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Hele-Shaw approximation for resin transfer molding

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A Hele-Shaw flow is studied as the limit of a Stokes flow. The velocity fields associated to each fluid solve a fluid dynamic problem with time dependent domain. The moving boundary formulation is based on a pseudo-concentration equation. Some existence of solution of this set of equations is given using the classical Leary-Schauder Theorem and the use of space-time integrated least square method for the transport equation.

1 Introduction

The Resin Transfer Molding (RTM) is a process used to make complex thin mouldings from monomers or oligomers with low viscosity [1, 7]. The mould is represented by a bounded domain W in $\mathbb{R}^3$, decomposed into two sub-domains $W_1(t)$ and $W_2(t)$. These domains are separated by the interface $G_i(t)$. The domain $W_1(t)$ represents the fluid part and $W_2(t)$ the air. (see Fig. 1). Let T be the filling time of the domain. It is assumed that the interface $G_i(t)$ remains sufficiently regular and connected and that the domain W has a piecewise C^1 boundary.

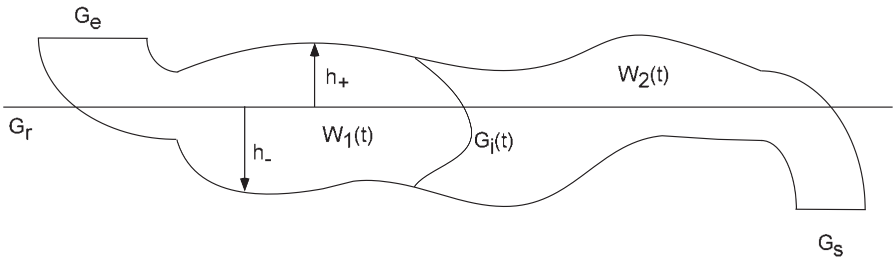

Fig. 1 Schematic mould

The fluid flow in W is driven by the Stokes equations. In order to represent the time evolution of the domains $W_1(t)$ and $W_2(t)$, a pseudo-concentration function $s(t, \xi)$ [10] is used. This function satisfies $s(t, \xi) = 1$ in $W_1(t)$ and $s(t, \xi) = 0$ in $W_2(t)$. So the free boundary $G_i(t)$ is fully described by the function s. This problem was studied by Nouri, Poupeau, and Demay in [21] for multi-fluid Stokes problems. In [1], existence results for the RTM moulding are proved, taking into account chemical reactions. The problem without chemical reaction is solved in [21].

The aim of this paper is to study this problem when the mould thickness goes to zero. Starting from a model where the flow is driven by the Stokes equation and the free boundary is described by the pseudo-concentration function s, an asymptotic *Hele-Shaw* model is obtained. For this, asymptotic expansions and models are used. Some existence results are then presented for the solution of the asymptotic model and some convergence results are given.

The Hele-Shaw model is widely used to describe injection moulding processes (see e.g. [14, 15, 17, 22–24]). In [14] the velocity field is a potential flow and the free boundary tracking is studied using variational inequalities. The two books [15,24] present a comprehensive study of free boundary problems via variational inequalities. In [24] a Hele-Shaw model is derived starting from the full Navier-Stokes equation and using a priori simplifications motivated by the geometry of the domain and the flow character. In [17] a Hele-Shaw model is derived starting form the porous media Darcy law. In [26], analytic results are given for Hele-Shaw in $\mathbb{R}^n$ with $n \geq 3$; these results generalize the classical 2D results. Let us also mention the web page http://www.maths.ox.ac.uk/˜howison/Hele-Shaw/, where a more than 600 references are available until 1998.

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In Sect. 2 a description of the problem is given. A scale change leads to a domain which is independent of the thickness of W. The Hele-Shaw model is then obtained. It has the advantage to transform the initial system into a very simple system of 2D equations. In Sect. 3 a variational formulation of the problem is given. The Sect. 4 is dedicated to the proof of the existence of solutions to the variational formulation of the problem. In Sect. 5 some estimations (similar to those in [8]) are presented. They allow to obtain Hele-Shaw equations. Instead of using renormalized solutions like in [21], a space-time least square method like in [4–6] is used to study the free boundary evolution. This method is well adapted to this case.

2 The problem description

Let $W \subset \mathbb{R}^3$ be the domain with sufficiently regular boundary defined by (see Fig. 1)

$$W = \left\{ \xi = (\xi_i) \in \mathbb{R}^3,\ (\xi_1, \xi_2) \in G_r,\ h_-(\xi_1, \xi_2) < \xi_3 < h_+(\xi_1, \xi_2) \right\} ,$$

where G_r is a reference plane and $h_-, h_+ : G_r \to \mathbb{R}$ are Lipschitz functions. For $T > 0$, and for $1 \le i \le 3$ set the strictly positive functions

$$\begin{aligned} \mu_i :\ [0,1] \times (0,T) \times W &\to [m_\mu, M_\mu] \\ (z, t, \xi) &\mapsto \mu_i(z, t, \xi) \end{aligned}$$

with $\mu_i(0,t,\xi) = \mu_i^a(t,\xi)$ and $\mu_i(1,t,\xi) = \mu_i^p(t,\xi)$; μ^a is the air viscosity and $\mu = \operatorname{diag}(\mu_1, \mu_2, \mu_3)$ is a diagonal anisotropic viscosity tensor. The tensor μ^p denotes the polymer viscosity. It is assumed that μ^a and μ^p are continuous functions. The face G_e describes the input of the mould and G_s its output. Finally set

$$G_v = G_e \cup G_s ,$$

and

$$G_h = \partial W - G_v .$$

In order to simplify, we assume that the faces G_v are not vertical.

2.1 The equations

With the previous notations, the air and polymer flows are driven by the following anisotropic equations (the inertial term is neglected compared with viscous terms [7]).

$$-\Delta_\mu v + \operatorname{grad} p = f \quad \text{in} \quad W , \tag{1}$$

$$\operatorname{div} v = 0 \quad \text{in} \quad W , \tag{2}$$

$$\partial_t s + (\, v \,|\, \operatorname{grad} s \,) = 0 \quad \text{in} \quad (0,T) \times W , \tag{3}$$

where $v = (v_1, v_2, v_3)$ represent the fluid velocity , p is the pressure, $f = (0, 0, -\rho g)$ is the gravity force, s is the pseudo-concentration function such that $\mu = \operatorname{diag}(\mu_1, \mu_2, \mu_3)$ is the diagonal turbulent viscosity tensor with $\mu_i(t,\xi) = \mu_i(s(t,\xi), t, \xi)$. The operator Δ_μ is defined by

$$\Delta_\mu \varphi = \sum_{i=1}^{3} \frac{\partial}{\partial \xi_i} \left(\mu_i \frac{\partial \varphi}{\partial \xi_i} \right) = \operatorname{div}(\mu \operatorname{grad} \varphi) .$$

Moreover the boundary conditions for the velocity v and the pseudo-concentration s are

$$v = 0 \quad \text{on} \quad G_h , \tag{4}$$

$$v = v_e \quad \text{and} \quad s = 1 \quad \text{on} \quad G_e , \tag{5}$$

$$v = v_s \quad \text{on} \quad G_s , \tag{6}$$

$$s(0,\xi) = 0 \quad \forall \xi \in W . \tag{7}$$

We assume that the velocity v satisfies $(\, v_s \,|\, n \,) > 0$ on G_s and $(\, v_e \,|\, n \,) < 0$ on G_e. As usual the vector n is the outward normal vector on the boundary of W.

Let d be the diameter of G_h, $h_0 = sup_{(\xi_1,\xi_2)\in G_r}|h_+(\xi_1,\xi_2) - h_-(\xi_1,\xi_2)|$ and define

$$\varepsilon = \frac{h_0}{d}\,.$$

Since h_0 is small compared to d, one has $0 < \varepsilon < 1$. From now it is assumed, without loss of generality, that $d = 1$. In order to make W independent of ε, let us do the following changes of variables and functions:

$$x_1 = \xi_1\,, \quad x_2 = \xi_2\,, \quad x_3 = \xi_3/\varepsilon\,,$$
$$u_1 = v_1\,, \quad u_2 = v_2\,, \quad u_3 = v_3/\varepsilon\,;$$

since p is large, we set

$$p = \frac{q}{\varepsilon^2}\,.$$

We then define

$$\begin{aligned}
\Gamma_r &= \{(x_1,x_2)\,;\quad (\xi_1,\xi_2)\in G_r\}\,,\\
\Omega &= \left\{x=(x_1,x_2,x_3)\,;\quad (x_1,x_2)\in\Gamma_r\,,\quad \frac{h_-(x_1,x_2)}{h_0} < x_3 < \frac{h_+(x_1,x_2)}{h_0}\right\}\,,\\
Q &= (0,T)\times\Omega\,,\\
\Gamma_h &= \left\{x=(x_i)\in\mathbb{R}^3,\ \xi=(\xi_i)\in G_h\right\}\,,\\
\Gamma_v &= \left\{x=(x_i)\in\mathbb{R}^3,\ \xi=(\xi_i)\in G_v\right\}\\
&= \Gamma_e\cup\Gamma_s\,.
\end{aligned}$$

With these changes, we obtain the following problem

$$\begin{aligned}
-\Delta_{\mu_{\varepsilon^2}} u_i + \frac{\partial q}{\partial x_i} &= 0\,,\quad i=1,2 \quad \text{in} \quad \Omega\,,\\
-\varepsilon^2\cdot\Delta_{\mu_{\varepsilon^2}} u_3 + \frac{\partial q}{\partial x_3} &= -\varepsilon^3\cdot\rho g \quad \text{in} \quad \Omega\,,\\
\operatorname{div} u &= 0 \quad \text{in} \quad \Omega\,,\\
\partial_t s + (\,u\,|\,\operatorname{grad} s\,) &= 0 \quad \text{in} \quad Q\,,
\end{aligned}$$

with the boundary and initial conditions

$$\begin{aligned}
u &= 0 \quad \text{on} \quad \Gamma_h\,,\\
u &= u_n \quad \text{on} \quad \Gamma_n\,,\\
s &= 1 \quad \text{on} \quad (0,T)\times\Gamma_e\,,\\
s(0,x) &= 0 \quad \text{in} \quad \Omega\,,
\end{aligned}$$

where $\mu_\delta = \operatorname{diag}(\delta\mu_1,\delta\mu_2,\mu_3)$.

Remark 1. As usual, after a modification of the right-hand side member in the above equation, we assume that $u = 0$ on $\partial\Omega$ and $s = 0$ on $(0,T)\times\Gamma_e$. More precisely, let b defined as $b = -\partial_t s_0 - (\,u\,|\,\operatorname{grad} s_0\,)$ where s_0 is a sufficiently regular extension to Q, of the function equal to 1 on Γ_e with $s_0(0,x) = 0$ on a maximal part of Ω and $0 \le s_0 \le 1$; then set $c = s - s_0$.

Hence we obtain

$$-\Delta_{\mu_{\varepsilon^2}} u_i + \frac{\partial q}{\partial x_i} = f_i\,,\quad i=1,2 \quad \text{in} \quad \Omega\,, \tag{8}$$

$$-\varepsilon^2\cdot\Delta_{\mu_{\varepsilon^2}} u_3 + \frac{\partial q}{\partial x_3} = \varepsilon^3\cdot f_3 \quad \text{in} \quad \Omega\,, \tag{9}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad \Omega\,, \tag{10}$$

$$\partial_t c + (\, u \,|\, \operatorname{grad} c \,) = b \quad \text{in} \quad Q\,, \tag{11}$$

with the boundary and initial conditions

$$u = 0 \quad \text{on} \quad \partial\Omega\,, \tag{12}$$

$$c = 0 \quad \text{on} \quad (0,T) \times \Gamma_e\,, \tag{13}$$

$$c(0,x) = 0 \quad \text{in} \quad \Omega\,. \tag{14}$$

2.2 The Hele Shaw limit case

Intuitively, when $\varepsilon \to 0$, we get the following problem

$$-\frac{\partial}{\partial x_3}\left(\mu_3 \frac{\partial u_1}{\partial x_3}\right) + \frac{\partial q}{\partial x_1} = 0 \quad \text{in} \quad \Omega\,, \tag{15}$$

$$-\frac{\partial}{\partial x_3}\left(\mu_3 \frac{\partial u_2}{\partial x_3}\right) + \frac{\partial q}{\partial x_2} = 0 \quad \text{in} \quad \Omega\,, \tag{16}$$

$$\frac{\partial q}{\partial x_3} = 0 \quad \text{in} \quad \Omega\,, \tag{17}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad \Omega\,, \tag{18}$$

$$\partial_t s + (\, u \,|\, \operatorname{grad} s \,) = 0 \quad \text{in} \quad Q\,. \tag{19}$$

with the boundary and initial conditions

$$u = 0 \quad \text{on} \quad \Gamma_h\,, \tag{20}$$

$$s = 1 \quad \text{on} \quad \Gamma_e \quad \text{ae}\; t \in (0,T)\,, \tag{21}$$

$$u_i \cdot n_3 = u_{v,i} \cdot n_3 \quad \text{on} \quad \Gamma_v\,, \quad i = 1,2\,, \tag{22}$$

$$s(0,x) = 0 \quad \text{in} \quad \Omega\,, \tag{23}$$

Eqs. (15)–(19) are called the *Hele-Shaw equations*. Eqs. (15)–(18) describe the velocity field and eq. (19) the evolution of the moving boundary for the limit problem. These equations can be modified in order to obtain a simplified 2D problem for the velocity u. Indeed eq. (17) shows that q is independent of x_3. In order to simplify, let us assume that μ_3 does not depend of x_3. Integrating eqs. (15)–(16) we get

$$u_i(x) = \frac{1}{2\mu_3}\frac{\partial q}{\partial x_i}(x) \cdot (x_3 - h_-)(x_3 - h_+)\,, \tag{24}$$

for $i = 1, 2$, and using (18)

$$\begin{aligned} u_3(x) = &-\frac{1}{12\mu_3}\left(\frac{\partial^2 q}{\partial {x_1}^2}(x) + \frac{\partial^2 q}{\partial {x_1}^2}(x)\right) \cdot (x_3 - h_-)(x_3 - h_+) \\ &+ \frac{1}{4\mu_3}\left(\frac{\partial q}{\partial x_1}\frac{\partial h_-}{\partial x_1} + \frac{\partial q}{\partial x_2}\frac{\partial h_-}{\partial x_2}\right)\big((x_3^2 - h_-^2) - 2h_+(x_3 - h_-)\big) \\ &+ \frac{1}{4\mu_3}\left(\frac{\partial q}{\partial x_1}\frac{\partial h_+}{\partial x_1} + \frac{\partial q}{\partial x_2}\frac{\partial h_+}{\partial x_2}\right)\big((x_3^2 - h_-^2) - 2h_-(x_3 - h_-)\big)\,. \end{aligned} \tag{25}$$

Let $\overline{u_i}$ be the mean value of u_i along the x_3 axis:

$$\overline{u_i} = \frac{1}{h_+ - h_-}\int_{h_-}^{h_+} u_i(x)dx_3\,, \quad i = 1,2,3\,,$$

we get

$$\overline{u_1} = -\frac{(h_+ - h_-)^2}{12\mu_3}\,\frac{\partial q}{\partial x_1}\,,$$

$$\overline{u_2} = -\frac{(h_+ - h_-)^2}{12\mu_3}\,\frac{\partial q}{\partial x_2}\,,$$

$$\overline{u_3} = 0\,,$$

so using (18)

$$\operatorname{div}\big((h_+ - h_-)^3\,\operatorname{grad} q\big) = 0\,. \tag{26}$$

The pressure q is obtained using eq. (26). Then we get the velocity components u_1, u_2 and u_3 using eqs. (24) and (25).

Remark 2. a) Usually the 2D elliptic Darcy eq. (26) is the starting point to study Hele-Shaw problems. Our aim is to prove that eqs. (15)–(19) admit a solution which is the limit when $\varepsilon \to 0$ of a solution of eqs. (8)–(11).

b) Remark also that the use of the pseudo-concentration function s to describe the free boundary allows to handle with moving boundary having less regularity than in usual methods.

3 Variational formulations

In this section we give a variational formulation of the problem (8)–(14). Moreover eq. (11) is solved using a least square method like in [4–6]. Some extensions of results given in the mentioned papers are presented.

3.1 The Hilbert spaces

As usual, let $H^s(\Omega)$ be the Sobolev space of order s, and $H_0^1(\Omega) = \big\{\varphi \in H^1(\Omega);\ \varphi = 0 \quad \text{on} \quad \partial\Omega\big\}$. Let us consider the following Hilbert spaces

$$V = H_0^1(\Omega)^3$$

equipped with the scalar product

$$(\,u\,|\,v\,)_1 = \int_\Omega (\,\nabla u\,|\,\nabla v\,)\,dx\,,$$

$$H(\Omega, \operatorname{div}) = \big\{w \in L^2(\Omega)^3,\ \operatorname{div} w \in L^2(\Omega)\big\}$$

equipped with the graph norm,

$$H_0(\Omega, \operatorname{div}) = \{w \in H(\Omega, \operatorname{div}),\ (\,w\,|\,n\,) = 0 \quad \text{on} \quad \partial\Omega\}\,,$$

$$H(\partial_3, \Omega) = \left\{\varphi \in L^2(\Omega),\ \frac{\partial\varphi}{\partial x_3} \in L^2(\Omega)\right\}$$

equipped with the graph norm,

$$H_0(\partial_3, \Omega) = \{\varphi \in H(\partial_3, \Omega),\ \varphi\, n_3 = 0 \quad \text{on} \quad \partial\Omega\}\,,$$

and for $\Gamma \subset \partial\Omega$ with non zero measure

$$H_0(\partial_3, \Omega, \Gamma) = \{\varphi \in H(\partial_3, \Omega),\ \varphi\, n_3 = 0 \quad \text{on} \quad \Gamma\}\,,$$

$$L_0^2(\Omega) = \left\{\varphi \in L^2(\Omega);\ \int_\Omega \varphi(x)\,dx = 0\right\}.$$

The space $H_0(\partial_3, \Omega)$ is equipped with the norm

$$|\varphi|_{3,\Omega} = \left\|\frac{\partial\varphi}{\partial x_3}\right\|_{L^2(\Omega)}$$

which is equivalent to the norm on $H(\partial_3, \Omega)$. Finally for $w \in L^\infty(Q)^3$, define $\widetilde{w}$ as

$$\widetilde{w} = (1, w_1, w_2, w_3)^t \in L^\infty(Q)^4$$

and for the sufficiently regular function φ defined on Q, set

$$\widetilde{\nabla}\varphi = \left(\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_3}\right)^t .$$

Then for $\varphi \in \mathcal{D}(\overline{Q})$, consider the norm

$$\|\varphi\|_{H(w,Q)} = \left(\|\varphi\|^2_{L^2(Q)} + \left\|\left(\,\widetilde{w}\,|\,\widetilde{\nabla}\varphi\right)\right\|^2_{L^2(Q)}\right)^{1/2}$$

(see [4–6,9]) and then define the space $H(w,Q)$ as the closure of $\mathcal{D}(\overline{Q})$ for this norm:

$$H(w,Q) = \overline{\mathcal{D}(\overline{Q})}^{H(w,Q)} .$$

If w is regular enough, it can be seen that

$$H(w,Q) = \left\{\varphi \in L^2(Q), \left(\,\widetilde{w}\,|\,\widetilde{\nabla}\varphi\right) \in L^2(Q)\right\}$$

(see e.g. [16,20]). For

$$\begin{aligned}\partial Q_- &= \{(t,x) \in \partial Q, (\,\widetilde{w}\,|\,\widetilde{n}\,) < 0\}\,,\\ &= (0,T) \times \Gamma_e \cup \{0\} \times \Omega\,,\end{aligned}$$

if $\Gamma \subset \partial Q_-$, let

$$H_0(w,Q,\Gamma) = \{\varphi \in H(w,Q), \varphi = 0 \quad \text{on} \quad \Gamma\}\ .$$

Let us define the linear form $\mathcal{A}c = \left(\,\widetilde{w}\,|\,\widetilde{\nabla}c\right)$. Then eq. (11) can be written as $\mathcal{A}c = b$. We now give an extension of the curved Poincaré inequality obtained in [4,5].

Theorem 1 (Curved Poincaré inequality)**.** *If* $\operatorname{div} w = 0$*, the semi-norm on* $H_0(w,Q,\Gamma)$ *defined by*

$$|\varphi|_{1,w} = \left(\int_Q (\mathcal{A}\varphi)^2 dtdx\right)^{1/2}$$

is a norm, equivalent to the norm on $H(w,Q)$.

Proof. We have to show that there is a constant C such that

$$\|\varphi\|_{L^2(Q)} \leq C \cdot |\varphi|_{1,w}$$

for all $\varphi \in \mathcal{D}(\overline{Q})$ with $\varphi = 0$ on ∂Q_-. Set

$$\partial Q_+ = \{(t,x) \in \partial Q, (\,\widetilde{w}\,|\,\widetilde{n}\,) > 0\}\ .$$

Since $\widetilde{\operatorname{div}}(\widetilde{w}) = 0$, we have

$$\int_Q \left[\left(\,\widetilde{w}\,|\,\widetilde{\nabla}\varphi\right) \cdot \xi + \varphi \cdot \left(\,\widetilde{w}\,|\,\widetilde{\nabla}\xi\right)\right] dtdx = \int_{\partial Q_+} \xi\varphi\,(\,\widetilde{w}\,|\,\widetilde{n}\,)\,d\widetilde{\sigma}$$

for all regular enough function ξ. Choose $\xi = (T-t)\varphi$, then

$$\int_Q \left[2\left(\,\widetilde{w}\,|\,\widetilde{\nabla}\varphi\right) \cdot (T-t)\varphi - \varphi^2\right] dtdx = \int_{\partial Q_+} (T-t)\varphi^2\,(\,\widetilde{w}\,|\,\widetilde{n}\,)\,d\widetilde{\sigma} \geq 0\,.$$

So using the Cauchy-Schwarz inequality we get

$$\|\varphi\|_{L^2(Q)} \leq 2T \cdot \left\|\left(\,\widetilde{w}\,|\,\widetilde{\nabla}\varphi\right)\right\|_{L^2(Q)}\ .$$

□

Henceforth the space $H_0(w,Q,\Gamma)$ is equipped with the norm $|\varphi|_{1,w}$.

Remark 3. As an easy consequence of the above arguments, for any $\varphi\in H_0(w,Q,\Gamma)$, the norm defined by:

$$|||\varphi||| = \left(\|\varphi\|^2_{L^2(Q)} + \int_{\partial Q_+} (\,\widetilde{w}\,|\,\widetilde{n}\,)\,(T-t)\varphi^2 d\widetilde{\sigma}\right)^{1/2}$$

verifies

$$\|\varphi\|_{L^2(Q)} \le |||\varphi||| \le 2T|\varphi|_{1,w}\,.$$

3.2 The non divergence free case

In this section, a generalization of Theorem 1 is presented when $u\in L^\infty(Q)^3$, $\operatorname{div} u\in L^\infty(Q)$. In this situation the advection eq. (11) has the form

$$\frac{\partial c}{\partial t} + \operatorname{div}(uc) = b \quad\text{in}\quad Q\,, \tag{27}$$

with the same boundary conditions as in Sect. 2.1 and where $u: Q\to\mathbb{R}$ is a given vector field. Define the operator $\mathcal{B}$ as

$$\mathcal{B}c = \frac{\partial c}{\partial t} + \operatorname{div}(u\,c)$$

and for $\varphi\in\mathcal{D}(\overline{Q})$, consider the norm

$$\|\varphi\|_{K(u,Q)} = \left(\|\varphi\|^2_{L^2(Q)} + \|\mathcal{B}\varphi\|^2_{L^2(Q)}\right)^{1/2}$$

and then define

$$K(u,Q) = \overline{\mathcal{D}(\overline{Q})}^{K(u,Q)}\,, \qquad\qquad K_0(u,Q,\partial Q_-) = \{\varphi\in K(u,Q), \varphi = 0 \quad\text{on}\quad \partial Q_-\}\,.$$

The following theorem is a generalization of Theorem 1.

Theorem 2. *If $u\in L^\infty(Q)^3$ and $\operatorname{div} u\in L^\infty(Q)$, the seminorm on $K_0(u,Q,\partial Q_-)$ defined by*

$$|\varphi|_{1,u} = \left(\int_Q (\mathcal{B}\varphi)^2 dt dx\right)^{1/2} \tag{28}$$

is a norm, equivalent to the norm given on $K(u,Q)$.

Proof. As in Theorem 1, we have to show that there is a constant C such that

$$\|\varphi\|_{L^2(Q)} \le C\cdot|\varphi|_{1,w}$$

for all $\varphi\in\mathcal{D}(\overline{Q})$ with $\varphi=0$ on ∂Q_-. We have

$$\int_Q \left[\left(\frac{\partial\varphi}{\partial t} + \operatorname{div}(u\,\varphi)\right)\cdot\xi + \varphi\cdot\left(\frac{\partial\xi}{\partial t} + (\,u\,|\,\nabla\xi\,)\right)\right] dt dx = \int_{\partial Q_+} \xi\varphi\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,d\widetilde{\sigma} \tag{29}$$

for all regular enough function ξ. For $\alpha:(0,T)\to\mathbb{R}$, choose $\xi=\alpha\cdot\varphi$, then

$$\frac{\partial\xi}{\partial t} + (\,u\,|\,\nabla\xi\,) = \alpha\cdot\left(\frac{\partial\varphi}{\partial t} + (\,u\,|\,\nabla\varphi\,)\right) + \alpha'\varphi = \alpha\cdot\left(\frac{\partial\varphi}{\partial t} + \operatorname{div}(u\varphi) - \varphi\operatorname{div} u\right) + \alpha'\varphi\,.$$

Let $v\in L^\infty(0,T)$ be defined by

$$v(t) = \sup_{x\in\Omega}|\operatorname{div}(u(t,x))|\,.$$

With the above choices, eq. (29) has the form

$$\int_Q \left[\left(\alpha' + \alpha v - \alpha(v + \operatorname{div} u)\right) \varphi^2 + 2\alpha\varphi \left(\frac{\partial \varphi}{\partial t} + \operatorname{div}(u\,\varphi) \right) \right] dtdx = \int_{\partial Q_+} \alpha \varphi^2 \,(\,\widetilde{u}\,|\,\widetilde{n}\,)\, d\widetilde{\sigma}\,. \tag{30}$$

Let α be the solution of the differential equation

$$\alpha' + \alpha v = -2\,, \quad \alpha(T) = 0\,.$$

An easy computation gives

$$\alpha(t) = 2\mathrm{e}^{-w(t)} \left(\int_0^T \mathrm{e}^{w(s)} ds - \int_0^t \mathrm{e}^{w(s)} ds \right) \geq 0\,,$$

with $w(t) = \int_0^t \mathrm{e}^{v(s)} ds$. Introducing this value in eq. (30) we obtain

$$\int_Q \left[-2\varphi^2 - \alpha(v + \operatorname{div} u)\varphi^2 + 2\alpha\varphi \left(\frac{\partial \varphi}{\partial t} + \operatorname{div}(u\,\varphi) \right) \right] dtdx = \int_{\partial Q_+} \alpha\varphi^2 \,(\,\widetilde{u}\,|\,\widetilde{n}\,)\, d\widetilde{\sigma} \geq\ 0\,.$$

Hence

$$\int_Q \varphi^2 dtdx \leq \int_Q \alpha\varphi \left(\frac{\partial \varphi}{\partial t} + \operatorname{div}(u\,\varphi) \right) dtdx\,,$$

and the result is proved using the Cauchy-Schwartz inequality. □

3.3 A weak formulation

With the above notations, a weak formulation of the problem (8–10, 12) is given by: Find $u \in L^\infty(0,T;V)$, $q \in L^\infty(0,T;L^2_0(\Omega))$ such that

$$\int_\Omega (\,\mu_{\varepsilon^2}\,\nabla u_i\,|\,\nabla v_i\,)\,dx - \int_\Omega q \cdot \frac{\partial v_i}{\partial x_i}\,dx = \int_\Omega f_i v_i\,dx\,, \quad i = 1,2\,, \tag{31}$$

$$\varepsilon^2 \cdot \int_\Omega (\,\mu_{\varepsilon^2}\,\nabla u_3\,|\,\nabla v_3\,)\,dx - \int_\Omega q \cdot \frac{\partial v_3}{\partial x_3}\,dx = \varepsilon^3 \cdot \int_\Omega f_3 v_3\,dx\,, \tag{32}$$

$$\int_\Omega \operatorname{div} u \cdot \psi\,dx = 0\,, \tag{33}$$

for all $v = (v_i) \in V$, and all $\psi \in L^2_0(\Omega))$.

In $L^2(Q))$, a solution of eq. (11) corresponds to a zero of the following convex, positive functional

$$J(c) = \tfrac{1}{2} \int_Q (\mathcal{A}c - b)^2 dxdt.$$

The Gâteau derivative of J is

$$DJ(c)\varphi = \int_Q (\mathcal{A}c - b)\mathcal{A}\varphi\,dtdx\,.$$

So a sufficient condition to get the least square solution of (11) is

$$\int_Q \mathcal{A}c \cdot \mathcal{A}\varphi\,dtdx = \int_Q b \cdot \mathcal{A}\varphi\,dtdx\,.$$

Hence a weak formulation of eqs. (11, 13, 14) is: Find $c \in H_0(u,Q,\partial Q_-)$ such that

$$\int_Q \left(\widetilde{u}\,|\,\widetilde{\nabla} c\right) \left(\widetilde{u}\,|\,\widetilde{\nabla}\varphi\right) dtdx = \int_Q b \cdot \left(\widetilde{u}\,|\,\widetilde{\nabla}\varphi\right) dtdx \tag{34}$$

for all $\varphi \in H_0(u,Q,\partial Q_-)$ (see [4–6, 9, 11, 12]) .

The next section is devoted to the proof of the following theorem.

Theorem 3. *The problem* (31)–(34) *has at least a solution.*

Like in [21], we will use a fixed point method, but without using the notion of renormalized solution.

4 Proof of Theorem 3

Let us first recall two well known results. With the notations and hypothesis of Sect. 3 we have

Theorem 4. *For some fixed* $t \in (0,T)$ *and* μ*, if* $f \in L^\infty(\Omega)^3$*, the problem* (31),(33) *has a unique solution, and* $u \in V \cap L^\infty(\Omega)^3$.

Proof. See e.g. [18,19,25]. □

Theorem 5. *For a fixed* $u \in L^\infty(Q)^3$ *such that* $\operatorname{div} u = 0$*, the problem* (34) *has a unique solution. Moreover*

$$|c|_{1,u} = \left\| \left(\widetilde{u} \,|\, \widetilde{\nabla} c \right) \right\|_{L^2(Q)} \le \|b\|_{L^2(Q)} .$$

This solution is the least square solution *of* (11).

Proof. This assertion is a consequence of the Curved Poincaré inequality (Theorem 1) and of the Lax-Milgram Theorem (see also [4]). □

Remark 4. For the numerical solution of eq. (34), a time marching approach can be used to avoid the consideration of all of Q (see e.g. [9,11,12]).

Corollary 6. *The solution* c *of eq.* (34) *belongs to the space*

$$X = L^2(Q) \cap L^2(\partial Q_+, (\,\widetilde{u} \,|\, \widetilde{n}\,)\, d\widetilde{\sigma})$$

equipped with the norm $|||c|||$.

The following theorem is a maximum principle for the solution of problem (34).

Theorem 7. *Let* s_0 *be the function defined in Sect.* 2.1*, Remark* 1*. Then the solution of* (34) *satisfies*

$$0 \le c + s_0 = s \le 1 .$$

Proof. For all $\varphi \in H_0(u, Q, \partial Q_-)$ we have

$$\int_Q \left(\widetilde{u} \,|\, \widetilde{\nabla}(c + s_0) \right) \left(\widetilde{u} \,|\, \widetilde{\nabla}\varphi \right) dtdx = 0 .$$

Choose $\varphi_1 = (c + s_0 - 1)^+$, then $\widetilde{\nabla}\varphi_1 = \widetilde{\nabla}(c + s_0)$ on the set $A_1 = \{(t,x) \in Q, c + s_0 - 1 > 0\}$. So we get

$$\int_{A_1} \left(\widetilde{u} \,|\, \widetilde{\nabla}(c + s_0) \right)^2 dtdx = 0 .$$

Hence the set A_1 has a zero measure, so $c + s_0 \le 1$. If we choose $\varphi_2 = (c + s_0)^-$, we show in the same way that $c + s_0 \ge 0$. □

Let us give a result concerning the operator

$$\begin{aligned} \mathcal{A} : H_0(u, Q, \partial Q_-) &\to L^2(Q) \\ \varphi &\mapsto \left(\widetilde{u} \,|\, \widetilde{\nabla}\varphi \right) . \end{aligned}$$

Lemma 8. *If* $u \in L^\infty(Q)^3$ *verifies* $\operatorname{div} u = 0$*, there exists a constant* $C > 0$ *such that*

$$\inf_{\varphi \in H_0(u,Q,\partial Q_-), \|\varphi\| \le 1} \ \sup_{\psi \in L^2(Q), \|\psi\| \le 1} \int_Q \left(\widetilde{u} \,|\, \widetilde{\nabla}\varphi \right) \cdot \psi \, dtdx \ge C , \tag{35}$$

and for all $\psi \in L^2(Q),\ \psi \ne 0$,

$$\sup_{\varphi \in H_0(u,Q,\partial Q_-), \|\varphi\| \le 1} \int_Q \left(\widetilde{u} \,|\, \widetilde{\nabla}\varphi \right) \cdot \psi \, dtdx > 0 . \tag{36}$$

Proof. The inequality (35) is a straightforward consequence of Theorem 1. Let us prove (36). Let $\psi \in L^2(Q)$ be of norm 1 and assume that inequality (36) is false. For all $\varphi \in \mathcal{D}(Q)$ we have

$$\int_Q \left(\widetilde{u} \,|\, \widetilde{\nabla}\varphi \right) \cdot \psi \, dtdx = 0\,.$$

Since

$$\int_Q \left(\widetilde{u} \,|\, \widetilde{\nabla}\varphi \right) \cdot \psi + \left(\widetilde{u} \,|\, \widetilde{\nabla}\psi \right) \cdot \varphi \, dt\, dx = 0$$

we get $\left(\widetilde{u} \,|\, \widetilde{\nabla}\psi \right) = 0$ in $\mathcal{D}'(Q)$, so in $L^2(Q)$, hence ψ belongs to $H(u, Q)$.

Now let us show that $\psi_{|\partial Q_+} = 0$. Let $\varphi \in \mathcal{D}(\overline{Q})$, $\varphi = 0$ on ∂Q_-. Since $\left(\widetilde{u} \,|\, \widetilde{\nabla}\psi \right) = 0$ belongs to $L^2(Q)$, we have

$$0 = \int_Q \left(\widetilde{u} \,|\, \widetilde{\nabla}\psi \right) \cdot \varphi \, dt\, dx = \int_{\partial Q_+} (\, \widetilde{u} \,|\, \widetilde{n} \,)\, \psi\varphi \, d\widetilde{\sigma}\,.$$

Therefore the support of the distribution $(\, \widetilde{u} \,|\, \widetilde{n} \,)\, \psi$ is a subset of ∂Q_-, so $\psi_{|\partial Q_+} = 0$. Using Theorem 1, with $\widetilde{w} = -\widetilde{u}$ we then get $\psi = 0$. □

The next theorem shows that the solution of (34) is also a weak solution.

Theorem 9. *For $f \in L^2(Q)$, let φ be the solution of*

$$\int_Q \left[\left(\widetilde{u} \,|\, \widetilde{\nabla}\varphi \right) - f \right] \cdot \left(\widetilde{u} \,|\, \widetilde{\nabla}\psi \right) \, dt\, dx = 0 \tag{37}$$

for all $\psi \in H_0(u, Q, \partial Q_-)$, i.e., the function φ is the least square solution of

$$\left(\widetilde{u} \,|\, \widetilde{\nabla}\varphi \right) = f \tag{38}$$
$$\varphi = 0 \quad on \quad \partial Q_-\,.$$

Then it is a weak solution:

$$\int_Q \left[\left(\widetilde{u} \,|\, \widetilde{\nabla}\varphi \right) - f \right] \cdot \psi \, dt\, dx = 0 \tag{39}$$

for all $\psi \in L^2(Q)$.

Proof. The solution of (37) verifies

$$\mathcal{A}^*(\mathcal{A}\varphi - f) = 0\,.$$

But by Lemma 8 we have $\operatorname{Ker} \mathcal{A}^* = \{0\}$, so

$$\mathcal{A}\varphi - f = 0\,,$$

and φ is a weak solution of (38). □

Finally the next result shows that any least square solution of the transport equation is a renormalized solution.

Definition 1. [13] For $u \in L^\infty(Q)^3$ such that $\operatorname{div} u = 0$, $\varphi \in L^\infty(Q)$ is a renormalized solution of

$$\left(\widetilde{u} \,|\, \widetilde{\nabla}\varphi \right) = f \quad \text{with} \quad \varphi = 0 \quad \text{on} \quad \partial Q_-$$

if for any $\beta \in \mathcal{C}^1(\mathbb{R})$, $\beta(0) = 0$, $\beta(\varphi)$ is a weak solution of

$$\left(\widetilde{u} \,|\, \widetilde{\nabla}\beta(\varphi) \right) = \beta(\varphi)' f\,.$$

We have the following

Theorem 10. *For $f \in L^2(Q)$, the solution of* (37) *is a renormalized solution.*

Proof. This is an easy consequence of Theorem 9. Indeed this theorem shows that φ is a solution of

$$\int_Q \left[\left(\widetilde{u} \,|\, \widetilde{\nabla}\varphi \right) - f \right] \cdot \psi \, dt\, dx = 0$$

for all $\psi \in L^2(Q)$. Let $\psi = \beta'(\varphi)w$ with $w \in \mathcal{D}(\bar{Q}),\ w_{|\partial Q_-} = 0$. We have

$$\int_Q \left[\left(\widetilde{u} \,|\, \widetilde{\nabla}\varphi \right) - f \right] \cdot \beta'(\varphi) w \, dt\, dx = \int_Q \left[\left(\widetilde{u} \,|\, \widetilde{\nabla}\beta(\varphi) \right) - \beta'(\varphi) f \right] \cdot w \, dt\, dx = 0\,.$$

□

Let X be the space defined in Corollary 6, equipped with the norm $|||\cdot|||$. We will use the following

Lemma 11. a) *Let H be a subspace of $H_0(\operatorname{div}, \Omega) \cap L^\infty(Q)^3$. Let $u, u_m \in H$ be such that u_m weakly converges to u in $L^2(\Omega)^3$ and $\operatorname{div} u_m = \operatorname{div} u = 0$. Let $s_m \in H_0(u_m, Q)$ (resp. $s \in H_0(u, Q)$) be the least square solution of $\left(\widetilde{u_m} \,|\, \widetilde{\nabla} s_m \right) = 0$, (resp. $\left(\widetilde{u} \,|\, \widetilde{\nabla} s \right) = 0$). Then s_m strongly converges to s in X.*

b) *Let $s, s_m \in L^\infty(Q)$ with s_m converging to s in $L^2(Q)$. If u and u_m are the solutions of* (31),(33) *associated with s and s_m respectively. Then u_m strongly converges to u in V.*

Proof. a) By Corollary 6, the sequence s_m is bounded in X, so we can assume that s_m weakly converges to $\widetilde{s}$ in X. Let us show that s_m strongly converges to $\widetilde{s}$.

Since s_m is a solution of $\widetilde{\operatorname{div}}(\widetilde{u_m} s_m) = 0$, for all $\varphi \in H^1(Q)$ we have

$$\begin{aligned} 0 &= \int_Q \left(\widetilde{u_m} \,|\, \widetilde{\nabla} s_m \right) \cdot \varphi(T-t)\, dtdx \\ &= -\int_Q \left(\widetilde{u_m} \,|\, \widetilde{\nabla}\varphi \right) (T-t) s_m - \varphi s_m \, dtdx + \int_{\partial Q} (\, \widetilde{u_m} \,|\, \widetilde{n}\,)\, s_m \varphi(T-t)\, d\widetilde{\sigma}\,. \end{aligned}$$

Choose $\varphi = 1$, since $u_m = u$ on ∂Q and $s_m = s_e$ on ∂Q_- then

$$Q_e = \int_{\partial Q_-} |\, (\, \widetilde{u} \,|\, \widetilde{n}\,)\, | s_e (T-t)\, d\widetilde{\sigma} = \int_Q s_m \, dtdx + \int_{\partial Q_+} (\, \widetilde{u} \,|\, \widetilde{n}\,)\, s_m (T-t)\, d\widetilde{\sigma}\,.$$

But $s_m \rightharpoonup \widetilde{s}$, so

$$Q_e = \int_Q \widetilde{s}\, dtdx + \int_{\partial Q_+} (\, \widetilde{u} \,|\, \widetilde{n}\,)\, \widetilde{s}(T-t)\, d\widetilde{\sigma}\,.$$

As $s_m \in \{0,1\}$, $s_m = s_m^2$ and $\widetilde{s} = \widetilde{s}^2$, so

$$Q_e = |||s_m|||^2 = |||\widetilde{s}|||^2\,,$$

therefore s_m strongly converges to $\widetilde{s}$ in X.

Now the function $\widetilde{s}$ is a solution of $\left(\widetilde{u} \,|\, \widetilde{\nabla}\widetilde{s} \right) = 0$ in distributions sense, indeed for all $\varphi \in \mathcal{D}(Q)$ and all n we have

$$-\int_Q \widetilde{s} \left(\widetilde{u} \,|\, \widetilde{\nabla}\varphi \right) dt\, dx = \int_Q (s_m - \widetilde{s}) \left(\widetilde{u} \,|\, \widetilde{\nabla}\varphi \right) dt\, dx - \int_Q s_m \left(\widetilde{u} - \widetilde{u_m} \,|\, \widetilde{\nabla}\varphi \right) dt\, dx\,.$$

which is arbitrarily small if m is sufficiently large, since s_m strongly converges to $\widetilde{s}$. So by uniqueness of s, we obtain $\widetilde{s} = s$.

b) For any m dropping the ε we have, for all $\varphi \in V$

$$\int_\Omega (\, \mu_m \nabla u_m \,|\, \nabla\varphi\,)\; dx = \int_\Omega (\, f \,|\, \varphi\,)\; dx\,,$$

in particular

$$\int_\Omega (\, \mu_m \nabla u_m \,|\, \nabla u\,)\; dx = \int_\Omega (\, f \,|\, u\,)\; dx\,,$$

so

$$\int_\Omega (\, \mu_m \nabla u_m \,|\, \nabla(u_m - u)\,)\; dx = \int_\Omega (\, f \,|\, (u_m - u)\,)\; dx = \int_\Omega (\, (\mu - \mu_m)\nabla u \,|\, \nabla u_m\,)\; dx\,.$$

Let $A_m = \{x \in \Omega, \mu - \mu_m \neq 0\}$. Since $s_m \to s$, we have $\mu_m \to \mu$. So the measure of A_m converges to 0 when $m \to \infty$. Hence

$$\int_\Omega (\, \mu_m \nabla u_m \,|\, \nabla(u_m - u)\,)\; dx \to 0$$

when $m \to \infty$. Therefore

$$\begin{aligned} \|u_m - u\|_V^2 &= \int_\Omega (\, \nabla(u_m - u) \,|\, \nabla(u_m - u)\,)\; dx \\ &\leq \frac{1}{m_\mu}\left[\left|\int_\Omega (\, \mu_m \nabla u_m \,|\, \nabla(u_m - u)\,)\; dx\right| + \left|\int_\Omega (\, \mu \nabla u \,|\, \nabla(u_m - u)\,)\; dx\right|\right] \\ &\to 0 \quad \text{if} \quad m \to \infty\,. \end{aligned}$$

□

Proof of Theorem 3. We follow the proof of [21] adapted to our situation. Let K be the convex part of $L^\infty(Q)$ defined by

$$K = \left\{ s \in L^2(Q), 0 \leq s \leq 1 \right\}\,.$$

For $s \in K$ let u be the solution of eqs. (31)–(33) associated to s. Define $\sigma = F(u)$ as the unique solution of eq. (34). Theorem 7 shows that the function σ is in K.

Let us show that the map F is continuous. Let s_n be a sequence in K that converges to s in $L^2(Q)$. The sequence u_n associated to eqs. (31)–(33) is bounded in $L^2(0,T;V)$, so it admits a subsequence weakly converging to u which is the solution of (31)–(33) for s. Let now σ_n be the solution of (34) associated with u_n. By lemma 11, $\sigma_n \in H_0(u, Q, \partial Q_-)$ and $\sigma_n \to \sigma$ in $H_0(u, Q, \partial Q_-)$.

Finally we show that the mapping F is compact. Let s_n be a sequence in K. The sequence u_n given by the solution of (31)–(33) is bounded in $L^2(0,T;V)$, so there is a subsequence that weakly converges to u. Let σ_n (resp. σ) be the solution of (34) associated with u_n (resp. u). Then as above σ_n converges to σ in $H_0(u, Q, \partial Q_-)$. □

5 Some estimations

In this section we give some estimations on the solution of eqs. (31)–(34) as ε goes to zero. A study of the behavior of the Stokes equation is first presented. Finally, using results obtained in Sect. 4, the existence of a weak solution of problem (15)–(19) is given.

5.1 Estimations for the Stokes equation

Let us fix a time $t \in (0,T)$. Moreover, in order to simplify the notations, we set $\partial_j \varphi = \frac{\partial \varphi}{\partial x_j}$ and $\lambda_j = \sqrt{\mu_j}$. In the sequel C denotes a generic constant which does not depend of ε. Let $(u,q) = (u^\varepsilon, q^\varepsilon) \in V \times L^2_0(\Omega)$ be the solution of the problem (31)–(33). If we choose $v = u$ and $\psi = q$ in these equations, we get

$$\begin{aligned} &\|\lambda_1\partial_1\, u_1\|^2_{L^2(\Omega)} + \|\lambda_2\partial_2\, u_1\|^2_{L^2(\Omega)} + \varepsilon^{-2}\,\|\lambda_3\partial_3\, u_1\|^2_{L^2(\Omega)} + \|\lambda_1\partial_1\, u_2\|^2_{L^2(\Omega)} + \|\lambda_2\partial_2\, u_2\|^2_{L^2(\Omega)} \\ &+ \varepsilon^{-2}\,\|\lambda_3\partial_3\, u_2\|^2_{L^2(\Omega)} + \varepsilon^2\,\|\lambda_1\partial_1\, u_3\|^2_{L^2(\Omega)} + \varepsilon^2\,\|\lambda_2\partial_2\, u_3\|^2_{L^2(\Omega)} + \|\lambda_3\partial_3\, u_3\|^2_{L^2(\Omega)} \\ &= \varepsilon^{-2}\left[\int_\Omega f_1 u_1 + f_1 u_1 + \varepsilon^3 f_3 u_3\, dx\right] \\ &\leq \varepsilon^{-2}\left[\|f_1\|_{L^2(\Omega)}\,\|u_1\|_{L^2(\Omega)} + \|f_2\|_{L^2(\Omega)}\,\|u_2\|_{L^2(\Omega)} + \varepsilon^3\,\|f_3\|_{L^2(\Omega)}\,\|u_3\|_{L^2(\Omega)}\right] \\ &\leq C\varepsilon^{-2}\left[\|f_1\|_{L^2(\Omega)}\,\|\lambda_3\partial_3\, u_1\|_{L^2(\Omega)} + \|f_2\|_{L^2(\Omega)}\,\|\lambda_3\partial_3\, u_2\|_{L^2(\Omega)} + \varepsilon^3\,\|f_3\|_{L^2(\Omega)}\,\|\lambda_3\partial_3\, u_3\|_{L^2(\Omega)}\right]. \end{aligned}$$

The last inequality is a consequence of the Poincaré-Friedrichs inequality in the x_3 direction. Using the next elementary inequality

$$b \cdot c \leq \tfrac{1}{2}\left(\eta b^2 + \frac{c^2}{\eta}\right)$$

for any $b, c, \eta > 0$, we obtain

$$\begin{aligned} &\|\lambda_1\partial_1 u_1\|^2_{L^2(\Omega)} + \|\lambda_2\partial_2 u_1\|^2_{L^2(\Omega)} + \varepsilon^{-2}\|\lambda_3\partial_3 u_1\|^2_{L^2(\Omega)} + \|\lambda_1\partial_1 u_2\|^2_{L^2(\Omega)} + \|\lambda_2\partial_2 u_2\|^2_{L^2(\Omega)} \\ &+ \varepsilon^{-2}\|\lambda_3\partial_3 u_2\|^2_{L^2(\Omega)} + \varepsilon^2\|\lambda_1\partial_1 u_3\|^2_{L^2(\Omega)} + \varepsilon^2\|\lambda_2\partial_2 u_3\|^2_{L^2(\Omega)} + \|\lambda_3\partial_3 u_3\|^2_{L^2(\Omega)} \leq C\cdot\varepsilon^{-2}\,. \end{aligned} \tag{40}$$

From this inequality we obtain the following

Theorem 12. *If* $u = u^\varepsilon \in V$ *is the solution of the problem* (31)–(33)*, we have*

a) $\left\|\partial_j u_i^\varepsilon\right\|_{L^2(\Omega)} \leq C\varepsilon^{-1} \quad i, j = 1, 2 \quad \textit{and} \quad \left\|\partial_j u_3^\varepsilon\right\|_{L^2(\Omega)} \leq C\varepsilon^{-2} \quad j = 1, 2\,,$
b) $\|\partial_3 u_i^\varepsilon\|_{L^2(\Omega)} \leq C \quad i = 1, 2 \quad \textit{and} \quad \|\partial_3 u_3^\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^{-1}\,.$

As a consequence of this theorem, we have

Corollary 13. *With the previous hypothesis we have*

a) $\|u_i^\varepsilon\|_{L^2(\Omega)} \leq C\,, \quad i = 1, 2\,.$
b) $\|\partial_i u_i^\varepsilon\|_{H^{-1}(\Omega)} \leq C\,, \quad i = 1, 2, 3\,.$

Proof. The first inequality follows from inequality b) of Theorem 12 and from the Poincaré-Friedrichs inequality in the x_3 direction. The second is a consequence of a), of the inequality $\|\partial_3 \varphi\|_{H^{-1}(\Omega)} \leq \|\varphi\|_{L^2(\Omega)}$ and $\operatorname{div} u = 0$. □

Let us now study the behavior of the pressure $q = q^\varepsilon$. Since

$$\left\|\partial_j\left(\lambda_j\partial_3 u_i\right)\right\|_{H^{-1}(\Omega)} \leq \left\|(\lambda_j\partial_j u_i\right\|_{L^2(\Omega)}$$

we get from eq. (8)

$$\|\partial_i q\|_{H^{-1}(\Omega)} \leq \varepsilon^2\|\lambda_1\partial_1 u_i\|_{L^2(\Omega)} + \varepsilon^2\|\lambda_2\partial_2 u_i\|_{L^2(\Omega)} + \|\lambda_3\partial_3 u_i\|_{L^2(\Omega)} + \|f_i\|_{L^2(\Omega)}\,,$$

so using Theorem 12

$$\|\partial_i q\|_{H^{-1}(\Omega)} \leq C \quad \text{for} \quad i = 1, 2\,.$$

In the same way we get

$$\|\partial_3 q\|_{H^{-1}(\Omega)} \quad \leq \quad C\varepsilon\,.$$

In particular $\|\nabla q\|_{H^{-1}(\Omega)^3} \leq C$ so, using ([19], Th. I.2.2), $q = q^\varepsilon$ is uniformly bounded in $L^2(\Omega)$, independently of ε. Hence the next theorem is proved.

Theorem 14. *Let* $(u^\varepsilon, q^\varepsilon) \in H \times L^2_0(\Omega)$ *be the solution of the problem* (31)–(33)*. Then*

$$\|\partial_3 q^\varepsilon\|_{H^{-1}(\Omega)} \;\to\; 0 \quad \textit{if} \quad \varepsilon \;\to\; 0$$

and q^ε *is uniformly bounded in* ε *in* $L^2_0(\Omega)$*.*

5.2 Weak solution of the limit problem

Let us now define the space

$$H = H_0(\partial_3, \Omega)^3 \cap H_0(\Omega, \operatorname{div})\,.$$

We use the following norm on H which is equivalent to the product norm:

$$\|u\| = \|u\|_H = \left(\|\partial_3 u_1\|^2_{L^2(\Omega)} + \|\partial_3 u_2\|^2_{L^2(\Omega)} + \|\operatorname{div} u\|^2_{L^2(\Omega)}\right)^{1/2}\,.$$

A weak formulation of problem (15)–(23) is:
Find $u \in L^\infty(0,T;H)$, $q \in L^\infty(0,T;L^2_0(\Omega))$ and $c \in H_0(u,Q,\partial Q_-)$ such that

$$\int_\Omega \mu_3\,\partial_3\,u_1\,\partial_3\,v_1\,dx + \int_\Omega \mu_3\,\partial_3\,u_2\,\partial_3\,v_2\,dx - \int_\Omega q\cdot\operatorname{div} v\,dx - \int_\Omega \operatorname{div} u\cdot r\,dx = \int_\Omega (g_1\,v_1 + g_2\,v_2)\,dx\,, \tag{41}$$

$$\int_Q \left(\widetilde{u}\,|\,\widetilde{\nabla} c\right)\left(\widetilde{u}\,|\,\widetilde{\nabla}\varphi\right)\,dt\,dx = \int_Q b\cdot\left(\widetilde{u}\,|\,\widetilde{\nabla}\varphi\right)\,dt\,dx \tag{42}$$

for all $v = (v_i) \in H$, all $r \in L^2_0(\Omega))$ and all $\varphi \in H_0(u,Q,\partial Q_-)$, where g_1 and g_2 are suitable functions taking the boundary conditions into account. We have

Theorem 15. *The problem* (41) *has a unique solution in* $H \times L^2_0(\Omega)$.

Proof. For $\varphi = (u,q)$ and $\psi = (v,r)$ in $X = H \times L^2_0(\Omega)$, let $a : X \times X \to \mathbb{R}$ be the symmetric and continuous bilinear form defined by

$$a(\varphi,\psi) = \int_\Omega \mu_3\,\partial_3\,u_1\,\partial_3\,v_1\,dx + \int_\Omega \mu_3\,\partial_3\,u_2\,\partial_3\,v_2\,dx - \int_\Omega q\cdot\operatorname{div} v\,dx - \int_\Omega \operatorname{div} u\cdot r\,dx\,.$$

In order to prove the theorem, it is sufficient to show that there exists $C > 0$ such that for all $\varphi \in X$

$$\sup_{\psi\neq 0} \frac{a(\varphi,\psi)}{\|\psi\|} \geq C\cdot\|\varphi\|\,, \tag{43}$$

i.e., the bilinear form a is *persuasive* (see [3,4]). Fix $\varphi = (u,q) \in X$, using the Brezzi-Babuska *inf-sup* condition [19], let $\beta > 0$, $v \in H^1_0(\Omega)^3$ and $r \in L^2_0(\Omega)$ be such that

$$\|v\|_{H^1_0(\Omega)^3} = \|q\|_{L^2(\Omega)}\,, \qquad (\,q\,|\,\operatorname{div} v\,)_{L^2(\Omega)} \geq \beta\,\|q\|^2_{L^2(\Omega)}\,, \qquad r = \operatorname{div} u\,. \tag{44}$$

In order to obtain $a(\varphi,\psi) \geq C\,\|\varphi\|\,\|\psi\|$, let us choose $\psi \in X$ of the form

$$\psi = \alpha\cdot(u,-q) - (v,0) - (0,r)\,.$$

If, as above, $\lambda_3 = \sqrt{\mu_3}$, some easy computations give

$$\begin{aligned} a(\varphi,\psi) \geq{}& \alpha\left(\|\lambda_3\partial_3\,u_1\|^2_{L^2(\Omega)} + \|\lambda_3\partial_3\,u_2\|^2_{L^2(\Omega)}\right) - (\,\lambda_3\partial_3\,u_1\,|\,\lambda_3\partial_3\,v_1\,)_{L^2(\Omega)} - (\,\lambda_3\partial_3\,u_2\,|\,\lambda_3\partial_3\,v_2\,)_{L^2(\Omega)} \\ &+ \beta\,\|q\|^2_{L^2(\Omega)} + \|\operatorname{div} u\|^2_{L^2(\Omega)} \\ \geq{}& \alpha C_1\left(\|\partial_3\,u_1\|^2_{L^2(\Omega)} + \|\partial_3\,u_2\|^2_{L^2(\Omega)}\right) - C_2\left((\,\partial_3\,u_1\,|\,\partial_3\,v_1\,)_{L^2(\Omega)} + (\,\partial_3\,u_2\,|\,\partial_3\,v_2\,)_{L^2(\Omega)}\right) \\ &+ \beta\,\|q\|^2_{L^2(\Omega)} + \|\operatorname{div} u\|^2_{L^2(\Omega)}\,, \end{aligned}$$

with C_1 and $C_2 > 0$. If $\alpha = 2C_2/C_1$, the inequality (43) is verified for $C = \max(C_2,\beta,1)$. □

As a consequence of Theorem 15, we get in the same way as the proof of Theorem 3 the following

Theorem 16. *The problem* (41)–(42) *has at least one solution.*

For any $\varepsilon > 0$, let $(u^\varepsilon, q^\varepsilon, c^\varepsilon)$ be a solution of eqs. (31)–(34). By Corollary 13 and Theorem 12,

- u^ε_i is bounded in $H_0(\partial_3,\Omega)$, $i = 1,2$.
- u^ε_3 is bounded in $H^{-1}(\Omega)$.

From Theorem 14, q^ε is bounded in $L^2(\Omega)$ and ∇q^ε is bounded in $H^{-1}(\Omega)^3$ with $\partial_3 q^\varepsilon \to 0$ if $\varepsilon \to 0$. Hence there exists a sequence $\varepsilon_n \to 0$ such that

- $u^{\varepsilon_n}_i \rightharpoonup u_i$ in $H_0(\partial_3,\Omega)$,
- $u^{\varepsilon_n}_3 \rightharpoonup u_3$ in $H^{-1}(\Omega)$,
- $q^{\varepsilon_n} \rightharpoonup q$ in $L^2(\Omega)$,
- $\nabla q^{\varepsilon_n} \rightharpoonup \nabla q$ in $H^{-1}(\Omega)^3$ with $\partial_3 q = 0$.

Moreover, since $\operatorname{div} u^{\varepsilon_n} = 0$ we get $\operatorname{div} u = 0$ in $H^{-1}(\Omega)$ and (u,q) is a solution of (41). In the same way, c^ε is bounded in X, so we can assume that $c^{\varepsilon_n} \rightharpoonup c$ in X and c is a solution of (42). So we have

Theorem 17. *The solution* $(u^\varepsilon, q^\varepsilon, c^\varepsilon)$ *of eqs.* (31)–(34) *weakly converges to* (u,q,c) *in* $H_0(\partial_3,\Omega) \times H_0(\partial_3,\Omega) \times H^{-1}(\Omega) \times L^2_0(\Omega) \times X$ *which is a solution of* (41)–(42).

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