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Priorean Strict Implication, Q and Related Systems*

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Abstract. We introduce a system PSI for a strict implication operator called *Priorean strict implication*. The semantics for PSI is based on partial Kripke models without accessibility relations. PSI is proved sound and complete with respect to that semantics, and Prior's system Q and related systems are shown to be fragments of PSI or of a mild extension of it.

Keywords: strict implication, system Q, partial Kripke models

1. Introduction

In standard Kripkean modal semantics, the following two clauses are equivalent:

(c_1) for every world v accessible from w, $\neg A \vee B$ is true at v;

(c_2) for every world v accessible from w, A is not true at v, or B is true at v.

They are indeed equivalent truth-clauses at world w for the formula $\Box(\neg A \vee B)$.

In some non-standard modal semantics this equivalence does not hold. Suppose for instance that we interpret some modal language L by means of *partial* Kripke models, i.e. by means of triples $\langle \mathcal{W}, R, V \rangle$, where $\langle \mathcal{W}, R \rangle$ is a standard frame and V is a partial function from $atoms(\mathsf{L}) \times \mathcal{W}$ to $\{0, 1\}$. Then consider such a partial Kripke model $\langle \{w, v\}, R, V \rangle$, where $R = \{\langle w, v \rangle\}$, and let A and B be two atoms such that A is undefined at v and $V(B, v) = 0$. Assuming that a negation is undefined if the negated formula is itself undefined, and that a disjunction is undefined if one disjunct is undefined and the other is false, $\neg A \vee B$ will be undefined at v. In that case, neither A nor $\neg A \vee B$ will be true at v, and consequently, (c_1) will be false and (c_2) true.

In such partial contexts, then, (c_1) and (c_2) define two distinct strict implication operators. The first is definable by means of negation, disjunction

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and a necessity operator $\Box$ with the standard truth-clause $\ulcorner\Box A$ is true at w iff A is true at every world accessible from $w\urcorner$. But the second is not so definable. So, assuming you already have a logic for a necessity operator interpreted by the standard truth-clause in a given class of partial Kripke models, there remains the question as to determining the logic for a strict implication operator interpreted by a truth-clause like (c_2) above in the same class of models.

There are at least two reasons why the study of strict implication operators with a (c_2)-like truth-clause is interesting. First, it is a "natural" truth-clause for necessary implication: intuitively, A necessarily implies B when B is true whenever A is true. Second, languages with such an operator have great expressive power. For instance, assuming that $\Rightarrow$ is interpreted by a clause like (c_2), the standard necessity operator $\Box$ is definable by $\Box A =_{\text{df}} t \Rightarrow A$ — with t a primitive truth constant (t is defined to be true at every world of every model).

There are many kinds of partial Kripke semantics. For given a language whose formulas are built up out of a given set of atoms, partial Kripke models provide interpretations for the atoms only; so, there remains the task of providing truth-clauses for the operators of the language, and even for the usual truth-functional connectives, we have several possibilities at disposal. In this paper, of course, I will not explore all these possibilities. Rather, I shall focus on a kind of partial Kripke semantics – the *Q-like semantics* – among which figures a semantics for Arthur Prior's system Q.[1]

The structure of the paper is the following. In section 2, I introduce the Q-like semantics. In section 3, I present system PSI, a system for a strict implication operator interpreted according to a truth-clause of type (c_2) in a Q-like semantics. The soundness and completeness of PSI is established in section 4. In section 5, it is shown that Prior's system Q and related systems presented in (Correia, 1999) are fragments of PSI or PSI^t (PSI augmented by a truth-constant t), with respect to natural translations. And finally in the last section, a variety of strict implication operators definable within PSI and PSI^t are presented and compared, and it is shown how PSI can be made to collapse into a standard S5-type system for strict implication.

[1] As we shall see in the next section, in Q-like semantics the truth-functional connectives are interpreted according to weak Kleene matrices. For a different kind of partial Kripke semantics, see (Thijsse, 1990) and (Jaspars and Thijsse, 1996), where the truth-functional connectives are interpreted according to the *strong* Kleene matrices.

2. Q-like Semantics

In this section, I define Q-like semantics in quite a general way, and I introduce some definitions and conventions which will be useful later on.

Q-like semantics are defined for languages of a certain type – which we shall simply call *languages*.

The vocabulary of a language consists of (i) a denumerable set of propositional letters called *atoms*, (ii) a set of propositional operators, among which are $\neg$ (negation) and $\wedge$ (conjunction), and (maybe) (iii) the true propositional letter or truth constant t (which is not counted as an atom). The formulas of a language are defined recursively in the usual way. The letters A, B, $\ldots$ (resp. p, q, $\ldots$) are always used for formulas (resp. atoms) of a language.

Given any language and formulas A and B of that language, we shall use

- $A \vee B$ for $\neg(\neg A \wedge \neg B)$,
- $A \rightarrow B$ for $\neg A \vee B$,
- $A \leftrightarrow B$ for $(A \rightarrow B) \wedge (B \rightarrow A)$, and
- $\widehat{A}$ for $A \vee \neg A$.

For the rest of this section, L is a fixed language whose set of atoms is at.

A *Q-model* for L is defined as a pair $\langle \mathcal{W}, V \rangle$, where $\mathcal{W}$ (worlds) is a non-empty set and V (valuation) is a partial function from $at \times \mathcal{W}$ to $\{0, 1\}$ – that is, a total function from some subset of $at \times \mathcal{W}$ to $\{0, 1\}$. Thus, Q-models are partial Kripke models as defined in the introduction, but without accessibility relations.

Let $\mathcal{M} = \langle \mathcal{W}, V \rangle$ be a Q-model for L and let w be in $\mathcal{W}$. A is said to be *defined* at w in $\mathcal{M}$ – in short: $d(A, w, \mathcal{M})$ – iff for every atom p in A, $V(p, w) = 0$ or $V(p, w) = 1$. Mention of the model will often be omitted, and $\ulcorner d(A, B, w) \urcorner$ will be used for $\ulcorner d(A, w)$ & $d(B, w) \urcorner$.

Under the intended interpretation of the semantics, being defined at a world is equivalent to being true or false at that world. Thus, the idea is that a complex formula is true or false at a world iff all its constituent formulas are true or false at that world. With this in mind, we see that negation and conjunction must be interpreted according to the weak Kleene matrices:

$\neg$	
0	1
1	0
2	2

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

where 0 is for falsehood, 1 for truth and 2 for undefinedness – or as Prior would say, for “unstatability”.

Relative to a Q-model $\langle \mathcal{W}, V \rangle$ for L, the truth-predicate $\vDash$ is defined recursively in the usual manner. In conformity with the previous matrices, the clauses for the atoms, negation and conjunction at $w \in \mathcal{W}$ are required to be the following:

– $w \vDash p$ iff $V(p, w) = 1$;
– $w \vDash \neg A$ iff $d(A, w)$ and $w \nvDash A$;
– $w \vDash (A \wedge B)$ iff $w \vDash A$ and $w \vDash B$.

We have then:

– $w \vDash (A \vee B)$ iff $d(A, B, w)$ and ($w \vDash A$ or $w \vDash B$);
– $w \vDash (A \rightarrow B)$ iff $d(A, B, w)$ and ($w \nvDash A$ or $w \vDash B$);
– $w \vDash (A \leftrightarrow B)$ iff $d(A, B, w)$ and
  ($w \nvDash A$ or $w \vDash B$) and ($w \nvDash B$ or $w \vDash A$).

If the language contains the truth-constant t, we add the further clause:

– $w \vDash t$.

If L contains any further n-ary operator $\circ$, then we require that its truth clause be such that $\circ A_1 ... A_n$ is true at world w only if $\circ A_1 ... A_n$ is defined at w, i.e. only if A_1, dots, and A_n are all defined at w. Given this assumption and the truth-clause for negation, one can then prove that:

– $d(A, w)$ iff ($w \vDash A$ or $w \vDash \neg A$),

which is in agreement with the intended interpretation of the semantics. (The left-to-right direction is a consequence of the clause for negation, and the proof for the other direction proceeds by induction on the structure of A.)

We close this presentation of Q-like semantics with the definitions of *Q-satisfiability* and *Q-validity*: a set Γ of formulas is Q-satisfiable iff there is a Q-model and a world w of that Q-model such that $w \vDash A$ for every $A \in \Gamma$; and a formula A is Q-valid iff $\{\neg A\}$ is not Q-satisfiable.

3. System PSI

The vocabulary of the language of PSI is given by (i) a denumerable set *at* of propositional letters (the atoms), and (ii) the operators $\neg$ (negation), $\wedge$ (conjunction) and $\Rightarrow$ (Priorean strict implication). We shall use

– $\Box A$ for $\widehat{A} \Rightarrow A$, and
– $\Diamond A$ for $\neg \Box \neg A$.

This language is given a Q-like semantics. The clause for $\Rightarrow$ is just:

– $w \models A \Rightarrow B$ iff $d(A, B, w)$ and $\forall v \in \mathcal{W}, v \not\models A$ or $v \models B$.

We have then:

– $w \models \Box A$ iff $d(A, w)$ and $\forall v \in \mathcal{W}\ v \not\models \neg A$;

– $w \models \Diamond A$ iff $d(A, w)$ and $\exists v \in \mathcal{W}\ v \models A$.

System PSI is defined by the following axioms and rules (here and elsewhere, at(A) is the set of all atoms in formula A):

Axioms:	
Every propositional tautology	
$(A \Rightarrow B \wedge B \Rightarrow C) \rightarrow (A \Rightarrow C)$	Transitivity
$(A \Rightarrow B \wedge A \Rightarrow C) \rightarrow [A \Rightarrow (B \wedge C)]$	Conjunction
$(A \Rightarrow C \wedge B \Rightarrow C) \rightarrow [(A \vee B) \Rightarrow C]$	Disjunction
$(A \Rightarrow B) \rightarrow (A \rightarrow B)$	**T**
$\Diamond(A \Rightarrow B) \rightarrow (A \Rightarrow B)$	**E**
$(A \wedge \neg A) \Rightarrow B$	Left-Irrelevance
Rules:	
$A, A \rightarrow B/B$	Modus ponens
$A \rightarrow B/A \Rightarrow B$, if at$(B) \subseteq$ at(A)	Necessitation

Of course, the labels **T**, **E** and Necessitation are used because of the analogy with their standard references.

4. Soundness and Completeness

THEOREM 1 (Soundness). *Every theorem of PSI is Q-valid.*

PROOF. It is easy to prove that every axiom of PSI is Q-valid. The following argument shows that both Modus ponens and Necessitation preserve Q-validity.

(A) Let $A \rightarrow B$ be Q-valid, and such that at$(B) \subseteq$ at(A). Let $\mathcal{M} = \langle \mathcal{W}, V \rangle$ be a Q-model with $v \in \mathcal{W}$. We want to show that $v \not\models \neg(A \Rightarrow B)$. This is trivially true if $A \Rightarrow B$ is not defined at v. So suppose that $A \Rightarrow B$ is defined at v, i.e. that $d(A, B, v)$. Now let w be any world in $\mathcal{W}$ such that $w \models A$. Since all atoms in B are in A, $d(A, B, w)$. So, since $A \rightarrow B$ is

Q-valid, $w \vDash A \to B$. Now given that $w \vDash A$, it follows that $w \vDash B$. Thus, $v \vDash A \Rightarrow B$. Consequently, $v \nvDash \neg(A \Rightarrow B)$.

(B) Let A and $A \to B$ be Q-valid, and let $\mathcal{M} = \langle \mathcal{W}, V \rangle$ be a Q-model with $v \in \mathcal{W}$. We want to show that $v \nvDash \neg B$. If B is not defined at v, this is trivially the case. Suppose that B is defined at v. We want to prove that B is true at v in $\mathcal{M}$. Suppose first that every atom in A is in B. Then both A and $A \to B$ are defined at v. So, since both A and $A \to B$ are Q-valid, they both are true at v in $\mathcal{M}$. And by the properties of truth-at-a-world, so is B. Suppose now that some atoms in A are not in B, and let $p_1, \ldots, p_n$ be these atoms. Consider the Q-model $\mathcal{N} = \langle \mathcal{W}, V' \rangle$ whose valuation V' is defined by:

- $V'(p_i, v) = 1$ for every $i \in \{1, \ldots, n\}$;
- $V'(p, w) = V(p, w)$ if $w \neq v$ or p is not one of $p_1, \ldots, p_n$.

One can then show (by induction on the length of C) that for every formula C whose atoms are all in B: $\forall w \in \mathcal{W}$, $\mathcal{N}, w \vDash C$ iff $\mathcal{M}, w \vDash C$. As a consequence, $\mathcal{N}, v \vDash B$ iff $\mathcal{M}, v \vDash B$. Now by construction, both A and $A \to B$ are true at v in $\mathcal{N}$, and so by the properties of truth-at-a-world, so is B. By the previous result, it follows that B is true at v in $\mathcal{N}$. ■

Useful for the completeness proof will be the:

PROPOSITION 1.

1. $\vdash A \Rightarrow A$
2. $\vdash \Box\neg A \to (A \Rightarrow B)$
3. $\vdash (A \Rightarrow B) \to \Box(A \Rightarrow B)$
4. *If* $\vdash A$, *then* $\vdash \Box A$
5. $\vdash (A \Rightarrow B) \to (\Diamond A \to \Diamond B)$
6. $\vdash \Diamond(A \wedge B) \to (\Diamond A \wedge \Diamond B)$
7. $\vdash \Diamond(A \wedge \widehat{B}) \to [\Diamond(A \wedge B) \vee \Diamond(A \wedge \neg B)]$
8. $\vdash (A \Rightarrow C) \to [(A \wedge B) \Rightarrow C]$
9. $\vdash [(A \Rightarrow B) \wedge (A' \Rightarrow B')] \to [(A \wedge A') \Rightarrow (B \wedge B')]$
10. $\vdash [((A \wedge B) \Rightarrow C) \wedge ((A \wedge B') \Rightarrow C)] \to [(A \wedge (B \vee B')) \Rightarrow C)]$
11. $\vdash [\Box(A \to B) \wedge A \Rightarrow \widehat{B}] \to (A \Rightarrow B)$
12. $\vdash A \Rightarrow \widehat{p}$, *if* $p \in \mathrm{at}(A)$
13. $\vdash (\widehat{p}_1 \wedge \cdots \wedge \widehat{p}_n) \Rightarrow \widehat{B}$, *if* $\mathrm{at}(B) = \{p_1, \ldots, p_n\}$

Its proof, though sometimes painful, does not present any special difficulty. It is omitted.

The canonical constructions in standard modal logics are based on the notion of maximal consistent sets of formulas. The property of maximality for sets in a canonical model corresponds in the semantics to the fact that every formula is either true or false at every world of every model. In the present context, maximality is too strong. We need a finer notion, that of *relative-maximality.* A set Γ of formulas will be said to be relatively maximal (*r-maximal* for short) when for every A such that $\mathrm{at}(A) \subseteq \mathrm{at}(\Gamma)$, $A \in \Gamma$ or $\neg A \in \Gamma$. ($\mathrm{at}(\Gamma)$ is defined to be the set of all atoms in all formulas in Γ.) And where Γ is a set of formulas and Σ a set of atoms, Γ will be said to be *Σ-maximal* when it is r-maximal and $\mathrm{at}(\Gamma) = \Sigma$.

Propositions 2, 3 and 4 below actually hold for any system containing classical propositional logic. Their proofs are easy and omitted.

PROPOSITION 2. *Let Γ be a consistent set of formulas, and let Σ be a set of atoms such that* $\mathrm{at}(\Gamma) \subseteq \Sigma$. *Then Γ has an extension which is both consistent and Σ-maximal. (The definition of consistency is standard: Γ is consistent iff it is not inconsistent; and Γ is inconsistent iff there are A_1, ..., A_n in Γ such that $\vdash \neg(A_1 \wedge \cdots \wedge A_n)$.)*

PROPOSITION 3. *For every r-maximal consistent set of formulas Γ:*

1. *If $\vdash A$ and* $\mathrm{at}(A) \subseteq \mathrm{at}(\Gamma)$, *then $A \in \Gamma$;*
2. *If $A \in \Gamma$ and $(A \to B) \in \Gamma$, then $B \in \Gamma$;*
3. *If $\vdash (A \to B)$ and $A \in \Gamma$ and* $\mathrm{at}(B) \subseteq \mathrm{at}(\Gamma)$, *then $B \in \Gamma$.*

PROPOSITION 4. *For every r-maximal consistent set of formulas Γ:*

1. *$\neg A \in \Gamma$ iff* $\mathrm{at}(A) \subseteq \mathrm{at}(\Gamma)$ *and $A \notin \Gamma$;*
2. *$(A \wedge B) \in \Gamma$ iff A, $B \in \Gamma$.*

Now, let α be an arbitrary consistent set of formulas, and let @ be a maximal (i.e. relative to the whole language) consistent extension of it. The aim is to build a Q-model on the basis of @, one of whose worlds is @, and such that relative to that Q-model, @ makes true all the members of α. The worlds of this Q-model will be all the r-maximal, consistent and *closed* sets of formulas. The most difficult part of the completeness proof will be to prove the crucial Proposition 14 below. It requires many steps to go through.

Where Γ is a set of formulas, the *closure* of Γ, $c\Gamma$, is the set $\{B : \exists A_1, \ldots, A_n \in \Gamma\ (A_1 \wedge \cdots \wedge A_n) \Rightarrow B \in @\}$. We shall say that Γ is *closed* when $c\Gamma \subseteq \Gamma$. It is easy to prove that Γ is closed iff if $A_1, \dots, A_n \in \Gamma$ and $(A_1 \wedge \cdots \wedge A_n) \Rightarrow B \in @$, then $B \in \Gamma$. So in particular, @ is closed. The closure operator c has the usual properties: $\Gamma \subseteq c\Gamma$; if $\Gamma \subseteq \Lambda$, then $c\Gamma \subseteq c\Lambda$; $\bigcup_{i \in I} c\Gamma_i \subseteq c \bigcup_{i \in I} \Gamma_i$; $c\Gamma$ is closed.

PROPOSITION 5. *Let Γ be a closed set of formulas. Then:*

1. *If* $\mathrm{at}(A) \subseteq \mathrm{at}(\Gamma)$*, then* $\widehat{A} \in \Gamma$*;*

2. *If* $\Box A \in @$ *and* $\mathrm{at}(A) \subseteq \mathrm{at}(\Gamma)$*, then* $A \in \Gamma$*.*

PROOF. 1. Let p be an atom such that $p \in \mathrm{at}(\Gamma)$. Then for some formula B, $p \in \mathrm{at}(B)$ and $B \in \Gamma$. By Proposition 1.12 and the closure of Γ, we have then $\widehat{p} \in \Gamma$. Now suppose $\mathrm{at}(A) \subseteq \mathrm{at}(\Gamma)$, and let $p_1, \dots, p_n$ be all the atoms in A. Then by the previous result and the closure of Γ, $(\widehat{p}_1 \wedge \cdots \wedge \widehat{p}_n) \in \Gamma$. So, by Proposition 1.13 and the closure of Γ, we have: $\widehat{A} \in \Gamma$.

2. By 1 above and the closure of Γ. ■

PROPOSITION 6. *Let Γ be a set of formulas such that $A \vee B \in \Gamma$. If $C \in c(\Gamma \cup \{A\})$ and $C \in c(\Gamma \cup \{B\})$, then $C \in c\Gamma$.*

PROOF. Suppose that $C \in c(\Gamma \cup \{A\})$ and $C \in c(\Gamma \cup \{B\})$. Then by the definition of closure and Proposition 1.8, we can find $D_1, \dots, D_n \in \Gamma$ such that $(D_1 \wedge \cdots \wedge D_n \wedge A) \Rightarrow C \in @$ and $(D_1 \wedge \cdots \wedge D_n \wedge B) \Rightarrow C \in @$. By Proposition 1.10, it follows that $(D_1 \wedge \cdots \wedge D_n \wedge (A \vee B)) \Rightarrow C \in @$. Now suppose that $A \vee B \in \Gamma$. Then by closure $C \in c\Gamma$. ■

A set Γ of formulas will be said to be *@-compatible* when for all $A_1, \dots, A_n$ in Γ, $\Diamond(A_1 \wedge \cdots \wedge A_n) \in @$.

PROPOSITION 7. *Every @-compatible set of formulas is consistent, and its closure is @-compatible.*

PROOF. To show that every @-compatible set of formulas is consistent, use Proposition 1.4. To show that the closure of every @-compatible set of formulas is @-compatible, let Γ be an @-compatible set of formulas, and suppose for a *reductio* that $c\Gamma$ is not @-compatible. Then we can find $A^1, \dots, A^n$ in $c\Gamma$ such that $\neg\Diamond(A^1 \wedge \cdots \wedge A^n) \in @$. Let $(A^1_i)_{1 \le i \le s(1)}, \dots, (A^n_i)_{1 \le i \le s(n)}$ be n families of elements of Γ such that for every k with $1 \le k \le n$, $(A^k_1 \wedge \cdots \wedge A^k_{s(k)}) \Rightarrow A^k \in @$. Let B be $[(A^1_1 \wedge \cdots \wedge A^1_{s(1)}) \wedge \cdots \wedge (A^n_1 \wedge \cdots \wedge A^n_{s(n)})]$, and let C be $(A^1 \wedge \cdots \wedge A^n)$. By Proposition 1.9, $B \Rightarrow C \in @$. So, by Proposition 1.5, it follows that $\neg\Diamond B \in @$. As a consequence, Γ is not @-compatible. Contradiction. So, $c\Gamma$ is @-compatible. ■

Proposition 8. *Let Γ be @-compatible, and suppose that $\widehat{A} \in \Gamma$. Then one of $\Gamma \cup \{A\}$ or $\Gamma \cup \{\neg A\}$ is @-compatible.*

Proof. Suppose both $\Gamma \cup \{A\}$ and $\Gamma \cup \{\neg A\}$ are not @-compatible. Then by the definition of @-compatibility and Proposition 1.6, we can find B_1, $\dots$, B_n in Γ such that $\neg\Diamond(B_1 \wedge \cdots \wedge B_n \wedge A) \in @$ and $\neg\Diamond(B_1 \wedge \cdots \wedge B_n \wedge \neg A) \in @$. Now, assuming that Γ is @-compatible and $\widehat{A} \in \Gamma$, we have $\Diamond(B_1 \wedge \cdots \wedge B_n \wedge A) \in @$. So by Proposition 1.7, $\Diamond(B_1 \wedge \cdots \wedge B_n \wedge A) \in @$ or $\Diamond(B_1 \wedge \cdots \wedge B_n \wedge \neg A) \in @$. Contradiction. ■

Where Γ and Λ are sets of formulas, Γ is a *friend of* Λ iff for every formula A, if $\widehat{A} \in \Lambda$, then $A \in \Gamma$ or $\neg A \in \Gamma$; and Γ is a *good friend of* Λ iff Γ is an @-compatible friend of Λ such that $\Lambda \subseteq \Gamma$.

Proposition 9. *Let $(\Gamma_n)_{n\in\mathbb{N}}$ be a series of sets of formulas and let Λ be $\bigcup_{n\in\mathbb{N}} \Gamma_n$. Consider the following 5 conditions:*

A. *$(\Gamma_n)_{n\in\mathbb{N}}$ is increasing (i.e each member of the series is a subset of its successor);*

B. *Each member of $(\Gamma_n)_{n\in\mathbb{N}}$ is @-compatible;*

C. *For every integer n, $c\Gamma_n \subseteq \Gamma_{n+1}$;*

D. *For every integer n, Γ_{n+1} is a friend of Γ_n;*

E. *For every integer n, Γ_{n+1} is a good friend of $c\Gamma_n$.*

Then:

1. *If* **A**, *then $c\Lambda = \bigcup_{n\in\mathbb{N}} c\Gamma_n$;*
2. *If* **A** *and* **B**, *then Λ is @-compatible;*
3. *If* **C**, *then Λ is closed;*
4. *If* **D** *and Λ is closed, then Λ is r-maximal;*
5. *If* **E**, *then Λ is r-maximal, consistent and closed.*

Proof. 1. Suppose that $A \in c\Lambda$. Then there are B_1, $\dots$, B_n in Λ such that $(B_1 \wedge \cdots \wedge B_n) \Rightarrow A \in @$. Since B_1, $\dots$, B_n are in Λ, we can find n integers $k(1)$, $\dots$, $k(n)$ such that $B_1 \in \Gamma_{k(1)}, \dots, B_n \in \Gamma_{k(n)}$. But then, if **A** is true, $B_1, \dots, B_n \in \Gamma_m$, where $m = \max\{k(1), \dots, k(n)\}$. In that case, $A \in c\Gamma_m \subseteq \bigcup_{n\in\mathbb{N}} c\Gamma_n$. Thus, if **A**, then $c\Lambda \subseteq \bigcup_{n\in\mathbb{N}} c\Gamma_n$. Conversely, by a general property of closure, $\bigcup_{n\in\mathbb{N}} c\Gamma_n \subseteq c\Lambda$.

2. Assume that each Γ_n is @-compatible, and for a *reductio*, suppose that Λ is not. Then we can find $B_1, \dots, B_n$ in Λ such that $\neg\Diamond(B_1 \wedge \cdots \wedge B_n) \in @$.

Then, assuming **A**, $B_1, \ldots, B_n$ are in some Γ_m. Now, given that $\neg\Diamond(B_1 \wedge \cdots \wedge B_n) \in @$, this means that Γ_m is not @-compatible. Which is false if we accept **B**. So, Λ is @-compatible.

3. Assume **C**. **C** entails **A**. So, $c\Lambda = \bigcup_{n\in\mathbb{N}} c\Gamma_n$. Now, a consequence of **C** is that $\bigcup_{n\in\mathbb{N}} c\Gamma_n \subseteq \bigcup_{n\in\mathbb{N}} \Gamma_{n+1} \subseteq \Lambda$. So, $c\Lambda \subseteq \Lambda$.

4. Assume that **D** and Λ is closed. Let A be such that $\mathrm{at}(A) \subseteq \mathrm{at}(\Lambda)$. Then by Proposition 5.1, $\widehat{A} \in \Lambda$. So, $\widehat{A}$ is a member of some Γ_m. Then by **D**, either A or $\neg A$ is a member of Γ_{m+1}, and so either A or $\neg A$ is a member of Λ.

5. It suffices to note that **E** entails **A**, **B**, **C** and **D**. ■

Proposition 10. *Each @-compatible set of formulas has a good friend.*

Proof. Let Γ be @-compatible. We assume a numbering of all the formulas $A_0, A_1, \ldots,$ and we build a series $(\Gamma_n)_{n\in\mathbb{N}}$ as follows:

- Γ_0 is Γ;
- Γ_{n+1} is $\begin{cases} \Gamma_n & \text{if } \widehat{A}_n \notin \Gamma, \\ \Gamma_n \cup \{A_n\} & \text{if } \widehat{A}_n \in \Gamma \text{ and } \Gamma_n \cup \{A_n\} \text{ is @-compatible,} \\ \Gamma_n \cup \{\neg A_n\} & \text{otherwise.} \end{cases}$

Obviously, $(\Gamma_n)_{n\in\mathbb{N}}$ is increasing. Moreover, by Proposition 8, for all $n \in \mathbb{N}$, if Γ_n is @-compatible, then so is Γ_{n+1}. So, since $\Gamma_0 = \Gamma$ is @-compatible, all the Γ_ns are @-compatible. Now let Λ be $\bigcup\limits_{n\in\mathbb{N}} \Gamma_n$. Then Λ is @-compatible (see Proposition 9.2). Moreover, Λ is obviously an extension of Γ. Finally, by construction, for every formula A, if $\widehat{A} \in \Gamma$, then $A \in \Lambda$ or $\neg A \in \Lambda$. So, Λ is a good friend of Γ. ■

Now we let $\mathcal{W}^C$ be the set of all r-maximal, consistent, and closed sets of formulas. Note that in particular, $@ \in \mathcal{W}^C$.

Proposition 11. *Every @-compatible set of formulas has an extension in* $\mathcal{W}^C$.

Proof. Let Γ be @-compatible. We build a series $(\Gamma_n)_{n\in\mathbb{N}}$ as follows:

- Γ_0 is Γ;
- Γ_{n+1} is $\varnothing$ if $c\Gamma_n$ has no good friend, and some good friend of $c\Gamma_n$ otherwise,

and we put $\Lambda = \bigcup_{n\in\mathbb{N}} \Gamma_n$. Obviously, $\Gamma = \Gamma_0$ is a subset of Λ. Furthermore, by Propositions 10 and 7, one can prove (by induction) that for every integer n, Γ_{n+1} is a good friend of $c\Gamma_n$. It follows by Proposition 9.5 that $\Lambda \in \mathcal{W}^C$. ■

PROPOSITION 12. *Every @-compatible set of formulas Γ such that $\widehat{A} \notin c\Gamma$ has a good friend Λ such that $\widehat{A} \notin c\Lambda$.*

PROOF. Let Γ be @-compatible and such that $\widehat{A} \notin c\Gamma$. We assume a numbering of all the formulas A_0, A_1, ..., and we build a series $(\Gamma_n)_{n\in\mathbb{N}}$ as follows:

– Γ_0 is Γ;

– Γ_{n+1} is $\begin{cases} \Gamma_n & \text{if } \widehat{A}_n \notin \Gamma, \\ \Gamma_n \cup \{A_n\} & \text{if } \widehat{A}_n \in \Gamma \text{ and } \Gamma_n \cup \{A_n\} \text{ is @-compatible} \\ & \text{and } \widehat{A} \notin c(\Gamma_n \cup \{A_n\}), \\ \Gamma_n \cup \{\neg A_n\} & \text{otherwise.} \end{cases}$

We shall now prove the following:

LEMMA 1. *For every integer n, if Γ_n is @-compatible and $\widehat{A} \notin c\Gamma_n$, then so it goes for Γ_{n+1}.*

PROOF. The hard part of the proof is to show that, assuming that:

1. Γ_n is @-compatible,
2. $\widehat{A} \notin c\Gamma_n$,
3. $\widehat{A}_n \in \Gamma$,

if either $\Gamma_n \cup \{A_n\}$ is not @-compatible or $\widehat{A} \in c(\Gamma_n \cup \{A_n\})$, then both $\Gamma_n \cup \{\neg A_n\}$ is @-compatible and $\widehat{A} \notin c(\Gamma_n \cup \{\neg A_n\})$. This is done in four steps (**A**, **B**, **C**, **D**). Assume 1–3 above.

A. Suppose $\Gamma_n \cup \{A_n\}$ is not @-compatible. Then by Proposition 8, since Γ_n is @-compatible and $\widehat{A}_n \in \Gamma \subseteq \Gamma_n$, $\Gamma_n \cup \{\neg A_n\}$ is @-compatible.

B. Suppose $\widehat{A} \in c(\Gamma_n \cup \{A_n\})$. Since $\widehat{A} \notin c\Gamma_n$ and $\widehat{A}_n \in \Gamma \subseteq \Gamma_n$ and Γ_n is @-compatible, it follows by Proposition 6 that $\widehat{A} \notin c(\Gamma_n \cup \{\neg A_n\})$.

C. Suppose $\widehat{A} \in c(\Gamma_n \cup \{A_n\})$. Assume for a *reductio* that $\Gamma_n \cup \{\neg A_n\}$ is not @-compatible. Then we can find B_1, ..., B_m in Γ_n such that $\neg\Diamond(B_1 \wedge \dots \wedge B_m \wedge \neg A_n) \in @$. Then by Proposition 1.2, $\widehat{A} \in c(\Gamma_n \cup \neg A_n)$. Now since $\widehat{A}_n \in \Gamma \subseteq \Gamma_n$ and Γ_n is @-compatible, it follows by Proposition 6 that $\widehat{A} \in c\Gamma_n$. But this contradicts assumption 2 above. So, $\Gamma_n \cup \{\neg A_n\}$ is @-compatible.

D. Using the same line of reasoning as in **C** above, one can prove that if $\widehat{A} \in c(\Gamma_n \cup \{\neg A_n\})$, then $\Gamma_n \cup \{A_n\}$ is @-compatible, which by contraposition gives the result. ■

Now, a consequence of the Lemma and the initial hypotheses on Γ is that for every integer n, Γ_n is @-compatible and $\widehat{A} \notin c\Gamma_n$. Let us put $\Lambda = \bigcup_{n \in \mathbb{N}} \Gamma_n$. Then Λ is a good friend of Γ (see the proof for Proposition 10). Moreover, $\widehat{A} \notin c\Lambda$ (for if $\widehat{A}$ were a member of $c\Lambda = \bigcup_{n \in \mathbb{N}} c\Gamma_n$, then $\widehat{A}$ would be a member of some $c\Gamma_n$, which as we just saw is excluded). ■

Proposition 13. *Let Γ be @-compatible and such that $\widehat{A} \notin c\Gamma$. Then Γ has an extension in $\mathcal{W}^C$, which does not contain $\widehat{A}$.*

Proof. This is a modified version of the proof for Proposition 11. Let Γ be @-compatible, and such that $\widehat{A} \notin c\Gamma$. We build a series $(\Gamma_n)_{n \in \mathbb{N}}$ as follows:

- Γ_0 is Γ;
- Γ_{n+1} is $\varnothing$ if $c\Gamma_n$ has no good friend Σ such that $\widehat{A} \notin c\Sigma$, and some good friend Σ of $c\Gamma_n$ such that $\widehat{A} \notin c\Sigma$ otherwise.

Using Proposition 12, we can then prove that for every integer n, Γ_{n+1} is a good friend of $c\Gamma_n$ and $\widehat{A} \notin c\Gamma_{n+1}$. It follows by Proposition 9.5 that $\bigcup_{n \in \mathbb{N}} \Gamma_n \in \mathcal{W}^C$. It also follows that $\bigcup_{n \in \mathbb{N}} \Gamma_n$ does not contain $\widehat{A}$. ■

Completeness is now easy to prove.

Proposition 14. *$A \Rightarrow B \in$ @ iff for every w in $\mathcal{W}^C$, either $A \notin w$ or $B \in w$.*

Proof. From left to right, by closure. For the other direction, suppose $A \Rightarrow B \notin$ @. By maximality, $\neg(A \Rightarrow B) \in$ @. Now, either $A \Rightarrow \widehat{B} \in$ @, or $A \Rightarrow B \notin$ @.

(a) Suppose $A \Rightarrow \widehat{B} \in$ @. By Proposition 1.9, it follows that, $\Diamond(A \wedge \neg B) \in$ @, and so $\{A, \neg B\}$ is @-compatible. By Proposition 11, it has an extension w in $\mathcal{W}^C$. Now, $A \in w$, and since $\neg B \in w$, $B \notin w$.

(b) Suppose now $A \Rightarrow \widehat{B} \notin$ @. Then $\widehat{B} \notin c\{A\}$. Moreover, since $\neg(A \Rightarrow B) \in$ @, by Proposition 1.2 we have $\Diamond A \in$ @. So, $\{A\}$ is @-compatible. By Proposition 13, then, $\{A\}$ has an extension w in $\mathcal{W}^C$ which does not contain $\widehat{B}$. Now, $A \in w$; and since $\widehat{B} \notin w$, $B \notin w$. ■

Proposition 15. *For every w in $\mathcal{W}^C$, $A \Rightarrow B \in w$ iff $A \Rightarrow B \in$ @ and $\mathrm{at}(A) \cup \mathrm{at}(B) \subseteq \mathrm{at}(w)$.*

Proof. (a) Suppose $A \Rightarrow B \in @$. Then by Proposition 1.3, $\Box(A \Rightarrow B) \in @$. So by Proposition 5.2, if $\mathrm{at}(A) \cup \mathrm{at}(B) \subseteq \mathrm{at}(w)$, then $A \Rightarrow B \in w$.

(b) Suppose now that $A \Rightarrow B \in w$. Then trivially $\mathrm{at}(A) \cup \mathrm{at}(B) \subseteq \mathrm{at}(w)$. Suppose that $A \Rightarrow B \notin @$. Then by maximality, $\neg(A \Rightarrow B) \in @$. So by Axiom **E**, $\Box\neg(A \Rightarrow B) \in @$. Now given that $\mathrm{at}(A) \cup \mathrm{at}(B) \subseteq \mathrm{at}(w)$, we have by Proposition 5.2: $\neg(A \Rightarrow B) \in w$, and so by consistency: $A \Rightarrow B \notin w$. Contradiction. So, $A \Rightarrow B \in @$. ■

Proposition 16. *For every w in $\mathcal{W}^C$, $A \Rightarrow B \in w$ iff $\mathrm{at}(A) \cup \mathrm{at}(B) \subseteq \mathrm{at}(w)$ and for every v in $\mathcal{W}^C$, either $A \notin v$ or $B \in v$.*

Proof. By Propositions 14 and 15. ■

We define the valuation V^C on $\mathcal{W}^C$ by: $V^C(p, w) = 0$ iff $\neg p \in w$; and $V^C(p, w) = 1$ iff $p \in w$. $\mathcal{M}^C$ is the Q-model $\langle \mathcal{W}^C, V^C \rangle$.

Proposition 17. *For every w in $\mathcal{W}^C$, $d(A, w, \mathcal{M}^C)$ iff $\mathrm{at}(A) \subseteq \mathrm{at}(w)$.*

Proof. By definition of d, $d(A, w, \mathcal{M}^C)$ iff for every atom p in A, $V^C(p, w) = 0$ or $V^C(p, w) = 1$. So, $d(A, w, \mathcal{M}^C)$ iff for every atom p in A, $\neg p \in w$ or $p \in w$. So (using Proposition 4.1), $d(A, w, \mathcal{M}^C)$ iff $\mathrm{at}(A) \subseteq \mathrm{at}(w)$. ■

Proposition 18. *For every w in $\mathcal{W}^C$: $\mathcal{M}^C, w \models A$ iff $A \in w$.*

Proof. By induction on the length of A, using Propositions 4, 16 and 17. ■

The empty set is trivially a member of $\mathcal{W}^C$. Let $\mathcal{N}^C$ be the Q-model obtained from $\mathcal{M}^C$ by withdrawing the empty set from $\mathcal{W}^C$. Then obviously, Proposition 18 holds of $\mathcal{N}^C$ as well. This fact will be used in section 6.2.

Theorem 2 (Completeness). *Every PSI-consistent set of formulas is Q-satisfiable.*

Proof. By Proposition 18 and the fact that $\alpha \subseteq @$, it follows that $@ \models \alpha$ (relative to $\mathcal{M}^C$). So, α is satisfiable. The result follows from the fact that α was an arbitrary consistent set of formulas. ■

Corollary 1. *Every Q-valid formula is a theorem of PSI.*

Proof. From Theorem 2. ■

In the next sections, we will need a mild extension of PSI. Let us enrich the language of PSI with a truth-constant t (remember from section 2: t is not to be counted as an atom, and t is true at every world of every Q-model). We define system PSI^t as PSI plus t as an axiom. Obviously, PSI^t is sound. Completeness is proved by modifying the previous proof in the obvious way.

5. PSI, Q and Related Systems

Let us consider the following three languages and associated Q-like semantics:

[L] Language L is a standard modal language with $\neg$, $\wedge$ and $\Box$ as primitive operators. The truth-clause for $\Box$ is the one given in section 3.

[L^{S}] Language L^{S} is L augmented with the monadic propositional operator S. The semantics for L^{S} is the same as for L, with the following extra truth-clause for S:

– $w \models \mathsf{S}A$ iff $\forall v \in \mathcal{W}$, $d(A, v)$.

Thus, S expresses necessary definedness – or as Prior would say, necessary "statability".

[$\mathsf{L}^{>}$] Language $\mathsf{L}^{>}$ is L augmented with the binary propositional operator $>$. The semantics for $\mathsf{L}^{>}$ is the same as for L, with the following extra truth-clause for $>$:

– $w \models A > B$ iff $d(A, B, w)$ and $\forall v \in \mathcal{W}$, if $d(A, v)$, then $d(B, v)$.

Thus, $>$ expresses a form of conditional statability.

In (Correia, 1999), the corresponding three logics (i.e. sets of Q-valid formulas) are given Hilbert-style axiomatizations, which are labeled respectively $\mathrm{S5}^{-}$, Q and $\mathrm{S5}^{>}$. Q is indeed nothing but Arthur Prior's modal logic Q.[2]

[2] System Q was first presented semantically by Prior in (Prior, 1957), ch. V, and first axiomatized by Bull in (Bull, 1964). It has been presented in various places and in several languages. In the original formulation of (Prior, 1957), two primitive non–classical operators are used: M (true is some world) and L (true in all worlds). In (Prior, 1959), he uses M and S (statable in all worlds). My S is identical to Prior's S. My $\Box$ is a form of weak necessity, equivalent to Prior's $\neg M \neg$. Conversely, M is definable as $\neg\Box\neg$, i.e. as the $\Diamond$ introduced in section 3. Prior's Lp is definable in Prior's terms as $\neg M \neg p \wedge Sp$, i.e. in my terms as $\Box p \wedge \mathsf{S}p$: being always true is equivalent to being never false and always statable. Finally, Sp is definable (in bastard notation) as $L(p \rightarrow p)$. The two necessity operators $\Box$ and L, of course, have different properties. L is strictly stronger than $\Box$ (what is always true is never false, but what is never false may fail to be always true: it may be sometimes unstatable). The rule Necessitation for $\Box$ preserves validity while it does not for L. $\Box$ and M are inter–definable, but L and M are not.

In (Correia, 1999), I used an axiomatization of Q in language L^{S}, with $\Diamond$ defined as $\neg\Box\neg$, consisting of all PC–tautologies, the rule Modus ponens and the following axiom schemas and rules:

– $\mathsf{S}A \rightarrow \mathsf{S}p$, for p any atom in A

– $(\mathsf{S}p_1 \wedge \cdots \wedge \mathsf{S}p_n) \rightarrow \mathsf{S}A$, where $p_1, \ldots, p_n$ are all the atoms in A

– $\Diamond\mathsf{S}A \rightarrow \mathsf{S}A$

Let us then consider the translation $*$ into the language of PSI or PSI^t defined by:

- $p^* = p$
- $(\neg A)^* = \neg A^*$
- $(A \wedge B)^* = A^* \wedge B^*$
- $(\Box A)^* = (A^* \vee \neg A^*) \Rightarrow A^*$
- $(\mathsf{S}A)^* = t \Rightarrow (A^* \vee \neg A^*)$
- $(A > B)^* = (A^* \vee \neg A^*) \Rightarrow (B^* \vee \neg B^*)$

It is easy to prove semantically that:

THEOREM 3.

1. $\forall A \in \mathsf{L}$, *A is a theorem of* S5$^-$ *iff* A^* *is a theorem of* PSI*;*
2. $\forall A \in \mathsf{L}^{\mathsf{S}}$, *A is a theorem of* Q *iff* A^* *is a theorem of* PSI^t*;*
3. $\forall A \in \mathsf{L}^{>}$, *A is a theorem of* S5$^>$ *iff* A^* *is a theorem of* PSI*.*

6. Miscellaneous

6.1. Strict Implication in PSI and PSI^t

Let us use the following abbreviations for formulas of the language of PSI^t:

- $\Box A$ for $\widehat{A} \Rightarrow A$;
- $\blacksquare A$ for $t \Rightarrow A$;
- $\mathsf{S}A$ for $t \Rightarrow \widehat{A}$.

- $(\mathsf{S}p_1 \wedge \cdots \wedge \mathsf{S}p_n \wedge \Box(A \to B) \wedge \Box A) \to \Box A$, where $p_1, \ldots, p_n$ are all the atoms of A not in B
- $\Box A \to A$
- $\Diamond A \to \Box\Diamond A$
- $A/\Box A$

This axiomatization is almost identical to the one to be found in (Prior & Fine, 1977), pp. 84–85, where Prior takes S and M (that is, S and $\Diamond$) as sole non–classical operators.

It should be noted that the semantics presented in (Correia, 1999) is in fact a bit different from the Q-like semantics presented here. The models have a distinguished (actual) world, and instead of a partial valuation, each model is endowed with two binary relations between worlds and atoms, one which determines which atom is defined at which world, and the other which determines which atom is true at which world. But this difference is inessential.

We have then:

- $w \vDash \Box A$ iff $d(A, w)$ and $\forall v \in \mathcal{W}\ v \nvDash \neg A$;
- $w \vDash \blacksquare A$ iff $\forall v \in \mathcal{W}\ v \vDash A$;
- $w \vDash \mathsf{S}A$ iff $\forall v \in \mathcal{W}\ d(A, v)$.

Thus, $\blacksquare$ expresses a strong form of necessity, as opposed to the weak form expressed by $\Box$.[3] S expresses necessary statability. Strong necessity is actually weak necessity plus necessary statability: $\blacksquare A \leftrightarrow (\Box A \wedge \mathsf{S}A)$ is a theorem.

In system PSI^t, one can naturally define two strict implication connectives besides the primitive $\Rightarrow$: $\Box(A \rightarrow B)$ and $\blacksquare(A \rightarrow B)$. Of course, only the first is definable in PSI. Their truth-clauses are given by:

- $w \vDash \Box(A \rightarrow B)$ iff (i) $d(A, B, w)$ and (ii) $\forall v \in \mathcal{W}$ if $d(A, B, v)$, then $v \vDash A \rightarrow B$;
- $w \vDash \blacksquare(A \rightarrow B)$ iff $\forall v \in \mathcal{W}\ \ v \vDash A \rightarrow B$.

The following are theorems of PSI^t, and show some relationships between our three strict implication operators:

- $\blacksquare(A \rightarrow B) \rightarrow (A \Rightarrow B)$;
- $(A \Rightarrow B) \rightarrow \Box(A \rightarrow B)$;
- $(A \Rightarrow B) \leftrightarrow \Box(A \rightarrow B) \wedge A \Rightarrow \widehat{B}$;
- $\blacksquare(A \rightarrow B) \leftrightarrow (A \Rightarrow B) \wedge \mathsf{S}A \wedge \mathsf{S}B$;
- $\blacksquare(A \rightarrow B) \leftrightarrow \Box(A \rightarrow B) \wedge \mathsf{S}A \wedge \mathsf{S}B$.

6.2. Collapsing into S5

The schema *Right-Irrelevance*:

- $A \Rightarrow \widehat{B}$

is not Q-valid. In fact, let Q-model $\langle \{v, w\}, V \rangle$ be such that $V(p, v) = V(p, w) = V(q, v) = 1$ and q is undefined at v. Then $\neg(p \Rightarrow \widehat{q})$ is true at v.

What happens if we add Right-Irrelevance to PSI? The new system is sound and complete with respect to the class $\mathcal{C}$ of all Q-models for PSI whose valuations are *total*. In fact, it is quite easy to see that Right-Irrelevance is

[3] $\blacksquare$ is equivalent to Prior's L. See footnote 2.

true at every world of any of these Q-models, so that PSI + Right-Irrelevance is sound with respect to $\mathcal{C}$. For completeness, it is sufficient to note that Q-model $\mathcal{N}^C$ of section 4 is in $\mathcal{C}$, provided that Right-Irrelevance be added to PSI (by Right-Irrelevance, every non-empty closed set of formulas becomes (*at*-) maximal). Similar remarks hold of system PSI^t, of course.

By adjoining Right-Irrelevance to PSI^t, thus, $\Box$ and $\blacksquare$ collapse into an S5 necessity operator. All the same, our three strict implication connectives collapse into an S5-type strict implication.

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