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Anomalies associated to infinitesimal diffeomorphisms of the base space of a trivial bundle: a purely geometrical approach

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Introduction

The realization by the physics community that the classical Yang-Mills field theory can be geometrically described in terms of quantities related to fibre bundles [35,13], lead to the discovery of interesting topological aspects of this theory, and revitalized the old idea of formulating a physical theory in terms of geometrical properties of some manifold.

The same geometrical setting combined with the path integral formalism to describe the quantum theory, allowed an elegant explanation of the presence of the gauge anomaly [3], a kind of quantum breaking of the classical gauge invariance.

Previous work on the algebraic aspects of those anomalies revealed a cohomological method to construct them [6,41,33]. Such an approach makes use of quantities defined on the total space of the fibre bundle upon which the classical theory is based.

In the present work we use the same cohomological setup to find the possible anomalies associated to symmetries that are related to diffeomorphisms of the base space of the fibre bundle.

We tried as much as possible to really solve the resulting cohomology, but at the end we had to content ourselves with some special examples of anomalies, leaving open the question whether they are trivial or not.

This work is organized as follows.

Chapter I is a collection of definitions and propositions of differential geometry, and of classical gauge field theory. In Chapter II we use this material to geometrically formulate a field theory and the problem of the occurrence of the anomalies.

In the first section of Chapter II we briefly explain what is meant by symmetry of a field theory, and we introduce the concept of anomaly.
In Section II.2 the axial and gauge anomalies of a theory based on a trivial bundle are treated as examples. The Wess-Zumino consistency condition for the gauge anomaly is then introduced.

Section II.3 begins with the definition of the coboundary operator associated to a representation of a Lie algebra and of the resulting cohomology. We then show how the gauge anomaly can be interpreted as a non-trivial element of some appropriate cohomology, and give a method of constructing it. The material contained in this section is more or less standard.

In Section II.4 we explain what we mean by symmetry related to diffeomorphisms of the base space of the fibre bundle upon which the theory is based. We then introduce the associated anomaly and show that it also verifies a natural consistency condition.

In Section II.5 we formulate the problem of finding the above anomaly in purely mathematical terms. Through a descent equation we relate the unintegrated anomaly with an element of a cohomology with coefficients in the algebra of functions depending on the gauge potential. The concepts of local cochains and their symbols are introduced there.

In Section II.6 we apply some techniques of differential analysis to characterize the symbols of the local cocycles. We then look for globally and naturally defined quantities whose symbols possess the same characteristics and determine which ones are non-trivial.

In Section II.7 we restrict our analysis to the affine transformations of the base space. Globally defined cocycles exist then; with them we construct the affine anomaly.

Section II.8 contains a geometrical treatment of the conformal anomaly. In the last subsection we show how the conformal anomaly of a two dimensional manifold with constant curvature can be interpreted as an affine anomaly.

In the Appendix we collected the proofs which were not written in the main text.

I Fibre bundles and gauge theories

I.1 Differential manifolds and differential geometry

We begin this chapter by recalling some well known definitions and concepts of differential geometry. Most of the material covered here is taken from the excellent book of *S.Kobayashi* and *K.Nomizu* [26]. See also [14,21,30,18].

I.1.1 Differential manifolds

Definition I.1.1 *A (real) manifold M of dimension n is a Hausdorff topological space with the property that every point x of M has a neighborhood U homeomorphic to $\mathbf{R}^n$.*

Denote by φ the homeomorphism $U \longrightarrow \mathbf{R}^n$. The couple (U, φ) is called a *chart* of the manifold M.
An *atlas* on M is a set of charts $\{(U_\alpha, \varphi_\alpha)\}$ such that the sets U_α cover M. From now on we provide M with a fixed atlas $\{(U_\alpha, \varphi_\alpha)\}$.

If M as a topological space is compact we call it a *compact* manifold. In this case the index α can be chosen such that it belongs to a finite set.

Definition I.1.2 *A (smooth) differential manifold M is a manifold with the property that the maps $\varphi_{\alpha\beta} \doteq \varphi_\alpha \circ \varphi_\beta^{-1}$ of $\mathbf{R}^n$ into $\mathbf{R}^n$ are smooth, i.e. all their partial derivatives exist.*

A manifold M is said to be *orientable* if there exists an atlas $\{(U_\alpha, \varphi_\alpha)\}$ such that for all α, β the Jacobian $|\frac{\partial \varphi_{\alpha\beta}}{\partial x}|$ is positive everywhere in $\varphi_\beta(U_\alpha \cap U_\beta)$.

Denote by x^μ the coordinate functions

$$x^\mu : \mathbf{R}^n \longrightarrow \mathbf{R} \quad , \mu = 1, \ldots, n$$
$$\vec{v} \longmapsto v^\mu$$

where $\vec{v} = (v^1, \ldots, v^n)$.

Definition I.1.3 *The system of functions $\varphi_\alpha^\mu \doteq x^\mu \circ \varphi_\alpha : U_\alpha \longrightarrow \mathbf{R}$ is called a local coordinate system in U_α .*

We shall simply write x^μ for $\varphi_\alpha^\mu(x)$ where no confusion arises.

Given two manifolds M and $\tilde{M}$ with atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(\tilde{U}_\alpha, \tilde{\varphi}_\alpha)\}$ a map $\phi : M \longrightarrow \tilde{M}$ is said to be smooth if its representatives

$$\tilde{\varphi}_\beta \circ \phi \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap \phi^{-1}(\tilde{U}_\beta)) \longrightarrow \tilde{\varphi}_\beta(\tilde{U}_\beta)$$

are smooth.
A map $\phi : M \longrightarrow \tilde{M}$ is called a *diffeomorphism* if it is a homeomorphism and both ϕ and ϕ^{-1} are smooth.
We shall denote by *Diff* M the group of all diffeomorphisms of a manifold M onto itself.

Definition I.1.4 *A one-parameter group of diffeomorphisms of M is a map*

$$\begin{aligned} \varphi: \quad & \mathbf{R} \times M \longrightarrow M \\ & (t,x) \longmapsto \varphi_t(x) \end{aligned}$$

such that i) $\varphi_t \in Diff\, M, \quad \forall t \in \mathbf{R}$
ii) $\varphi_t \circ \varphi_s = \varphi_{t+s}$

I.1.2 Tangent and cotangent spaces

Let $C^\infty(M)$ be the algebra of smooth functions on M:

$$C^\infty(M) = \{f : M \longrightarrow \mathbf{R},\ f \text{ smooth }\}$$

Definition I.1.5 *A tangent vector v_x of M at x is a linear map:*

$$v_x : C^\infty(M) \longrightarrow \mathbf{R}$$

satisfying the Leibniz rule:

$$v_x(fg) = (v_x f)g(x) + f(x)(v_x g), \quad f, g \in C^\infty(M)$$

The vector space TM_x consisting of all tangent vectors of M at x is called the *tangent space* of M at x. It can be proven that $dimTM_x = n = dimM$.

We can find a (local) basis for TM_x in the following way. Let $(U_\alpha, \varphi_\alpha)$ be a chart of M and define the tangent vector ∂_μ at $x \in U_\alpha \subset M$ by setting

$$\partial_\mu f = \frac{\partial}{\partial x^\mu}(f \circ \varphi_\alpha^{-1}), \quad f \in C^\infty(M)$$

It is easy to prove that the ∂_μ 's are linearly independent. Namely

$$\begin{aligned} a^\mu \partial_\mu &= 0 \text{ implies} \\ 0 &= a^\mu \partial_\mu \varphi_\alpha^\nu = a^\mu \frac{\partial}{\partial x^\mu}(\varphi_\alpha^\nu \circ \varphi_\alpha^{-1}) = a^\mu \frac{\partial}{\partial x^\mu} x^\nu = a^\nu \end{aligned} \tag{I.1}$$

The basis $\{\partial_\mu,\ \mu = 1, \ldots, n\}$ is called the *natural basis* of TM_x with respect to the local coordinate system $\{\varphi_\alpha^\mu\}$ in U_α.

Denote by $\{\partial'_\mu\}$ the natural basis with respect to the local coordinate system $\{\varphi_\beta^\mu\}$. In the overlapping region $U_\alpha \cap U_\beta$ we have

$$\partial'_\mu f = \frac{\partial}{\partial x'^\mu}(f \circ \varphi_\beta^{-1}) = \frac{\partial}{\partial x'^\mu}(f \circ \varphi_\alpha^{-1} \circ \varphi_{\alpha\beta}) = \frac{\partial}{\partial x^\nu}(f \circ \varphi_\alpha^{-1})\frac{\partial \varphi_{\alpha\beta}^\nu}{\partial x'^\mu}$$

$$\text{i.e. } \partial'_\mu = \frac{\partial \varphi_{\alpha\beta}^\nu}{\partial x'^\mu}\partial_\nu = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu \tag{I.2}$$

where

$$x'^\mu = \varphi_\beta^\mu(x) \text{ and } \varphi_{\alpha\beta}^\nu(x') = x^\nu = \varphi_\alpha^\nu(x), \quad x \in U_\alpha \cap U_\beta$$

The *components* $\{v_x^\mu\}$ of the tangent vector v_x at $x \in U_\alpha$ with respect to the local coordinate system in U_α are the coordinates of v_x with respect to the natural basis $\{\partial_\mu\}$: $v_x = v_x^\mu \partial_\mu$.
From equation (I.1) we immediately get $v_x^\mu = v_x \varphi_\alpha^\mu$.

Definition I.1.6 *Given a smooth map $\phi : M \longrightarrow \tilde{M}$ its derivative $d\phi_x$ (or $(\phi_*)_x$) at x is the linear map*

$$d\phi_x : TM_x \longrightarrow T\tilde{M}_{\phi(x)}$$

given by

$$(d\phi_x v_x)\tilde{f} = v_x(\tilde{f} \circ \phi), \quad \tilde{f} \in C^\infty(\tilde{M})$$

Definition I.1.7 *A covector ω_x of M at x is a linear map $TM_x \longrightarrow \mathbf{R}$.*

The vector space consisting of all covectors of M at x is therefore the dual of TM_x. It will be denoted by TM_x^* and it is called the *cotangent space* of M at x. We have $dim TM_x^* = dim TM_x = n$.
We shall designate the basis of TM_x^* dual to $\{\partial_\mu\}$ by $\{dx^\mu\}$ and call it the natural cobasis of TM_x^*. By definition $dx^\nu(\partial_\mu) = \delta_\mu^\nu$.

I.1.3 Tensor fields

Having defined the tangent and cotangent spaces at a point x of M we can form the tensor product

$$\bigotimes_q^p TM_x \hat{=} \underbrace{TM_x \otimes \cdots \otimes TM_x}_{p-times} \otimes \underbrace{TM_x^* \otimes \cdots \otimes TM_x^*}_{q-times}$$

It is called the tensor space of type (p,q) over TM_x.
It can be shown that the tensor space of type (p,q) is isomorphic to the space of all multilinear maps

$$\underbrace{TM_x \times \cdots \times TM_x}_{q-times} \times \underbrace{TM_x^* \times \cdots \times TM_x^*}_{p-times} \longrightarrow \mathbf{R}$$

Definition I.1.8 *A tensor field T_q^p of type (p,q) on M is a smooth assignment of an element of $\bigotimes_q^p TM_x$ to each point x of M.*

A tensor field of type $(p,0)$ resp. $(0,q)$ is called *contravariant* of degree p resp. *covariant* of degree q. By definition a tensor field of type (0,0) is a function on M.
We shall denote by $\mathcal{T}_q^p(M)$ the space of all tensor fields of type (p,q) over M.
The *components* of a tensor field T_q^p with respect to a local coordinate system in U_α are the functions $T_{\nu_1 \ldots \nu_q}^{\mu_1 \ldots \mu_p} \in C^\infty(U_\alpha)$ given by

$$T_q^p(x) = T_{\nu_1 \ldots \nu_q}^{\mu_1 \ldots \mu_p}(x)\, \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_p} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_q}, \quad x \in U_\alpha$$

where $\{\partial_\mu\}$ and $\{dx^\nu\}$ are the natural bases of TM_x resp. TM_x^*.
Denoting by the same symbol ∂_μ resp. dx^ν the tensor field on U_α which associates to each point $x \in U_\alpha$ the natural basis element $\partial_\mu \in TM_x$ resp. $dx^\nu \in TM_x^*$, a tensor field T_q^p on U_α can be expressed as

$$T_q^p = T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} \, \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_p} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_q} \tag{I.3}$$

Given two tensor fields T_q^p and $\tilde{T}_s^r$ their tensor product $T_q^p \otimes \tilde{T}_s^r$ is the tensor field of type (p+r,q+s) given by

$$(T_q^p \otimes \tilde{T}_s^r)(x) = T_q^p(x) \otimes \tilde{T}_s^r(x)$$

Definition I.1.9 *Given a smooth map $\phi : M \longrightarrow \tilde{M}$ and a covariant tensor field $\tilde{T}_q$ on $\tilde{M}$ its pull-back $\phi^*\tilde{T}_q$ is the tensor field on M defined by*

$$\phi^*\tilde{T}_q(x; v_x^1, \dots, v_x^q) = \tilde{T}_q(\phi(x); d\phi_x v_x^1, \dots, d\phi_x v_x^q), \quad v_x^i \in TM_x$$

We clearly have: $\phi^*(\tilde{T}_q \otimes \tilde{T'}_{q'}) = \phi^*\tilde{T}_q \otimes \phi^*\tilde{T'}_{q'}$.

Given a diffeomorphism $\phi : M \longrightarrow \tilde{M}$ and a contravariant tensor field T^p on M, the contravariant tensor field $\phi_* T^p$ on $\tilde{M}$ is defined by

$$\phi_* T^p(\tilde{x}; \tilde{\omega}_{\tilde{x}}^1, \dots, \tilde{\omega}_{\tilde{x}}^p) = T^p(x; \phi^*\tilde{\omega}_x^1, \dots, \phi^*\tilde{\omega}_x^p)$$

$$\text{where } \phi(x) = \tilde{x}, \quad \tilde{\omega}_{\tilde{x}}^i \in T\tilde{M}_{\tilde{x}}^*$$

It has to be noted that $\phi_* T^p$ can be defined only for a diffeomorphism ϕ.

I.1.4 Riemannian metrics

Definition I.1.10 *A (pseudo) Riemannian metric g on M is a covariant tensor field of type $(0,2)$ with the property that $g(x)$, considered as a bilinear map*

$$g(x) : TM_x \times TM_x \longrightarrow \mathbf{R}$$

is symmetric and (non-degenerate) positive definite.

A manifold M equipped with a (pseudo) Riemannian metric g is called a *(pseudo) Riemannian* manifold.

A Riemannian manifold M is called *Euclidean* if there exists a diffeomorphism $\phi : M \longrightarrow \mathbf{R}^n$ and the metric g verifies

$$(\phi^{-1})^* g = \delta$$

where δ is the canonical metric on $\mathbf{R}^n$.

A metric g on a manifold M defines an isomorphism between TM_x and TM_x^* in the following manner. Each tangent vector v_x determines a covector ${}^g v_x$ by

$${}^g v_x(w_x) = g(x; v_x, w_x), \quad \forall w_x \in TM_x$$

The application $v_x \longmapsto {}^g v_x$ is clearly linear, and since g_x is non-degenerate it is an isomorphism.
The components of the covector ${}^g v_x$ with respect to a local coordinate system in U_α are given by

$$ {}^g v_{x\,\mu} = g_{\mu\nu}(x) v_x^\nu, \quad x \in U_\alpha \tag{I.4} $$

The above isomorphism can be naturally extended to the tensor space leading to an identification of tensors of type (p,q) with tensors of type (p-k,q+k), (-q≤k≤p). The same applies (locally) to tensor fields.
When not necessary, we shall not actually distinguish a contravariant tensor field T^p from its associated covariant tensor field ${}^g T_p$.

I.1.5 Vector fields

Definition I.1.11 *A vector field on M is a contravariant tensor field of type* $(1,0)$.

The set of all vector fields on M is naturally a vector space with respect to the pointwise defined sum and scalar multiplication. This vector space will be denoted by $\chi(M)$.
In what follows we often identify a vector field X on M with a derivation on the algebra $C^\infty(M)$ by (see [14])

$$ \begin{aligned} X\ :\ C^\infty(M) &\longrightarrow C^\infty(M) \\ f &\longmapsto Xf \qquad \text{with } Xf(x) = X_x f\ ,\ X_x \equiv X(x) \end{aligned} \tag{I.5} $$

We clearly have $X(fg) = (Xf)g + f(Xg)$, i.e. $X \in Der\, C^\infty(M)$.

The *Lie bracket* of two vector fields X and Y on M is the vector field $[X,Y]$ defined by

$$ [X,Y]f = X(Yf) - Y(Xf), \quad f \in C^\infty(M) $$

The components of $[X,Y]$ with respect to a local coordinate system are given by

$$ [X,Y]^\mu = XY^\mu - YX^\mu \tag{I.6} $$

As can be easily verified the Lie bracket satisfies the Jacobi identity. Hence, equipped with this product, $\chi(M)$ becomes a Lie algebra.

Definition I.1.12 *Let $\phi : M \longrightarrow \tilde{M}$ be a smooth map. Two vector fields $X \in \chi(M)$ and $\tilde{X} \in \chi(\tilde{M})$ are said to be ϕ-related if*

$$ \tilde{X}(\phi(x)) = d\phi_x X(x), \quad \forall x \in M $$

i.e. if

$$ \tilde{X}\tilde{f} \circ \phi = X(\tilde{f} \circ \phi), \quad \forall \tilde{f} \in C^\infty(\tilde{M}) $$

The ϕ-relation satisfies the following condition:

Proposition I.1.1 *Let $X_i \in \chi(M)$ be ϕ-related to $\tilde{X}_i \in \chi(\tilde{M})$, $i = 1,2$. Then $[X_1, X_2]$ is ϕ-related to $[\tilde{X}_1, \tilde{X}_2]$.*

Given a diffeomorphism $\phi : M \longrightarrow \tilde{M}$ and a vector field X on M, the vector field $\phi_* X$ on $\tilde{M}$ (see above) can also be defined by

$$\phi_* X(\phi(x)) = d\phi_x X(x), \quad \forall x \in M \tag{I.7}$$

Clearly X and $\phi_* X$ are ϕ-related.

The most important property of a vector field on a compact manifold M is that it can be considered as the generator of a one-parameter group of diffeomorphisms of M. To see how the two concepts are related we first note that a one-parameter group of diffeomorphisms φ_t of a manifold M induces a vector field X by

$$X_x f = \lim_{t \to 0} \frac{1}{t}[f(\varphi_t(x)) - f(x)], \quad \forall x \in M, \quad f \in C^\infty(M) \tag{I.8}$$

where to simplify we have assumed $\varphi_{t=0}(x) = x$.
Conversely it can be proven [26] that given a vector field X on a *compact* manifold M, there exists a unique one-parameter group of diffeomorphisms φ_t, with $\varphi_0(x) = x$, which induces the given X. In this case φ_t is said to be generated by X.

I.1.6 Lie derivatives

Let φ_t be a one-parameter group of diffeomorphisms of a manifold M and X the induced vector field.

Definition I.1.13 *Given a covariant (resp. contravariant) tensor field T_q (resp. T^p) on M its Lie derivative with respect to X, $L_X T_q$ (resp. $L_X T^p$) is the covariant (resp. contravariant) tensor field on M defined by*

$$L_X T_q(x) = \lim_{t \to 0} \tfrac{1}{t}[\varphi_t^* T_q(x) - T_q(x)]$$

$$\textit{resp.} \quad L_X T^p(x) = \lim_{t \to 0} \tfrac{1}{t}[T^p(x) - \varphi_{t*} T^p(x)]$$

Even though we have defined the Lie derivative for covariant resp. contravariant tensor fields separately, it can be easily generalized to tensor fields of any type [26]. The Lie derivative has the following properties:

Proposition I.1.2
i) $L_X(T_q \otimes T^p) = L_X T_q \otimes T^p + T_q \otimes L_X T^p$
ii) $L_X f = X f, \quad f \in C^\infty(M)$
iii) $L_X Y = [X, Y], \quad X, Y \in \chi(M)$
iv) $L_{[X,Y]} = L_X \circ L_Y - L_Y \circ L_X$

To express the components of the Lie derivative of a tensor field we need the following Proposition.

Proposition I.1.3 *Let ∂_μ and dx^ν be the natural tensor fields on $U_\alpha \subset M$. For any vector field Z on M we have*

$$\begin{array}{ll} i) & L_Z\partial_\mu = -(\partial_\mu Z^\rho)\partial_\rho \\ ii) & L_Z dx^\nu = dx^\rho \partial_\rho Z^\nu \end{array}$$

where Z^ρ are the components of Z with respect to the local coordinate system in U_α.

From (I.3) and Propositions I.1.2 and I.1.3 it follows that the components of the tensor field $L_Z T^p_q$ of type (p,q) are given by

$$\begin{aligned} L_Z T^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_q} &= Z^\rho\partial_\rho T^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_q} + T^{\mu_1\cdots\mu_p}_{\rho\nu_2\cdots\nu_q}\partial_{\nu_1}Z^\rho + \cdots + T^{\mu_1\cdots\mu_p}_{\nu_1\cdots\rho}\partial_{\nu_q}Z^\rho \\ &\quad - T^{\rho\mu_2\cdots\mu_p}_{\nu_1\cdots\nu_q}\partial_\rho Z^{\mu_1}\cdots - T^{\mu_1\cdots\rho}_{\nu_1\cdots\nu_q}\partial_\rho Z^{\mu_p} \end{aligned} \tag{I.9}$$

It has to be remarked that on a Riemannian manifold we generally have (see (I.1.4))

$$^g(L_X T_q) \neq L_X\, {}^gT_q \tag{I.10}$$

I.1.7 Differential forms

Denote by $\bigwedge^q TM^*_x$ the space of all q-linear skew-symmetric maps ω^q_x

$$\omega^q_x : \underbrace{TM_x \times \cdots \times TM_x}_{q-times} \longrightarrow \mathbf{R}, \quad x \in M$$

$\bigwedge^q TM^*_x$ is called the q-th exterior power of TM^*_x. Since $dim TM_x = n = dim M$ we clearly have $\bigwedge^q TM^*_x \equiv \emptyset$ for $q > n$.
Note that even though the index q appears as a superscript, ω^q_x is actually a covariant tensor.

Let $\omega^q_x \in \bigwedge^q TM^*_x$ and $\tau^r_x \in \bigwedge^r TM^*_x$. Their *exterior product* $\omega^q_x \wedge \tau^r_x$ is the element of $\bigwedge^{q+r} TM^*_x$ defined by

$$\begin{aligned} &\omega^q_x \wedge \tau^r_x(v^1_x, \ldots, v^{q+r}_x) = \\ &= \frac{1}{q!r!}\sum_\pi sign\,\pi\, \omega^q_x(v^{\pi(1)}_x, \ldots, v^{\pi(q)}_x)\, \tau^r_x(v^{\pi(q+1)}_x, \ldots, v^{\pi(q+r)}_x), \quad v^i_x \in TM_x \end{aligned} \tag{I.11}$$

where the summation is over all permutations π of $(1, \ldots, q+r)$ and $sign\,\pi$ denotes the sign of the permutation π.
It is easy to verify that

$$\omega^q_x \wedge \tau^r_x = (-1)^{qr}\tau^r_x \wedge \omega^q_x \tag{I.12}$$

Definition I.1.14 *A q-form ω^q on M is a smooth assignment of an element of $\bigwedge^q TM^*_x$ to each point x of M.*

We shall denote by $\Omega^q(M)$ the (pointwise defined) vector space of all q-forms on M and by $\Omega(M)$ the graded vector space $\bigoplus_{q=0}^{n} \Omega^q(M)$, $dim M = n$, where by definition $\Omega^0(M) = C^\infty(M)$.
The exterior product of two forms is pointwise defined by

$$\omega^q \wedge \tau^r(x) = \omega^q(x) \wedge \tau^r(x)$$

With respect to this product $\Omega(M)$ becomes a graded anticommutative algebra.

To a q-form ω^q corresponds a q-linear (over $C^\infty(M)$) skew symmetric map

$$\hat{\omega}^q : \underbrace{\chi(M) \times \cdots \times \chi(M)}_{q-times} \longrightarrow C^\infty(M) \tag{I.13}$$

$$\hat{\omega}^q(f_1X_1, \ldots, f_qX_q) = f_1 \cdots f_q \hat{\omega}^q(X_1, \ldots, X_q), \quad f_i \in C^\infty(M)$$

given by

$$\hat{\omega}^q(X_1, \ldots, X_q)(x) = \omega^q(x; X_1(x), \ldots, X_q(x)), \quad \forall x \in M$$

It can actually be shown [26] that this correspondence is bijectiv. Henceforth we shall identify a q-form with such a map.

Definition I.1.15 *Given a q-form ω^q on M its exterior derivative is the (q+1)-form $d\omega^q$ given by*

$$d\omega^q(X_1, \ldots, X_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} X_i \omega^q(X_1, \ldots, \hat{\imath}, \ldots, X_{q+1})$$
$$+ \sum_{i<j} (-1)^{i+j} \omega^q([X_i, X_j], X_1, \ldots, \hat{\imath}, \ldots, \hat{\jmath}, \ldots, X_{q+1}), \quad X_i \in \chi(M)$$

where $\hat{\imath}$ means that the i-th term is omitted from the set $(X_1, \ldots, X_{q+1})$.

As a special case we have for a one-form ω

$$d\omega(X_1, X_2) = X_1\omega(X_2) - X_2\omega(X_1) - \omega([X_1, X_2])$$

The exterior derivation has the following properties:

Proposition I.1.4 *i)* $d^2 \equiv 0$
ii) *d is an antiderivation in the algebra $\Omega(M)$, i.e.* $d(\omega^q \wedge \tau^r) = d\omega^q \wedge \tau^r + (-1)^q \omega^q \wedge d\tau^r$
iii) $\phi^* \circ d = d \circ \phi^*$ *for every smooth map* ϕ

Definition I.1.16 *Let X be a vector field on M. The interior product $\imath_X$ with respect to X, is the antiderivation of degree -1 of $\Omega(M)$*

$$\imath_X : \Omega^q(M) \longrightarrow \Omega^{q-1}(M)$$

given by *i)* $\imath_X f \equiv 0, \quad f \in \Omega^0(M) \equiv C^\infty(M)$
ii) $(\imath_X \omega^q)(X_1, \ldots, X_{q-1}) = \omega^q(X, X_1, \ldots, X_{q-1}), \quad X_i \in \chi(M)$

Proposition I.1.5 *Given a q-form ω^q on M its Lie derivative $L_Z\omega^q$ with respect to $Z \in \mathcal{X}(M)$ is given by*

$$L_Z\omega^q(X_1,\ldots,X_q) = Z\omega^q(X_1,\ldots,X_q) - \sum_{i=1}^{q} \omega^q(X_1,\ldots,[Z,X_i],\ldots,X_q), \quad X_i \in \mathcal{X}(M)$$

Restricted to $\Omega(M)$ the Lie derivative fulfills the following conditions:

Proposition I.1.6 *i)* $L_Z(\omega^q \wedge \tau^r) = L_Z\omega^q \wedge \tau^r + \omega^q \wedge L_Z\tau^r$
ii) $L_Z = d \circ \imath_Z - \imath_Z \circ d$
iii) $L_{Z_1} \circ \imath_{Z_2} - \imath_{Z_2} \circ L_{Z_1} = \imath_{[Z_1,Z_2]}, \quad Z, Z_i \in \mathcal{X}(M)$

From condition ii) it follows

iv) $L_Z \circ d = d \circ L_Z$

The concept of form can be generalized to include the so called *vector-valued* forms.
Let $\bigwedge^q TM_x^* \otimes F$ be the space of all q-linear skew-symmetric maps, again denoted by ω_x^q,

$$\omega_x^q : \underbrace{TM_x \times \cdots \times TM_x}_{q-times} \longrightarrow F, \quad x \in M$$

where F is a finite dimensional vector space.
In analogy with a real valued form, a F-valued form is a smooth assignment of an element of $\bigwedge^q TM_x^* \otimes F$ to each point $x \in M$.
The space of all F-valued q-forms on M is denoted by $\Omega^q(M;F)$. It is isomorphic to the space of all q-linear (over $C^\infty(M)$) skew-symmetric maps

$$\omega^q : \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \longrightarrow C^\infty(M;F)$$

where $C^\infty(M;F)$ denotes the vector space of all smooth maps $M \longrightarrow F$.

All operations defined above are still meaningful for F-valued forms except for the exterior product, which can be defined only if F is an algebra.
In this case, if we denote the product in F by a dot, the exterior product $\wedge$ in $\Omega(M;F) = \bigoplus_{q=0} \Omega^q(M;F)$ is defined by

$$\omega^q \wedge \tau^r(X_1,\ldots,X_{q+r}) = \tag{I.14}$$
$$= \frac{1}{q!r!}\sum_{\pi} sign\,\pi\, \omega^q(X_{\pi(1)},\ldots,X_{\pi(q)}) \cdot \tau^r(X_{\pi(q+1)},\ldots,X_{\pi(q+r)}), \quad X_i \in \mathcal{X}(M)$$

In the special case where F is a Lie algebra L with product $[\,,\,]$, we shall denote the above product by $[\wedge]$

$$[\omega^q \wedge \tau^r](X_1,\ldots,X_{q+r}) =$$
$$= \frac{1}{q!r!}\sum_{\pi} sign\,\pi\, [\omega^q(X_{\pi(1)},\ldots,X_{\pi(q)}), \tau^r(X_{\pi(q+1)},\ldots,X_{\pi(q+r)})] \tag{I.15}$$

For a L-valued 1-form ω we have

$$[\omega \wedge \omega](X_1, X_2) = [\omega(X_1), \omega(X_2)] - [\omega(X_2), \omega(X_1)] = 2[\omega(X_1), \omega(X_2)] \tag{I.16}$$

which in general differs from 0.

It is easily checked that this product verifies:

$$\begin{aligned} &i) \quad [\omega^q \wedge \tau^r] = (-1)^{qr+1} [\tau^r \wedge \omega^q] \\ &ii) \quad (-1)^{qs} [\omega^q \wedge [\tau^r \wedge \sigma^s]] + (-1)^{qr} [\tau^r \wedge [\sigma^s \wedge \omega^q]] + (-1)^{rs} [\sigma^s \wedge [\omega^q \wedge \tau^r]] = 0 \end{aligned} \tag{I.17}$$

i.e. for a L-valued 1-form ω :

$$[\omega \wedge [\omega \wedge \omega]] \equiv 0 \tag{I.18}$$

Equipped with this product $\Omega(M; L)$ becomes a graded Lie algebra.

I.1.8 Integration on Riemannian manifolds

Let M be a compact, oriented, n-dimensional, Riemannian manifold with metric g. Since M is oriented the set of n-forms $\epsilon|_{U_\alpha}$ defined on $U_\alpha \subset M$

$$\epsilon|_{U_\alpha}(x) = \frac{1}{n!}\sqrt{\mathrm{g}(x)}\epsilon_{\mu_1\cdots\mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}, \quad x \in U_\alpha \tag{I.19}$$

where $\mathrm{g}(x)$ is the determinant of the "matrix" $g_{\mu\nu}(x)$, patch together to give a globally defined n-form ϵ on M. The n-form ϵ is called the *natural volume form* on M.

Definition I.1.17 *Given a q-form ω on M, the (n-q)-form $^*\omega$ whose components with respect to a local coordinate system in U_α are given by*

$$^*\omega_{\mu_1\cdots\mu_{n-q}}(x) = \frac{1}{q!}\sqrt{\mathrm{g}(x)}\epsilon_{\mu_1\cdots\mu_{n-q}\nu_1\cdots\nu_q} g^{\nu_1,\rho_1} \cdots g^{\nu_q,\rho_q}\omega_{\rho_1\cdots\rho_q}(x), \quad x \in U_\alpha$$

is called the (Hodge) dual form of ω.

On a n-dimensional manifold M the *$\star$-operator* verifies

$$^{**}\omega^q = (-1)^{q(n-q)}\omega, \quad \omega^q \in \Omega^q(M) \tag{I.20}$$

Definition I.1.18 *Given a vector field X on M its divergence $div\, X$ is the function on M given by*

$$L_X\epsilon = (div\, X)\epsilon, \quad X \in \mathcal{X}(M)$$

From the local expression of the volume form in a coordinate neighborhood U_α it can be easily verified that

$$(div\, X)(x) = \frac{1}{\sqrt{\mathrm{g}(x)}}\partial_\mu(\sqrt{\mathrm{g}(x)}X^\mu), \quad x \in U_\alpha \tag{I.21}$$

Proposition I.1.7 *The divergence of a vector field X on a n-dimensional Riemannian manifold M can also be expressed as*

$$div\, X = (-1)^{n+1}\, {}^{*}d^{*}X$$

where we have identified through the metric g the vector field X with a 1-form on M.

Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas on M. Given a general n-form ω on M we loosely define the *integral* of ω on M by

$$\int_M \omega = \sum_\alpha \int_{\varphi_\alpha(U_\alpha)} (\varphi_\alpha^{-1\,*}\omega_{1\cdots n})(x)\, d^n x$$

where $\omega_{1\cdots n}$ is the component of the form ω with respect to the local coordinate system in U_α. The integral on the right-hand side is a usual multiple integral on $\mathbf{R}^n$.

Observe that for a compact manifold the index α belongs to a finite set, hence the above sum is finite.

For a more mathematically precise definition of an integral on a manifold see e.g. [14].

As can be easily checked the n-form ω can be written as $\omega = {}^{*}\omega\, \epsilon$, where ${}^{*}\omega \in C^\infty(M)$. Therefore $\omega_{1\cdots n} = {}^{*}\omega\sqrt{g}$, and

$$\int_M \omega = \sum_\alpha \int_{\varphi_\alpha(U_\alpha)} \varphi_\alpha^{-1\,*}({}^{*}\omega\sqrt{g})(x) d^n x = \sum_\alpha \int_{\varphi_\alpha(U_\alpha)} {}^{*}\omega(x)\sqrt{g(x)}\, d^n x \qquad (1.22)$$

where in the last equation we have used the same symbol for a function on U_α and its pull-back on $\varphi_\alpha(U_\alpha)$.

Proposition I.1.8 (Stokes) *Let M be a n-dimensional, compact, oriented manifold with boundary ∂M. ∂M is a $(n-1)$-dimensional submanifold of M.*
For a $(n-1)$-form ω on M the following equality holds

$$\int_M d\omega = \int_{\partial M} \omega$$

where ∂M is oriented coherently with M.

Corollary I.1.1 *Let M be a n-dimensional, compact, oriented manifold without boundary, i.e. $\partial M \equiv \emptyset$. Then for any $(n-1)$-form ω on M, $\int_M d\omega = 0$.*

For a n-form ω on a n-dimensional manifold M we have

$$L_X\omega = d\imath_X\omega + \imath_X d\omega = d\imath_X\omega, \quad \forall X \in \mathcal{X}(M)$$

Therefore on a manifold M without boundary

$$\int_M L_X\omega = 0, \quad \forall X \in \mathcal{X}(M) \qquad (I.23)$$

I.1.9 Lie groups

Definition I.1.19 *A Lie group G is a manifold with a group structure such that the map*

$$G \times G \longrightarrow G$$
$$(g,h) \longmapsto g.h^{-1}$$

is smooth.

The identity element of G will be denoted by e.

Let L_g (resp. R_g) be the left (resp. right) multiplication operator

$$L_g : \quad G \longrightarrow G$$
$$h \longmapsto g.h \quad (\text{resp. } R_g h = h.g)$$

A vector field X on G is called *left invariant* (resp. *right invariant*) if it satisfies

$$L_{g*}X = X \qquad (\text{resp. } R_{g*}X = X) \quad \forall g \in G$$
$$\text{i.e.} \quad (dL_g)_h X_h = X_{g.h} \quad \forall g,h \in G$$

Definition I.1.20 *The Lie algebra LG of the Lie group G is the Lie algebra of all left invariant vector fields on G.*

Proposition I.1.9 *The vector space LG is isomorphic to TG_e, the tangent space of G at the identity e.*

Proof: Given an element X_e of TG_e we associate to it a left invariant vector field X by

$$X_g = (dL_g)_e X_e, \quad \forall g \in G$$

and viceversa $X_e = X(\mathrm{e})$. □

From the above Proposition it follows that $dim\, LG = dim\, G = n$.

Let $\{T_i, \quad i = 1, \ldots, n\}$ be a basis of $TG_e \equiv LG$. The *structure constant* c_{ij}^k of LG with respect to the basis $\{T_i\}$ are the numbers given by

$$[T_i, T_j] = c_{ij}^k T_k, \quad i,j,k = 1, \ldots, n \tag{I.24}$$

Denote by ad_g the following automorphism

$$ad_g : \qquad G \longrightarrow G$$
$$h \longmapsto g.h.g^{-1}$$

Definition I.1.21 *The action ad of G in G*

$$ad : \quad G \longrightarrow AutG$$
$$g \longmapsto ad_g$$

is called the adjoint action of G in G.

Given an element E of LG, the vector field $ad_{g*}E$ is again left invariant. Moreover we know that (see page 10)

$$ad_{g*}[E_1, E_2] = [ad_{g*}E_1, ad_{g*}E_2], \quad E_1, E_2 \in LG$$

The map ad_{g*} is therefore an automorphism of LG. To simplify we shall denote it also by ad_g.

Definition I.1.22 *The representation of G in LG given by*

$$\begin{aligned} ad \; : \; G &\longrightarrow Aut\, LG \\ g &\longmapsto ad_g \end{aligned}$$

is called the adjoint representation of G.

It can be shown that a left invariant vector field on G always generates a one-parameter group of automorphisms of G. Let φ_t be the one-parameter group of automorphisms of G generated by $E \in LG$ and denote by $\exp tE$ the group element $\varphi_t(e), \quad t \in \mathbf{R}$.

Definition I.1.23 $\exp tE$ *is called the one-parameter subgroup of G generated by $E \in LG$.*

The derivative of the adjoint representation ad of G, i.e. the associated representation of LG in LG, will be denoted by Ad, and it is called the adjoint representation of LG.

$$\begin{aligned} Ad \; : \; LG &\longrightarrow Der\, LG \\ E &\longmapsto Ad_E \end{aligned}$$

By definition

$$Ad_E H = \lim_{t \to 0} \frac{1}{t}[ad_{\exp tE} H - H], \quad E, H \in LG \tag{I.25}$$

Proposition I.1.10 *The representation Ad_E is explicitly given by*

$$Ad_E H = [E, H], \quad E, H \in LG$$

Assume G is a matrix Lie group. In this case its Lie algebra LG is isomorphic to a matrix Lie algebra, i.e. we have

$$[E_1, E_2] = E_1 E_2 - E_2 E_1, \quad E_i \in LG$$

where on the right-hand side the matrix multiplication is understood.
As set of matrices we moreover have $G \subset LG$. The adjoint representation of G is explicitly given by

$$ad_g E = g E g^{-1}, \quad g \in G, \quad E \in LG$$

I.2 Principal fibre bundles and associated vector bundles

I.2.1 Action of a Lie group on a manifold

Let M be a smooth manifold and G a Lie group.

Definition I.2.1 *A Lie group G is said to act on a manifold M on the right (resp. on the left) if to every element $g \in G$ it is associated a diffeomorphism a_g of M satisfying $a_{g.h} = a_h \circ a_g$ (resp. $a_{g.h} = a_g \circ a_h$) $\forall g, h \in G$ and if the map:*

$$\begin{aligned} a: \quad & G \times M \longrightarrow M \\ & (g, x) \longmapsto a_g(x) \end{aligned}$$

is smooth.

The action a is said to be *free* if $a_g(x) = x$ for some $x \in M$ implies $g = e \equiv$ identity of G.
The *orbit* of a point x of M is the set of points $\{a_g(x) \in M, g \in G\}$. If the action a is free then there is a bijection between the orbit of x and G.
To simplify the notation we shall write $a_g(x) = x \cdot g$ (resp. $a_g(x) = g \cdot x$) for a right (resp. left) action a.

As we saw before every element E of the Lie algebra LG of G generates a one-parameter subgroup $\exp tE$ of G. If G acts on a manifold M the vector field $E \in LG$ induces a vector field Z_E on M by

$$Z_E(x) f = \lim_{t \to 0} \frac{1}{t}[f(a_{\exp tE}(x)) - f(x)], \quad \forall x \in M, \quad f \in C^\infty(M) \tag{I.26}$$

Definition I.2.2 *The vector field Z_E on M induced by an element $E \in LG$ is called the fundamental vector field associated to E.*

Proposition I.2.1 *Let G act on a manifold M on the right (resp. on the left) and let Z_E and Z_F be the fundamental vector fields on M associated to E resp. $F \in LG$.*
Then we have

$$Z_{[E,F]} = [Z_E, Z_F] \qquad \qquad (\text{resp. } Z_{[E,F]} = -[Z_E, Z_F]\ )$$

I.2.2 Principal fibre bundles

Let P be a smooth manifold and G a Lie group acting on P on the right.
The action of G induces an equivalence relation in P by: $(x \sim y)$ if $y = x \cdot g$ for some $g \in G$, i.e. two points are equivalent if they are in the same orbit.
It can be proved [26] that under some not too restrictive conditions on P and a the quotient space P/G, i.e. the set of all orbits, has the structure of a smooth manifold.

Definition I.2.3 *A principal fibre bundle $(P \xrightarrow{\pi} M, G)$, or briefly P, over the manifold M consists of a manifold P and a free right action of a Lie group G on P*

such that:

1. *the projection π is smooth*
2. *π factors through P/G and induces a diffeomorphism $P/G \longrightarrow M$, i.e. M may be identified with the orbit manifold P/G*
3. *the manifold P is locally trivial, i.e. for every open set U_α of M, there exists a diffeomorphism ψ_α*

$$\begin{aligned} \psi_\alpha \,:\, \pi^{-1}(U_\alpha) &\longrightarrow U_\alpha \times G \\ z &\longmapsto (\pi(z), \phi_\alpha(z)) \end{aligned}$$

where $\phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow G$ satisfies $\phi_\alpha(z \cdot g) = \phi_\alpha(z) \cdot g$

The manifold P is called the *total space*, M the *base space* and G the *structure group* of the fibre bundle. The set of points $\pi^{-1}(x)$, $x \in M$, is called the *fibre* over x. The fibre over x is a manifold diffeomorphic to G.

Given an atlas $\{(U_\alpha, \varphi_\alpha)\}$ of M the family $\{(U_\alpha, \psi_\alpha)\}$ is said to be a *principal coordinate representation* for P.

The map $\psi_{\alpha\beta} \,:\, \pi^{-1}(U_\alpha \cap U_\beta) \longrightarrow G$

$$z \longmapsto \phi_\alpha(z).[\phi_\beta(z)]^{-1}$$

depends only on the orbit of z:

$$\psi_{\alpha\beta}(z \cdot g) = \psi_{\alpha\beta}(z)$$

This means that $\psi_{\alpha\beta}$ can be considered as a map on $U_\alpha \cap U_\beta$

$$\begin{aligned} \psi_{\alpha\beta} \,:\, U_\alpha \cap U_\beta &\longrightarrow G \\ x &\longmapsto \psi_{\alpha\beta}(z) \quad \text{with } \pi(z) = x \end{aligned}$$

The maps $\{\psi_{\alpha\beta}\}$ are called *transition functions* of the fibre bundle corresponding to the open covering $\{U_\alpha\}$ of M.

As an example consider the product manifold $P = M \times G$. A right action a of G on P is defined by: $a_g(x,h) = (x, h.g)$, $(x,h) \in M \times G$. The projection π is simply $\pi(x,h) = x$. Clearly $\pi^{-1}(U_\alpha) = U_\alpha \times G$.

The principal fibre bundle $(M \times G \xrightarrow{\pi} M, G)$ is called *trivial.*

Definition I.2.4 *A (global) section σ of a fibre bundle P is a smooth map $\sigma : M \longrightarrow P$ such that $\pi \circ \sigma$ is the identity transformation of M.*

We shall denote by $SecP$ the set of all sections of a bundle P.

Proposition I.2.2 *A principal fibre bundle P admits a global section iff it is the trivial bundle $M \times G$.*

Given a principal coordinate representation $\{(U_\alpha, \psi_\alpha)\}$ for P it is possible to find a family of *local* sections $\sigma_\alpha : U_\alpha \longrightarrow \pi^{-1}(U_\alpha)$ in the following way.
Consider the map $z \longmapsto z \cdot \phi_\alpha(z)^{-1}, \quad z \in \pi^{-1}(U_\alpha)$. As before this map depends only on the orbit of z, hence it determines a map

$$\begin{aligned} \sigma_\alpha : \quad & U_\alpha \longrightarrow \pi^{-1}(U_\alpha) \\ & x \longmapsto z \cdot \phi_\alpha(z)^{-1} \end{aligned} \tag{1.27}$$

where as usual $\pi(z) = x$.
Clearly $\pi(\sigma_\alpha(x)) = x$. The local sections $\{\sigma_\alpha\}$ are called the *natural (local) sections* associated to the principal coordinate representation $\{(U_\alpha, \psi_\alpha)\}$ for P.
On an overlapping region $U_\alpha \cap U_\beta$ we have

$$\begin{aligned} \sigma_\alpha(x) &= z \cdot \phi_\alpha(z)^{-1} = z \cdot (\phi_\beta(z)^{-1} . \psi_{\beta\alpha}(x)) = \\ &= \sigma_\beta(x) \cdot \psi_{\beta\alpha}(x), \quad x \in U_\alpha \cap U_\beta, \quad \pi(z) = x \end{aligned} \tag{1.28}$$

By Proposition I.2.1 the right action of G on P induces a homomorphism

$$\begin{aligned} LG &\longrightarrow \chi(P) \\ E &\longmapsto Z_E \end{aligned}$$

As the action of G transforms a fibre $\pi^{-1}(x)$ over x onto itself, the tangent vector $Z_E(z)$ lays in the tangent space of $\pi^{-1}(x)$ at z, $\pi(z) = x$.

Definition I.2.5 *The tangent space of the fibre $\pi^{-1}(x)$ at z is called the vertical subspace of TP_z and shall be denoted by V_z.*

A vector field Z on P is called *vertical* if $Z(z) \in V_z, \quad \forall z \in P$. As an example Z_E, $E \in LG$, is a vertical vector field. The set of all vertical vector fields, denoted by $\chi_v(P)$, is a subalgebra of $\chi(P)$.

Proposition I.2.3 *The set $\chi_v(P)$ of all vertical vector fields on P is isomorphic to $C^\infty(P; LG)$. The isomorphism being given by*

$$\begin{aligned} C^\infty(P; LG) &\longrightarrow \chi_v(P) \\ \varepsilon &\longmapsto Z_\varepsilon \end{aligned}$$

with $Z_\varepsilon(z) = Z_{\varepsilon(z)}(z)$ where $Z_{\varepsilon(z)}$ is the fundamental vector field associated to $\varepsilon(z) \in LG$.

A diffeomorphism ψ of P onto itself is called an *automorphism* of the principal fibre bundle $(P \xrightarrow{\pi} M, G)$ if it satisfies: $\psi(z \cdot g) = \psi(z) \cdot g, \quad \forall z \in P$.
An automorphism ψ of P induces a diffeomorphism j_ψ of M given by

$$j_\psi(x) = \pi(\psi(z)), \quad \pi(z) = x$$

An automorphism ψ is called *vertical* (or *based*) if j_ψ is the identity transformation of M. For such an automorphism we clearly have $\psi(z) = z \cdot \tilde{\gamma}(z)$ where $\tilde{\gamma} : P \longrightarrow G$. The condition $\psi(z \cdot g) = \psi(z) \cdot g$ implies $(z \cdot g) \cdot \tilde{\gamma}(z \cdot g) = (z \cdot \tilde{\gamma}(z)) \cdot g$. Since the action a is free this means that $\tilde{\gamma}$ must satisfy

$$\tilde{\gamma}(z \cdot g) = g^{-1}.\tilde{\gamma}(z).g = ad_{g^{-1}}\tilde{\gamma}(z) \tag{I.29}$$

We shall denote by $AutP$ (resp. Aut_0P) the group of all automorphisms (resp. vertical automorphisms) of P.

For the trivial bundle $P = M \times G$, Aut_0P is isomorphic to the group of smooth maps $C^\infty(M;G)$, the multiplication in $C^\infty(M;G)$ being defined pointwise:

$$\gamma_1.\gamma_2(x) = \gamma_1(x).\gamma_2(x), \quad \forall x \in M, \quad \gamma_1, \gamma_2 \in C^\infty(M;G)$$

An element $\gamma \in C^\infty(M;G)$ determines an element $\psi \in Aut_0P$ by

$$\psi(x,h) = (x, \gamma(x).h), \quad (x,h) \in P = M \times G$$

In this case $\tilde{\gamma}$ is given by $\tilde{\gamma}(x,h) = h^{-1}.\gamma(x).h$.
Denote by τ the following homomorphism

$$\begin{aligned} \tau \ : \ DiffM &\longrightarrow Aut\, C^\infty(M;G) \\ \phi &\longmapsto \tau_\phi \end{aligned}$$

where $\tau_\phi \ : \ C^\infty(M;G) \longrightarrow C^\infty(M;G)$

$$\gamma \longmapsto \tau_\phi \gamma = \gamma \circ \phi^{-1}$$

Proposition I.2.4 *Let P be a trivial bundle. Then $AutP$ is isomorphic to the semidirect product of $DiffM$ and Aut_0P relative to the homomorphism τ*

$$AutP \equiv DiffM \times_\tau Aut_0P$$

Proof: to an element $(\phi, \gamma) \in DiffM \times_\tau Aut_0P$ we associate an element $\psi \in AutP$ by

$$\psi(x,h) = (\phi(x), \gamma(\phi(x)).h), \quad (x,h) \in M \times G$$

We have

$$\begin{aligned} \psi_1 \circ \psi_2(x,h) &= \psi_1(\phi_2(x), \gamma_2(\phi_2(x)).h) = \\ &= (\phi_1 \circ \phi_2(x), \gamma_1(\phi_1 \circ \phi_2(x)).\gamma_2(\phi_2(x)).h) = \\ &= (\phi_1 \circ \phi_2(x), (\gamma_1.\tau_{\phi_1}\gamma_2)(\phi_1 \circ \phi_2(x)).h) \end{aligned}$$

i.e. to $\psi_1 \circ \psi_2$ corresponds the element $(\phi_1 \circ \phi_2, \gamma_1.\tau_{\phi_1}\gamma_2)$ of $DiffM \times_\tau Aut_0P$, which by definition is the semidirect product $(\phi_1, \gamma_1).{}_\tau(\phi_2, \gamma_2)$. □

I.2.3 Associated vector bundles

Given a principal fibre bundle P with structure group G and a representation ρ of G in the vector space F, we can associate with P a fibre bundle $P \times_\rho F$ in the following way.
In the cartesian product $P \times F$ we define an equivalence relation by setting

$$(z,v) \sim (z',v') \;\; \text{if } (z',v') = (z \cdot g, \rho(g^{-1})v) \text{ for some } g \in G$$

Denote the quotient space of $P \times F$ by the above relation by $P \times_\rho F$. An equivalence class in $P \times_\rho F$ will be denoted by $[z,v]$. By definition $[z,v] = [z \cdot g, \rho(g^{-1})v]$, $g \in G$. A projection π_F of $P \times_\rho F$ onto M is defined by $\pi_F([z,v]) = \pi(z) = x$.

Given a principal coordinate representation $\{(U_\alpha, \psi_\alpha)\}$ for P, the diffeomorphism $\psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times G$ induces a map

$$^F\psi_\alpha \; : \; \pi_F^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$$

given by

$$^F\psi_\alpha([z,v]) = (\pi(z), \rho(\phi_\alpha(z))v), \quad [z,v] \in \pi_F^{-1}(U_\alpha)$$

With the help of these maps a manifold structure can be introduced on $P \times_\rho F$, see e.g. [26].

Definition I.2.6 *$(P \times_\rho F \xrightarrow{\pi_F} M, F, G)$ is called the vector bundle associated with P by the representation ρ. It is a vector bundle with standard fibre F and structure group G.*

A F-valued q-form ω on P is called *equivariant* with respect to the representation ρ of G in F if

$$a_g^* \omega = \rho(g^{-1}) \circ \omega, \quad \forall g \in G \tag{I.30}$$

Lemma I.2.1 *To every equivariant map $\tilde{s} : P \longrightarrow F$ corresponds a global section s of $P \times_\rho F$ and viceversa. The section s is given by* $s : \begin{array}{l} M \longrightarrow P \times_\rho F \\ x \longmapsto s(x) = [z, \tilde{s}(z)] \end{array}$

with $\pi(z) = x$.

The set of all sections of the associated vector bundle $P \times_\rho F$ is naturally a vector space over $\mathbf{C}$ with respect to the following operations

$$\begin{array}{ll} (s_1 + s_2)(x) = [z, \tilde{s}_1(z) + \tilde{s}_2(z)], & \pi(z) = x \\ \alpha s_1(x) = [z, \alpha \tilde{s}_1(z)], & \alpha \in \mathbf{C} \end{array} \tag{I.31}$$

where $s_i \in Sec\, P \times_\rho F$ and $\tilde{s}_i$ denotes the corresponding equivariant map $P \longrightarrow F$.

I.2.4 Examples

The most important example of a principal fibre bundle is the so called *bundle of linear frames*. This was introduced by *E.Cartan* who called it "la méthode du repère mobile". Its importance is due to the fact that every field theory in a Riemannian manifold is implicitly formulated in terms of this fibre bundle.

A *linear frame* u_x of a manifold M is an ordered basis of the tangent space TM_x at $x \in M$

$$u_x = (e_1(x), \ldots, e_n(x)), \quad e_i(x) \in TM_x, \quad n = dim\, M$$

Two linear frames u_x and u'_x are related by a change of basis

$$e'_i(x) = e_\mu(x) E^\mu_i \quad \text{briefly } u'_x = u_x \cdot E \tag{I.32}$$

where E belongs to the general linear group $GL(n, \mathbf{R})$.

Denote by LM_x the set of all linear frames of M at x, and by LM the disjoint union of all LM_x, $x \in M$. By associating to a linear frame $u_x \in LM_x$ its base point x, we trivially define a map $\pi : LM \longrightarrow M$.

Given an atlas $\{(U_\alpha, \varphi_\alpha)\}$ on M and the natural basis $\{\partial_\mu, \mu = 1, \ldots, n\}$ of TM_x, $x \in U_\alpha$, each linear frame u_x can be expressed as

$$u_x = (E^\mu_1(x)\partial_\mu, \ldots, E^\mu_n(x)\partial_\mu), \quad E(x) \in GL(n, \mathbf{R})$$

This shows that there is a bijection

$$\begin{aligned} \psi_\alpha \;:\; \pi^{-1}(U_\alpha) &\longrightarrow U_\alpha \times GL(n, \mathbf{R}) \\ u_x &\longmapsto (x, E(x)) \end{aligned}$$

With the help of the maps ψ_α we can introduce a manifold structure on LM.

If we let $GL(n, \mathbf{R})$ act on LM as a change of basis

$$\begin{aligned} a \;:\; LM \times GL(n, \mathbf{R}) &\longrightarrow LM \\ (u_x, E) &\longmapsto u_x \cdot E, \qquad u_x \in LM_x \end{aligned}$$

we obtain a free right action of $GL(n, \mathbf{R})$ on LM.

Definition I.2.7 *The so obtained principal fibre bundle* $(LM \xrightarrow{\pi} M, GL(n, \mathbf{R}))$ *is called the bundle of linear frames over* M.

Definition I.2.8 *A manifold* M *is called parallelizable if the bundle of linear frames over* M *is trivial.*

If M is a Riemannian manifold with metric g, then we can choose an *orthonormal* (with respect to g) frame $o_x = (e_1(x), \ldots, e_n(x))$ at $x \in M$:

$$g(x; e_i(x), e_j(x)) = \delta^i_j, \quad e_i(x), e_j(x) \in TM_x$$

Two orthonormal frames o_x and o'_x are related by an orthogonal change of basis, i.e. $o'_x = o_x \cdot O$ where $O \in O(n, \mathbf{R})$.
The set of all orthonormal frames at x will be denoted by OM_x. Analogously to the bundle of linear frames we can construct the *bundle of orthonormal frames* $(OM \xrightarrow{\pi} M, O(n, \mathbf{R}))$ over M.
Given an atlas $\{(U_\alpha, \varphi_\alpha)\}$ on M the basis elements $e_i(x)$ of an orthonormal frame $o_x = (e_1(x), \ldots, e_n(x))$ at $x \in U_\alpha$ can still be expressed as (see eq. I.32)

$$e_i(x) = E_i^\mu(x)\partial_\mu, \quad i, \mu = 1, \ldots, n$$

where $\{\partial_\mu\}$ is the natural basis of TM_x.
From the orthonormality of the $e_i(x)$'s it follows that in this case the inverse matrix of E, denoted by e, is given by

$$e^i_\mu(x) = g_{\mu\nu}(x) E_i^\nu(x), \quad x \in U_\alpha \tag{I.33}$$

Definition I.2.9 *The functions e^i_μ on U_α are called vielbeins. In the case that M is a 4-dimensional manifold they are called vierbeins or tetrads.*

If the Riemannian manifold M is oriented we can consider the oriented orthonormal frames. Two such frames at $x \in M$ are related by

$$o'_x = o_x \cdot O \quad \text{where in this case } O \in SO(n, \mathbf{R})$$

We shall denote by $(OM^+ \xrightarrow{\pi} M, SO(n, \mathbf{R}))$ the resulting bundle of oriented orthonormal frames.

As an example of an associated vector bundle, consider the canonical representation ι of $GL(n, \mathbf{R})$ in $\mathbf{R}^n$

$$(\iota(E)\vec{v})^i = E^i_j v^j \quad \text{or } \iota(E)\vec{v} = E\vec{v}, \quad \vec{v} = (v^1, \ldots, v^n) \in \mathbf{R}^n$$

and the associated vector bundle $(LM \times_\iota \mathbf{R}^n \xrightarrow{p} M, \mathbf{R}^n, GL(n, \mathbf{R}))$.

Given a linear frame $u_x = (e_1(x), \ldots, e_n(x))$ at $x \in M$, every tangent vector $v_x \in TM_x$ can be written as

$$v_x = v^i(x) e_i(x)$$

where $v^i(x)$ are the coordinates of v_x with respect to the linear frame u_x. As usual the coordinates of v_x with respect to the frame $u'_x = u_x \cdot E$ are given by

$$v'^i(x) = e^i_\nu(x) v^\nu(x)$$

Identifying the coordinates of a tangent vector with respect to a linear frame with a vector in $\mathbf{R}^n$, we obtain an isomorphism

$$\begin{array}{ccl} TM_x & \xrightarrow{\simeq} & p^{-1}(x) \\ v_x & \longmapsto & [u_x, \vec{v}(x)] = [u_x \cdot E, e\,\vec{v}(x)] = [u'_x, \vec{v}'(x)] \end{array}$$

where $\vec{v}(x) = (v^1(x), \ldots, v^n(x))$.

Definition I.2.10 *The associated vector bundle $LM \times_\iota \mathbf{R}^n$ is denoted by TM and it is called the tangent bundle of M.*

Since a vector field on M is a smooth assignment of an element of TM_x to each $x \in M$, we can identify $\chi(M)$ with $Sec\,TM$.

Definition I.2.11 *The $\mathbf{R}^n$-valued 1-form θ on LM given by*

$$\theta(u_x; w) = u_x^{-1}(d\pi_{u_x} w), \quad w \in T\,LM_{u_x}$$

where $u_x^{-1}(d\pi_{u_x} w)$ are the coordinates of the tangent vector $d\pi_{u_x} w \in TM_x$ with respect to the linear frame u_x, is called the solder (or canonical) form of LM.

If u_α is a local section of LM over U_α then we clearly have

$$(u_\alpha^* \theta)_x = e^i_\mu dx^\mu \, \hat{e}_i, \quad x \in U_\alpha \tag{I.34}$$

where $u_\alpha(x) = \{E_i^\mu(x)\partial_\mu\}$ and $\{\hat{e}_i\}$ is the canonical basis of $\mathbf{R}^n$.

Definition I.2.12 *The tensor bundle TM^p_q of type (p,q) over M is the associated vector bundle*

$$TM^p_q = LM \times_\rho (\mathbf{R}^n)^p \otimes (\mathbf{R}^{n*})^q$$

where the representation ρ of $GL(n, \mathbf{R})$ in $(\mathbf{R}^n)^p \otimes (\mathbf{R}^{n*})^q$ is given by

$$\rho(E)[(\vec{v}_1, \ldots, \vec{v}_p) \otimes (\vec{\omega}_1, \ldots, \vec{\omega}_q)] = (E\vec{v}_1, \ldots, E\vec{v}_p) \otimes (\tilde{e}\vec{\omega}_1, \ldots, \tilde{e}\vec{\omega}_q)$$

where $E \in GL(n, \mathbf{R})$, $\vec{v}_i \in \mathbf{R}^n$, $\vec{\omega}_j \in \mathbf{R}^{n*}$, and $\tilde{e}$ denotes the transposed matrix of $e = E^{-1}$.

As can be easily verified the fibre over x of this vector bundle is isomorphic to $\bigotimes^p_q TM_x$. Thus the space $T^p_q(M)$ of all tensor fields of type (p,q) over M can be identified with $Sec\,TM^p_q$.

I.3 Connections in a principal fibre bundle

I.3.1 Definition of a connection

Let $(P \xrightarrow{\pi} M, G)$ be a principal fibre bundle over a n-dimensional manifold M. The fibre $\pi^{-1}(x)$ over $x \in M$ is a submanifold of P, therefore we can consider its tangent space at a point $z \in \pi^{-1}(x)$. We defined it (see Definition I.2.5) as the vertical subspace V_z at $z \in P$. Clearly $V_z \subset TP_z$.
Another subspace H_z of TP_z, such that $TP_z = V_z \oplus H_z$, is called *horizontal.*

Definition I.3.1 *A connection Γ in the principal fibre bundle $(P \xrightarrow{\pi} M, G)$ is a smooth assignment of a horizontal subspace H_z to each point z of P, such that*

$$H_{z \cdot g} = (da_g)_z H_z, \quad \forall z \in P$$

where a is the right action of G on P.

A tangent vector belonging to H_z is called a *horizontal vector.*
From the above Definition it follows that given a connection Γ in P, any vector field Z on P can be uniquely decomposed as

$$Z = Z_v + Z_h \text{ with } Z_v(z) \in V_z \text{ and } Z_h(z) \in H_z, \quad \forall z \in P$$

i.e. $\mathcal{X}(P) = \mathcal{X}_v(P) \oplus \mathcal{X}_h(P)$, where the notation used is evident.

We shall denote by h the projection

$$h : \mathcal{X}(P) \longrightarrow \mathcal{X}_h(P) \tag{1.35}$$

and by h_z its local version $h_z : TP_z \longrightarrow H_z$.

Definition I.3.2 *A r-form τ on P satisfying*

$$\tau(Z_v, Z_1, \ldots, Z_{r-1}) \equiv 0, \quad \forall Z_v \in \mathcal{X}_v(P), \quad \forall Z_i \in \mathcal{X}(P)$$

is called a horizontal form.

Proposition I.3.1 *Let τ be an invariant, i.e. $a_g^* \tau = \tau$, $\forall g \in G$, horizontal r-form on P. Then $\tau = \pi^* \bar{\tau}$, where $\bar{\tau}$ is a uniquely defined r-form on M.*

I.3.2 The connection form and its curvature form

In the physics literature a connection is usually introduced through a gauge potential A, which is LG-valued 1-form on the space-time manifold. We shall see in the next section how the gauge potential comes about. For the moment let us introduce a LG-valued 1-form ω on P associated to the connection Γ.
By Proposition I.2.3 any vertical vector field Z_v on P can be written as $Z_v = Z_\varepsilon$, where $\varepsilon \in C^\infty(P; LG)$. We define ω by

$$\omega(Z) = \omega(Z_v + Z_h) = \omega(Z_v) = \varepsilon, \quad \forall Z \in \mathcal{X}(P)$$

i.e. by

$$\begin{array}{ll} i) & \omega(Z_\varepsilon) = \varepsilon, \quad \varepsilon \in C^\infty(P; LG) \\ ii) & \omega(Z_h) = 0, \quad \forall Z_h \in \chi_h(P) \end{array} \tag{I.36}$$

Observe that ω implicitly depends on the connection Γ through the decomposition $Z = Z_v + Z_h$.
The so defined LG-valued 1-form ω is called the *connection form* on P.

Proposition I.3.2 *The connection form ω satisfies*

$$a_g^* \omega = ad_{g^{-1}} \circ \omega$$

i.e. ω is equivariant with respect to the adjoint representation of G.

Proposition I.3.3 *Given an automorphism ψ of P and a connection form ω on P, the form $\psi^*\omega$ is still a connection form.*

The utility of a connection Γ in a principal fibre bundle P is that it allows the introduction of a so-called exterior covariant derivative and more important, of a covariant derivation which acts on the sections of any vector bundle associated with P (see below).

Given a r-form τ on P, its *exterior covariant derivative*, denoted by $D\tau$, is the (r+1)-form given by

$$D\tau(Z_1, \ldots, Z_{r+1}) = d\tau(hZ_1, \ldots, hZ_{r+1}), \quad Z_i \in \chi(P) \tag{I.37}$$

where h is the projection (I.35). By definition $D\tau$ is clearly horizontal.
The LG-valued 2-form $D\omega$ deserves special attention.

Definition I.3.3 *Given a connection form ω on P, the LG-valued 2-form $D\omega$ is called the curvature form of ω. It will be denoted by Ω.*

Proposition I.3.4 *The curvature form Ω satisfies*

$$\begin{array}{lll} i) & \Omega = d\omega + \frac{1}{2}[\omega \wedge \omega] & \textit{(structure equation of Maurer-Cartan)} \\ ii) & D\Omega \equiv 0 & \textit{(Bianchi's identity)} \end{array}$$

Choosing a basis $\{T_i,\, i = 1, \ldots, dimG\}$ of LG, we can write

$$\omega = \omega^i T_i, \quad \Omega = \Omega^i T_i, \quad \text{where } \omega^i \in \Omega^1(P), \Omega^i \in \Omega^2(P)$$

Denoting by c_{ij}^k the structure constants of LG with respect to the basis $\{T_i\}$, the structure equation of Maurer-Cartan can be expressed as

$$\Omega^i = d\omega^i + \frac{1}{2} c_{jk}^i\, \omega^j \wedge \omega^k \tag{I.38}$$

I.3.3 Horizontal lifts

Given a connection Γ in $(P \xrightarrow{\pi} M, G)$ and a vector field X on the base space M, there is a unique horizontal vector field X^* on P such that

$$d\pi_z X^*(z) = X(\pi(z)), \quad \forall z \in P$$

Definition I.3.4 *The vector field X^* on P is called the (horizontal) lift with respect to the given connection Γ, of the vector field X on M.*

It can be shown that the lift X^* is *invariant*, i.e.

$$a_{g*}X^* = X^*, \quad \forall g \in G \tag{I.39}$$

If X generates a one-parameter group of diffeomorphisms of M, then X^* generates (locally) a one-parameter group of automorphisms of P, called the *parallel displacement.*

I.3.4 The covariant derivative

As we said before the importance of a connection Γ in the principal fibre bundle $(P \xrightarrow{\pi} M, G)$ is that it determines a linear map between the sections of any vector bundle associated with P. This map is defined as follows.
By Lemma 1.2.1 to a section s of the vector bundle $P \times_\rho F$ we can associate an equivariant map $\bar{s} : P \longrightarrow F$ and viceversa. Given a vector field X on the base space M, the map $X^*\bar{s} : P \longrightarrow F$, where X^* is the lift of X with respect to a given connection Γ in P, is still equivariant. Hence we can associate to it a section $\nabla_X s$ of $P \times_\rho F$

$$\nabla_X s(x) = [z, X^*\bar{s}(z)], \quad \pi(z) = x$$

Definition I.3.5 *The section $\nabla_X s$ of $P \times_\rho F$ associated to the equivariant map $X^*\bar{s}$ is called the covariant derivative of s with respect to the vector field X on M.*

The covariant derivative ∇_X satisfies the following conditions.

Proposition I.3.5 *i)* $\nabla_{X+Y} s = \nabla_X s + \nabla_Y s, \quad X, Y \in \mathcal{X}(M)$

ii) $\nabla_{fX} s = f\nabla_X s, \quad f \in C^\infty(M)$
i.e. $\nabla_{fX} s(x) = f(x)\nabla_X s(x)$

iii) $\nabla_X fs = (Xf)s + f\nabla_X s$

By conditions *i)* and *ii)* above and given a section s of $P \times_\rho F$, we can define a $(Sec\, P \times_\rho F)$-valued 1-form ∇s on M by setting

$$\nabla s(X) = \nabla_X s, \quad X \in \mathcal{X}(M) \tag{I.40}$$

Definition I.3.6 *The $(Sec\, P \times_\rho F)$-valued 1-form ∇s on M is called the covariant differential of $s \in Sec\, P \times_\rho F$.*

Suppose we are given a principal coordinate representation for P and the associated set of natural sections $\{\sigma_\alpha\}$ (see 1.27). To a section s of the associated vector bundle $P \times_\rho F$ corresponds an equivariant map $\tilde{s} : P \longrightarrow F$, hence a set of maps $\{s_\alpha\}$, where $s_\alpha = \tilde{s} \circ \sigma_\alpha : U_\alpha \longrightarrow F$. We can consider the covariant derivative ∇_X as acting on the map s_α by defining

$$\nabla_X s_\alpha = X^* \tilde{s} \circ \sigma_\alpha \tag{I.41}$$

Proposition I.3.6 *Let $\{\sigma_\alpha\}$ be the set of natural local sections of the principal fibre bundle $(P \xrightarrow{\pi} M, G)$,and X^* the lift with respect to a connection Γ in P of the vector field X on M. Given a map $\tilde{s} : P \longrightarrow F$, equivariant with respect to a representation ρ of G in the vector space F, the map $\nabla_X s_\alpha = X^* \tilde{s} \circ \sigma_\alpha$ is explicitly given by*

$$\nabla_X s_\alpha(x) = X_x s_\alpha + d\rho[(\sigma_\alpha^* \omega)(x; X_x)] s_\alpha(x), \quad x \in M$$

where $d\rho$ is the derivative of the representation ρ, which is a representation of LG in F, and ω is the connection form of the connection Γ.

Proof: denote by z the point $\sigma_\alpha(x)$ of P and by E the element of LG given by

$$E = \omega(z; (d\sigma_\alpha)_x X_x) = (\sigma_\alpha^* \omega)(x; X_x)$$

The tangent vector $(d\sigma_\alpha)_x X_x$ can be expressed as

$$(d\sigma_\alpha)_x X_x = X_z^* + Z_E(z)$$

where Z_E is the fundamental vector field associated to $E \in LG$.
Whence

$$\begin{aligned} X^* \tilde{s}(z) = X_z^* \tilde{s} &= [(d\sigma_\alpha)_x X_x] \tilde{s} - Z_E(z) \tilde{s} = \\ &= X_x(\tilde{s} \circ \sigma_\alpha) - \lim_{t \to 0} \frac{1}{t} [\tilde{s}(z \cdot \exp tE) - \tilde{s}(z)] \end{aligned}$$

As $\tilde{s}$ is equivariant, the last term in the above equation is equal to

$$\begin{aligned} &\lim_{t \to 0} \frac{1}{t} [\rho(\exp -tE) \tilde{s}(z) - \tilde{s}] = \\ &\quad = -\lim_{t \to 0} \frac{1}{t} [\rho(\exp tE) - \mathbf{1}] \tilde{s}(z) = -d\rho(E) \tilde{s}(z) \end{aligned}$$

Thus

$$X^* \tilde{s}(z) = X_x s_\alpha + d\rho(E) \tilde{s}(z), \quad z = \sigma_\alpha(x)$$

which proves the Proposition. □

Corollary I.3.1 *By considering, analogously to (I.40), ∇s_α to be a F-valued 1-form on U_α we explicitly have*

$$\nabla s_\alpha = ds_\alpha + d\rho(\sigma_\alpha^* \omega) s_\alpha$$

It is usually in this form that the connection Γ appears in a field theory. We shall see later on that the map s_α can be interpreted as a matter field of the theory.

I.3.5 Linear connections

An important example of connections are those in the principal fibre bundle LM (or OM) (see Subsection I.2.4).

Definition I.3.7 *A connection in LM (resp. OM) is called a linear connection (resp. a Riemannian connection) of the manifold M.*

Consider the associated vector bundle $TM = LM \times_\iota \mathbf{R}^n$, where as usual $n = dim\, M$. We saw in Subsection I.2.4 that its sections can be considered as vector fields on M. Hence, in this case the covariant derivative ∇_X maps the vector field $Y \in \mathcal{X}(M)$ into the vector field $\nabla_X Y$.
Given a local coordinate system in $U_\alpha \subset M$, we can write $X = X^\mu \partial_\mu$, $Y = Y^\mu \partial_\mu$. Therefore, by Proposition I.3.5

$$\begin{aligned} \nabla_X Y &= \nabla_X Y^\nu \partial_\nu = (XY^\nu)\partial_\nu + Y^\nu \nabla_X \partial_\nu = \\ &= (XY^\nu)\partial_\nu + X^\mu Y^\nu \nabla_{\partial_\mu} \partial_\nu \end{aligned}$$

Denoting the components of the vector field $\nabla_{\partial_\mu}\partial_\nu$ by $\Gamma^\rho_{\mu\nu}$, we finally obtain

$$\nabla_X Y = (XY^\rho + X^\mu Y^\nu \Gamma^\rho_{\mu\nu})\partial_\rho \quad \text{in } U_\alpha \subset M \tag{I.42}$$

Definition I.3.8 *The functions $\Gamma^\rho_{\mu\nu}$ on U_α are called the components of the linear connection with respect to the local coordinate system in U_α. They are also called Christoffel's symbols.*

Definition I.3.9 *A Riemannian connection is called Levi-Civita (or symmetric) if $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$.*

Proposition I.3.7 *The components of the unique Levi-Civita connection of a Riemannian manifold M are given by*

$$\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2}\, g^{\alpha\beta}\,(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu})$$

where g is the metric tensor of M.

Given three vector fields X, Y, Z on M, we can form the following vector fields

$$\begin{aligned} T(X,Y) &= \nabla_X Y - \nabla_Y X + [X,Y] \\ \text{and} \quad R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \end{aligned} \tag{I.43}$$

It is easy to check that

$$\begin{aligned} T(f_1 X, f_2 Y) &= f_1 f_2 T(X,Y) \\ \text{and} \quad R(f_1 X, f_2 Y) f_3 Z &= f_1 f_2 f_3 R(X,Y)Z, \quad f_i \in C^\infty(M) \end{aligned}$$

From this it follows [26] that the map T resp. R is actually a tensor field of type (1,2) resp. (1,3) on M.

Definition I.3.10 *The tensor field T resp. R is called the torsion resp. the curvature of the linear connection of M.*

Proposition I.3.8 *The components of the torsion T resp. curvature R with respect to a local coordinate system are given by*

$$\begin{aligned} T^{\alpha}_{\mu\beta} &= \Gamma^{\alpha}_{\mu\beta} - \Gamma^{\alpha}_{\beta\mu} \\ R^{\alpha}_{\beta\mu\nu} &= \partial_{\mu}\Gamma^{\alpha}_{\nu\beta} - \partial_{\nu}\Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\tau}\Gamma^{\tau}_{\nu\beta} - \Gamma^{\alpha}_{\nu\tau}\Gamma^{\tau}_{\mu\beta} \end{aligned}$$

where by definition $R(X,Y)^{\alpha}_{\beta} = X^{\mu}Y^{\nu}R^{\alpha}_{\beta\mu\nu}$.

Definition I.3.11 *The tensor field of type (0,2), denoted by Ric, whose components are locally given by*

$$Ric_{\beta\nu} = R^{\alpha}_{\beta\alpha\nu}$$

is called the Ricci tensor field.
If M is a Riemannian manifold with metric g then the function $\mathcal{R}$ which is locally given by

$$\mathcal{R} = g^{\mu\nu} Ric_{\mu\nu}$$

is called the scalar curvature of M.

I.4 Geometrical interpretation of a gauge theory

In this section we shall show how the fields of a gauge theory can be interpreted as geometric quantities related to a principal fibre bundle and its associated vector bundles.
To simplify matters we shall take the structure group G to be a matrix Lie group, as e.g. $SU(r)$.

I.4.1 Gauge transformations and changes of section

Let $(P \xrightarrow{\pi} M, G)$ be a principal fibre bundle with a principal coordinate representation $\{(U_\alpha, \psi_\alpha)\}$. Given a connection Γ in P with connection form ω, we can pull-back ω on U_α with the help of the natural local section σ_α. Denote this pull-back by $A_\alpha = \sigma_\alpha^* \omega$. A_α is a LG-valued 1-form on U_α. The relation between two such local forms A_α and A_β on an overlapping region $U_\alpha \cap U_\beta$ is given by the following Proposition.

Proposition I.4.1 *Let $\{(U_\alpha, \psi_\alpha)\}$ be a principal coordinate representation of a principal fibre bundle P and $\{\sigma_\alpha\}$ the associated set of natural local sections. Given a connection Γ in P with connection form ω, the forms $A_\alpha = \sigma_\alpha^* \omega$ verify the following compatibility condition*

$$A_{\beta x} = \psi_{\alpha\beta}(x)^{-1} A_{\alpha x} \psi_{\alpha\beta}(x) + \psi_{\alpha\beta}(x)^{-1} d\psi_{\alpha\beta x}, \quad x \in U_\alpha \cap U_\beta$$

where $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$ are the transition functions corresponding to the given principal coordinate representation and the matrix multiplication is understood.

The first important consequence of Proposition I.4.1 is that due to their transformation properties on an overlapping region, the forms A_α don't patch together to give a globally defined 1-form on the base space M. Secondly, the above compatibility condition involves only the transition functions $\psi_{\alpha\beta}$, which relate the sections σ_α and σ_β by (see I.24)

$$\sigma_\beta(x) = \sigma_\alpha(x) \cdot \psi_{\alpha\beta}(x), \quad x \in U_\alpha \cap U_\beta$$

or briefly

$$\sigma_\beta = \sigma_\alpha \cdot \psi_{\alpha\beta} \quad \text{in } U_\alpha \cap U_\beta$$

We can thus immediately formulate the following

Corollary I.4.1 *Let $\{\sigma_\alpha\}$ and $\{\sigma'_\alpha\}$ be two sets of natural local sections related by*

$$\sigma'_\alpha = \sigma_\alpha \cdot \gamma_\alpha$$

where γ_α is a smooth map $U_\alpha \longrightarrow G$.
Given a connection form ω on P, the forms $A_\alpha = \sigma_\alpha^* \omega$ and $A'_\alpha = \sigma'^*_\alpha \omega$ on U_α satisfy

$$A'_\alpha = \gamma_\alpha^{-1} A_\alpha \gamma_\alpha + \gamma_\alpha^{-1} d\gamma_\alpha \tag{1.44}$$

Condition (I.44) is what physicists call a *gauge transformation.* It arises from the change of section $\sigma_\alpha \to \sigma_\alpha \cdot \gamma_\alpha$. It's easily verified that the set of all gauge transformations forms a group isomorphic to $C^\infty(U_\alpha; G)$.

In physical terms a 1-form on the space-time manifold "transforming" as A_α is a so called *gauge potential.* Hence, a first step in finding a geometrical formulation of a classical gauge theory is to identify the space-time manifold with an open set U_α of the base space M of a principal fibre bundle $(P \xrightarrow{\pi} M, G)$, and the gauge potential with the pull-back by a natural local section σ_α over U_α of a connection form ω on P.

The fact that the space-time must be considered a subset of a larger manifold is welcome for the following argument. If we take the space-time to be the Euclidean manifold $\mathbf{R}^n$ and require the action functional of the gauge theory to be finite, then, due to the behaviour of the gauge potential at infinity, we can construct in a natural way a principal fibre bundle with base space $\mathbf{R}^n \cup U_\infty$, where U_∞ is a neighborhood of the point at infinity (see e.g. [16]). As it is well known $\mathbf{R}^n \cup U_\infty$ is diffeomorphic, through a stereographic projection, to the n-dimensional sphere S^n. Hence, the constructed fibre bundle is equivalent to a fibre bundle over S^n. In other words, the finiteness of the action functional forced us to *compactify* the space-time to S^n.

Taking this argument into account, the appropriate geometrical framework to mathematically describe a classical gauge theory is thus the principal fibre bundle $(P \xrightarrow{\pi} S^n, G)$. An atlas on S^n is given by two charts $\{(U_\alpha, \varphi_\alpha), \quad \alpha = 1,2\}$, where U_1 resp. U_2 is the southern resp. northern hemisphere of S^n. The above mentioned stereographic projection identifies U_1 with the space-time $\mathbf{R}^n$. In this framework the gauge potential is therefore the pull-back by a natural section σ_1 over U_1 of a connection form ω on $(P \xrightarrow{\pi} S^n, G)$.

Henceforth we won't limit ourselves to the specific case $M = S^n$ but consider instead a general compact base space M without boundary. We shall generally call gauge potential the pull-back by a natural section σ_α over $U_\alpha \subset M$ of a connection form ω on $(P \xrightarrow{\pi} M, G)$.

Returning to the gauge transformations, we interpreted them as changes of natural section $\sigma_\alpha \to \sigma_\alpha \cdot \gamma_\alpha$. The disadvantage of this interpretation is that it is local, i.e. we can perform different gauge transformations in different open sets U_α of M. To avoid this we induce a change in any natural section by composing it with a vertical automorphism ψ of P

$$\sigma_\alpha \to \sigma'_\alpha = \psi \circ \sigma_\alpha$$

Since ψ can be expressed as $\psi(z) = z \cdot \tilde{\gamma}(z)$, $z \in P$ (see page 21) we obtain

$$\sigma'_\alpha(x) = \psi(\sigma_\alpha(x)) = \sigma_\alpha(x) \cdot \tilde{\gamma}(\sigma_\alpha(x)) = \sigma_\alpha(x) \cdot \gamma_\alpha(x) \tag{I.45}$$

where $\gamma_\alpha = \tilde{\gamma} \circ \sigma_\alpha$.
The transformed gauge potential A'_α on U_α is given by

$$A'_\alpha = \sigma'^*_\alpha \omega = (\psi \circ \sigma_\alpha)^* \omega = \sigma^*_\alpha \circ \psi^* \omega$$

As ψ is an automorphism of P, by Proposition I.3.3 $\psi^*\omega$ is still a connection form. Hence, from an active point of view, A'_α can be considered as being the pull-back by the natural section σ_α of the transformed connection form $\psi^*\omega$.
We naturally arrive at the following definition (see [2])

Definition I.4.1 *A gauge transformation in the principal fibre bundle P is a vertical automorphism of P.*

This definition is very useful in a mathematical formulation of a gauge theory since it is perfectly global.

By the above Definition the group of all gauge transformations, denoted by $\mathcal{G}$, is isomorphic to the group of vertical automorphisms of P, i.e. $\mathcal{G} \equiv Aut_0P$. Every element ψ of Aut_0P determines a map $\tilde{\gamma} : P \longrightarrow G$ satisfying (see (I.29))

$$\tilde{\gamma}(z \cdot g) = ad_{g^{-1}}\tilde{\gamma}(z), \quad \forall g \in G$$

This means that $\tilde{\gamma}$ is equivariant with respect to the adjoint representation of G in G. By Lemma I.2.1 this is equivalent to say that $\tilde{\gamma}$ is a section of the associated vector bundle $P \times_{ad} G$ (G is a matrix Lie group, hence a vector space). Observe that we can pointwise define a group multiplication in $Sec(P \times_{ad} G\ )$ by setting (see Subsection I.2.3)

$$\gamma_1 \cdot \gamma_2(x) = [z, \tilde{\gamma}_1(z) \cdot \tilde{\gamma}_2(z)], \quad \gamma_1, \gamma_2 \in Sec(P \times_{ad} G\ ), \quad z \in P$$

and that to $\psi_1 \circ \psi_2$ in Aut_0P corresponds $\gamma_1 \cdot \gamma_2$ in $Sec(P \times_{ad} G\ )$.
We heuristically proved

Proposition I.4.2 *The group of all gauge transformations $\mathcal{G} \equiv Aut_0P$ is isomorphic to the group $Sec(P \times_{ad} G\ )$.*

It can be shown [31] that $Sec(P \times_{ad} G\ )$ is an infinite dimensional Lie group. Its Lie algebra is the infinite dimensional Lie algebra of all sections of the associated vector bundle $P \times_{ad} LG$, where ad is the adjoint representation of G (see Definition I.1.22). The algebraic product in $Sec(P \times_{ad} LG\ )$ is defined pointwise as above.

A one-parameter subgroup γ_t of $Sec(P \times_{ad} G\ )$ generated by $\varepsilon \in Sec(P \times_{ad} LG\ )$ is denoted by $\gamma_t = \exp t\varepsilon$, and by this it is meant

$$\gamma_t(x) = [z, \exp t\tilde{\varepsilon}(z)], \quad \pi(z) = x$$

where $\tilde{\varepsilon} : P \longrightarrow LG$ is the equivariant map corresponding to the section ε.

From Proposition I.4.2 it follows that the Lie algebra $\mathcal{LG}$ of the group of gauge transformations $\mathcal{G}$ is isomorphic to $Sec(P \times_{ad} LG\ )$.

By Proposition I.2.3 the equivariant map $\tilde{\varepsilon} \in C^\infty(P; LG)$ uniquely determines a vertical vector field $Z_{\tilde{\varepsilon}}$. Given any connection form ω on P, $\tilde{\varepsilon}$ is given by $\tilde{\varepsilon} = \omega(Z_{\tilde{\varepsilon}})$ (see (I.36)).
Since $\tilde{\varepsilon}$ is equivariant with respect to the adjoint representation of G we must have

$$\omega(z \cdot g; Z_{\tilde{\varepsilon}}(z \cdot g)) = ad_{g^{-1}}\omega(z; Z_{\tilde{\varepsilon}}(z))$$

By Proposition I.3.2 this implies

$$\omega(z\cdot g; Z_{\tilde\varepsilon}(z\cdot g)) = (a_g^*\omega)(z; Z_{\tilde\varepsilon}(z)) = \omega(z\cdot g; (da_g)_z Z_{\tilde\varepsilon}(z)), \quad \forall z\in P,\ g\in G$$

As $(da_g)_z Z_{\tilde\varepsilon}(z)$ is vertical from the above equation it follows that the vertical vector field $Z_{\tilde\varepsilon}$ associated to the equivariant map $\tilde\varepsilon$ is invariant, i.e. $a_{g*}Z_{\tilde\varepsilon} = Z_{\tilde\varepsilon}$ (see (I.7)). The vector field $Z_{\tilde\varepsilon}$ is the one induced by the one-parameter group ψ_t of vertical automorphisms of P given by

$$\psi_t(z) = z\cdot\tilde\gamma_t(z) = z\cdot\exp t\tilde\varepsilon(z)$$

To avoid a cumbersome notation we shall use the same symbol ε for ε and $\tilde\varepsilon$. Denote by $\mathcal{X}_v^I(P)$ the set of all vertical invariant vector fields on P. $\mathcal{X}_v^I(P)$ is a subalgebra of $\mathcal{X}(P)$.

Proposition I.4.3 *The Lie algebra* $Sec\,P\times_{ad}LG \cong \mathcal{L}\mathcal{G}$ *is isomorphic to the Lie algebra* $\mathcal{X}_v^I(P)$ *with product given by minus the Lie bracket. In other words*

$$Z_{[\varepsilon_1,\varepsilon_2]} = -[Z_{\varepsilon_1}, Z_{\varepsilon_2}]$$

where ε_i *is the equivariant map corresponding to the section* $\varepsilon_i\in Sec\,P\times_{ad}LG$.

Proof: we must only prove the stated equality. Denote by Z_i the vertical invariant vector field Z_{ε_i}, $i=1,2$. To the Lie bracket $[Z_1,Z_2]$ corresponds the equivariant map

$$\omega([Z_1,Z_2]) = Z_1\omega(Z_2) - L_{Z_1}\omega(Z_2)$$

where L_Z is the Lie derivative.
As Z_1 is induced by a one-parameter group ψ_t of automorphisms of P and since $\psi_t^*\omega$ is still a connection form, the 1-form

$$L_{Z_1}\omega = \lim_{t\to 0}\frac{1}{t}[\psi_t^*\omega - \omega]$$

is horizontal. Hence $\omega([Z_1,Z_2]) = Z_1\omega(Z_2)$ or

$$\begin{aligned}\omega([Z_1,Z_2]) &= \tfrac{1}{2}\{Z_1\omega(Z_2) - Z_2\omega(Z_1)\} =\\ &= \tfrac{1}{2}\{d\omega(Z_1,Z_2) + \omega([Z_1,Z_2])\} =\\ &= \tfrac{1}{2}\{\Omega(Z_1,Z_2) - [\omega(Z_1),\omega(Z_2)] + \omega([Z_1,Z_2])\}\end{aligned}$$

where Ω is the curvature form of ω. As the Z_i are vertical, $\Omega(Z_1,Z_2)$ vanishes. We thus obtain

$$\omega([Z_1,Z_2]) = -[\omega(Z_1),\omega(Z_2)] = -[\varepsilon_1,\varepsilon_2] \quad \square$$

The above minus sign is reminiscent of Proposition I.2.1. In fact the group $Sec(P\times_{ad}G) \cong Aut_0P$ acts on P on the left

$$a_{\psi_1\circ\psi_2}(z) = \psi_1\circ\psi_2(z) = a_{\psi_1}\circ a_{\psi_2}(z), \quad \psi_i\in Aut_0P$$

See also [29], where this situation is explained for the general case of the diffeomorphism group of a manifold.

Under an infinitesimal gauge transformation represented by $\gamma = \exp \epsilon\varepsilon$, $\epsilon \ll 1$, $\varepsilon \in Sec(P \times_{ad} LG\ )$, the gauge potential $A_\alpha = \sigma_\alpha^* \omega$ transforms as follows

$$A_\alpha \to A'_\alpha = A_\alpha + \epsilon([A_\alpha, \varepsilon_\alpha] + d\varepsilon_\alpha) + o(\epsilon^2) \tag{I.46}$$

where $\varepsilon_\alpha = \varepsilon \circ \sigma_\alpha : U_\alpha \longrightarrow LG$.
By Corollary I.3.1 and Proposition I.1.10 the infinitesimal gauge transformation (I.46) is equivalent to

$$A_\alpha \to A'_\alpha = A_\alpha + \epsilon\nabla\varepsilon_\alpha + o(\epsilon^2) \tag{I.47}$$

where $\nabla\varepsilon_\alpha$ is the covariant differential of ε_α.
From the active point of view the infinitesimal gauge transformation (I.47) is equivalent to

$$\omega \to \omega' = \omega + \epsilon\, L_{Z_\epsilon}\omega + o(\epsilon^2) \tag{I.48}$$

where Z_ϵ is the vertical invariant vector field associated to $\varepsilon \in Sec\, P \times_{ad} LG$. Thus

$$\sigma_\alpha^* L_{Z_\epsilon}\omega = \nabla\epsilon_\alpha = \nabla\sigma_\alpha^*\epsilon \tag{I.49}$$

Let Ω be the curvature form of the connection form ω on P.

Definition I.4.2 *Given a set of natural sections $\{\sigma_\alpha\}$ of P, the LG-valued 2-form $\sigma_\alpha^*\Omega$ on U_α is called the gauge field strength form.*

The gauge field strength form will be denoted by F_α. Clearly

$$F_\alpha = dA_\alpha + \tfrac{1}{2}[A_\alpha \wedge A_\alpha] \tag{I.50}$$

Under the gauge transformation represented by $\gamma \in Sec(P \times_{ad} G\ )$ the field strength F_α transforms as

$$F_\alpha \to F'_\alpha = \gamma_\alpha^{-1} F_\alpha \gamma_\alpha \quad \text{where } \gamma_\alpha = \tilde{\gamma} \circ \sigma_\alpha \tag{I.51}$$

I.4.2 Matter fields

Usually in a gauge theory so called *matter fields* are also present. These are fields defined on the space-time manifold which transform under a gauge transformation according to an irreducible representation ρ of the structure group G.

Consider the associated vector bundle $P \times_\rho F$, where F is the irreducible vector space corresponding to ρ. Given an equivariant map

$$\tilde{\phi} : P \longrightarrow F, \quad \tilde{\phi}(z \cdot g) = \rho(g)^{-1}\tilde{\phi}(z), \quad \forall g \in G, z \in P$$

and a set of natural sections $\{\sigma_\alpha\}$ of P, the map

$$\phi_\alpha = \tilde{\phi} \circ \sigma_\alpha = \sigma_\alpha^*\tilde{\phi} : U_\alpha \longrightarrow F$$

transforms under the gauge transformation represented by $\gamma \in Sec\, P \times_{ad} G$ as (see (I.45))

$$\begin{aligned} \phi'_\alpha(x) &= \tilde{\phi} \circ \sigma'_\alpha(x) = \tilde{\phi}(\sigma_\alpha(x) \cdot \tilde{\gamma}(\sigma_\alpha(x)) = \\ &= \tilde{\phi}(\sigma_\alpha(x) \cdot \gamma_\alpha(x)) = \rho(\gamma_\alpha(x))^{-1} \phi_\alpha(x), \quad x \in U_\alpha \end{aligned} \tag{I.52}$$

where as usual $\gamma_\alpha = \tilde{\gamma} \circ \sigma_\alpha$.
From this it follows that the matter fields of the theory can be considered as the pull-back by a natural section of the equivariant maps $\tilde{\phi} : P \longrightarrow F$. As for the gauge potential A_α, the matter field ϕ_α is defined only locally on $U_\alpha \subset M$.

We saw in Subsection I.3.4 how the covariant derivative ∇_X is defined to act on ϕ_α (see I.41)

$$\nabla_X \phi_\alpha = X^* \tilde{\phi} \circ \sigma_\alpha$$

where X^* is the horizontal lift of $X \in \mathcal{X}(M)$. Since $X^* \tilde{\phi}$ is still equivariant, $\nabla_X \phi_\alpha$ transforms under a gauge transformation as a matter field

$$(\nabla_X \phi_\alpha)'(x) = X^* \tilde{\phi} \circ \sigma'_\alpha(x) = \rho(\gamma_\alpha(x))^{-1} \nabla_X \phi_\alpha(x) \tag{I.53}$$

It is for this reason that the covariant derivative is so useful.

The most important example of matter fields are the so called *spinor fields* or *spinors*. To introduce them consider the bundle of oriented orthonormal frames $(OM^+ \xrightarrow{\pi} M, SO(n, \mathbf{R}))$ of an orientable Riemannian manifold M (see Subsection I.2.4). As it is well known the special orthogonal group $SO(n, \mathbf{R})$ possesses for n even two inequivalent complex irreducible representations of dimension $d = 2^{\frac{n}{2}-1}$. Denote them by D^+ resp. D^-.

Definition I.4.3 *A righthanded resp. lefthanded Weyl spinor is the pull-back by a natural section e_α of OM^+ of a map $\tilde{\psi}_+$ resp. $\tilde{\psi}_-$: $OM^+ \longrightarrow \mathbf{C}^d$, equivariant with respect to the representation D^+ resp. D^- of $SO(n, \mathbf{R})$ in $\mathbf{C}^d$, $d = 2^{\frac{n}{2}-1}$.*

Definition I.4.4 *A Dirac spinor is the pull-back by a natural section e_α of OM^+ of a map $\tilde{\psi}$: $OM^+ \longrightarrow \mathbf{C}^{2d}$, equivariant with respect to the representation $D^+ \oplus D^-$ of $SO(n, \mathbf{R})$ in $\mathbf{C}^{2d}$, $d = 2^{\frac{n}{2}-1}$.*

It has to be noted that in order to globally define a spinor field on a manifold M, the latter must also satisfy a special topological condition, namely its second Stiefel-Whitney class must vanish (see [14]).

Definition I.4.5 *An orientable, Riemannian manifold whose second Stiefel-Whitney class vanishes is called a spin manifold.*

By taking the tensor product of a spinor field and a general matter field defined on the same open set U_α

$$e_\alpha^* \tilde{\psi} \otimes \sigma_\alpha^* \tilde{\phi} : U_\alpha \longrightarrow \mathbf{C}^d \otimes F$$

we clearly obtain a spinor which transforms also under a gauge transformation in the principal fibre bundle $(P \xrightarrow{\pi} M, G)$.

I.4.3 The symmetry group of a gauge theory

A gauge field theory with gauge group G is usually formulated in terms of fields defined on the space-time manifold. Geometrically the latter is identified with a subset of the base space M of a principal fibre bundle $(P \xrightarrow{\pi} M, G)$. From what we have seen in the preceding subsections, the geometrical objects corresponding to the fields of the theory are all pull-backs by natural sections of quantities globally defined on the total space P. Since the physics doesn't specify which particular natural section has to be used, we are in the presence of a degree of freedom which translates in a symmetry of the theory. As we have seen, this gauge symmetry is induced by vertical automorphisms of P, hence Aut_0P is an invariance group of the theory.

Clearly this is not the only possible invariance. The action functional of the theory is usually constructed in such a way that it is invariant also under transformations which are related to the geometric properties of the space-time manifold. As an example, a gauge theory formulated in a 4-dimensional Minkowski space is assumed to be invariant under Poincaré transformations, which form the so called isometry group of the Minkowski space. We shall assume that this extra invariance group, denoted by $\mathcal{G}_M$, can be identified with a subgroup of the diffeomorphism group $Diff M$ of the base space of the principal fibre bundle upon which the theory is based.

In the special case that this principal fibre bundle is trivial we can combine the gauge invariance group Aut_0P with $\mathcal{G}_M$ to form the semi-direct product $\mathcal{G}_M \times_\tau Aut_0P$ (see Proposition I.2.4). Therefore the total invariance group, denoted by $\mathcal{G}_T$, of a gauge theory based on a trivial fibre bundle P is isomorphic to a subgroup of the automorphism group $AutP$.

$$\mathcal{G}_T = \mathcal{G}_M \times_\tau Aut_0P \subset AutP \tag{I.54}$$

II Anomalies

In this chapter we shall consider a gauge field theory based on the trivial bundle $P = M \times G$. Even though we shall later work with the bundle of linear frames LM, we don't explicitly consider the gauge theory based on it, i.e. the gravitational theory. In what follows P is always different from LM.

The fact that P is trivial implies by Proposition I.2.2 that it admits a global section σ, which in turn allows us to pull-back on M every geometrical quantity defined on P. In the trivial bundle case we can thus extend to the whole base space M the gauge theory originally formulated on an open set of M. Henceforth we shall assume that M is a compact, connected, spin manifold of even dimension $n = 2k$ and without boundary. In other words we forget about the physical reality of our space-time and consider a gauge theory as a purely geometrical setting.

For a trivial bundle P the group of all vertical automorphisms Aut_0P is isomorphic to the group $C^\infty(M;G)$ (see page 21). For a trivial P we thus have (cf. Subsection I.4.1)

$$\mathcal{G} \equiv Aut_0P \cong Sec\,P \times_{ad} G \;\cong C^\infty(M;G)$$

and (see Proposition I.4.3)

$$\mathcal{L}\mathcal{G} \cong \chi_v^I(P) \cong Sec\,P \times_{ad} LG \;\cong C^\infty(M;LG)$$

A gauge transformation (see Definition I.4.1) is therefore represented by an element $\gamma \in C^\infty(M;G)$. γ plays the rôle of the locally defined map $\gamma_\alpha \in C^\infty(U_\alpha;G)$ of Section I.4.

II.1 Classical and quantum symmetries

II.1.1 Global formulation of a classical field theory

A classical field theory is usually defined by an action functional S, which is given by an integral on the space-time manifold of a Lagrangian $\mathcal{L}$. The Lagrangian is a function of the fields, generally denoted by ϕ, and their derivatives $d\phi$ (see [34]). By classical we mean that no factors of $\hbar$ should appear in $\mathcal{L}$. This can always be accomplished by appropriately rescaling the fields ϕ. According to the above we replace the space-time manifold by the manifold M.

$$S[\phi] = \int_M \mathcal{L}(\phi, d\phi) = \int_M \epsilon\, {}^*\mathcal{L}(\phi, d\phi)$$

where ϵ is the volume form on M (see Subsection I.1.8).
We shall limit ourselves to *local* field theories, i.e. those where the n-form $\mathcal{L}$ at a point x of M depends on the fields taken at x only. In this case the action functional S is called local.

The *field equations* of the theory are obtained by requiring that the action functional be stationary under an arbitrary variation $\delta\phi$ of the fields. By assuming the field ϕ is a q-form and by implicitly taking into account the signs which arise from commuting $\delta\phi$ resp. $\delta d\phi$ to the right in $\delta\mathcal{L}$, we can write

$$\begin{aligned}
\delta S &= \int_M \delta\mathcal{L} = \int_M \{\frac{\partial\mathcal{L}}{\partial\phi} \wedge \delta\phi + \frac{\partial\mathcal{L}}{\partial d\phi} \wedge \delta d\phi\} = \\
&= \int_M \{\frac{\partial\mathcal{L}}{\partial\phi} \wedge \delta\phi + (-1)^{n-q-1}\,[d(\frac{\partial\mathcal{L}}{\partial d\phi} \wedge \delta\phi) - d\frac{\partial\mathcal{L}}{\partial d\phi} \wedge \delta\phi]\} = \\
&= \int_M [\frac{\partial\mathcal{L}}{\partial\phi} + (-1)^{n-q}\, d\frac{\partial\mathcal{L}}{\partial d\phi}] \wedge \delta\phi + (-1)^{n-q-1} \int_{\partial M} \frac{\partial\mathcal{L}}{\partial d\phi} \wedge d\phi
\end{aligned}$$

where $n = dim\, M$.
Since the variation $\delta\phi$ is arbitrary, and by assumption $\partial M \equiv \emptyset$, the condition $\delta S = 0$ implies

$$\frac{\partial\mathcal{L}}{\partial\phi} + (-1)^{n-q}\, d\frac{\partial\mathcal{L}}{\partial d\phi} = 0 \tag{II.1}$$

The theory is said to be invariant under the transformation $\phi \to \phi'$ if the value of the action doesn't change, i.e. if

$$S[\phi] = S[\phi']$$

To obtain the above equality is sufficient that the Lagrangian changes by an exterior derivative

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L}(\phi', d\phi') = \mathcal{L} + d\,{}^*X \tag{II.2}$$

where X is a vector field on M, identified through the metric with a 1-form.
By applying the star operation on both sides of (II.2), we see that under a symmetry transformation ${}^*\mathcal{L}$ changes at most by a divergence (see Proposition I.1.7)

$${}^*\mathcal{L} \to {}^*\mathcal{L}' = {}^*\mathcal{L} + (-1)^{n+1}\, div\, X \tag{II.3}$$

Consider an infinitesimal symmetry transformation $\phi \to \phi' = \phi + \delta_o\phi$. By repeating the same steps as in the derivation of the field equations (II.1), we obtain

$$\begin{aligned}
\delta_o\mathcal{L} &= \{[\frac{\partial\mathcal{L}}{\partial\phi} + (-1)^{n-q}\, d\frac{\partial\mathcal{L}}{\partial d\phi}] \wedge \delta_o\phi\} + (-1)^{n-q-1} d(\frac{\partial\mathcal{L}}{\partial d\phi} \wedge \delta_o\phi) \\
&\overset{!}{=} d\,{}^*X
\end{aligned}$$

Therefore, for fields verifying the field equations (II.1), i.e. *on shell*, the $(n-1)$-form

$${}^*j = -{}^*X + (-1)^{n-q-1}\, \frac{\partial\mathcal{L}}{\partial d\phi} \wedge \delta_o\phi \tag{II.4}$$

is closed, i.e. $d\,{}^*j = 0$.
This is just a simplified form of Noether's theorem, which states that to each one-parameter invariance group corresponds a conserved current j, which is the vector

field on M given by (see (I.20))

$$j = -X + (-1)^q\, {}^*(\frac{\partial \mathcal{L}}{\partial d\phi} \wedge \delta_o \phi) \tag{II.5}$$

Conserved means: $div\, j = 0$ for fields verifying (II.1).

II.1.2 The quantum case: the effective action

The quantum field theory corresponding to a classical action S, if it exists, is specified by a set of vacuum expectation values of time-ordered products of field's operators, so called *Green functions*, with the help of which every observable of the theory can in principle be computed (see e.g. [24]).
In the path integral formulation of the quantum theory, the vacuum expectation value $< O >$ of an operator O is given by the weighted mean

$$< O > = \int [d\phi]\, O(\phi)\, e^{-\frac{1}{\hbar} S[\phi]} \tag{II.6}$$

where the symbol $\int [d\phi]$ means that we must "sum" over all possible classical field's configurations, and we set $W[0] = \int [d\phi]\, e^{-\frac{1}{\hbar} S[\phi]} = 1$. $O(\phi)$ is the explicit form of the operator O in function of the fields ϕ.
By assuming the operator O is a p-form, equation (II.6) can be written as

$$\begin{aligned} < O > &= -\hbar \frac{\delta}{\delta J} \int [d\phi]\, e^{-\frac{1}{\hbar}\{S[\phi] + \int_M J \wedge O(\phi)\}} \Big|_{J=0} = \\ &= -\hbar \frac{\delta}{\delta J} W_O[J] \mid_{J=0} = -\hbar \frac{\delta}{\delta J} \ln W_O[J] \mid_{J=0} \end{aligned} \tag{II.7}$$

where J is an external (n-p)-form.
It is in the form (II.7) that vacuum expectation values usually appear in the path integral formalism, i.e. as functional derivatives of so called *generating functionals*.
Consider the generating functional W of the Green's functions

$$W[J] = \int [d\phi]\, e^{-\frac{1}{\hbar}\{S[\phi] + \int_M J \wedge \phi\}} \tag{II.8}$$

where in this case J is a (n-q)-form, ϕ being a q-form.
We can express it as

$$W[J] = e^{-\frac{1}{\hbar} Z[J]} \tag{II.9}$$

where $Z[J]$ is the generating functional of the connected Green functions [24].
By performing a functional Legendre transformation on $Z[J]$ we can define a new functional Γ by

$$\Gamma[\phi] = Z[J] - \int_M J \wedge \phi \quad \text{with } \phi = \frac{\delta Z}{\delta J} \tag{II.10}$$

Definition II.1.1 *The functional Γ is called the effective action of the theory.*

In general the effective action is a non-local functional of ϕ, which can formally be given by an expansion in powers of $\hbar$ [24,32]

$$\Gamma[\phi] = \sum_{k=0} \hbar^k \, \Gamma^{(k)}[\phi] \tag{II.11}$$

A term of order k in this expansion describes processes which correspond to Feynman diagrams with k loops. We shall assume that each term in (II.11) gives a finite contribution, in other words we shall assume that the effective action has been *regularized.*
It can be shown that the 0-th order term in (II.11) is the classical action S

$$\Gamma^{(0)}[\phi] = S[\phi]$$

In analogy to the classical case we say that the quantum theory is invariant under the transformation $\phi \to \phi'$ if

$$\Gamma[\phi'] = \Gamma[\phi]$$

Hence a classical symmetry is still a symmetry of the quantum theory if

$$\Gamma^{(k)}[\phi'] = \Gamma^{(k)}[\phi], \quad \forall k \geq 1 \tag{II.12}$$

If this is the case then the vacuum expectation value of Noether's current j (II.5) is conserved, $< div\, j >= 0$, independently of any field equations.
As can be expected it is not always possible to find a regularized effective action such that (II.12) is verified. In that case we say that the theory possesses an *anomaly.*

II.2 Axial and gauge anomalies

In this section we consider gauge theories on a parallelizable manifold M whose matter fields ψ are Dirac spinors transforming under a gauge transformation according to a r-dimensional, fundamental representation of the matrix Lie group $SU(r)$. As it is well known there exist two such representations, which are conjugate to each other. Writing $\psi = \psi_M \otimes \psi_P$, (see Subsection I.4.2), this means that ψ_P is the pull-back by the global section σ of $(P \xrightarrow{\pi} M, SU(r))$ of an equivariant map $\tilde{\psi}_P$

$$\tilde{\psi}_P : P \longrightarrow \mathbf{C}^r, \quad \tilde{\psi}_P(z \cdot g) = \rho(g)^{-1}\tilde{\psi}_P(z), \quad \forall z \in P, g \in SU(r) \tag{II.13}$$

where $\rho(g) = g$ or g^*.
The covariant derivative $\nabla_X \psi$, $X \in \mathcal{X}(M)$, is thus given by (see Proposition I.3.6)

$$\begin{aligned} \nabla_X \psi &= X\psi + A(X)\psi = X\psi_M \otimes X\psi_P + \psi_M \otimes A(X)\psi_P \quad \text{if } \rho(g) = g \\ \text{resp. } \nabla_X \psi &= X\psi + A(X)^*\psi \quad \text{if } \rho(g) = g^* \end{aligned} \tag{II.14}$$

where A is the gauge potential, a 1-form on M taking values in the Lie algebra $su(r)$ of $SU(r)$.
Observe that A is antihermitian, i.e. $A(x; v_x)^\dagger = -A(x; v_x)$, $\forall v_x \in TM_x$.
Instead of taking the real and imaginary parts of ψ as independent matter fields, we take ψ and $\bar{\psi} = \psi^\dagger \gamma^0 = \psi_M^\dagger \gamma^0 \otimes \psi_P^\dagger$, where $\{\gamma^i, \quad i = 0, \ldots, n-1\}$, $n = 2k = dim\, M$, are the well known γ-matrices.
The decomposition of the Dirac spinor ψ as a sum of right- and left-handed spinors

$$\begin{aligned} &\psi = \psi_+ + \psi_- = (\psi_{M+} + \psi_{M-}) \otimes \psi_P \\ &\psi_{M\pm} = (\tfrac{1 \pm \gamma_5}{2})\psi_M, \quad \gamma_5 = i^{\frac{n}{2}}\, \gamma^0 \cdots \gamma^{n-1}, \quad \gamma_5^2 = 1 \end{aligned} \tag{II.15}$$

will be frequently used.

II.2.1 The axial anomaly

Consider the gauge theory defined by the following classical Lagrangian

$$^*\mathcal{L} = \tfrac{i}{2}\bar{\psi}\, \nabla\!\!\!/ \psi + h.c. = \tfrac{i}{2}\bar{\psi}(\partial\!\!\!/ + A\!\!\!/)\psi + h.c. \tag{II.16}$$

where the operator $\nabla\!\!\!/$ is locally given by $\nabla\!\!\!/ \psi(x) = \gamma^i E_i^\mu(x) \nabla_\mu \psi(x)$, $x \in M$, E_i^μ being the inverse vielbeins (see (I.33)), and $+h.c.$ means that the hermitian conjugate of the preceding expression has to be added.
A part from being invariant under the gauge transformations

$$\begin{aligned} \psi(x) &\rightarrow \gamma(x)^{-1}\psi(x) = \psi_M(x) \otimes \gamma(x)^{-1}\psi_P(x) \\ \bar{\psi}(x) &\rightarrow \bar{\psi}\gamma(x) \\ A_x &\rightarrow \gamma(x)^{-1}A_x\gamma(x) + \gamma(x)^{-1}d\gamma_x, \quad \gamma \in C^\infty(M; SU(r)) \end{aligned} \tag{II.17}$$

the Lagrangian (II.16) is also invariant under the following (global) chiral transformations

$$\begin{array}{rcl} \psi_+(x) & \to & U_+\psi_+(x) \\ \psi_-(x) & \to & U_-\psi_-(x) \end{array}, \quad U_\pm \in U(1) \qquad \text{(II.18)}$$

This can be easily verified by noting that ${}^*\mathcal{L}$ can be written as

$${}^*\mathcal{L} = \tfrac{i}{2}\{\bar{\psi}_+ \not\nabla \psi_+ + \bar{\psi}_- \not\nabla \psi_-\} + h.c.$$

Are these symmetries still present in the quantum theory? The answer is no, in the sense that either the gauge invariance or the global chiral invariance has to be abandoned at the quantum level.
In a gauge theory the gauge invariance is clearly more fundamental than the global chiral invariance. We thus choose to retain the gauge symmetry. The price we pay is that the resulting effective action Γ is not invariant under the full group of chiral transformations (II.18). The pathological transformations are the so called *axial* transformations

$$\begin{array}{rcl} \psi_+(x) & \to & U\psi_+(x) \\ \psi_-(x) & \to & U^{-1}\psi_-(x) \end{array}, \quad U \in U(1) \qquad \text{(II.19)}$$

By writing $U = e^{i\alpha}$, $\alpha \in \mathbf{R}$, (II.19) is equivalent to

$$\psi(x) \to e^{i\alpha\gamma_5}\,\psi(x)$$

Under an infinitesimal axial transformation, $\alpha \ll 1$, the change in the effective action is found to be

$$\Gamma[\psi, \bar{\psi}, A] \to \Gamma[\psi, \bar{\psi}, A] + \alpha\hbar\Delta[A] + o(\alpha^2\hbar^2)$$

where $\Delta[A]$ is the *axial anomaly*. In four dimensions it is explicitly given by [20,41]

$$\Delta[A] = -\frac{i}{4\pi^2}\int_M Tr(F \wedge F) \qquad \text{(II.20)}$$

where $F = dA + \frac{1}{2}[A \wedge A] = dA + A \wedge A$ is the gauge field strength form, and Tr denotes the trace.
Since $Tr(A \wedge A \wedge A \wedge A) = 0$ we have

$$Tr(F \wedge F) = d\,Tr(dA \wedge A + \tfrac{2}{3}A \wedge A \wedge A)$$

The axial anomaly actually vanishes for a *trivial* bundle over a manifold M without boundary.

II.2.2 The gauge anomaly

In a physical theory there is no reason for the fields ψ_+ and ψ_- to transform under a gauge transformation according to the same irreducible representation of the gauge group G. In a chiral theory it is usually assumed that the right-handed fermion field ψ_+ transforms according to a representation ρ, whereas the left-handed one, ψ_-, is assumed to transform according to the conjugate representation ρ^*.

For the case $G = SU(r)$ and $\rho(g) = g$, $g \in SU(r)$, the gauge invariant Lagrangian for such a theory is given by the following expression

$$\begin{aligned} {}^*\mathcal{L} &= \tfrac{i}{2}\{\bar{\psi}_+ \nabla\!\!\!/_+\psi_+ + \bar{\psi}_- \nabla\!\!\!/_-\psi_-\} + h.c. = \\ &= \tfrac{i}{2}\{\bar{\psi}_+(\partial\!\!\!/ + A\!\!\!/)\psi_+ + \bar{\psi}_-(\partial\!\!\!/ + A\!\!\!/^*)\psi_-\} + h.c. \end{aligned} \qquad (II.21)$$

where locally $A\!\!\!/^*(x) = \gamma^i E_i^\mu(x) A_\mu^*(x)$.
By using the decomposition (II.15) the above Lagrangian is equivalent to

$${}^*\mathcal{L} = \tfrac{i}{2}\bar{\psi}(\partial\!\!\!/ + v\!\!\!/ + \gamma_5\, a\!\!\!/)\psi + h.c. \qquad (II.22)$$

where $v\!\!\!/ = \frac{1}{2}(A\!\!\!/ + A\!\!\!/^*)$ and $a\!\!\!/ = \frac{1}{2}(A\!\!\!/ - A\!\!\!/^*)$.
As it turns out it is not possible to find a gauge invariant effective action Γ corresponding to the Lagrangian II.21. The best one can do is to give a Γ which, under an infinitesimal gauge transformation represented by $\gamma = \exp \epsilon\varepsilon$, $\epsilon \ll 1$, $\varepsilon \in C^\infty(M; su(r))$ (see (I.47))

$$\begin{aligned} \psi_+ &\rightarrow \psi'_+ = \psi_+ - \epsilon\,\varepsilon\psi_+ + o(\epsilon^2) \\ \psi_- &\rightarrow \psi'_- = \psi_- - \epsilon\,\varepsilon^*\psi_- + o(\epsilon^2) \\ A &\rightarrow A' = A + \epsilon\nabla\varepsilon + o(\epsilon^2) \end{aligned} \qquad (II.23)$$

varies as follows

$$\Gamma[\psi', \bar{\psi}', A'] = \Gamma[\psi, \bar{\psi}, A] + \hbar\epsilon\Delta[A;\varepsilon] + o(\epsilon^2\hbar^2) \qquad (II.24)$$

In four dimensions the *gauge anomaly* $\Delta[A;\varepsilon]$ is given by [41]

$$\Delta[A;\varepsilon] = \frac{i}{4\pi^2}\int_M Tr\{\varepsilon\, d(dA \wedge A + \tfrac{1}{2}A \wedge A \wedge A)\} \qquad (II.25)$$

Observe that the gauge anomaly doesn't depend on the matter fields ψ. This will be important in the subsequent algebraic analysis of the anomalies.

II.2.3 The Wess-Zumino consistency condition

By defining the infinitesimal variation of the gauge potential A under a gauge transformation by $\delta_0^\varepsilon A = \nabla\varepsilon$ and considering two subsequent infinitesimal gauge transformations $\gamma_i = \exp\epsilon\varepsilon_i$, $\varepsilon_i \in C^\infty(M; su(r))$, $i = 1,2$, we find

$$\delta_0^{\varepsilon_1}\delta_0^{\varepsilon_2}A - \delta_0^{\varepsilon_2}\delta_0^{\varepsilon_1}A = \delta_0^{[\varepsilon_1,\varepsilon_2]}A \qquad (II.26)$$

This is a direct consequence of the Lie group structure of the group of gauge transformations $\mathcal{G}$.

Let us define the variation $\vartheta(\varepsilon)$ of a functional W depending on the gauge potential A by

$$(\vartheta(\varepsilon)W)[A] = \lim_{t\to 0}\frac{1}{t}\{W[A+t\nabla\varepsilon] - W[A]\} = \frac{d}{dt}W[A+t\delta_0^\varepsilon A]|_{t=0} \qquad \text{(II.27)}$$

By definition we have

$$\begin{aligned}\vartheta(\varepsilon_1)\vartheta(\varepsilon_2)W[A] &= \frac{d}{dt}(\vartheta(\varepsilon_2)W)\ [A+t\delta_0^{\varepsilon_1}A]|_{t=0} = \\ &= \frac{d}{dt}\frac{d}{d\bar{t}}\,W[A+t\delta_0^{\varepsilon_1}A+\bar{t}\delta_0^{\varepsilon_2}A+t\bar{t}\delta_0^{\varepsilon_1}\delta_0^{\varepsilon_2}A]\Big|_{t=\bar{t}=0} = \\ &= \frac{d}{dt}\{[\frac{\partial}{\partial\bar{t}}+t\frac{\partial}{\partial t\bar{t}}]W[A+t\delta_0^{\varepsilon_1}A+\bar{t}\delta_0^{\varepsilon_2}A+t\bar{t}\delta_0^{\varepsilon_1}\delta_0^{\varepsilon_2}A]\}|_{t=\bar{t}=0} = \\ &= \frac{d}{dt}\frac{d}{d\bar{t}}\,W[A+t\delta_0^{\varepsilon_1}A+\bar{t}\delta_0^{\varepsilon_2}A]\Big|_{t=\bar{t}=0}+\frac{d}{dt}\,W[A+t\delta_0^{\varepsilon_1}\delta_0^{\varepsilon_2}A]|_{t=0}\end{aligned}$$

Thus by making use of (II.26) we easily find

$$(\vartheta(\varepsilon_1)\vartheta(\varepsilon_2) - \vartheta(\varepsilon_2)\vartheta(\varepsilon_1))W = \vartheta([\varepsilon_1,\varepsilon_2])W \qquad \text{(II.28)}$$

If we drop the dependence of the effective action Γ on the matter fields we can express (II.24) as

$$\vartheta(\varepsilon)\Gamma[A] = \hbar\Delta[A;\varepsilon] \qquad \text{(II.29)}$$

From equation (II.28) we immediately get the *Wess-Zumino consistency condition* [36] for the gauge anomaly Δ

$$\vartheta(\varepsilon_1)\Delta[A;\varepsilon_2] - \vartheta(\varepsilon_2)\Delta[A;\varepsilon_1] = \Delta[A;[\varepsilon_1,\varepsilon_2]] \qquad \text{(II.30)}$$

Equation (II.30) is the starting point for the algebraic construction of the anomalies. If a gauge anomaly exists, as given by a variation of an effective action, then it must satisfy (II.30). Such anomalies are called *consistent* to distinguish them from the so called *covariant* anomalies [4].
Observe that no consistency condition exists for the axial anomaly (II.20). This is due to the abelian character of the axial symmetry.

A method to find a possible consistent anomaly is thus to solve equation (II.30), i.e. to look for all functionals Δ depending on the gauge potential A and on the map $\varepsilon \in C^\infty(M;LG)$, which verify (II.30). A trivial solution to equation (II.30) is given by

$$\Delta[A;\varepsilon] = \vartheta(\varepsilon)\tilde{\Delta}[A] \qquad \text{(II.31)}$$

where $\tilde{\Delta}$ is a *local* functional of the gauge potential A.
If the gauge anomaly were given by an expression like (II.31) then by subtracting $\hbar\tilde{\Delta}$ from the effective action Γ, which is equivalent to modifying the classical Lagrangian $\mathcal{L}$ by terms of order $\hbar$, so called *counter terms*, we could define a gauge invariant effective action $\tilde{\Gamma}$

$$\vartheta(\varepsilon)\tilde{\Gamma} = \vartheta(\varepsilon)(\Gamma - \hbar\tilde{\Delta}) = 0$$

Therefore the *non-trivial* consistent anomalies are given by the solutions of (II.30) modulo solutions of the form (II.31). This is almost a cohomological problem.

II.3 Cohomological construction of the gauge anomalies

II.3.1 Representations of a Lie algebra and the associated coboundary operators

In this subsection we introduce the algebraic machinery which will allow us to identify the non-trivial consistent anomalies with representatives of appropriate cohomological classes. As reference we used [22].

Let L be a Lie algebra and ϑ a representation of L in the vector space F

$$\begin{aligned} \vartheta : \ & L \longrightarrow L(F) \\ & E \longmapsto \vartheta(E) \quad \text{with } \vartheta([E_1, E_2]) = \vartheta(E_1) \circ \vartheta(E_2) - \vartheta(E_2) \circ \vartheta(E_1),\ E_i \in L \end{aligned}$$

where $L(F)$ is the Lie algebra of all linear transformations of F.
Denote by $\bigwedge^k(L; F)$ the vector space of all k-linear skew-symmetric maps λ^k

$$\lambda^k : L^k = \underbrace{L \times \cdots \times L}_{k-times} \longrightarrow F$$

Definition II.3.1 *An element $\lambda^k \in \bigwedge^k(L; F)$ is called a k-cochain with values in F.*

Given a k-cochain λ^k we define a (k+1)-cochain $\delta_\vartheta \lambda^k$ by

$$\begin{aligned} \delta_\vartheta \lambda^k(E_1, \ldots, E_{k+1}) = & \sum_{i=1}^{k+1} (-1)^{i+1} \vartheta(E_i)[\lambda^k(E_1, \ldots, \hat{\imath}, \ldots, E_{k+1})] + \\ & + \sum_{i<j} (-1)^{i+j} \lambda^k([E_i, E_j], E_1, \ldots, \hat{\imath}, \ldots, \hat{\jmath}, \ldots, E_{k+1}), \quad E_i \in L \quad \text{(II.32)} \end{aligned}$$

Definition II.3.2 *The operator $\delta_\vartheta : \bigwedge^k(L; F) \longrightarrow \bigwedge^{k+1}(L; F)$ is called the (Chevalley) coboundary operator associated to the representation ϑ of L in F.*

Observe the similarity of (II.32) with Definition I.1.15 for the exterior derivative. In fact a q-form ω^q is a special example of a q-cochain for the case (see (I.13))
$L = \chi(M)$, $\quad F = C^\infty(M)$ and $\vartheta(X)f = Xf$, $\quad X \in \chi(M)$, $\quad f \in C^\infty(M)$.
Thus $\Omega^q(M) \subset \bigwedge^q(\chi(M); C^\infty(M))$.

Proposition II.3.1 *The coboundary operator δ_ϑ verifies $\delta_\vartheta^2 = 0$.*

Definition II.3.3 *A k-cochain λ^k is called a k-cocycle if $\delta_\vartheta \lambda^k = 0$.*
The cocycle λ^k is called a k-coboundary if $\lambda^k = \delta_\vartheta \tilde{\lambda}^{k-1}$ for some (k-1)-cochain $\tilde{\lambda}^{k-1}$.

Denote by $Z^k(L; F)$ resp. $N^k(L; F)$ the subspaces of all k-cocycles resp. k-coboundaries. Clearly $N^k(L; F) \subset Z^k(L; F)$.

Definition II.3.4 *The quotient space $H^k(L; F) = Z^k(L; F)/N^k(L; F)$ is called the k-th cohomology (space) of L with coefficients in F.*

In the special case

$$Z^q(M) \doteq Z^q(\chi(M); C^\infty(M)) \cap \Omega^q(M), \quad N^q(M) \doteq N^q(\chi(M); C^\infty(M)) \cap \Omega^q(M)$$

the space $Z^q(M)/N^q(M)$, denoted by $H^q_{deRham}(M)$, is called the q-th de Rham cohomology of M. A q-form in $Z^q(M)$ resp. $N^q(M)$ is called *closed* resp. *exact*.

If F is an algebra with product $\cdot$ we can define, analogously to the exterior product (I.11), a product $\wedge$ in $\bigwedge(L; F) = \oplus_{k=0} \bigwedge^k(L; F)$ by

$$\lambda^k \wedge \mu^l(E_1, \ldots, E_{k+l}) = \tag{II.33}$$
$$= \frac{1}{k!l!} \sum_\pi sign\, \pi\, \lambda^k(E_{\pi(1)}, \ldots, E_{\pi(k)}) \cdot \mu^l(E_{\pi(k+1)}, \ldots, E_{\pi(k+l)})$$

where $E_i \in L$ and $\lambda^k, \mu^l \in \bigwedge(L; F)$.
As should be expected (cf. Proposition I.1.4), the coboundary operator δ_ϑ is an antiderivation in the algebra $\bigwedge(L; F)$ with product (II.33)

$$\delta_\vartheta(\lambda^k \wedge \mu^l) = \delta_\vartheta \lambda^k \wedge \mu^l + (-1)^k \lambda^k \wedge \delta_\vartheta \mu^l \tag{II.34}$$

By (II.34) the subspaces $Z(L; F) = \oplus_{k=0} Z^k(L; F)$ and $N(L; F) = \oplus_{k=1} N^k(L; F)$ are subalgebras of $\bigwedge(L; F)$. Moreover $N(L; F)$ is an ideal of $Z(L; F)$, i.e.
$Z(L; F) \wedge N(L; F) \subset N(L; F)$.
Hence the cohomology $H(L; F) = \oplus_{k=0} H^k(L; F)$ inherits the algebra structure of $\bigwedge(L; F)$.

II.3.2 Cohomological construction of the gauge anomalies

In this subsection we apply the formalism of the preceding subsection to the case $L = \mathcal{LG} \cong C^\infty(M; LG)$, the Lie algebra of the group of gauge transformations $\mathcal{G}$ [6].
To begin with we take the vector space F as the space of all functionals W which depend on the gauge potential A. We denote this space by $\mathcal{F}[A]$.
We define a representation of $\mathcal{LG}$ in $\mathcal{F}[A]$ by the variation $\vartheta(\varepsilon)$ introduced in (II.27)

$$\vartheta : \mathcal{LG} \longrightarrow L(\mathcal{F}[A])$$
$$\varepsilon \longmapsto \vartheta(\varepsilon)$$

$$\text{with} \quad (\vartheta(\varepsilon)W)[A] = \lim_{t \to 0} \frac{1}{t}\{W[A + t\,\nabla\varepsilon] - W[A]\} \tag{II.35}$$

That ϑ is a representation is evident from (II.28).

By considering the effective action Γ as a 0-cochain, we see that the gauge anomaly Δ is the 1-cochain given by

$$\Delta[A; \varepsilon] = \frac{1}{\hbar}(\delta_\vartheta \Gamma)[A; \varepsilon]$$

In view of Proposition II.3.1, the anomaly Δ satisfies

$$\delta_\vartheta \Delta = 0 \tag{II.36}$$

which by (II.32) is equivalent to the Wess-Zumino consistency condition (II.30)

$$0 = (\delta_\vartheta \Delta)[A;\varepsilon_1,\varepsilon_2] = \vartheta(\varepsilon_1)\Delta[A;\varepsilon_2] - \vartheta(\varepsilon_2)\Delta[A;\varepsilon_1] - \Delta[A;[\varepsilon_1,\varepsilon_2]]$$

A consistent gauge anomaly is thus a 1-cocycle belonging to $Z^1(\mathcal{LG};\mathcal{F}[A])$.

By the discussion following (II.31), to cohomologically describe a non-trivial anomaly we restrict ourselves to the space $\mathcal{F}_{loc}[A]$ of local functionals depending on the gauge potential A. Is the resulting space $Z^1(\mathcal{LG};\mathcal{F}_{loc}[A])$ big enough to contain all gauge anomalies? This seems to be the case [7], as can also be seen from the explicit 4-dimensional example (II.25). Thus by assuming that all gauge anomalies are *local*, we can identify the non-trivial consistent ones with elements of $H^1(\mathcal{LG};\mathcal{F}_{loc}[A])$.

By definition a local anomaly can be expressed as

$$\Delta[A;\varepsilon] = \int_M \mathcal{D}^{1,n}(A;\varepsilon) \tag{II.37}$$

where $\mathcal{D}^{1,n}(A;\varepsilon)$ is a n-form on M depending on the gauge potential A, the map ε and possibly their derivatives. $\mathcal{D}^{1,n}(A;\varepsilon)$ must be local, i.e. at a point $x \in M$ it must depend on its arguments taken at x only.

Observe that by Corollary I.1.1 we can modify the n-form $\mathcal{D}^{1,n}(A;\varepsilon)$ by adding to it an exact form without changing the anomaly $\Delta[A;\varepsilon]$.

Denote by $\Omega^q_A(M)$ the space of all local q-forms on M depending on the gauge potential A. We can easily generalize the representation (II.35) of $\mathcal{LG}$ in $\mathcal{F}[A]$ to a representation, denoted by ϑ_{loc}, of $\mathcal{LG}$ in the vector space $\Omega^q_A(M)$

$$(\vartheta_{loc}(\varepsilon)\mathcal{D}^q)(A) = \lim_{t\to 0}\frac{1}{t}[\mathcal{D}^q(A + t\,\nabla\varepsilon) - \mathcal{D}^q(A)] = \frac{\partial \mathcal{D}^q}{\partial A} \wedge \nabla\varepsilon, \quad \mathcal{D}^q(A) \in \Omega^q_A(M)$$

The n-form $\mathcal{D}^{1,n}(A;\varepsilon)$ can thus be interpreted as a 1-cochain in $\bigwedge^1(\mathcal{LG};\Omega^n_A(M))$.

Clearly

$$\vartheta(\varepsilon_1)\Delta[A;\varepsilon_2] = \int_M \vartheta_{loc}(\varepsilon_1)\mathcal{D}^{1,n}(A;\varepsilon_2)$$

and therefore also

$$(\delta_\vartheta\Delta)[A;\varepsilon_1,\varepsilon_2] = \int_M (\delta_{loc}\mathcal{D}^{1,n})(A;\varepsilon_1,\varepsilon_2) \tag{II.38}$$

where δ_{loc} is the coboundary operator associated to the representation ϑ_{loc}.

In view of Corollary I.1.1 condition (II.36) is therefore equivalent to

$$(\delta_{loc}\mathcal{D}^{1,n})(A;\varepsilon_1,\varepsilon_2) = d\,(\mathcal{D}^{2,n-1}(A;\varepsilon_1,\varepsilon_2)) \tag{II.39}$$

where d is the exterior derivative and $\mathcal{D}^{2,n-1}$ is a 2-cochain with values in $\Omega_A^{n-1}(M)$. Similarly, the 1-cochain $\mathcal{D}^{1,n}(A;\varepsilon)$ gives rise to a non-trivial anomaly provided

$$\mathcal{D}^{1,n}(A;\varepsilon) \neq (\delta_{loc}\mathcal{D}^n)(A;\varepsilon) \quad \text{(modulo an exact form)} \tag{II.40}$$

where $\mathcal{D}^n(A) \in \Omega_A^n(M)$.
We shall find a solution to equation (II.39) by means of Proposition I.4.3, which identifies $\mathcal{LG}$ with the Lie algebra $\chi_v^I(P)$ of vertical invariant vector fields on P with product given by minus the Lie bracket. See also [7] where a similar approach has been adopted.

By Proposition I.4.3 the map $\varepsilon \in C^\infty(M;LG) \cong \mathcal{LG}$ can be expressed as

$$\varepsilon = \omega(Z_\varepsilon)\circ\sigma = \sigma^*(\omega(Z_\varepsilon))$$

where $Z_\varepsilon \in \chi_v^I(P)$, ω is the connection form and σ a global section of the trivial bundle P.
As $A = \sigma^*\omega$ we can write

$$\mathcal{D}^{1,n}(A;\varepsilon) = \mathcal{D}^{1,n}(\sigma^*\omega;\sigma^*(\omega(Z_\varepsilon))) \doteq \mathcal{D}_\sigma^{1,n}(\omega;Z_\varepsilon) \tag{II.41}$$

From eq. (I.49) and the fact that the 1-form $L_{Z_\varepsilon}\omega$ is horizontal (see the proof of Proposition I.4.3), we have

$$\mathcal{D}^{1,n}(A + t\,\nabla\varepsilon_1;\varepsilon_2) = \mathcal{D}^{1,n}(\sigma^*(\omega + t\,L_{Z_{\varepsilon_1}}\omega);(\omega + t\,L_{Z_{\varepsilon_1}}\omega)(Z_{\varepsilon_2})\circ\sigma)$$

Whence

$$\begin{aligned}(\vartheta_{loc}(\varepsilon_1)\mathcal{D}^{1,n})(A;\varepsilon_2) &= \lim_{t\to 0}\frac{1}{t}[\mathcal{D}_\sigma^{1,n}(\omega + t\,L_{Z_{\varepsilon_1}}\omega;Z_{\varepsilon_2}) - \mathcal{D}_\sigma^{1,n}(\omega;Z_{\varepsilon_2})] = \\ &\doteq (\tilde{\vartheta}(Z_{\varepsilon_1})\mathcal{D}_\sigma^{1,n})(\omega;Z_{\varepsilon_2})\end{aligned} \tag{II.42}$$

Henceforth we shall simply write Z_i for the vertical invariant vector field Z_{ε_i}, $\varepsilon_i \in \mathcal{LG}$.

Denote by $\chi_-(P)$ the Lie algebra of all vector fields on P with product given by minus the Lie bracket.

Proposition II.3.2 *The operation $\tilde{\vartheta}$ defined in* (II.42) *is a representation of the Lie algebra $\chi_-(P)$ in the vector space $\Omega_\omega^q(P)$ of all q-forms on P depending on a connection form.*

Proof: observe that by definition $\tilde{\vartheta}$ satisfies

$$(\tilde{\vartheta}(Z_1)\tilde{\vartheta}(Z_2)\tilde{\mathcal{D}}^q)(\omega) = \frac{d}{dt}(\tilde{\vartheta}(Z_2)\tilde{\mathcal{D}}^q)(\omega + tL_{Z_1}\omega)|_{t=0} =$$

$$= \frac{d}{dt}\frac{d}{dt'}\tilde{\mathcal{D}}^q(\omega + tL_{Z_1}\omega + t'L_{Z_2}(\omega + tL_{Z_1}\omega))|_{t=t'=0} = \frac{\partial\tilde{\mathcal{D}}^q}{\partial\omega}\wedge L_{Z_2}L_{Z_1}\omega;\quad \tilde{\mathcal{D}}^q \in \Omega_\omega^q(P)$$

Therefore, by Proposition I.1.2

$$[\tilde{\vartheta}(Z_1)\tilde{\vartheta}(Z_2) - \tilde{\vartheta}(Z_2)\tilde{\vartheta}(Z_1)]\tilde{\mathcal{D}}^q(\omega) = \frac{\partial\tilde{\mathcal{D}}^q}{\partial\omega} \wedge (L_{Z_2}L_{Z_1}\omega - L_{Z_1}L_{Z_2}\omega) =$$

$$= \frac{\partial\tilde{\mathcal{D}}^q}{\partial\omega} \wedge L_{-[Z_1,Z_2]}\omega = (\tilde{\vartheta}(-[Z_1, Z_2])\tilde{\mathcal{D}}^q)(\omega) \quad \square$$

Given a q-form $\tilde{\mathcal{D}}^q(\omega) \in \Omega^q_\omega(P)$, we can define a k-cochain, denoted by $\imath^k\tilde{\mathcal{D}}^q$, belonging to $\bigwedge^k(\mathcal{X}^I_v(P); \Omega^{q-k}_\omega(P))$ by

$$\begin{aligned} \imath^k\tilde{\mathcal{D}}^q \ : \ & \mathcal{X}^I_v(P) \times \cdots \times \mathcal{X}^I_v(P) \longrightarrow \Omega^{q-k}_\omega(P) \\ & Z_1, \ldots, Z_k \longmapsto \imath(Z_1, \ldots, Z_k)\tilde{\mathcal{D}}^q(\omega) \end{aligned} \tag{II.43}$$

where $\imath(Z_1, \ldots, Z_k) \doteq \imath_{Z_k} \circ \cdots \circ \imath_{Z_1}$, $\imath_{Z_i}$ being the interior product with respect to $Z_i \in \mathcal{X}^I_v(P)$ (see Definition I.1.16).
Assume now we are given a *polynomial* q-form $\tilde{\mathcal{D}}^q_{poly}(\omega)$. By this we mean a form constructed out of exterior products of ω and $d\omega$ only. A polynomial form is obviously local. By Proposition I.1.6 we clearly have

$$\tilde{\vartheta}(Z_1)\tilde{\mathcal{D}}^q_{poly}(\omega) = L_{Z_1}\tilde{\mathcal{D}}^q_{poly}(\omega) \tag{II.44}$$

Denote by $\tilde{\delta}$ the coboundary operator associated to the representation $\tilde{\vartheta}$ of $\mathcal{X}_v(P)$.

Proposition II.3.3

$$\tilde{\delta}(\imath^k\tilde{\mathcal{D}}^q_{poly}) = \imath^{k+1}d\,\tilde{\mathcal{D}}^q_{poly} + (-1)^k\,d\,\imath^{k+1}\tilde{\mathcal{D}}^q_{poly}$$

where d is the exterior derivative on P, and $d\,\imath^{k+1}\tilde{\mathcal{D}}^q_{poly}$ is the (k+1)-cochain given by

$$d\,\imath^{k+1}\tilde{\mathcal{D}}^q_{poly}(\omega; Z_1, \ldots, Z_{k+1}) \doteq d\,\imath(Z_1, \ldots, Z_{k+1})\tilde{\mathcal{D}}^q_{poly}(\omega)$$

Proof: by definition

$$\tilde{\delta}(\imath^k\tilde{\mathcal{D}}^q_{poly})(\omega; Z_1, \ldots, Z_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}\,\tilde{\vartheta}(Z_i)[\imath^k\tilde{\mathcal{D}}^q_{poly}(\omega; Z_1, \ldots, \hat{\imath}, \ldots, Z_{k+1})]$$

$$+ \sum_{i<j}(-1)^{i+j}\,\imath^k\tilde{\mathcal{D}}^q_{poly}(\omega; -[Z_i, Z_j], Z_1, \ldots, \hat{\imath}, \ldots, \hat{\jmath}, \ldots, Z_{k+1}) =$$

$$= \sum_{i=1}^{k+1}(-1)^{i+1}\,\imath(Z_1, \ldots, \hat{\imath}, \ldots, Z_{k+1})L_{Z_i}\,\tilde{\mathcal{D}}^q_{poly}(\omega)$$

$$+ \sum_{i<j}(-1)^{i+j}\,\imath(-[Z_i, Z_j], Z_1, \ldots, \hat{\imath}, \ldots, \hat{\jmath}, \ldots, Z_{k+1})\tilde{\mathcal{D}}^q_{poly}(\omega)$$

where the caret means omission.

Lemma II.3.1 *Let X_i, $i = 1, \ldots, k+1$, be any vector field. Then*

$$\begin{aligned} d\imath(X_1, \ldots, X_{k+1}) &= (-1)^{k+1}\imath(X_1, \ldots, X_{k+1})d \\ &+ (-1)^k\{\sum_{i=1}^{k+1}(-1)^{i+1}\imath(X_1, \ldots, \hat{\imath}, \ldots, X_{k+1})L_{X_i} \\ &+ \sum_{i<j}(-1)^{i+j}\imath(-[X_i, X_j], X_1, \ldots, \hat{\imath}, \ldots, \hat{\jmath}, \ldots, X_{k+1})\} \end{aligned}$$

For the proof of this Lemma see the Appendix, where most of the proofs of Chapter II are handwritten.
Thus

$$\begin{aligned} \tilde{\delta}(\imath^k\tilde{\mathcal{D}}^q_{poly})(\omega; Z_1, \ldots, Z_{k+1}) &= (-1)^k d(\imath^{k+1}\tilde{\mathcal{D}}^q_{poly}(\omega; Z_1, \ldots, Z_{k+1})) \\ &\quad + \imath^{k+1}d\tilde{\mathcal{D}}^q_{poly}(\omega; Z_1, \ldots, Z_{k+1}) \end{aligned}$$

which proves the proposition. □

Let $\sigma^*\imath^k\tilde{\mathcal{D}}^q$ be the "pull-back" of the k-cochain (II.43)

$$\sigma^*\imath^k\tilde{\mathcal{D}}^q(\omega; Z_1, \ldots, Z_k) \doteq \sigma^*\imath(Z_1, \ldots, Z_k)\tilde{\mathcal{D}}^q(\omega) \qquad \text{(II.45)}$$

For a polynomial q-form $\tilde{\mathcal{D}}^q_{poly}(\omega)$, the k-cochain $\sigma^*\imath^k\tilde{\mathcal{D}}^q_{poly}(\omega)$ clearly belongs to $\bigwedge^k(\mathcal{LG}; \Omega^{q-k}_A(M))$. (The only problem could arise from $\sigma^*\imath_Z d\omega$. But $\sigma^*\imath_Z d\omega =$ $= \sigma^*\imath_Z(\Omega - \frac{1}{2}[\omega \wedge \omega]) = -\sigma^*[\omega(Z), \omega] = -[\varepsilon, A]$).
As can be easily checked we have

$$\delta_{loc}(\sigma^*\imath^k\tilde{\mathcal{D}}^q_{poly}) = \sigma^*(\tilde{\delta}\imath^k\tilde{\mathcal{D}}^q_{poly})$$

Thus, by Proposition II.3.3 and Proposition 1.1.4

$$\delta_{loc}\sigma^*\imath^k\tilde{\mathcal{D}}^q_{poly} = \sigma^*\imath^{k+1}d\tilde{\mathcal{D}}^q_{poly} + (-1)^k d(\sigma^*\imath^{k+1}\tilde{\mathcal{D}}^q_{poly})$$

A solution to eq. (II.39) is therefore given by the 1-cochain

$$\mathcal{D}^{1,n}_{poly}(A; \varepsilon) = \sigma^*\imath_Z\tilde{\mathcal{D}}^{n+1}_{poly}(\omega), \quad \epsilon = \omega(Z) \qquad \text{(II.46)}$$

where the polynomial (n+1)-form $\tilde{\mathcal{D}}^{n+1}_{poly}(\omega)$ has to satisfy

$$\imath(Z_1, Z_2)d\tilde{\mathcal{D}}^{n+1}_{poly}(\omega) = 0 \qquad \text{(II.47)}$$

This means that $d\tilde{\mathcal{D}}^{n+1}_{poly}(\omega)$ has to be horizontal. The only horizontal polynomial forms on P are the forms polynomial in Ω, the curvature form of ω, and these are the building blocks of the so called *characteristic classes* of vector bundles (see e.g. [27]).

Denote by $\bigvee^k(LG)$ the space of all symmetric k-linear maps $LG^k \longrightarrow \mathbf{R}$. Given an element $\mathcal{S}^k$ of this space, a horizontal polynomial 2k-form on P is given by

$$\mathcal{S}^k(\Omega) = \Omega^{i_1} \wedge \cdots \wedge \Omega^{i_k}\mathcal{S}^k(T_{i_1}, \ldots, T_{i_k})$$

where $\{T_i,\ i=1,\dots,dim G\}$ is a basis of the Lie algebra LG.
Let S^k be Ad-invariant, i.e.

$$\sum_{i=1}^{k} S^k(E_1,\dots,[E,E_i],\dots,E_k)=0,\quad \forall E,E_i\in LG \tag{II.48}$$

Definition II.3.5 *The $(2k-1)$-form $TS^k(\omega)$ on P defined by*

$$\begin{aligned} TS^k(\omega) &= k\int_0^1 dt\,\{\omega^{i_1}\wedge\Omega_t^{i_2}\wedge\cdots\Omega_t^{i_k}\}\,S^k(T_{i_1},\dots,T_{i_k})= \\ &= k\int_0^1 dt\,S^k_{\omega,\Omega_t,\dots,\Omega_t}\end{aligned}$$

where

$$\Omega_t=t\Omega+\tfrac{1}{2}(t^2-t)[\omega\wedge\omega]=t\,d\omega+\tfrac{t^2}{2}[\omega\wedge\omega],\quad t\in\mathbf{R}$$

is called the Chern transgression (or Chern-Simons) form associated to the Ad-invariant map S^k.

Proposition II.3.4 *Let $S^k: LG^k\longrightarrow\mathbf{R}$ be an Ad-invariant k-linear symmetric map. Then*

$$S^k(\Omega)=d\,TS^k(\omega)$$

Proof: we have

$$\begin{aligned} d\Omega_t &= d(t\,d\omega+\tfrac{1}{2}t^2[\omega\wedge\omega])=t^2[d\omega\wedge\omega]=t^2[\Omega\wedge\omega]= \\ &= t\,[\Omega_t\wedge\omega],\qquad\text{since}\quad[\omega\wedge[\omega\wedge\omega]]=0\end{aligned}$$

Thus,

$$\begin{aligned} d\,TS^k(\omega) &= k\int_0^1 dt\{S^k_{d\omega,\Omega_t,\dots,\Omega_t}-\sum_{m=2}^{k}S^k_{\omega,\Omega_t,\dots,\underset{(m)}{t[\Omega_t\wedge\omega]},\dots,\Omega_t}\}= \\ &= k\int_0^1 dt\{S^k_{d\omega,\Omega_t,\dots,\Omega_t}+S^k_{t[\omega\wedge\omega],\Omega_t,\dots,\Omega_t}\}= \\ &= k\int_0^1 dt\{S^k_{\frac{d}{dt}\Omega_t,\Omega_t,\dots,\Omega_t}=\int_0^1 dt\frac{d}{dt}S^k_{\Omega_t,\dots,\Omega_t}=S^k(\Omega)\qquad\square\end{aligned}$$

By Proposition II.3.4 a (n+1)-form satisfying (II.47) is given by

$$\tilde{\mathcal{D}}^{n+1}_{poly}(\omega)=TS^{\frac{n}{2}+1}(\omega)$$

We've finally found a solution to eq. (II.39), namely

$$\mathcal{D}^{1,n}_{poly}(A;\varepsilon)=\sigma^{*}\imath_Z TS^{\frac{n}{2}+1}(\omega),\quad \epsilon=\omega(Z) \tag{II.49}$$

Explicitly (cf. eq. [41])

$$\mathcal{D}^{1,n}_{poly}(A;\varepsilon)=(\tfrac{n}{2}+1)\int_0^1 dt\,\{S^{\frac{n}{2}+1}_{\epsilon,F_t,\dots,F_t}-\tfrac{n}{2}(t^2-t)\,S^{\frac{n}{2}+1}_{A,[\epsilon,A],F_t,\dots,F_t}\} \tag{II.50}$$

where $F_t = \sigma^* \Omega_t = t F + \frac{1}{2}(t^2 - t)[A \wedge A]$.
For a matrix Lie algebra an example of a k-linear symmetric Ad-invariant map $\mathcal{S}^k$ is the symmetrized trace Str

$$Str(E_1, \ldots, E_k) = \sum_{\pi} Tr(E_{\pi(1)}, \ldots, E_{\pi(k)}) \tag{II.51}$$

As can be easily checked the 4-dimensional anomaly (II.25) is proportional to the integral of (II.50) with the choice Str for $\mathcal{S}^3$.

What about condition (II.40)? The only thing one can say is that if the (n+1)-form $\tilde{\mathcal{D}}^{n+1}_{poly}(\omega)$ in (II.46) is exact, i.e.

$$\tilde{\mathcal{D}}^{n+1}_{poly}(\omega) = d\,\tilde{\mathcal{D}}^{n}_{poly}(\omega)$$

then the 1-cochain (II.46) doesn't verify (II.40). Indeed

$$\begin{aligned} \mathcal{D}^{1,n}_{poly}(A;\varepsilon) &= \sigma^* \imath_Z d\,\tilde{\mathcal{D}}^{n}_{poly}(\omega) = \\ &= \sigma^* \{L_Z \tilde{\mathcal{D}}^{n}_{poly}(\omega) - d \imath_Z \tilde{\mathcal{D}}^{n}_{poly}(\omega)\} = \\ &= \vartheta_{loc}(\varepsilon) \sigma^* \tilde{\mathcal{D}}^{n}_{poly}(\omega) \quad \text{(modulo an exact form)} \end{aligned}$$

That the form $T\mathcal{S}^{\frac{n}{2}+1}(\omega)$ is not exact is part of Chevalley's theorem [12] which states that $T\mathcal{S}^{\frac{n}{2}+1}(\omega) \in H^{\frac{n}{2}+1}_{deRham}(P)$.

II.4 Base space anomalies

II.4.1 Introduction

In this section we introduce the *base space anomalies*, i.e. the possible, non-trivial, consistent anomalies associated to transformations of the base space M which are symmetries of a classical gauge field theory (see Subsection I.4.3). As usual we take M to be a compact, $n = 2k$-dimensional spin manifold without boundary. In what follows we shall explicitly show the dependence of the theory on the metric tensor g. Obviously g is considered as an external classical field, i.e. we shall work in a background gravitational field.

We suppose the classical theory is given by an action functional S which can be expressed by an integral on M of a Lagrangian $\mathcal{L}$. The theory so formulated is global, i.e. it is chart independent. This means that the classical theory is automatically invariant under general coordinate transformations or, from an active point of view, diffeomorphisms of M. Now we impose that the quantum theory, if it exists, possesses this "symmetry" too; in other words anomalies associated to (infinitesimal) diffeomorphisms (or Einstein transformations), so called *gravitational anomalies*, should be absent.

Having established the general covariance of the theory we may now choose an atlas on M and work in a coordinate neighborhood. This would allow us to work with quantities, as the Christoffel's symbols and the vielbeins, which are defined locally on M.

Gauge invariance and general covariance are usually not the only symmetries of a classical theory. A theory can also be invariant under transformations which are so to speak only in part of geometrical nature. By this we mean that a symmetry can result from transforming some fields according to their geometrical character and the remaining ones in a suitable way.

The simplest example thereof is the free Yang-Mills theory in a 4-dimensional Riemannian space with metric g. The action functional is given by

$$S[F,g] = \frac{1}{2}\int_M Tr({}^*F \wedge F)$$

where F is the gauge field strength form and ${}^*F$ is its dual form (see Definition I.1.17).
General covariance implies $S[F',g'] = S[F,g]$, where

$$F' = F + \epsilon L_X F \quad \text{and} \quad g' = g + \epsilon L_X g, \quad \epsilon \ll 1$$

X being the vector field on M generating the infinitesimal diffeomorphism corresponding to the (infinitesimal) change of coordinates (see below).
But we also have $S[F',g] = S[F,g]$, if X generates an infinitesimal conformal transformation of the 4-dimensional Riemannian manifold M. This can be checked by

noting that for an infinitesimal conformal transformation X we have (cf. Proposition II.8.5)

$$L_X {}^*\tau = {}^*L_X\tau + (1 - \frac{2q}{n})\, divX\, {}^*\tau$$

τ being a q-form on a n-dimensional Riemannian manifold.
Hence,

$$\begin{aligned} S[F',g] &= S[F,g] + \frac{\epsilon}{2}\int_M Tr\{{}^*L_X F \wedge F + {}^*F \wedge L_X F\} + o(\epsilon^2) = \\ &= S[F,g] + \frac{\epsilon}{2}\int_M L_X\, Tr({}^*F \wedge F) + o(\epsilon^2) \end{aligned}$$

The last term in the above equation vanishes if M is without boundary.
At the quantum level this symmetry is not maintained [25]. The associated anomaly is called *conformal* (or *trace*, or *Weyl*) anomaly.

The question we shall try to answer is whether there are other possible anomalies of this kind in a gauge field theory.

II.4.2 Infinitesimal diffeomorphisms and the consistency condition for the base space anomalies

In Subsection I.1.5 we saw that a vector field X on a compact manifold M, i.e. a section of the tangent bundle TM, generates a one-parameter group of diffeomorphisms of M. Since the vector space $\mathcal{X}(M)$ of all smooth vector fields on M is a Lie algebra with respect to the product given by the Lie bracket, the natural conclusion would be to identify $\mathcal{X}(M)$ with the infinite dimensional Lie algebra of the diffeomorphism group $DiffM$. Due to the infinite dimension things are not so simple. First of all not all diffeomorphisms close to the identity lie on a one-parameter group of diffeomorphisms. So we must find another way to parametrize the identity component of $DiffM$. This can still be done using $\mathcal{X}(M)$, but more structure on M is needed, i.e. a linear connection. For details see [31].
Secondly, similarly to what we explicitly showed in Proposition I.4.3, the group multiplication in $DiffM$, i.e. the composition of maps, induces a product in $\mathcal{X}(M)$ which is the negative of the Lie bracket.
Keeping in mind these remarks we set $\mathcal{L}DiffM \cong \mathcal{X}_-(M)$, where the subscript - means that the product in $\mathcal{X}(M)$ is minus the Lie bracket.

Under a diffeomorphism ϕ, a covariant tensor field T_q on M transforms according to

$$T_q \longrightarrow T'_q = \phi^* T_q \tag{II.52}$$

where $\phi^* T_q$ is the pull-back of T_q (cf. Definition I.1.9).
Observe that the pull-back is an anti-representation of $DiffM$ in the space of all covariant tensors.
The infinitesimal variation $\delta_0 T_q$ of T_q under an infinitesimal diffeomorphism belonging to the one-parameter subgroup generated by $X \in \mathcal{X}_-(M)$, briefly under

the infinitesimal diffeomorphism X, is given by (cf. Definition I.1.13)

$$\delta_0 T_q = T'_q - T_q = L_X T_q \tag{II.53}$$

where L_X is the Lie derivative with respect to X.
A spinor field ψ transforms as a scalar under the action of the diffeomorphism group [15]

$$\psi' = \psi \circ \phi = \phi^* \psi, \quad \phi \in Diff M \tag{II.54}$$

Thus, under an infinitesimal diffeomorphism X

$$\delta_0 \psi = L_X \psi, \qquad \text{locally } \delta_0 \psi(x) = X^\mu \partial_\mu \psi(x)$$

The assumed absence of gravitational anomalies for a gauge theory on a Riemannian manifold means (see Subsection II.1.2)

$$\Gamma^{(k)}[\psi', \bar{\psi}', A', g'] = \Gamma^{(k)}[\psi, \bar{\psi}, A, g], \quad \forall k$$

where $\psi' = \psi + \epsilon L_X \psi$, etc., $\epsilon \ll 1$, $X \in \mathcal{X}(M)$.

Suppose now the classical theory is invariant under the following transformation of the fields

$$\begin{array}{lcl} \psi & \rightarrow & \psi' = \psi + \epsilon \delta_0^X \psi \\ A & \rightarrow & A' = A + \epsilon \delta_0^X A = A + \epsilon L_X A \;, \quad X \in \mathcal{X}_-(M) \\ g & \rightarrow & g' = g \end{array} \tag{II.55}$$

where the variation $\delta_0^X \psi$ is not given by the Lie derivative.

Analogously to the gauge anomalies we shall be interested in finding local nontrivial anomalies Δ associated to the transformations (II.55), which depend on the gauge potential A and the metric g only. At the one-loop level Δ is given by

$$\epsilon \Delta[A, g; X] = \Gamma^{(1)}[\psi', \bar{\psi}', A', g] - \Gamma^{(1)}[\psi, \bar{\psi}, A, g]$$

or (cf. (II.27) and (II.29))

$$\Delta[A, g; X] = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \{\Gamma^{(1)}[A + \epsilon L_X A, g] - \Gamma^{(1)}[A, g]\} \doteq \vartheta(X) \Gamma^{(1)}[A, g] \tag{II.56}$$

where we have neglected the dependence of the one-loop effective action on the matter fields.
As can be easily verified we have (cf. Proposition II.3.2)

$$(\vartheta(X_1)\vartheta(X_2) - \vartheta(X_2)\vartheta(X_1))\Gamma^{(1)} = \vartheta(-[X_1, X_2])\Gamma^{(1)}$$

get st5Therefore, the consistent base space anomaly has to satisfy the consistency condition

$$\vartheta(X_1)\Delta[A, g; X_2] - \vartheta(X_2)\Delta[A, g; X_1] = \Delta[A, g; -[X_1, X_2]] \tag{II.57}$$

This is similar to the consistency condition for the gravitational anomalies [36]. Again we shall look for non-trivial anomalies, i.e. those which are not variations of local functionals

$$\Delta[A, g; X] \neq \vartheta(X)\tilde{\Delta}[A, g]$$

The problem being identical to the one considered in the preceding section, we directly formulate it in cohomological terms.

II.5 Cohomological construction of the base space anomalies

In this section we state the problem of finding possible, non-trivial, consistent, local base space anomalies, in the most possible appropriate mathematical form.

As we know a local anomaly $\Delta[A,g;X]$ is given by an integral on M of a local n-form $\mathcal{D}^{1,n}(A,g;X)$. By means of the variation $\vartheta(X)$ introduced in (II.56), we define a representation ϑ of $\mathcal{X}.(M)$ in the vector space $\Omega_A^q(M)$ of all local q-forms on M depending on a gauge potential A (cf. Proposition II.3.2). By assumption $\vartheta(X)$ doesn't act on the metric g; we shall therefore omit the possible dependence of the local forms on it.

Denote by δ the coboundary operator associated to the representation ϑ. A non-trivial, consistent, local anomaly, if it exists, is given by a local n-form $\mathcal{D}^{1,n}(A;X)$ satisfying (cf. (II.39) and (II.40))

$$\begin{aligned} &(\delta\mathcal{D}^{1,n})(A;X_1,X_2) = d(\mathcal{D}^{2,n-1}(A;X_1,X_2)) \\ \text{and}\quad &\mathcal{D}^{1,n}(A;X) \neq (\delta\mathcal{D}^{n})(A;X) \quad \text{(modulo an exact form)} \end{aligned} \tag{II.58}$$

In the next subsection we show how the solutions of the above equations are related to elements of $H^{n+1}(\mathcal{X}.(M);C_A^\infty(M))$, where $C_A^\infty(M) = \Omega_A^0(M)$ is the space of all local functions on M depending on a gauge potential A.

II.5.1 Local cochains and the descent equation

We begin this subsection by giving a more precise definition of the space $\Omega_A^q(M)$.

The gauge potential A belongs to $\Omega^1(M;LG)$, LG being the Lie algebra of the structure group of the trivial bundle upon which the gauge theory is based.
Denote by $\mathsf{V}_{loc}^m(\Omega^1(M;LG);\Omega^q(M))$ the vector space of all symmetric, m-linear, *local* maps $\mathcal{D}_m^q$ [6]

$$\begin{aligned} \mathcal{D}_m^q \; : \; &\underbrace{\Omega^1(M;LG)\times\cdots\times\Omega^1(M;LG)}_{m-times} \longrightarrow \Omega^q(M) \\ &A_1,\ldots,A_m \longmapsto \mathcal{D}_m^q(A_1,\ldots,A_m) \end{aligned} \tag{II.59}$$

The term local is taken here in its mathematical meaning, i.e. the map $\mathcal{D}_m^q$ is such that

$$supp\,\mathcal{D}_m^q(A_1,\ldots,A_m) \subset suppA_1 \cap\cdots\cap suppA_m$$

where $suppA$ is the support of the 1-form A [14].
It is a theorem of differential analysis which identifies the mathematical with the physical meaning of local. In fact for a local map $\mathcal{D}_m^q$ the q-form $\mathcal{D}_m^q(A_1,\ldots,A_m)$ is local, i.e. at a point $x\in M$ it depends on its arguments and their derivatives taken at x only [11].
Since $\Omega^1(M;LG)\cong\Omega^1(M)\otimes LG$, the local map $\mathcal{D}_m^q$ can be written as

$$\mathcal{D}_m^q = \bar{\mathcal{D}}_m^q \otimes \psi_m \tag{II.60}$$

where $\tilde{\mathcal{D}}^q_m$ is a local map $\Omega^1(M)^m \longrightarrow \Omega^q(M)$ and ψ_m a covariant tensor of degree m on LG.
A local q-form $\mathcal{D}^q(A) \in \Omega^q_A(M)$ can clearly be expressed as

$$\mathcal{D}^q(A) = \sum_{m=0}^{s} \mathcal{D}^q_m(A,\dots,A) \tag{II.61}$$

where s is finite.
In the following we shall identify through (II.61) the space $\Omega^q_A(M)$ with the direct sum $\bigoplus_{m=0}^{s} \mathrm{V}^m_{loc}(\Omega^1(M;LG);\Omega^q(M))$, s being a fixed finite integer.

Definition II.5.1 *The "exterior" product of two local maps $\mathcal{D}^p_l$ and $\mathcal{D}^q_m$ is the local map $\mathcal{D}^p_l \wedge \mathcal{D}^q_m$ defined by*

$$\begin{aligned}(\mathcal{D}^p_l \wedge \mathcal{D}^q_m)(A_1,\dots,A_{l+m}) &= \\ &\doteq \frac{1}{(l+m)!}\sum_{\pi} \mathcal{D}^p_l(A_{\pi(1)},\dots,A_{\pi(l)}) \wedge \mathcal{D}^q_m(A_{\pi(l+1)},\dots,A_{\pi(l+m)})\end{aligned}$$

Obviously, $\mathcal{D}^p_l \wedge \mathcal{D}^q_m = (-1)^{pq}\mathcal{D}^q_m \wedge \mathcal{D}^p_l$.
The so defined product makes the direct sum $\Omega^{\cdot}_A(M) \doteq \bigoplus_{q=0}^{n} \Omega^q_A(M)$ into a graded anticommutative algebra.

Definition II.5.2 *The derivative of the local map $\mathcal{D}^q_m$ is the $\Omega^{q+1}(M)$-valued local map $d^{\cdot}\mathcal{D}^q_m$, given by*

$$d^{\cdot}\mathcal{D}^q_m \doteq d \circ \mathcal{D}^q_m$$

$$i.e., \qquad d^{\cdot}\mathcal{D}^q_m(A_1,\dots,A_m) = d(\mathcal{D}^q_m(A_1,\dots,A_m))$$

where on the right-hand side d is the ordinary exterior derivative.

As can be easily verified the operation $d^{\cdot}$ is an antiderivation in $\Omega^{\cdot}_A(M)$.

Proposition II.5.1 *The linear transformation $\vartheta^{\vee}(X)$, $X \in \mathcal{X}.(M)$, which maps the local map $\mathcal{D}^q_m$ to the map $\vartheta^{\vee}(X)\mathcal{D}^q_m$ given by*

$$(\vartheta^{\vee}(X)\mathcal{D}^q_m)(A_1,\dots,A_m) = \sum_{i=1}^{m} \mathcal{D}^q_m(A_1,\dots,L_X A_i,\dots,A_m), \quad A_i \in \Omega^1(M;LG)$$

defines a representation $\vartheta^{\vee}$ of $\mathcal{X}.(M)$ in the graded differential algebra $(\Omega^{\cdot}_A(M), d^{\cdot})$.

Proof: that the map $\vartheta^{\vee}\mathcal{D}^q_m$ is local follows from $supp\, L_X A \subset suppA \cap suppX$ (see [21], vol.I, ch.IV).
From

$$(\vartheta^{\vee}(X_1)\vartheta^{\vee}(X_2)\mathcal{D}^q_m)(A_1,\dots,A_m) = \sum_{i=1}^{m}(\vartheta^{\vee}(X_2)\mathcal{D}^q_m)(A_1,\dots,L_{X_1}A_i,\dots,A_m) =$$

$$= \sum_{i\neq j}^{m} \mathcal{D}^q_m(A_1,\dots,L_{X_1}A_i,\dots,L_{X_2}A_j,\dots,A_m) + \sum_{i=1}^{m} \mathcal{D}^q_m(A_1,\dots,L_{X_2}L_{X_1}A_i,\dots,A_m)$$

it follows $\vartheta^\vee(X_1)\vartheta^\vee(X_2) - \vartheta^\vee(X_2)\vartheta^\vee(X_1) = \vartheta^\vee(-[X_1, X_2])$.
As can be easily checked $\vartheta^\vee(X)$ is a derivation in $\Omega_A^\cdot$ and trivially $\vartheta^\vee(X)d^\cdot =$ $= d^\cdot\vartheta^\vee(X)$. □

The representation $\vartheta^\vee$ defined in Proposition II.5.1 corresponds to the representation ϑ introduced in (II.56) (see above)

$$(\vartheta(X)\mathcal{D}^q)(A) = \sum_{m=0}^{s} (\vartheta^\vee(X)\mathcal{D}_m^q)(A, \ldots, A) \tag{II.62}$$

We can now define a *local* k-cochain $\mathcal{D}^{k,\cdot} = \sum_{m=0}^{s} \mathcal{D}_m^{k,\cdot}$ with values in $\Omega_A^\cdot(M)$ as a k-linear, skew-symmetric map

$$\mathcal{D}^{k,\cdot} : \underbrace{\mathcal{X}_\cdot(M) \times \cdots \times \mathcal{X}_\cdot(M)}_{k-times} \longrightarrow \Omega_A^\cdot(M) \tag{II.63}$$

such that

$$supp\, \mathcal{D}_m^{k,\cdot}(X_1, \ldots, X_k)(A_1, \ldots, A_m) \subset suppX_1 \cap \cdots \cap suppX_k \cap suppA_1 \cap \cdots \cap suppA_m$$

Observe that the cochain $\mathcal{D}_m^{k,q}$ can be considered as a local map

$$\mathcal{D}_m^{k,q} : \mathcal{X}_\cdot(M)^k \times \Omega^1(M; LG)^m \longrightarrow \Omega^q(M)$$

Again, the so obtained q-form $\mathcal{D}_m^{k,q}(X_1, \ldots, X_k)(A_1, \ldots, A_m)$ is local.
As usual we denote by $\bigwedge(\mathcal{X}_\cdot(M); \Omega_A^\cdot(M))$ the algebra, with product (II.33), of all local cochains.

We finally arrive at the identification of the local n-form $\mathcal{D}^{1,n}(A; X)$ giving rise to an anomaly with the local 1-cochain $\mathcal{D}^{1,n}$ (cf. (II.61))

$$\mathcal{D}^{1,n}(A; X) = \sum_{m=0}^{s} \mathcal{D}_m^{1,n}(X)(A, \ldots, A)$$

Denote by $\delta^\vee$ the coboundary operator associated to the representation $\vartheta^\vee$. From eq. (II.62) it follows

$$(\delta\mathcal{D}^{1,n})(A; X_1, X_2) = \sum_{m=0}^{s} \delta^\vee\mathcal{D}_m^{1,n}(X_1, X_2)(A, \ldots, A)$$

$$\text{or briefly} \quad \delta\mathcal{D}^{1,n} = \sum_{m=0}^{s} \delta^\vee\mathcal{D}_m^{1,n} = \delta^\vee\mathcal{D}^{1,n}$$

Definition II.5.3 *The derivative of the local k-cochain $\mathcal{D}^{k,q}$ is the local cochain $d\mathcal{D}^{k,q}$ given by*

$$d\mathcal{D}^{k,q} \doteq d^\cdot \circ \mathcal{D}^{k,q}$$

where the derivative on the right-hand side is defined in Definition II.5.2.

Since $\vartheta^\vee$ is a representation in the graded differential algebra $\Omega_A^\cdot(M)$, in particular $\vartheta^\vee \circ d^\cdot = d^\cdot \circ \vartheta^\vee$, it can be easily verified that

$$\delta^\vee \circ d = d \circ \delta^\vee \tag{II.64}$$

Condition (II.58) is thus equivalent to

$$\begin{aligned} &\delta^\vee \mathcal{D}^{1,n} = d\mathcal{D}^{2,n-1} \\ \text{and} \quad &\mathcal{D}^{1,n} \neq \delta^\vee \mathcal{D}^{n} + d\mathcal{D}^{1,n-1} \end{aligned} \tag{II.65}$$

Application of the nilpotent operator $\delta^\vee$ on both sides of the first equation in (II.65) and use of (II.64) gives

$$d\,\delta^\vee \mathcal{D}^{2,n-1} = 0$$

A theorem by De Wilde [38] states that a local map $L^{k,q}$ with values in the closed q-forms, (q<n), of a n-dimensional manifold M can be expressed as $L^{k,q} = d \circ L^{k,q-1}$, where $L^{k,q-1}$ is a local map with values in $\Omega^{q-1}(M)$. By generalizing this theorem to the local cochains $\mathcal{D}^{k,q}$ (see [6]), we get

$$\delta^\vee \mathcal{D}^{2,n-1} = d\mathcal{D}^{3,n-2}$$

which is similar to the equation we started from. Thus, the first equation in (II.65) implies the existence of local k-cochains, $1 \leq k \leq n+1$, satisfying

$$\delta^\vee \mathcal{D}^{k,n-k+1} = d\mathcal{D}^{k+1,n-k}, \quad 1 \leq k \leq n+1 \tag{II.66}$$

Equation (II.66) is a so called *descent equation*. It always arises when one of the two cohomologys of a double differential complex, in the present case $\{\bigwedge(\mathcal{X}_-(M); \Omega_A^\cdot(M));\ \delta^\vee, d\}$, is trivial [10].
Similarly, by applying $\delta^\vee$ on both sides of the second equation in (II.65) and using the first one, we get

$$d\mathcal{D}^{2,n-1} \neq d\delta^\vee \mathcal{D}^{1,n-1}$$

or, by the cited theorem,

$$\mathcal{D}^{2,n-1} - \delta^\vee \mathcal{D}^{1,n-1} \neq d\mathcal{D}^{2,n-2}$$

Thus, the k-cochains in (II.66) must be such that

$$\mathcal{D}^{k,n-k+1} \neq \delta^\vee \mathcal{D}^{k-1,n-k+1} + d\mathcal{D}^{k,n-k}, \quad 1 \leq k \leq n+1 \tag{II.67}$$

The bottom equations in the systems (II.66) and (II.67) read

$$\begin{aligned} &\delta^\vee \mathcal{D}^{n+1,0} = 0 \\ &\mathcal{D}^{n+1,0} \neq \delta^\vee \mathcal{D}^{n,0} \end{aligned} \tag{II.68}$$

This means that the cochain $\mathcal{D}^{n+1,0}$ is a cocycle belonging to $H^{n+1}(\mathcal{X}_{\cdot}(M);\Omega^0_A(M)) =$

$= H^{n+1}(\mathcal{X}_{\cdot}(M);C^\infty_A(M))$.
Therefore, if it exists a local 1-cochain $\mathcal{D}^{1,n}$ verifying (II.65) then it must exist a local (n+1)-cochain $\mathcal{D}^{n+1,0}$ verifying (II.68).
It has to be emphasized that the converse is not true: it is not always possible to ascend eq. (II.66) [6].
Observe that $\delta^\vee\mathcal{D}^{n+1,0} = \sum_{m=0}^{s}\delta^\vee\mathcal{D}^{n+1,0}_m = 0$ implies $\delta^\vee\mathcal{D}^{n+1,0}_m = 0$, $\forall m$, since $\delta^\vee$ doesn't change the number of the A_l's.
The problem of solving (II.65) is now reduced to finding $H^{n+1}(\mathcal{X}_{\cdot}(M);C^\infty_A(M))$ and ascending eq. (II.66).

Remark that a similar procedure for finding the consistent gauge anomalies has been used in reference [6]. In our treatment of that case (see Subsection II.3.2) a rôle similar to the descent equation is played by Proposition II.3.3.

II.5.2 The local expression of the local cochains and their symbols

In this subsection we introduce the notation used to locally describe differential operators [14,9]. The symbols of the local cochains are then defined [39].

As we said, the local k-cochain $\mathcal{D}^{k,0}_m$ can be considered as a local map

$$\mathcal{D}^{k,0}_m : \mathcal{X}_{\cdot}(M)^k \times \Omega^1(M;LG)^m \longrightarrow C^\infty(M)$$

antisymmetric in its first k arguments (vector fields), and symmetric in the remaining m ones (LG-valued 1-forms). We are interested in the local form of the function $\mathcal{D}^{k,0}_m(X_1,\ldots,X_k)(A_1,\ldots,A_m)$ in a coordinate neighborhood $U \subset M$. In the following we shall omit the second superscript in $\mathcal{D}^{k,0}_m$.

Notation Let $\{x^\mu,\ \mu = 1,\ldots,n\}$ be a local coordinate system in U. We denote by D_α the following partial differential operator

$$D_\alpha \doteq \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial x^{1\,\alpha_1}\cdots\partial x^{n\,\alpha_n}};\quad \alpha_\mu \in \mathbf{N} \tag{II.69}$$

The subscript α in (II.69) is a so called *multi-index*. It stands for $\alpha = (\alpha_1,\ldots,\alpha_n)$. Given two multi-indices α and β their sum $\alpha+\beta$ is the multi-index given by $\alpha+\beta = (\alpha_1+\beta_1,\ldots,\alpha_n+\beta_n)$. The number $\alpha_1+\cdots+\alpha_n$ will be denoted by $|\alpha|$. Let $\vec{\xi} = (\xi_1,\ldots,\xi_n)$, $\xi_i \in \mathbf{R}$, be a n component object. We denote by ξ^α the number given by

$$\xi^\alpha \doteq \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n} = \prod_{\mu=1}^{n}\xi_\mu^{\alpha_\mu},\quad \alpha = (\alpha_1,\ldots,\alpha_n)$$

As can be easily verified the operator D_α satisfies a generalized Leibniz rule

$$D_\alpha(f\,g) = \sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}(D_{\alpha-\beta}f)(D_\beta g),\quad f,g \in C^\infty(U) \tag{II.70}$$

where $\sum_{\beta=0}^{\alpha}$ stands for $\sum_{\beta_1=0}^{\alpha_1} \cdots \sum_{\beta_n=0}^{\alpha_n}$ and $\alpha! = \alpha_1! \cdots \alpha_n!$.

We are ready to explicitly write down the local expression of a general local cochain $\mathcal{D}^k_m$, namely

$$[\mathcal{D}^k_m(X_1, \dots, X_k)(A_1, \dots, A_m)](x) = \tag{II.71}$$
$$= \sum_{\alpha^1,\dots,\alpha^k} \sum_{\beta^1,\dots,\beta^m} \mathbf{a}(x)^{\alpha^1 \dots \alpha^k; \beta^1 \dots \beta^m} (D_{\alpha^1}\mathbf{X}_1, \dots, D_{\alpha^k}\mathbf{X}_k; D_{\beta^1}\mathbf{A}_1, \dots, D_{\beta^m}\mathbf{A}_m)$$

where $\mathbf{X}_i = (X_i^1, \dots, X_i^n)$, $1 \le i \le k$, resp. $\mathbf{A}_l$, $1 \le l \le m$, are the components of the vector field X_i, resp. the 1-form A_l, with respect to the local coordinate system in U. $\mathbf{a}(x)^{\alpha^1 \dots \alpha^k; \beta^1 \dots \beta^m}$ is a linear map from $(\mathbf{R}^n)^k \times (\mathbf{R}^n)^m \times LG^m$ to $\mathbf{R}$.
Choosing a basis $\{T_p,\, p = 1, \dots, dimG\}$ of LG we can express (II.71) as

$$[\mathcal{D}^k_m(X_1, \dots, X_k)(A_1, \dots, A_m)](x) = \tag{II.72}$$
$$= \sum_{\alpha^1,\dots,\alpha^k} \sum_{\beta^1,\dots,\beta^m} \mathbf{d}(x)^{\alpha^1 \dots \alpha^k; \beta^1 \dots \beta^m} (D_{\alpha^1}\mathbf{X}_1, \dots, D_{\alpha^k}\mathbf{X}_k; D_{\beta^1}\mathbf{A}_1^{p_1}, \dots, D_{\beta^m}\mathbf{A}_m^{p_m}) \times$$
$$\times \psi_m(T_{p_1}, \dots, T_{p_m})$$

where $\mathbf{d}(x)^{\alpha^1 \dots \alpha^k; \beta^1 \dots \beta^m}$ is a linear map from $(\mathbf{R}^n)^k \times (\mathbf{R}^n)^m$ to $\mathbf{R}$ and ψ_m is a covariant tensor of degree m on LG (cf. (II.60)).

Definition II.5.4 *The total order of the local cochain $\mathcal{D}^k_m$ is the maximal value of $|\alpha^1| + \dots + |\alpha^k| + |\beta^1| + \dots + |\beta^m|$ for which the map $\mathbf{d}(x)^{\alpha^1 \dots \alpha^k; \beta^1 \dots \beta^m}$ is not vanishing identically.*

Let ξ_i, $1 \le i \le k$, and ζ_l, $1 \le l \le m$, be covectors belonging to TM_x^*, $x \in U$. Locally, e.g. $\xi_i = \xi_{i\mu} dx^\mu$.

Definition II.5.5 *The characteristic polynomial of the local cochain $\mathcal{D}^k_m$, denoted by $P_{\mathcal{D}}$, is the polynomial given by*

$$P_{\mathcal{D}}(\xi_1, \dots, \xi_k; X_{1x}, \dots, X_{kx} | \zeta_1, \dots, \zeta_m; A_{1x}^{p_1}, \dots, A_{mx}^{p_m}) =$$
$$= \sum_{\alpha^i} \sum_{\beta^l} \xi_1^{\alpha^1} \cdots \xi_k^{\alpha^k} \zeta_1^{\beta^1} \cdots \zeta_m^{\beta^m} \mathbf{d}(x)^{\alpha^1 \dots \alpha^k; \beta^1 \dots \beta^m} (\mathbf{X}_{1x}, \dots, \mathbf{X}_{kx}; \mathbf{A}_{1x}^{p_1}, \dots, \mathbf{A}_{mx}^{p_m})$$

where, e.g. $\xi_i^{\alpha^i} = \prod_\mu \xi_{i\mu}^{\alpha^i_\mu}$.

Observe that the total degree of $P_{\mathcal{D}}$ is given by the total order of $\mathcal{D}$ plus (k+m). When no confusion arises we simply write X_i resp. $A_l^{p_l}$ for X_{ix} resp. $A_{lx}^{p_l}$.

Proposition II.5.2 *The characteristic polynomial $P_{\mathcal{D}}$ possesses the following symmetry*

$$P_{\mathcal{D}}(\ \ , \xi_{i_1}, \ \ , \xi_{i_2}, \ \ ; \ \ , X_{i_1}, \ \ , X_{i_2}, \ \ | \cdots) =$$
$$= -P_{\mathcal{D}}(\ \ , \xi_{i_2}, \ \ , \xi_{i_1}, \ \ ; \ \ , X_{i_2}, \ \ , X_{i_1}, \ \ | \cdots)$$

Proof: this follows from the antisymmetry of the local cochain $\mathcal{D}^k_m$ in the X_i. □

Definition II.5.6 *The degree of a monomial in $P_{\mathcal{D}}$ is the $(k+m)$-tuple*

$$(|\alpha^1|, \ldots, |\alpha^k|; |\beta^1|, \ldots, |\beta^m|)$$

where $|\alpha^i| = \sum_\mu \alpha^i_\mu$, $|\beta^l| = \sum_\mu \beta^l_\mu$, and the α^i_μ resp. β^l_μ are the partial degrees of the monomial with respect to ξ^i_μ resp. ζ^l_μ, $\mu = 1, \ldots, n$.

Assume that in the characteristic polynomial $P_{\mathcal{D}}$ there are terms of degree

$$(a_1, \ldots, a_{k_0}, 1, \ldots, \underset{\underset{(k_1)}{\uparrow}}{1}, 0, \ldots, 0; b_1, \ldots, b_m), \quad 0 \leq k_0 \leq k_1 \leq k \tag{II.73}$$

By Proposition II.5.2 we can assume $a_i \geq a_{i+1} \geq 2$.

Definition II.5.7 *The symbol of the local cochain $\mathcal{D}^k_m$, denoted by $\sigma_{\mathcal{D}}$, consists of those terms of $P_{\mathcal{D}}$ of degree (II.73), where k_1 is the largest possible, $\sum_{l=1}^m b_l$ is maximal for this value of k_1, and the a_i's are maximal for these values of k_1 and $\sum_{l=1}^m b_l$.*

Definition II.5.8 *The order of the local cochain $\mathcal{D}^k_m$ is the degree of one monomial in the symbol of $\mathcal{D}^k_m$.*

In the next section we shall express the symbol of the coboundary $\delta^\vee \mathcal{D}^k_m$ by means of the symbol of $\mathcal{D}^k_m$. This will allow us to determine the form of the symbols of the cocycles, and thus of the possible non-trivial elements of $H(\mathcal{X}_-(M); C^\infty_A(M))$.

II.6 Cohomology of $\mathcal{X}_-(M)$ with coefficients in $C^\infty_A(M)$

In this section we solve the cohomology $H^{n+1}(\mathcal{X}_-(M); C^\infty_A(M))$ by a method similar to the one used by the authors of reference [39] to compute $H(\mathcal{X}(M); \Omega^\cdot(M))$, where the representation of $\mathcal{X}(M)$ in $\Omega^\cdot(M)$ is given by the Lie derivative.

II.6.1 The relation between the symbols $\sigma_{\delta^\vee \mathcal{D}}$ and $\sigma_{\mathcal{D}}$

By definition, the coboundary operator $\delta^\vee$ associated to the representation $\vartheta^\vee$ defined in Proposition II.5.1 acts on the local k-cochain $\mathcal{D}^k_m$ as

$$\begin{aligned}
&\delta^\vee \mathcal{D}^k_m(X_1,\ldots,X_{k+1})(A_1,\ldots,A_m) = \\
&= \sum_{i=1}^{k+1}(-1)^{i+1}\sum_{l=1}^{m}\mathcal{D}^k_m(X_1,\ldots,\hat{\imath},\ldots,X_{k+1})(A_1,\ldots,L_{X_i}A_l,\ldots,A_m) \\
&+ \sum_{i<j}(-1)^{i+j}\,\mathcal{D}^k_m(-[X_i,X_j],X_1,\ldots,\hat{\imath},\ldots,\hat{\jmath},\ldots,X_{k+1})(A_1,\ldots,A_m)
\end{aligned} \tag{II.74}$$

The components of the Lie derivative $L_X A^p$ of the 1-form A^p with respect to a local coordinate system in U, are given by (cf. (I.10))

$$(L_X A^p)_\mu = X^\nu \partial_\nu A^p_\mu + A^p_\nu \partial_\mu X^\nu \tag{II.75}$$

Therefore the action of the representation $\vartheta^\vee(X)$ on the local cochain $\mathcal{D}^k_m$ transforms a term $D_\beta \mathbf{A}$ in the local expression (II.71) into

$$\begin{aligned}
D_\beta \mathbf{L_X A}^p &= \{D_\beta(X^\nu\partial_\nu A^p_\mu + A^p_\nu \partial_\mu X^\nu)\} \\
&= \{\sum_{\delta=0}^{\beta}\binom{\beta}{\delta}[(D_\delta X^\nu)(D_{\beta-\delta}\partial_\nu A^p_\mu) + (D_{\beta-\delta}A^p_\nu)(D_\delta\partial_\mu X^\nu)]\}
\end{aligned}$$

The characteristic polynomial of $\vartheta^\vee(X_i)\mathcal{D}^k_m(X_1,\ldots,\hat{\imath},\ldots,X_{k+1})(A_1,\ldots,A_m)$ is thus given by

$$\begin{aligned}
&\sum_{l=1}^{m}\sum_{\ldots\beta^l\ldots}\sum_{\delta^l=0}^{\beta^l}\binom{\beta^l}{\delta^l}\{<\zeta_l,X_i>\xi_i^{\delta^l}\zeta_l^{\beta^l-\delta^l}\cdots\mathbf{d}(x)^{\cdots}(\cdots\hat{\imath}\cdots;\mathbf{A}^{p_1}_{1x},\ldots,\mathbf{A}^{p_m}_{mx}) \\
&\quad + <A^{p_l}_l,X_i>\xi_i^{\delta^l}\zeta_l^{\beta^l-\delta^l}\cdots\mathbf{d}(x)^{\cdots}(\cdots\hat{\imath}\cdots;\mathbf{A}^{p_1}_{1x},\ldots,\vec{\xi_i},\ldots,\mathbf{A}^{p_m}_{mx})\} = \\
&= \sum_{l=1}^{m}\sum_{\ldots\beta^l\ldots}\{<\zeta_l,X_i>(\xi_i+\zeta_l)^{\beta^l}\cdots\mathbf{d}(x)^{\cdots}(\cdots\hat{\imath}\cdots;\mathbf{A}^{p_1}_{1x},\ldots,\mathbf{A}^{p_m}_{mx}) \\
&\quad + <A^{p_l}_l,X_i>(\xi_i+\zeta_l)^{\beta^l}\cdots\mathbf{d}(x)^{\cdots}(\cdots\hat{\imath}\cdots;\mathbf{A}^{p_1}_{1x},\ldots,\underset{\uparrow\,(l)}{\vec{\xi_i}},\ldots,\mathbf{A}^{p_m}_{mx})\}
\end{aligned} \tag{II.76}$$

where $<\zeta_l, v_x>$ is the value of the covector $\zeta_l \in TM^*_x$ applied on the tangent vector $v_x \in TM_x$.
By analogously treating the term $D_\alpha[X_i, X_j]$ we arrive at the following Proposition

Proposition II.6.1 *The characteristic polynomial $P_{\delta^\vee\mathcal{D}}$ of the local coboundary $\delta^\vee\mathcal{D}^k_m$ is given by*

$$P_{\delta^\vee\mathcal{D}}(\xi_1,\ldots,\xi_{k+1};X_1,\ldots,X_{k+1}|\zeta_1,\ldots,\zeta_m;A_1^{p_1},\ldots,A_m^{p_m}) =$$

$$= \sum_{i=1}^{k+1}(-1)^{i+1}\sum_{l=1}^{m} < \zeta_l, X_i > P_{\mathcal{D}}(\cdots\hat{i}\cdots;|\zeta_1\ldots\zeta_l+\xi_i\ldots\zeta_m;A_1^{p_1}\ldots A_m^{p_m})$$

$$+\sum_{i=1}^{k+1}(-1)^{i+1}\sum_{l=1}^{m} < A_l^{p_l}, X_i > P_{\mathcal{D}}(\cdots\hat{i}\cdots;|\zeta_1\ldots\zeta_l+\xi_i\ldots\zeta_m;A_1^{p_1}\ldots\vec{\xi_i}\ldots A_m^{p_m})$$

$$+\sum_{i<j}(-1)^j < \xi_j, X_i > P_{\mathcal{D}}(\ldots\underset{(i)}{\xi_i+\xi_j}\ldots\hat{j}\ldots;\ldots\underset{(i)}{X_j}\ldots\hat{j}\ldots|\cdots)$$

$$+\sum_{i<j}(-1)^{j+1} < \xi_i, X_j > P_{\mathcal{D}}(\ldots\underset{(i)}{\xi_i+\xi_j}\ldots\hat{j}\ldots;\ldots\hat{j}\ldots|\cdots)$$

where $P_{\mathcal{D}}$ is the characteristic polynomial of the cochain $\mathcal{D}^k_m$.

From Proposition II.6.1 it follows that a monomial of degree $(a_1,\ldots,a_k;b_1,\ldots,b_m)$ gives rise to terms in $P_{\delta^\vee\mathcal{D}}$ of degrees

$$\begin{array}{ll}(a_1,\ldots,a_{i-1},d_l,a_i,\ldots,a_k;b_1,\ldots,b_l-d_l+1,\ldots,b_m) & 0\le d_l\le b_l\\ (a_1,\ldots,a_{i-1},d_l+1,a_i,\ldots,a_k;b_1,\ldots,b_l-d_l,\ldots,b_m) & \\ (a_1,\ldots,a_i-c_i,\ldots,a_{j-1},c_i+1,a_j,\ldots,a_k;b_1,\ldots,b_m) & 0\le c_i\le a_i\\ (a_1,\ldots,a_i-c_i+1,\ldots,a_{j-1},c_i,a_j,\ldots,a_k;b_1,\ldots,b_m) & \end{array} \qquad \text{(II.77)}$$

Assume the order of the local cochain $\mathcal{D}^k_m$ is

$$(a_1,\ldots,a_{k_0},1,\ldots,\underset{(k_1)}{1},0,\ldots,0;b_1,\ldots,b_m) \qquad \text{(II.78)}$$

From (II.77) and Definition II.5.8 we get that in this case the order of $\delta^\vee\mathcal{D}^k_m$ is

$$(a_1,\ldots,a_{k_0},1,\ldots,\underset{(k_1+1)}{1},0,\ldots,0;b_1,\ldots,b_m) \qquad \text{(II.79)}$$

Terms of this degree in $P_{\delta^\vee\mathcal{D}}$, i.e. the symbol of $\delta^\vee\mathcal{D}^k_m$, can arise only from the symbol of $\mathcal{D}^k_m$ or from terms of degree

$$(a_1,\ldots,a_{k_0},1,\ldots,\underset{(i)}{0},\ldots,\underset{(k_1+1)}{1},0,\ldots,0;b_1,\ldots,b_m), \quad k_0+1\le i\le k_1$$

which are so to speak already contained in $\sigma_{\mathcal{D}}$ through antisymmetrization.
It has to be emphasized that this is valid only if the order of $\mathcal{D}^k_m$ is (II.78). As can be seen from (II.77) terms of degree (II.79) can also result from terms of degree "larger" then (II.78), which are absent by the assumption on the order of $\mathcal{D}^k_m$.
By taking into account all possible contributions of degree (II.79) in $P_{\delta^\vee\mathcal{D}}$ we obtain

Proposition II.6.2 *Let $\sigma_{\mathcal{D}}$ be the symbol of the local cochain $\mathcal{D}^k_m$ of order (II.78). Then the symbol $\sigma_{\delta^\vee\mathcal{D}}$ of the local coboundary $\delta^\vee\mathcal{D}^k_m$ is given by*

$$\begin{aligned}
&\sigma_{\delta^\vee\mathcal{D}}(\xi_1,\dots,\xi_{k_1+1};X_1,\dots,X_{k+1}|\zeta_1,\dots,\zeta_m;A_1^{p_1},\dots,A_m^{p_m}) = \\
&= \sum_{k_0+1\le i\le k_1+1}(-1)^{i+1}\sum_{l=1}^{m} <\zeta_l,X_i>\xi_i\cdot D_{\zeta_l}\sigma_{\mathcal{D}}(\dots\hat{\imath}\dots;\dots\hat{\imath}\dots|\cdots) \\
&+ \sum_{k_0+1\le i\le k_1+1}(-1)^{i+1}\sum_{l=1}^{m} <A_l^{p_l},X_i>\xi_i\cdot D_{A_l^{p_l}}\sigma_{\mathcal{D}}(\dots\hat{\imath}\dots;\dots\hat{\imath}\dots|\cdots) \\
&+ \sum_{\substack{i<j \\ k_0+1\le j\le k_1+1}}(-1)^{j} <\xi_j,X_i>X_j\cdot D_{X_i}\sigma_{\mathcal{D}}(\dots\hat{\jmath}\dots;\dots\hat{\jmath}\dots|\cdots) \\
&+ \sum_{\substack{i<j \\ k_0+1\le j\le k_1+1}}(-1)^{j+1} <\xi_i,X_j>\xi_j\cdot D_{\xi_i}\sigma_{\mathcal{D}}(\dots\hat{\jmath}\dots;\dots\hat{\jmath}\dots|\cdots) \\
&+ \sum_{i=k_0+1}^{k_1+1}\sum_{j=k_1+2}^{k+1}(-1)^{i} <\xi_i,X_j>X_i\cdot D_{X_j}\sigma_{\mathcal{D}}(\dots\hat{\imath}\dots;\dots\hat{\imath}\dots|\cdots)
\end{aligned}$$

where we have introduced the operation $y\cdot D_x f(x) \doteq \frac{d}{dt}f(x+ty)|_{t=0}$.

In the next subsection we interpret the symbol of a local cochain itself as a cochain, and show that the symbol of the coboundary $\delta^\vee\mathcal{D}^k_m$ is a suitable coboundary of the symbol of $\mathcal{D}^k_m$.

II.6.2 Algebraic interpretation of the symbols

Definition II.6.1 *Let F be a vector space and $\{e_i^*\}$ a basis of the dual space F^*. A function $p : F \longrightarrow \mathbf{R}$ is called polynomial if it can be expressed as a polynomial of the e_i^*'s.*

Denote by $P^{q,p}(TM_x)$ the space of all polynomial functions p

$$p : TM_x^{*q}\times TM_x^p \longrightarrow \mathbf{R}$$

As an example the characteristic polynomial $P_{\mathcal{D}}$ of a local cochain $\mathcal{D}^k_m$ is a polynomial function belonging to $P^{k+2m,k}(TM_x)$.
Consider the representation ρ of the group $GL(n,\mathbf{R})$ in $P^{q,p}(TM_x)$ defined by

$$(\rho(E)p)(\xi_1,\dots,\xi_q;X_1,\dots,X_p) = p(\tilde{E}\xi_1,\dots,\tilde{E}\xi_q;E^{-1}X_1,\dots,E^{-1}X_p)$$

where in a basis $\tilde{E}\xi_i = (\tilde{E}\vec{\xi_i})_\mu dx^\mu$, $\xi_i\in TM_x^*$, $\tilde{E}$ being the transposed of the matrix E.
The corresponding representation ρ_* of the Lie algebra $gl(n,\mathbf{R})$ is given by

$$\rho_*(U)p = \frac{d}{dt}[\rho(\exp tU)p]|_{t=0},\quad U\in gl(n,\mathbf{R})$$

We explicitly have

$$(\rho_*(U)p)(\xi_1,\ldots,\xi_q;X_1,\ldots,X_p) = \sum_{i=1}^q(\tilde{U}\xi_i)\cdot D_{\xi_i}p(\xi_1,\ldots,\xi_q;X_1,\ldots,X_p)$$
$$-\sum_{j=1}^p(UX_j)\cdot D_{X_j}p(\xi_1,\ldots,\xi_q;X_1,\ldots,X_p) \tag{II.80}$$

where the operation $y\cdot D_x$ was introduced in Proposition II.6.2.
In the special case where $U = \vec{X}\otimes\vec{\xi}\in gl(n,\mathbf{R})$, i.e. $U^\mu_\nu = X^\mu\xi_\nu$, $\mu,\nu = 1,\ldots,n$, we have

$$\tilde{U}\xi_i = <\xi_i,X>\xi \quad \text{resp.} \quad UX_j = <\xi,X_j>X \tag{II.81}$$

As usual we introduce r-cochains C^r, which are r-linear, skew-symmetric maps

$$C^r : gl(n,\mathbf{R})^r \longrightarrow P^{q,p}(TM_x)$$

Since by assumption the symbol $\sigma_{\mathcal{D}}$ of a local cochain $\mathcal{D}^k_m$ is linear in X_i resp. ξ_i for $k_0+1\leq i\leq k_1$, and by Proposition II.5.2 antisymmetric in the pairs (ξ_i,X_i), we can consider it as a $(k_1$-$k_0)$-cochain $C^{k_1-k_0}_{\mathcal{D}}$

$$C^{k_1-k_0}_{\mathcal{D}} : gl(n,\mathbf{R})^{k_1-k_0} \longrightarrow P^{k_0+2m,k_0+k-k_1}(TM_x)$$

given by

$$C^{k_1-k_0}_{\mathcal{D}}(U_1\ldots U_{k_1-k_0})(\xi_1\ldots\xi_{k_0},\zeta_1\ldots\zeta_m,A_1^{p_1}\ldots A_m^{p_m};X_1\ldots X_{k_0},X_{k_1+1}\ldots X_k) =$$
$$\doteq \sigma_{\mathcal{D}}(\xi_1\ldots\xi_{k_1};X_1\ldots X_k|\zeta_1\ldots\zeta_m;A_1^{p_1}\ldots A_m^{p_m}) \tag{II.82}$$

where $U_s = \vec{X}_{k_0+s}\otimes\vec{\xi}_{k_0+s}$, $1\leq s\leq k_1-k_0$.
Denote by δ_* the coboundary operator associated to the representation ρ_* of $gl(n,\mathbf{R})$ in $P^{q,p}(TM_x)$.

Proposition II.6.3 *Let $\sigma_{\mathcal{D}}$ be the symbol of a local cochain $\mathcal{D}^k_m$ of order (II.78), and $\sigma_{\delta^\vee\mathcal{D}}$ the symbol of the coboundary $\delta^\vee\mathcal{D}^k_m$. Considered as cochains $C^{k_1-k_0}_{\mathcal{D}}$ resp. $C^{k_1-k_0+1}_{\delta^\vee\mathcal{D}}$, as in (II.82), they satisfy*

$$\sigma_{\delta^\vee\mathcal{D}} = (-1)^{k_0}\,\delta_*\,\sigma_{\mathcal{D}}$$

This Proposition will allow us to determine the form of the symbols of the local cocycles and to restrict the search of the non-trivial ones.
Observe that even though our cochains and symbols differ from those in reference [39], Proposition II.6.3 is formally equivalent to equation (3.4) of the same reference.

With a procedure similar to the one used by the authors of [39] we can restrict ourselves to local cochains of order (II.78) with

$$\sum_{i=1}^{k_0}a_i + \sum_{l=1}^{m}b_l + m - [k-(k_1-k_0)] = 0 \tag{II.83}$$

As a consequence of (II.83) we obtain

$$\rho_*(\mathbf{1})C_{\mathcal{D}}^{k_1-k_0}(U_1,\ldots,U_{k_1-k_0})=0,\quad U_s=\vec{X}_{k_0+s}\otimes\vec{\xi}_{k_0+s},\ 1\le s\le k_1-k_0 \quad \text{(II.84)}$$

where $\mathbf{1}$ is the identity element of $gl(n,\mathbf{R})$.
This can be easily verified by using (II.80) and noting that $\xi\cdot D_\xi\,\xi^\alpha=|\alpha|\,\xi^\alpha$.

Proposition II.6.4 *Let $\tilde{P}^{q,p}(TM_x)$ be the space of polynomial functions $\tilde{p}$ verifying $\rho_*(\mathbf{1})\tilde{p}=0$. Then*

$$H^r(gl(n,\mathbf{R});\tilde{P}^{q,p}(TM_x))=\bigwedge_I^r(gl(n,\mathbf{R}))\times P^{q,p}_{\rho_*=0}(TM_x)$$

where $\bigwedge_I^r(gl(n,\mathbf{R}))$ is the space of all r-linear, skew-symmetric, Ad-invariant maps $gl(n,\mathbf{R})^r\longrightarrow\mathbf{R}$, and $P^{q,p}_{\rho_*=0}(TM_x)$ is the subspace of all polynomial functions invariant under the representation ρ_*.

Proof: see §5 of reference [39].

Proposition II.6.5 *The elements of $P^{q,p}_{\rho_*=0}(TM_x)$ are given by ordinary polynomials in $<\xi_i,X_j>$, $1\le i\le q$, $1\le j\le p$.*

Proof: see [37].

We are ready to write down the general form of the symbol of a local cocycle of order (II.78). For a cocycle $\mathcal{D}^k_m$ we clearly have $\sigma_{\delta\circ\mathcal{D}}=0$. By Proposition II.6.3 this implies $\delta_*\sigma_{\mathcal{D}}=0$. Moreover we want at least that $\mathcal{D}^k_m$ is not the coboundary of a local cochain $\tilde{\mathcal{D}}^{k-1}_m$ of order $(a_1,\ldots,a_{k_0};1,\ldots,\underset{(k_1-1)}{1},0,\ldots,0;b_1,\ldots,b_m)$, i.e. we require that

$$\sigma_{\mathcal{D}}\neq(-1)^{k_0}\delta_*\sigma_{\tilde{\mathcal{D}}}$$

It follows that the symbol $\sigma_{\mathcal{D}}$, identified with the $(k_1$-$k_0)$-cochain $C_{\mathcal{D}}^{k_1-k_0}$ (II.82), belongs to

$$H^{k_1-k_0}(gl(n,\mathbf{R});\tilde{P}^{k_0+2m,k_0+k-k_1}(TM_x))$$

By assumption $\sigma_{\mathcal{D}}$ is linear in the X_i, $1\le i\le k_1$, resp. $A_l^{p_l}$, $1\le l\le m$, $C^\infty(M)$-linear in the X_j, $k_1\le j\le k$, and of degree $a_i\ge 2$ in the ξ_i, $1\le i\le k_0$. By Propositions II.6.4 and II.6.5 it must therefore have the following form

$$\mathcal{A}^{k_1-k_0}(X_{k_0+1}\otimes\xi_{k_0+1},\ldots,X_{k_1}\otimes\xi_{k_1})S^{k_0}(\xi_1,\ldots,\xi_{k_0};X_1,\ldots,X_{k_0})\times \quad \text{(II.85)}$$

$$\times\imath(X_{k_1+1},\ldots,X_k)\xi_1\wedge\cdots\wedge\xi_{k_0}\wedge\zeta_1\wedge\cdots\wedge\zeta_{l_1}\wedge A_1^{p_1}\wedge\cdots\wedge A_m^{p_m}$$

with $k-k_1=k_0+l_1+m\le n$.
In the above formula $\mathcal{A}^{k_1-k_0}$ is an element of $\bigwedge_I^{k_1-k_0}(gl(n,\mathbf{R}))$ and S^{k_0} is a polynomial linear in the ξ_i and X_i and, by Proposition II.5.2, symmetric in the pairs (ξ_i,X_i), $1\le i\le k_0$.
S^{k_0} can therefore be considered as the restriction to the elements $\vec{X}_i\otimes\vec{\xi}_i$ of a k_0-linear, symmetric map $gl(n,\mathbf{R})^{k_0}\longrightarrow\mathbf{R}$.

By Proposition II.6.4 $\mathcal{S}^{k_0}$ must be invariant under the representation $\rho_\cdot$, which on monomials of the form $\vec{X}_i \otimes \vec{\xi}_i$ is equivalent to minus the adjoint representation of $gl(n, \mathbf{R})$ (cf. (II.80)). Hence $\mathcal{S}^{k_0}$ is an element of $\bigvee_I^{k_0}(gl(n, \mathbf{R}))$, the space of all k_0-linear, symmetric, Ad-invariant maps $gl(n, \mathbf{R})^{k_0} \longrightarrow \mathbf{R}$.
Observe that in (II.85) we omitted the vector symbol on the arguments of $\mathcal{A}^{k_1-k_0}$ since its Ad-invariance implies that its value doesn't depend on the chosen basis of TM_x resp. TM_x^*.
From (II.85) it is evident that the order of a cocycle $\mathcal{D}_m^k$ is

$$(2,\ldots,\underset{\underset{(k_0)}{\uparrow}}{2},1,\ldots,\underset{\underset{(k_1)}{\uparrow}}{1},0,\ldots,0;1,\ldots,\underset{\underset{(l_1)}{\uparrow}}{1},0,\ldots,0), \quad k-k_1 = k_0+l_1+m \leq n \quad \text{(II.86)}$$

In the next subsections we shall look for globally defined local cocycles $\mathcal{D}_m^{n+1}$ with symbol given by (II.85), in particular for those which are not coboundaries.

II.6.3 Cochains with values in $\mathcal{T}_1^1{}_A(M)$

To properly handle the quantities we shall work with in what follows, we need some intermediate concepts.

Analogously to $\Omega_A^\cdot(M)$ we can define $\mathcal{T}_q^p{}_A$, the space of tensor fields of type (p,q) depending locally on the gauge potential A (cf. Subsection II.5.1). The representation $\vartheta^\vee$ of $\mathcal{X}_\cdot(M)$ in $\Omega_A^\cdot(M)$ is easily generalized to a representation, denoted also by $\vartheta^\vee$, in $\mathcal{T}_q^p{}_A(M)$. Its associated coboundary operator $\delta^\vee$ acts on local k-cochains
$\tau^k \in \bigwedge^k(\mathcal{X}_\cdot(M); \mathcal{T}_q^p{}_A(M))$ as in (II.74).

Proposition II.6.6 *Let TM_1^1 be the tensor bundle of type (1,1) over M. We have*

$$TM_1^1 \simeq LM \times_{ad} gl(n, \mathbf{R})$$

where LM is the bundle of linear frames over M and ad is the adjoint representation of $GL(n, \mathbf{R})$ (cf. Subsection I.2.4).

Proof: by definition $gl(n, \mathbf{R})$ is the group of endomorphisms of $\mathbf{R}^n$, which is isomorphic to $\mathbf{R}^n \otimes \mathbf{R}^{n*}$. We have

$$ad_E(\vec{v} \otimes \vec{\omega}) = E(\vec{v} \otimes \vec{\omega})E^{-1} = (E\vec{v} \otimes \tilde{E}^{-1}\vec{\omega}), \quad \vec{v} \in \mathbf{R}^n, \ \vec{\omega} \in \mathbf{R}^{n*}$$

The proposition then follows from Definition I.2.12. □

We pointwise define a product in $\mathcal{T}_1^1(M) = Sec\, TM_1^1$ by setting

$$(T_1 \cdot T_2)(x) \doteq [u_x, \mathbf{T}_1(x)\mathbf{T}_2(x)], \quad T_i \in \mathcal{T}_1^1(M), \ u_x \in LM_x, \ x \in M$$

where $\mathbf{T}_i(x) \in gl(n, \mathbf{R})$ are the components of T_i with respect to the linear frame u_x. The components of the tensor field $T_1 \cdot T_2$ with respect to a local coordinate system are clearly given by

$$(T_1 \cdot T_2)^\mu_\nu(x) = T_1{}^\mu{}_\rho(x) T_2{}^\rho{}_\nu(x), \quad x \in M \qquad \text{(II.87)}$$

The product

$$[T_1, T_2] \doteq T_1 \cdot T_2 - T_2 \cdot T_1 \tag{II.88}$$

obviously makes $\mathcal{T}_1^1(M)$ into a Lie algebra.

Given an element S^p of $\bigvee_I^p(gl(n, \mathbf{R}))$ we denote by the same symbol the map

$$S^p : \mathcal{T}_1^1(M) \times \cdots \times \mathcal{T}_1^1(M) \longrightarrow C^\infty(M)$$

given by

$$S^p(T_1, \ldots, T_p)(x) = S^p(\mathbf{T}_1(x), \ldots, \mathbf{T}_p(x)), \quad x \in M \tag{II.89}$$

where $\mathbf{T}_i(x) \in gl(n, \mathbf{R})$ are the components of T_i with respect to *any* linear frame. The products (II.87) and (II.88) induce the products $\wedge$ (II.33) and $[\wedge]$ (cf. (I.15)) in the space of cochains with values in $\mathcal{T}_1^1{}_A(M)$.

Remember that we have

$$\tau^k \wedge \tau^l = (-1)^{kl} \tau^l \wedge \tau^k \tag{II.90}$$

$$\delta^\vee(\tau^k \wedge \tau^l) = \delta^\vee \tau^k \wedge \tau^l + (-1)^k \tau^k \wedge \delta^\vee \tau^l \tag{II.91}$$

The map (II.89) induces a map, also denoted by S^p,

$$S^p : \bigwedge^{k_1}(\mathcal{X}_\cdot(M); \mathcal{T}_1^1{}_A(M)) \times \cdots \times \bigwedge^{k_p}(\mathcal{X}_\cdot(M); \mathcal{T}_1^1{}_A(M)) \to \bigwedge^{k_1+\cdots+k_p}(\mathcal{X}_\cdot(M); C_A^\infty(M))$$

given by

$$S^p(\tau^{k_1}, \ldots, \tau^{k_p})(X_1, \ldots, X_{k_1+\cdots+k_p}) = \tag{II.92}$$
$$\doteq \frac{1}{k_1! \cdots k_p!} \sum_\pi sign\, \pi\, S^p(\tau^{k_1}(X_{\pi(1)}, \ldots), \ldots, \tau^{k_p}(\ldots, X_{\pi(k_1+\cdots+k_p)}))$$

Proposition II.6.7

$$\delta^\vee S^p(\tau^{k_1}, \ldots, \tau^{k_p}) = \sum_{i=1}^p (-1)^{k_1+\cdots+k_{i-1}} S^p(\tau^{k_1}, \ldots, \delta^\vee \tau^{k_i}, \ldots, \tau^{k_p})$$

Proof: locally we can write

$$S^p(\tau^{k_1}, \ldots, \tau^{k_p}) =$$
$$= \tau^{k_1\, \mu_1}_{\nu_1} \wedge \cdots \wedge \tau^{k_p\, \mu_p}_{\nu_p} S^p(\partial_{\mu_1} \otimes dx^{\nu_1}, \ldots, \partial_{\mu_p} \otimes dx^{\nu_p})$$

where $\tau^{k_i\, \mu_i}_{\nu_i}$ is a k_i-cochain with values in $C_A^\infty(M)$. The statement then follows from (II.91). □

Proposition II.6.8 *Let $S^p \in \bigvee_I^p(gl(n, \mathbf{R}))$. Then for any k-cochain τ^k*

$$\sum_{i=1}^p (-1)^{k(k_1+\cdots+k_{i-1})} S^p(\tau^{k_1}, \ldots, [\tau^k \wedge \tau^{k_i}], \ldots, \tau^{k_p}) = 0$$

Proof: let $\{E^\alpha_\beta,\ \alpha,\beta = 1,\dots,n\}$ be the canonical basis of $gl(n,\mathbf{R})$, i.e. $(E^\alpha_\beta)^\mu_\nu = \delta^\alpha_\nu \delta^\mu_\beta$. Then

$$\sum_{i=1}^{p}(-1)^{k(k_1+\cdots+k_{i-1})}\mathcal{S}^p(\tau^{k_1},\dots,[\tau^k \wedge \tau^{k_i}],\dots,\tau^{k_p}) =$$

$$= \sum_{i=1}^{p}(-1)^{k(k_1+\cdots+k_{i-1})}\tau^{k_1\,\mu_1}_{\nu_1}\wedge\cdots\tau^{k\,\mu}_{\nu}\wedge\tau^{k_i\,\mu_i}_{\nu_i}\cdots\wedge\tau^{k_p\,\mu_p}_{\nu_p}\mathcal{S}^p(E^{\nu_1}_{\mu_1},\dots,[E^\nu_\mu,E^{\nu_i}_{\mu_i}],\dots,E^{\nu_p}_{\mu_p}) =$$

$$= \tau^{k\,\mu}_{\nu}\wedge\tau^{k_1\,\mu_1}_{\nu_1}\cdots\wedge\tau^{k_p\,\mu_p}_{\nu_p}\sum_{i=1}^{p}\mathcal{S}^p(E^{\nu_1}_{\mu_1},\dots,[E^\nu_\mu,E^{\nu_i}_{\mu_i}],\dots,E^{\nu_p}_{\mu_p}) = 0$$

since $\mathcal{S}^p$ is Ad-invariant. □

II.6.4 The naturally defined differential cochains

Let M be a compact, connected, n-dimensional manifold. Henceforth we provide M with a linear connection Γ, which doesn't need to be Levi-Civita nor Riemannian.

Proposition II.6.9 *Let Γ be a linear connection of a manifold M and ∇_X the associated covariant derivative with respect to the vector field X on M. Then*

$$Q_X \doteq L_X - \nabla_X$$

is a tensor field of type (1,1) on M.

Proof: see page 235 of reference [26].

Proposition II.6.10 *The components of Q_X with respect to a local coordinate system are given by*

$$[Q_X]^\alpha_\beta = -\partial_\beta X^\alpha - X^\rho\Gamma^\alpha_{\rho\beta}$$

where $\Gamma^\alpha_{\rho\beta}$ are the components of the linear connection Γ.

Proof: it suffices to show that

$$[Q_X(Y)]^\alpha = (L_X Y - \nabla_X Y)^\alpha = -Y^\beta(\partial_\beta X^\alpha + X^\rho\Gamma^\alpha_{\rho\beta})$$

and this follows from (I.10) and (I.42). □

In the terminology of the preceding subsection the map

$$\begin{aligned} Q:\ \mathcal{X}.(M) &\longrightarrow T^1_{1A}(M) \\ X &\longmapsto Q_X \end{aligned} \qquad \text{(II.93)}$$

is thus a local 1-cochain with values in $T^1_{1A}(M)$. Observe that Q can depend on the gauge potential A only if the linear connection Γ does. We shall see in the next section examples thereof. In this subsection we limit ourselves to A-independent 1-cochains Q.

The symbol of Q is given by

$$\sigma_Q(\xi, X) = -\vec{X} \otimes \vec{\xi} \tag{II.94}$$

As its order is $(1; \cdots)$ we call it a 1-*differential* cochain.

Denote by S the local 2-cochain with values in $\mathcal{T}^1_{1A}(M)$ defined by

$$S = \delta^{\vee} Q - \tfrac{1}{2}[Q \wedge Q] \tag{II.95}$$

Since Q doesn't depend on the gauge potential we can write

$$S(X_1, X_2) = Q_{[X_1, X_2]} - [Q_{X_1}, Q_{X_2}], \quad X_1, X_2 \in \mathcal{X}_c(M)$$

Observe the similarity of (II.95) with the structure equation of Maurer-Cartan for the curvature form (cf. Proposition I.3.4).

Lemma II.6.1 *Let $\Gamma^{\rho}_{\mu\nu}$ be the components of a linear connection Γ. Under an infinitesimal diffeomorphism $X \in \mathcal{X}_c(M)$ they transform according to*

$$\begin{aligned} \Gamma'^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\mu\nu} &\doteq (L_X \Gamma)^{\rho}_{\mu\nu} = \\ &= X^{\sigma}\partial_{\sigma}\Gamma^{\rho}_{\mu\nu} + \Gamma^{\rho}_{\sigma\nu}\partial_{\mu}X^{\sigma} + \Gamma^{\rho}_{\mu\sigma}\partial_{\nu}X^{\sigma} - \Gamma^{\sigma}_{\mu\nu}\partial_{\sigma}X^{\rho} + \partial_{\mu}\partial_{\nu}X^{\rho} \end{aligned}$$

Proof: see reference [40].

Proposition II.6.11 *The components of the tensor field $S(X_1, X_2)$ are given by*

$$S(X_1, X_2)^{\alpha}_{\beta} = -X^{\rho}_1 (L_{X_2}\Gamma)^{\alpha}_{\rho\beta} + X^{\rho}_2 (L_{X_1}\Gamma)^{\alpha}_{\rho\beta} - R(X_1, X_2)^{\alpha}_{\beta}, \quad X_1, X_2 \in \mathcal{X}_c(M)$$

where R is the curvature tensor field of the linear connection Γ.

Proof: by making use of Proposition II.6.10 and (II.87) we can straightforwardly compute $S(X_1, X_2)^{\mu}_{\nu}$. The proposition then follows by comparing the result with Lemma II.6.1 and Proposition I.3.7. □

From the above proposition and Lemma II.6.1 it follows that the symbol of the 2-cochain S is

$$\sigma_S(\xi_1; X_1, X_2) = (\vec{X}_1 \otimes \vec{\xi}_1) < \xi_1, X_2 > \tag{II.96}$$

S is thus of order $(2, 0; \cdots)$; i.e. it is a 2-differential cochain.

Since $[Q \wedge [Q \wedge Q]] \equiv 0$, from (II.95) we get

$$\delta^{\vee} S = [Q \wedge \delta^{\vee} Q] = [Q \wedge S] \tag{II.97}$$

Consider the following $2p$-cochain with values in $C^{\infty}(M)$

$$\mathcal{S}^p(S) \doteq \mathcal{S}^p(S, \ldots, S) \tag{II.98}$$

where $\mathcal{S}^p \in \bigvee^p_I(gl(n, \mathbf{R}))$.

Its symbol is

$$\mathcal{S}^p(X_1 \otimes \xi_1, \ldots, X_p \otimes \xi_p)\, \imath(X_{p+1}, \ldots, X_{2p})\xi_1 \wedge \cdots \wedge \xi_p \tag{II.99}$$

We have thus found a local cochain whose symbol has the form (II.85) necessary to be a cocycle. In fact,

$$\delta^\vee S^p(S) = \sum_{i=1}^{p} S^p(S,\ldots,\underset{(i)}{\underset{\uparrow}{\delta^\vee S}},\ldots,S) = \sum_{i=1}^{p} S^p(S,\ldots,\underset{(i)}{\underset{\uparrow}{[Q,S]}},\ldots,S) = 0,$$

but,

Proposition II.6.12

$$S^p(S) = \delta^\vee p \int_0^1 dt\, S^p(Q,S_t,\ldots,S_t)$$

where $S_t = t\,\delta^\vee Q - \frac{t^2}{2}[Q\wedge Q] = t\,S + \frac{1}{2}(t-t^2)[Q\wedge Q]$, $t\in\mathbf{R}$.

Proof: analogous to the proof of Proposition II.3.4.

Since $(L_X\Gamma)$ is the only naturally and globally defined quantity on M whose local expression contains a term like $\partial_\mu\partial_\nu X^\rho$, we obtain

Corollary II.6.1 *There are no non-trivial cocycles of order* $(2,\ldots,2,0,\ldots,0\,;\cdots)$*, i.e.*

$$H_{2\text{-}diff}(\mathcal{X}.(M);C_A^\infty(M)) = 0$$

It remains to us to consider the 1-differential cochains whose symbols are given by the first factor in (II.85).
Observe that the 2-cochain S_t defined in Proposition II.6.12 is of order $(1,1\,;\cdots)$. Its symbol is the symbol of $\frac{1}{2}(t-t^2)[Q\wedge Q]$, i.e.

$$(t-t^2)[\vec{X}_1\otimes\vec{\xi}_1,\vec{X}_2\otimes\vec{\xi}_2]$$

It follows that the $(2p-1)$-cochain

$$TS_Q^p \doteq p\int_0^1 dt S^p(Q,S_t,\ldots,S_t) \qquad \text{(II.100)}$$

is of order $(1,\ldots,\underset{(2p-1)}{\underset{\uparrow}{1}};\cdots)$ and that its symbol is given by

$$p\int_0^1 dt(t-t^2)^{p-1}\frac{-1}{2^{p-1}}\sum_\pi sign\,\pi S^p(U_{\pi(1)},[U_{\pi(2)},U_{\pi(3)}],\ldots,[U_{\pi(2p-2)},U_{\pi(2p-1)}]) \qquad \text{(II.101)}$$

where $U_i = \vec{X}_i\otimes\vec{\xi}_i$, $i=1,\ldots,2p-1$.
Now for any Lie algebra L there exists a linear map

$$\rho_L : \mathsf{V}_I^p(L) \longrightarrow \Lambda_I^{2p-1}(L),$$

called the *Cartan map*, which is explicitly given by [22]

$$(\rho_L S^p)(E_1,\ldots,E_{2p-1}) = \tag{II.102}$$
$$= \frac{(-1)^{p-1}(p-1)!}{2^{p-1}(2p-1)!}\sum_\pi sign\,\pi S^p(E_{\pi(1)},[E_{\pi(2)},E_{\pi(3)}],\ldots,[E_{\pi(2p-2)},E_{\pi(2p-1)}])$$

We thus obtain that the symbol of TS^p_Q is

$$(-1)^p p!\,(\rho_{gl}S^p)(X_1\otimes\xi_1,\ldots,X_{2p-1}\otimes\xi_{2p-1}), \tag{II.103}$$

which has exactly the same form as the first factor in (II.85).
Unfortunately, by Proposition II.6.12,

$$\delta^\vee TS^p_Q = S^p(S)\neq 0$$

The case $p=1$ deserves special attention.
By using Proposition II.6.11 and Lemma II.6.1 it is straightforward to compute $S^1(S)=Tr(S)$. Namely

$$Tr\,S(X_1,X_2) = L_{X_1}Tr\,Q_{X_2} - L_{X_2}Tr\,Q_{X_1} + Tr\,R(X_1,X_2)$$

Denote by Tr_Q the cochain given by $Tr_Q(X)\doteq Tr\,Q_X$. From the above formula we get

$$(\delta^\vee Tr_Q)(X_1,X_2) = L_{X_1}Tr\,Q_{X_2} - L_{X_2}Tr\,Q_{X_1} + Tr\,R(X_1,X_2) \tag{II.104}$$

This result will be used later on to construct non-trivial (n+1)-cocycles.

Since Q_X is the only naturally and globally defined quantity on M whose local expression contains a term like $\partial_\mu X^\nu$, we can state the following

Corollary II.6.2 $\quad H^p_{1\text{-}diff}(\mathcal{X}.(M);C^\infty_A(M))=0,\quad p\neq n+1$

II.6.5 The 0-differential cochains

By definition the 0-differential cochains are those of order $(0,\ldots,0\,;\,\cdots)$.
From (II.85) we see that the 0-diff. cocycles must have the following form

$$\imath(X_1,\ldots,X_k)\mathcal{D}^k_{poly}(A)\doteq \imath^k\mathcal{D}^k_{poly}(A;X_1,\ldots,X_k) \tag{II.105}$$

where $\mathcal{D}^k_{poly}(A)$ is a polynomial k-form depending on the gauge potential A, i.e. an element of $\mathsf{V}^m_{loc}(\Omega^1(M;LG);\Omega^k(M))$ (cf. Subsections II.3.2 and II.5.1).
Similarly to (II.44) we have

$$(\vartheta^\vee(X)\mathcal{D}^k_{poly})(A) = L_X\,\mathcal{D}^k_{poly}(A) \tag{II.106}$$

Since $\imath(X_1,\ldots,X_{k+1})\mathcal{D}^k_{poly}(A)\equiv 0$, Lemma II.3.1 implies

$$(\delta^\vee\imath^k\mathcal{D}^k_{poly}(A))(X_1,\ldots,X_{k+1}) = \imath(X_1,\ldots,X_{k+1})\,d\,\mathcal{D}^k_{poly}(A) \tag{II.107}$$

Therefore, to give rise to a cocycle the k-form $\mathcal{D}^k_{poly}(A)$ must be closed. From (II.107) it is also evident that $\imath^k\mathcal{D}^k_{poly}(A)$ is a non-trivial cocycle only if $\mathcal{D}^k_{poly}(A)\in H^k_{deRham}(M)$. We proved

Corollary II.6.3 $\quad H_{0\text{-}diff}(\mathcal{X}.(M); C^\infty_A(M)) = H^A_{deRham}(M)$

where $H^A_{deRham}(M)$ *is the subalgebra of* $H_{deRham}(M)$ *consisting of polynomial forms in the gauge potential* A. *In particular we have*

$$H^{n+1}_{0\text{-}diff}(\mathcal{X}.(M); C^\infty_A(M)) = 0, \quad n = dim\, M$$

II.6.6 The 1-differential (n+1)-cocycles

Consider the (n+1)-cochain $\imath^n \mathcal{D}^n_{poly}(A) \wedge Tr_Q$. As $\mathcal{D}^n_{poly}(A)$ is closed, we obtain

$$\delta^\vee(\imath^n \mathcal{D}^n_{poly}(A) \wedge Tr_Q) = (-1)^n \imath^n \mathcal{D}^n_{poly}(A) \wedge \delta^\vee Tr_Q$$

Proposition II.6.13 *Let* τ *be any n-form on* M. *Then the (n+2)-cochain* $\imath^n \tau \wedge \delta^\vee Tr_Q$ *vanishes identically.*

Proof: see the Appendix.

The above constructed (n+1)-cochain is therefore a cocycle. That it is non-trivial is evident, since to construct the polynomial form $\mathcal{D}^n_{poly}(A)$ no use has been made of the linear connection of M.

Corollary II.6.4 $\quad H^{n+1}_{1\text{-}diff}(\mathcal{X}.(M); C^\infty_A(M)) \simeq \Omega^n_{A\,poly}(M)$

As can be verified by a direct calculation, it is possible to ascend eq. (II.66) starting with the cocycle $\imath^n \mathcal{D}^n_{poly}(A) \wedge Tr_Q$ only if the curvature of the linear connection Γ of M satisfies

$$Tr\, R(X_1, X_2) = 0, \quad \forall X_i \in \mathcal{X}.(M)$$

In that case we find the following n-form which integrated can give rise to a possible base space anomaly

$$\mathcal{D}^{1,n}_{poly}(A; X) = \mathcal{D}^n_{poly}(A)\, Tr\, Q_X, \quad X \in \mathcal{X}.(M) \qquad (II.108)$$

Notice that the above condition on the curvature is satisfied if Γ is a Riemannian connection [21].

In the next sections we shall find more 1-differential (n+1)-cocycles by restricting the cochains to subalgebras of $\mathcal{X}.(M)$.

II.7 Affine anomalies

Let M be a compact, n-dimensional, orientable manifold equipped with a linear connection Γ.
In this section we find a special class of cocycles belonging to $\bigwedge^{n+1}(a(M); C_A^\infty(M))$, where $a(M)$ denotes the subalgebra of $\mathcal{X}.(M)$ consisting of infinitesimal affine transformations of M, and ascend equation (II.66), thus finding a local n-form giving rise to a possible base space anomaly associated to the infinitesimal affine transformations or briefly an *affine anomaly*.

II.7.1 The $a(M)$-restricted, 1-differential cochains

Definition II.7.1 *A vector field X on M is called an infinitesimal affine transformation if $(L_X\Gamma) = 0$, where Γ represents the components of the linear connection of M.*

We denote by $a(M)$ the space of all infinitesimal affine transformations of M. Clearly $a(M)$ is a subalgebra of $\mathcal{X}.(M)$. Observe that $a(M)$ implicitly depends on the linear connection Γ of M.

Let S be the 2-cochain with values in $T^1_{1A}(M)$ defined in (II.95). In view of Proposition II.6.11 we have

$$S(X_1, X_2) = -R(X_1, X_2) \quad \text{for } X_i \in a(M)$$

where R is the curvature tensor field of Γ.
Briefly,

$$S|_{a(M)} = (\delta^\vee Q - \tfrac{1}{2}[Q \wedge Q])|_{a(M)} = -R|_{a(M)} \tag{II.109}$$

Whence,

$$TS^p_Q|_{a(M)} = (-1)^{p-1}p\int_0^1 dt\, S^p(Q, tR + \tfrac{1}{2}(t^2 - t)[Q \wedge Q], \dots)|_{a(M)} \tag{II.110}$$

where TS^p_Q is the $(2p-1)$-cochain (II.100).

Definition II.7.2 *Let X be a vector field on M which generates the 1-parameter group of diffeomorphisms φ_t. Its natural lift, denoted by $\tilde{X}$, is the vector field on LM induced by the 1-parameter group of diffeomorphisms φ_{*t} which maps the linear frame $u_x = \{e_i(x)\}$, $i = 1, \dots, n$, into $\varphi_{*t}(u_x) = \{(d\varphi_t)_x e_i(x)\}$, $x \in M$.*

Proposition II.7.1 *i) The natural lift $\tilde{X}$ is invariant, i.e. $a_{g*}\tilde{X} = \tilde{X}$, $\forall g \in GL(n, \mathbf{R})$*

ii) $\tilde{X}$ and X are π-related, where π is the projection $LM \longrightarrow M$ (cf. Definition I.1.12).

iii) $\widetilde{[X_1, X_2]} = [\tilde{X}_1, \tilde{X}_2]$, $X_i \in \mathcal{X}.(M)$

Proof: see reference [26].

Let ω be the connection form associated to the linear connection Γ. By definition ω is a $gl(n,\mathbf{R})$-valued 1-form on LM. Denote by Ω its curvature form.

Proposition II.7.2 *The companents of the tensor field $Q_X = L_X - \nabla_X$ resp. $R(X,Y)$ with respect to the linear frame u_x are given by*

$$[Q_X]^i_j(x) = -\omega^i_j(u_x;\tilde{X}) \qquad \text{resp.} \quad R(X,Y)^i_j(x) = \Omega^i_j(u_x;\tilde{X},\tilde{Y}), \quad x \in M$$

where $\tilde{X}$ and $\tilde{Y}$ are the natural lifts of X and $Y \in \mathcal{X}(M)$.

Proof: see the Appendix.

As S^p is Ad-invariant we can write

$$\begin{aligned}\pi^*\left[TS^p_Q(X_1,\ldots,X_{2p-1})\right] &= \\ &= (-1)^p \imath(\tilde{X}_1,\ldots,\tilde{X}_{2p-1})\, p\int_0^1 dt\, S^p(\omega,\Omega_t,\ldots,\Omega_t) = \\ &= (-1)^p \imath^{2p-1} TS^p(\omega;\tilde{X}_1,\ldots,\tilde{X}_{2p-1}), \quad X_i \in a(M)\end{aligned} \tag{II.111}$$

where π is the projection $LM \to M$ and $\Omega_t = t\,\Omega + \frac{1}{2}(t^2-t)[\omega\wedge\omega]$.
Observe that $TS^p(\omega)$ is the Chern transgression form on the bundle LM (cf. Definition II.3.5).
By Proposition II.6.12 and (II.109) we have

$$\delta^\vee TS^p_Q|_{a(M)} = (-1)^p S^p(R)|_{a(M)} \tag{II.112}$$

For $S^p \in \bigvee^p_I(gl(n,\mathbf{R}))$, $S^p(R)$ is a 2p-form on M which is a combination of characteristic classes of the tangent bundle TM [21]. Assuming that M is of even dimension and setting $p = \frac{n}{2}+1$ in (II.112) we obtain

$$\delta^\vee TS^{\frac{n}{2}+1}_Q|_{a(M)} = (-1)^{\frac{n}{2}+1} S^{\frac{n}{2}+1}(R)|_{a(M)} \equiv 0 \tag{II.113}$$

The (n+1)-cochain $TS^{\frac{n}{2}+1}_Q$ is therefore a cocycle of $\delta^\vee$ when restricted to $a(M)$. That it is non-trivial follows from the following argument. Its order is $(1,\ldots,\underset{(n+1)}{\underset{\uparrow}{1}};0)$, therefore by equation (II.77) it may only be the coboundary of a cochain of order $(1,\ldots,\underset{(n)}{\underset{\uparrow}{1}};0)$, but this is excluded by the discussion following Proposition II.6.5.

Remember that the Chern transgression $TS^{\frac{n}{2}+1}(\omega)$ was used in Subsection II.3.2 to construct the gauge anomaly of a theory based on a trivial bundle P. As it is well known Einstein's theory of gravity can be geometrically interpreted as a gauge theory based on the bundle LM (or OM^+ if spinors are present) which by definition is trivial only if M is parallelizable. The gauge anomaly of this special gauge theory is called the *Lorentz anomaly*.

As we have already said in Subsection II.4.1 a classical theory formulated in terms of a Lagrangian $\mathcal{L}$ is automatically invariant under the diffeomorphism group of M. The possible anomaly associated to this invariance is the gravitational anomaly. Due to the tight relation between the bundle LM and its base space M, in particular between an infinitesimal diffeomorphism X and its natural lift $\tilde{X}$, the gravitational anomaly, too, can be constructed starting with the transgression form $TS^{\frac{n}{2}+1}(\omega)$, where ω represents a linear connection of M. It suffices to replace the vertical invariant vector fields Z_i of Subsection II.3.2 with the natural lifts $\tilde{X}_i$ [7]. As this fact suggests there is a relation between the gravitational and the Lorentz anomaly. Indeed, for a parallelizable manifold, one can shift the anomaly from the gravitational to the Lorentz form by adding a Bardeen-Zumino counter term to the Lagrangian $\mathcal{L}$ [36,1,7]. To be definite we shall assume that all theories we consider are free from gravitational anomalies.

As can be seen from (II.111) the cocycle $\pi^* TS_Q^{\frac{n}{2}+1}|_{a(M)}$ is nothing else than the restriction to $a(M)$ of the (n+1)-cochain $\imath^{n+1}TS^{\frac{n}{2}+1}(\omega)$ out of which the gravitational anomaly is constructed. Therefore, the form of the affine anomaly we are looking for must be the same as the one of the gravitational anomaly restricted to $a(M)$.

It has to be emphasized that this doesn't mean that the affine anomaly *is* a gravitational anomaly. First of all the infinitesimal transformations of the fields (II.55) we assume are not given by the (natural) action of the diffeomorphism group; we can therefore not say a priori that the two kinds of anomalies coincide. In Subsection II.7.3 we shall give the only (at least to us) known example of a theory possessing an affine anomaly. Unfortunately for this theory the affine anomaly is equivalent to a Lorentz anomaly and thus, according to the above, equivalent to a gravitational anomaly. Secondly, the gravitational anomaly is usually computed by taking as linear connection the Levi-Civita connection of the assumed parallelizable Riemannian manifold M; as we shall see below our construction of the affine anomaly is not limited to this case and moreover it enables us to introduce a gauge potential-dependence.

II.7.2 The descent equation for the $a(M)$-restricted 1-diff. cochains

To ascent equation (II.66) we shall mainly work with quantities, as the Chern transgression form, defined on the total space LM. We begin this subsection by stating two lemmas valid for any principal fibre bundle $(P \xrightarrow{\pi} M, G)$.

Let h be the horizontal projection (I.35) associated to a connection Γ in P. We denote by h^* the projection

$$h^* : \Omega^*(P;F) \longrightarrow \Omega^*(P;F)$$
$$\psi \longmapsto h^*\psi$$

given by $h^*\psi(Z_1,\ldots,Z_q) \doteq \psi(hZ_1,\ldots,hZ_q), \quad \psi \in \Omega^q(P;F), \; Z_i \in \mathcal{X}(P)$.
The exterior covariant derivative D (I.37) associated to the connection Γ can thus

be expressed as

$$D = h^{*} \circ d$$

Let ω be the connection form on P associated to Γ and $\{E_i,\ i = 1, \ldots, dim\, G\}$ a basis of the Lie algebra LG.

Lemma II.7.1 *Denote by $\Omega = \Omega^i E_i$ the curvature form of ω. Then*

$$D \circ h^{*} = h^{*} \circ [d - \Omega^i \wedge \imath(Z_{E_i})]$$

where Z_{E_i} is the fundamental vector field induced by $E_i \in LG$ (cf. Definition I.2.2).

Proof: see the Appendix.

Given an element $\mathcal{S}^p \in \bigvee_I^p(LG)$ we denote by $T\mathcal{S}^p_{\omega;l_1,\ldots,l_m}$, $0 \leq m \leq p-1$, the following invariant $(2p-2m-1)$-form on P

$$T\mathcal{S}^p_{\omega;l_1,\ldots,l_m} \doteq (-1)^p \frac{p!}{(p-m-1)!} \int_0^1 dt\, (1-t)^m\, \mathcal{S}^p(\omega, \Omega_t, \ldots, \Omega_t, E_{l_1}, \ldots, E_{l_m}) \tag{II.114}$$

Lemma II.7.2

$$d\, T\mathcal{S}^p_{\omega;l_1,\ldots,l_m} = \sum_{i=1}^{m} \imath(Z_{E_{l_i}})\, T\mathcal{S}^p_{\omega;l_1,\ldots,\hat{l}_i,\ldots,l_m}; \quad m \geq 1$$

Proof: see the Appendix.

Suppose now we are given another connection $\tilde{\Gamma}$ in P with connection form $\tilde{\omega}$.
In Subsection II.3.2 we defined the representation $\tilde{\vartheta}$ of $\mathcal{X}_\cdot(P)$ in $\Omega^q_\omega(P)$, the space of local q-forms on P depending on a connection form ω (cf. Proposition II.3.2).

Given the (2p-2m-1)-form (II.114), a k-cochain, denoted by $T\mathcal{S}^{k,2p-k-1}_{\omega,\tilde{\omega};m}$, belonging to $\bigwedge^k(\mathcal{X}_\cdot(P); \Omega^{2p-k-1}_{\tilde{\omega}}(P))$ is defined by

$$\begin{aligned} T\mathcal{S}^{k,2p-k-1}_{\omega,\tilde{\omega};m}(Z_1, \ldots, Z_k) &= \\ &\doteq \frac{1}{m!} \Omega^{l_1} \wedge \cdots \wedge \Omega^{l_m} \wedge h^{*}[\imath(Z_1, \ldots, Z_k) T\mathcal{S}^p_{\tilde{\omega};l_1,\ldots,l_m}] = \\ &= \frac{1}{m!} \Omega^{l_1} \wedge \cdots \wedge \Omega^{l_m} \wedge h^{*}[\imath^k T\mathcal{S}^p_{\tilde{\omega};l_1,\ldots,l_m}(Z_1, \ldots, Z_k)], \quad Z_i \in \mathcal{X}_\cdot(P) \end{aligned} \tag{II.115}$$

where Ω and h^{*} are the curvature form and the projection associated to Γ, and $T\mathcal{S}^p_{\tilde{\omega};l_1,\ldots,l_m}$ is constructed out of the connection form $\tilde{\omega}$.
Clearly, the k-cochain (II.115) doesn't vanish identically only if $1 \leq k \leq 2p-2m-1$.

Lemma II.7.3 *Let $\tilde{\delta}$ be the coboundary operator associated to the representation $\tilde{\vartheta}$ of $\mathcal{X}_\cdot(P)$. Then*

$$\begin{aligned} D \sum_{m=0}^{p-1} T\mathcal{S}^{k,2p-k-1}_{\omega,\tilde{\omega};m}(Z_1, \ldots, Z_k) &= (-1)^{p+k}\, \imath(Z_1, \ldots, Z_k) \mathcal{S}^p(\tilde{\Omega}) + \\ &+ (-1)^{k+1} \sum_{m=0}^{p-1} \frac{1}{m!} \Omega^{l_1} \wedge \cdots \wedge \Omega^{l_m} \wedge h^{*}[(\tilde{\delta}\, \imath^{k-1} T\mathcal{S}^p_{\tilde{\omega};l_1,\ldots,l_m})(Z_1, \ldots, Z_k)] \end{aligned}$$

Proof: see the Appendix.

We now return to the case $P = LM$ and restrict the cochains $TS^{k,q}_{\omega,\tilde{\omega};m}$ to the subalgebra of $\mathcal{X}_-(LM)$ consisting of natural lifts of infinitesimal affine transformations of M.

Observe that the q-form $TS^{k,q}_{\omega,\tilde{\omega};m}(\tilde{X}_1, \dots, \tilde{X}_k)$ is invariant and by definition horizontal. By Proposition I.3.1 it can thus be identified with a form on M.

Generally the connection form $\tilde{\omega}$ is given by $\omega + \kappa$, where κ is a horizontal, ad-equivariant, $gl(n, \mathbf{R})$-valued 1-form on LM. Any tensor field K of type (1,2) on M determines such a 1-form κ (and viceversa) by

$$\kappa(u_x; w) \doteq \mathbf{K}(x; d\pi_{u_x} w, \cdot), \quad u_x \in LM_x, \quad w \in T\,LM_{u_x}$$

where $\mathbf{K}(x; d\pi_{u_x} w, \cdot)$ are the components of the tensor $K(x; d\pi_{u_x} w, \cdot)$ of type (1,1) with respect to the linear frame u_x (cf. Proposition II.6.6).

In what follows we shall identify K with a tensor-valued 1-form on M [21] by setting

$$K(x; v) \doteq K(x; v, \cdot), \quad v \in TM_x, \tag{II.116}$$

and by abuse of notation simply write $\kappa = \pi^* K$.

In view of Proposition II.7.2 and the fact that $h^*\tilde{\omega} = \kappa = \pi^* K$, we can write (cf. eq. (II.111))

$$TS^{k,q}_{\omega,\tilde{\omega};m}(\tilde{X}_1, \dots, \tilde{X}_k) = \pi^* [TS^{k,q}_{Q,K;m}(X_1, \dots, X_k) \tag{II.117}$$

where the k-cochain $TS^{k,q}_{Q,K;m}$ belongs to $\bigwedge^k(a(M); \Omega^q(M))$.

Lemma II.7.4 *Let K be a tensor field of type (1,2) on M and κ the associated horizontal, ad-equivariant, $gl(n, \mathbf{R})$-valued 1-form on LM. Then to $L_X K$ is associated $L_{\tilde{X}}\kappa$, where $\tilde{X}$ is the natural lift of $X \in \mathcal{X}(M)$.*

Proof: see the Appendix.

Assume now the tensor field K belongs to T^1_{2A} (cf. Subsection II.6.3), and that it verifies

$$(\vartheta^\vee(X)K)(A) = L_X K(A), \quad \forall X \in a(M) \tag{II.118}$$

Since $L_{\tilde{X}}\omega = 0$ for $X \in a(M)$ [26], by Lemma II.7.4 and condition (II.118) we have

$$L_{\tilde{X}}\tilde{\omega}(A) = L_{\tilde{X}}\kappa(A) = (\vartheta^\vee(X)\kappa)(A), \quad X \in a(M)$$

Therefore by Proposition II.7.1 *iii*) we can write for such a A-dependent connection form $\tilde{\omega}$

$$\frac{1}{m!}\Omega^{l_1} \wedge \cdots \wedge \Omega^{l_m} \wedge h^*[\bar{\delta}\,\imath^{k-1} TS^p_{\tilde{\omega}(A);l_1,\dots,l_m}(\tilde{X}_1, \dots, \tilde{X}_k)] =$$
$$= \pi^* [\delta^\vee TS^{k-1,2p-k}_{Q,K(A);m}(X_1, \dots, X_k)]$$

Denote by $TS^{k,n+1-k}_Q(A)$ the following k-cochain belonging to $\bigwedge^k(a(M); \Omega^{n+1-k}_A(M))$

$$TS^{k,n+1-k}_Q(A) = \sum_{m=0}^{\frac{n}{2}} TS^{k,n+1-k}_{Q,K(A);m}$$

where $n = dim\, M$.

Proposition II.7.3 *Let $K(A)$ be a tensor field of type (1,2) on M, depending on the gauge potential A and satisfying condition (II.118). Then the k-cochains $TS_Q^{k,n+1-k}(A)$, $1 \leq k \leq n+1$, verify the descent equation*

$$d\, TS_Q^{k,n+1-k}(A) = (-1)^{k+1}\, \delta^\vee\, TS_Q^{k-1,n+2-k}(A)$$

Proof: for $p = \frac{n}{2}+1$ we have $S^p(\tilde{\Omega}) = \pi^* S^p(\tilde{R}) \equiv 0$. Therefore by Lemma II.7.3 and eq. (II.117)

$$D\,\pi^* TS_Q^{k,n+1-k}(A; X_1, \ldots, X_k) = (-1)^{k+1}\pi^*[\delta^\vee\, TS_Q^{k-1,n+2-k}(A; X_1, \ldots, X_k)]$$

Lemma 2 of Chapter XII of reference [27] states that for any q-form φ on M

$$D\,\pi^*\varphi = d\,\pi^*\varphi = \pi^* d\,\varphi$$

The assertion then follows by noting that π^* is injective. □

For $k = n+2$ the statement of Proposition II.7.3 reads

$$\delta^\vee\, TS_Q^{n+1,0}(A) = 0,$$

with (cf. (II.110))

$$\begin{aligned} TS_Q^{n+1,0}(A) &= TS^{n+1,0}_{Q,K(A);0} = \\ &= (-1)^{\frac{n}{2}}(\tfrac{n}{2}+1)\int_0^1 dt\, S^{\frac{n}{2}+1}(\hat{Q}, t\,\tilde{R} + \tfrac{1}{2}(t^2-t)[\hat{Q}\wedge\hat{Q}], \ldots) \end{aligned} \quad \text{(II.119)}$$

where $\hat{Q}_X = Q_X - K(A;X)$ and $\tilde{R}$ is the curvature tensor field of the linear connection $\tilde{\Gamma}(A)$.
Note that for $K \equiv 0$ the (n+1)-cocycle (II.119) is equivalent to the restricted cocycle $TS_Q^{\frac{n}{2}+1}|_{a(M)}$ (cf. (II.110)). It can actually be proved that for a tensor field $K(A)$ verifying (II.118) we have

$$\hat{S}|_{a(M)} = (\delta^\vee\hat{Q} - \tfrac{1}{2}[\hat{Q}\wedge\hat{Q}])|_{a(M)} = -\tilde{R}|_{a(M)} \quad \text{(II.120)}$$

which is identical to (II.109). Therefore, in general

$$TS_Q^{n+1,0}(A) = TS_{\hat{Q}}^{\frac{n}{2}+1}|_{a(M)}$$

Is the cocycle $TS_Q^{n+1,0}(A)$ non-trivial? To answer this question we must know its order and use eq. (II.77) to check if it can not be a coboudary. Unfortunately, due to its restriction to $a(M)$ and its A-dependence, its order can not be directly read from (II.119) but must be determined case by case.
Assuming that $TS_Q^{n+1,0}(A)$ is not trivial, a possible unintegrated, affine anomaly is therefore given by the n-form $TS_Q^{1,n}(A; X)$, $X \in a(M)$. Explicitly

$$TS_Q^{1,n}(A;X) = \sum_{m=0}^{\frac{n}{2}} TS^{1,n}_{Q,K(A);m}(X) = (-1)^{\frac{n}{2}} \sum_{m=0}^{\frac{n}{2}} \frac{(\frac{n}{2}+1)!}{m!\,(\frac{n}{2}-m)!} \int_0^1 dt\,(1-t)^m$$

$$\{\mathcal{S}^{\frac{n}{2}+1}(Q_X, t\hat{R} + \tfrac{1}{2}(t^2-t)[K \wedge K], \ldots, \underset{(\frac{n}{2}-m+2)}{\underset{\uparrow}{R}}, \ldots, R)$$

$$- m\,\mathcal{S}^{\frac{n}{2}+1}(K, t\hat{R} + \tfrac{1}{2}(t^2-t)[K \wedge K], \ldots, \underset{(\frac{n}{2}-m+2)}{\underset{\uparrow}{\imath_X R}}, \ldots, R) \tag{II.121}$$

$$-(\tfrac{n}{2}-m)\mathcal{S}^{\frac{n}{2}+1}(K, (t^2-t)[Q_X, K], t\hat{R} + \tfrac{1}{2}(t^2-t)[K \wedge K], \ldots, \underset{(\frac{n}{2}-m+2)}{\underset{\uparrow}{R}}, \ldots, R)\}$$

If $K \equiv 0$ then the above expression simplifies to

$$TS_Q^{1,n}(X) = (-1)^{\frac{n}{2}}(\frac{n}{2}+2)\,\mathcal{S}^{\frac{n}{2}+1}(Q_X, R, \ldots, R) \tag{II.122}$$

We shall discuss in the next subsection the physical implications of the affine anomaly (II.121).

II.7.3 Physical interpretation of the affine anomaly

Consider a gauge theory based on a trivial bundle $P = M \times G$ whose base space M is a compact, 2k-dimensional spin manifold without boundary. As usual in a physical theory we assume that M is provided with the Levi-Civita connection.

Definition II.7.3 *A vector field X on a Riemannian manifold M is called an infinitesimal isometry (or a Killing vector field) if $L_X g = 0$, where g is the metric tensor of M.*

We denote by $\imath(M)$ the space of all infinitesimal isometries of M. $\imath(M)$ is a subalgebra of $\mathfrak{X}_-(M)$.

Proposition II.7.4 *Let M be a compact Riemannian manifold equipped with the Levi-Civita connection. Then $a(M) = \imath(M)$.*

Proof: see Corollary 3.9 of Ch.VI of [26].

From the above Proposition it follows that the anomaly we found in the preceding subsection could appear in a theory invariant under the infinitesimal isometries. Are there any classical field theories which possess such a symmetry and which can also be anomalous?

Since by assumption all theories we consider are invariant under *any* infinitesimal diffeomorphism provided one transforms the fields according to their geometrical character, the invariance should result from varying some of the fields regardless

of their intrinsic nature.
An example thereof is the chiral $SU(r)$-gauge theory whose Lagrangian $\mathcal{L}$ is locally given by (cf. (II.16))

$$\mathcal{L}|_{U_\alpha} = \tfrac{i}{2}\,\bar{\psi}_{+\alpha}\gamma^i\, {}^*e^i \wedge (\, d - \tfrac{1}{4}\sigma^j_k\Gamma^k_j + A\,)\psi_{+\alpha} + h.c. \tag{II.123}$$

As usual $\psi_{+\alpha} = (\frac{1+\gamma_5}{2})\psi_{M\alpha} \otimes \psi_P$ denotes a right-handed spinor field, $\psi_{M\alpha}$ being the pull-back by a local section $o_\alpha = \{E_{i\alpha}\}$, $i = 1, \ldots, n$, of OM^+ over $U_\alpha \subset M$ of an equivariant map $\tilde{\psi}_M : OM^+ \to \mathbf{C}^{2d}$, $d = 2^{\frac{n}{2}-1}$ (see Definition I.4.4), and ψ_P the pull-back by a global section of $P = M \times SU(r)$ of an equivariant map $\tilde{\psi}_P : P \to \mathbf{C}^r$ (see (II.13)).
In (II.123) ${}^*e_\alpha = {}^*e^i_\alpha \hat{e}_i$ is the Hodge dual of the pull-back by o_α of the solder form θ of LM (see Definition I.2.11), $\Gamma = \Gamma^i_j T^j_i$ is the pull-back of the $so(n, \mathbf{R})$-valued connection form ω corresponding to the Levi-Civita connection of M [26], A is the gauge potential, and $\sigma^i_j = \frac{1}{2}[\gamma^i, \gamma^j]$.

Even though to locally define $\mathcal{L}$ one needs a local section o_α it can be verified that its definition is actually section-independent. This is a special case of a gauge invariance, the so called *local Lorentz* invariance. As a consequence of this symmetry the n-forms $\mathcal{L}|_{U_\alpha}$ patch together to give a globally defined Lagrangian $\mathcal{L}$ and therefore a theory which is general covariant (cf. Subsection II.4.1). This has to be contrasted with the Lagrangian given by (II.16), which is general covariant but *not* local Lorentz invariant.

As independent fields of the theory we take the sets $\{\psi_{+\alpha}\}$, $\{e^i_\alpha\}$, and the gauge potential A. The metric tensor g is then given by $g = \{\sum_i^n e^i_\alpha \otimes e^i_\alpha\}$ (see eqs. (I.33) and (I.34)), from which the Levi-Civita connection can be constructed [26].

According to the above, the classical action $S[\{\psi_{+\alpha}\}, \{\bar{\psi}_{+\alpha}\}, \{e^i_\alpha\}, A]$ is invariant under the (infinitesimal) local Lorentz rotations

$$\begin{array}{rcl} \psi_{+\alpha} & \to & \psi_{+\alpha} + \frac{1}{4}\epsilon\, \varepsilon^i_{j\alpha}\sigma^j_i\psi_{+\alpha} \\ e^i_\alpha & \to & e^i_\alpha - \epsilon\, \varepsilon^i_{j\alpha} e^j_\alpha \\ A & \to & A \end{array} \tag{II.124}$$

where $\epsilon \ll 1$ and $\varepsilon_\alpha = \varepsilon^i_{j\alpha} T^j_i \in C^\infty(U_\alpha, so(n, \mathbf{R}))$. This transformation corresponds to the change of local section $o_\alpha \to o_\alpha \cdot \exp \epsilon\varepsilon_\alpha$.
The action S is also invariant under the infinitesimal diffeomorphisms

$$\begin{array}{rcl} \psi_{+\alpha} & \to & \psi_{+\alpha} + \epsilon X\psi_{+\alpha} \\ e^i_\alpha & \to & e^i_\alpha + \epsilon L_X e^i_\alpha \\ A & \to & A + \epsilon L_X A \end{array} \quad , \quad X \in \mathcal{X}(M) \tag{II.125}$$

Clearly S is also invariant under the usual gauge transformations (II.17), but in this subsection we shall not consider this symmetry.

Proposition II.7.5 *Let $X \in \imath(M)$, and $o_\alpha = \{E_{i\alpha},\ i = 1, \ldots, n\}$, be a local section of OM^+. Denote by $[X, E_{i\alpha}]^j$, $j = 1, \ldots, n$, the components of the vector* *field* $[X, E_{i\alpha}]$ *with respect to the linear frame* o_α. *Then the matrix* $[X, E_\alpha](x)$, $x \in U_\alpha$, *whose elements are given by* $[X, E_{i\alpha}]^j(x)$, *belongs to* $so(n, \mathbf{R})$.

Proof: see the Appendix.

According to the above proposition, an orthonormal frame $o_\alpha = \{E_{i\alpha}\}$ is transformed by an infinitesimal isometry X into the *orthonormal* frame

$$o'_\alpha = \{E_{i\alpha} + \epsilon L_X E_{i\alpha}\} = \{E_{j\alpha}(\delta_i^j + \epsilon [X, E_{i\alpha}]^j)\} = o_\alpha \cdot (\mathbf{1} + \epsilon [X, E_\alpha])$$

We can thus interpret an infinitesimal isometry X as a change of local section represented by the map $[X, E_\alpha] \in C^\infty(U_\alpha; so(n, \mathbf{R}))$.

By combining an infinitesimal isometry with the inverse of the local Lorentz rotation corresponding to it, we can define the following infinitesimal *affine transformations* of the fields appearing in the action S

$$\begin{array}{rcl} \psi_{+\alpha} & \rightarrow & \psi_{+\alpha} + \epsilon\,(X\psi_{+\alpha} - \frac{1}{4}[X, E_{j\alpha}]^i\,\sigma_i^j\psi_{+\alpha}) \\ e^i_\alpha & \rightarrow & e^i_\alpha \\ A & \rightarrow & A + \epsilon L_X A \end{array} \quad , \quad X \in \imath(M) \qquad (\text{II}.126)$$

As the classical theory defined by the Lagrangian (II.123) is separately invariant under local Lorentz rotations and infinitesimal diffeomorphisms, it is clearly also invariant under the combined transformations (II.126). Notice that the transformation law (II.126) has the same form as the one postulated in (II.55). Therefore, the formalism we developed to construct the base space anomalies is applicable and gives the unintegrated affine anomaly (II.121).

That such an anomaly does exist for the theory we are considering, follows from the fact that the local Lorentz invariance can not be maintained at the quantum level [36,1]. This is not surprising since, as we said before, local Lorentz invariance is just a special kind of gauge invariance.

Thus, the affine anomaly

$$\Delta_\imath[A, g; X] = \int_M TS_Q^{1,n}(A; X), \quad X \in \imath(M) \qquad (\text{II}.127)$$

where the g-dependence comes from the Levi-Civita connection used to construct $TS_Q^{1,n}$, should be equivalent to the Lorentz anomaly associated to the infinitesimal change of local section represented by the map $[X, E_\alpha] \in C^\infty(U_\alpha; so(n, \mathbf{R}))$, $X \in \imath(M)$.

What can we say about the tensor field $K(A)$ appearing in the general expression of $TS_Q^{1,n}(A; X)$? Since $L_X g = 0$ for $X \in \imath(M)$, the following tensor fields of type (1,2) satisfy condition (II.118)

$$K_1(A) \doteq Tr\,({}^g A \otimes F), \qquad K_2(A) \doteq Tr\,(A \otimes {}^g F)$$

where locally ${}^g F = (g^{\mu\nu} F_{\nu\rho})\partial_\mu \otimes dx^\rho$.

If the structure group G of the trivial bundle upon which the theory is based is the $U(1)$-group, then the tensor field

$$K_3(A) \doteq \mathbf{1} \otimes A$$

where $\mathbf{1}$ is the identity tensor field on M, fulfills condition (II.118) for all $X \in \mathcal{X}_{\cdot}(M)$.
Which one of the above tensor fields has to be used in constructing $TS_Q^{1,n}(A;X)$ is not determined by the present analysis.

II.8 Conformal anomalies

Let M be a compact, n=2k-dimensional, oriented, Riemannian manifold without boundary. In this section we restrict ourselves to the subalgebra of $\mathcal{X}_\cdot(M)$ consisting of infinitesimal conformal transformations and give special examples of n-forms $\mathcal{D}_c^{1,n}(X)$ which represent possible, unintegrated, consistent, conformal anomalies.

II.8.1 Geometrical preliminaries

Proposition II.8.1 *Let M be an oriented Riemannian manifold equipped with the natural volume form ϵ (see (I.19)) and a Riemannian connection. Then, for every vector field X on M, we have*

$$div\, X = -Tr\, Q_X$$

where $div\, X$ denotes the divergence of X (see Definition I.1.18) and $Tr\, Q_X$ the contraction locally given by $[Q_X]^\alpha_{\cdot\alpha}$ (cf. Subsection II.6.4).

Proof: see Appendix 6 of [26].

Corollary II.8.1 *The hypotheses being the same as in the previous proposition, we locally have*

$$div\, X = \partial_\alpha X^\alpha + X^\mu \Gamma^\alpha_{\mu\alpha} = \nabla_\alpha X^\alpha + X^\mu T^\alpha_{\mu\alpha}$$

where $\Gamma^\alpha_{\mu\beta}$ and $T^\alpha_{\mu\beta}$ are the components of the connection resp. the torsion with respect to a local coordinate system.

Proof: this is immediate from Propositions II.6.10 and I.3.7.

Corollary II.8.2 *With the same hypotheses as those in Proposition II.8.1 we have*

$$div\, [X,Y] = L_X(div\, Y) - L_Y(div\, X), \quad \forall X,Y \in \mathcal{X}_\cdot(M)$$

Proof: this follows from eq. (II.104) by noting that for a Riemannian connection we have $Tr\, R(X,Y) = 0$, $\forall X,Y \in \mathcal{X}_\cdot(M)$.

Definition II.8.1 *A vector field Z on M is called an infinitesimal conformal transformation if $L_Z g = \sigma\, g$, where g is the metric tensor of M and $\sigma \in C^\infty(M)$.*

We denote by $c(M)$ the subalgebra of all infinitesimal conformal transformations of M. Clearly $\imath(M) \subset c(M)$ (cf. Definition II.7.3).

Proposition II.8.2 *Assume M is provided with the Levi-Civita connection. Then the infinitesimal conformal transformation Z induces infinitesimal changes on the curvature R, the Ricci tensor field Ric and the scalar curvature $\mathcal{R}$ (see Definition I.3.10).*

They are locally given by

$$
\begin{aligned}
(L_Z R)^{\alpha}_{\beta\mu\nu} &= -\tfrac{1}{2}\{\,\delta^{\alpha}_{\mu}\nabla_\nu\nabla_\beta\sigma - \delta^{\alpha}_{\nu}\nabla_\mu\nabla_\beta\sigma - g_{\mu\beta}\nabla_\nu\nabla^{\alpha}\sigma + g_{\nu\beta}\nabla_\mu\nabla^{\alpha}\sigma\,\}\\
(L_Z Ric)_{\beta\nu} &= -\tfrac{1}{2}\{\,(n-2)\nabla_\beta\nabla_\nu\sigma + g_{\beta\nu}\Box\sigma\,\}\\
L_Z\mathcal{R} &= -\{\,\sigma\mathcal{R} + (n-1)\Box\sigma\,\}
\end{aligned}
$$

where $\Box$ *denotes the Laplacian, which is locally given by* $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$.

Proof: see [40].

Definition II.8.2 *Let* M *be a Riemannian manifold of dimension* $n\geq 4$ *equipped with the Levi-Civita connection. The tensor field* W *of type (1,3) whose components are given by*

$$
\begin{aligned}
W^{\alpha}_{\beta\mu\nu} &= R^{\alpha}_{\beta\mu\nu} - \frac{1}{(n-2)}\,(\delta^{\alpha}_{\mu}Ric_{\beta\nu} - \delta^{\alpha}_{\nu}Ric_{\beta\mu} - g_{\beta\mu}g^{\alpha\rho}Ric_{\rho\nu} - g_{\beta\nu}g^{\alpha\rho}Ric_{\rho\mu})\\
&\quad + \frac{\mathcal{R}}{(n-1)(n-2)}(\delta^{\alpha}_{\mu}g_{\beta\nu} - \delta^{\alpha}_{\nu}g_{\beta\mu})
\end{aligned}
$$

is called the Weyl conformal curvature tensor.

Proposition II.8.3 *Let* $Z \in c(M)$. *Then* $L_Z W = 0$.

Proof: see [40].

Proposition II.8.4 *Assume the manifold* M *is provided with the natural volume form* ϵ. *Then the function* σ *of Definition II.8.1 is given by*

$$\sigma = \frac{2}{n}(div\, Z), \quad Z \in c(M)$$

Proof: the proposition is easily proved by using the local expression (I.21) for the divergence of Z.

From the above proposition it follows that for an infinitesimal isometry X we have $div\, X = 0$.

Proposition II.8.5 *Let* τ *be a q-form on* M. *Then for any infinitesimal conformal transformation* Z *we have*

$$L_Z{}^{*}\tau = {}^{*}L_Z\tau + (1 - \frac{2q}{n})(div\, Z)\,{}^{*}\tau$$

where ${}^{*}\tau$ *denotes the Hodge dual form of* τ *(see Definition I.1.17).*

Proof: see the Appendix.

II.8.2 Special solutions of $H^{n+1}(c(M); C^{\infty}_{A}(M))$

In this subsection the Riemannian manifold M is provided with the natural volume form ϵ and a Riemannian connection.

Let $\mathcal{D}^n_c(A)$ be an element of $\Omega^n_A(M)$, i.e. a local n-form depending on the gauge potential A, satisfying

$$(\vartheta^{\vee}(Z)\mathcal{D}^n_c)(A) = L_Z \mathcal{D}^n_c(A) + a\,(div\, Z)\mathcal{D}^n_c(A), \quad \forall Z \in c(M), \quad a \in \mathbf{R} \qquad \text{(II.128)}$$

From Proposition II.8.5 and Lemma II.3.1 it follows that the n-cochain $\imath^n\mathcal{D}^n_c(A)$ verifies

$$\delta^{\vee}\imath^n\mathcal{D}^n_c(A)|_{c(M)} = -a\,Tr_Q \wedge \imath^n\mathcal{D}^n_c(A)|_{c(M)} \qquad \text{(II.129)}$$

As in Subsection II.6.6 consider the (n+1)-cochain

$$\mathcal{D}^{n+1,0}_c(A) \doteq \imath^n\mathcal{D}^n_c(A) \wedge Tr_Q|_{c(M)} \qquad \text{(II.130)}$$

From (II.129) and Proposition II.6.13 it follows that it is a cocycle.
Using (II.128) and Lemma II.3.1 it can be verified that

$$\mathcal{D}^{n+1,0}_c(A) = \frac{(-1)^n}{a}\delta^{\vee} d\,\imath^{n-1}\mathcal{D}^n_c(A)|_{c(M)}, \quad a \neq 0$$

Therefore, the cocycle $\mathcal{D}^{n+1,0}_c(A)$ is a non-trivial element of $H^{n+1}(c(M); C^{\infty}_A(M))$ if the n-form $\mathcal{D}^n_c(A)$ verifies

$$(\vartheta^{\vee}(Z)\mathcal{D}^n_c)(A) = L_Z\mathcal{D}^n_c(A), \quad \forall Z \in c(M) \qquad \text{(II.131)}$$

Notice that this condition is automatically fulfiled by the polynomial n-forms, i.e. those forms constructed out of exterior products of A and dA only (cf. (II.106)). If the n-form $\mathcal{D}^n_c(A)$ is polynomial then the unrestricted cocycle (II.130) is equivalent to the general one considered in Subsection II.6.6. In the following $\mathcal{D}^n_c(A)$ stands for a *non*-polynomial n-form.

If $\mathcal{D}^n_c(A)$ meets condition (II.131) we can ascend eq. (II.66) starting with $\mathcal{D}^{n+1,0}_c(A)$. The resulting n-form $\mathcal{D}^{1,n}_c(A;Z)$ is given by

$$\mathcal{D}^{1,n}_c(A;Z) = \mathcal{D}^n_c(A)\,Tr\,Q_Z = -\mathcal{D}^n_c(A)(div\,Z), \quad Z \in c(M) \qquad \text{(II.132)}$$

which is similar to the general one found in Subsection II.6.6.
In what follows we identify through the $*$-operator (see Definition I.1.17) the n-form $\mathcal{D}^n_c(A)$ with the function $\mathcal{D}^0_c(A) \doteq {}^*\mathcal{D}^n_c(A)$. By Proposition II.8.5, condition (II.131) becomes

$$(\vartheta^{\vee}(Z)\mathcal{D}^0_c)(A) = {}^*(\vartheta^{\vee}(Z)\mathcal{D}^n_c)(A) = L_Z\mathcal{D}^0_c(A) + (div\,Z)\,\mathcal{D}^0_c(A), \quad \forall Z \in c(M) \qquad \text{(II.133)}$$

The problem of constructing the possible A-dependent conformal anomalies is now reduced to the one of finding functions $\mathcal{D}^0_c(A)$ verifying (II.133). This suffices from a mathematical point of view. Physically, the function $\mathcal{D}_c^0(A)$ must fulfil also a dimensionality requirement, as we now explain.

In units where $\hbar = c = 1$ the action functional S, and therefore the anomaly, must be dimensionless. Since the natural cobasis elements dx^μ are assumed to have dimension of length, the components of the metric tensor are dimensionless. Hence, the natural volume element ϵ has dimension (length)^n. By assumption the components of the gauge potential A have dimension (length)$^{-1}$, whereas the components of a vector field X have dimension of length, so that $div\, X$ is dimensionless. Clearly the exterior derivative reduces the dimension by one. From this it follows that in units where $\hbar = c = 1$ the n-form $\mathcal{D}_c^n(A)$ must be dimensionless or, equivalently, the function $\mathcal{D}_c^0(A)$ must have dimension (length)$^{-n}$.

To construct a function meeting condition (II.133) and of the right dimensionality, namely (length)$^{-n}$, we have the metric tensor g, the gauge potential A, its dual ${}^*A$, and their exterior derivatives. Below we give some special examples of such functions in two and four dimensions. To simplify the notation we express them locally.

2-dimensions. The simplest example is

$$\mathcal{D}_c^0(A) = {}^*[\gamma\,({}^*A \wedge A)] = \gamma\,(A_\mu A^\mu)$$

where γ denotes the Killing form of the Lie algebra of the gauge group G. The resulting conformal anomaly is given by the integral of

$$\mathcal{D}_c^{1,2}(A; Z) = -\gamma({}^*A \wedge A)(div\, Z), \quad Z \in c(M)$$

If the gauge group is $U(1)$, then the "divergence"

$$\mathcal{D}_c^0(A) = \nabla_\mu A^\mu + A^\mu T^\alpha_{\mu\alpha} = -{}^*d{}^*A$$

satisfies (II.133) and is of dimension (length)$^{-2}$. Its associated unintegrated conformal anomaly is given by

$$\mathcal{D}_c^{1,2}(A; Z) = (d{}^*A)(div\, Z), \quad Z \in c(M)$$

It is unlikely that the integrals of these 2-forms will ever appear as the conformal anomalies of any physical theory because they are not gauge invariant.

If $G = U(1)$, then the following polynomial cochain (cf. Subsection II.6.6)

$$\mathcal{D}_{poly}^{1,2}(A; Z) = F\,(div\, Z)$$

is gauge invariant and possesses all the requirements to be an anomaly.

4-dimensions. In this case the simplest example is

$$\mathcal{D}_c^0(A) = \gamma(F_{\mu\nu}F^{\mu\nu})$$

which is also gauge invariant.
The corresponding 4-form $\mathcal{D}_c^{1,4}(A;Z)$ is given by

$$\mathcal{D}_c^{1,4}(A;Z) = -\gamma({}^*F \wedge F)(div\, Z), \quad Z \in c(M)$$

The integral of this form is the gauge contribution to the conformal anomaly [19] of the free Yang-Mills theory (cf. Subsection II.4.1).

II.8.3 The A-independent conformal anomalies

In this subsection we use Proposition II.8.2 to directly construct special n-forms which combined with the 1-cochain Tr_Q as in (II.132) give n-forms $\mathcal{D}_c^{1,n}(Z)$ verifying (cf. Subsection II.5.1)

$$\left.\begin{array}{ll} & (\delta^\vee \mathcal{D}_c^{1,n})(Z_1, Z_2) = d(\mathcal{D}_c^{2,n-1}(Z_1, Z_2)) \\ \text{and} & \mathcal{D}_c^{1,n}(Z_1) \neq (\delta^\vee \mathcal{D}_c^{0,n})(Z_1) \quad \text{(modulo an exact form)} \end{array}\right\}, \quad Z_i \in c(M) \tag{II.134}$$

In what follows the Riemannian manifold M is provided with the natural volume form ϵ and the Levi-Civita connection.

Let τ be any A-independent n-form. By Proposition II.8.1 and Corollary II.8.2 the 1-cochain $\tau\, Tr_Q$ with values in the n-forms verifies

$$\begin{aligned}(\delta^\vee \tau\, Tr_Q)(Z_1, Z_2) &= \tau\, Tr\, Q([Z_1, Z_2]) = \\ &= d\{(\imath_{Z_1}\tau)\, Tr\, Q(Z_2) - (\imath_{Z_2}\tau)\, Tr\, Q(Z_1)\} \\ &\quad - (L_{Z_1}\tau)\, Tr\, Q(Z_2) + (L_{Z_2}\tau)\, Tr\, Q(Z_1), \quad \forall Z_i \in \mathcal{X}(M)\end{aligned}$$

Therefore, to verify the first equation in (II.134) we must find a n-form τ_c such that

$$(L_{Z_1}\tau_c)\, div\, Z_2 - (L_{Z_2}\tau_c)\, div\, Z_1 = d(\text{something}), \quad \forall Z_i \in c(M) \tag{II.135}$$

Observe that the second equation in (II.134) is always satisfied since in our formalism, for an A-independent 0-cochain $\mathcal{D}_c^{0,n}$, we have $\delta^\vee \mathcal{D}_c^{0,n} \equiv 0$, and $\tau_c\, Tr\, Q_Z = -\tau_c(div\, Z) \neq d(\text{something})$ if τ_c is not the volume form ϵ.

As in the previous subsection we identify the n-form τ_c with the function $f_c \doteq {}^*\tau_c$. By Propositions II.8.5 and I.1.7, condition (II.135) is equivalent to

$$(L_{Z_1} f_c)\, div\, Z_2 - (L_{Z_2} f_c)\, div\, Z_1 = div\,(\text{something}), \quad \forall Z_i \in c(M)$$

or

$$\int_M \epsilon\, \{(L_{Z_1} f_c)\, div\, Z_2 - (L_{Z_2} f_c)\, div\, Z_1\} = 0, \quad \forall Z_i \in c(M) \tag{II.136}$$

The only geometrical quantities we have at our disposal to construct a function f_c meeting condition (II.136) are the metric tensor g, the volume form ϵ, the curvature tensor field R and its covariant differentials. As explained in the preceding subsection, the functions f_c must also fulfil the physical requirement of being of dimension $(\text{length})^{-n}$. Obviously, in our units, the curvature has dimension $(\text{length})^{-2}$.

A class of solutions to eq. (II.136) consists of those functions f_c verifying

$$L_Z f_c = a\,(div\,Z)\,f_c, \quad \forall Z \in c(M), \quad a \in \mathbf{R}$$

Examples thereof are the total contractions of tensor products of the Weyl tensor W. We denote such a contraction by f_{pW}, where p is the number of times the tensor W appears in its expression. Since any contraction of W vanishes [14], the minimal value for p is 2. The function f_{2W} is nothing else than the square of the norm $\| W \|$ of W, which is locally defined by

$$\| W \|^2 \doteq W_{\alpha\beta\mu\nu} W^{\alpha\beta\mu\nu}$$

We have

$$L_Z \| W \|^2 = -\frac{4}{n}(div\,Z)\,\| W \|^2\,, \quad Z \in c(M)$$

Generally,

$$L_Z f_{pW} = -\frac{2p}{n}(div\,Z)\,f_{pW}\,. \quad Z \in c(M) \tag{II.137}$$

As the dimension of W is $(\text{length})^{-2}$, p must be equal to $k = \frac{n}{2}$.

A possible A-independent conformal anomaly is therefore given by the integral of

$${}^{W}\mathcal{D}_c^{1,n}(Z) = {}^{*}f_{kW}\,(div\,Z), \quad k = \frac{n}{2}, \quad n \geq 4, \quad Z \in c(M) \tag{II.138}$$

Observe that the Weyl tensor vanishes identically for the 4-sphere S^4 [14].

Clearly, the task of constructing more functions f_c of dimension $(\text{length})^{-n}$ and meeting condition (II.136) becomes harder with increasing dimension. Below we shall limit ourselves to functions that are linear combinations of total contractions of tensor products of the curvature tensor field only, and to manifolds of dimension two and four. Notice that by definition the Ricci tensor and the scalar curvature are already contractions of R.

2-dimensions. By using Proposition II.8.2 and partial integration we obtain

$$\int_M \epsilon\,[(L_{Z_1}\mathcal{R})\,\sigma_2 - (L_{Z_2}\mathcal{R})\,\sigma_1] = (n-1)\int_M \epsilon\,[\sigma_1 \Box \sigma_2 - \sigma_2 \Box \sigma_1] = 0, \quad Z_i \in c(M), \quad \forall n$$

where $\sigma_i = \frac{2}{n}(div\,Z_i)$.

Hence, the integral of the 2-form

$$\mathcal{D}_c^{1,2}(Z) = {}^{*}\mathcal{R}\,(div\,Z), \quad Z \in c(M) \tag{II.139}$$

fulfils all the requirements to be a consistent conformal anomaly, and indeed it is one [19]. Observe that $\mathcal{R}$ is the only function of dimension $(\text{length})^{-2}$ we can build

with g and R.
We shall see in Subsection II.8.5 that in the case where $\mathcal{R}$ is constant the 2-dimensional conformal anomaly can be identified with an affine anomaly and thus be derived with the formalism of the previous section.

4-dimensions. Consider the squared norms $\| R \|^2$ and $\| Ric \|^2$. They are of dimension $(\text{length})^{-4}$. By Proposition II.8.2 they verify

$$\begin{aligned} L_Z \| R \|^2 &= -\frac{4}{n}\sigma \| R \|^2 - \frac{8}{n} Ric^{\beta\nu}\nabla_\beta\nabla_\nu\sigma \\ L_Z \| Ric \|^2 &= -\frac{4}{n}\sigma \| Ric \|^2 - \frac{2}{n}(n-2) Ric^{\beta\nu}\nabla_\beta\nabla_\nu\sigma - \frac{2}{n}\mathcal{R}\Box\sigma, \quad Z \in c(M) \end{aligned}$$

where $\sigma = \frac{2}{n}(div\, Z)$.
By using partial integration and the contracted Ricci identity [14]

$$\nabla_\beta Ric^\beta_\nu = \frac{1}{2}\nabla_\nu \mathcal{R}$$

we get

$$\int_M \epsilon \{(L_{Z_1} \| R \|^2)\sigma_2 - (L_{Z_2} \| R \|^2)\sigma_1\} = \frac{4}{n}\int_M \epsilon\, \mathcal{R}(\sigma_1\Box\sigma_2 - \sigma_2\Box\sigma_1)$$

$$\int_M \epsilon \{(L_{Z_1} \| Ric \|^2)\sigma_2 - (L_{Z_2} \| Ric \|^2)\sigma_1\} = \int_M \epsilon\, \mathcal{R}(\sigma_1\Box\sigma_2 - \sigma_2\Box\sigma_1)$$

We also have

$$\int_M \epsilon \{(L_{Z_1}\mathcal{R}^2)\sigma_2 - (L_{Z_2}\mathcal{R}^2)\sigma_1\} = \frac{4}{n}(n-1)\int_M \epsilon\, \mathcal{R}(\sigma_1\Box\sigma_2 - \sigma_2\Box\sigma_1)$$

where $Z_i \in c(M)$.
Combining these three results we find that the linear combination

$$a \| R \|^2 + b \| Ric \|^2 + c\mathcal{R}^2$$

fulfils condition (II.136) if $\frac{4}{n}a + b + \frac{4(n-1)}{n}c = 0$.
The 4-form

$$\mathcal{D}^{1,4}_c(Z) = {}^*(a \| R \|^2 + b \| Ric \|^2 + c\mathcal{R}^2)(div\, Z), \quad Z \in c(M) \tag{II.140}$$

is thus a possible unintegrated conformal anomaly if

$$a + b + 3c = 0 \tag{II.141}$$

This result is the same as the one first obtained by Duff [19] using dimensional regularization techniques, and later by Bonora et al. [8] with a cohomological method.
It has to be emphasized that our derivation differs from the one of reference [8]

in two aspects. First of all, we consider the action of the group of (infinitesimal) conformal transformations and not that of the abstract Weyl rescaling group (see below), and secondly, this action is assumed to leave invariant the metric tensor g and its derived Levi-Civita quantities (cf. Subsection II.4.1). As a consequence, the coformal anomaly

$$\Delta_c[g;Z] = \int_M \epsilon\,(div\,Z)\,\Box\mathcal{R}, \quad Z \in c(M) \tag{II.142}$$

which was obtained by the authors of reference [17] and proved to be cohomological trivial in [8], doesn't even need to be considered in our approach since, by using partial integration and Proposition II.8.2, we obtain

$$\int_M \epsilon\,(div\,Z)\,\Box\mathcal{R} = -\frac{n}{2}\int_M \epsilon\,\sigma\,\Box\mathcal{R} = \frac{n\,(4-n)}{8\,(n-1)}\int_M \epsilon\,\sigma\,\mathcal{R}^2$$

i.e. the anomaly (II.142) vanishes identically in four dimensions.
Clearly, it has still to be shown that a classical field theory invariant under the transformations (II.55) with $Z \in c(M)$ does exist. We do this next.

II.8.4 The classical, conformal invariant, scalar field theory

Let ϕ be a function on M, in physical terms a scalar field, and consider the following classical action

$$S[\phi,g] = \int_M \{{}^*d\phi \wedge d\phi + \tfrac{n-2}{4(n-1)}\,{}^*\phi^2\,\mathcal{R}\} \tag{II.143}$$

This is the so called *conformally coupled*, massless, scalar field theory [5].
Apart from being invariant under the infinitesimal diffeomorphism transformations, this action is also invariant under the (Weyl) rescalings

$$\begin{array}{rcl} \phi & \to & \phi' \doteq \Omega^{\frac{2-n}{n}}\phi \\ g & \to & g' \doteq \Omega^2 g \end{array} \;;\quad \Omega \in C^\infty(M;\mathbf{R}^+) \tag{II.144}$$

It is these transformations that are physically interpreted as conformal transformations and are supposed to give rise to the above discussed conformal anomaly [8].

Two remarks are in order here. First of all, the Weyl rescaling group, which is isomorphic to $C^\infty(M;\mathbf{R}^+)$, is abelian. Hence, analogously to the axial symmetry case (cf. Subsection II.2.1), no natural consistency condition exists for the associated Weyl or conformal anomaly. Secondly, the transformations (II.144) can not be described geometrically and so they don't fit our symmetry scheme (cf. Subsection I.4.3). We therefore look for another formulation of the Weyl rescaling transformations (II.144).

By making use of Propositions II.8.2 and II.8.5, it is a straightforward calculation to show that the action (II.143) is invariant under the following (geometrical) infinitesimal, conformal transformations

$$\begin{array}{rcl} \phi & \to & \phi' \doteq \phi + \epsilon\,\{L_Z\phi + \frac{2-n}{2n}(div\,Z)\,\phi\} \\ g & \to & g' = g \end{array} \;,\quad Z \in c(M), \quad \epsilon \ll 1 \tag{II.145}$$

which are of the same kind as those postulated in (II.55).
Thus, the above construction of the conformal anomaly is a posteriori justified. Notice that for $Z \in \imath(M) \subset c(M)$ the transformations (II.145) are just infinitesimal diffeomorphisms and that the conformal anomalies we found all vanish. This is consistent with the assumption of the absence of gravitational anomalies.

II.8.5 The 2-dimensional conformal anomaly as an affine anomaly

As it is well known [27] two dimensional real manifolds possess special properties. In this subsection we exploit them to relate the conformal anomaly (II.139) of a 2-dimensional, orientable, Einstein manifold to an affine anomaly.

Definition II.8.3 *An Einstein manifold is a Riemannian manifold M with the property that the Ricci tensor field of its Levi-Civita connection verifies*

$$Ric = k\, g$$

where k is a constant and g the metric tensor of M.

Definition II.8.4 *An almost complex structure on a manifold M is a tensor field J of type (1,1) satisfying*

$$J(x; J(x; v_x)) = -v_x, \quad \forall v_x \in TM_x, \quad \forall x \in M$$

Briefly,

$$J^2 = -\mathbf{1} \quad \text{or in components} \quad J^\mu_\nu J^\nu_\rho = -\delta^\mu_\rho$$

Not all manifolds admit an almost complex structure. As an example only the two and six dimensional spheres possess such a structure.

Proposition II.8.6 *Let ϵ be the natural volume form of a two dimensional, oriented, Riemannian manifold M. Then the tensor field*

$$J_\epsilon \doteq {}^g\epsilon$$

is an almost complex structure of M.

Proof: The components of J_ϵ are given by $J^\alpha_\beta = g^{\alpha\mu}\epsilon_{\mu\beta} = \sqrt{g}g^{\alpha\mu}\epsilon_{\mu\beta}$.
Clearly, $J^\alpha_\beta J^\beta_\gamma = J^{\alpha\beta}J_{\beta\gamma} = \epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = -\delta^\alpha_\gamma$ □

Notice that every 2-dimensional, orientable manifold admits an almost complex structure since all such manifolds are complex [27].

As can be easily verified, the almost complex structure J_ϵ is parallel with respect to the Levi-Civita connection of M, i.e.

$$\nabla_X J_\epsilon = 0, \quad \forall X \in \mathcal{X}(M) \tag{II.146}$$

and it is invariant under the infinitesimal conformal transformations of M, i.e.

$$L_Z J_\epsilon = 0, \quad \forall Z \in c(M) \tag{II.147}$$

Proposition II.8.7 *Let Z be an infinitesimal conformal transformation of a 2-dimensional, oriented, Einstein manifold. Then Z can be uniquely written as*

$$Z = X + J_\epsilon Y$$

where X and Y are infinitesimal isometries.

Proof: see [40].

In the following we assume that every infinitesimal conformal transformation of a 2-dimensional, oriented, Einstein manifold is decomposed as in the above proposition.

By the remark at the end of the preceding subsection, the isometric part X of an infinitesimal conformal transformation Z of a 2-dimensional, oriented, Einstein manifold doesn't contribute to the conformal anomaly. Moreover, the vector field $J_\epsilon(Z - X) = -Y$ is an infinitesimal isometry; it can thus be used to generate the affine anomaly (II.122).
As Y depends linearly on Z, we can define the following unintegrated, conformal anomaly

$$\mathcal{D}_c^{1,2}(Z) \doteq TS_Q^{1,2}(J_\epsilon(Z - X)) = 3\,S^2(Q_Y, R) \tag{II.148}$$

Notice that by (II.147) the Lie bracket of two infinitesimal conformal transformations $Z_i = X_i + J_\epsilon Y_i$, $i = 1, 2$, is decomposed as

$$[Z_1, Z_2] = [X_1, X_2] - [Y_1, Y_2] + J_\epsilon\{[X_1, Y_2] + [Y_1, X_2]\}$$

Therefore (cf. Proposition II.7.3),

$$\begin{aligned}
\delta^\vee \mathcal{D}_c^{1,2}(Z_1, Z_2) &= \mathcal{D}_c^{1,2}([Z_1, Z_2]) = \\
&= -TS_Q^{1,2}([X_1, Y_2] + [Y_1, X_2]) = -\delta^\vee TS_Q^{1,2}(X_1, Y_2) - \delta^\vee TS_Q^{1,2}(Y_1, X_2) = \\
&= d\,\{TS_Q^{2,1}(X_1, Y_2) + TS_Q^{2,1}(X_1, Y_2)\}
\end{aligned}$$

which shows that the conformal anomaly given by the integral of (II.148) is consistent.

As the Levi-Civita connection is torsionless we have (cf. Proposition II.6.10)

$$Q_X = -\nabla X, \quad \forall X \in \mathcal{X}(M)$$

From (II.146) it therefore follows that

$$Q_{J_\epsilon X} = J_\epsilon Q_X, \quad \forall X \in \mathcal{X}(M) \tag{II.149}$$

If we take Tr as the bilinear, symmetric, Ad-invariant map S^2, then the unintegrated, conformal anomaly (II.148) is locally given by

$$\mathcal{D}_c^{1,2}(Z) = -3\,Tr\,\{Q_{J_\epsilon(Z-X)}R\} = -\tfrac{3}{2}\,[J_\epsilon]^\alpha_\beta [Q_{(Z-X)}]^\beta_\gamma R^\gamma_{\alpha\mu\nu} dx^\mu \wedge dx^\nu$$

where use has been made of (II.149).

Proposition II.8.8 *Let M be a two dimensional, Riemannian manifold with metric g. The components of the curvature tensor field R of the Levi-Civita connection of M are then given by*

$$R^{\alpha}_{\beta\mu\nu} = \frac{1}{2}\mathcal{R}\,(\delta^{\alpha}_{\mu} g_{\beta\nu} - \delta^{\alpha}_{\nu} g_{\beta\mu})$$

where $\mathcal{R}$ is the scalar curvature.

Proof: see [30].

Observe that by the above proposition a 2-dimensional, Einstein manifold has a constant scalar curvature.
With the help of Proposition II.8.8 we can explicitly write down the expression of $\mathcal{D}^{1,2}_c(Z)$ in terms of Z. Namely,

$$\mathcal{D}^{1,2}_c(Z) = -\tfrac{3}{4}{}^*\mathcal{R}[Q_{(Z-X)}]^{\alpha}_{\alpha} = \tfrac{3}{4}{}^*\mathcal{R}\,(div\, Z), \quad Z \in c(M) \tag{II.150}$$

where we used the fact that $div\, X = 0$ for $X \in \imath(M)$.
The unintegrated, 2-dimensional, conformal anomaly (II.150) is the same as the one given in the preceding subsection (cf. (II.139)).

It would clearly be nice if the decomposition of an infinitesimal conformal transformation as given in Proposition II.8.7 were valid for any two dimensional manifold. We believe that this is the case, but we didn't find any proof of this.

Conclusions

In this work we investigated the possibility of the appearance in a quantum field theory of consistent anomalies associated to symmetry transformations that are related to infinitesimal diffeomorphisms of the manifold M upon which the classical theory is based.

The most important result we found is the affine anomaly (II.121). Even its presence in the only known affine invariant field theory is masked by the Lorentz anomaly (see Subsection II.7.3), we believe that there are other instances where it describes a "proper" anomaly. As we saw in Subsection II.8.5 this is the case of the conformal anomaly of a two dimensional Einstein manifold.

To be honest, we don't consider our treatment of the 2-dimensional conformal anomaly very satisfying from the topological point of view. A more elegant way of interpreting *all* conformal anomalies as affine anomalies would be the construction of a natural, conformal invariant connection in a bundle over M. Such a connection exists [28]. It is the *normal conformal connection* in the bundle of second order frames over M [28,23]. However, our present knowledge of this subject is not deep enough to allow us realizing this program, and we must content ourselves with the brute force construction of the conformal anomalies given in Subsections II.8.2 and II.8.3.

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The Appendix

In this appendix we give the proofs that were not explicitly written in the main text.

Lemma II.3.2 *Let X_i, $i = 1, \ldots, k+1$, be any vector field. Then*

$$
\begin{aligned}
d\imath(X_1, \ldots, X_{k+1}) &= (-1)^{k+1}\imath(X_1, \ldots, X_{k+1})d \\
&+ (-1)^k \{ \sum_{i=1}^{k+1} (-1)^{i+1} \imath(X_1, \ldots, \hat{\imath}, \ldots, X_{k+1}) L_{X_i} \\
&+ \sum_{i<j} (-1)^{i+j} \imath(-[X_i, X_j], X_1, \ldots, \hat{\imath}, \ldots, \hat{\jmath}, \ldots, X_{k+1}) \}
\end{aligned}
$$

Proof: We have (see [21])

$$
d\imath(X_1,..,X_{k+1}) = (-)^{k+1}\Big[\imath(X_1,..,X_{k+1})d - \sum_{i=1}^{k+1} (-)^{i+1} \imath(X_{i+1},..,X_{k+1}) L_{X_i} \imath(X_1,..,X_{i-1})
$$

From $[L_{X_1}, \imath_{X_2}] = \imath_{[X_1,X_2]}$ it follows

$$
L_{X_i}\imath(X_1,..,X_{i-1}) = L_{X_i}\imath_{X_{i-1}}\circ \ldots \circ \imath_{X_1} = \imath_{X_{i-1}} \circ L_{X_i} \circ \imath_{X_{i-2}} \circ \ldots \circ \imath_{X_1} +
$$

$$
+ \imath_{[X_i, X_{i-1}]} \circ \imath_{X_{i-2}} \circ \ldots \circ \imath_{X_1} = \ldots =
$$

$$
= \sum_{j=1}^{i-1} \imath(X_1,..,[X_i, X_j],..,X_{i-1}) + \imath(X_1,..,X_{i-1}) L_{X_i}
$$

Therefore

$$
d\imath(X_1,..,X_{k+1}) = (-)^{k+1}\imath(X_1,..,X_{k+1})d +
$$

$$
+ (-)^k \Big[\sum_{i=1}^{k+1} (-)^{i+1} \sum_{j=1}^{i-1} \imath(X_{i+1},..,X_{k+1})\imath(X_1,..,[X_i,X_j],..,X_{i-1}) +
$$

$$
+ \sum_{i=1}^{k+1} (-)^{i+1} \imath(X_{i+1},..,X_{k+1})\imath(X_1,..,X_{i-1}) L_{X_i} \Big] =
$$

$$
= (-)^{k+1}\imath(X_1,..,X_{k+1})d +
$$

$$
+ (-)^k \Big[\sum_{i=1}^{k+1} (-)^{i+1} \imath(X_1,..,\hat{\imath},..,X_{k+1}) L_{X_i} - \sum_{i<j} (-)^{i+j} \imath([X_i, X_j], X_1,..,\hat{\imath},..\hat{\jmath},..X_{k+1}) \Big]
$$

□

Proposition II.6.13 *Let τ be any n-form on M. Then the $(n+2)$-cochain $\imath^n\tau \wedge \delta^\vee Tr_Q$ vanishes identically.*

Proof: We have

$$(\imath^n \tau \wedge \delta^\vee Tr_Q)(X_1, \ldots, X_{n+2}) =$$

$$= \frac{1}{2n!} \sum_{\pi} \text{sign}\, \pi \;\; \tau(X_{\pi(1)} \ldots X_{\pi(n)}) \Big[L_{X_{\pi(n+1)}} Tr\, G_{X_{\pi(n+2)}} - L_{X_{\pi(n+2)}} Tr\, G_{X_{\pi(n+1)}}$$

$$+ Tr\, R(X_{\pi(n+1)}, X_{\pi(n+2)}) \Big] =$$

$$= \sum_{i=1}^{n+2} (-)^{i+n+2} \imath(X_1 \ldots \hat{i} \ldots X_{n+2})(\tau \wedge d\, Tr_{G_{X_i}})$$

$$+ \imath(X_1 \ldots X_{n+2})(\tau \wedge Tr\, R)$$

Since $(\tau \wedge d\, Tr_{G_{X_i}})$ is a $(n+1)$-form, $\forall X_i \in \mathcal{X}(M)$, and $(\tau \wedge Tr\, R)$ is a $(n+2)$-form, the above expression vanishes identically on a n-dimensional manifold. $\square$

Proposition II.7.2 *The components of the tensor field $D_X = L_X - \nabla_X$ resp. $R(X,Y)$ with respect to the linear frame u_x are given by*

$$
\begin{aligned}
&[D_X]^\alpha_\beta(x) = -\omega^\alpha_\beta(u_x;\tilde{X}) \\
\textit{resp.}\quad &R^\alpha_\beta(X,Y)(x) = \Omega^\alpha_\beta(u_x;\tilde{X},\tilde{Y})
\end{aligned}
\quad ; \quad x \in M
$$

where $\tilde{X}$ and $\tilde{Y}$ are the natural lifts of X and $Y \in \mathfrak{X}(M)$.

Proof: Let u_x be given by $u_x = \{E_i^\mu(x)\partial_\mu\}$, where $\{\partial_\mu\}$ is the natural basis of TM_x, and take (x^μ, E_i^μ) as a local coordinate system in $\pi^{-1}(U_\alpha)$, $x \in U_\alpha \subset M$.
If X is a vector field on M whose natural components are $\{X^\mu\}$ then its natural lift is locally given by

$$\tilde{X}(u_x) = X^\mu \partial_\mu - E_i^\mu \partial_\mu X^\nu \frac{\partial}{\partial E_i^\nu}$$

We also know that the local expression of the connection form ω is (see [26])

$$\omega^i_j(u_x) = e^i_\alpha \left[\Gamma^\alpha_{\mu\beta} E^\beta_j dx^\mu + dE^\alpha_j \right]$$

where $e^i_\alpha = [E^{-1}]^i_\alpha$

Hence,

$$\omega^i_j(u_x;\tilde{X}(u_x)) = e^i_\alpha X^\mu \Gamma^\alpha_{\mu\beta} E^\beta_j - e^i_\alpha (\partial_\beta X^\alpha) E^\beta_j =$$

$$= -e^i_\alpha [D_X]^\alpha_\beta E^\beta_j$$

where $[D_X]^\alpha_\beta$ are the natural components of the tensor field D_X.
The second part of the Proposition is just a definition of the curvature tensor field (see [26]) $\square$

Lemma II.7.1 *Denote by $\Omega = \Omega^i E_i$ the curvature form of ω. Then*

$$D \circ h^* = h^* \circ [d - \Omega^i \wedge \imath(Z_{E_i})]$$

where Z_{E_i} is the fundamental vector field induced by $E_i \in LG$ (cf. Definition 1.2.2).

Proof: let φ be a p-form on P and $X_i \in \mathcal{X}(P)$, $i = 1, \ldots, p+1$.

We have

$$(D \circ h^*)\varphi(X_1, \ldots, X_{p+1}) = (h^* \circ d \circ h^*)\varphi(X_1, \ldots, X_{p+1}) =$$

$$= dh^*\varphi(hX_1, \ldots, hX_{p+1}) =$$

$$= \sum_{i=1}^{p+1} (-1)^{i+1}\, hX_i\, h^*\varphi(hX_1, \ldots, \hat{i}, \ldots, hX_{p+1}) - \sum_{i<j} (-1)^{i+j}\, h^*\varphi([hX_i, hX_j], \ldots) =$$

$$= \sum_{i=1}^{p+1} (-1)^{i+1}\, hX_i\, \varphi(hX_1, \ldots, \hat{i}, \ldots, hX_{p+1}) + \sum_{i<j} (-1)^{i+j}\, \varphi(h[hX_i, hX_j], hX_1, \ldots, hX_{p+1})$$

Since $(1-h)[hX_i, hX_j] = -\Omega^k(X_i, X_j) Z_{E_k}$ (see [21], vol II)

we obtain

$$(D \circ h^*)\varphi(X_1, \ldots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1}\, hX_i\, \varphi(hX_1, \ldots, \hat{i}, \ldots, hX_{p+1}) +$$

$$- \sum_{i<j} (-1)^{i+j}\, \varphi([hX_i, hX_j], hX_1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, hX_{p+1}) +$$

$$+ \sum_{i<j} (-1)^{i+j}\, \Omega^k(X_i, X_j)\, \varphi(Z_{E_k}, hX_1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, hX_{p+1}) =$$

$$= h^* d\varphi(X_1, \ldots, X_{p+1}) +$$

$$- \frac{1}{2(p-1)!} \sum_{\pi} \operatorname{sign} \pi\, \Omega^k(X_{\pi(1)}, X_{\pi(2)})\, \varphi(Z_{E_k}, hX_{\pi(3)}, \ldots, hX_{\pi(p+1)})$$

$\square$

Lemma II.7.2

$$dTS^p_{\omega;l_1,\dots,l_m} = \sum_{i=1}^{m} \imath(Z_{E_{l_i}})\, TS^p_{\omega;l_1\dots\hat{l}_i\dots l_m}$$

Proof: we have

$$\imath(Z_{E_{l_i}})\, T\mathcal{S}^p_{\omega;l_1\dots\hat{l}_i\dots l_m} =$$

$$= (-1)^p \frac{p!}{(p-m)!} \int_0^1 dt\, (1-t)^{m-1} \Big\{ \mathcal{S}^p(E_{l_i}, \Omega_t, \dots, \Omega_t, E_{l_1} \dots \hat{l}_i \dots E_{l_m})$$

$$- (p-m)\, \mathcal{S}^p(\omega, \imath(Z_{E_{l_i}})\Omega_t, \Omega_t, \dots, \Omega_t, \dots \hat{l}_i \dots) \Big\} =$$

$$= (-1)^p \frac{p!}{(p-m)!} \int_0^1 dt\, (1-t)^{m-1} \Big\{ \mathcal{S}^p(\Omega_t, \dots, \Omega_t, E_{l_1}, \dots, E_{l_m})$$

$$- (p-m)\, \mathcal{S}^p(\omega, (t^2-t)[E_{l_i}, \omega], \Omega_t, \dots, \Omega_t, \dots \hat{l}_i \dots) \Big\}$$

Thus,

$$\sum_{i=1}^{m} \imath(Z_{E_{l_i}})\, T\mathcal{S}^p_{\omega;l_1\dots\hat{l}_i\dots l_m} =$$

$$= (-1)^p \frac{p!}{(p-m)!} \int_0^1 dt\, \Big\{ \Big[-\frac{d}{dt}(1-t)^m\Big] \mathcal{S}^p(\Omega_t, \dots, \Omega_t, E_{l_1}, \dots, E_{l_m})$$

$$- (p-m)\,(1-t)^m\, \mathcal{S}^p(t[\omega, \omega], \Omega_t, \dots, \Omega_t, E_{l_1}, \dots, E_{l_m})$$

$$+ (p-m)(p-m-1)\,(1-t)^m\, \mathcal{S}^p(\omega, t[\omega, \Omega_t], \Omega_t, \dots, \Omega_t, E_{l_1}, \dots, E_{l_m}) \Big\} =$$

$$= (-1)^p \frac{p!}{(p-m)!} \int_0^1 dt\, \frac{d}{dt} \Big[-(1-t)^m\, \mathcal{S}^p(\Omega_t, \dots, \Omega_t, E_{l_1}, \dots, E_{l_m}) \Big]$$

$$+ (-1)^p \frac{p!}{(p-m)!} \int_0^1 dt \, (1-t)^m (p-m) \Big\{ \varphi^p \Big(\frac{d\Omega_t}{dt}, \Omega_t, \ldots, \Omega_t, E_{\ell_1}, \ldots, E_{\ell_m} \Big)$$

$$- \varphi^p \big(t[\omega, \omega], \Omega_t, \ldots, \Omega_t, E_{\ell_1}, \ldots, E_{\ell_m} \big)$$

$$- (p-m-1)\, \varphi^p \big(\omega, d\Omega_t, \Omega_t, \ldots, \Omega_t, E_{\ell_1}, \ldots, E_{\ell_m} \big) \Big\} =$$

$$= (-1)^p \frac{p!}{(p-m-1)!} \int_0^1 dt \, (1-t)^m \, d\, \varphi^p (\omega, \Omega_t, \ldots, \Omega_t, E_{\ell_1}, \ldots, E_{\ell_m})$$

since $\Omega_t = -t\, d\omega - \frac{t^2}{2}[\omega, \omega]$.

$\square$

Lemma II.7.3 *Let $\tilde\delta$ be the coboundary operator associated to the representation $\tilde\vartheta$ of $\chi.(P)$. Then*

$$D\sum_{m=0}^{p-1} TS^{k,2p-k-1}_{\omega,\tilde\omega;m}(Z_1,\ldots,Z_k) = (-)^{p+k}\,\imath(Z_1,\ldots,Z_k)S^p(\tilde\Omega)+$$

$$+\,(-)^{k+1}\sum_{m=0}^{p-1}\frac{1}{m!}\Omega^{l_1}\wedge\cdots\wedge\Omega^{l_m}\wedge h^*[(\tilde\delta\,\imath^{k-1}TS^p_{\tilde\omega;l_1,\ldots,l_m})(Z_1,\ldots,Z_k)]$$

Proof:

as $D\Omega = 0$ we have

$$D\sum_{m=0}^{p-1} TS^{k,2p-k-1}_{\omega,\tilde\omega,m}(Z_1,\ldots,Z_k) =$$

$$= \sum_{m=0}^{p-1}\frac{1}{m!}\Omega^{l_1}\wedge\ldots\wedge\Omega^{l_m}\wedge D\circ h^*\,\imath(Z_1,\ldots,Z_k)TS^p_{\tilde\omega,l_1,\ldots,l_m}$$

By Lemma II.7.1 and Lemma II.3.2 we have

$$(D\circ h^*)\imath(Z_1,\ldots,Z_k) = h^*\Big[d\,\imath(Z_1,\ldots,Z_k) - \Omega^s\wedge\imath(Z_{E_s})\imath(Z_1,\ldots,Z_k)\Big] =$$

$$= (-1)^k h^*\Big[\imath(Z_1,\ldots,Z_k)d - \Omega^s\wedge\imath(Z_1,\ldots,Z_k)\imath(Z_{E_s}) +$$

$$-\sum_{i=1}^{k}(-1)^{i+1}\imath(Z_1,\ldots,\hat{i},\ldots,Z_k)L_{Z_i} + \sum_{i<j}(-1)^{i+j}\imath([Z_i,Z_j],Z_1,\ldots,Z_k)\Big]$$

From Lemma II.7.2 and Proposition II.3.4 it follows that

$$d\,TS^p_{\tilde\omega;l_1,\ldots,l_m} = \begin{cases} (-)^p S^p(\tilde\Omega) & \text{for } m=0 \\ \sum_{i=1}^{m}\imath(Z_{E_{l_i}})TS^p_{\tilde\omega;l_1,\ldots,\hat{l_i},\ldots,l_m} & \text{for } m\geq 1\end{cases}$$

Therefore

$$D\sum_{m=0}^{p-1} TS^{k,2p-k-1}_{\omega,\tilde\omega,m}(Z_1,\ldots,Z_k) = (-1)^{k+p}\imath(Z_1,\ldots,Z_k)S^p(\tilde\Omega) +$$

$$+ (-1)^{k} \sum_{m=1}^{p-1} \frac{1}{m!} \Omega^{\ell_1} \wedge \cdots \wedge \Omega^{\ell_m} \wedge h^* \iota(z_1, \ldots, z_k) \sum_{\alpha=1}^{m} \iota(z_{E_{\ell_\alpha}}) T\varphi^{p}_{\tilde{\omega};\ldots \hat{\ell}_\alpha \ldots} +$$

$$+ (-1)^{k+1} \sum_{m=0}^{p-1} \frac{1}{m!} \Omega^{\ell_1} \wedge \cdots \wedge \Omega^{\ell_m} \wedge \Omega^{s} \wedge h^* \iota(z_1, \ldots, z_k) \iota(z_{E_s}) T\varphi^{p}_{\tilde{\omega}, \ell_1 \ldots \ell_m} +$$

$$+ (-1)^{k+1} \sum_{m=0}^{p-1} \frac{1}{m!} \Omega^{\ell_1} \wedge \cdots \wedge \Omega^{\ell_m} \wedge h^* \left[\left(\tilde{\delta}\, \iota^{k-1} T\varphi^{p}_{\tilde{\omega}; \ell_1, \ldots, \ell_m} \right)(z_1, \ldots, z_k) \right]$$

where use has been made of the fact that $T\varphi^{p}_{\tilde{\omega}, \ell_1 \ldots \ell_m}$ is a polynomial form and therefore $\hat{\theta}(z_i) T\varphi^{p}_{\tilde{\omega}; \ell_1 \ldots \ell_m} = L_{z_i} T\varphi^{p}_{\tilde{\omega}, \ell_1 \ldots \ell_m}$ (see page 51)

Rearranging the terms in the second and third summation we get

$$D \sum_{m=0}^{p-1} T\varphi^{k, 2p-k-1}_{\omega, \tilde{\omega}, m}(z_1, \ldots, z_k) = (-1)^{k+p} \iota(z_1, \ldots z_k)\, \varphi^{p}(\Omega) +$$

$$+ (-1)^{k+1} \sum_{m=0}^{p-1} \frac{1}{m!} \Omega^{\ell_1} \wedge \cdots \wedge \Omega^{\ell_m} \wedge h^* \left[\left(\tilde{\delta}\, \iota^{k-1} T\varphi^{p}_{\tilde{\omega}, \ell_1, \ldots, \ell_m} \right)(z_1, \ldots z_k) \right] +$$

$$+ (-1)^{k+1} \frac{1}{p!} \sum_{\alpha=1}^{p} \Omega^{\ell_1} \wedge \cdots \wedge \Omega^{\ell_p} \wedge h^* \iota(z_1, \ldots, z_k) \iota(z_{E_{\ell_\alpha}}) T\varphi^{p}_{\tilde{\omega}, \ell_1 \ldots \hat{\ell}_\alpha \ldots \ell_p}$$

where the last term vanishes for $k \geq 1$ because $T\varphi^{p}_{\tilde{\omega}; \ell_1, \ldots \ell_{p-1}}$ is a 1-form

□

Lemma II.7.4 *Let K be a tensor field of type (1,2) on M and κ the associated horizontal, ad-equivariant, $gl(n,\mathbf{R})$-valued 1-form on LM. Then to $L_X K$ is associated $L_{\tilde{X}}\kappa$, where $\tilde{X}$ is the natural lift of $X \in \mathcal{X}(M)$.*

Proof: let $u_x = (e_1(x), \ldots e_n(x))$ be a linear frame at $x \in M$ and denote by $\{e^{*i}(x)\}$ the dual basis of TM^*_x.

By definition the components of the tensor field K with respect to the linear frame u_x are

$$K(x)^i_{jk} = K(x; e^{*i}(x), e_j(x), e_k(x)), \quad x \in M.$$

We therefore have

$$L_X K(x, d\pi_{u_x} w)^i_j = L_X K(x; d\pi_{u_x} w, e^{*i}(x), e_j(x)) =$$

$$= \lim_{t\to 0} \frac{1}{t} \Big[K(\varphi_t(x); (d\varphi_t)_x d\pi_{u_x} w, \varphi_t^{-1*} e^{*i}(\varphi_t(x)), (d\varphi_t)_x e_j(x)) -$$

$$- K(x; d\pi_{u_x} w, e^{*i}(x), e_j(x)) \Big], \quad w \in TLM_{u_x}$$

where φ_t is the one-parameter group of diffeomorphisms of M generated by X.

The basis $\{\varphi_t^{-*} e^{*i}(\varphi_t(x))\}$ of $TM^*_{\varphi_t(x)}$ is clearly dual to the basis $\{(d\varphi_t)_x e_i(x)\} = \varphi_{*t}(u_x)$ (see Definition II.7.2)

Therefore

$$L_X K(x, d\pi_{u_x} w)^i_j =$$

$$= \lim_{t\to 0} \frac{1}{t} \Big[\kappa^i_j(\varphi_{*t}(u_x); (d\varphi_{*t})_{u_x} w) - \kappa^i_j(u_x; w) \Big] =$$

$$= L_{\tilde{X}} \kappa^i_j(u_x)$$

where we used the definition of the 1-form κ and the fact that $\varphi_t \circ \pi = \pi \circ \varphi_{*t}$. $\square$

Proposition II.7.5 *Let $X \in \imath(M)$ and $o_\alpha = \{E_{i\alpha}\}$, $i = 1, \ldots, n$, be a local section of OM^+. Denote by $[X, E_{i\alpha}]^j$, $j = 1, \ldots, n$, the components of the vector field $[X, E_{i\alpha}]$ with respect to the linear frame o_α. Then the matrix $[X, E_\alpha](x)$, $x \in U_\alpha$, whose elements are given by $[X, E_{i\alpha}]^j(x)$, belongs to $so(n, \mathbf{R})$.*

Proof: let g be the metric tensor field on M.
For any vector fields Y_i, $i = 1, 2, 3$, on M we have (see [26])

$$L_{Y_1} g(Y_2, Y_3) = Y_1 g(Y_2, Y_3) - g([Y_1, Y_2], Y_3) - g(Y_2, [Y_1, Y_3])$$

In particular for $X \in \imath(M)$ and $E_{i\alpha}$, $E_{j\alpha}$

$$0 = L_X g(E_{i\alpha}, E_{j\alpha}) = -g([X, E_{i\alpha}], E_{j\alpha}) - g(E_{i\alpha}, [X, E_{j\alpha}])$$

Since the components of g with respect to the orthonormal frame o_α are δ^i_j the above equation implies

$$[X, E_{i\alpha}]^j(x) + [X, E_{j\alpha}]^i(x) = 0$$

$\square$

Proposition II.8.5 *Let τ be a q-form on M. Then for any infinitesimal conformal transformation Z we have*

$$L_Z {}^*\tau = {}^*L_Z\tau + (1 - \frac{2q}{n})(div\ Z)\,{}^*\tau$$

where ${}^*\tau$ denotes the Hodge dual form of τ (see Definition I.1.17).

Proof: by definition we have (see Definition I.1.17)

$$(L_Z {}^*\tau)_{\mu_1 \ldots \mu_{m-q}} =$$

$$= \frac{1}{q!}\varepsilon_{\mu_1\ldots\mu_{m-q}\nu_1\ldots\nu_q}\left[(L_Z\sqrt{g})\,\tau^{\nu_1\ldots\nu_q} - \sqrt{g}\,(L_Z\tau)^{\nu_1\ldots\nu_q}\right]$$

where

$$L_Z\sqrt{g} \doteq \lim_{t\to 0}\frac{1}{t}\left\{\sqrt{\det[(\varphi_t^* g)_{\mu\nu}]} - \sqrt{\det[g_{\mu\nu}]}\right\}$$

φ_t being the one-parameter group of diffeomorphisms generated by Z. Since $\sqrt{g}$ is a scalar density of weight -1 (see [40]) we locally have

$$L_Z\sqrt{g} = Z^s\partial_s\sqrt{g} + \sqrt{g}\,\partial_s Z^s = \partial_s(Z^s\sqrt{g}) =$$

$$= \sqrt{g}\ div\ Z \qquad (\text{see I.21})$$

Moreover (see I.9)

$$(L_Z\tau)^{\nu_1\ldots\nu_q} = L_Z\left(g^{\nu_1 s_1}\cdots g^{\nu_q s_q}\tau_{s_1\ldots s_q}\right) =$$

$$= \sum_{i=1}^{q} g^{\nu_1 s_1}\cdots (L_Z g)^{\nu_i s_i}\cdots g^{\nu_q s_q}\tau_{s_1\ldots s_q} +$$

$$+ g^{\nu_1 s_1}\cdots g^{\nu_q s_q}(L_Z\tau)_{s_1\ldots s_q}$$

For an infinitesimal conformal transformation Z we have

$$(L_Z g)^{\mu\nu} = -\frac{2}{m}(\operatorname{div} Z)\, g^{\mu\nu}$$

Therefore

$$(L_Z \tau)^{\nu_1 \cdots \nu_q} = -\frac{2q}{m}(\operatorname{div} Z)\, \tau^{\nu_1 \cdots \nu_q} + g^{\nu_1 \rho_1} \cdots g^{\nu_q \rho_q} (L_Z \tau)_{\rho_1 \cdots \rho_q}$$

from which the Proposition easily follows.

$\square$