Published in Travaux de logique 17, 102-145, 2004
which should be used for any reference to this work

J.-Y.Beziau, A.Costa-Leite and A.Facchini (eds)
Aspects of Universal Logic, 102-145

Geometry for Modalities? Yes: Through n-Opposition Theory

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Abstract

We present here some new results about the fundamental relations between modal logic and geometry. The key of our approach is a general renewed theory of "logical n-opposition", a strong geometrical generalisation of Aristotle's classical "opposition theory". The question of the possible relations between (modal) logic and geometry, in some sense brightly foreseen by the russian logician N.A. Vasil'ev around 1910, happens to be still quite a mysterious one and can be shown to be of great relevance for contemporary research in many fields (for instance, but not only, in cognitive science). The most famous (seminal) entanglement between logic and geometry is given by Aristotle's "logical square of oppositions", a very poor structure in terms of further mathematical developments. Concrete applications of this structure, although not uninteresting, have been rather mean as well (cf. J. Piaget, J.A. Greimas and J. Lacan). Some recent results, however, have clearly shown that this sad and poor field (opposition theory as a limited and unclear encounter of modal logic and elementary geometry) is in fact *much bigger* than it was thought. This is strongly and mainly suggested, as we will recall, by the surprising (and almost unnoticed) discovery in 1953 - inside "opposition theory" - of a "logical hexagon" (Blanché), and then by that, in 2003, of a 3-dimensional "logical tetradecahedron" (Béziau and Moretti), which both are logical "implicational" structures (expressing opposition relations) furnished with a notably high degree of geometrical symmetries. These discoveries are clearly open to further development and

generalization, as we will briefly sketch and demonstrate (we will present here many new such structures). The problem thus arising nowadays is a lack of comprehension of the kind of (new) modal "language" we are speaking while developing such structures. Haven't we entered a radically new logical field? The bulk of this paper will consist in establishing that, in fact, this mysterious emerging field is really a totally *new* one, and that it happens to be huge (it has to do with science, not with history), and (at least partly) already precisely shaped : there are noticeable guidelines, or trends, of it - the principal one consisting in the discovery of a geometrical treatment of 'contradiction' in terms of poly-dimensional symmetries. To make such a shape the more explicit we can, as a new research framework in modal logic, we develop here a general logical-geometrical theory of "n-opposition", consisting mainly in a theory of geometrical "$\alpha n(m)$-structures" and in a theory of modal "$n(m)$-graphs" (the two working closely together). Both theories are based on the notion of "simplex of dimension n". This framework succeeds in systematising and in explaining all previous results and, more than this, it leads to infinitely many new ones, as we will detail in the paper. After sketching the proof of a useful general theorem, we end by showing that an interesting (by now difficult) problem (a conjecture), very promising if solved, is left open, hopefully solvable by a suited theory still to come, which we call the problem of a possible theory of $\beta n(m)$-structures.

1 Previous results : from the square to the 4 hexagons and, finally, to the tetradecahedron

We recall here Aristotle's basic doctrine. "**Opposition**" consists in a complex ordering, expressed geometrically by the "logical square" (of oppositions), of four different ways for two terms to be "opposed" one to the other[1]. These ways are : (1) *contradiction*, defined for two terms as, simultaneously, the impossibility to be both true *and* the impossibility to be both false ; (2) *contrariety*, defined for two terms as, simultaneously, the impossibility to be both true *but* the possibility to be both false ; (3) *sub-contrariety*, defined for two terms as, simultaneously, the possibility to be both true *but* the impossibility to be both false ; (4) *sub-alternation* (or

[1]In some sense, opposition theory can be said to be the logical theory of "difference".

implication), defined for an ordered couple of terms as the impossibility of having the first without having the second (so that, in some sense, it *contains* the fourth combinatorial case, i.e., *simultaneously*, the possibility of being both true *and* the possibility of being both false - plus the possibility that the first is false while the second is true).

As we see, the 4 kinds of oppositions exhaust the combinatorial possibilities of combined truth and/or falsity of two simultaneous terms. In the square, these 4 kinds constituting the concept of opposition are represented not by the 4 points (the vertices, the corners of the square) but by the lines (the square's 4 edges and 2 diagonals). For simplicity's sake, we will use the following convention : contradiction will generally be represented in red, contrariety will be in blue, sub-contrariety will be in green, sub-alternation (i.e. implication) will be in grey (sometimes in black) (sorry for colour-blind people!). The four points (corners) delimiting the square are *variables*, empty places to be fulfilled with modalities. **Classical opposition theory** is thus modal logic (at least the "core" of it, the universal Lewis system $S5$) entangled in some (simple) way, as we saw, with geometry (the square with its 4 vertices, 4 edges and 2 diagonals, cf. figure 1).

Many parts of Aristotle's logic have been abandoned or strongly revised during the "logical turn" of the second half of the nineteenth century (from Boole to Russell) : not the logical square[2]. This structure, in fact, if poor, seems nevertheless impressive by its incredible generality : it expresses graphically the fundamental quantificational relations (holding for $\forall$, $\exists$, $\neg\forall$, $\neg\exists$) and thus - modal logic being related to quantification theory, as we know now through "possible worlds semantics" - it expresses also the fundamental modal relations, at least those of the 4 "non-naked" modalities among the 6 basic ones of $S5$ (cf. figure 1).

But scholars in the history of logic have soon remarked the presence, in Aristotle's theory of modality, of two incompatible theoretical positions : the aforementioned logical square (AEOI) and the less known "logical triangle" (of contrarieties, AEY). The two do not fit together. Where does this unresolved ambiguity come from? It has been shown that Aristotle oscillates between two different notions of "possibility" : a "unilateral" one (expressed by the square of oppositions) and a "bilateral" one (expressed by

[2]For instance, Frege keeps it (with a mistake concerning subcontrariety, unduly identified to contrariety !) in his 1879 masterwork *Begriffsschrift*, where he feels obliged to give his own, reformulated version of it (*Begriffsschrift*, §12, p. 24 of the original edition ; cf. figure 1).

the triangle of contrarieties, here in blue).[3] They are different in this sense that the "unilateral possible" ("I") is incompatible with the impossibility ("E") but compatible with necessity ("A") - it is its consequence -, whereas the "bilateral possible" ("Y") is incompatible both with impossibility *and* with necessity (cf. figure 2).

After almost 2500 years this almost unnoticed historical and logical-geometrical riddle (two "possibles" are possible, the square and the triangle being seemingly doomed to remain nastily unrelated) was elegantly solved, from a logical point of view, by R. Blanché (1953) : by adding in the logic and geometric plan of opposition theory a "triangle of sub-contrarieties" ("IUO", here in green) - dual of the triangle of contrarieties -, totally unknown to Aristotle, and thus getting a new geometrical-logical figure, the "logical hexagon"[4]. Effectively, the square emerges then, in a very elegant way, as a simple "by-product" of the two triangles, and in fact we have now not one ("AEOI"), but 3 symmetric logical squares (according to the 3 symmetry axes furnished by the 3 red diagonals of the hexagon) : AEOI, EYIU and YAUO (cf. figure 2). Aristotle's aporia is then solved : the two distinct notions of possibility (bilateral and unilateral) are now explained nicely and interrelated (the "bilateral possible" is equivalent to the conjunction of the "unilaterally possible that" and the "unilaterally possible that not" : $Y \leftrightarrow I \wedge O$). This new solution, the logical hexagon, is mathematically much more interesting than the old strange "logical square" (the symmetries are now much richer). From a philosophical point of view, there is a reason, apart from his aversion for geometry and mathematics in general, why Aristotle, as a logician, missed one of the 6 relevant positions (the topmost "U") of the hexagon, thus being prevented from discovering this last general structure : this position (U = "$A \vee E$", "$\Box \vee \neg\Diamond$", "$\forall \vee \neg\exists$") reflects in fact the necessitarian position of Diodoros Cronos ("necessary or impossible, no third way"), the determinist position of the Megarian school, excluding the possibility of an open future (this last Aristotelian indeterminist position being represented by the dual position, "Y") : on a philosophical basis, Aristotle fights precisely against this option (by his doctrine of the "contingent futures", cf. *De Interpretatione*, 9). So the logical square alone is incomplete. One needs to consider the logical hexagon. However, this strange story of the (up to now cautious) intercourse of logic and geometry does not rest here.

Fifty years after Blanché, in 2003 J.-Y. Béziau discovers two further

[3]Cf. [10], p. 16-17.
[4]Cf. [5], [6], [7], [8] and [10].

possible simple modal decorations of the abstract geometrical-implicational structure of the hexagon : he claims that one is "paracomplete" (i.e. intuitionist), the other "paraconsistent"[5], all this in addition to Blanché's solution which Béziau calls retrospectively "classical" (classical, paracomplete and paraconsistent are said with respect to the modal properties of the respective negations, cf. [2] and [3]). This result, the existence of more than one hexagon, is possible by virtue of the fact that even modally "naked" propositions (say α), i.e. propositions without modal modifiers ($\Box$, $\Diamond$, $\neg\Box$, $\neg\Diamond$), as well as their negations (say $\neg\alpha$), still are modalities.[6]

Following Béziau's then unpublished (but shared) new results, A. Moretti and H. Smessaert discover (independently from each other) a fourth possible modal decoration of the abstract oppositional hexagon, here in blue (which I propose to call "emergent"). And, finally, Béziau and Moretti discover together a three-dimensional ordering of the four hexagons, the "logical tetradecahedron" (or "cube-octahedron"), a 3-dimensional polyhedron with 12 vertices and with 14 sides, 8 triangular and 6 square sides : the four hexagons are displayed according to an (invisible) inner tetrahedron, so that each vertex of the tetradecahedron lays at the intersection of 2 hexagons and no vertex of no hexagon remains non-intersected. This structure is geometrically striking (cf. figure 4).

This will be the starting point of the present paper.[7] Past discoveries seem to lay a strange problem : how comes that the beautiful logical order expressed geometrically by the "logical tetradecahedron" is so far related to nothing else similar in modal logic ? Is this really an isolated meaningless result ? Before any further consideration on the (mysterious) nature of this geometrical-logical structure, we want to quickly recall how the question of the possible uses of the tetradecahedron in (modal) logic has already been partly tackled.

[5]A formal system is said to be "paraconsistent" if it is inconsistent but not trivial.

[6]They are "zero degree modalities", according, for instance, to W.A. Carnielli and C. Pizzi, *Modalità e multimodalità*, FrancoAngeli, Milano, 2001 p.11.

[7]Note that in what follows the metalanguage will be classical, and we will not study paracomplete or paraconsistent aspects of the logic of the hexagons.

2 Possible uses of these start(l)ing results : the tetradecahedron as a translation rule between modal logic and geometry

For reasons to appear later, we will start by calling the implicitely well known graphs exhibiting the relations between basic modalities in the usual systems of modal logic (as, for instance, in [9], p.149-157 ; cf. figure 5 here)"**modal graphs**".

The discovery of the tetradecahedron makes additional sense when we see it as a **translation rule** between modal logic and some kind of geometry. Such a translation is rather precise : it relates the modal graph of $S5$ to the arrowed tetradecahedron (we use here the modal variables x_1, x_2, x_3, ..., x_6, cf. figure 6). Still for reasons to appear later, we call this last an instance of a "β-structure", whereas we will call the four hexagons it orders instances of "α-structures" (for short, a β-structure is something n-dimensional organising nicely some (n-1)-dimensional α-structures).

Now, the point is that this translation rule, which relates, modality by modality, the modal graph of $S5$ and the logical tetradecahedron containing the four hexagons seen before (cf. figure 6), is not a "dead horse". First, it belongs to an infinite (fractal) series of such translation rules, the series of the n-hyper-tetradecahedra (cf. [11]). And second, it allows to study many other classical modal graphs from a geometrical, tetradecahedron-based point of view (for such a study concerning the system $S4$, cf. [4] and [13]). We give here a hint to the simple technique allowing (given transitivity of the arrows in $S4$) to start a geometrical analysis of $S4$ through the translation rule (cf. fig. 7)[8].

As just mentioned (about the existence of an infinite fractal series of translation rules), a spectacular result concerns what we will call the "linear modal graphs" (as $S5$, modal graphs without branchings or outer unarrowed points). We show in [11] a general "fractal" property, determining in an algorithmic way the number, quality and structure of the geometrical n-dimensional figures ruling the logical space of linear modal graphs (it is a

[8] This technique allows the discovery of 17 inner tetradecahedra of $S4$. Then a further examination starts, allowing to discover, from these, some further hexagons ("emergent" ones) and, collecting them, further tetradecahedra (emergent ones), and so on (5-dimensional "hyper-tetradecahedra" ...) until closure, i.e. until the full determination of the complex shape of the geometrical structure of the logical space of $S4$ (cf. [4] and [13]).

fractal generalisation of the translation rule - going into bigger and bigger spatial dimensions - of the logical tetradecahedron). As such, this result seems to justify this kind of fertile investigations.

Then, the geometrical study of branching modal graphs (as the ones of $S4$ and of the $K5$-systems, cf. figure 5) will be another topic. It can be shown that one could bring up this study systematically on the basis of the aforementioned examination of linear graphs (a branching modal graph can be seen as a combination of two or more linear graphs ; we study this in [12]).

But we can say no more in this place about such specialized branches of the arising theory of the interrelations between modal logic and geometry. For our goal, in the present paper, is (more fundamentally) to give a general framework to the oppositional figures seen so far (square, hexagon and tetradecahedron) from the point of view of the problem which originated them all, that is to say the problem handled by opposition theory.

3 Which comprehensive frame ? The viable idea of a generalized "theory of n-opposition"

So we have this strange logical tetradecahedron. Said with some humor, two options seem possible : (1) the tetradecahedron is just an accident, a mind-pitfall for stupid Platonists (I am one), opposition theory (the essence of transcendental logic) does not change ("long live Aristotelianism !") ; or (2) the tetradecahedron (and the hexagon) is the sign of a major change to come inside good old opposition theory, it means some kind of primacy of infinite mathematics over "transcendental logic" ("down with Aristotelianism !")[9]. Our bet will be that this beautiful structure is not an "hapax legomenon" (i.e. an isolated, meaningless event related to nothing determinable). On the contrary, we take it as the sign of an underlying complex theory (whose first elements we will try to bring here into light), a theory comprising it as a fragment.[10] This conjecture of ours, predicting a

[9] For a philosophical position of this kind, we rely mainly on A. Badiou's proposal of a (atheistic) "Platonism of multiplicity", as in A. Badiou, *L'être et l'événement*, Seuil, Paris, 1988 ; *Conditions*, Seuil, Paris, 1991 ; *Court traité d'ontologie transitoire*, Seuil, Paris, 1998.

[10] For the very idea of "daring" look for extensions from "3-terms" oppositions (hexagonal case) to mysterious x-ones (with four terms, and perhaps more !), I wish to thank J.-Y. Béziau, whose interest in square-related "boring" stuff is pioneering and unusual among professional logicians, and from which I benefited, besides constant personal sup-

considerable extension of opposition theory, is not trivial : since Aristotle until now, opposition theory is considered a stupid fragment of classical mathematical logic (an object for history, not for scientific research), and concrete applications of this structure have been rather mean.[11] Our conjecture, if verified, would falsify this judgement and would, more than this, presumably bring new light on the fundamental relations between logic and geometry. Such relations, quite mysterious until now, are a central issue in contemporary thought (cf. for instance P. Gärdenfors' theory of "conceptual spaces" in cognitive science, opposing very convincingly geometry to logic as a paradigm for conceptual modelling ; as well as I. Matte Blanco' theory, a logical-geometrical approach to the "unconscious" features of the human mind).[12] And our bet (our conjecture) happens to be won (proved) : *there is* a global theory superseding Aristotle's one and explaining (and containing) the emergence of such strange logical structures as the tetradecahedron. Which we are now going to explain.

Considering that classical opposition theory (as emended by Blanché) is 3-opposition theory (it deals with two *triangles*, relating *three* contrary blue terms, and *three* subcontrary green terms), we want to explore the idea of a possible generalization of it by some kind of "n-opposition theory". By analogy with what we know already ($n = 3$, the hexagon's case), it will show up that the global n-theory has one geometrical side and one modal side. Effectively, one problem is that of looking for a good geometrical model of n-opposition. But the attempt to decorate with modalities such

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port, several funny and passionating discussions about syrens, dragons, monsters and the like.

[11]Cf., for instance, Piaget's theory of the "I.N.R.C. group" as a cognitive model of children intelligence growth, or Greimas' theory of the "semiotic square" as a general structuralist theory of meaning as such, not to mention Lacan's psychanalytical "sexuation formulas" on the complex structure of human sexual identity (cf. J. Piaget, *Six études de psychologie*, Gonthier, Genève, 1964, chapters 3, 5, 6 ; *Le structuralisme*, PUF, Paris, 1968 ; *L'épistémologie génétique*, PUF, Paris, 1970, p. 51-58 ; A.J. Greimas and F.R. Rastier, "Le jeu des contraintes sémiotiques" (1968) and Greimas, A.J., "Éléments d'une grammaire narrative" (1969), both in : Greimas, J.A., *Du sens. Essais sémiotiques*, Seuil, Paris, 1970 ; "Carré sémiotique" in : A.J. Greimas and J. Courtés, *Sémiotique. Dictionnaire raisonné de la théorie du langage*, Hachette, Paris, 1993 ; Dor, J., *Introduction à la lecture de Lacan. 2. La structure du sujet*, Denoël, Paris, 1992, chapters 12-15 ; Lacan, J., *Le séminaire - livre XX. Encore*, Seuil, Paris, 1975, chapter 7).

[12]P. Gärdenfors, *Conceptual Spaces. The Geometry of Mind*, The MIT Press, Cambridge MA and London, 2000 ; I. Matte Blanco, *The Unconscious as Infinite Sets. An Essay on Bi-logic*, Duckworth, London, 1975 ; E. Rayner, *Unconscious Logic. An Introduction to Matte Blanco's Bi-Logic and its Uses*, Routledge, London, 1995.

a n-oppositional model is yet another problem. Both must be tackled. We will first face the *geometrical* problem (how to express n-opposition geometrically beyond the hexagon). Then, armed with new formal tools, we will face the *logical* problem, that of decorating modally the geometrical expression of opposition.

3.1 First result (geometrical) : the notion of oppositional "n-structure" (the realm of αn-structures)

For simplicity, we begin by focussing on contrariety alone (we forget momentarily the other 3 kinds of oppositions). The starting point is a project of ours of generalizing the expression, by a (blue) triangle, of the relation of contrariety. The equilateral triangle expressed the fact that the 3 contrary terms under examination are "on a same plan", each one equally distant from the other two (they are, two by two, equally different). How to express the same thing with 4 terms (say : 4-contrariety) ? The obvious geometrical answer is, if we want to keep our geometrical equidistancy metaphor : "with a (blue) tetrahedron". Each point is then equally distant from the other 3, we are rescued by the use of a 3-dimensional, instead of a 2-dimensional, space. But then, how to express contrariety for 5 terms (5-contrariety) ? It can be shown that no 3-dimensional figure allows a distribution in the space of 5 points so that each point is equally distant from the other 4. Are we obliged to drop our equidistancy-criterium for contrariety, if we want to use geometry ? We are not : we only need to pursue our n-dimensional ascent. Here comes, in fact, the first important result in our "quest" : n-dimensional geometry allows a geometrical representation of n-contrariety (not yet n-opposition) - as "being n different guys at the same distance one from the other", or n-equidistancy -, by taking into account the mathematical series of the "simplexes of dimension n-1" (each "simplex of dimension n-1" is a (n-1)-dimensional structure whose n vertices have the same distance between any 2 of them, cf. [1] ; cf. figure 8).

Now, in order to have opposition (conceived as a combinatorially exhaustive theory, in the sense sketched above), we have to add to contrariety the three other opposition relations : sub-contrariety, contradiction and subalternation (i.e. implication). And this is geometrically possible ! Our geometrical leading remark (the existence of the series of the simplexes of dimension n) *does allow* the construction of the required opposition relations. It suffices, by analogy with the case of the hexagon, to combine, according to a suited symmetry, two simplexes (one blue for contrarieties, and one green for subcontrarieties) so that each vertex of the blue one is contradictory to the vertex of the green one symmetric to it (cf. figures 9-11).

We call "logical ***bi-simplex*** of dimension n-1" the blue-green (n-1)-dimensional structure thus obtained (having $2n$ vertices). Contrariety is assumed by construction (the blue simplex). Subcontrariety is obtained easily by symmetry (the green simplex). Contradiction is expressed by the n biggest diagonals (in red, when expressed) of the bi-simplex (as it was in the hexagon by the 3 red diagonals, and in the square by the 2 red diagonals, cf. figure 9). And subalternation can be easily expressed by adding a grey arrow between each couple of non-contradictory blue and green vertices (from the blue one to the green one).

This constructive device works out neatly even at the next stage (we move from $n = 3$ to $n = 4$, cf. figure 10). And this leads to the discovery of a new "logical oppositional structure", the "logical bi-tetrahedron" (or, more classically, "stella octangula") which, once the arrows expressing its subalternation relations (from blue to green vertices) are drawn (in grey), reveals itself to be a "**logical cube**" of oppositions.

And the algorithmic magic of this is that we can go further. If we step now from 4-opposition (the logical cube) to 5-opposition, combining two symmetric simplexes of dimension 4 (one blue for 5-contrariety, the other green for 5-subcontrariety), we get a new oppositional figure, which we call for simplicity (the geometry of a 4-dimensional space becoming slightly counter-intuitive) the logical "**α5-structure**" (cf. figure 11). Retrospectively, the logical hexagon and the logical cube will be named respectively **α3-structure** and **α4-structure**.

It seems to be rather easy to give a proof (by induction) of the generality of this geometric method, with a recursion based on the notion of simplex of dimension n-1 (such simplexes never stop, they are available for any n). However, we omit here to give such a proof. We move instead to the question of the graphical (geometrical) expression of the general case. If we look for a graphical expression of this efficient general algorithm, we can compare graphically the oppositional αn-structures, say for $n = 3$, 4, 5, 6, 7 (cf. figure 12). The drawing suggests a (very) small proviso to this general algorithm, according to the fact that n is odd or even. When n is odd it will be easy to lay the two simplexes, blue and green, "face to face", in order to let appear clearly in a 2-dimensional drawing the appropriate symmetry of the contradictory vertices (related by a red diagonal). We call this circular representation, which simply alternates blue and green terms, duly arrowed, "doughnut" or "big wheel" - in honour to Bart.

When n is even, however, the symmetry of the contradictory vertices cannot be expressed as easily in a 2-dimensional drawing (put in a circle, $2n$ points alternatively blue and green, with n even, do not allow blue and green points to be opposite - each blue point will be centrally symmetric to a blue point, each green to a green one, which is not geometrically satisfactory). So we adopt the convention of representing it in the way depicted in the lower part of our schema (of figure 12, or at the right end of figure 13), that is with 4 niveaux, the topmost one with a green term (as we will see, it will be called "head"), than a row of n-1 blue terms, then a row of n-1 green terms (as we will see, these two rows constitute the "body"), and finally, at the bottom, a blue term (as we will see, it will be called the "tail"), all this duly arrowed (we call this representation "hamburger" - or "merry-go-round", in honour to Homer).

The **general graphical algorithm** (for n odd or even) is finally the following : for each **αn-structure** - a "bi-simplex of dimension (n-1)", resulting from the superposition of two simplexes of dimension (n-1) (one blue, the other green) - all blue points are contrary to each other, all green points are subcontrary to each other, each blue point is contradictory to one green point and one only (and reciprocally), each blue point implies n-1 green points (all except its contradictory), each green point is implied by n-1 blue points (all except its contradictory). Geometrically speaking, we are done!

3.2 Second result (logical) : the notion of modal "$n(m)$-graph" (from αn-structures to $\alpha n(m)$-structures)

As we have now fulfilled the task of expressing *geometrically*, in a way generalized to any finite integer n, the logical relations constituting opposition (i.e. contradiction, contrariety, sub-contrariety and sub-alternation - viz implication), we want now to give "modal flesh" to our "geometrical skeleton". In other words, the important thing now is, once a decoration with modalities is applied to an αn-structure (i.e. once each vertex of the geometrical structure receives a modality), to check the structure's validity by checking the validity of the (grey) arrows (the checked oppositional αn-structure is **valid** iff *all* its arrows do obtain). One has to check them one by one, to see if the implication (subalternation) they mean of a modality (the arrow's ending point) by another (the arrow's starting point) obtains (i.e. it is true), and this depends on the context given by the chosen "modal graph". As we will see, outside the known case of the hexagons (in fact, the case $n = 3$), this question of the "checking" is not yet straightforward. As it happens, some difficulties must be tackled, something new must be created at this point in order to solve 2 new simultaneous tasks : (1) attributing modalities to the vertices (i.e. to the variables, to the empty places) of the αn-structures in a non-trivial way (there is no use in attributing modalities in a way that never - or always - works, there must be some working and some non-working decorations, as for the square and for the hexagon) ; (2) and then - inside a given decoration - judging, for each of the arrows, if it does obtain for that decoration (if the implication it expresses is valid for that decoration).

The case $n = 3$: "3-opposition theory". We will first examine the already known case of $n = 3$ (that is to say, how to decorate modally the logical hexagon). For reasons to appear later, we introduce the use of **"alphabetical variables"** for modalities : instead of using usual modal operators ($\Box$, $\Diamond$, $\neg\Box$, $\neg\Diamond$, ...), we will adopt Greek letters for the left side (positive modalities) and Arabic letters for the right side (negative modalities, cf. figure 14). The unique rule to be observed here is that the alphabetical order common to the two alphabets (A, B, G, D, E, F, K, L, M, N, ...) is implicative, i.e. each term of each of the two series implies the term, inside the same family, next to him alphabetically (A $\rightarrow$ B, B $\rightarrow$ G, etc.)[13].

We need now to make the already mentioned notion of **modal graph** more precise. A *modal graph* is an arrowed structure furnished with some symmetries. In this starting case (the classical modal graphs, as in [9], p. 149-157) the symmetry seems to be a left-right one, holding between a left side where "positive modalities" (i.e. without negations before them) lay and a right side where "negative modalities" (i.e. modalities preceded by a negation) lay.[14] This classical symmetry has 2 main features : (1) it

[13]Clearly, our alphabetical order is a special one, resulting from some sort of a compromise between different diverging classical alphabets (we have, for instance, G instead of C, etc.).

[14]Truly speaking, even for arrows it is a *central* symmetry (the only restriction to such a centrality : the arrows always follow a top-down direction, i.e. the symmetry does not reverse the arrows' orientation). But given the fact that, in standard modal logic, each side of the modal graphs (left as right) has, for shapes, a further inner *vertical* symmetry (a top-bottom one) the central symmetry seems to reduce, for shapes, to a simpler horizontal symmetry (the mentioned left-right one). However, the central

seems to work like a mirror as for arrows' geometrical concatenations (the same concatenation at both sides via a left-right symmetry) ; (2) it ties each element (i.e. each modality, each arrow's extremity) of one side to an element of the other side, its contradictory negation (this last symmetry being central - left-top corresponds to right-bottom, left-center corresponds to right-center, and so on cf. figure 15).

Additionally, modal graphs have ***layers*** (that is, the number of terms "A", "B", ..., "M", adopted for that graph in each alphabetical family), parametred by an integer m. For $m = 2$, you have A (alpha and alif) and B (beta and ba) ; for $m = 3$ you have A (alpha and alif), B (beta and ba) and G (gamma and jim) ; and so on (cf. figures 15 and 16). For reasons to appear later on, we call the linear, non-branching modal graphs of classical modal logic (think of $S5$) "**modal 3-graphs**" and, in order to take into account the parameter m, we call them still more precisely "**modal 3(m)-graphs**".

All this can also be expressed symbolically (i.e. without arrows) by sets of "**3(m)-relations**", saying, for each element, which element is its (contradictory) negation. The implications previously expressed by the arrows in the modal 3-graph are here encoded by the alphabetical order relative to each alphabetical family (A→B, B→G, etc. ; we assume here transitivity of the arrows, cf. figures 15 and 16).

So, modal graphs constitute a **decoration *method*** in the following sense. To decorate (modally) an hexagon you procede as follows : you give it a Greek value for a blue vertex, an Arabic value for another blue vertex ; the rest of the decoration follows automatically : their respective negations are given to two precise green vertices (the ones centrally symmetric to the previous blue vertices) ; as its value the remaining green vertex receives the disjunction of the two first blue vertices ; as its value the remaining blue vertex receives the conjunction of the first two green vertices (i.e. the conjunction of the negations of the first 2 blue vertices). This can be tried for every possible combination of 2 elements of the modal graph (one Greek, the other Arabic).[15] This method thus suffices : (1) for decorating vertices

symmetry remains anyway observable as far as contradictory negations are concerned (e.g., if "$\Box\alpha$" is left-top, "$\neg\Box\alpha$" is right-bottom, and so on), and such a central symmetry could be brought back into study concerning the arrows themselves (i.e. breaking the mentioned inner vertical symmetry), as we study elsewhere, cf. [12].

[15]Nevertheless, **this decoration method is restrictive**: it always *forces* one green term of the hexagon (and thus, in a dual way, its contradictory, centrally symmetric blue term) to *be identical* to the disjunction of the two blue terms adjacent to it (we have in this case "strong hexagons"). This is coherent with what did Blanché and

(we will see) in a non-useless way and (2) for deciding the truth (or triviality, or falsity) of any such finite decoration of $\alpha 3$-structures (their number is m^2), going far beyond the 4 hexagons of the "logical tetradecahedron". This is not a trivial result, because in this way we can build many more hexagons, true, trivial or false, relative to each of the $3(m)$-graphs - the four hexagons (of figures 3 and 4) were just the "by-product" of the 3(3)-graphs. We need to introduce the following terminology : an $\alpha 3$-structure decorated via a modal $3(m)$-graph will be called an **$\alpha 3(m)$-structure**. Of course, this being an abstract generalization of modal logic (we do not care yet about boxes, diamonds, indexed boxes, indexed diamonds and the like), a further work, not even touched in this paper, will be to interpret with "concrete" modalities the alphabetical variables (but this can be done with no harm, cf. [4] [13]). In this theory we offer thus a very abstract, modally uninterpreted framework.

We give in figure 17 a sample of nine instances of $\alpha 3(m)$-structures, three (one true, one trivial, one false) for each value of m (m = 3, 4, 5). Red $\top$ means that the so-denoted formula is a tautology, red $\perp$ means contradiction, red arrows mean false implications. Remark that the true hexagon of $\alpha 3(3)$ in figure 17 (the one formed with "beta" and "alif") corresponds in fact to Béziau's paracomplete hexagon mentioned before (cf. figure 3)[16]. One important feature to be remarked is that the value of a possible decoration of an $\alpha 3(m)$-structure "evolves" through the values of m, thus meaning that it is relative to the context of evaluation given by the modal $3(m)$-graph. Remark in this sense that the hexagon formed with "gamma" and "ba" is false in $\alpha 3(3)$, trivial in $\alpha 3(4)$ and true in $\alpha 3(5)$. Similarly, the hexagon formed with "alpha" and "jim" is trivial in $\alpha 3(3)$

Béziau. But you could as well accept hexagons where one green term (and, in a dual way, its contradictory blue term) *is implied by*, but *is different from*, the disjunction of the two blue terms adjacent to it (in such case, not taken into account in this paper, we would have "weak hexagons"). Such an implicative but not identical green term (and its blue dual) would still satisfy the logical constraints of the hexagon's arrows. In other terms, strong hexagons are defined by two parameters, weak hexagons by three. Such unrestricted view, studied by Pellissier in [16], leads to many interesting results. In particular, Pellissier proves that strong and weak hexagons together constitute *all* hexagons (which applies to higher αn-structures). In generalizing Blanché and Béziau's intuitive restriction to the strong hexagons by keeping it ourselves, we show, as we will see, that this move, even if restrictive, leads to a viable and fertile theory of the modal decoration of graphs, the "head-body-tail" theory. We argue that this sub-theory, which alone makes possible to elaborate the notion of modal $n(m)$-graph, is the backbone of the general complex one, and is necessary to understand the latter's inner geometrical structure (cf. [11]).

[16]By "$\alpha n(m)$" we mean the logical space of the $\alpha n(m)$-structures.

but true in $\alpha3(4)$. In the same way, the hexagon formed with "delta" and "ba" is false in $\alpha3(4)$ but true in $\alpha3(5)$ (for a richer study of $3(m)$-graphs and $\alpha3(m)$-structures and some very nice results, cf. [11]).

It would be interesting to extend this kind of examination to the case of modal 3-graphs containing branchings (why restrict ourselves to linearity?). This task will not be undertaken here (cf. [12]). The important point to be handled now is rather this one : how to generalize to 4-opposition what was seen so far for 3-opposition ? How to step from the decoration of the $\alpha3$-structure to that of the $\alpha4$-structure (the "logical cube") ? It will show up that the answer to this involves the two following new steps.

First step towards the extended idea of modal $n(m)$-graph ($n \geq 4$) : from "bi-dimensional contradiction" to "multi-dimensional contradiction". The problem is now the following : if we want to use such $3(m)$-graphs to decorate modally the $\alpha4$-structure (the bi-tetrahedron, or cube) we generally fail : such classical $3(m)$-graphs, do not seem to be an adequate tool in order to decorate modally the geometrical $\alpha4$-structure in a sensible (i.e. non trivial) way.[17] Such decorations will generally appear to be violating some of the logical constraints expressed geometrically by the $\alpha4$-structure : no clear model seems possible, the structure does not seem to hold in this way (we omit to give here a proof - combinatorial in nature - of this affirmation, the question is treated fully in [16]). The problem seems to remain even if you add to the $3(m)$-graph complications such as the ones of the modal graphs of $S4$ or of the $K5$-systems (i.e. even if you add branching, instead of linearity). It is very hard to find non-trivial decorations this way (if only possible). This simply means that standard modal logic (as long as it is identified, as it usually is, to the use of 3-graphs, linear or branching) is not fit to decorate "naturally" $\alpha4$-structures. In other words, something new must be done here, if we are to decorate it properly (i.e. non-trivially and easily). And indeed we must : if we do not, the $\alpha4$-structure (and a fortiori any αn-structure, $n \geq 4$) could be thrown away as modally useless.

Now, our new leading remark to solve this problem will be that, despite the fact that we are used to it, in modal graphs contradictions *need not*

[17]For a very deep and clear discussion of this complex point, duly complicating (and clarifying) our present statement, by means of a powerful set-theoretical decorating procedure, cf. Pellissier [16]. This opens to a larger, *in fieri* scope of n-opposition theory, here untouched.

always relate points to points (why should they ?) : they could relate points to lines, points to surfaces, points to 3-dimensional volumes (cf. figure 18), ..., points to n-dimensional volumes, etc., so as to open modal logic to n-dimensional geometry (provided that we interpret such geometrical entities as sets of logically related points - in fact *disjunctions* of such points). And this is precisely all we need to do.

Hence we get the required idea concerning how to change usefully the shape of the modal graphs. We open classical modal graphs to **multi-dimensional contradiction**, by "expanding" the dimensional symmetry of their frame (i.e. by complexifying the simple left-right symmetry). First, we add an additional alphabet (so that there are three : Greek, Arabic and now Hebrew). Then we take a stack of m vertically parallel (black) *triangles* (m is the number of "layers", i.e. the number of triangles). The stack of triangles is entangled with three independent vertical columns of m-1 arrows each (each arrow relates 2 elements belonging to 2 adjacent layers or triangles). To decorate the points (joints) of each column (points which are triangle's vertices), we chose one different alphabet for each column, so that in every triangle, so to say, one vertex is Greek, another is Arabic and the third is Hebrew. Finally, for each term of each alphabet (and, which is the same, for each vertex in a triangle) its contradiction will be defined as the disjunction of the two terms (i.e. a triangle's edge relating them) - of the other alphabets - corresponding to it (this edge happens to be centrally symmetric to the starting term - a vertex -, all modal $4(m)$-graphs being geometrically obtained so that the central symmetry relates each vertex to an edge and each edge to a vertex, cf. figure 21).

So, presumably, in the general case (n = any integer) it suffices to generalize these two moves : (1) generalize the use of the "alphabetical variables" (one alphabet for each n-1 oppositional family of terms : Greek, Arabic, Hebrew, Indian, Japanese, Russian, ...), a simple device that will reveal itself useful in building the new theory (alphabetical terms must be seen as abstract modalities, each family being equally opposed to all the others as, in classical modal logic, positive left-handed modalities are "opposed" to negative right-handed modalities) ; (2) and generalize the use of (black) triangles : for $n = 5$ it will be (black) tetrahedra, ... , etc. (for each n, it will be a black simplex of dimension n-2). This reflects our idea of an extended treatment of contradiction. We pass from a 1-dimensional (left-right) "point-point" treatment of contradiction to a poly-dimensional one. And this brings us back, as we will see later in more detail, to the now familiar series of the simplexes of dimension n (cf. figure 18).

Second step towards the extended $n(m)$-graphs ($n \geq 4$) : opposition terms can be "heads", "bodies" or "tails". We made previously a *geometrical* model of n-opposition theory (cf. section 3.1 above) without spending a word about the number and shape of the "composite modalities" (as seen for $n = 3$). Effectively, for $n = 3$, we saw that each hexagon has one "or" term and one "and" term (think of Blanché's discovery : the U and the Y vertices). The crucial point is then : how to generalize this ? Given that (1) it can be shown that in each αn-structure there must be at least 1 composite term (this is what Blanché has implicitely shown against Aristotle : not the square but 2 triangles forming a hexagon) and given that (2) it can't be shown that there can be no more than 1 such composite term in each αn-structure (as testified by Moretti and Smessaert's fourth emerging hexagon, and more deeply by Pellissier, cf. [16]), it seems that there is room for the taking of decisions. At the present, the most natural solution seems to consist in keeping, in each αn-structure, one and one only such "or" term and one and one only such "and" term (for any n, the composite "or-term" will be a green disjunction of n-1 blue simple terms, whereas the composite "and-term" will be a blue conjunction of n-1 green simple terms).[18] This solution works. We call it the **"head-body-tail theory"**.

So, in each n-structure, among the $2n$ opposite terms (n contraries and, contradictory to them, n subcontraries), 2(n-1) should constitute some kind of basis (so they must be named with n-1 different alphabets), while the n-th couple of contradictory (i.e. centrally symmetric) terms is just the composition of the previous n-1 (more precisely, the singular green composite term is the disjunction of the n-1 basic - or "pure" - blue contraries, and the singular blue composite term is the conjunction of the n-1 basic - or "pure" - green sub-contraries). We call the singular blue composite term "**tail**", the singular green composite term "**head**", the rest of the blue terms "**(blue) body**", and the rest of the green terms "**(green) body**". Each αn-structure has, finally, 1 head (green), 2(n-1) terms in the body (n - 1 being blue, n - 1 being green) and 1 tail (blue) (cf. figure 19).

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[18] This choice of ours results in concentrating on a simpler family of logical hexagons (and further αn-structures), the "purest" in some sense, among all the possible ones ("strong hexagons" instead of "weak hexagons", according to a terminology we owe to Pellissier, cf. [16]). This restrictive choice, as we will see, allows to find beautiful orderings, leaving for further investigations more complex explorations of the general field, where the present prima facie articulations will reveal themselves very useful to structure the "peripheral" knowledge.

The case $n = 4$: there is room for a "4-opposition theory". In order to decorate the $\alpha4(m)$-structures we want to consider now the set of possible modal $4(m)$-graphs (m is the number of "layers"). First we introduce a new alphabetical family, Hebrew (cf. figure 20).

According to what was previously said, each $4(m)$-graph is a column composed of m triangles. Each triangle is made out of 3 terms (its vertices) belonging to the three opposed modal families (here : one "Greek", one "Arabic', one "Hebrew", cf. figure 21). In each triangle each edge relating two vertices will be read as the (inclusive) logical disjunction of these two vertices.

The big change is that now contradiction, for each term X on the modal graph, is defined as the *disjunction* of the two terms Y and W most far from it (the truth of Y or the truth of W suffice to make X false ; X is true *iff* both Y and W are false). As we will now illustrate by some examples, each of such $4(m)$-graphs suffices to decorate the $\alpha4$-structure, but each does it in a different way, thus specifying it in an $\alpha4(m)$-structure (cf. figure 21).

The relations depicted graphically by the $4(m)$-graphs can be expressed symbolically by sets of "$4(m)$-relations", the implications previously depicted by the arrows being implicitely contained in the alphabetical order inside each oppositional family : A implies B, which implies G, which implies D, ..., which implies M (more particularly, inside each alphabetical family : alpha implies beta, alif implies ba, aleph implies beth, etc., cf. figure 22).

As already said, the use of a modal $4(m)$-graph relatively to an $\alpha4(m)$-structure is twofold. First, it shows which decorations (with modalities) are possible from a purely combinatorial point of view : all possible triples of terms so that one is Greek, another is Arabic and the other is Hebrew (all other terms of the decoration being then mechanically determined : the 3 respective green negations of the 3 blue elements constituting the starting triple, the green disjunction of three mentioned blue terms, and the blue conjunction of the three green negations). The number of such possibilities in $\alpha4(m)$ is m^3. Second, it provides a criterium as to which decorations, among all possible ones, are logically viable (there will be true, trivial and false decorations). In this respect one can see modal $4(m)$-graphs as some kind of 3-dimensional modal oppositional "truth tables", on which to rely in order to check the validity of oppositional modal formulas.

We give here two examples of such truth-value calculations via the modal graphs. In the first calculation we want to see if "alpha or alif or beth" is a tautology inside $\alpha 4(3)$. Using the modal 4(3)-graph we see that a counter-example of it is possible ("not alpha and not alif and not beth").[19] So the starting formula is not a tautology of $\alpha 4(3)$ (cf. figure 23).

In the second calculation we propose here, we want to check the validity of the formula "beta or ba or guimel" inside the logical space of $\alpha 4(4)$. Using the modal 4(4)-graph corresponding to it, we see that it is impossible to draw a counter-model of it without contradiction ("not beta and not ba and not guimel"), hence the starting formula is valid, it is a tautology of $\alpha 4(4)$ (cf. figure 24).[20]

Enjoying this new modal tool, we can check all possible cases inside $\alpha 4(m)$. As an example of this, we will give, first, some instances of possible decorations of the $\alpha 4(3)$-structure by the modal 4(3)-graph, that is, the graph formed by a stack of 3 black triangles (the 3 "layers"). Among the 27 possible decorations one can find three kinds of issues : an $\alpha 4(3)$-structure can be true, trivial or false. It is ***true*** when all arrows obtain *and* when the head is not a tautology (equivalently : the tail is not a contradiction). It is ***trivial*** when all arrows obtain *but* the head is a tautology (equivalently : the tail is a contradiction). It is false when some arrows are false (cf. figure 25).

If we consider now a further layer ($m = 4$) in the modal $4(m)$-graph, the reasoning is the same. There will be 64 possible cases here, some true, some trivial and some false. We give, in the figure, an instance of each type within the possible decorations of the $\alpha 4$-structure by the modal 4(4)-graph. Remark that one decoration which was trivial in $\alpha 4(3)$ ("beta or ba or beth") is true in $\alpha 4(4)$. Remark also that not all false decorations inside a given $\alpha n(m)$ make the head tautological (equivalently : the tail contradictory) : we see here that the decoration with "alpha", "jim" and "guimel" makes the $\alpha 4(4)$-structure false without making its head ("alpha or jim or guimel") tautological (equivalently : without making its tail "not alpha and not jim and not guimel" contradictory, cf. figure 26).

Same story, again, for a modal 4-graph with now five layers (i.e. a stack of 5 black triangles), the modal 4(5)-graph necessary to decorate the $\alpha 4(5)$-

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[19]Explanation of the graphical deductions depicted in figure 23. One passes from (3) to (4) because something false (as in (3)) cannot be implied by something true (as would be in (4)); then "(1) and (2)" imply (7), "(2) and (4)" imply (5), and "(4) and (1)" imply (6) because of the definition of contradictory negation inside 4-graphs.

[20]Explanation of the graphical deductions in figure 24. (1) and (2) imply (4) by the definition of contradictory negation in 4-graphs; but (4) and (3) lead to contradiction.

structure. Among the 125 possible decorations (i.e. 125 possible instances of $\alpha 5(5)$-structures) a check of the arrows and of the heads via the 4(5)-graph will give the three usual kinds of results (true ones, trivial ones and false ones, cf. figure 27).[21] Here, as well, we remark that a decoration (the A G G)[22] which was bad (false) with a trivial head in $\alpha 4(3)$ and bad (false) with a non trivial head in $\alpha 4(4)$ is fine (true) in $\alpha 4(5)$ (cf. figure 27).

So we take act that $\alpha 4(m)$-graphs constitute an adequate (i.e. non-trivial) tool to decorate the $\alpha 4(m)$-structures with modalities. The $\alpha 4$-structure (the "logical cube" presented in this paper) is thus neither trivial nor useless. In this way, 4-opposition theory works. But can opposition theory be developped further, considering the case $n = 5$?

The case $n = 5$: there is room for 5-opposition theory. [23]

What happens with $n = 5$? Again, it can be shown that in some sense neither 3-graphs nor 4-graphs can decorate the $\alpha 5$-structure. This means that if we want to go to the next step ($n = 5$), we just need to be able to deal with *four* (instead of three) oppositional families of abstract modal terms (four families of concatenated arrows, if m is bigger than 1 - for $m = 1$ there are no arrows) : the fifth term is a head-tail pair (we still change the geometrical quality of contradiction, we have, this time, a **point-surface contradictory negation**, cf. figure 29).

So we introduce $5(m)$-graphs in the usual way, by the adjunction (to the previous case) of a fourth term (welcome to the Indian-Sanskrit guys), passing thus from a (black) triangle to a (black) tetrahedron. The number of such (black) tetrahedra constituting (in a "column" or stack) the $5(m)$-graph is m (there are m "layers", so to speak, cf. figure 29). As we will show by some examples, each of such modal $5(m)$-graphs suffices to decorate the $\alpha 5$-structure with modalities.

[21] Truly speaking, inside $\alpha 4$-structures and higher, there can be more than 3 issues, if inside the set of false decorations one makes an inner distinction according to the number of false arrows and the quality of the head, tautological or not. But this bears no consequences at our level, results thereupon will be given elsewhere.

[22] By "A G G" we mean, of course, the ordered triple consisting in the A-like Greek term, the G-like Arabic term and the G-like Hebrew term (the same kind of lecture ruling, naturally, also when other capitals are available, or when we deal with a different number of them).

[23] This case could have been omitted (as boringly similar to the previous one). We give its explicit development in order to familiarize the reader with handling α-structures more than 3-dimensional (as the 4-dimensional $\alpha 5(m)$-structures here, and in order to familiarize her/him with the handling of modal graphs of increasing geometrical complexity.)

Here as well, the relations depicted geometrically by the $5(m)$-graphs (as in figure 29) can also be expressed symbolically by a set of "$5(m)$-relations" (cf. figure 30, where we omit, however, the expression of the implications - we could state them one by one -, implicitely encoded in the alphabetical order of each series of variables).

As in the previous case ($n = 4$), the use of a modal $5(m)$-graph is double. First, it shows which decorations (with modalities) are possible from a combinatorial point of view. That is to say, simply all possible 4-tuples of terms such that one is Greek, another is Arabic, another is Hebrew, a fourth is Indian (these first 4 will be blue ; all other terms of the decoration being then automatically determined : the 4 green negations of the first mentioned 4 blue, their green disjunction and the blue conjunction of the green negations). The number of the possible combinations in $\alpha5(m)$ is m^4. Second, it provides a criterium as to which decorations, among all possible ones, are logically viable (as before, there will be true, trivial and false decorations). In this respect one can see a modal $5(m)$-graph as some kind of 4-dimensional modal oppositional "truth table", on which to rely in order to check the validity of oppositional modal formulas (easy to use in its 2-dimensional paper projection, cf. figures 29, 31, 32).

As before, we give here two examples of such truth-value calculations via the modal graphs. In the first calculation we want to see if "alpha or alif or beth or ba" is a tautology inside $\alpha5(3)$. Using the modal 5(3)-graph we see that a counter-example of it is possible ("not alpha and not alif and not beth and not ba"). So the starting formula is not a tautology of $\alpha5(3)$ (cf. figure 31).[24]

In the second calculation proposed, we want to check the validity of the formula "alpha or ba or ba or da" inside the logical space of $\alpha5(4)$. Using the modal 5(4)-graph corresponding to it we see that it is impossible to draw a counter-model of it without contradiction ("not alpha and not ba and not ba and not da"), hence the starting formula is valid, it is a tautology of $\alpha5(4)$ (cf. figure 32)[25].

Enjoying this new modal tool, we can check all possible cases inside

[24]Explanation of figure 31. (3) implies (5), and (4) implies (6) because something false cannot be implied by something true. "(1), (2) and (5)" imply (10), "(2), (5) and (6)" imply (7), "(5), (6) and (1)" imply (8), "(6), (1) and (2)" imply (9) because of the definition of contradictory negation in 5-graphs.

[25]Explanation of figure 32. (2) implies (5), and (3) implies (6) because something false cannot be implied by something true; "(1), (5) and (6)" imply (7) because of the definition of contradictory negation in 5-graphs; but (7) and (4) lead to contradiction.

$\alpha 5(m)$. As an example, we will first give some instances of possible decorations of the $\alpha 5(3)$-structure by the modal 5(3)-graph, that is, the graph formed by a stack of 3 black tetrahedra (the 3 "layers"). Among the 81 possible decorations here one can find three kinds of issues : an $\alpha 5(3)$-structure can be true, trivial or false. As before, it is *true* when all its arrows obtain *and* when its head is not a tautology (equivalently : its tail is not a contradiction). It is *trivial* when all arrows obtain *but* the head is a tautology (equivalently : the tail is a contradiction). It is *false* when some arrows are false (cf. figure 33).

If we consider now a further layer ($m = 4$) in the modal graph, the reasoning is the same. There will be 256 possible cases here, some true, some trivial and some false. We give, in the figure, an instance of each type within the possible decorations of the $\alpha 5$-structure by the modal 5(4)-graph. Remark that the decoration with "alpha", "ba", "beth" and "ga", which was false in $\alpha 5(3)$, is now true (cf. figure 34).

We give, thirdly, some instances of possible decorations of the $\alpha 5$-structure by the modal 5(5)-graph. We have here 625 possible cases, among which true ones, trivial ones and false ones. Remark that the "gamma", "jim", "guimel" and "ga" decoration, which was false in $\alpha 5(4)$, is now trivial (cf. in figure 35).

We stop here this list of modal $n(m)$-graphs. It is easy to see that it can go on with no limit (however, we give here no proof of this sentence - we will give it fully elsewhere).

So there is room for generalized n-opposition theory, $n \geq 4$. It seems that all which we have seen can be generalized to any finite n. A **modal $n(m)$-graph** (i.e. a modal n-graph with m layers) is defined as a stack of m (black) simplexes of dimension n-2 (the "gems", cf. figures 36 and 37), each containing n-1 terms belonging to n-1 different alphabetical families ; between each couple of adjacent black simplexes lays a set of n-1 arrows, each arrow relating each upper alphabetical term to the lower one corresponding alphabetically to it. Each αn-structure deserves a n-graph in order to be decorated usefully, and each $n(m)$-graph specifies the αn-structure into a multiplicity of **$\alpha n(m)$-structures** (same geometrical shape, but different truth-value of the same decorations). And $n(m)$-graphs do work nicely : they are some kind of (n-1)-dimensional modal oppositional "truth-tables" (easy to use in their 2-dimensional paper projection). And given that for any n there is an αn-structure and a modal n-graph, for any n there is an adequate n-opposition : this is "**n-opposition theory**", the general framework we were looking for. This double-sided algorithm works for any finite integer values of n and m. It uses twice the series of the simplexes (of dimension n-1 and of dimension n-2, respectively) and one can imagine a rather simple proof, by recursion, of this generality relying (twice) precisely on the notion of simplex of dimension N (however, we must omit here to give the precise proof of this).

3.3 Some theorems of n-opposition theory

Before concluding, we give here some simple and intuitive definitions and theorems, useful to make quicker calculations for $\alpha n(m)$-structures within the frame of modal $n(m)$-graphs.

Simple theorems concerning the graphic treatment of modal $n(m)$-graphs in terms of their "gems". Inside modal $n(m)$-graphs it is useful to introduce the general notion of "gem" (i.e. the black structure - a simplex - in each layer of a $n(m)$-graph).

Definition 1 : we call "**$n(m)$-gem**" (for short, in what follows: "gem") each of the m "simplexes of dimension (n-2)" characterising a modal $n(m)$-graph.[26]

Definition 2 : a gem is "**lower**" than another *iff* it can be reached from this last by means of the oriented arrows of the $n(m)$-graph, i.e. *iff* each element of the second gem implies, by a finite series of concatenated arrows, one and only one element of the first gem, leaving no element of the first not implied.

Definition 3 : a gem is "**higher**" than another *iff* it is neither lower than it nor identical to it (there are no unordered gems inside a modal $n(m)$-graph).

Definition 4 : a gem is "**symmetric**" with respect to another gem *iff*, for each of the two gems, each of its n-1 terms is defined as the negation of the disjunction of the other n-2 terms of the other gem which are not of the same alphabetical family as the first term (the truth or falsity of an

[26] Differently from a blue simplex (of dimension n-1) of contrariety, which has n terms and is (n -1)-dimensional (resp. differently from a green simplex of sub-contrariety ...), a $n(m)$-gem has (n -1) terms and is (n -2)-dimensional : it just has simple terms, i.e. "mono-alphabetical" ones, without propositional binary connectives : it does not have the composite term "tail" "$\neg a \wedge \neg b \wedge \ldots \wedge \neg(n-1)$" (resp. it does not have the composite term "head" "$a \vee b \vee \ldots \vee (n\text{-}1)$").

element of a given gem is always to be checked in the symmetric gem, as being the negation of the disjunction of the elements - of the families other than the first - of this last).

Definition 5 : we call "**descendant**" of an element of a given gem every element of every gem lower than the first one such that that element belongs to the same alphabet as the first element (i.e. it comes from the first by a finite series of implications).

Definition 6 : we call "**ancestor**" of an element of a given gem every element of every gem higher than the first one such that that element belongs to the same alphabet as the first element (i.e. it leads to the first by a finite series of implications).

Definition 7 : we have a "**true gem**" (green) *iff* all its elements are true (green).

Definition 8 : we have a "**false gem**" (red) *iff* all its elements are false (red).

Definition 9 : we have a "**normal gem**" *iff* some of its elements are true (green) and some of its elements are false (red).

Theorem 1 : if an element of a gem is true (green) then all its descendants are true (green).
(Proof : something true cannot imply something false).

Theorem 2 : if an element of a gem is false (red) then all its ancestors are false (red).
(Proof : something false cannot be implied by something true).

Theorem 3 : if a gem is true (green) then all lower gems are true (green).
(Proof : suppose that, with respect to a green gem, some lower gem is not green. Then at least one of the elements of this gem is false. But then all ancestors of this element must be false, including the one belonging to the starting green gem, which is impossible).

Theorem 4 : if a gem is false (red) than all higher gems are false (red).
(Proof : suppose that, with respect to a red gem, some higher gem is not red. Then at least one of the elements of this gem is true. But then all descendants of this element must be true, including the one belonging to the starting red gem, which is impossible).

Definition 10 : a gem is "**central**" *iff* one of the following equivalent conditions obtains : (1) when it lays at the same distance from the first and from the last gem of the modal $n(m)$-graph (m has to be odd) ; (2) when each of its elements is defined as the negation of the disjunction of the other elements (of the same gem) ; (3) when it is symmetrical to itself.

Definition 11 : a gem is "**peripheral**" when it is not central.

Theorem 5 : if a gem is central, it contains one and only one true element (green), all other elements of that central gem being false (red).
(Proof : (a) suppose it contains no true elements : then it is impossible to satisfy the constraint of poly-dimensional contradiction, defining, for every element of the gem, the falsity of that element (there is no contradictory green element available for that), which leads to contradiction. (b) Suppose the gem contains only true elements : than it is impossible to satisfy the constraint defining, for every element of the gem, the truth of that element : there are no red elements available for that, which again leads to overt contradiction. (c) Suppose there is, in that central gem, at least one false (red) element and at least two true (green) elements : by definition of truth inside modal $n(m)$-graphs, no element can be true (for an element of a gem, in order to be true, the disjunction of all other elements of its symmetric gem should be false), which leads, once more, to open contradiction).

Definition 12 : a gem is "**superior**" when it is not central and it belongs to the higher half of the stack (i.e., if there is a central gem, when it is higher with respect to the central gem).

Definition 13 : a gem is "**inferior**" when it is not central and belongs to the lower half of the stack (i.e., if there is a central gem, when it is lower with respect to the central gem).

Theorem 6 : if a gem is superior, then it is not true (not all green).
(Proof : if that superior gem were true (green), all implied elements (and thus all lower gems) should be true (green). But then there would be no false (red) elements in the symmetric gem, which must be inferior, and therefore no element of the first gem could be true (green), which leads to contradiction).

Theorem 7 : if a gem is inferior, then it is not false (not all red).
(Proof : suppose an inferior gem is red ; then all its ancestors - and thus all the gems higher than this first - should be false (red). But then there would be no true (green) elements in the symmetric gem which must be superior, and therefore no element of the first gem could be false (red), which leads to contradiction).

Theorem 8 : if a superior gem is false (red), then its symmetric gem is true (green).
(Proof : suppose a superior gem is red and its symmetric inferior gem is not green, i.e. it is red or normal. Then at least one element of the inferior gem is false (red), which is impossible, because there are no green elements available in the superior red gem to support this).

Theorem 9 : if an inferior gem is true (green), then its symmetric is

false (red).
(Proof : suppose an inferior gem is green while its symmetric superior gem is not red - i.e. it is green or normal. Then at least one element of that superior gem is true (green), which is impossible, because there are no red elements in the inferior green gem to support that).

Theorems and terminology of this kind allow quicker calculations and deductions with modal $n(m)$-graphs. As examples of applications of theorem 8, cf. figures 23 and 31. As examples of applications of theorem 2, cf. figures 23, 31 and 32.

A more general theorem concerning modal oppositional implications. There is one big useful theorem ruling all simple implications of one alphabetical term by another belonging to a different alphabetical family.

General Theorem : inside each $n(m)$-graph (that is, inside each $\alpha n(m)$-structure) the value (true or false) of all possible simple implications of one simple alphabetical modality by another one (the two belonging to two different alphabets) is given by a matrix (as the one given in figure 38) having m rows and m columns, such that in the place determined by the i^{th} row and the j^{th} column it contains the implication of a consequent "$\neg J$" by an antecedent "I". Such an implication is true *iff* it belongs to the "upper left" triangular half of the matrix (diagonal included), otherwise it is false.

Explanation : We give a graphical expression of it, as a matrix with m rows and m columns (in each place of the matrix there is a simple implication of two simple alphabetical modalities, cf. figure 38). The prime signs suffixing the consequent mean that this last term belongs to an alphabetical family different from the one of the antecedent (anyone different from it). So, for instance, "A $\rightarrow \neg B$' " means that any term "A" (i.e. alpha, alif, aleph, ...) implies the negation of any term "B" of the families other than the first one (the Greek "A" implies the negation of Arabic, Hebrew, Indian, ... "B" ; the Arabic "A" implies the negation of Greek, Hebrew, Indian, ... "B", and so on). The theorem covers all possible simple cases of implications. For each $n(m)$-graph, in order to check simple implications, one has to draw the adequate matrix (one with m rows and m columns) and then just read it! The proof of this theorem is simple but a bit tedious, we will give the full version of it elsewhere. Here we will just give a sketch of it (almost all steps are done by recursion).

Sketch of the proof : First, one has to prove that for all $\alpha n(m)$ we have

A→¬M'. Then, one generalizes that result by proving that for all $\alpha n(m)$ and for all p (with $0 \leq p \leq [m\text{-}1]$) we have (A+$p$)→¬(M-$p$)'.[27] This last proves that in each adequate matrix the terms on the left-bottom/right-top diagonal are true (true simple implications). The last two steps consist then in proving that in each row of the matrix all the implications *preceding* (i.e. at the left side of) the one on the diagonal are true and, lastly, that for each row of the matrix all the implications *following* (i.e. at the right side of) the true one in the diagonal are false, which ends the proof of the theorem. The first is done, as usual, by recursion (in fact, a finite series of embedded recursions). First we prove that for all $\alpha n(m)$ we have (A→¬M')→(A→¬K') (with [K]≤[M]) and thus, by modus ponens, that for all $\alpha n(m)$ we have A→¬K' (for all K such that K≤M). We generalize that by proving that for all $\alpha n(m)$, (A+p)→¬K' (with [K]≤[M-p] and $0 \leq p \leq$[M-1]) (the left-top triangular half of the matrix contains the true simple implications). Similarly, for the last point, we demonstrate that for all $\alpha n(m)$ ¬((B+p)→¬(M-p)') (with $0 \leq p \leq$[M-2]), which we generalize by proving that for all $\alpha n(m)$ ¬((M-p)→¬K') (with $0 \leq p \leq$[M-2]) (the "right-bottom" triangular half of the matrix contains the false simple implications).

4 Conclusion and perspectives

It is difficult to judge a theory which is new, especially if you are the author. Nevertheless it is time for us to try to draw some general guidelines. Our theory of n-opposition offers (or reveals) a possible geometrical side to modal logics. But is it the only possible - or the best - one ? As such this question is too wide, we won't be able to treat it here. In order to sum up about the general question of the relations between (modal) logic and geometry *in as much the present theory is concerned and can bring some lights*, we will briefly recall what has been done here, then we will evoke what should or could be done next, ending with some more philosophical considerations.

[27] By "(A+p)" we mean the alphabetical letter whose (numerical) rank in the alphabet is the one of "A" plus the integer "p". So, for instance, in the alphabet used here (cf. figure 14) "(B+3)" is "E" ; "(C+1)" is "D", and so on. By "[M+p]" we mean the the number given by the sum of the rank of "M" (in our chosen alphabet of figure 14) and the number "p".

What has been achieved here. We indicated in which way such a new theory commands a necessarily two-folded approach of opposition, in terms of geometrical "$\alpha n(m)$-structures" first, to cope with the more geometrical part of the treatment of opposition, and then in terms of modal linear "$n(m)$-graphs", in order to cope with the more logical side of the job, consisting in decorating with abstract modalities the open series of geometrical multi-dimensional αn-structures. The key ingredient, in both parts of the theory, happens to be the mathematical notion of "simplex of dimension n" (giving the "bi-simplexes of dimension n-1" in the first case, the "$n(m)$-gems" in the second).

The theory produces then two noticeable novelties in the logical knowledge. First, it shows that the number of instances of logical squares, hexagons and tetradecahedra (the already known structures) is potentially infinite, inside 3-opposition, with respect to the possible modal decorations (this is generalized $3(m)$-opposition theory). This result is perhaps more spectacular than it seems, it produces for instance an infinite series of $\beta 3(m)$-structures of which the beautiful tetradecahedron is just the first and simplest element (we develop this in [11]). Second, it shows that there are, outside the two known α-structures ($\alpha 2$ and $\alpha 3$, i.e. square and hexagon), an infinity of other αn-structures ($\alpha 4$ or "cube", $\alpha 5$, $\alpha 6$, ..., αn, ..., we showed here the first three new ones, but gave the general intuitive law).

It must be noticed that in the present paper **we restricted ourselves quite much** by considering only "strong" hexagons (i.e. by imposing our "head-body-tail" construction principle over the decoration of the α-structures), instead of considering all possible "weak" hexagons (we owe this terminology to Pellissier, cf. [16]). This was a necessary move in order to elaborate and highlight the useful notion of "modal n-graph". But after reading a pre-final draft of the present paper, and relying on it, Pellissier investigated the more general case (weak hexagons) - by means of a set-theoretical decorating technique he elaborated - and found very interesting and very strong results, generalising ours. In particular, he has shown that, in $\alpha 3(3)$ *all* hexagons (including ones with modal terms composed of several binary propositional connectors - the 2 first of which had been discovered independently from Pellissier by Hans Smessaert) do collect themselves in a 3-dimensional "logical tetraisocahedron" (which, according to us, is composed of three tetradecahedra, among which Beziau and Moretti's one). This result is very important, because this elegant figure is some kind of real closure of the field we described for n=3 and m=3. Besides, Pellissier has found more general results, concerning the whole of modal graphs theory (cf. [16]).

Again, the **principal known application so far** seems to be the establishment of a series of translation rules (between modal logic and solid geometry), by now all inside 3-opposition (in this paper we saw just the simplest one and discussed some possible issues, but in [11] we expand it considerably). This result is already nice (it can, for instance, be suitably adapted to branching inside standard modal logic), but its lack of generality - we could have, but we still have not, translation rules based upon βn-structures with $n > 3$ - brings us to the following questioning remark.

What should or could be done next. There are two further crucial distinctions, one inside the structures, the other inside the graphs. Inside the geometrical structures we distinguish between α-structures and β-structures (as already said, β-structures are higher-dimensional structures gathering together nicely a multiplicity of α-structures, as the tetradecahedron does with respect to the four hexagons). Inside the modal graphs we distinguish between *linear* $n(m)$-graphs and *branching* $n(m)$-graphs (branching graphs, as the one of $S4$, can be obtained as combinations of partially different linear graphs, as the one of $S5$, these last ones being translatable into β-structures, as $S5$ is translatable into the tetradecahedron).[28]

The theory would be almost "perfect" in its architectural harmony if it was not for a persisting lack, the fact that we still know of no equivalent of the tetradecahedron (which belongs to 3-opposition) for 4-opposition (the case of the logical cubes) and beyond. In other terms, the **open question** sounds : is there, as well, an open (infinite) series of βn-structures (each term of this series presumably "fractalized", itself, by m into an infinite series of $\beta n(m)$-structures) ? A positive answer to this question would, according to us, definitely assess the theory and the novelty and legitimacy of this "strange new field" of modal logic. But such a positive answer is not yet available. The first step in answering this, the question of the existence of a 4-dimensional $\beta 4$-structure ordering nicely the 3-dimensional logical cubes (or $\alpha 4$-structures) is still open (the problem is more difficult in this case than in the tetradecahedron's case, because contradiction is now defined as a conjunction of negations, instead of as an unique negation, and this makes things a bit harder). If the answer to this general further question were "no", meaning that the tetradecahedron (and its infinite fractal

[28] The geometry of branching graphs is yet another topic, cf. [12] and [13].

series inside $3(m)$-opposition, with a varying m) is a poor lone boy (or girl), then n-opposition theory would seem much less elegant and balanced. If not the tetradecahedron itself, its whole family of $(p+3)$-dimensional "p-hyper-tetradecahedra" (as we call them in [11]) would be, again, an "hapax legomenon", a strange geometrical fragment lost in a not so geometrical world of modal logics. We tend to believe that, despite the difficulties up to now in establishing general "βn-structure theory", there is probably one such theory not far, which someone will bring someday into light.

Among **more distant questions** we can mention:

1) the question whether there is a meaning in extending our present theory of n-opposition to the case where n belongs to Z (can "negative n-opposition" be meaningful?);

2) the question wether it is possible to conceive a similar theory with a number of oppositions different from 4 (i.e. contradictions, contrarieties, sub-contrarieties and sub-alternations);

3) the question wether we can conceive some non standard version of the present theory, that is, taking in the metalevel (or metalanguage) some non standard logic, instead of classical logic (these last two questions I owe to discussions with J.-Y. Béziau);

4) the question of the possible relations of n-opposition theory to n-categories theory. As it seems, n-opposition (modal graphs) is contained into 2-categories, but can be developed so as to become n-categorical (Moretti and Pellissier, joint paper to come);

5) the question of the relation of this theory to other "logical-geometrical" issues, such as linear logic or multi-dimensional modal logic (these last reflection I owe to Alexandre Costa-Leite).

Perspectives. It is now legitimate to go back to the principal question which originated our paper: do we have, when dealing with the logical "squares", "hexagons" and "tetradecahedron", a new field in (modal) logic (aren't such new structures just curious but irrelevant cases ?) ? Our paper was an attempt (we believe successful) to answer by a strong "YES" to that question. We testimony the birth of a new field of (modal) logic, strongly interrelated to geometry, where strange entities as the logically arrowed squares, hexagons and tetradecahedra do pullulate happily. We tried to show that these three emerging structures really need a reformulation of old opposition theory and that such a reformulation, in terms of our own theory of n-opposition, is possible and, by now, effective.

Again, the other important general open question concerns the status of the relations between (modal) logic and geometry (i.e. what can be done maximally in this direction).

More philosophical remarks From a philosophical point of view, the relevant question is, according to us, twofold:

1) does n-opposition theory bring new insights on the foundations of logic? In some sens it seems it does: it changes Aristotle's (largely shared) views (think of H. Slater's "sacred" use of Aristotle for criticizing paraconsistent logics); it shows some constitutive links to the problem of the meaning of negation (in particular paraconsistent negation) and thus to Béziau's elaboration of a "universal logic" (cf. [2], [3]); it is a very abstract version of modal logic; it incorporates some of Vasil'ev's most essentials ideas about the relations between logic and geometry (cf. [15]); it has important links to category theory and thus to topos theory (thus being possibly related to the ambitious and impressive contemporary philosophical project of A. Badiou).

2) Does n-opposition theory open a viable answer to P. Gardenfors radical objection to the modeling pretensions of logic over concepts? This question, at this stage, remains open, but it will be interesting, in the future, to see if our theory, somehow extended, will be able - being logical *and* geometrical - to express the logically untractable "conceptual spaces".

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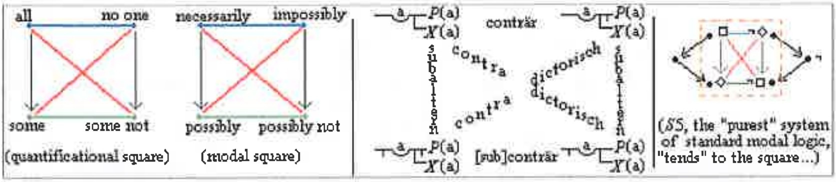

Figure 1. Instances of the "logical square" by Aristotle, Frege and C.I. Lewis

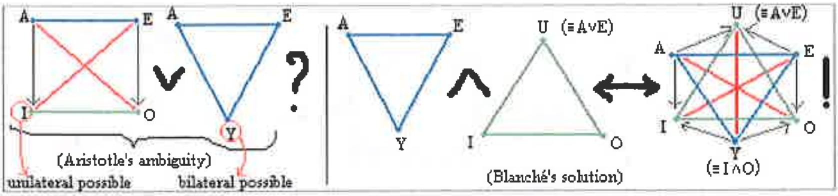

Figure 2. Aristotle's aporia and Blanché's solution (the "logical hexagon")

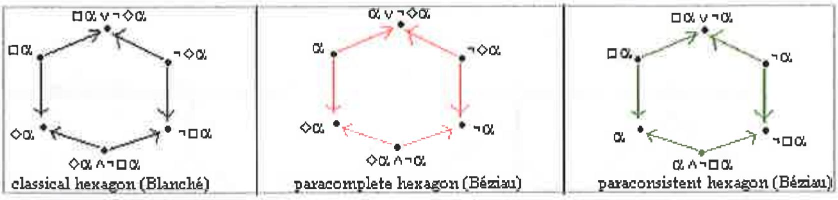

Figure 3. Béziau's two new modal hexagons (paracomplete and paraconsistent)

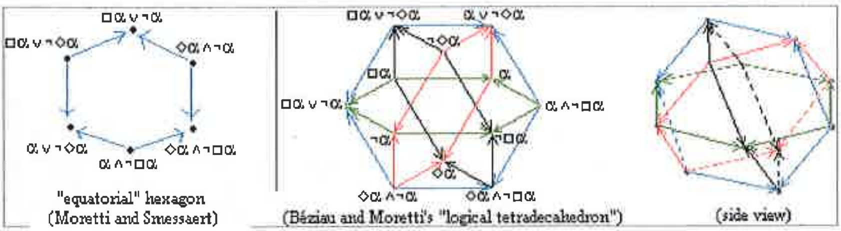

Figure 4. Moretti and Smessaert's fourth hexagon (in blue), Béziau and Moretti's "logical tetradecahedron" (ordering the four hexagons)

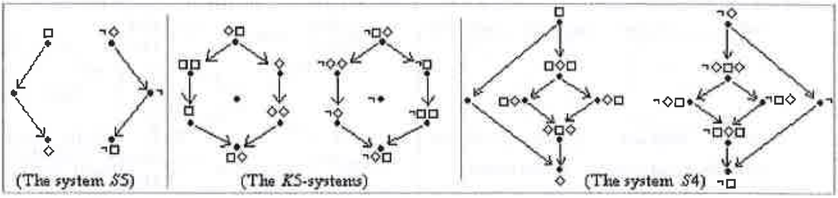

Figure 5. Three examples of classical "modal graphs" (from Chellas modified)

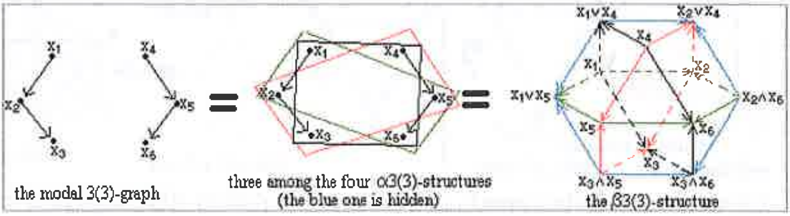

Figure 6. Our translation rule between the modal and the geometrical spaces

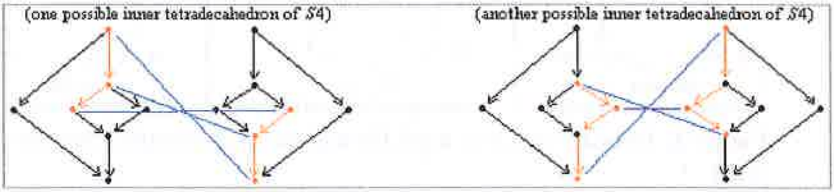

Figure 7. How many "non-emergent" tetradecahedra in $S4$? Just count them!

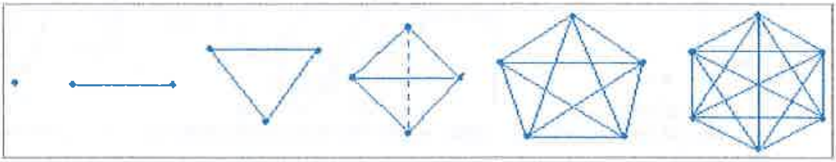

Figure 8. The series of the simplexes of dimension n ($n = 0, 1, 2, 3, 4, 5$)

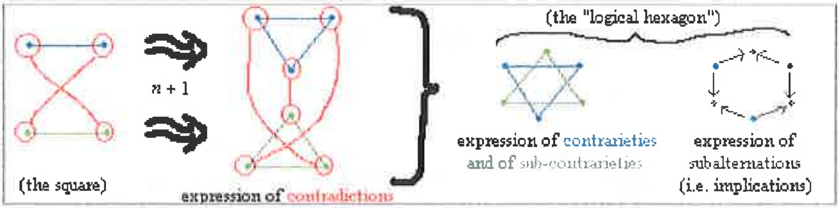

Figure 9. The opposition relations inside the "α3-structure" (or logical hexagon)

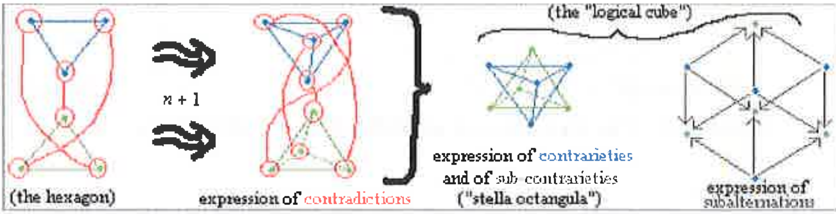

Figure 10. The opposition relations inside the "α4-structure" (or "logical cube")

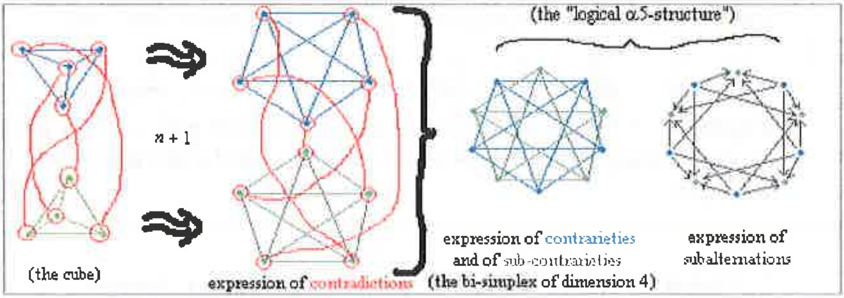

Figure 11. The opposition relations inside the "α5-structure"

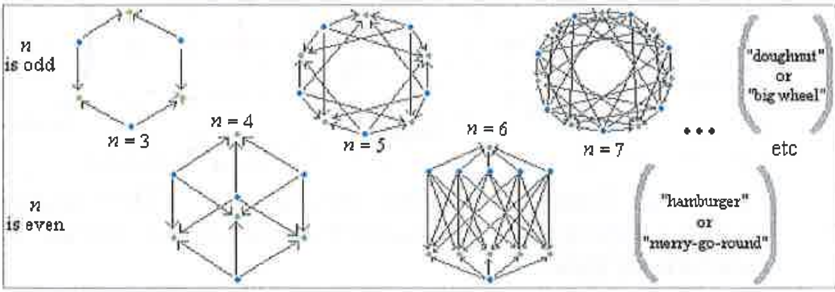

Figure 12. Some oppositional αn-structures ($n = 3, 4, 5, 6, 7$)

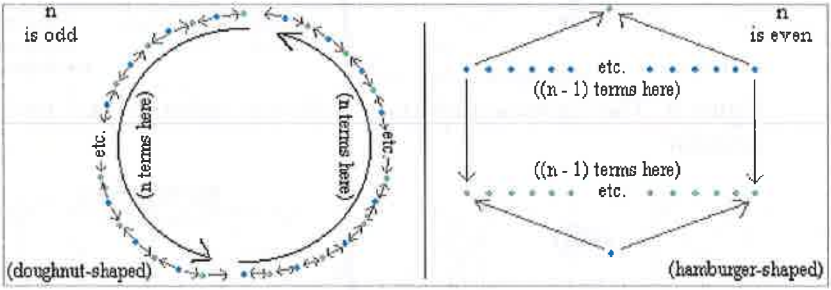

Figure 13. The two general models of the n-structures : for n is odd or even

A	B	G	D	E	F	K	L	M	N	(etc.)
α (alpha)	β (beta)	γ (gamma)	δ (delta)	ε (epsylon)	φ (phi)	κ (kappa)	λ (lambda)	μ (mu)	ν (nu)	(etc.)
ا (alif)	ب (ba)	ج (jim)	د (dal)	ع ('ain)	ف (fa)	ك (kaf)	ل (lam)	م (mim)	ن (nun)	(etc.)

Figure 14. The two alphabetical families (Greek and Arabic) used here to deal abstractly with modalities (left side and right side of the modal 3-graph)

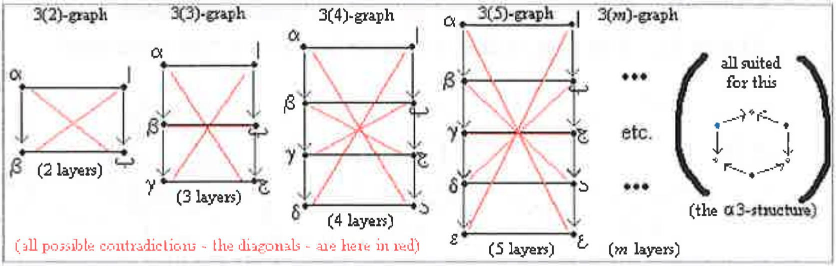

Figure 15. Examples of modal "3(m)-graphs". They can support geometrical decoration for α3-structures (hexagons), but not for α4-structures or higher

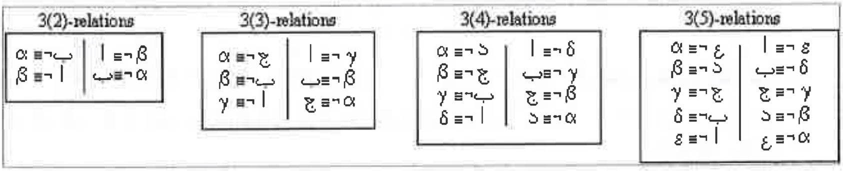

Figure 16. The $3(m)$-relations corresponding to the modal $3(m)$-graphs

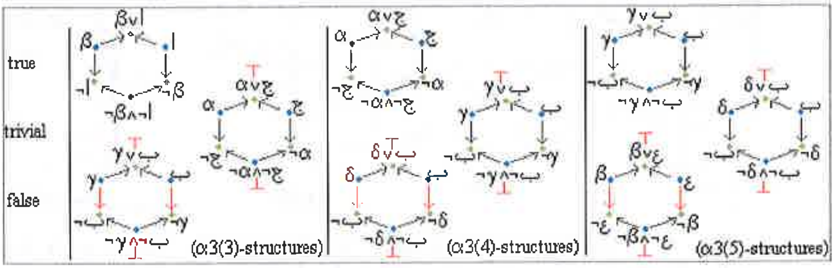

Figure 17. A sample of the possible $\alpha 3(m)$-structures, $m = 3, 4, 5$

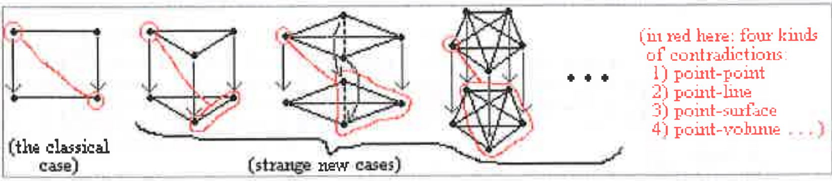

Figure 18. Some multi-dimensional alternatives in the ways to express "contradiction" geometrically (inside a $n(2)$-modal graph, $n = 3, 4, 5, 6, \ldots$)

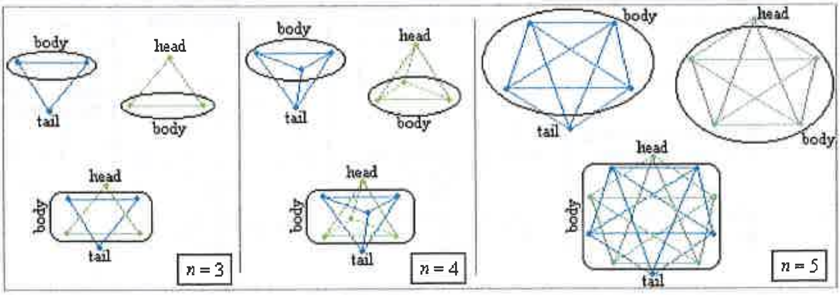

Figure 19. "Anatomy" of the αn-structures (or bi-simplexes of dimension n-1)

A	B	G	D	E	F	K	L	M	N	(etc.)
א	ב	ג	ד	ה	פ	כ	ל	מ	נ	
(aleph)	(beth)	(guimel)	(daleth)	(he)	(pe)	(kap)	(lamed)	(mem)	(nun)	(etc.)

Figure 20. The third alphabetical family, Hebrew, used to deal with 4-opposition

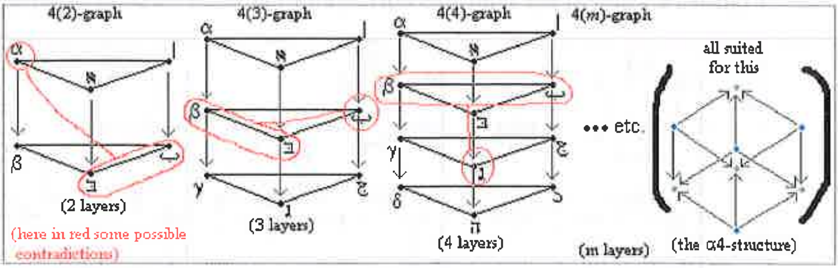

Figure 21. Examples of modal "4(m)-graphs". They can support modal decoration for the oppositional α4-structure, but not for the α5-structure or higher

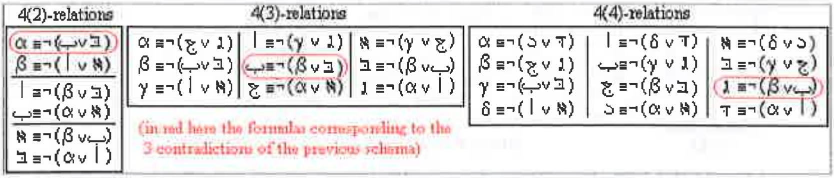

Figure 22. The 4(m)-relations corresponding to the 4(m)-graphs, m = 2. 3. 4

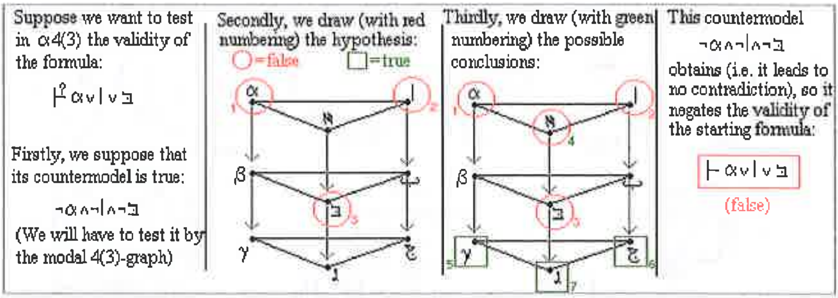

Figure 23. A possible calculation on an α4(3)-structure via the modal 4(3)-graph

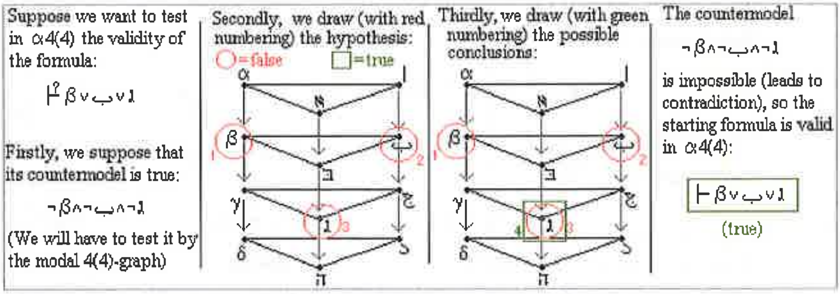

Figure 24. A possible calculation on a $\alpha 4(4)$-structure via the modal 4(4)-graph

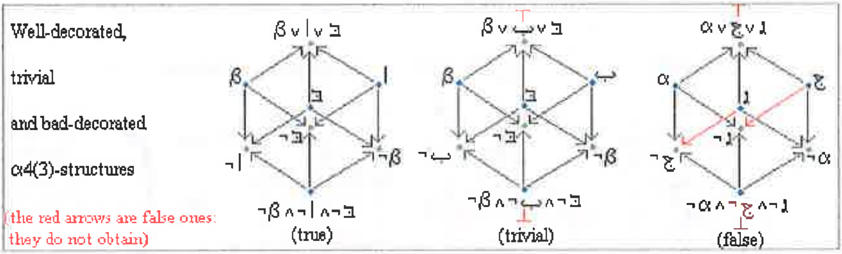

Figure 25. Some instances of possible decorations of the $\alpha 4(3)$-structure by the modal 4(3)-graph

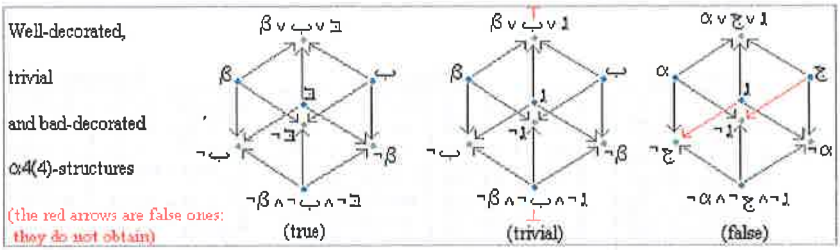

Figure 26. Some instances of possible decorations of the $\alpha 4(4)$-structure by the modal 4(4)-graph

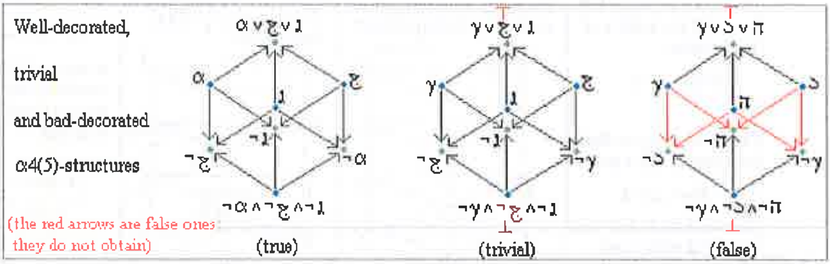

Figure 27. Some instances of possible decorations of the $\alpha 4(5)$-structure by the modal 4(5)-graph

A	B	G	D	E	F	K	L	M	N	(etc.)
अ	ब	ग	द	ए	फ	क	ल	म	न	
(a)	(ba)	(ga)	(da)	(e)	(fa)	(ka)	(la)	(ma)	(na)	(etc.)

Figure 28.The fourth alphabetical family, Sanskrit (used for 5-opposition)

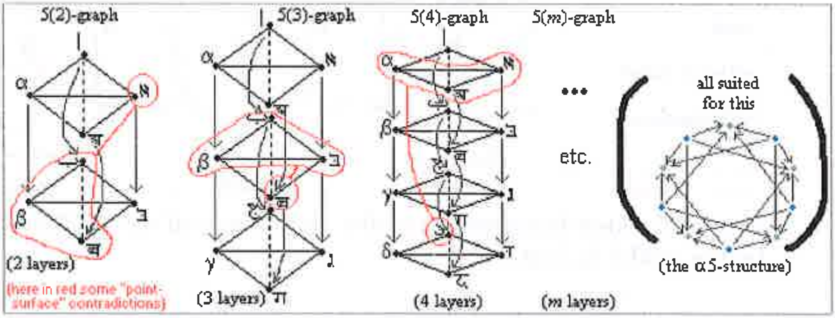

Figure 29. Examples of $5(m)$-graphs. They can support modal decorating for the oppositional $\alpha 5$-structure, but not for the $\alpha 6$-structure or higher

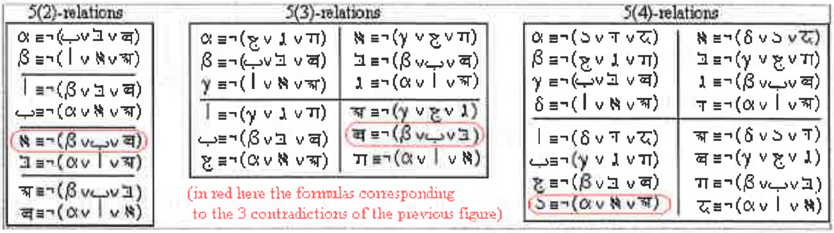

Figure 30. The $5(m)$-relations corresponding to the $5(m)$-graphs, m = 2, 3, 4

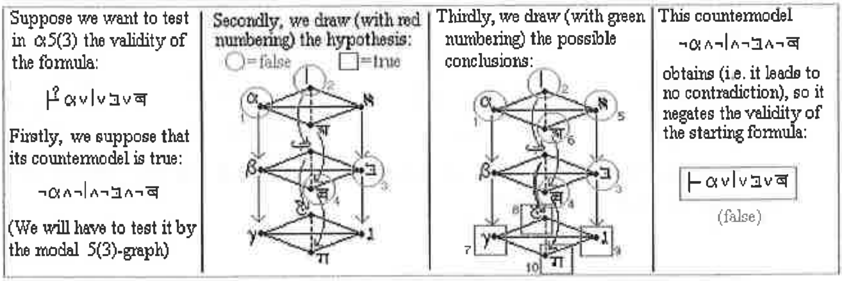

Figure 31. A possible calculation on an $\alpha 5(3)$-structure via the modal 5(3)-graph

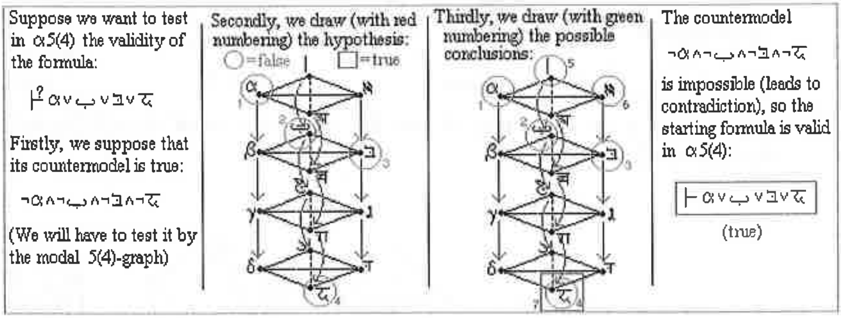

Figure 32. A possible calculation on an $\alpha 5(4)$-structure via the modal 5(4)-graph

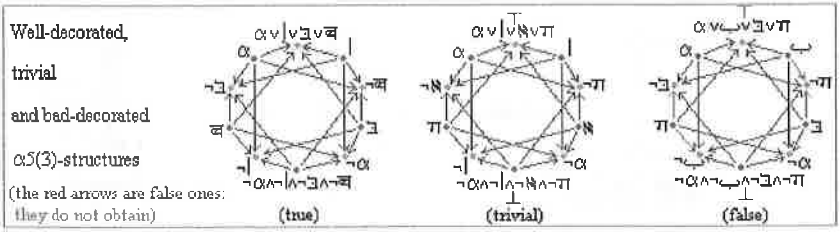

Figure 33. Some instances of possible decorations of the $\alpha 5(3)$-structure by the modal 5(3)-graph

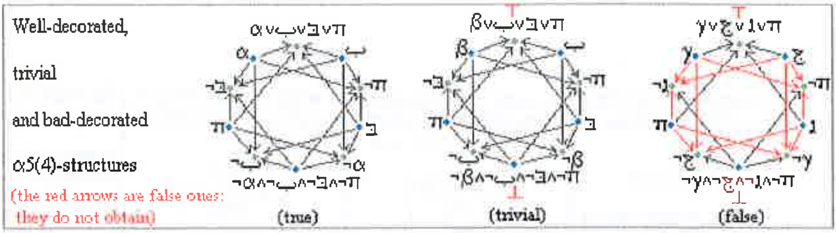

Figure 34. Some instances of possible decorations of the $\alpha 5(4)$-structure by the modal 5(4)-graph

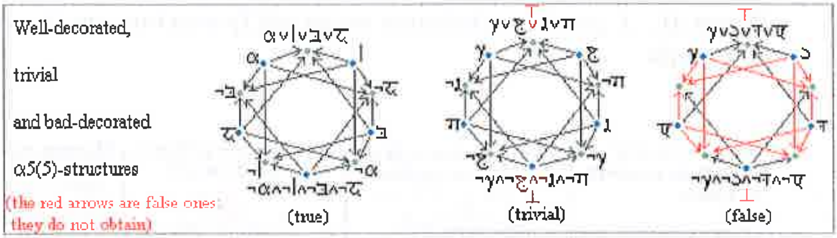

Figure 35. Some instances of possible decorations of the $\alpha 5(5)$-structure by the modal 5(5)-graph

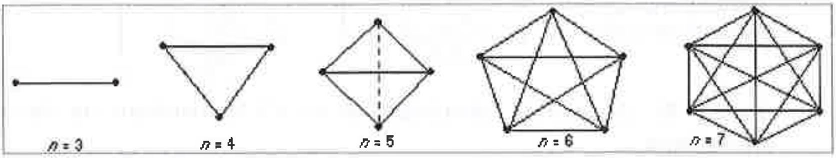

Figure 36. The series of the of the (black) "n-gems" ($n = 3, 4, 5, 6, 7$)

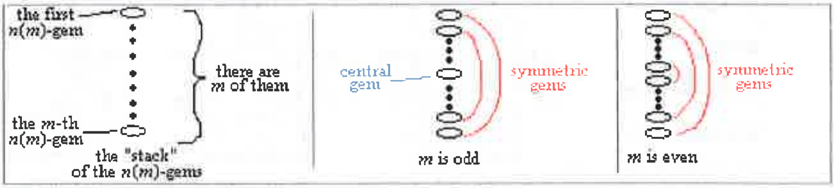

Figure 37. The shape of the modal $n(m)$-graphs in terms of the $n(m)$-gems

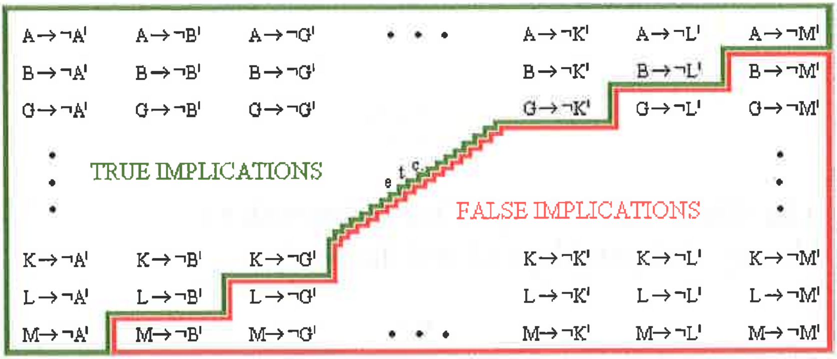

Figure 38. The general theorem detailing which simple implications are valid inside the modal $n(m)$-graph (the matrix must have m rows and m columns)