Published in Proceedings of the Cambridge Philosophical Society 69, 315-318, 1969
which should be used for any reference to this work

Quantum mechanics and the gravitational red shift

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Abstract. It is shown that the formula for the gravitational red shift predicted by the theory of general relativity can also be derived by classical quantum mechanics combined with relativistic arguments. The agreement between the two derivations is a consequence of the separability of the time-dependent wave function, and of the first-order time differential in the wave equation.

1. *Introduction.* The gravitational red shift is one of the three classical tests of general relativity. It relates the frequencies ν and ν' of two identical clocks placed in different gravitational fields, and observed in one of them. If they are observed in a weak field, where ν is created, the red shift is given by

$$\nu' = \nu\left(1-\frac{\gamma M}{c^2R_0}\right),$$

or

$$\Delta\nu = \nu'-\nu = -\frac{\gamma M}{c^2R_0}\nu, \tag{1}$$

where
γ is the universal gravitational constant,
M the mass creating the gravitational field,
R_0 the distance from M at which ν' is emitted,
c the speed of light.

In the classical derivation of equation (1) by Einstein (1), prior to the development of quantum mechanics, and in subsequent expositions (Tolman (2) and Møller (3), for example), one uses only geometrical transformations of first-order time differentials, and nothing is said about the systems themselves. Although relation (1) has been confirmed experimentally, it seems that for the sake of consistency between general relativity and quantum mechanics, one should be able to derive the red-shift formula by means of an argument dealing with the atomic systems involved.

2. *The Schwarzschild metric and local frames of observation.* An empty space with a gravitational mass M at the origin has a line element known as the Schwarzschild metric

$$ds^2 = c^2\left(1-\frac{2m}{R}\right)dt_0^2-\left(1-\frac{2m}{R}\right)^{-1}dR^2-R^2(d\theta^2+\sin^2\theta\, d\phi^2) \tag{2}$$

in polar coordinates, and where $m = \gamma M/c^2$. As shown in Tolman (2), this metric can also be written in isotropic coordinates:

$$ds^2 = c^2\left(1-\frac{2m}{R}\right)dt_0^2-\left(1+\frac{2m}{R}\right)(dx_0^2+dy_0^2+dz_0^2), \tag{3}$$

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where $$R = \sqrt{(x_0^2+y_0^2+z_0^2)}.$$

Relation (3) holds for distances R such that $(m/R)^2$ and higher terms may be neglected, and it is more practical for the following analysis than the usual expression (2) in polar coordinates.

Consider now a local coordinate system (x, y, z, t) at a point P_1, itself at a distance R_0 from the centre of M.

If x, y and $z \ll R_0$, then $\{1 \pm (2m/R_0)\} = cte$ near P_1 and the metric becomes

$$ds_1^2 = c^2\left(1-\frac{2m}{R_0}\right)dt^2 - \left(1+\frac{2m}{R_0}\right)(dx^2+dy^2+dz^2). \tag{4}$$

If P_1 goes to infinity, we have another metric

$$ds_2^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{5}$$

and using the transformations

$$\left.\begin{aligned} dt'^2 &= \left(1-\frac{2m}{R_0}\right)dt^2,\\ dx'^2 &= \left(1+\frac{2M}{R_0}\right)dx^2,\\ dy'^2 &= \left(1+\frac{2m}{R_0}\right)dy^2,\\ dz'^2 &= \left(1+\frac{2m}{R_0}\right)dz^2, \end{aligned}\right\} \tag{6}$$

the metric (4) becomes

$$ds_1^2 = c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2. \tag{7}$$

Expressions (7) and (5) are now the metrics of two local Euclidean spaces tangent to the general relativistic space at distances R_0 and infinity. The scaling factors for their coordinates can be obtained from relations (6).

3. *Physical situation and quantized systems.* Let the mass M be the sun (radius R_0) and P_1 a point on its surface. Consider two identical atomic systems S_1 and S_2 located at P_1 and on the earth, respectively. All observations are made near reference system S_2. In view of the astronomical and atomic distances involved, we may assume that (*a*) the earth is practically the point at infinity, and (*b*) the atomic systems S_1 and S_2 may be described in the locally tangent Euclidean spaces.

The quantized systems S_1 and S_2 obey Schrödinger's time-dependent wave equation, and so we have

$$H_1\psi(x', y', z', t') = i\hbar\frac{\partial}{\partial t'}\psi(x', y', z', t') \tag{8}$$

for S_1 on the sun, and

$$H_2\psi(x, y, z, t) = i\hbar\frac{\partial}{\partial t}\psi(x, y, z, t) \tag{9}$$

for S_2 on the earth. Also note that

$$\left.\begin{aligned} H_1 &= H(x', y', z'),\\ H_2 &= H(x, y, z). \end{aligned}\right\} \tag{10}$$

Let us first concentrate on reference system S_2. Assuming that $\psi(x,y,z,t)$ is separable, i.e. that

$$\psi(x,y,z,t) = \psi(x,y,z).f(t) \tag{11}$$

(9) becomes

$$H_2\psi(x,y,z) = i\hbar\psi(x,y,z)\frac{1}{f(t)}\frac{\partial}{\partial t}f(t) \tag{12}$$

and with

$$f(t) = \exp(-iEt/\hbar) \tag{13}$$

we get the classical relation

$$H_2\psi_n(x,y,z) = E_n\psi_n(x,y,z). \tag{14}$$

For S_1 we get in a similar way

$$\psi(x',y',z',t') = \psi(x',y',z')f(t') = \psi(x',y',z').F(t) \tag{15}$$

and

$$H_1\psi(x',y',z') = i\hbar\psi(x',y',z')\frac{1}{F(t)}\frac{\partial}{\partial t'}F(t), \tag{16}$$

if we want to observe S_1 in the time coordinate t of S_2. Using relation (6) we get

$$\frac{\partial}{\partial t'} = \frac{1}{\{1-(2m/R_0)\}^{\frac{1}{2}}}\frac{\partial}{\partial t} \tag{17}$$

and with

$$F(t) = \exp(-iE't/\hbar) \tag{18}$$

we have

$$H_1\psi_n(x',y',z') = \frac{E'_n}{\{1-(2m/R_0)\}^{\frac{1}{2}}}\psi_n(x',y',z'). \tag{19}$$

E'_n represents now the energy eigenvalues of S_1, as seen in the time coordinate of the reference system S_2. Comparing (14) and (19), and in view of (10), we see that

$$E'_n = E_n\left(1-\frac{2m}{R_0}\right)^{\frac{1}{2}}. \tag{20}$$

This expression is independent of the Hamiltonian of the atomic system considered.

It follows from (20) that

$$\Delta E'_n = \Delta E_n\left(1-\frac{2m}{R_0}\right)^{\frac{1}{2}} \simeq \Delta E_n\left(1-\frac{m}{R_0}\right) \tag{21}$$

and consequently

$$\nu' = \nu\left(1-\frac{m}{R_0}\right),$$

$$\Delta\nu = \nu'-\nu = -\frac{\gamma M}{c^2R_0}\nu. \tag{22}$$

This is precisely the red-shift formula (1), obtained in a different way.

4. *Conclusions.* We see from the previous section that the agreement between the two derivations of the red-shift formula is essentially due to two factors in the quantum mechanical approach:

(*a*) the assumption that the time-dependent wave function $\psi(x,y,z,t)$ is separable, and

(*b*) the presence of a first-order time differential only, in Schrödinger's equation. As mentioned earlier, the general relativistic derivation uses first-order time differentials to get to the red-shift formula (1).

The first condition also ensures that the red-shift formula is independent of the Hamiltonian of the system considered, as it is implicitly assumed in the classical derivation. It is interesting to note that if we had used Dirac's equation instead of Schrödinger's equation in section 3, the red-shift formula would have been the same, because both equations contain the first-order time differential.

The author wishes to express his gratitude to Professor C. A. Coulson, F.R.S. and to Dr H. Davies, of the Mathematical Institute Oxford, for helpful comments on the manuscript, and also to the Fondation des Bourses en Chimie (Basle, Switzerland) for financial support.

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