_v A \vee (B \wedge C)$ iff $\Gamma \vDash_v (A \vee B) \wedge (A \vee C)$. By (c) and (e), if $A \vee (B \wedge C), (A \vee B) \wedge (A \vee C) \vDash_v \Gamma$, then $A \vee (B \wedge C) \vDash_v \Gamma$; and the converse holds by proposition 4.1. By (d) and (e), if $A \vee (B \wedge C), (A \vee B) \wedge (A \vee C) \vDash_v \Gamma$, then $(A \vee B) \wedge (A \vee C) \vDash_v \Gamma$; and once again the converse holds by proposition 4.1. So, $A \vee (B \wedge C) \Vdash_v (A \vee B) \wedge (A \vee C)$ and $(A \vee B) \wedge (A \vee C) \Vdash_v A \vee (B \wedge C)$.

2. (for (AC7)) Suppose that both $A \Vdash_v B$ and $B \Vdash_v A$, i.e. that for every finite set of TFFs Γ the following three conditions hold:

- (a) $\Gamma \vDash_v A$ iff $\Gamma \vDash_v B$;
- (b) $A \vDash_v \Gamma$ iff $A, B \vDash_v \Gamma$;
- (c) $B \vDash_v \Gamma$ iff $A, B \vDash_v \Gamma$.

Then let Γ be a finite set of TFFs. (i) Suppose that $\Gamma \vDash_v \neg A$. Then $A \vDash_v \Gamma^*$, where $\Gamma^* = \{\neg C : C \in \Gamma\}$. By (b) and (c), it follows that $B \vDash_v \Gamma^*$, and so that $\Gamma \vDash_v \neg B$. (ii) Suppose now that $\neg A \vDash_v \Gamma$. Then $\Gamma^* \vDash_v A$. By (a), it follows that $\Gamma^* \vDash_v B$, and thus $\neg B \vDash_v \Gamma$. By proposition 4.1, we have then $\neg A, \neg B \vDash_v \Gamma$. (iii) Suppose that $\neg A, \neg B \vDash_v \Gamma$. Then $\Gamma^*, \neg A \vDash_v B$, and so by (a) $\Gamma^*, \neg A \vDash_v A$, i.e. $\neg A \vDash_v \Gamma$. By (i), (ii) and (iii), $\neg A \Vdash_v \neg B$.

3. (for (AC8)) Suppose that $A \Vdash_v B$, i.e. that for every finite set of TFFs Γ ,

- (a) if $\Gamma \vDash_v A$, then $\Gamma \vDash_v B$;
- (b) $A \vDash_v \Gamma$ iff $A, B \vDash_v \Gamma$.

Then let Γ be a finite set of TFFs. (i) Suppose that $\Gamma \vDash_v A \vee C$. Then $\Gamma, \neg C \vDash_v A$, and so by (a), $\Gamma, \neg C \vDash_v B$. Consequently, $\Gamma \vDash_v B \vee C$. (ii) Suppose that $A \vee C \vDash_v \Gamma$. Then both $A \vDash_v \Gamma$ and $C \vDash_v \Gamma$. By (b), $A, B \vDash_v \Gamma$. By proposition 4.1, we also have $A, C \vDash_v \Gamma$ and $B, C \vDash_v \Gamma$. It follows that $A \vee C, B \vee C \vDash_v \Gamma$. (iii) Suppose that $A \vee C, B \vee C \vDash_v \Gamma$. Then $A, B \vDash_v \Gamma$ and $C \vDash_v \Gamma$. By (b) and $A, B \vDash_v \Gamma$, we have $A \vDash_v \Gamma$. So, $A \vee C \vDash_v \Gamma$. By (i), (ii) and (iii), $A \vee C \Vdash_v B \vee C$.

4. (for (AC9)) Suppose that both $A \Vdash_v B$ and $B \Vdash_v C$, i.e. that for every finite set of TFFs Γ ,
- (a) if $\Gamma \vDash_v A$, then $\Gamma \vDash_v B$;
 - (b) $A \vDash_v \Gamma$ iff $A, B \vDash_v \Gamma$;
 - (c) if $\Gamma \vDash_v B$, then $\Gamma \vDash_v C$;
 - (d) $B \vDash_v \Gamma$ iff $B, C \vDash_v \Gamma$.

Then let Γ be a finite set of TFFs. (i) By (a) and (c), if $\Gamma \vDash_v A$, then $\Gamma \vDash_v C$. (ii) Suppose that $A \vDash_v \Gamma$. Then by (b), $A, B \vDash_v \Gamma$, and so $B \vDash_v \Gamma, \neg A$. By (d), it follows that $B, C \vDash_v \Gamma, \neg A$, and so $A, B, C \vDash_v \Gamma$. From this and the fact that $A \vDash_v \Gamma$, by proposition 4.1 it follows that $A, C \vDash_v \Gamma$. (iii) Suppose that $A, C \vDash_v \Gamma$. Then $A \vDash_v \Gamma, \neg C$. By (b), it follows that $A, B \vDash_v \Gamma, \neg C$, and so $B, C \vDash_v \Gamma, \neg A$. By (d), it follows that $B \vDash_v \Gamma, \neg A$, and so $A, B \vDash_v \Gamma$. Finally by (b), $A \vDash_v \Gamma$. By (i), (ii) and (iii), $A \Vdash_v C$. ■

We can then conclude:

PROPOSITION 4.3. (*Soundness*) For all TFFs A and B , if $A \vdash B$, then $A \Vdash B$.

5. Completeness

Let Γ be an arbitrary non-empty, finite set of TFFs. We define the notion of a Γ -formula recursively as follows:

- All TFFs in Γ are Γ -formulas;
- If A and B are Γ -formulas, then so is $A \vee B$.

A *maximal* Γ -formula is a Γ -formula containing all TFFs in Γ (a given TFF may appear several times in a maximal Γ -formula). It is clear that in system AC, any two maximal Γ -formulas A and B are such that $A \dashv\vdash B$. We shall call this principle DISJUNCTIVE EQUIVALENCE.

To every non-empty finite set of TFFs Γ , we associate a given maximal Γ -formula we call Γ^+ (say, the first such formula according to an initial numbering of all the TFFs). We put $\Gamma^- = \{\neg A : A \in \Gamma\}^+$. In what follows, every occurrence of an expression like ' $\Gamma^- \vee A$ ' or ' $A \vee \Gamma^-$ ' should be understood as ' A ' if the set Γ is empty. And similarly, every occurrence of an expression like ' $\Delta^+ \vee A$ ' or ' $A \vee \Delta^+$ ' should be understood as ' A ' if the set Δ is empty.

For every TFF A , let $v(A)$ be the set defined as follows:

- $\langle \Gamma, \Delta \rangle \in v(A)$ iff $\Gamma \cup \Delta \neq \emptyset$, $\Gamma \cup \Delta$ is a finite set of atoms, and $A \vdash (\Gamma^- \vee \Delta^+)$.

$v(A)$ is obviously a valuation.

PROPOSITION 5.1. *For all finite sets of TFFs Γ and Δ such that $\Gamma \cup \Delta \neq \emptyset$, $\Gamma \vDash_{v(A)} \Delta$ iff $A \vdash (\Gamma^- \vee \Delta^+)$.*

PROOF. In order to prove this, it is enough to prove the following four claims:

- $A \vdash \Gamma^- \vee (\Delta, \neg B)^+ \text{ iff } A \vdash (\Gamma, B)^- \vee \Delta^+$;
- $A \vdash (\Gamma, \neg B)^- \vee \Delta^+ \text{ iff } A \vdash \Gamma^- \vee (\Delta, B)^+$;
- $A \vdash \Gamma^- \vee (\Delta, B \vee C)^+ \text{ iff } A \vdash \Gamma^- \vee (\Delta, B, C)^+$;
- $A \vdash (\Gamma, B \vee C)^- \vee \Delta^+ \text{ iff } A \vdash (\Gamma, B)^- \vee \Delta^+ \text{ and } A \vdash (\Gamma, C)^- \vee \Delta^+$.

Given (AC9) and (T10), it is enough to prove the following:

1. $\Gamma^- \vee (\Delta, \neg B)^+ \dashv\vdash (\Gamma, B)^- \vee \Delta^+$;
2. $(\Gamma, \neg B)^- \vee \Delta^+ \dashv\vdash \Gamma^- \vee (\Delta, B)^+$;
3. $\Gamma^- \vee (\Delta, B \vee C)^+ \dashv\vdash \Gamma^- \vee (\Delta, B, C)^+$;
4. $(\Gamma, B \vee C)^- \vee \Delta^+ \dashv\vdash [(\Gamma, B)^- \vee \Delta^+] \wedge [(\Gamma, C)^- \vee \Delta^+]$.

Points 1 and 3 directly follow from DISJUNCTIVE EQUIVALENCE. For the remaining two points:

- By DISJUNCTIVE EQUIVALENCE, $(\Gamma, \neg B)^- \vee \Delta^+ \dashv\vdash \neg \neg B \vee (\Gamma^- \vee \Delta^+)$. By (T18) and (AC9), it follows that $(\Gamma, \neg B)^- \vee \Delta^+ \dashv\vdash B \vee (\Gamma^- \vee \Delta^+)$. By DISJUNCTIVE EQUIVALENCE, $\Gamma^- \vee (\Delta, B)^+ \dashv\vdash B \vee (\Gamma^- \vee \Delta^+)$. The result follows from (AC9).
- By DISJUNCTIVE EQUIVALENCE, $(\Gamma, B \vee C)^- \vee \Delta^+ \dashv\vdash \neg(B \vee C) \vee (\Gamma^- \vee \Delta^+)$. By (T18), (AC9) and the definition of conjunction, it follows that $(\Gamma, B \vee C)^- \vee \Delta^+ \dashv\vdash (\neg B \wedge \neg C) \vee (\Gamma^- \vee \Delta^+)$, and thus by DISJUNCTIVE EQUIVALENCE and (AC9), $(\Gamma, B \vee C)^- \vee \Delta^+ \dashv\vdash (\Gamma^- \vee \Delta^+) \vee (\neg B \wedge \neg C)$. By (AC6a), (AC6b) and (AC9), we have then $(\Gamma, B \vee C)^- \vee \Delta^+ \dashv\vdash ((\Gamma^- \vee \Delta^+) \vee \neg B) \wedge ((\Gamma^- \vee \Delta^+) \vee \neg C)$. Now by DISJUNCTIVE EQUIVALENCE, $((\Gamma^- \vee \Delta^+) \vee \neg B) \dashv\vdash ((\Gamma, B)^- \vee \Delta^+)$ and $((\Gamma^- \vee \Delta^+) \vee \neg C) \dashv\vdash ((\Gamma, C)^- \vee \Delta^+)$. So by (T11) and (AC9), we have $(\Gamma, B \vee C)^- \vee \Delta^+ \dashv\vdash ((\Gamma, B)^- \vee \Delta^+) \wedge ((\Gamma, C)^- \vee \Delta^+)$. ■

PROPOSITION 5.2. *For all TFFs A and B , $A \vdash B$ iff $\vDash_{v(A)} B$.*

PROOF. By proposition 5.1, $\models_{v(A)} B$ iff $A \vdash \{B\}^+$. But by DISJUNCTIVE EQUIVALENCE and (AC9), $A \vdash \{B\}^+$ iff $A \vdash B$. ■

PROPOSITION 5.3. $v(A)$ is an AC-valuation.

PROOF. Suppose that $\langle \Gamma, \Delta \rangle \in v(A)$ and $\langle \Gamma' \cup \Gamma'', \Delta' \cup \Delta'' \rangle \in v(A)$. Then $\Gamma \cup \Delta \neq \emptyset$ and $\Gamma' \cup \Gamma'' \cup \Delta' \cup \Delta'' \neq \emptyset$. It also follows that

1. $A \vdash \Gamma^- \vee \Delta^+$, and
2. $A \vdash (\Gamma', \Gamma'')^- \vee (\Delta', \Delta'')^+$.

In order to prove that $\langle \Gamma \cup \Gamma', \Delta \cup \Delta' \rangle \in v(A)$, we have to establish that $A \vdash (\Gamma, \Gamma')^- \vee (\Delta, \Delta')^+$.

CASE 1: $\Gamma' \cup \Delta' = \emptyset$. The result is immediate.

CASE 2: $\Gamma'' \cup \Delta'' = \emptyset$ (and so, $\Gamma' \cup \Delta' \neq \emptyset$). By (1), (2), (T12), (T3) and (AC9), $A \vdash (\Gamma^- \vee \Delta^+) \vee (\Gamma'^- \vee \Delta'^+)$. CASE 2a: $\Gamma \cup \Gamma' \neq \emptyset$ and $\Delta \cup \Delta' \neq \emptyset$. By DISJUNCTIVE EQUIVALENCE and (AC9), it follows that $A \vdash (\Gamma, \Gamma')^- \vee (\Delta, \Delta')^+$. CASE 2b: $\Gamma \cup \Gamma' = \emptyset$ (and so, $\Delta \cup \Delta' \neq \emptyset$). Then $A \vdash \Delta^+ \vee \Delta'^+$. By DISJUNCTIVE EQUIVALENCE and (AC9), it follows that $A \vdash (\Delta, \Delta')^+$. CASE 2c: $\Delta \cup \Delta' = \emptyset$ (and so, $\Gamma \cup \Gamma' \neq \emptyset$). Then $A \vdash \Gamma^- \vee \Gamma'^-$. By DISJUNCTIVE EQUIVALENCE and (AC9), it follows that $A \vdash (\Gamma, \Gamma')^-$.

CASE 3: $\Gamma' \cup \Delta' \neq \emptyset$ and $\Gamma'' \cup \Delta'' \neq \emptyset$. From (2), DISJUNCTIVE EQUIVALENCE and (AC9), it follows that $A \vdash (\Gamma'^- \vee \Delta'^+) \vee (\Gamma''^- \vee \Delta''^+)$. From this, (1) and (T13), it follows that $A \vdash (\Gamma^- \vee \Delta^+) \vee (\Gamma'^- \vee \Delta'^+)$. The result follows from this, DISJUNCTIVE EQUIVALENCE and (AC9). ■

PROPOSITION 5.4. (Completeness) For all TFFs A and B , if $A \Vdash B$, then $A \vdash B$.

PROOF. Suppose $A \not\vdash B$. Then by proposition 5.2, $\not\models_{v(A)} B$. Now $A \vdash A$, and so by DISJUNCTIVE EQUIVALENCE and (AC9), $A \vdash \{A\}^+$. By proposition 5.2, it follows that $\models_{v(A)} A$. Thus by proposition 5.3, $A \not\vdash B$. ■

REMARK I. From this completeness result and the proof of the soundness of AC, one can deduce that $A \Vdash B$ iff for every AC-valuation v , $A \Vdash_v B$. So we might have adopted a different semantics for AC, defining ‘ A semantically contains B ’ as ‘for every AC-valuation v , $A \Vdash_v B$ ’. The original notion of semantic containment is simpler, hence my choice of it. But when dealing with Angell’s 1977 system, we shall need the more complicated notion (with restricted quantification on AC-valuations). The use of the later notion,

however, seems to be in any case quite natural. For following Angell's intuitions, which seem to be sound, p analytically contains q iff p and $p \wedge q$ carry exactly the same information. But if two sentences p and q carry the same information, then the truth of any sentence containing p implies the truth of the corresponding sentence obtained by replacing one or more occurrence of p by q , and conversely the falsity of any sentence containing p implies the falsity of the corresponding sentence obtained by replacing one or more occurrence of p by q . Now putting these intuitions together, it becomes quite natural to adopt the more complicated notion of semantic containment in order to model analytic containment. For $A \Vdash_v B$ iff for all finite sets of TFFs Γ and Δ , (a) $\Gamma \vDash_v A, \Delta$ iff $\Gamma \vDash_v A \wedge B, \Delta$, and (b) $\Gamma, A \vDash_v \Delta$ iff $\Gamma, A \wedge B \vDash_v \Delta$.

REMARK II. Let *Min* be the system whose axioms are:

- (AC6a),
- (AC6b),
- all formulas $A \rightarrow B$ with A and B maximal Γ -formulas for some set Γ , and
- all formulas $A \rightarrow A[\neg\neg B/B]$ and $A \rightarrow A[B/\neg\neg B]$,

and whose rules are:

- (AC9),
- $A \rightarrow B \wedge C / A \rightarrow B$,
- $A \rightarrow B \wedge C / A \rightarrow C$,
- $A \rightarrow B, A \rightarrow C / A \rightarrow B \wedge C$, and
- $A \rightarrow B, C \rightarrow D / A \wedge C \rightarrow B \wedge D$.

Let us say that formula A *semantically contains** formula B iff B is true in every valuation where A is true. It is quite easy to establish that if $A \rightarrow B$ is a theorem of *Min*, then A semantically contains* B . The converse holds because in the above completeness proof for AC, only principles of *Min* have been used except in the proof for proposition 5.3.

6. Getting New Consequence Relations

As we saw, Anderson and Belnap's system of tautological entailments E_{fde} is AC plus $A \rightarrow A \vee B$. Given the previous results, it is quite easy to get a characterization of E_{fde} in the present semantical framework. Let a *TE-valuation* be a valuation v which satisfies the following condition:

- For all finite sets of atoms Γ, Γ', Δ and Δ' , if $\langle \Gamma, \Delta \rangle \in v$, then $\langle \Gamma \cup \Gamma', \Delta \cup \Delta' \rangle \in v$.

This property transmits to arbitrary TFFs, i.e. if v is a TE-valuation, then if $\Gamma \vDash_v \Delta$, then $\Gamma, \Gamma' \vDash_v \Delta, \Delta'$ for all finite sets of TFFs Γ, Γ', Δ and Δ' . (A proof of this may be given following the method used in the proof of proposition 4.1.) The semantics of TE-valuations validates $A \rightarrow A \vee B$, i.e. for all TFFs A and B and every TE-valuation v , if A is true in v , then so is $A \vee B$. Given that TE-valuations are AC-valuations, E_{fde} is sound with respect to the semantics of TE-valuations. Completeness can be established by noting that $v(A)$ is a TE-valuation provided that axiom $A \rightarrow A \vee B$ is accepted.

By adding axiom $A \rightarrow B \vee \neg B$ to E_{fde} , one gets a system PC for classical consequence ($A \vdash B$ iff B is a classical consequence of A).⁶ Let a PC-valuation be a TE-valuation which satisfies the following condition:

- For every finite non-empty set of atoms Γ , $\langle \Gamma, \Gamma \rangle \in v$.

This property transmits to arbitrary TFFs, i.e. if v is a PC-valuation, then $\Gamma \vDash_v \Gamma$ for every finite non-empty set of TFFs Γ . (As above, a proof may be given following the method used in the proof of proposition 4.1.) The semantics of PC-valuations validates $A \rightarrow B \vee \neg B$: for every TFF B and every PC-valuation v , $B \vee \neg B$ is true in v . So, PC is sound with respect to the semantics of PC-valuations. Completeness can be established by noting that $v(A)$ is a PC-valuation if axiom $A \rightarrow B \vee \neg B$ is accepted.

Here is another modelling of classical consequence which involves certain valuations which will be useful in the next section. Where c is any set of atoms, we let \tilde{c} be the valuation defined by $\tilde{c} = \{ \langle \Gamma, \Delta \cup \{A\} \rangle : A \in c, \Gamma \cup \Delta \text{ finite set of atoms} \} \cup \{ \langle \Gamma \cup \{A\}, \Delta \rangle : A \notin c, \Gamma \cup \Delta \text{ finite set of atoms} \}$. It is easy to see that \tilde{c} is a PC-valuation.

PROPOSITION 6.1. *For every subset c of At , and for all TFFs A and B , $\vDash_{\tilde{c}} \neg A$ iff $\not\vDash_{\tilde{c}} A$, and $\vDash_{\tilde{c}} A \vee B$ iff $\vDash_{\tilde{c}} A$ or $\vDash_{\tilde{c}} B$.*

PROOF. (1) We first prove that for all finite sets of TFFs Γ and Δ such that $\Gamma \cup \Delta \neq \emptyset$, $\Gamma \vDash_{\tilde{c}} \Delta$ iff $\exists A \in \Gamma \ A \vDash_{\tilde{c}}$ or $\exists A \in \Delta \ \not\vDash_{\tilde{c}} A$ —which will give us the second part of proposition 6.1. Given that \tilde{c} is a PC-valuation,

⁶ One may wonder whether $B \wedge \neg B \rightarrow A$ should not also be added. This is not needed, since E_{fde} has the rule $A \rightarrow B / \neg B \rightarrow \neg A$. This rule can be shown to hold in $\text{AC} + A \rightarrow A \vee B$ as follows. Suppose $A \vdash B$. Then by the rules of AC we get $A \vee B \vdash B$, and by the rules of $\text{AC} + A \rightarrow A \vee B$ we get $B \vdash A \vee B$. So $B \vdash A \vee B$. By the rules of AC it follows that $\neg B \vdash \neg(A \vee B)$, and so $\neg B \vdash \neg A \wedge \neg B$, and so $\neg B \vdash \neg A$.

the right-to-left direction of the proposition is secured, so we just have to prove the other direction of the proposition, which we shall call \square . (a) By definition of \tilde{c} , \square holds for Γ and Δ sets of atoms. (b) Suppose that it holds for all sets Γ and Δ of degree $\leq n$, and let Γ and Δ be such that $\Gamma \cup \Delta$ is of degree $n + 1$ and $\Gamma \vDash_{\tilde{c}} \Delta$. We then decompose each of Γ and Δ into three components as in the proof of proposition 4.1, and we have:

- $\Gamma_1, \Gamma_2, \Gamma_3 \vDash_{\tilde{c}} \Delta_1, \Delta_2, \Delta_3$,

i.e., following the notational conventions of the proof of proposition 4.1:

- $\forall x \in \overrightarrow{\Gamma_3} \quad \Gamma_1, x, \overline{\Delta_2} \vDash_{\tilde{c}} \Delta_1, \widehat{\Delta_3}, \overline{\Gamma_2}$.

By induction hypothesis, it follows that at least one of the following 3 conditions hold:

1. $\exists A \in \Delta_1 \cup \widehat{\Delta_3} \cup \overline{\Gamma_2} \quad \vDash_{\tilde{c}} A$;
2. $\exists A \in \Gamma_1 \cup \overline{\Delta_2} \quad A \vDash_{\tilde{c}}$;
3. $\forall x \in \overrightarrow{\Gamma_3} \quad \exists A \in x \quad A \vDash_{\tilde{c}}$.

By (1), $\exists A \in \Delta_1 \vDash_{\tilde{c}} A$, or $\exists A \in \Delta_3 \vDash_{\tilde{c}} A$ (the fact that c is a PC-valuation can be used to prove this), or $\exists A \in \Gamma_2 \quad A \vDash_{\tilde{c}}$. By (2), $\exists A \in \Gamma_1 \quad A \vDash_{\tilde{c}}$, or $\exists A \in \Delta_2 \vDash_{\tilde{c}} A$. Finally, by (3) $\exists A \in \Gamma_3 \quad A \vDash_{\tilde{c}}$ (this result is not immediate; a proof by induction on the cardinal of Γ_3 can be used to prove it).

(2) The first part of proposition 6.1 can easily be proved by induction on the degree of A , making use of the second part of the proposition which has just been proved. ■

From this result it is clear that B is a classical consequence of A iff for every subset c of At , B is true in \tilde{c} if A is.

The semantics of valuations captures properties of the truth-functional connectives which are so general that it can be used to provide a characterization for a number of consequence relations, by imposing suitable conditions on valuations.

7. The System AC*

I shall call my version of Angell's 1977 system for analytic containment 'AC*'.

The system is formulated in a classical propositional language L with primitive operators \neg and \vee , augmented by the binary sentential operator \rightarrow

representing analytic containment. The formulas of the system are defined as follows: all the TFFs of L are formulas; if A and B are TFFs, then $A \rightarrow B$ is a formula; all truth-functional compounds of formulas are formulas. I shall use (as before) A, B, \dots for TFFs, and X, Y, \dots for arbitrary formulas. Conjunction (\wedge) and material implication (\supset) are defined as usual in terms of \neg and \vee , and $A \leftrightarrow B$ will be short for $A \rightarrow B \wedge B \rightarrow A$.

The system is defined by the following axiom schemata and rules:

- $$\begin{array}{ll}
 A \leftrightarrow \neg\neg A & (\text{AC}^*1) \\
 A \rightarrow A \wedge A & (\text{AC}^*2) \\
 A \wedge B \rightarrow A & (\text{AC}^*3) \\
 A \vee B \rightarrow B \vee A & (\text{AC}^*4) \\
 A \vee (B \vee C) \leftrightarrow (A \vee B) \vee C & (\text{AC}^*5) \\
 A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C) & (\text{AC}^*6) \\
 (A \leftrightarrow B) \supset (\neg A \rightarrow \neg B) & (\text{AC}^*7) \\
 (A \rightarrow B) \supset (A \vee C \rightarrow B \vee C) & (\text{AC}^*8) \\
 (A \rightarrow B) \supset ((B \rightarrow C) \supset (A \rightarrow C)) & (\text{AC}^*9) \\
 (A \rightarrow B) \supset (A \supset B) & (\text{AC}^*10) \\
 X, X \supset Y / Y & (\text{AC}^*11)
 \end{array}$$

AC* contains classical propositional logic as well as AC. For a comparison, Angell's original system in [2] is formulated as follows (with mutual analytic containment \leftrightarrow taken as primitive, $X \rightarrow Y$ defined as $X \leftrightarrow (X \wedge Y)$; remember that in the 1977 system, the formation rules are unrestricted, so that a formula may contain any number of nested \leftrightarrow):

- $$\begin{array}{l}
 X \leftrightarrow \neg\neg X \\
 X \leftrightarrow X \wedge X \\
 X \wedge Y \leftrightarrow Y \wedge X \\
 X \wedge (Y \wedge Z) \leftrightarrow (X \wedge Y) \wedge Z \\
 X \vee (Y \wedge Z) \leftrightarrow (X \vee Y) \wedge (X \vee Z) \\
 (X \leftrightarrow Y) \supset (\neg X \leftrightarrow \neg Y) \\
 (X \leftrightarrow Y) \supset (X \wedge Z \leftrightarrow Y \wedge Z) \\
 (X \leftrightarrow Y) \supset ((Y \leftrightarrow Z) \supset (X \leftrightarrow Z)) \\
 (X \leftrightarrow Y) \supset (Y \leftrightarrow X) \\
 (X \rightarrow Y) \supset (X \supset Y) \\
 X, X \supset Y / Y
 \end{array}$$

Let a *model* be any pair $\langle c, \mathcal{V} \rangle$, where c is a subset of At and \mathcal{V} is a set of AC-valuations containing \tilde{c} (see the previous section for the definition of \tilde{c}). Where $m = \langle c, \mathcal{V} \rangle$ is a model, truth in m —for short, \models_m —is defined as follows:

1. For A an atom, $\models_m A$ iff $A \in c$;
2. $\models_m \neg X$ iff $\not\models_m X$;
3. $\models_m X \vee Y$ iff $\models_m X$ or $\models_m Y$;
4. $\models_m A \rightarrow B$ iff for every v in \mathcal{V} , $A \Vdash_v B$.

(Reminder: $A \Vdash_v B$ iff for every set of TFFs Γ , (a) if $\Gamma \models_v A$, then $\Gamma \models_v B$, and (b) $A \models_v \Gamma$ iff $A, B \models_v \Gamma$.) The clauses for conjunction and material implication are then, unsurprisingly:

- $\models_m X \wedge Y$ iff $\models_m X$ and $\models_m Y$;
- $\models_m X \supset Y$ iff $\not\models_m X$ or $\models_m Y$.

We shall say that X is *valid*—for short, $\models X$ —iff X is true in every model.

From proposition 6.1, it follows that:

PROPOSITION 7.1. *For every model $m = \langle c, \mathcal{V} \rangle$ and every TFF A , $\models_m A$ iff $\models_{\tilde{c}} A$.*

Given the proof of the soundness of AC, it is clear that axioms (AC*1)–(AC*9) are valid. It is also obvious that rule (AC*11) preserves validity. The soundness of AC* with respect to the proposed semantics follows from the fact that axiom (AC*10) is valid, which follows from proposition 7.1. (For suppose that $A \rightarrow B$ is true in model $m = \langle c, \mathcal{V} \rangle$. Then since $\tilde{c} \in \mathcal{V}$, $\not\models_{\tilde{c}} A$ or $\models_{\tilde{c}} B$, and so by proposition 7.1, $\not\models_m A$ or $\models_m B$, i.e. $A \supset B$ is true in m .) Hence:

PROPOSITION 7.2. *(Soundness) Every theorem of AC* is valid.*

For completeness, suppose that formula X_0 is not a theorem of the system, and let $@$ be a maximal consistent extension of $\{\neg X_0\}$. Where A is any TFF, we let $v(A)$ be the set defined as follows:

- $\langle \Gamma, \Delta \rangle \in v(A)$ iff $\Gamma \cup \Delta \neq \emptyset$, $\Gamma \cup \Delta$ is a finite set of atoms, and $A \rightarrow (\Gamma^- \vee \Delta^+) \in @$.

Each $v(A)$ is obviously a valuation, and we can prove that it is indeed an AC-valuation (adapt the proof of proposition 5.3). We then put $c = At \cap @$, $\mathcal{V} = \{v(A) : A \in L\} \cup \{\tilde{c}\}$, and $m = \langle c, \mathcal{V} \rangle$. m is then a model.

By a classical result, for all formulas X and Y , $\neg X \in @$ iff $X \notin @$, and $X \vee Y \in @$ iff $X \in @$ or $Y \in @$. Moreover:

PROPOSITION 7.3. *For all TFFs A and B , $A \rightarrow B \in @$ iff for every v in \mathcal{V} , $A \Vdash_v B$.*

PROOF. Adapting the proof of proposition 5.1, we can prove that for all finite sets of TFFs Γ and Δ such that $\Gamma \cup \Delta \neq \emptyset$, and every TFF C , $\Gamma \vDash_{v(C)} \Delta$ iff $C \rightarrow (\Gamma^- \vee \Delta^+) \in @$. (1) Suppose that $A \rightarrow B \in @$, and let v be in \mathcal{V} . Then if v is \tilde{c} , $A \Vdash_v B$ can be proved using proposition 6.1. Suppose now that v is $v(C)$ for some TFF C . $A \Vdash_v B$ can be proved using the fact that both $(A \rightarrow B) \wedge (C \rightarrow D \vee A) \supset (C \rightarrow D \vee B)$ and $(A \rightarrow B) \wedge (C \rightarrow \neg A \vee D) \supset (C \rightarrow (\neg A \vee \neg B) \vee D)$ are theorems of AC^* . (2) Suppose now that $A \rightarrow B \notin @$. Then $\not\vDash_{v(A)} B$, and since $\vDash_{v(A)} A$ by the theoremhood of $A \rightarrow A$, it follows that it is not the case that $A \Vdash_{v(A)} B$. ■

Thus we can establish that for every formula X , $\vDash_m X$ iff $X \in @$. Since $\neg X_0 \in @$, $\not\vDash_m X_0$, and so X_0 is not valid. So since X_0 was arbitrary:

PROPOSITION 7.4. *(Completeness) Every valid formula is a theorem of AC^* .*

Concluding Remark

Interesting developments of the previous material should go in at least the following two directions: (i) building systems for analytic containment in which formulas with nested arrows are taken into account, and specific axioms about such formulas are added, and (ii) getting a better understanding of the kind of valuations which have been introduced in this paper.

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