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Some Facets of M-Theory Compactifications

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par

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Prof. Jean-Pierre Derendinger (directeur de thèse)
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Prof. Ignatios Antoniadis
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IMPRIMATUR POUR LA THESE

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UNIVERSITE DE NEUCHATEL

FACULTE DES SCIENCES

La Faculté des sciences de l'Université de Neuchâtel sur le rapport des membres du jury,

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autorise l'impression de la présente thèse.

Neuchâtel, le 26 septembre 2001

Le doyen:

J.-P. Derendinger

To my parents

Contents

Physics is beautiful. It makes me sad beyond words to know that so many people think of the physical sciences as barren, boring, bone-dry. Not so: when you lie outside in the grass on a clear dark night and look up at the stars, what you see is splendid. It is also physics. Understanding can lift you off the Earth, safer and faster and further than any rocket. The mind can travel among the stars, even enter them to see what causes those fires inside. To the beauty of seeing, we can add the beauty of understanding. And there is another level of beauty beyond that: the beauty of discovery, of creation, of doing *physics.*

(VINCENT ICKE,
The force of symmetry.)

The Milky Way around the Southern Cross
by GREG BOCK, Queensland, Australia, March 1996

Preface

This document can roughly be divided in two parts. The first three chapters are devoted to a very brief introduction to string theory. They are intended to give an idea of the framework in which the more technical following chapters will lie. The original material presented in the second part concerns M-theory compactifications and has been extracted from three articles [31, 60, 61] which are the visible aspects of productive collaborations with Adel Bilal and Jean-Pierre Derendinger. To emphasize the collective origin of the findings at the heart of this work, I shall always use below the personal plural pronoun "we".

In Chapter 1, we give some motivation for the study of string theory and a brief survey of its history up to recent non-perturbative developments. Chapter 2 begins by an enumeration of the five consistent perturbative superstring theories in ten dimensions. We then explain a possible way to go beyond the perturbative formulation by focusing on the so-called solitonic BPS states which are extended solutions (named branes) to the effective low-energy field theories (supergravity theories). In the light of the perturbative approach, the string multiplets may actually contain three classes of states: the fundamental strings, the solitonic branes and the Dirichlet branes. These extended objects are useful in establishing connections between the five consistent superstring theories in ten dimensions. These connections, or dualities, form the topic of Chapter 3. Some equivalences are already manifest at the perturbative level, whereas others involve manipulations (e.g. inversion) of the string coupling constant and can only be conjectured from a basic knowledge of non-perturbative properties of the theories under consideration. Interestingly enough, this discussion leads naturally to evidence for the existence of a consistent quantum theory in eleven dimensions. A few facets of this still largely mysterious M-theory and of its connections to the superstring theories in dimensions below than eleven constitute the subject of the second part of this document.

Our overview of string theory is far from being exhaustive. In particular, recent insights (like Matrix theory [15], the Maldacena conjecture [141], non-commutative geometry [48, 199], large radius compactifications [36, 10] or the Randall–Sundrum proposition [177, 178]) are omitted because they are not directly related to the core of the thesis. Fortunately, the literature on the subject is well supplied. The common references for superstring theories in the perturbative domain are the textbooks written by Michael B. Green, John H. Schwarz and Edward Witten [107, 108] (two volumes) and by Dieter Lüst and Stefan Theisen [140]. Many of the developments up to 1998 are covered in the lectures notes prepared by Elias Kiritsis [128] and in the textbook written by Joseph Polchinski [171, 172] (two volumes).

In Chapter 4, we study some aspects of M-theory on the orbifold $S^1/\mathbb{Z}_2$, a compactification which is thought to be related to the $E_8 \times E_8$ heterotic superstring theory. We work in the "upstairs" (boundary-free) formalism and insist on properly defining all fields on the circle. We carefully resolve the modified Bianchi identity for the four-form field strength of eleven-dimensional supergravity, collecting information from the invariance of this field strength under small and large gauge and local Lorentz transformations, and from the cancellation of the one-loop anomaly. In presence of M-five-branes, we find that there is an unexpected additional contribution to the anomaly inflow from the eleven-dimensional topological term. We also consider the quantization of the four-form flux and the small-radius limit in which the perturbative heterotic superstring theory is recovered. The corresponding heterotic anomaly-cancelling terms, as inherited from the M-theory approach, are shown to differ from the usual ones by the addition of a well-defined local counterterm.

In the low-energy limit, M-theory on the orbifold $S^1/\mathbb{Z}_2$ is formulated in terms of Bianchi identities with sources localized at the two orbifold singularities and anomaly-cancelling counterterms. A further compactification on a six-dimensional Calabi–Yau manifold leads to $N = 1$ local supersymmetry in four dimensions. In Chapter 5, we present a formulation of the effective supergravity which explicitly relates four-dimensional supergravity multiplets and field equations with these fundamental M-theory features. This formalism proves particularly convenient for the introduction in the effective supergravity of non-perturbative M-theory contributions. Indeed, in Chapter 6 we complete our analysis of the effective $N = 1$ four-dimensional supergravity of M-theory by the introduction of M-five-branes. One of the massless supermultiplets generated by these extended objects describes the modulus associated with the position of the branes along the circle S^1. Starting from the dynamics of the five-brane modes obtained by reduction and supersymmetrization of the covariant five-brane bosonic action, we derive the effective four-dimensional supergravity of this multiplet and its coupling to bulk moduli and to Yang–Mills and charged matter multiplets located on the $\mathbb{Z}_2$ fixed planes. Our construction respects all symmetries of M-theory, including the self-duality of the antisymmetric tensor living on the brane world-volume (a self-duality property which strongly constrains the corrections to gauge couplings). The five-brane contribution to the effective scalar potential turns out to be formally similar to a renormalization of the dilaton, and the vacuum structure is not modified. Altogether, the impact of the five-brane modulus on the effective supergravity is reminiscent of string one-loop corrections produced by standard compactification moduli.

We end with four appendices. Appendix A gives a summary of our conventions and notations. Appendix B contains some well-known useful facts about massless field representations in various dimensions. In Appendix C, we introduce the concept of anomaly polynomials and collect explicit expressions for the one-loop anomalies in ten and six dimensions. Finally, Appendix D provides some information about the superconformal tensor calculus for $N = 1$ four-dimensional supergravity.

List of acronyms

BPS	Bogomol'nyi–Prasad–Sommerfield [32, 174]
CJS	Cremmer, Julia and Scherk [51]
GSW	Green, Schwarz and Witten [107, 108]
HW	Hořava and Witten [116, 117]
KK	Kaluza–Klein [126, 129]
LOW	Lukas, Ovrut and Waldram [136, 137, 138, 139]
NS	Neveu–Schwarz
PST	Pasti, Sorokin and Tonin [164, 165, 166, 167, 13]
QCD	Quantum Chromodynamics
QED	Quantum Electrodynamics
QFT	Quantum Field Theory
R	Ramond
srl	small-radius limit
sugra	supergravity
vev	vacuum expectation value
wv	world-volume

Expliquer, comprendre, pénétrer quelque chose au moins du mystère du monde, soulever au moins un coin du voile d'Isis, il n'est pas, dans le domaine des choses de l'esprit, de joie plus solide et de plus enivrant bonheur que d'avoir pu, fût-ce une seule fois, dans le plus humble domaine et sur le plus infime détail, y parvenir.

(Théodore Monod,
Méharées.)

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It is a great pleasure to thank all the postdocs and students I met during the working out of this thesis. They all contributed one way or another to its materialization. A partial name list contains, in random order, Fernando Quevedo, Giovanni Amelino-Camelia, Philippe Page, Matteo Mombelli, Konstadinos Sfetsos, Massimiliano Capezzali, Stefan Weigert, Chong-Sun Chu, Stephan Stieberger, Dimitris Matalliotakis, Rodolfo Russo, Roland Lüscher, Raymond Frésard, Matthias Hengsberger, Claudio Lucchesi, Stéphane Blaser, Philippe Devaud, Roland Kirchhofer, Philippe Curty, Nicolas Moeller, Alain Sewer, Nicolas Miéville, Olivier Laesser, Luca Carlevaro and Aldous Zaugg. I address special warm thanks to Philippe Jacquod, Gianni Di Domenico, Maxime Bagnoud, Oliver Gallus and Matthieu Karrer, the five supercolleagues with whom I had the fortune to share an office (as well as many non-academic activities).

The members of the administrative and technical staffs of our Institute should also have been included to this enumeration. They solve any problem with kindness and efficiency. I am also delighted to cite Greg Bock for giving me the chance to use one of his marvelous photographs of the southern sky. At a more abstract digital level, I should not forget the designers of the Macintosh computers for keeping on “thinking different” and the Linux community for providing such exceptional operating systems. On the financial side, I acknowledge the Swiss National Science Foundation, the Swiss Office for Education and Science, and the European Union for supporting the research that went into this document.

Finally, I would like to express my invaluable gratitude to my parents, and to Cédric, Célestin, Didier, Erhard, Marie, Philippe and Chantal for their irrepressible vitality.

Part I

Introduction

Chapter 1

A first step towards a unified theory

Experimenters have accumulated over the years a large amount of evidence that all the interactions taking place in our Universe can be described in terms of four fundamental forces: the gravitational force, the electromagnetic force, the weak force and the strong force. In our all-day life, we most of the time are only aware of the gravitational and electromagnetic forces. Gravity has the distinctive features to be universal (it acts on all particles), always attractive and long range. It shares this last characteristic with the electromagnetic force which acts only on charged particles and is not always attractive. The nuclear forces, on the contrary, are of short range and their effects are confined within a very small region (of a typical diameter less than 10^{-15} m) around their sources. The strong force is in particular responsible for binding together the quarks inside hadrons and for keeping neutrons and protons within the nucleus, while a manifestation of the weak interaction is the radioactive β-decay.

In the present understanding, forces between matter constituents (fermions with half-integer spin: quarks and leptons) are seen as exchanges of force carriers (bosons with integer spin). For instance, the electromagnetic and gravitational forces are, respectively, transmitted via the photon (the electromagnetic boson) and the graviton (a massless spin-2 messenger). The weak interaction is carried by the (weak gauge) bosons W^+, W^- and Z^0, while in the strong interaction quarks interact through the exchange of eight kinds of gluons. The non-gravitational interactions are described by quantum field theories (QFT) based on a common principle: *symmetry*. During the sixties and seventies, it appeared to be possible to draw a local gauge theory, now known as the Standard Model, which incorporates the theory of strong interactions (quantum chromodynamics or QCD with the gauge group (color group) $SU(3)_c$) and the unified theory of weak and electromagnetic interactions (Glashow–Weinberg–Salam electroweak theory with the gauge group $SU(2)_L \times U(1)_Y$[1]). This model, whose symmetry is essentially based on the compact gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ times the (non-compact global) Poincaré group, is in impressive agreement with experiment. However, it is not completely satisfactory for at least two reasons. First, it unifies only

[1] The index L indicates that the fermions with a left chirality are the only one to transform under the weak isospin group, and the index Y is related to the weak hypercharge. The electromagnetic gauge group $U(1)_{\mathrm{e.m.}}$ contained in the direct product $SU(2)_L \times U(1)_Y$ is obtained through spontaneous symmetry breaking (Higgs mechanism). This mechanism generates also the masses of the weak gauge bosons.

partially the strong and electroweak interactions, does not fix the number of generations of quarks and leptons, and contains (in the case of three generations with massless neutrinos) 19 arbitrary parameters[2]. Secondly, this theory takes no heed of Albert Einstein's general relativity which is nevertheless a fundamental pillar of modern physics. The point is that the current formulations of the Standard Model and of general relativity are incompatible. The first one is based on quantum mechanics, while the second one is a classical field theory (that can be regarded as a gauge theory based on the Poincaré symmetry group) from which it is apparently impossible to build a consistent (renormalizable) quantum theory. These two models have proved to be stunningly adapted to the description of phenomena involving objects that are either tiny and light (like electrons or quarks) or huge and heavy (like stars or galaxies), but their capacity to give an account for situations involving objects that are both small and very massive (like the beginning of our Universe or the inside of a black hole) is quite limited. One is then tempted to look for a more constrained unified theory which would have the Standard Model and general relativity as particular limits.

With this aim in mind, one can try to impose an enlarged symmetry which would include in a non-trivial way[3] the Poincaré group and an internal group[4]. *Supersymmetry* appears to be the only extension of this kind compatible with a quantum field theory. The corresponding Lie superalgebra has a fermionic sector with anticommutation relations between the supersymmetry generators (spin-1/2 "charges")[5]. Supersymmetry, unlike any other symmetry, relates bosonic and fermionic particles, and a set of fields realizing this symmetry is called a supermultiplet. One of the advantages of the introduction of supersymmetry is that it stabilizes the disparity between the characteristic scale of the Standard Model (of the order of $M_{W^\pm} \cong M_{Z^0} \cong 10^2\,\text{GeV}$) and the relevant scale in quantum gravity (which is given by the Planck energy $E_{\text{Pl}} = \sqrt{\hbar c^5/G} \cong 10^{19}\,\text{GeV}$, where G is the Newton's gravitational constant). In most non-supersymmetric quantum field theories one has to do some unappealing fine-tuning to prevent the mixing and equalization of the two scales by radiative corrections (this is the so-called "gauge hierarchy problem"). Only a few proposals, such as the technicolor model, may solve the gauge hierarchy problem without the help of supersymmetry.

Another interesting feature of supersymmetry is the close relationship between internal and space-time generators in the Lie superalgebra. In particular, if one considers *local* supersymmetry transformations, one is bound to consider also local Poincaré transformations. This means that a theory invariant under local supersymmetry transformations necessarily includes gravity. Such theories are usually named *supergravity theories*. In these models, one has to introduce one supersymmetry gauge field for each supersymmetry generator. These

[2] A possible choice for these free parameters amounts to take the three gauge coupling constants, the masses of the leptons e, μ, τ, the masses of the quarks u, d, s, c, b, t, the mass of the Higgs boson, the mass of the weak gauge boson $W^\pm$, the four parameters of the Cabibbo–Kobayashi–Maskawa matrix and an angle θ related to the non-perturbative structure of the QCD ground state inferred from the existence of instantons. Note that a better comprehension of the non-perturbative aspects of the theory may change the number of free parameters.

[3] That is, not by a simple direct product like in the Standard Model.

[4] A symmetry group which commutes with space-time transformations is called internal.

[5] More precisely, in four dimensions, each fundamental generator is a Majorana (or Weyl) spinor with four real degrees of freedom (supercharges). The ratio of the number of supercharges to the number of real degrees of freedom in the smallest spinor representation is denoted by N.

vector spinor fields are called gravitinos because they are in the same supermultiplet as the graviton. Their presence cures some of the infinities coming from graviton exchanges in the quantum theory, but unfortunately supergravity theories remain non-renormalizable theories. One may then rightfully ask oneself whether the consistent unification of the fundamental interactions can really be achieved in the framework of quantum field theory. Presently, the most encouraging way to solve the short-distance problem of quantum gravity is to build a quantum relativistic mechanics describing unidimensional entities (strings).[6] Loosely speaking, the basic idea is to identify the various vibrational modes of these mathematical curves (which can be open or closed, oriented or unoriented) with the different elementary particles found in our Universe. To understand the essence of *string theory* and how this model contributed to the development of supersymmetry and supergravity, it seems useful to give a brief survey of its history.

An historical sketch of perturbative string theory

The study of string theory began at the end of the sixties, soon after the formulation of the first principles of a unified theory for the weak and electromagnetic interactions. At that time no known renormalizable quantum field theory was able to describe all the peculiarities revealed by experiments involving strongly interacting particles. A major problem was the proliferation of various hadronic states with increasing spin and mass. The masses m of the lightest hadronic resonances of spin J appeared to be more or less given by an equation of the form

$$m^2 = \frac{1}{\alpha'} J + \text{constant}, \tag{1.1}$$

where the constant $\alpha' \cong 1\,(\text{GeV})^{-2}$ was called the Regge slope. In 1968, Gabriele Veneziano wrote an amplitude [224] to reproduce this Regge behaviour. His proposition also agreed with the so-called "duality hypothesis" for the hadronic scattering amplitudes which stated that the s- and t-channels[7] give an alternative description of the same physics.

Two years later, Yoichiro Nambu [151], Holger Nielsen [156] and Leonard Susskind [212] discovered that the basis of the Veneziano amplitude for mesons was most probably a theory of a vibrating open (or closed) *bosonic quantized string*. The action for such a string propagating in a d-dimensional space-time along a two-dimensional trajectory (called world-sheet by analogy with the world-line of a relativistic point particle) reads[8]

$$S_{\text{P}} = -\frac{T_1}{2} \int d\tau d\sigma\, \sqrt{h} h^{\hat{m}\hat{n}} g_{MN} (\partial_{\hat{m}} X^M)(\partial_{\hat{n}} X^N), \tag{1.2}$$

[6] Besides string theory, a certain number of models have been proposed as candidates for a coherent description of the quantum properties of space-time, the most popular being certainly *loop quantum gravity*. Unfortunately, our lack of knowledge of these alternatives to string theory prevents us to give a precise account of the current state of these researches. The interested reader may as a starting point consult Ref. [182] which gives a critical review of the various approaches to the quantization of gravity and contains a rather complete list of references.

[7] Here s and t denote the usual Mandelstam variables.

[8] The subscript P refers to the fact that this way of writing the action is appropriate to a Feynman path-integral quantization of the string (a point of view adopted in particular by Polyakov). When $g_{MN} \neq \eta_{MN}$, the action S_{P} corresponds to a non-trivial two-dimensional quantum field theory (known as a non-linear sigma model).

where

- T_1 is a coefficient (with dimension $[\text{mass}]^2$) that can be assimilated to the string tension; it is related to the Regge slope by $T_1 = (2\pi\alpha')^{-1}$.
- τ and σ are the two parameters used to describe a point on the string world-sheet: τ is in some sense the "proper time" of the string and σ is a space-like compact coordinate chosen such that $\sigma \in [0, \pi]$.
- $h_{\hat{m}\hat{n}}(\tau, \sigma)$ (with $\hat{m}, \hat{n} = 1, 2$ associated to the two world-sheet parameters) is the string world-sheet metric tensor and $h = |\det(h_{\hat{m}\hat{n}})|$.
- $X^M(\tau, \sigma)$ (with $M = 1, \ldots, d$) map the string world-sheet on space-time.
- $g_{MN}(X^O)$ is the space-time metric tensor; it is often supposed to be the Minkowski metric tensor η_{MN}.

The action (1.2) is proportional to the area swept out by the string. A physical string trajectory corresponds then to an extremum of this area.[9] It is interesting to note that consistency at the quantum level (for instance the absence of negative-norm states) dictates that the string is propagating in a 26-dimensional space-time. The spectrum contains a fundamental state with an imaginary mass (a scalar particle usually called "tachyon"), a massless sector, and an infinite number of massive excitations which were thought to reproduce the Regge behaviour of the hadronic resonances.

In 1971, the desire to describe also space-time fermions led Pierre Ramond [176] and André Neveu and John H. Schwarz [155] to complete the model by the introduction of fermionic degrees of freedom. To prevent the appearance of negative norm states, they extended the symmetry of the theory by requiring the invariance of the string world-sheet action under transformations mixing bosons and fermions. This *supersymmetry* associates a two-dimensional fermionic Majorana field (with two real components describing right and left propagating modes) to each space-time coordinate X^M (each bosonic field being decomposed into two functions corresponding to right and left propagating modes). The *fermionic string* (or *superstring*) was born. It turned out that to ensure the vanishing of boundary terms appearing when one varies the action to obtain the Euler-Lagrange equations, one has to impose certain conditions on the fermionic fields.[10] In the open string case, there are two possible boundary conditions: Ramond (R) conditions which are appropriate to the description of space-time fermionic states and Neveu–Schwarz (NS) conditions which are suited to the depiction of space-time bosonic states. In the closed string case, the conditions for the two fermionic components can be chosen independently and the theory has four distinct sectors: $\text{R} \otimes \text{R}$, $\text{NS} \otimes \text{NS}$, $\text{R} \otimes \text{NS}$ and $\text{NS} \otimes \text{R}$. The two first sectors are related to space-time bosonic states, while the two last sectors correspond to space-time fermionic states. The spectrum still contains a tachyon (in the NS sector for the open string and in

[9] Note the analogy with the point particle trajectories (geodesics) which are curves of extremal (minimal) length.

[10] Similar conditions must be satisfied by the bosonic fields. They will enter the discussion of a particular class of non-perturbative states in section 2.3.3.

the NS $\otimes$ NS sector for the closed string), a massless sector, and an infinite tower of massive excitations. Consistency at the quantum level requires that the superstring is propagating in a 10-dimensional space-time.

In spite of these attractive new developments, the physics community was not very enthusiastic about string theory because a number of recently conducted high-energy fixed angle scattering experiments showed results in direct conflict with the predictions of the Veneziano model. On the other hand, quantum chromodynamics was being constructed and it soon appeared to be a much more promising candidate for the theory of strong interactions. Only a few theorists were still believing in the beauty of string theory. In 1974, two of them, Joël Scherk and John H. Schwarz, suggested [184] that, maybe, string theory could lead to a unification of all known interactions (including gravity) in a consistent quantum theory. Their starting observation was that the massless closed string spectrum contains a traceless symmetric 2-index tensor. This state is unknown from hadronic physics, but it is tempting to identify it with the graviton, the quantum of the gravity field. This belief is confirmed by the fact that, at low energies, the field theory action which reproduces the string scattering amplitudes for this massless particle is given by the usual Einstein–Hilbert action of general relativity. The string parameter α' (or equivalently the string tension T_1) must then apparently be rescaled from the strong interaction scale to the characteristic scale in quantum gravity (Planck energy or Planck mass). Finally, following the intuitions of Theodor Kaluza and Oskar Klein [126, 129] (their papers were published in 1921 and 1926!), one has to compactify the ten-dimensional space-time down to four dimensions.

In 1976, Ferdinando Gliozzi, Joël Scherk and David Olive [103, 104] introduced the now-called GSO projection on the spectrum which removes the tachyon and leads to space-time supersymmetry. This truncation appears naturally when discussing the one-loop modular invariance[11] of the fermionic string partition function. In the (un)oriented open string case, it leads to $N = 1$ supersymmetry. In the closed string case, there are two possible choices for the GSO projection. The corresponding oriented theories have $N = 2$ supersymmetry and were dubbed type IIA or type IIB string theories depending on the relative chiralities of the two gravitinos present in the spectrum (the two chiralities are opposed in the type IIA string, while they are the same in the type IIB string). Moreover, one has the possibility to construct an unoriented version of the type IIB string with only $N = 1$ supersymmetry.

At the end of the seventies, three superstring theories free of tachyons and apparently consistent at the perturbative level were known: the two type II theories of oriented closed superstrings and another theory which includes unoriented open strings as well as a subset of the type IIB closed strings.[12] This last model was called type I superstring because its

[11] The world-sheet topology for the perturbation theory at one-loop is the one of a torus. Each torus can be described by a single complex number τ (often called Teichmüller parameter) obeying special equivalences. The modular transformations are actually large coordinate transformations (they cannot be obtained by infinitesimal reparametrizations) that leave the torus invariant but modify τ. These diffeomorphisms generate the group $SL(2,\mathbb{Z})$:

$$\tau \longmapsto \tau' = \frac{a\tau + b}{c\tau + d}, \quad \text{with } a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1.$$

[12] The presence of closed strings is necessary in the quantum theory because the endpoints of open strings must be allowed to join, that is to say open strings must be allowed to form closed strings.

spectrum contains a single gravitino.[13] The closed superstrings were not very promising phenomenologically because they do not (at the perturbative level) incorporate a non-Abelian gauge group and the generation of such a group by compactification appeared to be a difficult task. Non-Abelian gauge symmetries can however be introduced in the third superstring theory by placing internal degrees of freedom at the endpoints of the open strings.[14] However, a consistent quantum theory has in particular to be free of gauge and gravitational anomalies and it was not obvious that this requirement was satisfied by the type I superstring theory.[15] The breakthrough came in 1984 when Michael B. Green and John H. Schwarz [105, 106] showed that anomaly cancellation and finiteness in the type I superstring fixes the gauge group to be $SO(32)$. More precisely, these two authors, Jean Thierry-Mieg [215, 216] and others, then realized that from an effective theory point of view, another gauge group, namely $E_8 \times E_8$, was allowed, but on the other hand, it was known that type I superstring cannot incorporate such an exceptional group.[16] These developments were for the most part based on the fact that at low energy the massless modes of the string are described in good approximation by an effective field theory which is a supergravity.[17] This supergravity can be compared to the non-renormalizable Fermi theory proposed for the description of the weak interaction and which is a reasonable approximation of the standard electroweak model at energies well below the mass of the gauge boson $W^{\pm}$.

In 1985, David Gross, Jeffrey Harvey, Emil Martinec and Ryan Rohm [109, 110] constructed a new type of closed ten-dimensional string deriving from a combination between a fermionic and a bosonic strings. Since the bosonic string is living in a 26-dimensional space-time, one "compactifies" 16 spatial dimensions on an internal space (which is chosen to be a 16-dimensional torus). The resulting massless spectrum has $N = 1$ supersymmetry and contains non-Abelian gauge bosons.[18] Interestingly enough, the requirement of one-loop modular invariance fixes the corresponding gauge group to be $SO(32)$ or $E_8 \times E_8$. These two hybrid models, baptized *heterotic strings*, immediately sounded to be particularly well-adapted to the inclusion of gauge interactions (especially the $E_8 \times E_8$ model) and there was a hope to recover in a way or another the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge group of the Standard Model. To make contact with four-dimensional phenomenology, one has to reduce the dimensionality of space-time by curling up the extra coordinates into a certain 6-dimensional

[13]Since an (un)oriented open superstring has $N = 1$ supersymmetry, one can only consistently couple it to an unoriented type IIB superstring, and there remains only one supersymmetric partner of the graviton. This explains why this unoriented open plus closed string theory is called "type I".

[14]These degrees of freedom are called *Chan–Paton degrees of freedom*. They were originally introduced to reproduce an $SU(3)$ flavor symmetry (with a quark and an antiquark attached at the ends of the string). This idea, as well as many other developments that occurred during the first ten years of string theory, is covered in Ref. [187].

[15]As shown in appendix C, gravitational anomalies cancel in type II superstrings.

[16]The gauge groups the endpoint of an open string can accommodate are restricted to $U(n)$ (in the case of an oriented open string) and $SO(n)$ or $USp(2n)$ (in the case of an unoriented open string) [186, 142].

[17]All the massive string excitations have a mass squared proportional to the tension T_1, i.e. of the order of the Planck mass. We are then mainly interested in the limit $\alpha' \to 0$ (zero-slope limit) in which all the massive modes of the string decouple. Besides the graviton, the remaining massless sector contains many other particles with lower spin.

[18]The local gauge symmetry is coming from the invariance of the theory under local transformations of the internal degrees of freedom.

space. This compactification scheme is particularly promising when the compact space is a Calabi–Yau manifold [41, 40][19] or an orbifold [66, 67][20] because supersymmetry can then be broken from $N = 1$ in ten dimensions to $N = 1$ in four dimensions (instead of the less realistic unbroken $N = 4$ one would for instance obtains in a compactification on a six-dimensional torus)[21]. A single unbroken four-dimensional supersymmetry allows, contrarily to any extended four-dimensional supersymmetry (with $N > 1$), the chirality of the fermion representations as observed in Nature, and it has the potential to solve the gauge hierarchy problem. This remaining supersymmetry can then eventually be broken through, for instance, a non-perturbative mechanism like gaugino condensation [57, 64].

The infancy of a non-perturbative formulation of string theory

The discovery of the heterotic string (and the period that followed) is now regarded as the *first superstring revolution*. The attractive features of this superstring theory (natural inclusion of gravity, non-Abelian gauge interactions and chirality, finiteness and existence of compactification schemes leading to $N = 1$ supersymmetry in four dimensions, to name a few) have stimulated a huge amount of work during the end of the eighties. Progress however have not been as impressive as expected and some interrogations and criticism have raised during the beginning of the last decade:

- String theory is a perturbative theory based on the quantization of the two-dimensional string world-sheet action. This first quantization yields a QFT with signature $(1,1)$ and allows to compute scattering amplitudes (S-matrix elements) between string states. A second quantization in terms of a string field theory may lead to a more general formulation of the model with the potential to go beyond the perturbative expansion.
- There are five consistent perturbative superstring theories in ten dimensions and it seems difficult to decide which theory one should favour. More dramatically, there are many sensible spaces on which one can compactify any of these superstring theories, and nothing seems to tell us that one has to compactify them down to four dimensions (and not down to, for example, five or six dimensions). All the expected predictive power of the model is manifestly lost.
- The unification of the gauge and gravitational coupling strengths as one goes to higher and higher energies is not clearly established.
- There is no explanation to the smallness of the cosmological constant (the energy density of the vacuum) observed in Nature.
- The procedure that has to be used to break the remnant supersymmetry in four dimensions is still very speculative.

[19] A Calabi–Yau manifold of complex dimension D is a compact, complex, Kähler manifold with vanishing first Chern class. As conjectured by Eugenio Calabi and proved by Shing-Tung Yau, it admits a Ricci-flat metric of $SU(D)$ holonomy, a crucial property for the preservation of some unbroken supersymmetry.

[20] An orbifold is a quotient space $\Gamma \equiv \mathcal{M}/G$, where $\mathcal{M}$ is a manifold and G a discrete symmetry group which does not act freely (the orbifold has fixed points).

[21] A toroidal compactification does not break any supersymmetry.

Fortunately, the past five years or so have witnessed a great improvement in the understanding of the non-perturbative face of string theory and a few answers to some of the questions above have been provided. In particular, the discovery of a web of connections (or dualities) between the five superstring theories suggests that these models are actually five different descriptions of the same physics. Each superstring theory can apparently be seen as a specific perturbative expansion of a single (yet unknown) underlying theory near a particular point of the space of the various conceivable quantum vacua.[22] Furthermore, there is evidence that this abstract space contains at least another significant point corresponding to a theory in eleven dimensions which has been baptised *M-theory*.[23] According to the point of view of John H. Schwarz [192], this intrinsically non-perturbative theory seems to be on an equal footing with the type IIB theory (or perhaps even with a conjectured twelve-dimensional theory christened *F-theory*[24] [221, 147, 148, 231]). However, many people, following the ideas of Michael Duff [72], are convinced that M-theory (and through it eleven-dimensional supergravity, which was for a long time completely disconnected from string theory) is actually a cornerstone in the process of unification.

To fully assess the significance of this *second superstring revolution*, one has to understand the non-perturbative content of the string spectrum. The next chapter is then devoted to a description of some non-perturbative states and a more detailed discussion of the web of dualities is postponed until Chapter 3.

[22] It is interesting to note the analogy between this vision of string theory and the usual definition of a manifold (each known superstring theory being compared to an open set and the relations (dualities) which express their non-perturbative equivalence corresponding to the transition functions). This parallel was proposed by Cumrun Vafa at the CERN workshop in June 1996.

[23] Very little is known about the precise formulation of this theory, and it is even not clear if M should be standing for “Magic”, “Matrix”, “Membrane”, “Mother” or “Mysterious”.

[24] With F standing for “Father”. Apart from F-theory, it is worth noticing that some papers propose the existence of other non-standard supergravity-like theories in dimensions higher than eleven (most of the time these theories have non-standard signatures, for instance $(10,2)$ in F-theory). In particular, a theory in thirteen dimensions (baptised *S-theory*) has been speculated [16, 17].

Chapter 2

Non-perturbative states in string theory

The aim of this chapter is to see that mathematical curves are not the only objects that play a role in string theory. After having briefly reviewed some aspects of the perturbative approach to string theory, we focus on classical non-perturbative solutions in the framework of supersymmetric four-dimensional gauge theories. In particular, we discuss the existence of a special class of solitons, called BPS states, which preserve part of the underlying supersymmetry and lead to formulas believed to be exact at any value of the gauge coupling. We then extend this discussion to each of the ten- or eleven-dimensional supergravity which seems to be relevant as an effective low-energy approximation to string theory. In this case, it turns out that BPS states are also present in the form of infinitely extended solutions called p-branes (a 1-brane being a BPS string). The presence of these macroscopic solitons in the non-perturbative spectrum gives a new perspective on string theory. Non-perturbatively, none of these entities is really more fundamental than another, and each appears to be useful in a particular domain of the theory. For example, we shall see that 2- and 5-branes are essential to the understanding of the strong coupling regime of the type IIA superstring theory. What distinguishes strings is that they allow a consistent perturbative expansion.

2.1 "Old" perturbative string theory

Until recently, string theory has been essentially studied through perturbation theory. From this point of view, the fundamental objects are one-dimensional strings (which can be open or closed, oriented or unoriented) sweeping out a surface (the world-sheet) as they moved in a d-dimensional embedding (target) space-time. The particles in space-time are then identified with the oscillator excitations of the first-quantized strings. String scattering amplitudes are built from a fundamental string interaction that can be viewed as a process in which a single string splits into two or in which two strings join to give a single one. The strength of this interaction is governed by the dimensionless string coupling g_{s}. The amplitude describing a peculiar process is then developed according to a power series in the expansion parameter g_{s} (Feynman sum over histories). The interesting point in string theory is that the number of topologically distinct Feynman string diagrams (two-dimensional surfaces) is quite limited.

For instance, in the closed oriented superstring theories, there is only one Feynman string diagram at each order of the expansion.

The string coupling constant g_s is related to a massless real scalar field $\varphi = \varphi(X^M)$ (called the *dilaton*) present in all superstring theories with a coupling

$$S_\varphi = \frac{1}{4\pi} \int d\tau d\sigma \, \sqrt{h} \varphi R^{(2)}. \tag{2.1}$$

In this formula, $h_{\hat{m}\hat{n}}$ is the world-sheet metric and $R^{(2)} = R^{(2)}(h)$ is the scalar curvature on the world-sheet [hence the (2)]. The so-called Gauss–Bonnet term (two-dimensional Einstein–Hilbert action) $\chi = \frac{1}{4\pi} \int d\tau d\sigma \, \sqrt{h} R^{(2)}$ is a topological invariant called the Euler number: $\chi = 2(1-g) - b - c$, where g, b and c are respectively the genus (the number of handles), the number of boundaries (holes) and the number of cross-caps (in the case of an unoriented surface) characterizing completely the topology of the two-dimensional surface under consideration. Such a term can be added to the action (1.2) because it has the required classical symmetries (two-dimensional coordinate and Weyl invariances and Poincaré invariance). Note that an extra term is required in order to preserve the Weyl invariance if the world-sheet has a boundary [171]. The vacuum expectation value (vev) of the dilaton, $\langle\varphi\rangle$, determines the string coupling through the equation

$$g_\mathrm{s} = e^{\langle\varphi\rangle}. \tag{2.2}$$

More precisely, in perturbation theory around a certain vacuum the dilaton field is written as $\varphi = \langle\varphi\rangle + \varphi_\mathrm{f}$, where $\langle\varphi\rangle$ is the constant part (vev) of the field and φ_f its quantum fluctuation. Now, the perturbative expansion in terms of Feynman string diagrams is based on the Polyakov path-integral formalism which involves the exponential e^{-S}, where S is a sum containing in particular the actions (1.2) and (2.1). One observes that the vev $\langle\varphi\rangle$ will naturally appear in a prefactor $e^{-\chi\langle\varphi\rangle}$, which can be interpreted as $g_\mathrm{s}^{-\chi}$, with χ giving the order of the expansion in the world-sheet topologies.

The dilaton is a modulus: it corresponds to a flat direction (a line of degenerate minima) of the effective potential of the superstring theory.[1] The coupling g_s is then a free parameter and in particular there is no known reason that it should be small. However, the perturbation expansion is expected to give an arbitrarily good approximation insofar as we take the constant g_s sufficiently small[2] and string perturbation theory must be restricted to the domain of the (moduli) space of vacua for which $g_\mathrm{s} < 1$. For g_s of order 1 or greater, string perturbation theory can, at best, provide a very poor approximation of the real world.

[1]In string theory, there are several dynamical fields, called *moduli*, whose vacuum expectation values determine the parameters of the theory. To leading order in perturbation theory, these fields have no potential which could lift (by minimization) the degeneracy of their vev, and in many cases (especially if supersymmetry is unbroken), they cannot have a potential exactly (i.e. even when all non-perturbative effects are taken into account). Apart from the dilaton, typical examples of such moduli are the dynamical fields which determine the dimension and the shape of the compact space used in a string theory compactification (like for instance the radius in a compactification on a circle).

[2]In QED for example, the parameter measuring the coupling of photons and electrons is the dimensionless fine structure "constant",

$$\alpha = \frac{e^2}{4\pi} \cong \frac{1}{137}$$

(the number 1/137 corresponds to the asymptotic value (low energy limit) of the running coupling constant

2.1.1 Five well-defined perturbative theories in ten dimensions

To be consistent at the quantum level a perturbative superstring theory has to satisfy a number of stringent constraints such as modular invariance of the partition function and space-time supersymmetry. *Five* theories satisfying these requirements have been constructed. They all live in ten (or less by compactification) space-time dimensions (nine dimensions of space and one of time) and their essential features can be summarized as follows:

- The **type IIA** theory describes *closed oriented strings* and possesses a *space-time supersymmetry* $N = 2$.

  The two Majorana-Weyl supersymmetry generators (with 16 supercharges each) can be seen as coming from a single Majorana spinor (with 32 supercharges).[3] They are of opposite chirality and the theory displays a left-right symmetry (it is a *non-chiral theory*).[4]

  There is no freedom to introduce a Yang–Mills gauge group.

- The **type IIB** theory describes *closed oriented strings* and has a *space-time supersymmetry* $N = 2$.

  The two Majorana-Weyl supersymmetry generators (with 16 supercharges each) are of the same chirality and the theory displays a left-right asymmetry (it is a *chiral theory*).

  There is no freedom to introduce a Yang–Mills group.

- The **heterotic** theories describe *closed oriented strings* and have a *space-time supersymmetry* $N = 1$.

  There is a gauge group associated with the internal degrees of freedom following from the existence of a bosonic sector. It is constrained to be either $SO(32)$ or $E_8 \times E_8$ [109, 110, 111].

- The **type I** theory describes the interactions of *open and closed unoriented strings* and has a *space-time supersymmetry* $N = 1$.

  There is a gauge group associated with the possibility to add a discrete non-dynamical[5] degree of freedom at the two special points (the endpoints) of an open string. Assuming there are n possibilities at each endpoint, open string states are labelled by a pair of numbers $1 \leq i, j \leq n$, and consistency at the quantum level allows only one gauge group: $SO(n)$, with $n = 32$ [186, 142, 105].

α; the QED coupling increases gently with momentum transfer). Thanks to the smallness of the constant α, the first few terms of a power series expansion give already a very good approximation and so far the perturbative approach has been successful.

[3]This is a first hint about the existence of an eleven-dimensional theory. We will come back to this point in subsections 2.4.2 and 3.2.3.

[4]We could rephrase this remark by saying that the two gravitinos present in the superstring spectrum have opposite handedness.

[5]An endpoint prepared in a given state will remain in the same state.

The particle (field) content of each theory is found by quantizing a single string. The Table 2.1 displays the various massless bosonic fields present in each spectrum. For the type I and type II superstring theories, the origin of the fields as coming from the boundary conditions of the world-sheet fermions (i.e. as coming from the NS $\otimes$ NS or R $\otimes$ R sectors) has been indicated. It is worth noting that the graviton g_{AB}, the antisymmetric tensor B_{AB} and the dilaton φ are present in each spectrum. Each massless particle is associated with a field in the effective theory (supergravity) and there is a unique low energy supergravity for each superstring theory. These ten-dimensional effective supergravity theories will be discussed in section 2.4.

2.2 Solitons in field theory

In field theory, a solitonic solution is a configuration of fields which has finite energy and is localized in space and stable (it has thus the potentiality to describe physical particles). The 't Hooft–Polyakov monopole of the $SU(2)$ Georgi–Glashow (Yang–Mills–Higgs) gauge theory is a popular example of such a solution. The two main properties of a soliton can be summarized as follows:[6]

- A soliton is *non-perturbative*. It is a non-singular solution of non-linear field equations which cannot be obtained perturbatively from the linearized field equations. Its mass is inversely proportional to a positive power of the coupling constant and thus becomes large at weak coupling.
- A soliton carries a *conserved topological charge* (rather than a Noether one) related to the non-trivial topology of the vacuum of the theory.

2.2.1 Electric-magnetic duality and BPS states in supersymmetric four-dimensional gauge theories

In 1977, the properties of dyons (solitonic states carrying electric and magnetic charges) found in $SU(2)$ gauge theories led Montonen and Olive to suggest the existence of an electric-magnetic duality [145].[7] According to their proposition, there are two complementary formulations of these gauge theories and these two perspectives are related by the exchange of electric and magnetic (i.e. of fundamental and solitonic) degrees of freedom and by the inversion of the gauge coupling (i.e. by the exchange of weak and strong coupling). However, this duality was essentially justified by the classical mass spectrum and it was not clear how it would survive the introduction of quantum corrections. The precise matching of the fundamental and solitonic states was also problematic because the particles had different spins. Consideration of generalized versions of the model with extended supersymmetry clarified the situation [234, 162]. In 1994, Sen gave an non-trivial test of an extension of the original electric-magnetic duality to an $SL(2,\mathbb{Z})$ duality in $N = 4$ super-Yang–Mills theories [201]. This duality extension was motivated by the possibility to add a θ-term in the Yang–Mills

[6] A good introduction to solitons is the textbook by R. Rajaraman [175].

[7] This duality is nothing but a generalization of the duality present in source-free Maxwell's equations.

Perturbative string theory's name	Bosonic spectrum
Type IIA	g_{AB}, B_{AB}, φ (NS $\otimes$ NS) A_A, C_{ABC} (R $\otimes$ R)
Type IIB	$g_{AB}, B^{(1)}_{AB}, \varphi^{(1)}$ (NS $\otimes$ NS) $C^+_{ABCD}, B^{(2)}_{AB}, \varphi^{(2)}$ (R $\otimes$ R)
Heterotic $E_8 \times E_8$	g_{AB}, B_{AB}, φ A^a_A (gauge bosons of $E_8 \times E_8$)
Heterotic $SO(32)$	g_{AB}, B_{AB}, φ A^a_A (gauge bosons of $SO(32)$)
Type I	g_{AB}, φ (NS $\otimes$ NS) B_{AB} (R $\otimes$ R) A^a_A (gauge bosons of $SO(32)$)

Table 2.1: Massless bosonic sector of the five ten-dimensional perturbative string theories. Note that the gauge bosons A^a_A are always in the *adjoint* representation of the non-Abelian group considered and that C^+_{ABCD} denotes the components of a rank-4 antisymmetric tensor with a self-dual rank-5 field strength. The subscripts A, B, C, D go from 1 to 10.

Lagrangian (this angle θ is the one already mentioned in Chapter 1 at footnote 2) which shifts the allowed values of the electric charge ("Witten effect" [225]).[8] The $SL(2,\mathbb{Z})$ duality is usually dubbed *S-duality* because in the context of $N = 4$ Yang–Mills theories obtained by compactification of the heterotic string on a six-dimensional torus it acts on a field named S (dilaton-axion field) [91, 179]. We will meet this field when discussing four-dimensional effective actions in Chapters 5 and 6. During the same year 1994, the idea of electromagnetic duality in supersymmetric gauge theories culminated with the work of Seiberg and Witten [197, 198]. Using supersymmetry and $SL(2,\mathbb{Z})$ duality, these authors were able to obtain an *exact* expression for the low-energy effective action of the $N = 2$ super-Yang–Mills theory with gauge group $SU(2)$ and to tackle interesting non-perturbative phenomena (like electric charge confinement induced by condensation of magnetic monopoles).

These impressive developments in the comprehension of supersymmetric gauge theories would probably not have been possible without the presence in the spectrum of states whose essential properties are thought to be exact at any value of the gauge coupling.

BPS states in field theories

Semiclassically, one finds [32, 174] that the mass of a finite-energy solution of the Georgi–Glashow equations with electric and magnetic charges q_e and q_m has a lower bound (*Bogomol'nyi–Prasad–Sommerfield bound* or simply *BPS bound*) according to

$$M_\mathrm{dyon} \geq v\sqrt{q_\mathrm{e}^2 + q_\mathrm{m}^2}, \tag{2.3}$$

where v is the mean Higgs value. This inequality tells us that a charged particle is always massive. States for which this inequality is in fact an equality are said *BPS saturated.* The application of the relation (2.3) in the case of the 't Hooft–Polyakov monopole which has only a magnetic charge gives

$$M_\mathrm{m} \geq v|q_\mathrm{m}|. \tag{2.4}$$

In the limit of a vanishing Higgs potential, it is possible to find a solution saturating the bound (the *BPS monopole*).

An inequality similar to (2.3) or (2.4) can still hold in presence of quantum corrections in gauge theories with extended supersymmetry. Actually, the extended supersymmetry algebra ($N > 1$) for point-like states in four dimensions contains the commutators[9]

$$\begin{aligned} \{Q^i_\alpha, \overline{Q}^j_{\dot\beta}\} &= \delta^{ij}\sigma^\mu_{\alpha\dot\beta}P_\mu, \\ \{Q^i_\alpha, Q^j_\beta\} &= \epsilon_{\alpha\beta}Z^{ij}, \end{aligned} \tag{2.5}$$

where the Q^i are the N supersymmetry generators of the Lie algebra and P_μ is the momentum operator. The $N(N-1)/2$ numbers $Z^{ij} = -Z^{ji}$ are called central charges. They are combinations of gauge charges and scalars expectation values. Using a representation of the

[8] A detailed exposition of the original Montonen–Olive conjecture and of its subsequent refinements can be found for instance in the review written by Jeffrey A. Harvey [114].

[9] The Greek indices are spinor indices. We adopt here the two-component notation which is for instance used in Ref. [89].

algebra in terms of a massive particle in its rest frame $P^\mu = (m, 0, 0, 0)$ and taking advantage of the $U(N)$ internal symmetry of the algebra, one gets the inequality [89]

$$m \geq Z_\lambda, \tag{2.6}$$

where Z_λ is any eigenvalue of the central charge matrix Z^{ij} ($\lambda = 1, \ldots, N/2$ for even N or $\lambda = 1, \ldots, (N-1)/2$ for odd N). If the relation (2.6) is strictly satisfied ($m > Z_\lambda$ for all Z_λ), the massive multiplets (representations) contain 2^{2N} states constructed from the action of $2N$ fermionic creation operators on a Clifford vacuum. If n of the Z_λ are equal to one another and to m, the number of the non-trivial (non-zero) operators is decreased to $2N - 2n$. The representation has then only $2^{2(N-n)}$ states and is called a *short* or *BPS representation*. It is invariant under the supersymmetry transformations produced by the trivial generators. The extreme case in which all the Z_λ are equal to m corresponds to an *ultra-short representation* with a dimension (2^N for N even) equal to the one of the massless representation. In summary, a *BPS state* is invariant under a non-trivial part of the supersymmetry transformations and it lies in a smaller supersymmetry multiplet than a non-BPS state (some of the supersymmetry generators are represented by zero and cannot be used to create new states). The important point is that it is believed that a given multiplet cannot become another multiplet (with a different number of states) under a smooth change in the scalar fields of the theory (one assumes that the number of states does not vary discontinuously). As a consequence, the *BPS mass formula* that relates the mass and the conserved charges of a BPS state is then in particular supposed to be true (after renormalization) even at strong coupling.

2.3 BPS states in string theory

Following the example of gauge theories, string theory has low-energy field equations that admit various non-trivial solutions in the form of extended objects. In particular, the classical field equations of the effective ten- and eleven-dimensional supergravity theories admit BPS solutions that leave unbroken a certain fraction of (usually half) the supersymmetry of the underlying supergravity.[10] These BPS solutions (also called "extreme solutions") stretch on p spatial dimensions and on one of time.[11] They are localized in all the other $(d-1-p)$ spatial coordinates, where d is the dimension of the embedding space-time[12], and are commonly called *p-branes*: a 2-brane is a membrane, a 1-brane is a string, and when $p = 0$ the solitonic state has many characteristics of a point particle. A p-brane can be interpreted as an extended source for an Abelian $(p+1)$-form gauge potential A_{p+1} with $(p+2)$-form field strength F_{p+2},[13] and, as we shall see in subsection 2.4, each supergravity has a particular set of such forms.

[10] States preserving a different portion of supersymmetry (for example 1/4 or 1/8) can also be constructed, but they appear to be less fundamental and interesting than the states preserving half the supersymmetry.

[11] In other words, they have a $(p+1)$-dimensional world-volume.

[12] Of course, this has a sense provided $(p+1) \leq d$.

[13] For instance, the 0-branes (particles) are sources of the Maxwell's tensor F_2 (a 2-form), the 1-branes (strings) are sources of a 3-form field strength F_3, etc.

2.3.1 "Electrodynamics" of the n-forms

In classical four-dimensional electrodynamics, the 4-vector potential $(A^\mu) = (\phi, \vec{A})$ can be denoted as a 1-form $A_1 \equiv A_\mu dx^\mu$ from which one defines the electromagnetic 2-form by $F_2 = dA_1$[14]. The 2-form F_2 is invariant under the (gauge) transformation $A_1 \to A_1 + d\Lambda$, where Λ is an arbitrary 0-form. The homogeneous Maxwell's equations, $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{E} + \partial \vec{B}/\partial t = 0$, are unified in the identity[15]

$$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0 \quad \Leftrightarrow \quad dF_2 = 0 \tag{2.7}$$

which is known as the *Bianchi identity*. Using the 1-form $j_{1,\mathrm{e}} \equiv j_{\mu,\mathrm{e}} dx^\mu$ obtained from the (electric) 4-"current" $(j^\mu_\mathrm{e}) = (\rho_\mathrm{e}, \vec{j}_\mathrm{e})$, the inhomogeneous Maxwell's equations, $\vec{\nabla} \cdot \vec{E} = \rho_\mathrm{e}$ and $\vec{\nabla} \times \vec{B} - \partial \vec{E}/\partial t = \vec{j}_\mathrm{e}$, read

$$\partial_\mu F^{\mu\nu} = -j^\nu \quad \Leftrightarrow \quad d * F_2 = - * j_{1,\mathrm{e}}, \tag{2.8}$$

where $*F_2$ is the 2-form dual to F_2 with components $F^*_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$.[16] The Lagrangian for the electromagnetic field is

$$\mathcal{L}_\mathrm{em} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu j^\mu_\mathrm{e}, \tag{2.9}$$

and the corresponding action in terms of differential forms reads

$$S_\mathrm{em} = \int \Big(-\frac{1}{2} F_2 \wedge *F_2 - A_1 \wedge *j_{1,e} \Big). \tag{2.10}$$

The electric charge contained inside a surface $\partial\mathcal{M}$ is given by the integral[17]

$$Q_\mathrm{e}(\mathcal{M}) = \int_\mathcal{M} - * j_{1,\mathrm{e}} = \int_{\partial\mathcal{M}} *F_2, \tag{2.11}$$

where Stokes' theorem and the relation (2.8) were used.

[14] The corresponding antisymmetric tensor (field strength) is given by the components $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

[15] As usual, the electric and magnetic fields are expressed in terms of the 4-vector potential (A_μ) as

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi \quad \text{and} \quad \vec{B} = \vec{\nabla} \times \vec{A}.$$

[16] In this subsection, space-time is supposed to be flat. Note that our definition for the dual of a form [see Eq. (A.13)] differs from the one adopted in some textbooks (like for instance [144]). This explains the "non-conventional" sign in the relation between $d * F_2$ and $*j_{1,\mathrm{e}}$.

[17] In more common notations, the Gauss's law for point electric charges leads to:

$$Q_\mathrm{e}(\mathcal{M}) = \int_\mathcal{M} d^3x\, \rho_\mathrm{e}(x) = \int_{\partial\mathcal{M}} d\vec{S} \cdot \vec{E}.$$

In presence of magnetic monopoles [65], one introduces a magnetic 4-current $(k^\mu_{\rm m}) = (\sigma_{\rm m}, \vec{k}_{\rm m})$ and the associated 1-form $k_{1,{\rm m}}$ or the dual 3-form $j_{3,{\rm m}} = *k_{1,{\rm m}}$. The electrodynamic law (2.8) is not altered, but the Bianchi identity (2.7) becomes

$$dF_2 = -j_{3,{\rm m}}. \tag{2.12}$$

The magnetic current is then locally conserved ($dj_{3,{\rm m}} = 0$) and the magnetic charge inside the surface $\mathcal{M}$ is represented by the integral

$$Q_{\rm m}(\mathcal{M}) = \int_{\mathcal{M}} -j_{3,{\rm m}} = \int_{\partial\mathcal{M}} F_2. \tag{2.13}$$

Since the Bianchi identity is not $dF_2 = 0$ anymore, the vector potential cannot be globally defined. Actually, it must at least be singular on a curve, going from the pole to infinity. This curve is called the Dirac string. The coherence of the quantum description of an electrically charged particle (charge $q_{\rm e}$) moving in the field of a magnetic monopole (charge $q_{\rm m}$) imposes a condition on the product of the two charges [65]:[18]

$$q_{\rm e}q_{\rm m} = 2\pi k, \quad \text{with } k \in \mathbb{Z}. \tag{2.14}$$

This condition is called the Dirac quantization condition. One of its most important consequences is that the existence of a single monopole in the Universe implies charge quantization.

In the following of this subsection, we extend this classical electrodynamics to more general gauge interactions involving gauge fields (antisymmetric tensors) represented in terms of differential forms

$$A_n = \frac{1}{n!} A_{M_1\ldots M_n} dx^{M_1} \wedge \cdots \wedge dx^{M_n}, \tag{2.15}$$

where the components $A_{M_1\ldots M_n}$ are totally antisymmetric. A gauge transformation corresponds then to the addition of an exact form to the n-form potential A_n:

$$A_n \to A_n + d\Lambda_{n-1}, \tag{2.16}$$

where the parameter of the gauge transformation Λ_{n-1} is an arbitrary $(n-1)$-form, and the gauge invariant field strength reads

$$F_{n+1} = dA_n. \tag{2.17}$$

"Electric" p-branes

Replacing the 1-form potential A_1 of the usual electrodynamics by an n-form A_n leads to an Abelian gauge theory based on the group $U(1)$ in which the role of the electrically charged point particles is played by extended objects of spatial dimension $p = n - 1$ [213].

[18] More precisely, this condition comes from the requirement that the wave function of the electrically charged particle should be single valued. Wu and Yang have proved the Dirac quantization condition using another, more geometrical, point of view [235]. They have described the magnetic monopole using two non-singular potentials, valid in two different regions of space and related by a gauge transformation on their overlapping domain.

An electric p-brane has a world-volume (wv) with $(p+1) = n$ dimensions and the fact that it can be assimilated to an electric source follows from the presence, in the action describing the interaction of one extended object with the dynamical field $A_{M_1 \dots M_n}$, of a term of the form

$$q_{\mathrm{e}} \int_{\mathrm{wv}} \hat{A}_n \equiv \frac{1}{n!} q_{\mathrm{e}} \int_{\mathrm{wv}} dy^{\hat{m}_1} \wedge \cdots \wedge dy^{\hat{m}_n} \hat{A}_{\hat{m}_1 \dots \hat{m}_n}(y), \tag{2.18}$$

where q_{e} is a constant interpreted as the electric charge (or the brane tension as we will discuss below), $y^{\hat{m}}$ (with $\hat{m} = 1, \dots, n$) are coordinates on the n-dimensional world-volume of the p-brane and $\hat{A}_{\hat{m}_1 \dots \hat{m}_n}(y)$ is the gauge field "pulled back" onto the world-volume

$$\hat{A}_{\hat{m}_1 \dots \hat{m}_n}(y) = \frac{\partial z^{M_1}}{\partial y^{\hat{m}_1}} \cdots \frac{\partial z^{M_n}}{\partial y^{\hat{m}_n}} A_{M_1 \dots M_n}(z(y)), \tag{2.19}$$

$z^M(y)$ (with $M = 1, \dots, d$) being a parameterization in space-time of the n-dimensional history of the p-brane. The integral (2.18) can be seen as a generalization of the coupling $\int A_1 \wedge * j_{1,e}$ of Maxwell's theory, with a p-brane current

$$j_{\mathrm{e}}^{M_1 \dots M_n}(x) = q_{\mathrm{e}} \int_{\mathrm{wv}} dz^{M_1} \wedge \cdots \wedge dz^{M_n} \delta^{(d)}(x - z(y)). \tag{2.20}$$

In this expression, $\delta^{(d)}(x - z(y))$ is a d-dimensional delta-function and x is a generic point in space-time. Inserting the Hodge dual form of this current,

$$* j_{n,\mathrm{e}} = \frac{q_{\mathrm{e}}}{n!(d-n)!} \epsilon_{L_1 \dots L_{d-n} M_1 \dots M_n} \int_{\mathrm{wv}} dz^{M_1} \wedge \cdots \wedge dz^{M_n} \delta^{(d)}(x - z(y))\, dx^{L_1} \wedge \cdots \wedge dx^{L_{d-n}}, \tag{2.21}$$

into the d-dimensional integral $\int A_n \wedge * j_{n,\mathrm{e}}$, one recovers (up to a factor $(-1)^{n(d-n)}$) the world-volume term (2.18) . The "electromagnetic" action for the gauge field A_n can then be written as

$$\begin{aligned} S_{\mathrm{em}} &= \int d^d x \Big(- \frac{1}{2(n+1)!} F_{M_1 \dots M_{n+1}} F^{M_1 \dots M_{n+1}} + A_{M_1 \dots M_n} j_{\mathrm{e}}^{M_1 \dots M_n} \Big) \\ &= \int \Big(- \frac{1}{2} (-1)^{(n+1)(d-n-1)} F_{n+1} \wedge * F_{n+1} + (-1)^{n(d-n)} A_n \wedge * j_{n,\mathrm{e}} \Big), \end{aligned} \tag{2.22}$$

and the equations of motion read

$$\partial_{M_1} F^{M_1 M_2 \dots M_{n+1}} = - j_{\mathrm{e}}^{M_2 \dots M_{n+1}} \quad \Leftrightarrow \quad d * F_{n+1} = (-1)^{d-n} * j_{n,\mathrm{e}}. \tag{2.23}$$

The p-brane current is then locally conserved,

$$\partial_{M_1} j_{\mathrm{e}}^{M_1 \dots M_n}(x) = 0 \quad \Leftrightarrow \quad d * j_{n,\mathrm{e}} = 0, \tag{2.24}$$

and the expression for the total electric charge[19] inside a "surface" $(\partial \Sigma)_{d-n-1}$ (typically a

[19] More precisely, for the total electric charge per unit of p-volume (since the current $j_{n,\mathrm{e}}$ contains a charge per unit of $(d-1)$-(spatial)-volume).

sphere S^{d-n-1}) is

$$\begin{aligned} Q_{\mathrm{e}}(\Sigma) &= \int_{\Sigma_{d-n}} - * j_{n,\mathrm{e}} = (-1)^{d-n-1} \int_{(\partial\Sigma)_{d-n-1}} *F_{n+1} \\ &= (-1)^{d-p-2} \int_{(\partial\Sigma)_{d-p-2}} *F_{p+2}. \end{aligned} \tag{2.25}$$

Moreover, the conservation law (2.24), applied to the explicit expression (2.20) for the current $j_{\mathrm{e}}^{M_1 \ldots M_n}(x)$, implies that the p-brane world-volume has no boundary. This means that the directions of the p-brane must be either closed (compact) or infinitely extended. But the integral giving the charge $Q_{\mathrm{e}}(\Sigma)$ receives only contributions from the (0-dimensional) intersections between the $(p+1)$-dimensional world-volume and Σ_{d-p-1}. As illustrated in Fig. 2.1 for a particular case ($d = 4, p = 1$), a p-brane which is closed in at least one direction will then have an equal number of positive and negative contributions. In other words, the higher-dimensional analogue of a point-particle with a non-zero charge in four-dimensional electrodynamics is an object with a world-volume extended to infinity in all its directions. Similar considerations hold for the "magnetic" p-branes portrayed in the following paragraph.

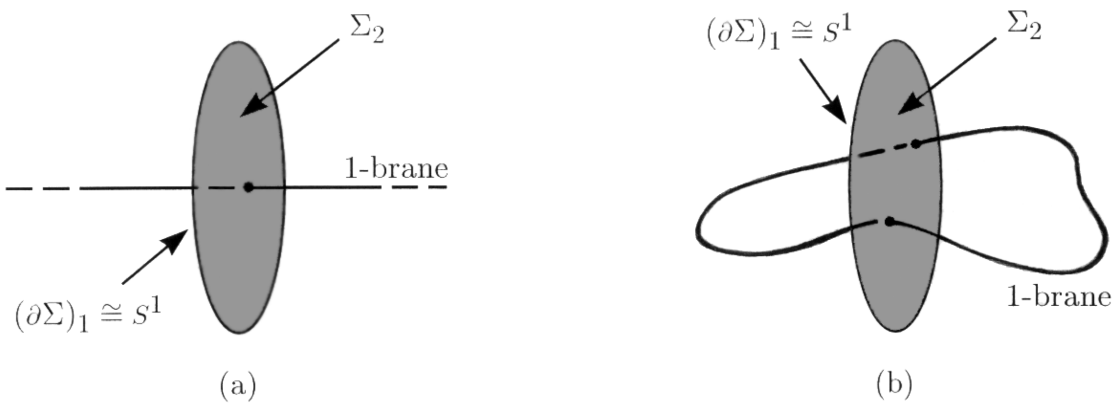

Figure 2.1: Illustration of the calculation of the charge $Q_{\mathrm{e}}(\Sigma)$ for $d = 4$ and $p = 1$.
By charge conservation, the world-volume of the string (1-brane) has no boundary, so that the 1-brane is either infinitely extended or closed. (a) The infinite 1-brane has one intersection with Σ_2 and its charge is non-zero. (b) The closed 1-brane has two intersections with Σ_2 and these contributions cancel each other.

Two examples of electric p-branes are the ten-dimensional heterotic strings ($p = 1$) for which

$$Q_{\mathrm{e}}(\Sigma) = -\int_{(\partial\Sigma)_7} *H_3, \tag{2.26}$$

where $H_3 = dB_2$ is the 3-form field strength for the 2-form potential B_2 from the massless bosonic sector of the string spectrum, and the ten-dimensional type II superstrings ($p = 1$) which are sources for the 2-form gauge potential from the NS $\otimes$ NS sector.

“Magnetic” p-branes

It is also possible to consider a generalization of the Dirac magnetic pole which is dual to the “electric” p-brane and is therefore associated with the Hodge dual of F_{p+2}. In a d-dimensional space-time, such a “magnetic” brane has dimension $\tilde{p} = (d-n-3) = (d-p-4)$, as can be verified from the following diagram:

$$\begin{array}{ccccc} p & \longrightarrow & A_{p+1} & \longrightarrow & F_{p+2} \\ & & & & \downarrow * \\ \tilde{p} \overset{\text{def.}}{=} d-p-4 & \longleftarrow & A_{d-p-3} & \longleftarrow & (*F)_{d-p-2} \end{array}$$

One should not confuse this object with the ’t Hooft–Polyakov monopole which is a static localized and non-singular solution in the framework of non-Abelian gauge theories. In fact, non-Abelian Yang–Mills gauge theories for extended objects coupling to p-forms cannot be fulfilled [213].

In presence of a dual $\tilde{p}$-brane described by a current $j_{p+3,\mathrm{m}}$, the Bianchi identity becomes

$$dF_{p+2} = (-1)^{dp+1} j_{p+3,\mathrm{m}}, \tag{2.27}$$

so that the “magnetic” charge per unit of $\tilde{p}$-volume is given by

$$Q_{\mathrm{m}}(\Sigma) = \int_{\Sigma_{p+3}} -j_{p+3,\mathrm{m}} = (-1)^{dp} \int_{(\partial\Sigma)_{p+2}} F_{p+2}, \tag{2.28}$$

where $(\partial\Sigma)_{p+2}$ is a “surface” (typically a sphere S^{p+2}) surrounding the $\tilde{p}$-brane.

Dyonic p-branes

A few p-branes are dyonic: they carry both electric and magnetic charges. Starting from the kinetic action for the gauge field

$$-\frac{1}{2(p+2)!} \int d^d x\, F_{M_1 \ldots M_{p+2}} F^{M_1 \ldots M_{p+2}} \tag{2.29}$$

which should be a pure number, it is easy to deduce that the dimension of the components $A_{M_1 \ldots M_{p+1}}$ is $[\text{mass}]^{\frac{d}{2}-1}$. The dimension of the electric charge (2.25) associated with a p-brane is then

$$[Q_{\mathrm{e}}] = [\text{mass}]^{-\frac{d}{2}+2+p}, \tag{2.30}$$

while the corresponding magnetic charge (2.28) has a dimension

$$[Q_{\mathrm{m}}] = [\text{mass}]^{\frac{d}{2}-2-p}. \tag{2.31}$$

In particular, the dimension of these charges is the same when $p = (d-4)/2$ (Q_{e} and Q_{m} are then actually dimensionless) and dyonic p-branes are thus conceivable.[20] So, there are dyons (0-branes) in four dimensions [94, 97], dyonic strings (1-branes) in six dimensions [80, 73], dyonic membranes (2-branes) in eight dimensions [20, 124], and dyonic 3-branes in ten dimensions [78].

[20] The relation $d = 2(n+1)$ gives also the dimension for which F_{n+1} and $*F_{n+1}$ have the same degree.

Quantization condition

Usually, the electric and magnetic charges $Q_{\rm e}$ and $Q_{\rm m}$ are not dimensionless. However, their product is always a pure number satisfying a generalization of the Dirac quantization condition (2.14) [154, 214]:[21]

$$Q_{\rm e}Q_{\rm m} = 2\pi k, \quad k \in \mathbb{Z}. \tag{2.32}$$

2.3.2 General structure of an extreme p-brane solution

To discuss the general structure of a basic p-brane solution in supergravity theories, it is sufficient to consider a classical theory in d dimensions including a metric g_{MN}, a scalar field φ (dilaton) and a single n-form gauge potential A_n (with corresponding field strength F_{n+1}). This "bosonic truncated action" (with respect to the full supergravity theory), written in the Einstein frame, is of the form[22]

$$S = \frac{1}{2\kappa_d^2}\int d^dx\,\sqrt{-g}\Big(-R - \frac{1}{2}(\partial_M\varphi)(\partial^M\varphi) - \frac{e^{a\varphi}}{2(n+1)!}F_{M_1\ldots M_{n+1}}F^{M_1\ldots M_{n+1}}\Big), \tag{2.33}$$

where a denotes a constant and $R = R(g)$ is the scalar curvature defined by $R = g^{MN}R_{MN}$ with a Ricci curvature tensor given, in terms of the Christoffel symbols

$$\Gamma^O_{MN} = \frac{1}{2}g^{OP}(\partial_M g_{NP} + \partial_N g_{MP} - \partial_P g_{MN}), \tag{2.34}$$

by

$$R_{MN} = \partial_N\Gamma^O_{OM} - \partial_O\Gamma^O_{MN} + \Gamma^P_{OM}\Gamma^O_{PN} - \Gamma^O_{MN}\Gamma^P_{OP}. \tag{2.35}$$

In the Einstein frame, the Einstein–Hilbert term has the usual form $-(2\kappa_d^2)^{-1}\sqrt{-g}R$. In ten dimensions, there is at least another relevant frame corresponding to the metric redefinition (Weyl rescaling) $g'_{AB} = e^{\varphi/2}g_{AB}$. This frame is often called the string frame because the rescaled metric g'_{AB} is the one which appears naturally in the string world-sheet action (1.2). In the string frame, the theory (2.33) takes the particular form

$$S = \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-g'}e^{-2\varphi}\Big(-R(g') + 4(\partial_A\varphi)(\partial^A\varphi) - \frac{e^{a'\varphi}}{2(n+1)!}F_{A_1\ldots A_{n+1}}F^{A_1\ldots A_{n+1}}\Big), \tag{2.36}$$

where the constant a' is equal to $+2$ if the n-form potential is coming from the $\mathrm{R}\otimes\mathrm{R}$ sector and to 0 otherwise.

The application of the variational principle to the action (2.33) gives the following equa-

[21] To find this condition, one can follow an approach close to the one adopted in Ref. [65] (introducing in particular a higher-dimensional analogue of the Dirac string on which the gauge field is singular).

[22] Note that here we have, in contrast with the preceding subsection, introduced a dimensionful gravitational coupling κ_d, so that the potential $A_{M_1\ldots M_n}$ is now dimensionless.

tions of motion for g_{MN}, φ and A_n:

$$
\begin{aligned}
0 &= R_{MN} + \frac{1}{2}(\partial_M \varphi)(\partial_N \varphi) + T_{MN}, \\
0 &= \partial_M(\sqrt{-g} g^{MN} \partial_N \varphi) - \frac{a}{2(n+1)!} \sqrt{-g} e^{a\varphi} F_{M_1 \ldots M_{n+1}} F^{M_1 \ldots M_{n+1}}, \\
0 &= \partial_{M_1}(\sqrt{-g} e^{a\varphi} F^{M_1 \ldots M_{n+1}}),
\end{aligned}
\tag{2.37}
$$

with

$$
T_{MN} = \frac{1}{2(n+1)!} e^{a\varphi} \Big((n+1) F_{MO_1 \ldots O_n} F_N{}^{O_1 \ldots O_n} - \frac{n}{d-2} g_{MN} F_{O_1 \ldots O_{n+1}} F^{O_1 \ldots O_{n+1}} \Big). \tag{2.38}
$$

Moreover, the fact that F_{n+1} is constructed from the n-form potential A_n results in the Bianchi identity

$$
\partial_{[M_1} F_{M_2 \ldots M_{n+2}]} = 0. \tag{2.39}
$$

To find a p-brane solution to these equations, one first splits the d space-time coordinates into $p+1$ world-volume coordinates $y^{\hat{m}}$ and $d-p-1$ transverse spatial coordinates z^a. One then makes a simplifying assumption by requiring a Poincaré symmetry on the world-volume of the brane and an $SO(d-p-1)$ symmetry in the transverse space. The metric and the dilaton are supposed to be of the form

$$
\begin{aligned}
ds^2 &= e^{A(r)} \eta_{\hat{m}\hat{n}} dy^{\hat{m}} dy^{\hat{n}} + e^{B(r)} \delta_{ab} dz^a dz^b, \\
\varphi &= \varphi(r),
\end{aligned}
\tag{2.40}
$$

where $r = \sqrt{\delta_{ab} z^a z^b}$ is the radial distance in the transverse space. This "ansatz" is invariant under Poincaré transformations on the brane and under rotations in the transverse space. For the gauge potential and the field strength, one assumes the expressions

$$
A_{\hat{m}_1 \ldots \hat{m}_{p+1}} = \epsilon_{\hat{m}_1 \ldots \hat{m}_{p+1}} e^{C(r)} \quad \text{and} \quad F_{a\hat{m}_1 \ldots \hat{m}_{p+1}} = \epsilon_{\hat{m}_1 \ldots \hat{m}_{p+1}} \partial_a e^{C(r)} \tag{2.41}
$$

(all the other components being set to zero) which, in the light of the preceding subsection, corresponds to an "electric" p-brane ansatz with $p = n-1$. It is also possible to construct a solitonic ($\tilde{p} = d-n-3$)-brane solution coupling to the dual of the gauge field A_n. The field strength of this "magnetic" ansatz reads

$$
F_{a_1 \ldots a_{d-\tilde{p}-2}} = \lambda \epsilon_{a_1 \ldots a_{d-\tilde{p}-2} b} \frac{z^b}{r^{d-\tilde{p}-1}}, \tag{2.42}
$$

where λ is an integration constant. All the other components are set to zero. It is worth noticing that supergravity theories usually contain Chern–Simons terms involving a wedge product between the potential and its field strength. Such terms have been omitted in the simplified action (2.33) because they do not contribute to the field equations due to the particular structure of the basic solutions (2.41) and (2.42) one considers here.

To completely determine the functions $A(r)$, $B(r)$, $C(r)$ and $\varphi(r)$, one imposes that the solution is asymptotically flat at transverse infinity ($A(r) = B(r) = 0$ for $r \to \infty$) and preserves half the supersymmetry of the untruncated theory ("extreme" solution). A complete derivation of these functions can be found for instance in the lecture notes written by K. S. Stelle [207] and will not be discussed here. In section 2.4 we will however give the precise form of the solitonic five-brane in eleven-dimensional supergravity.

Source terms and p-brane dynamics

The third equation (2.37) is nothing but a generalization of the Maxwell's equations (2.8) in absence of sources. It seems then natural to add p-brane source terms to the simplified supergravity action (2.33). These additional contributions are localized on the p-brane world-volume and, as anticipated in subsection 2.3.1, they induce the appearance of delta functions in the equations of motion (2.37).

The source terms associated to the presence of a p-brane must correspond to an action invariant under world-volume reparameterizations. In a first approximation, one can write

$$S_{p\text{-brane}} = -T_p \int_{\mathrm{wv}} d^{p+1}y \sqrt{-\det \hat{g}_{\hat{m}\hat{n}}} - q_p \int_{\mathrm{wv}} \hat{A}_{p+1} + \cdots , \tag{2.43}$$

where

$$\hat{g}_{\hat{m}\hat{n}}(y) = \frac{\partial X^M}{\partial y^{\hat{m}}} \frac{\partial X^N}{\partial y^{\hat{n}}} g_{MN}(X(y)) \tag{2.44}$$

is the induced metric on the $(p+1)$-dimensional world-volume of the p-brane. The potential $\hat{A}_{p+1}$ is also obtained as a pull-back of the $(p+1)$-form A_{p+1} present in the d-dimensional theory, and the p-brane tension T_p (i.e. the p-brane energy (mass) per unit of volume) is a dimensionful constant ($[T_p] = [\mathrm{mass}]^{p+1}$). The dots denote possible extra couplings to the d-dimensional fields, as well as terms describing the dynamics of the scalar, vector or tensor fields living solely on the p-brane.

The first term in $S_{p\text{-brane}}$ is proportional to the world-volume spanned by the trajectory of the p-brane. In the string case, it is simply a rewriting of the action (1.2) called the Nambu–Goto action. In ten dimensions, the metric used in (2.43) is assumed to be the so-called p-brane metric for which each term of the supergravity truncated action (2.36) has the same dependence on the dilaton. Written in the string frame, the generalization of the Nambu–Goto action is, most of the time, dilaton-dependent. Actually, the only exception concerns the fundamental string which is the appropriate source for a field strength coming from an NS $\otimes$ NS-like sector[23] and characterized in action (2.36) by a constant $a' = 0$. The Nambu–Goto action of its magnetic dual, the solitonic 5-brane, appears to include a dilaton prefactor $e^{-2\varphi}$, while other p-branes which couple to field strengths with $a' = +2$ have an intermediate dependence $e^{-\varphi}$ [76, 55]. In other words, the physical tension of the only fundamental extended objects in perturbative string theory, the strings, have no dependence on the string coupling, while their solitonic partner have a physical tension given by $T_5 g_{\mathrm{s}}^{-2}$. For large values of the string coupling constant g_{s}, these solitons are the lightest states and they dominate the dynamics of the theory. We will soon see that the intermediate p-branes with a physical tension $T_p g_{\mathrm{s}}^{-1}$ admit a precise description in terms of open strings.

BPS bound

A p-brane solution as described above represents a natural generalization of the BPS states (0-branes) we were talking about earlier. The supersymmetry algebra now contains a rank-p

[23] See the discussion at the beginning of the next subsection.

tensor which plays the role of a central charge and gives a lower bound on the mass per unit of p-volume M_p (or tension T_p) of the solution [96, 218]:

$$M_p \geq c_p|q_{\rm p}| \qquad (\text{or } \; T_p \geq c_p|q_{\rm p}|\,), \tag{2.45}$$

where c_p is some constant whose value is dictated by the underlying supergravity and the value of p. The bound (2.45) is saturated by the (extreme) p-brane solutions that preserves half the supersymmetry. These solutions belong to reduced supersymmetry multiplets and, as in the 0-brane case, are believed to have features that remain exact in the (unknown) quantum theory even at strong coupling (in particular the saturated relation between their tension and their charge is thought to be preserved).

2.3.3 D-p-branes or the sources for the Ramond-Ramond fields

In Table 2.1, we have briefly mentioned that the type I and type II superstring theories contain two distinct sectors of massless space-time bosons constructed from a tensor product between a left and a right propagating mode: the Neveu–Schwarz/Neveu–Schwarz (NS $\otimes$ NS) and Ramond/Ramond (R $\otimes$ R) states, the latter being products of fermions. The NS $\otimes$ NS and R $\otimes$ R gauge fields have then a quite different origin, and the examination of the vertex operators in the two sectors shows that the fundamental strings may carry a charge corresponding to the space-time gauge symmetry NS$\otimes$NS but are always neutral under the R$\otimes$R symmetry. In other words, no R $\otimes$ R sources can apparently be displayed in perturbative string theory. For some time, the non-perturbative states carrying R $\otimes$ R charges have been described as extreme p-branes of the type discussed above with a black hole geometry in the transverse directions. These extended versions of black holes were called black p-branes [118, 122, 228]. In 1995, Polchinski observed in the context of the type II superstring theories that one can find a better description of the solutions carrying R $\otimes$ R charges as soon as one adds an open string sector to these closed superstring theories [169].

To understand the gist of Polchinski's analysis, it is sufficient to consider a bosonic open string sector. In the conformal gauge[24], the bosonic part of the Polyakov action for a free fundamental open string in flat space-time is

$$S_{\rm P} = -\frac{T_1}{2}\int d\tau d\sigma\, \eta^{\hat m\hat n}(\partial_{\hat m}X_M)(\partial_{\hat n}X^M). \tag{2.46}$$

The variation of $S_{\rm P}$ under a general transformation $X^M \to X^M + \delta X^M$ reads

$$\delta S_{\rm P} = T_1\int d\tau d\sigma\, \delta X_M\, \partial_{\hat m}\partial^{\hat m}X^M - T_1\int d\tau\, (\delta X^M\partial_\sigma X_M)|_{\sigma=0}^{\sigma=\pi}\,, \tag{2.47}$$

and one sees that $\delta S_{\rm P}$ will vanish only if the X^M verify the free wave equation in two dimensions $\partial_{\hat m}\partial^{\hat m}X^M = 0$. Moreover, one has to impose either Neumann or Dirichlet conditions to the endpoints of the open string in order to cancel the boundary contribution.

[24]The conformal gauge corresponds to a particular choice of the intrinsic world-sheet metric $h_{\hat m\hat n}$:

$$(h_{\hat m\hat n}) = (\eta_{\hat m\hat n}) = (\eta^{\hat m\hat n}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Neumann conditions $\partial_\sigma X^M|_{\sigma=0,\pi} = 0$ are the most commonly used conditions because they respect Poincaré invariance and hence in particular conservation of momentum. The Dirichlet conditions $\delta X^M|_{\sigma=0,\pi} = 0$, on the other hand, break the invariance under translations and are usually associated with defects of space-time.[25] In fact, one has the freedom to choose different boundary conditions for different directions. For instance, one can consider the quantization of an open string whose endpoints satisfy Dirichlet boundary conditions for $(d-p-1)$ space-time coordinates:

$$\begin{aligned} M &= 1,\ldots,p+1 \;:\quad \left.\frac{\partial X^M}{\partial\sigma}\right|_{\sigma=0,\pi} = 0 && \text{(Neumann conditions)} \\ M &= p+2,\ldots,d \;:\quad X^M|_{\sigma=0,\pi} = Y^M_{\sigma=0,\pi} \ (= \text{constant}) && \text{(Dirichlet conditions).} \end{aligned} \tag{2.48}$$

The endpoints of the string can move freely in a $(p+1)$-dimensional hyperplane, and they are fixed at positions $Y^M_{\sigma=0,\pi}$ in the other coordinates. The extended object on which the endpoints can live is called a Dirichlet p-brane (also named D-p-brane or even D-brane): a D-0-brane is a particle, a D-1-brane is a string, a D-2-brane is a membrane, and so on.

Such hyperplanes are present in type I and type II superstring theories,[26] and Polchinski showed that these extended objects become dynamic at strong coupling and form a complete set of sources for electric and magnetic $\mathrm{R}\otimes\mathrm{R}$ charges (a D-p-brane is a source for a $(p+1)$-form $\mathrm{R}\otimes\mathrm{R}$ gauge potential). The effective action for an "electric" D-p-brane coupling to a $p+1$ gauge potential A_{p+1} in a type II superstring theory is given, in the string frame, by

$$S_{\text{D-}p\text{-brane}} = -T_p \int_{\mathrm{wv}} d^{p+1}y\, e^{-\varphi} \sqrt{-\det(\hat{g}_{\hat{m}\hat{n}} + \hat{B}_{\hat{m}\hat{n}} + 2\pi\alpha' F_{\hat{m}\hat{n}})} - q_p \int_{\mathrm{wv}} \hat{A}_{p+1}, \tag{2.49}$$

where $\hat{g}_{\hat{m}\hat{n}}$, $\hat{B}_{\hat{m}\hat{n}}$ and $\hat{A}_{p+1}$ are the pull-backs of the ten-dimensional metric and antisymmetric $\mathrm{NS}\otimes\mathrm{NS}$ and $\mathrm{R}\otimes\mathrm{R}$ tensors. In the perturbative regime, the D-p-brane dynamics can essentially be understood in terms of the open strings ending on it, and the massless fields living on the D-p-brane correspond to the original massless spectrum of an open string split in a $U(1)$ gauge field $A_{\hat{m}}$ with a field strength $F_{\hat{m}\hat{n}}$ and $10-p-1$ scalars describing the fluctuations of the extended object in space-time. Due to its open string origin, the action (2.49) has a particular dilaton dependence which implies that the physical tension of the brane, denoted by τ_p, is proportional to the inverse string coupling constant:

$$\tau_p \overset{\text{def.}}{=} T_p e^{-\langle\varphi\rangle} \sim \frac{1}{g_{\mathrm{s}}}. \tag{2.50}$$

Thus, a D-p-brane appears to be rigid and extremely massive at weak coupling. It is a non-perturbative object which, as already pointed out in the preceding subsection, can be considered as intermediate between a solitonic p-brane and a fundamental (perturbative) string. Moreover, as can be seen from the computation of the interaction energy between two parallel identical D-p-branes [12], the coupling constant (charge density) q_p equals the tension T_p. The net force resulting from gravitons, dilatons and $\mathrm{R}\otimes\mathrm{R}$ tensor field exchanges

[25] Notice that Dirichlet conditions can also be written $\partial_\tau X^M|_{\sigma=0,\pi} = 0$. Such conditions were first discussed in a perturbative context in type I open string toroidal compactifications [52].

[26] A theory in which open strings and heterotic strings interact has not been constructed yet.

vanishes[27] and the equality $T_p = q_p$ can be interpreted as the saturation of a BPS bound. Actually, in the presence of the D-p-brane half the supersymmetry of the original type II superstring theory is broken.

The description of these BPS states in terms of perturbative open strings ending on them remains relatively simple and has found many applications, especially in the study of the nature of space-time at the shortest distance scales. In particular, there were many mysteries related to the fact, that to a large extent, four-dimensional black holes behave like thermodynamic objects. For example, they admit an entropy following the usual laws of thermodynamics and emit particles in a similar way as black bodies. Solitonic black holes have been identified with certain configurations of D-branes, and this quantum-mechanical description has allowed the counting of the fundamental states (something which was impossible so far) [210].

2.3.4 Direct dimensional reduction and wrapping of p-branes

To end this section, we discuss two procedures that are useful when comparing the non-perturbative content of supergravity theories in various dimensions.

Imagine we have a p-brane solution of the classical equations of a supergravity theory in d dimensions. It is possible to construct new p'-branes solutions in $(d-q)$ dimensions following two different procedures called *direct dimensional reduction* and *wrapping* (*double dimensional reduction*):

- Direct dimensional reduction gives a p-brane in $(d-q)$ dimensions. This procedure involves a dimensional reduction in q space-like directions belonging to the transverse space of the p-brane. It rests on the BPS property which allows to form an infinite array of parallel identical p-branes in d dimensions (to restore translational symmetry in the compact directions), so that a periodic identification gives a single p-brane in $(d-q)$ dimensions. The tension T_p of the p-brane remains unchanged.

- Wrapping gives a $(p-q)$-brane in $(d-q)$ dimensions. This procedure involves a dimensional reduction in q space-like directions belonging to the world-volume of the p-brane. The tension of the reduced $(p-q)$-brane is $T_{p-q} = V_q T_p$, where V_q is the volume of the compact q-dimensional space.

It is also possible to consider direct dimensional reduction and wrapping simultaneously. We then obtain a $(p-r)$-brane in $(d-r-s)$ dimensions, where r and s are respectively the number of wrapped and directly reduced directions. The reduction does not preserve any supersymmetry except if the compact space on which the brane is wrapped is a torus or a supersymmetric cycle of a Calabi–Yau manifold. There are two types of supersymmetric cycles, the most common corresponding to complex (holomorphic) submanifolds of the Calabi–Yau manifold [19, 29, 230, 161].

[27] The forces related to gravitons and dilatons exchanges are attractive, while the one originating from the antisymmetric tensor field is repulsive. This kind of net force cancellation is a general feature of BPS states, and it allows to consider intricate configurations of BPS branes.

2.4 Supergravity theories and their brane content

There are five ten-dimensional anomaly-free supergravity, but only four of them can be interpreted as effective field theories for the five ten-dimensional superstring theories, that is to say as approximations of the string theories in the infinite tension limit (or equivalently in the limit $\alpha' \to 0$) when the only relevant fields are the massless ones:

$d = 10$:

1)	Type IIA	$\longrightarrow$	Type IIA (non-chiral) $N = 2$ supergravity
2)	Type IIB	$\longrightarrow$	Type IIB (chiral) $N = 2$ supergravity
3)	Heterotic $E_8 \times E_8$	$\longrightarrow$	$N = 1$ supergravity – Yang–Mills with $E_8 \times E_8$ as gauge group
4)	Heterotic $SO(32)$ and type I	$\longrightarrow$	$N = 1$ supergravity – Yang–Mills with $SO(32)$ as gauge group
5)	?	$\longrightarrow$	$N = 1$ supergravity – Yang–Mills with $U(1)^{496}$ as gauge group.

In eleven dimensions, there is a unique supergravity which is believed to be relevant as the effective (low-energy) field theory for the enigmatic M-theory:

$$d = 11 \quad \text{M-theory} \quad \longrightarrow \quad N = 1 \text{ eleven-dimensional supergravity}$$

In this section, we are going to briefly discuss these supergravity theories and enumerate their BPS brane solutions. According to the discussion in subsection 2.3.1, the only consistent BPS p-branes are the ones for which a $(p+1)$-form or a $(d-p-3)$-form (dual form) gauge potential appears in the effective Lagrangian. Since almost all solutions in supergravity theories with $d < 11$ can be related to the BPS p-branes of $d = 11$ supergravity, we will first address the eleven-dimensional case.

2.4.1 Eleven-dimensional supergravity

Eleven-dimensional supergravity is a classical field theory suggested [149] and built [51] more than twenty years ago. For a long time, it remained a kind of enigma because the consistent superstring theories were only able to provide a "justification" for the existence of supergravity theories in ten (or less) space-time dimensions. In the present understanding, the eleven-dimensional $N = 1$ supergravity is conjectured to be the effective low-energy field theory for M-theory. It has three massless fields:[28]

- the graviton g_{MN} (a traceless symmetric tensor) with $\frac{1}{2} \cdot 9 \cdot 10 - 1 = 44$ physical degrees of freedom;
- the potential C_{MNO} (a completely antisymmetric gauge tensor) with $9!/(3!6!) = 84$ physical degrees of freedom;

[28]Details about the physical polarizations (components) of massless fields are compiled in Appendix B.

- the gravitino ψ_M (a Majorana vector spinor field)[29] with $9 \cdot 16 - 16 = 128$ physical degrees of freedom.

As required by supersymmetry, the number of bosonic physical degrees of freedom just equals the number of fermionic physical degrees of freedom.

The eleven-dimensional supergravity action can be written [51][30]

$$\begin{aligned} S_{\mathrm{CJS}} &= -\frac{1}{2\kappa_{11}^2}\int d^{11}x\, e_{11} R - \frac{1}{96\kappa_{11}^2}\int d^{11}x\, e_{11} G_{MNPQ}G^{MNPQ} \\ &\quad -\frac{1}{12\kappa_{11}^2}\frac{1}{3!4!4!}\int d^{11}x\, \epsilon^{M_1\dots M_{11}} C_{M_1M_2M_3}G_{M_4M_5M_6M_7}G_{M_8M_9M_{10}M_{11}} \\ &\quad + S_{11\mathrm{dferm.}} \\ &= -\frac{1}{2\kappa_{11}^2}\int d^{11}x\, e_{11}R - \frac{1}{4\kappa_{11}^2}\int G_4 \wedge *G_4 - \frac{1}{12\kappa_{11}^2}\int C_3 \wedge G_4 \wedge G_4 \\ &\quad + S_{11\mathrm{dferm.}}\,, \end{aligned} \tag{2.51}$$

where κ_{11} is the eleven-dimensional gravitational coupling, $e_{11} = \sqrt{-\det g_{MN}}$, R is the scalar curvature and $G_4 = dC_3$. The Chern–Simons (topological) interaction term,

$$S_{\mathrm{top}} = -\frac{1}{12\kappa_{11}^2}\int C_3 \wedge G_4 \wedge G_4, \tag{2.52}$$

will be important when discussing anomaly cancellation in Chapter 4. The symbol $S_{11\mathrm{dferm.}}$ denotes the fermionic piece of the action which includes the so-called Rarita–Schwinger term $\overline{\psi}_M\Gamma^{MNO}D_N\psi_O$ (usual kinetic term for the gravitino), as well as $\overline{\psi}_M\Gamma^{MNOPQR}\psi_R G_{NOPQ}$ and $\overline{\psi}^N\Gamma^{OP}\psi^Q G_{NOPQ}$ couplings [51]. In the following, we will not need the precise form of these fermionic terms. They can always be deduced from the requirement of supersymmetry. A similar remark holds for the ten-dimensional supergravity actions considered below. In particular, we will implicitly assume that the possible symmetries of the bosonic sector can be extrapolated to the full supergravity theory.

One expects to find two branes related to the potential C_{MNO}:

- an *electric 2-brane* (membrane) [25, 26, 81],
- a *magnetic 5-brane* (magnetic dual of the 2-brane) [112].

[29] We omit here the spinor index.

[30] The relation between the normalization used by Cremmer, Julia and Scherk [51] and the one we have chosen is

$$\kappa_{11}^{\mathrm{CJS}} = \frac{1}{\sqrt{2}}\kappa_{11}, \quad C_{MNP}^{\mathrm{CJS}} = \frac{1}{\sqrt{2}\kappa_{11}}C_{MNP} \quad \text{and} \quad G_{MNPQ}^{\mathrm{CJS}} = \frac{1}{\sqrt{2}\kappa_{11}}G_{MNPQ}.$$

Similarly, the link with the fields used by Hořava and Witten [117] is provided by

$$C_{MNP}^{\mathrm{HW}} = \frac{1}{6\sqrt{2}}C_{MNP} \quad \text{and} \quad G_{MNPQ}^{\mathrm{HW}} = \frac{1}{\sqrt{2}}G_{MNPQ}.$$

Each of these extended solutions preserves half the supersymmetry. In the literature, these branes are collectively called *M-branes*. Intersections of M-branes leads to many other BPS solutions preserving part of the supersymmetry [163]. Note also that open membranes can end on M-5-branes. In a sense, the M-5-branes are then analogous to D-branes.

The ideas collected in subsection 2.3.2 can be applied to find the explicit solution corresponding to the magnetic 5-brane. First, we note that there is no dilaton φ (or coupling parameter) in eleven-dimensional supergravity.[31] Imposing the preservation of half the supersymmetry, it is possible to determine the precise form of the metric and magnetic ansätze (2.40) and (2.42). The result is:

$$\begin{aligned} ds^2 &= (1+kr^{-3})^{-1/3}\eta_{\hat{m}\hat{n}}dy^{\hat{m}}dy^{\hat{n}} + (1+kr^{-3})^{2/3}\delta_{ab}dz^a dz^b, \\ G_{a_1a_2a_3a_4} &= 3k\epsilon_{a_1a_2a_3a_4b}\frac{z^b}{r^5} \quad \text{(with all other components set to zero)}, \end{aligned} \tag{2.53}$$

where k is an integration constant which, in presence of an explicit 5-brane source, can be expressed in terms of the tension T_5. The geometry of this black p-brane solution interpolates between a flat space at $r \to \infty$ and an $(\text{AdS})_7 \times S^4$ region at $r \to 0$ (horizon) [207]. It has however no singularity.

Finally, since the single length scale in eleven-dimensional supergravity is $l_{11} \overset{\text{def.}}{=} (\kappa_{11}^2)^{1/9}$, the M-brane tensions are of the form

$$\begin{aligned} T_2^{\text{M}} &\sim l_{11}^{-3} = \kappa_{11}^{-2/3}, \\ T_5^{\text{M}} &\sim l_{11}^{-6} = \kappa_{11}^{-4/3}. \end{aligned} \tag{2.54}$$

2.4.2 Ten-dimensional type IIA supergravity

The type IIA supergravity in ten dimensions is the effective low-energy field theory appropriate for the type IIA superstring. This model contains two Majorana–Weyl spinors and two Majorana–Weyl gravitinos (the pairs being of opposite chirality), so that the total number of massless fermionic states is 128 (the same number as in eleven-dimensional supergravity). In addition, the type IIA supergravity has five massless bosonic fields representing 128 physical degrees of freedom. Three are coming from the "universal $N = 1$ ten-dimensional supergravity sector"

$$\varphi, B_{AB}, g_{AB} \quad (\text{NS} \otimes \text{NS sector}),$$

and the two remaining are the gauge potentials

$$A_A \text{ and } C_{ABC} \quad (\text{R} \otimes \text{R sector}).$$

[31] This absence of dilaton and of any moduli fields that would justify a perturbation theory makes difficult the construction of a membrane theory. There are other problems related to the quantization of the membrane world-volume theory which have been partially tackled in the context of matrix theory (for an introduction, see Ref. [30]).

The bosonic field equations can be deduced from the following action (written in the string frame):

$$
\begin{aligned}
S_{\mathrm{IIA}} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\, e_{10} e^{-2\varphi}\Big(- R + 4(\partial_A\varphi)(\partial^A\varphi) - \frac{1}{12}F_{ABC}F^{ABC}\Big) \\
&\quad -\frac{1}{4\kappa_{10}^2}\int d^{10}x\, e_{10}\Big(\frac{1}{2}F_{AB}F^{AB} + \frac{1}{24}\underline{F}_{ABCD}\underline{F}^{ABCD}\Big) \\
&\quad -\frac{1}{4\kappa_{10}^2}\int B_2\wedge F_4\wedge F_4,
\end{aligned}
\tag{2.55}
$$

where κ_{10} is the ten-dimensional gravitational coupling, $e_{10} = \sqrt{-\det g_{AB}}$, R is the ten-dimensional scalar curvature, and the field strengths are given by $F_2 = dA_1$, $F_3 = dB_2$, $F_4 = dC_3$ and $\underline{F}_4 = dC_3 - A_1\wedge F_3$. The type IIA supergravity has a global non-compact $U(1)$ symmetry [185, 39].

The type IIA action (2.55) in ten dimensions can be seen as coming from the Kaluza–Klein reduction of eleven-dimensional supergravity (2.51) on a circle S^1. The metric $g^{(11)}_{MN}$ in eleven dimensions gives rise to the metric g_{AB} in ten dimensions, to a vector potential A_A and to a dilaton φ. Explicitly, one defines

$$
\begin{aligned}
ds^2 &= g^{(11)}_{MN}dx^M dx^N \\
&= e^{-\frac{2}{3}\varphi}g_{AB}dx^A dx^B + e^{\frac{4}{3}\varphi}\left(dx^{11} + A_A dx^A\right)^2,
\end{aligned}
\tag{2.56}
$$

where the fields φ, g_{AB} and A_A depend only on the non-compact coordinates x^A (this independence in the compact coordinate x^{11} is the usual assumption in a dimensionally reduced theory). The metric g_{AB} of type IIA supergravity is not equal to the corresponding part $g^{(11)}_{AB}$ in the eleven-dimensional metric. Note also that the radius r_{11} of the compact dimension, measured with the string metric g_{AB} and expressed in terms of the string coupling constant $g_{\mathrm{IIA}} = e^{\langle\varphi\rangle}$, reads

$$
r_{11} = \sqrt{\alpha'}g_{\mathrm{IIA}}, \tag{2.57}
$$

where we took the liberty to introduce the typical string length $\sqrt{\alpha'} = (\kappa_{10}^2)^{1/8}$. It is important to stress that the effective value $\kappa_{10}^{\mathrm{eff}}$ of the gravitational constant is modified by the vev of the dilaton φ. According to the action (2.55), $\kappa_{10}^{\mathrm{eff}}$ is of the form $\kappa_{10}^{\mathrm{eff}} = \alpha'^2 g_{\mathrm{IIA}}$. This constant can also be seen as coming from its eleven-dimensional counterpart κ_{11}, and this gives the relation between the units in eleven and ten dimensions. Assuming that $(\kappa_{10}^{\mathrm{eff}})^2 = \kappa_{11}^2/(2\pi r_{11})$ where $2\pi r_{11}$ is the "volume" (length) of the compact space and using (2.57), one obtains

$$
l_{11}^9 \stackrel{\text{def.}}{=} \kappa_{11}^2 = 2\pi\alpha'^{9/2}g_{\mathrm{IIA}}^3. \tag{2.58}
$$

On the other hand, the rank-3 tensor C_{MNO} in eleven dimensions becomes a rank-3 tensor C_{ABC} and a rank-2 tensor B_{AB} in ten dimensions:

$$
\begin{aligned}
C_3 &= \frac{1}{6}C_{MNO}\, dx^M\wedge dx^N\wedge dx^O \\
&= \frac{1}{6}C_{ABC}\, dx^A\wedge dx^B\wedge dx^C + \frac{1}{2}B_{AB}\, dx^A\wedge dx^B\wedge dx^{11}.
\end{aligned}
\tag{2.59}
$$

The fields C_{ABC} and B_{AB} are assumed to be independent of the compact coordinate x^{11}.

The type IIA supergravity theory has seven kinds of p-branes [79]:

– a *fundamental string*,

– a *solitonic 5-brane* [37, 38],

– five *D-p-branes* with *even* values of p ($p = 0, 2, 4, 6$ and 8),

where the p-branes with $p = 0, 6$ are sources for the 1-form A_1, the ones with $p = 1, 5$ are related to the 2-form B_2, whereas the solutions with $p = 2, 4$ are connected with the 3-form C_3.[32] The D-8-brane (a domain wall solution)[33] is only relevant in the *massive* type IIA supergravity constructed by Romans [181]. This generalization of the type IIA supergravity contains a cosmological constant and cannot directly be obtained by reducing the CJS eleven-dimensional supergravity [18]. The D-8-brane is supposed to couple to the dual of the cosmological constant [22]. With the exception of this last Dirichlet brane, all the IIA p-branes have a simple interpretation in terms of the M-2- and M-5-branes:

- The IIA D-2-brane, respectively the IIA solitonic 5-brane, corresponds to the direct dimensional reduction of the M-2-brane, respectively the M-5-brane, on S^1.
- The IIA 1-brane (fundamental string), respectively the IIA D-4-brane, corresponds to the wrapping of the M-2-brane, respectively the M-5-brane, on S^1.
- The D-0-brane and the D-6-brane couple to the Kaluza–Klein gauge field A_1 and it seems natural to call them Kaluza–Klein p-branes.

In the light of subsection 2.3.4, it is possible, with the knowledge of the dependence of the M-brane tensions on the string coupling constant $g_{\rm IIA}$ [see Eqs. (2.54) and (2.58)] and of the expression (2.57) for the compact radius, to deduce the general behaviour of four IIA brane physical tensions:[34]

$$\begin{aligned} \mathcal{T}_1^{\rm IIA} &\sim \alpha'^{-1}, \\ \mathcal{T}_5^{\rm IIA} &\sim \frac{1}{g_{\rm IIA}^2}\alpha'^{-3}, \\ \mathcal{T}_{\text{D-}p}^{\rm IIA} &\sim \frac{1}{g_{\rm IIA}}\alpha'^{-(p+1)/2}, \quad p = 2, 4. \end{aligned} \tag{2.60}$$

The physical tensions associated to the remaining D-p-branes with $p = 0, 6, 8$ have also expressions proportional to $g_{\rm IIA}^{-1}$. We will come back to the case of the D-0-brane in the next chapter.

[32] The 0- and 2-branes are electric p-branes, and the 6- and 4-branes are their magnetic duals.

[33] The D-8-brane is of spatial codimension one, and, by definition, a domain wall solution separates the space-time into two regions.

[34] The precise numerical factors can be computed starting from the usual fundamental string tension $T_1^{\rm IIA} = (2\pi\alpha')^{-1}$ and using the conjectured dualities among the type II superstring theories.

2.4.3 Ten-dimensional type IIB supergravity

The type IIB supergravity in ten dimensions is the effective low-energy field theory appropriate for the type IIB superstring. This model contains two Majorana–Weyl spinors and two Majorana–Weyl gravitinos (the pairs being of the same chirality), so that the total number of massless fermionic states is 128. In addition, the type IIB supergravity has six massless bosonic fields representing 128 physical degrees of freedom. Three are coming from the "universal $N = 1$ ten-dimensional supergravity sector"

$$\varphi^{(1)}, B^{(1)}_{AB}, g_{AB} \quad (\mathrm{NS} \otimes \mathrm{NS}\ \text{sector}),$$

and the three remaining are the gauge fields

$$\varphi^{(2)}, B^{(2)}_{AB},\ \text{and}\ C^{+}_{ABCD} \quad (\mathrm{R} \otimes \mathrm{R}\ \text{sector}).$$

The 4-form C^{+}_{4} is what is called a *chiral p-form* because it has a self-dual field strength: $F^{+}_{5} = dC^{+}_{4} = *dC^{+}_{4}$. This self-duality property complicates the construction of a complete Lorentz invariant action for the type IIB supergravity [188, 28, 166], and the procedure commonly followed [20] is to temporarily ignore the self-duality condition and consider the bosonic action

$$\begin{aligned} S_{\mathrm{IIB}} \;=\; & \frac{1}{2\kappa_{10}^{2}} \int d^{10}x\, e_{10} e^{-2\varphi^{(1)}} \Big(-R + 4(\partial_{A}\varphi^{(1)})(\partial^{A}\varphi^{(1)}) - \frac{1}{12} F^{(1)}_{ABC} F^{(1)ABC} \Big) \\ & - \frac{1}{4\kappa_{10}^{2}} \int d^{10}x\, e_{10} \Big(F_{A}F^{A} + \frac{1}{6} \underline{F}_{ABC} \underline{F}^{ABC} + \frac{1}{240} \underline{F}_{A_1 \ldots A_5} \underline{F}^{A_1 \ldots A_5} \Big) \\ & - \frac{1}{4\kappa_{10}^{2}} \int C^{+}_{4} \wedge F^{(1)}_{3} \wedge F^{(2)}_{3}, \end{aligned} \tag{2.61}$$

where κ_{10} is the ten-dimensional gravitational coupling, $e_{10} = \sqrt{-\det g_{AB}}$, R is the ten-dimensional scalar curvature, and the field strengths are given by $F_{1} = d\varphi^{(2)}$, $F^{(1)}_{3} = dB^{(1)}_{2}$, $F^{(2)}_{3} = dB^{(2)}_{2}$, $\underline{F}_{3} = F^{(2)}_{3} - \varphi^{(2)} F^{(1)}_{3}$ and $\underline{F}_{5} = F^{+}_{5} - \frac{1}{2} B^{(2)}_{2} \wedge F^{(1)}_{3} + \frac{1}{2} B^{(1)}_{2} \wedge F^{(2)}_{3}$. Remarking the identity $d\underline{F}_{5} = F^{(1)}_{3} \wedge F^{(2)}_{3}$, the equation of motion for the field C^{+}_{4} leads to $d{*}\underline{F}_{5} - d\underline{F}_{5} = 0$. This expression is compatible with the self-duality condition $*\underline{F}_{5} = \underline{F}_{5}$ which can then be added as a constraint to the field equations following from the variation of the action S_{IIB}. We will meet another chiral p-form (a 2-form) in Chapter 6 when discussing the six-dimensional world-volume field theory which describes the dynamics of an M-5-brane. In this case, there is a way to write down a covariant action at the cost of the introduction of an auxiliary scalar field.

The type IIB supergravity theory has eight kinds of p-branes:

- a *fundamental string*,
- a *solitonic 5-brane* [37, 38],
- six *D-p-branes* with odd values of p ($p = -1, 1, 3, 5, 7$ and 9).

The 0-form $\varphi^{(2)}$ is connected to a D-(-1)-brane and to its magnetic dual (a D-7-brane), each of them being quite special [95]. In particular, a D-(-1)-brane has a point-like "world-volume" with Dirichlet conditions on all coordinates. In other words, it is localized in space and time and, after a Wick rotation, it can be interpreted as a kind of instanton. A D-(-1)-brane is then sometimes called *D-instanton*. The presence of the 4-form C_4^+ gives rise to a self-dual D-3-brane that has identical (up to a sign) electric and magnetic charges since $\int_{S^5} *F_5^+ = \int_{S^5} F_5^+$ [78]. Thus, only one type of source is related to the 4-form C_4^+. The two 2-forms $B_2^{(1)}$ and $B_2^{(2)}$ have each the possibility to couple to an electric 1-brane or to a magnetic 5-brane.[35] It is here important to mention that the type IIB supergravity has a global $SU(1,1)$ and a local $U(1)$ symmetries [196]. The symmetry $SU(1,1)$, which is a non-compact form of $SU(2)$, is isomorphic to $SL(2,\mathbb{R})$, and its subgroup $SL(2,\mathbb{Z})$ has led to a duality conjecture exchanging the NS $\otimes$ NS and R $\otimes$ R 2-forms [189]. Finally, the D-9-brane has no associated R $\otimes$ R potential. This degenerate case represents the ten-dimensional Minkowski space-time (the open strings can end anywhere).[36]

2.4.4 Ten-dimensional supergravity coupled to super Yang–Mills $E_8 \times E_8$ or $SO(32)$

The massless sector of the three consistent $N = 1$ superstring theories in ten dimensions admits $N = 1$ supergravity coupled to super Yang–Mills matter $E_8 \times E_8$ or $SO(32)$ as effective low-energy theory. The bosonic fields of the $N = 1$ supergravity – Yang–Mills theory are the usual 64 physical degrees of freedom corresponding to the dilaton φ, the graviton g_{AB} and the rank-2 antisymmetric tensor B_{AB}, supplemented by the 496 gauge bosons A_A^a of the groups $E_8 \times E_8$ or $SO(32)$. Their supersymmetric partners are a Majorana–Weyl spinor, a Majorana–Weyl gravitino and 496 gauginos (Majorana–Weyl spinors).

The heterotic supergravity bosonic action has the form

$$\begin{aligned} S_{\text{Het.}} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\, e_{10} e^{-2\varphi}\Big(-R + 4(\partial_A\varphi)(\partial^A\varphi) - \frac{1}{12}\underline{H}_{ABC}\underline{H}^{ABC}\Big) \\ & -\frac{1}{4\lambda^2}\int d^{10}x\, e_{10} e^{-2\varphi}\mathrm{tr}(F_{AB}F^{AB}), \end{aligned} \tag{2.62}$$

where λ is the gauge coupling constant, $F_2 = dA_1 + A_1^2$ is the Yang–Mills field strength with an associated gauge group $E_8 \times E_8$ or $SO(32)$, and the 3-form $\underline{H}_3$ is defined by[37]

$$\underline{H}_3 = dB_2 - \frac{\kappa_{10}^2}{\lambda^2}(\Omega_{3,\text{YM}} - \Omega_{3,\text{L}}) \tag{2.63}$$

[35] For instance, the type IIB fundamental string is electrically charged with respect to the 2-form potential $B_2^{(1)}$ from the NS $\otimes$ NS sector.

[36] Note that the presence of a D-9-brane (or of any other D-p-brane with *p odd*) in the type IIA superstring theory is forbidden by consistency of the boundary conditions satisfied by fermions.

[37] The gravitational contribution $\Omega_{3,\text{L}}$ involves more than one derivative and is usually omitted (assuming one restricts oneself to Lagrangians leading to classical equations of motion with at most second order derivatives). It is however essential when discussing anomaly cancellations.

with the Chern–Simons 3-forms

$$\begin{aligned}\Omega_{3,\mathrm{YM}} &= \mathrm{tr}(A_1 \wedge dA_1 + \frac{2}{3}A_1^3), \\ \Omega_{3,\mathrm{L}} &= \mathrm{tr}(\Omega_1 \wedge d\Omega_1 + \frac{2}{3}\Omega_1^3)\end{aligned} \tag{2.64}$$

(Ω_1 is the spin connection, the gauge field of the Lorentz group).

On the other hand, the bosonic sector of the type I supergravity action reads

$$\begin{aligned}S_{\mathrm{I}} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\, e_{10} e^{-2\varphi}\Big(- R + 4(\partial_A\varphi)(\partial^A\varphi) - \frac{1}{12}e^{2\varphi}\underline{H}_{ABC}\underline{H}^{ABC}\Big) \\ &-\frac{1}{4\lambda^2}\int d^{10}x\, e_{10} e^{-\varphi}\mathrm{tr}(F_{AB}F^{AB}),\end{aligned} \tag{2.65}$$

where the definitions given above in the heterotic case still apply, the 2-form B_2 coming now from the $\mathrm{R}\otimes\mathrm{R}$ sector. The only admissible gauge group is $SO(32)$.

The heterotic supergravity has an electric *1-brane* (which is nothing else than the fundamental heterotic string) and its magnetic dual, a *solitonic 5-brane*, which are sources for the 2-form potential B_2 [209].

The type I supergravity has a $\mathrm{R}\otimes\mathrm{R}$ rank-2 antisymmetric tensor that can consistently couple to two non-perturbative BPS states: a *D-1-brane* and a *D-5-brane* [99].[38] This model contains also a *D-9-brane* representing the usual open strings moving in the ten-dimensional space-time.

2.5 Summary

During our brief survey of the non-perturbative states in string theory, we have encountered three distinct sorts of extended BPS solutions of the ten-dimensional supergravity theories:

- *the fundamental p-branes*

  A fundamental p-brane is necessarily a string ($p = 1$) and has a physical tension with no dependence on the string coupling constant g_{s}. This object will then survive in the limit $g_{\mathrm{s}} \to 0$, and such branes are the fundamental degrees of freedom in a perturbative framework.

- *the solitonic p-branes*

  A solitonic p-brane is in some sense the analogue of the 't Hooft–Polyakov monopole in quantum field theory. This magnetic dual of the fundamental string exists only for $p = 5$ and has a physical tension proportional to $1/g_{\mathrm{s}}^2$.

[38]Note that the fundamental type I open string does not couple to the $\mathrm{R}\otimes\mathrm{R}$ potential. An infinitely extended string of this kind is not stable and therefore cannot be a BPS state.

- *the Dirichlet-p-branes*

  A Dirichlet-p-brane has no known direct counterpart in quantum field theory. It can be considered as an intermediate solution between a fundamental brane and a solitonic excitation. In particular, its physical tension is proportional to $1/g_s$.

The fundamental strings and the solitonic 5-branes carry the electric and magnetic charges of the 2-form potential B_2 coming from the NS $\otimes$ NS sector. Such solutions are present for all ten-dimensional supergravity theories except for the type I supergravity. The Dirichlet-p-branes on the other hand are the sources for the R $\otimes$ R fields. They are present in all ten-dimensional supergravity theories except the heterotic ones.

The single supergravity in eleven dimensions admits an electric 2-brane and a magnetic 5-brane as sources for the only gauge potential present in the spectrum.

All these basic p-branes are listed in Table 2.2. We have already made a few comments on the close relation between the M- and type IIA-branes, but the significance of the existence of the objects displayed in this table will really become apparent in the next chapter.

	φ $p=-1$	A_1 0	B_2 1	C_3 2	C_4^+ 3	4	5	6	7	8	9
11d sugra				e			m				
Type IIA		D	f	D		D	s	D		D	
Type IIB	D		f and D		D^+		s and D		D		D
Heterotic			f				s				
Type I			D				D				D

Table 2.2: Summary of the known basic extended solutions of the supergravity theories in eleven and ten dimensions. The symbols "f" and "s" denotes the fundamental and solitonic p-branes, while "D" is used for D-p-branes. The excitations "e" and "m" are the electric membrane and magnetic 5-brane in eleven-dimensional supergravity. For the "electric" p-branes, we have also indicated the associated gauge potentials.

Chapter 3

Duality in string theory

In this chapter, we briefly present the concept of duality in string theory. The term *duality* is used to define an equivalence between two different mathematical descriptions of the same physical phenomena. Two well-known examples of dualities in modern physics are the wave $\leftrightarrow$ particle duality and the electric $\leftrightarrow$ magnetic duality. In string theory, there are others *conjectured* dualities between...

...superstring theories with different compactification radii
(length $\leftrightarrow$ length^{-1} duality),

...superstring theories with different coupling constants
(strong $\leftrightarrow$ weak coupling duality),

...a superstring theory and a new theory in eleven dimensions,

...etc.

Most of the dualities which have been discovered so far in string theory fit roughly into three classes [192], called respectively

T-duality, *S-duality* and *U-duality.*

Each of these dualities represents an *exact conjectured quantum equivalence.* We discuss them in turn using as a tool the symmetries of the low-energy effective supergravity theories and the present knowledge of the non-perturbative string spectrum. Our intention is not to give a precise account of the web of all the known dualities in string theory, so that we restrict ourselves to the description of some general ideas. The last section contains a summary of the main string duality conjectures in various dimensions.

3.1 T-duality ("Target-space" duality)

Consistent theories in d dimensions ($d < 10$) can be obtained from the five ten-dimensional superstring theories by compactification on an internal $(10 - d)$-dimensional manifold. In order to understand T-duality, it is instructive to study in some detail a compactification on the simplest internal manifold, namely the circle S^1.

3.1.1 Compactification on a circle

We consider a theory of closed strings in ten dimensions. After the compactification on a circle $S^1 = \mathbb{R}/(2\pi R\,\mathbb{Z})$, there are nine space-time coordinates X^μ satisfying the closed string boundary conditions

$$X^\mu(\tau, \sigma + \pi) = X^\mu(\tau, \sigma), \quad \mu = 1, \ldots, 9, \tag{3.1}$$

while the internal space-like coordinate X^{10} as a periodicity such that

$$X^{10}(\tau, \sigma + \pi) = X^{10}(\tau, \sigma) + 2\pi R w, \quad w \in \mathbb{Z}, \tag{3.2}$$

where w is the *winding number* (the string wraps w times around the circle). The radius R of the circle can be seen as the vev of a massless scalar field v arising from the nine-dimensional decomposition of the ten-dimensional metric:

$$g_{AB} \stackrel{\text{on } S^1}{\longrightarrow} g_{\mu\nu}, g_{\mu 10}, g_{1010} = e^{2v}, \tag{3.3}$$

and $R = \sqrt{\alpha'} e^{\langle v \rangle}$. Following the example of the dilaton, the scalar field v is dynamical (it is a modulus) and, in absence of a specific potential, it can take any value.

The coordinate $X^{10}(\tau, \sigma) = X_R^{10}(\tau - \sigma) + X_L^{10}(\tau + \sigma)$ which is solution of the equation of motion (free wave equation in two dimensions) derived from the Polyakov action (1.2) with a flat metric and which verifies (3.2) reads

$$X^{10}(\tau, \sigma) = x^{10} + 2\alpha' p^{10} \tau + 2Rw\sigma + \cdots, \tag{3.4}$$

where the dots denote terms of the Fourier expansion which are proportional to $e^{-2in(\tau \pm \sigma)}$ with $n \in \mathbb{Z}$, $n \neq 0$. Quantum mechanically, the position (at $\tau = 0$) x^{10} and momentum p^{10} of the string center of mass are interpreted as operators satisfying the usual canonical commutation relation $[x^{10}, p^{10}] = i$ and p^{10} generates the translations of x^{10}. The identification (3.2) requires then that the quantum wave function $e^{ip^{10}x^{10}}$ is invariant under $x^{10} \to x^{10} + 2\pi R$, so that the internal momentum is restricted to discrete values

$$p^{10} = \frac{k}{R}, \quad \text{with } k \in \mathbb{Z}. \tag{3.5}$$

The mass spectrum (as seen by an observer in nine dimensions) is given by

$$M^2 = \underbrace{\frac{k^2}{R^2}}_{\substack{\text{Kaluza–Klein} \\ \text{excitations}}} + \underbrace{\frac{w^2 R^2}{\alpha'^2}}_{\substack{\text{winding} \\ \text{excitations}}} + \frac{2}{\alpha'}(N_R + N_L - 2), \tag{3.6}$$

where $\alpha' = 1/(2\pi T)$ and N_R (N_L) is the number operator of the right (left) moving (internal and external) oscillator excitations. We notice that this spectrum is symmetric under the transformation

$$R \to \frac{\alpha'}{R} \quad \text{and} \quad w \leftrightarrow k \tag{3.7}$$

which exchanges the Kaluza–Klein and winding excitations. This invariance of the mass spectrum is called *T-duality.* It can be extended to the string interactions if one assumes that [150, 102]

$$X_R^{10} \to -X_R^{10} \quad \text{and} \quad X_L^{10} \to X_L^{10}. \tag{3.8}$$

World-sheet supersymmetry implies then an analog correspondence for the fermionic partner:

$$\psi_R^{10} \to -\psi_R^{10} \quad \text{and} \quad \psi_L^{10} \to \psi_L^{10}. \tag{3.9}$$

The fact that a compactification on a circle of radius R is closely related to a compactification on a circle of radius α'/R suggests that there is a minimal length in string theory and the radius $R = \sqrt{\alpha'}$ (self-dual point) seems to be the smallest physically observable radius.[1] It is also interesting to note that for the particular radius $R = \sqrt{\alpha'}$ the gauge symmetry of a bosonic closed string theory is enlarged from $U(1) \times U(1)$ (which corresponds to the two Abelian Kaluza–Klein gauge bosons $g_{\mu 10}$ and $B_{\mu 10}$) to the non-Abelian gauge group $SU(2) \times SU(2)$ (because there are then four additional massless gauge bosons).

Naively the spectrum of a theory of open strings cannot exhibit the invariance described above (because of the absence of winding excitations). However, once non-perturbative extended objects (D-branes) are taken into account, a T-duality becomes also manifest in an open string theory [52, 2, 170, 12].

3.1.2 A rough definition of T-duality

Two theories A and B are said *T-dual* if theory A compactified on a space whose volume is large is physically equivalent to theory B compactified on a space whose volume is small (and vice versa). In particular, this means that a scalar field v whose exponential determines the volume of the compactified dimensions satisfies $v_A = -v_B$.

In most cases, T-duality can be considered as a perturbative equivalence. It relates then the weak coupling regime of a theory to the weak coupling regime of another theory and can be verified at each order in the string loop expansion. To ascertain its validity for the full (non-perturbative) theories, one has to find its action on the non-perturbative spectra of the two theories [101].

3.1.3 Examples

For some time, it has been known [52, 63] that the type IIA and type IIB superstring theories, following the example of the $E_8 \times E_8$ and $SO(32)$ heterotic superstring theories [100, 152, 153], are connected by T-dualities below ten dimensions.

- The $SO(32)$ heterotic superstring theory compactified on a circle of radius R with gauge group broken to $SO(16) \times SO(16)$ using Wilson lines[2] is equivalent to the $E_8 \times E_8$

[1]Remark that using non-perturbative objects like D-branes as a probe, it seems possible to sensibly discuss distance scales shorter than the string scale [203].

[2]Wilson lines are constant backgrounds for the gauge field A^a_{10}. Locally, this field is pure gauge, but it has non-trivial holonomy around non-trivial paths (non-contractible loops) in space-time. A comprehensive discussion of symmetry breaking by Wilson lines can be found in Ref. [108].

heterotic superstring theory compactified on a circle of radius α'/R with Wilson lines breaking the gauge group to $SO(16) \times SO(16)$. Thanks to the introduction of Wilson lines a continuous interpolation between the two compactified theories is possible [153]. In the absence of any Wilson lines, each theory has a $R \to \alpha'/R$ self-duality symmetry.

- A compactification of the type IIA superstring theory on a circle of radius R gives the same theory as the type IIB superstring theory compactified on a circle of radius α'/R.

  One can give some pieces of evidence for such a T-duality:

  (i). The massless bosonic spectra of the type IIA and type IIB superstring theories (as coming from the two distinct theories in ten dimensions) look rather equivalent in nine dimensions:[3]

$$\text{Type IIA} \left\{ \begin{array}{lcll} g_{AB}, B_{AB}, \varphi & \overset{\text{on } S^1}{\longrightarrow} & g_{\mu\nu}, g_{\mu 10}, g_{1010}, B_{\mu\nu}, B_{\mu 10}, \varphi & (\text{NS} \otimes \text{NS}) \\ A_A, C_{ABC} & & A_\mu, A_{10}, C_{\mu\nu\rho}, C_{\mu\nu 10} & (\text{R} \otimes \text{R}) \end{array} \right.$$

$$\text{Type IIB} \left\{ \begin{array}{lcll} g_{AB}, B^{(1)}_{AB}, \varphi^{(1)} & \overset{\text{on } S^1}{\longrightarrow} & g_{\mu\nu}, g_{\mu 10}, g_{1010}, B^{(1)}_{\mu\nu}, B^{(1)}_{\mu 10}, \varphi^{(1)} & (\text{NS} \otimes \text{NS}) \\ C^{+}_{ABCD}, B^{(2)}_{AB}, \varphi^{(2)} & & C_{\mu\nu\rho 10}, B^{(2)}_{\mu\nu}, B^{(2)}_{\mu 10}, \varphi^{(2)} & (\text{R} \otimes \text{R}) \end{array} \right.$$

  (ii). The transformation (3.9) corresponds to a switch of chirality for the right-moving fermions. Moreover, a Weyl (chirality) condition is not allowed when the dimension of space-time is odd.

  (iii). In ten dimensions, there are two $N = 2$ supergravity theories [121]: the non-chiral type IIA and the chiral type IIB. In nine dimensions, it turns out that there is only one $N = 2$ supergravity, and the type II superstring low-energy effective theories have been shown to agree [23].

Thus, the two heterotic (type II) theories seem to be located at different points of the moduli space of a single heterotic (type II) superstring theory.

Note also that it has been conjectured [211] that T-duality is related (via fiberwise application of $R \to \alpha'/R$ transformations) to some observed equivalences (mirror symmetries) between superstring theories compactified on different Calabi–Yau spaces. Such "geometric aspects of mirror symmetry" are described in a recent review by D. R. Morrison [146].

3.2 S-duality and evidence for a new dimension

After this discussion about T-duality, one is still faced with three apparently inequivalent superstring theories: heterotic, type II and type I. Evidence for the existence of another kind of duality, christened S-duality, will allow us to make new interesting connections between these formulations.

[3]Notice that because of the self-duality condition on the field strength of the type IIB rank-4 tensor, $C_{\mu\nu\rho\sigma}$ is not a nine-dimensional independent field.

3.2.1 A rough definition of S-duality

S-duality usually refers to an equivalence which involves the dilaton φ, that is, the string coupling constant g_s. In contrast with T-duality, S-duality does not concern moduli related to the geometry of compact spaces (like radii). Loosely speaking, two theories A and B which are perturbatively distinct are said *S-dual* if the behaviour of the theory A at strong coupling is given by the weakly coupled theory B (and vice versa). The dilatons of the two theories are then related by the equality $\varphi_A = -\varphi_B$, and the perturbative excitations of the theory A are mapped to the non-perturbative excitations of the theory B (and vice versa).

Originally, S-duality was defined as a symmetry between two regimes of the same model generalizing the electric-magnetic duality in supersymmetric four-dimensional gauge theories. It goes beyond string perturbation theory, and it is therefore presently impossible to prove its existence doing explicit (perturbative) calculations. Fortunately, the presence of BPS states in the string spectrum sometimes allows to accord a certain credibility to such a non-perturbative equivalence.

3.2.2 Two examples in ten dimensions

Two significant examples of S-duality in ten dimensions are the self-duality of the type IIB superstring theory [122, 189] and the equivalence between the type I and the $SO(32)$ heterotic superstring theories [228, 173, 6].

- The equations of motion of the type IIB ten-dimensional supergravity have an explicit $SL(2,\mathbb{R})$ symmetry. The conjecture is that the subgroup $SL(2,\mathbb{Z})$ represents an exact symmetry of the superstring theory. A justification for the breaking of the $SL(2,\mathbb{R})$ symmetry to its subgroup $SL(2,\mathbb{Z})$ holds in the fact that $SL(2)$ acts on the pair of 2-forms $B_2^{(1)}$ and $B_2^{(2)}$, as well as on the corresponding charges which are known to be quantized (only $SL(2,\mathbb{Z})$ transformations are then allowed). The $SL(2,\mathbb{Z})$ symmetry would relate the strong and weak coupling regimes of the type IIB theory.

- There is a strong-weak coupling duality between the type I and the $SO(32)$ heterotic superstring theories.

  One can give some pieces of evidence for such an S-duality:

  (i). The equivalence of the massless bosonic spectra (see Table 2.1).

  (ii). As can be seen from subsection 2.4.4, the two theories admit an analogous description in term of an effective low-energy field theory [$N = 1$ ten-dimensional pure supergravity coupled to $N = 1$ ten-dimensional super Yang–Mills $SO(32)$] provided one assumes the identifications

$$\begin{aligned} g^{\rm H}_{AB} &= e^{-\varphi_{\rm I}} g^{\rm I}_{AB}, & \varphi_{\rm H} &= -\varphi_{\rm I}, \\ \underline{H}^{\rm H}_3 &= \underline{H}^{\rm I}_3, & A^{\rm H}_1 &= A^{\rm I}_1. \end{aligned} \tag{3.10}$$

  (iii). The mapping of the branes:

$$\text{Type I} \left\{ \begin{array}{lcl} \text{D-1-brane} & \longleftrightarrow & \text{fund. string [173]} \\ \text{D-5-brane} & \longleftrightarrow & \text{solit. 5-brane} \end{array} \right\} \text{Heterotic } SO(32)\,.$$

3.2.3 Connection with a mysterious eleven-dimensional theory

So far we have met S-dual partners for the type I, $SO(32)$ heterotic and type IIB superstring theories. For the two remaining superstring theories, it seems difficult to find an appropriate ten-dimensional candidate. Surprisingly, it appears that the type IIA [217, 228, 191] and $E_8 \times E_8$ heterotic [116, 230, 117] superstring theories have an eleventh (space-like) dimension which becomes perceptible at strong coupling. More precisely, if one compares the low-energy effective action of the ten-dimensional type IIA superstring to the eleven-dimensional supergravity reduced on a circle S^1 (this reduction was outlined in subsection 2.4.2), one obtains a perfect agreement if the following relation between the radius r_{11} of the circle measured with the string metric and the type IIA string coupling constant is satisfied:

$$r_{11} = \sqrt{\alpha'} g_{\rm IIA}. \tag{3.11}$$

If r_{11} is large, the type IIA superstring is strongly coupled and its behavior is governed by eleven-dimensional supergravity on S^1 which is then supposed to be an effective field theory of a mysterious eleven-dimensional consistent quantum theory baptized *M-theory.* A relation analog to (3.11) holds for the $E_8 \times E_8$ heterotic string when its effective ten-dimensional action is compared to eleven-dimensional supergravity reduced on the orbifold (the line interval) $S^1/\mathbb{Z}_2$.[4]

We have gathered below some pieces of evidence for the existence of this extra dimension which is completely hidden in string perturbation theory.

Evidence for M-theory from the point of view of the type IIA superstring theory

(i). For a long time, $N = 1$ eleven-dimensional supergravity compactified on a circle is known to give the non-chiral $N = 2$ ten-dimensional supergravity.

(ii). The type IIA superstring theory contains an infinity of Dirichlet-particles (D-0-branes) with a physical mass

$$M \sim \frac{K}{\sqrt{\alpha'} g_{\rm IIA}}, \tag{3.12}$$

where K is an integer ($K \in \mathbb{N}$). The mass formula (3.12) is supposed to be an exact relation which remains valid when $g_{\rm IIA} \to \infty$ because the D-0-branes are BPS states. It is interesting to note that the structure of (3.12) is similar to the formula for the momentum states in a Kaluza–Klein compactification ($p^{11} = k/r_{11}$, $k \in \mathbb{Z}$). Thus, $g_{\rm IIA} \to \infty$ seems to be similar to $r_{11} \to \infty$ and the type IIA theory looks like an eleven-dimensional theory compactified on a circle S^1 whose radius r_{11}, when expressed in the type IIA metric, goes as $g_{\rm IIA}$.

(iii). Actually, it is possible to find an eleven-dimensional origin to the p-branes of the

[4] Here, one implicitly assumes that it makes sense to compactify M-theory on an orbifold. This conjectured eleven-dimensional quantum theory probably shares such a property with superstring theories.

non-chiral $N = 2$ ten-dimensional supergravity:

$$\text{Type IIA} \left\{ \begin{array}{lcl} \text{D-2-brane} & \stackrel{D}{\longleftarrow} & \text{M-2-brane} \\ \text{solit. 5-brane} & \stackrel{D}{\longleftarrow} & \text{M-5-brane} \\ & & \\ \text{fund. string} & \stackrel{W}{\longleftarrow} & \text{M-2-brane [75]} \\ \text{D-4-brane} & \stackrel{W}{\longleftarrow} & \text{M-5-brane [74]} \\ & & \\ \text{D-0-brane} & \longleftrightarrow & \text{KK gauge field [217, 228]} \\ & & \\ \text{D-6-brane} & \longleftrightarrow & \begin{array}{l}\text{KK monopole of the} \\ \text{compactification [217, 228]}\end{array} \end{array} \right\} \text{M-theory on } S^1$$

Here "D" means "direct dimensional reduction", while "W" symbolizes "wrapping" (double dimensional reduction). The type IIA D-8-brane present in the type IIA superstring spectrum has been omitted because its fate in the type IIA strong coupling limit is quite speculative. It was first discussed by Polchinski and Witten [173, 170], and more recent papers (see for instance Refs. [27, 183] and references therein) suggest the possible existence of an M-9-brane in the context of a massive version of eleven-dimensional supergravity with only partial Lorentz invariance [24].

Evidence for M-theory from the point of view of the $E_8 \times E_8$ heterotic superstring theory

(i). The discrete symmetry group $\mathbb{Z}_2$ acts on the compact coordinate $x^{11} \in [-\pi r_{11}, \pi r_{11}] \cong S^1$ by $x^{11} \to -x^{11}$. In other words, the points x^{11} and $-x^{11}$ are identified, so that there are two fixed points at values $x^{11} = 0$ and $x^{11} = \pi r_{11}$. Requiring the eleven-dimensional Lagrangian (2.51) to be invariant under $\mathbb{Z}_2$, one can clarify the situation for the two eleven-dimensional massless bosonic supergravity fields compactified on S^1:

$$g_{MN}, C_{MNO} \quad \stackrel{\text{on } S^1}{\longrightarrow} \quad \begin{array}{ll} g_{AB}, C_{AB11}, g_{1111} & (\text{even under } \mathbb{Z}_2) \\ g_{A11}, C_{ABC} & (\text{odd under } \mathbb{Z}_2) \end{array}$$

The states g_{A11} and C_{ABC}, which are odd under $\mathbb{Z}_2$, do not survive on the orbifold. On the fermionic side, the eleven-dimensional $N = 1$ supergravity compactified on S^1 gives a $N = 2$ non-chiral theory, but dividing by $\mathbb{Z}_2$ kills half the supersymmetries and leads to a ten-dimensional chiral supergravity with $N = 1$. The $\mathbb{Z}_2$ invariant spectrum in ten dimensions is then identical to the massless spectrum of a heterotic or type I superstring up to the gauge degrees of freedom. Now, one expects the existence of extra massless states arising at the two fixed points of the orbifold $S^1/\mathbb{Z}_2$ (the presence of such "twisted sectors" is a common fact in string theory where they are necessary in particular for modular invariance). As we will observe in the following chapter, anomaly cancellation by a generalized Green–Schwarz mechanism requires the introduction of 496 vector supermultiplets (each containing one gauge field and one gaugino) in the twisted sectors and the anomaly must be equally supported by the two fixed points (ten-dimensional hyperplanes). Thus, there are 248 vector supermultiplets propagating on each hyperplane, and 248 is the dimension of E_8. Since the radius of compactification is proportional to the string coupling constant $g_{E_8\times E_8}$ and gives (up to a factor π) the distance between the two fixed points, the two ten-dimensional hyperplanes meet in the limit $g_{E_8\times E_8} \to 0$ and the weakly coupled $E_8 \times E_8$ heterotic formulation is recovered.

(ii). The mapping of the branes:

$$\text{Heterotic } E_8 \times E_8 \left\{ \begin{array}{lcl} \text{fund. string} & \stackrel{W}{\longleftarrow} & \text{M-2-brane} \\ \text{solit. 5-brane} & \stackrel{D}{\longleftarrow} & \text{M-5-brane} \end{array} \right\} \text{M-theory on } S^1/\mathbb{Z}_2.$$

The direct dimensional reduction of the M-2-brane and the double dimensional reduction (wrapping) of the M-5-brane are eliminated by the $\mathbb{Z}_2$ projection (this can be seen as a consequence of the removal of the rank-3 tensor C_{ABC}).

3.3 U-duality

Two theories A and B are said *U-dual* if the theory A compactified on a space whose volume is large (or small) is equivalent to the theory B strongly (or weakly) coupled. In this case, one has $v_A = \pm\varphi_B$. More generally, a U-duality mixes the couplings (the dilatons) and the other dynamical fields [160].

The concept of U-duality appears for instance in the context of the type II superstring theory below ten dimensions.[5] In fact, the type II superstring theory is known to have a T-duality symmetry which can be embedded, together with the $SL(2,\mathbb{Z})$ symmetry already present in ten dimensions (S-duality of the type IIB superstring theory), in a larger symmetry group (the U-duality group) [122, 228].

3.4 Connection with a conjectured twelve-dimensional theory (F-theory)

As mentioned earlier in subsection 3.2.2, the ten-dimensional type IIB superstring theory is self-dual. Its $SL(2,\mathbb{Z})$ symmetry acts in particular on the parameter $\tau = \varphi^{(2)} + ie^{-\varphi^{(1)}}$ built from the two scalars $\varphi^{(1)}$ and $\varphi^{(2)}$ present in the massless spectrum. Vafa proposed a geometric interpretation of this symmetry in terms of a hypothetical (10+2)-dimensional quantum theory (baptized *F-theory*) compactified on a torus (τ being identified to the complex parameter of the torus) [221]. Compactifications to dimensions lower than 10 must then involve manifolds (elliptic fibrations) on which the parameter τ varies (whereas on "conventional perturbative" compact manifolds it is usually taken to be constant) and lead to interesting new models. F-theory is certainly not as fundamental as M-theory, but in some cases Vafa's geometric approach has proven useful and instructive [147, 148, 231].

[5] Recall that the type IIA and type IIB superstring theories are perturbatively equivalent below ten dimensions.

3.5 Summary

We end this chapter by listing some of the main duality conjectures. Figure 3.1 focuses on theories near the critical dimension ten. Evidence for other interesting dualities in lower dimensions can also be obtained, and features of M-theory compactifications have been derived using equivalences involving the type IIA or heterotic $E_8 \times E_8$ superstring theories. For example, a way to get information about M-theory on some internal space $\mathcal{K}$ is, loosely speaking, to compactify it further on a circle of radius R and take advantage of the identification with the type IIA theory compactified on $\mathcal{K}$. In the limit $R \to \infty$, one then recovers M-theory on $\mathcal{K}$ [228].

$$d=10 \begin{cases} \text{Type IIB} & \longleftrightarrow \quad \text{F-theory on } T^2 \text{ [221]}, \\ \text{Type IIA} & \longleftrightarrow \quad \text{M-theory on } S^1 \text{ [217, 228, 191]}, \\ \text{Heterotic } E_8 \times E_8 & \longleftrightarrow \quad \text{M-theory on } S^1/\mathbb{Z}_2 \text{ [116, 230, 117]}, \\ \text{Type IIB} & \longleftrightarrow \quad \text{Type IIB [122, 189]}, \\ \text{Heterotic } SO(32) & \longleftrightarrow \quad \text{Type I [228, 173]}, \end{cases}$$

where the orbifold $S^1/\mathbb{Z}_2$ can be regarded as a line segment I.

$$d=9 \begin{cases} \text{Type IIB on } S^1 & \longleftrightarrow \quad \text{M-theory on } T^2 \text{ [190, 191, 192]}, \\ \text{Type I on } S^1 \text{ or Heterotic } SO(32) \text{ on } S^1 & \longleftrightarrow \quad \text{M-theory on } \mathcal{C} \text{ [192]}, \\ \text{Heterotic } SO(32) \text{ on } S^1 & \longleftrightarrow \quad \text{Heterotic } E_8 \times E_8 \text{ on } S^1 \text{ [100, 152, 153]}, \\ \text{Type IIA on } S^1 & \longleftrightarrow \quad \text{Type IIB on } S^1 \text{ [52, 63]}, \end{cases}$$

where $\mathcal{C} = I \times S^1 = S^1/\mathbb{Z}_2 \times S^1$ is a cylinder.

$$d=7 \quad \text{Heterotic on } T^3 \longleftrightarrow \text{M-theory on K3 [228]}.$$

$$d=6 \begin{cases} \text{Heterotic on } T^4 & \longleftrightarrow \quad \text{Type IIA on K3 [122, 228]}, \\ \text{Type IIB on K3} & \longleftrightarrow \quad \text{M-theory on } T^5/\mathbb{Z}_2 \text{ [53, 229]}. \end{cases}$$

$$d=5 \quad \text{Heterotic on } T^4 \times S^1 \longleftrightarrow \text{Type IIB on K3} \times S^{1'} \text{ [228]}.$$

$$d=4 \begin{cases} \text{Heterotic on } T^6 & \longleftrightarrow \quad \text{Heterotic on } T^6 \ (SL(2,\mathbb{Z}) \text{ duality}) \text{ [91, 179, 194, 195, 200]}, \\ \text{Heterotic on } T^6 & \longleftrightarrow \quad \text{Type IIA on K3} \times T^2 \text{ [88]}. \end{cases}$$

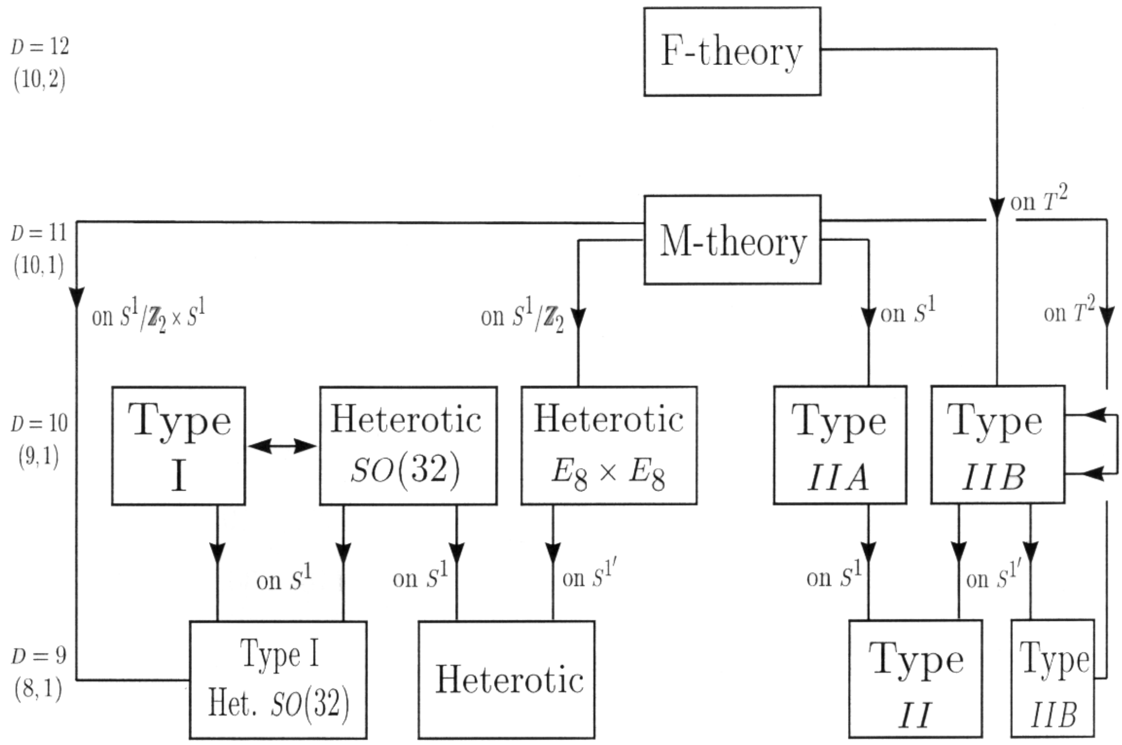

Figure 3.1: Illustrative view of some duality conjectures with a particular emphasis on the ten-dimensional superstring theories.

The manifold K3 is the only topologically distinct Calabi–Yau manifold of complex dimension 2. Its basic properties were unraveled by the German mathematician (and physicist) Ernst Eduard Kummer, and the name "K3", which refers to Kummer, Kähler and Kodaira, was proposed soon after the first climb of the K2 in July 1954. A compactification on the manifold K3 breaks half the supersymmetries of the underlying theory. The strong-weak coupling duality between the heterotic superstring compactified on T^4 and the type IIA superstring compactified on K3 is sometimes called *string-string duality* [122]. It is possible to deduce the last duality above from this conjecture by further compactification on T^2 [88, 202]. A rather detailed account of compactifications on K3 in the framework of string duality can be found in Ref. [11].

Part II

A few examples of M-theory compactifications

Chapter 4

M-theory on the orbifold $S^1/\mathbb{Z}_2$

Considerations of anomaly cancellation have played an essential role in the understanding of the connections between M-theory and the perturbative superstring theories. In particular, an important ingredient in constructing the strongly-coupled $E_8 \times E_8$ heterotic superstring from M-theory compactified on $S^1/\mathbb{Z}_2$ [116, 117] was the observation that the gauge and gravitational anomaly polynomial of the $E_8 \times E_8$ heterotic superstring can be written as a sum of two terms, each being associated with one E_8 factor, and that each of these terms separately factorises as required by anomaly cancellation through a Green–Schwarz mechanism. This enables local anomaly cancellation on each of the two ten-dimensional $S^1/\mathbb{Z}_2$ fixed planes in the M-theory picture [116, 117, 54].

The anomaly-cancellation mechanism in M-theory has essentially two known origins. One is a *Green–Schwarz term* $\int G_4 \wedge X_7$, where G_4 is the field strength of the three-form C_3 and X_7 is a purely gravitational Chern–Simons type seven-form. The presence of this anomaly-cancelling term is well-known from string duality [222, 77] and from M-five-brane anomaly cancellations [77, 229, 233]. The other possible root is *anomaly inflow* from the topological interaction term $\int C_3 \wedge G_4 \wedge G_4$ of eleven-dimensional supergravity. In uncompactified M-theory in eleven dimensions, both terms are gauge and local Lorentz invariant and no anomaly needs to be cancelled (this is not surprising in this odd dimension). However, a compactification on the orbifold $S^1/\mathbb{Z}_2$ involves chiral projections on the even-dimensional fixed planes and there are then chiral anomalies which need to be cancelled. It has been shown in Ref. [117] that closure of the supersymmetry algebra for the $S^1/\mathbb{Z}_2$ orbifold implies a modification of the Bianchi identity $dG_4 = 0$, and this is why the terms $\int G_4 \wedge X_7$ and $\int C_3 \wedge G_4 \wedge G_4$ have non-vanishing anomalous transformations under gauge and local $SO(9,1)$ Lorentz transformations on the fixed planes.

In Ref. [117], the modified Bianchi identity $dG_4 \neq 0$ was solved in a particular way and it was concluded that anomaly cancellation requires a certain fixed ratio λ^3/κ_{11}^2 between the gauge coupling λ and the gravitational coupling κ_{11}. Subsequent analyses [71, 49, 135, 84, 85, 113, 86] have emphasized the fact that there is actually a one-parameter family of solutions $G_4(b)$, $b \in \mathbb{R}$, to the modified Bianchi identity.[1] It was stated that anomaly cancellation

[1] Of course, to any solution $G_4 = dC_3 + \cdots$ of the Bianchi identity $dG_4 = \cdots$ one can add a four-form dA_3 with A_3 any three-form. The point is that most of these A_3 can be reabsorbed into C_3. The only relevant A_3 must be made from the gauge and Lorentz Chern–Simons three-form Ω_3 on the fixed planes, so

alone does not fix the ratio λ^3/κ_{11}^2 but relates it to a cubic polynomial in the parameter b. It was argued that b and hence λ^3/κ_{11}^2 can be fixed if one also takes into account the quantization of the flux of G_4 [232].

In this chapter, we carefully reanalyse the issues of solving the modified Bianchi identity, of cancelling the anomaly and of the flux quantization for M-theory on the orbifold $S^1/\mathbb{Z}_2$. We adopt the "upstairs approach" in which one works on the boundary-free circle S^1 and imposes a $\mathbb{Z}_2$ projection on the fields. We insist on defining properly all fields on S^1 (all fields should be periodic). In particular, the modified Bianchi identity for G_4 involves a delta function $\delta(x^{11})$ and the periodicity immediately rules out the use of the "step function" $\theta(x^{11})$ as a primitive. Using instead a correctly periodically defined function $\varepsilon(x^{11})$ such that $d\varepsilon/dx^{11} = 2\delta(x^{11}) - 1/\pi$ turns out to be crucial to obtain a consistent solution to the modified Bianchi identity.

We investigate gauge and local Lorentz invariance of G_4 and insisting on invariance under large transformations, that is, insisting on having a globally well-defined field strength G_4, appears to be very powerful. It not only implies the well-known cohomology condition [230], but it also fixes the parameter b of the solution G_4 provided there are topologically non-trivial gauge or gravity configurations. The same value of b stands out in the presence of five-branes, and it is also the only one which allows a safe truncation to the perturbative heterotic superstring: although the zero-modes on the circle give the desired relation $\underline{H}_3 = dB_2 - \kappa_{10}^2\lambda^{-2}(\Omega_{3,\mathrm{YM}} - \Omega_{3,\mathrm{L}})$ for all values of b, the neglected higher modes are gauge and local Lorentz invariant only if b takes a specific value. It is worth noting that these arguments are not a consequence of G_4-flux quantization.

Looking at anomaly cancellations also yields some surprises. We do the analysis for an arbitrary parameter b. Again, the use of the periodic function $\varepsilon(x^{11})$ is crucial, and the constant term in $d\varepsilon/dx^{11} = 2\delta(x^{11}) - 1/\pi$ plays a major role. Due to this term, the relation between the parameter b and the ratio λ^3/κ_{11}^2 is drastically modified with respect to previous analyses [117, 71, 49, 135, 84, 85, 113, 86]: the terms cubic in b cancel, and one is left with a quadratic equation for b. This equation will have important consequences when considering more complicated orbifold compactifications [86], e.g. compactifications on $S^1/\mathbb{Z}_2 \times T^4/\mathbb{Z}_2$, where the analysis has encountered certain difficulties.[2] As a consistency check of our solution we show how the combination of the Green–Schwarz term and the topological term in eleven dimensions leads to the Green–Schwarz term of the heterotic superstring. We also describe how the discussion of anomalies is affected by the presence of five-branes. The Green–Schwarz term $\int G_4 \wedge X_7$ automatically ensures cancellation of the five-brane one-loop anomalies without any need of further modification, but we note that anomaly inflow from the topological term induces a non-vanishing anomaly on the five-brane world-volumes.

Another subtle point is the flux quantization of G_4. In the "downstairs approach", where one considers M-theory compactified on the interval $I = S^1/\mathbb{Z}_2$ (the eleven-dimensional space-time has then two ten-dimensional boundary planes), there is no modification of the Bianchi identity $dG_4 = 0$, but there is a non-trivial boundary condition on G_4 [117] enabling the necessary anomaly inflow to cancel the one-loop anomaly on the boundary planes. This

that the only freedom is essentially $A_3 \sim b\Omega_3$, with one real parameter b.

[2]We are grateful to D. Lüst for sharing his insights on this point.

boundary condition does not admit a free parameter like b (in some sense, it is equivalent to a fixed value of b). In this formalism, Witten concluded [232] that the flux of G_4 over any four-cycle has to be integer or half-integer. In particular, this ensures that the membrane functional integral is well-defined. Naively, the latter would seem to be well-defined only for integer flux, but it was shown in Ref. [232] that in the case of half-integer flux the three-dimensional membrane functional integral has a parity anomaly [3] which precisely cancels the sign ambiguity due to the half-integrality of the G_4-flux. In the boundary-free upstairs approach where the Bianchi identity for G_4 is modified, this same flux quantization should appear as a consequence of our solution. We show that this is indeed the case for four-cycles not wrapping the circle S^1. There is no analogue of the four-cycle wrapping the circle in the downstairs approach and we cannot draw any conclusions on quantization from the results of Ref. [232]. For such cycles, one is left with an integral which is closely related to the flux $\int \underline{H}_3 = \int (dB_2 - \kappa_{10}^2 \lambda^{-2}(\Omega_{3,\mathrm{YM}} - \Omega_{3,\mathrm{L}}))$ in the heterotic superstring, but we cannot say anything interesting about the flux of the four-form G_4.

This chapter has the following structure. In section 4.1, we discuss the orbifold $S^1/\mathbb{Z}_2$, the modification of the Bianchi identity for G_4, and some technical details on the periodic "step function" $\varepsilon(x^{11})$ on the circle. In section 4.2, we solve the modified Bianchi identity and study gauge and local Lorentz transformations to refine the solution. As a consistency check, we also consider the reduction to the effective supergravity Lagrangian of the ten-dimensional $E_8 \times E_8$ heterotic superstring. In section 4.3, we broach the subject of anomaly cancellations and obtain the relation between b^2 and λ^6/κ_{11}^4. The relevant anomaly polynomials are presented in Appendix C. Section 4.4 is devoted to the issue of G_4-flux quantization. In section 4.5, we spell out the heterotic anomaly-cancelling terms as inherited from the present M-theory approach. Our main results are summarized in section 4.6.

4.1 The orbifold $S^1/\mathbb{Z}_2$

4.1.1 The modified Bianchi identity for G_4

By definition, the orbifold $S^1/\mathbb{Z}_2$ is the quotient space of the circle S^1 by the discrete symmetry group $\mathbb{Z}_2$ which does not act freely on S^1 (there are two fixed points). In this thesis, we are working in the so-called "upstairs approach" on a smooth eleven-dimensional manifold $\mathcal{M}_{11}$ without boundary. $\mathcal{M}_{11}$ is just the product

$$\mathcal{M}_{11} = \mathcal{M}_{10} \times S^1, \tag{4.1}$$

where $\mathcal{M}_{10}$ is a ten-dimensional manifold and S^1 a circle. The $\mathbb{Z}_2$ symmetry is required afterwards. Its action on the compact coordinate[3]

$$x^{11} \in [-\pi r, \pi r] \cong S^1 \tag{4.2}$$

[3]For notational convenience, we will in the following most of the time consider a circle of unit radius. An arbitrary radius r will be reintroduced in subsection 4.2.5 and section 4.5 for the discussion of the perturbative heterotic limit (small-radius limit).

is simply $x^{11} \to -x^{11}$. In other words, the points x^{11} and $-x^{11}$ are identified, and there are then two fixed points at $x^{11} = 0 \stackrel{\text{def.}}{=} x_1^{11}$ and $x^{11} = \pi \stackrel{\text{def.}}{=} x_2^{11}$. From the point of view of the whole space-time $\mathcal{M}_{11}$, they actually correspond to two parallel ten-dimensional invariant hypersurfaces. The idea of Hořava and Witten [117] is to allow for an E_8 super Yang–Mills theory on each of these orbifold planes and to couple this model to eleven-dimensional supergravity on $\mathcal{M}_{11}$ submitted to the $\mathbb{Z}_2$ symmetry.

The minimal Yang–Mills action for the ten-dimensional E_8 gauge fields A_A^a and gauginos χ^a in the adjoint representation (index a) is[4]

$$S_{\text{10dgauge}} = -\frac{1}{\lambda^2} \int d^{10}x \, e_{10}^{(11)} \, \text{tr} \left(\frac{1}{4} F_{AB} F^{AB} + \frac{1}{2} \overline{\chi} \Gamma^A D_A \chi \right), \tag{4.3}$$

where λ is the gauge coupling constant, $e_{10}^{(11)} = \sqrt{-\det g_{AB}^{(11)}}$ ($g_{AB}^{(11)}$ is the restriction of the eleven-dimensional metric g_{MN} to the $S^1/\mathbb{Z}_2$ fixed planes[5]) and $F_{\mathit{2}} = dA_{\mathit{1}} + A_{\mathit{1}}^2$. The gamma matrices are 32×32 real matrices verifying the anticommutation relation $\{\Gamma_A, \Gamma_B\} = 2g_{AB}^{(11)} I_{32}$, and the symbol D_A denotes the Yang–Mills and Lorentz covariant derivative. The addition of extra terms describing interactions between eleven- and ten-dimensional fields has been done in Ref. [117].

In the eleven-dimensional supergravity action (2.51), one has to eliminate all the fields which are odd under $\mathbb{Z}_2$. Requiring the invariance of the topological action S_{top} implies that the three-form $C_{\mathit{3}} = \frac{1}{3!} C_{M_1 M_2 M_3} dx^{M_1} \wedge dx^{M_2} \wedge dx^{M_3}$ is odd under $\mathbb{Z}_2$. Since the component C_{ABC} is not associated to any dx^{11}, it does not survive, while the component C_{AB11} must be even and is therefore kept. The $\mathbb{Z}_2$ projection also breaks half of the thirty-two supersymmetries. The preservation of the sixteen remaining supersymmetries requires a modification of the Bianchi identity $dG_{\mathit{4}} = 0$. This modification involves the Yang–Mills curvatures and, as usual in string effective actions, anomaly cancellation in turn requires the presence of Lorentz curvatures [117][6]:

$$dG_{\mathit{4}} = -\frac{\kappa_{11}^2}{\lambda^2} \sum_{i=1}^{2} \delta_{\mathit{1},i} \wedge \left(\text{tr} F_{\mathit{2},i}^2 - \frac{1}{2} \text{tr} R_{\mathit{2}}^2 \right), \tag{4.4}$$

where we have used the one-forms $\delta_{\mathit{1},1}$ and $\delta_{\mathit{1},2}$ defined by[7]

$$\delta_{\mathit{1},1} = \delta(x^{11}) dx^{11} \quad \text{and} \quad \delta_{\mathit{1},2} = \delta(x^{11} - \pi) dx^{11}. \tag{4.5}$$

[4]χ is a Majorana–Weyl spinor describing eight physical modes. Since the vector A_A has $10-2$ transverse polarizations, there is an equal number of bosonic and fermionic modes.

[5]Explicitly, one defines

$$g_{MN} \stackrel{\text{def.}}{=} \begin{pmatrix} g_{AB}^{(11)} & 0 \\ 0 & g_{1111}^{(11)} \end{pmatrix}.$$

[6]A conventional supersymmetry calculation only yields the $\text{tr} F_{\mathit{2},i}^2$ term. The $\text{tr} R_{\mathit{2}}^2$ term is a higher-order effect (this contribution cannot be derived from supersymmetry of effective actions with up to two derivatives).

[7]Note that these one-forms are well-defined on the circle S^1 by definition:

$$\delta_{\mathit{1},2} = \delta(x^{11} - \pi) dx^{11} \equiv \delta(x^{11} + \pi) dx^{11}.$$

4.1.2 Construction of a periodic "step function" on the circle

To find an expression for G_4 that satisfies the modified Bianchi identity (4.4), we need zero-forms ε_i such that $d\varepsilon_i$ includes $\delta_{1,i}$. Determining a primitive of a delta function on a circle requires a minimum of care: a simple "step function" $\theta(x^{11})$ is not periodic and therefore not appropriated. In fact, the existence of a periodic function $f(x^{11}) = f(x^{11} + 2\pi)$ on the circle with $f'(x^{11}) = \delta(x^{11})$ [8] would imply[9]

$$1 = \int_{S^1} \delta_{1,1} = \int_{-\pi}^{\pi} dx^{11}\, \delta(x^{11}) = \int_{-\pi}^{\pi} dx^{11}\, f'(x^{11}) = f(\pi) - f(-\pi) = 0. \tag{4.6}$$

One can however define

$$\varepsilon_1(x^{11}) = \text{sign}(x^{11}) - \frac{x^{11}}{\pi}, \quad x^{11} \in [-\pi, \pi], \tag{4.7}$$

which is a periodic function (for instance, $\varepsilon_1(\pi) = \varepsilon_1(-\pi) = 0$) odd under $\mathbb{Z}_2$. The graph of this function is shown in Figure 4.1 together with the one of its counterpart

$$\varepsilon_2(x^{11}) = \varepsilon_1(x^{11} \pm \pi). \tag{4.8}$$

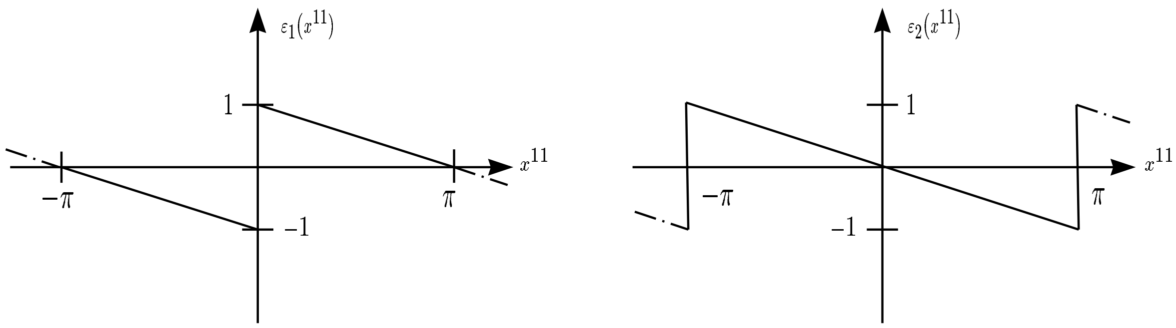

Figure 4.1: Graphs of the periodic "step functions" ε_1 and ε_2 on the circle.

These two functions ε_1 and ε_2 satisfy

$$\varepsilon_1'(x^{11}) = 2\delta(x^{11}) - \frac{1}{\pi} \quad \text{and} \quad \varepsilon_2'(x^{11}) = 2\delta(x^{11} - \pi) - \frac{1}{\pi}. \tag{4.9}$$

In terms of forms, these definitions can be concisely written as

$$d\varepsilon_i = 2\delta_{1,i} - \frac{1}{\pi} dx^{11}, \quad i = 1, 2. \tag{4.10}$$

[8] The prime $'$ denotes a derivative with respect to x^{11}.

[9] Since the integrand is periodic, one can integrate over any interval of length 2π. However, in order to have each orbifold singularity exactly once, it is preferable not to integrate from $-\pi$ to π but to slightly shift the interval of integration. Thus, for instance, $\int_{-\pi}^{\pi} dx^{11}$ is meant to be $\lim_{\eta\to 0} \int_{-\pi+\eta}^{\pi+\eta} dx^{11}$.

In the following sections, we will have to integrate combinations of the functions ε_i and δ over the circle. Since the functions ε_i are odd under $\mathbb{Z}_2$, one has

$$\int_{S^1} dx^{11}\,\varepsilon_i = 0, \quad i = 1, 2. \tag{4.11}$$

Integrals involving products of the functions ε_i read

$$\int_{S^1} dx^{11}\,\varepsilon_i\varepsilon_j = \pi\left(\delta_{ij} - \frac{1}{3}\right), \quad i, j = 1, 2, \tag{4.12}$$

where δ_{ij} denotes the Kronecker symbol. It is also useful to consider integrals of products of one delta function and two ε_i functions. These $\mathbb{Z}_2$-even distributions are ill-defined and have to be regularized. Any sensible regularization should preserve the relation (4.10) and the $\mathbb{Z}_2$-oddness of the ε_i. For an expression like $\delta_{1,1}\varepsilon_1\varepsilon_1$, one can think that in the vicinity of $x^{11} = 0$ the linear piece $-x^{11}/\pi$ in ε_1 is unimportant and $\varepsilon_1'(x^{11}) \simeq 2\delta(x^{11})$ so that

$$\begin{aligned} \delta(x^{11})\varepsilon_1(x^{11})\varepsilon_1(x^{11}) &\simeq \frac{1}{2}\varepsilon_1'(x^{11})\varepsilon_1(x^{11})\varepsilon_1(x^{11}) = \frac{1}{6}\frac{d}{dx^{11}}\left(\varepsilon_1(x^{11})\right)^3 \\ &\simeq \frac{1}{6}\frac{d}{dx^{11}}\varepsilon_1(x^{11}) \simeq \frac{1}{3}\delta(x^{11}). \end{aligned} \tag{4.13}$$

This computation can be rigorously verified. For instance, we may take the regularized $\varepsilon_1^\eta(x^{11})$, $\eta \in \mathbb{R} > 0$, to be the continuous function which coincides with $\varepsilon_1(x^{11})$ for $x^{11} \notin [-\eta, \eta]$ and equals $(1/\eta - 1/\pi)\,x^{11}$ for $x^{11} \in [-\eta, \eta]$. The corresponding $\delta_{1,1}^\eta(x^{11})$ is defined according to Eq. (4.10) and vanishes everywhere except inside the interval $[-\eta, \eta]$ where it equals $(2\eta)^{-1}$. It is then easy to compute the three integrals

$$\begin{aligned} \int_{S^1} \delta_{1,1}^\eta\varepsilon_1^\eta\varepsilon_1^\eta &= \frac{1}{3}\left(1 - \frac{\eta}{\pi}\right)^2 \overset{\eta\to 0}{\longrightarrow} \frac{1}{3}, \\ \int_{S^1} \delta_{1,1}^\eta\varepsilon_1^\eta\varepsilon_2^\eta &= -\frac{1}{3}\frac{\eta}{\pi}\left(1 - \frac{\eta}{\pi}\right) \overset{\eta\to 0}{\longrightarrow} 0, \\ \int_{S^1} \delta_{1,1}^\eta\varepsilon_2^\eta\varepsilon_2^\eta &= \frac{1}{3}\left(\frac{\eta}{\pi}\right)^2 \overset{\eta\to 0}{\longrightarrow} 0, \end{aligned} \tag{4.14}$$

and similarly for $\delta_{1,2}^\eta$. In other words, one has

$$\int_{S^1} \delta_{1,i}^\eta\varepsilon_j^\eta\varepsilon_k^\eta \overset{\eta\to 0}{\longrightarrow} \frac{1}{3}\delta_{ij}\delta_{ik}, \tag{4.15}$$

and more generally, for any smooth function $f(x^{11})$,

$$\int_{S^1} \delta_{1,i}^\eta\varepsilon_j^\eta\varepsilon_k^\eta f \overset{\eta\to 0}{\longrightarrow} \frac{1}{3}\delta_{ij}\delta_{ik}\int_{S^1}\delta_{1,i}f. \tag{4.16}$$

On the other hand, $\delta_{1,i}^\eta\varepsilon_j^\eta$ is an odd function of $x^{11} \in [-\pi, \pi]$ and

$$\int_{S^1} \delta_{1,i}^\eta\varepsilon_j^\eta f = 0. \tag{4.17}$$

To summarize, the important result is that we can safely make the replacements

$$\delta_{1,i}\varepsilon_j\varepsilon_k \to \frac{1}{3}\delta_{1,i}(\delta_{ij}\delta_{ik}) \quad \text{and} \quad \delta_{1,i}\varepsilon_j \to 0. \tag{4.18}$$

4.2 Solution of the modified Bianchi identity

4.2.1 A family of solutions

In this subsection, we would like to solve the modified Bianchi identity

$$dG_4 = -(4\pi)^2 \frac{\kappa_{11}^2}{\lambda^2} \sum_{i=1}^{2} \delta_{1,i} \wedge I_{4,i}, \tag{4.19}$$

where we have rewritten the content of the parenthesis appearing in (4.4) in terms of the four-form[10]

$$I_{4,i} = \frac{1}{(4\pi)^2} \left(\mathrm{tr} F_{2,i}^2 - \frac{1}{2} \mathrm{tr} R_2^2 \right). \tag{4.20}$$

The one-form $\delta_{1,i}$ has support only on the ith fixed plane, so that only the values of the smooth four-form $I_{4,i}$ on this fixed plane will contribute. Moreover, $\delta_{1,i}$ contains dx^{11} and only the components of $I_{4,i}$ not including dx^{11} are relevant. To emphasize this fact, we are going to use the following convention: a tilde on a p-form $A_{p,i}$ means that $\widetilde{A}_{p,i}$ is obtained from $A_{p,i}$ by dropping the dx^{11} components and by setting the argument equal to x_i^{11}, i.e. equal to 0 or π depending on whether $i = 1$ or 2. In this context, the exterior derivative d applied to a tilde p-form produces a tilde $(p+1)$-form: $d\widetilde{A}_{p,i} = \widetilde{A}_{p+1,i}$. In the Bianchi identity (4.19) we can then replace $I_{4,i}$ by $\widetilde{I}_{4,i}$. This has no effect on the gauge term $\mathrm{tr} F_{2,i}^2$ (because the gauge fields are living solely on the planes), but in the gravitational part R_2 is replaced by $\widetilde{R}_{2,i} \equiv \widetilde{R}_2|_{x^{11}=x_i^{11}}$. Explicitly, one has $\widetilde{R}_{2,i} = d\widetilde{\Omega}_{1,i} + \widetilde{\Omega}_{1,i} \wedge \widetilde{\Omega}_{1,i}$, where $\widetilde{\Omega}_{1,i}$ is the correspondingly mutilated spin connection[11]. With the help of the spin connections $\widetilde{\Omega}_{1,i}$, one defines the Chern–Simons three-forms

$$\widetilde{\omega}_{3,i} = \frac{1}{(4\pi)^2} \left(\mathrm{tr}(A_{1,i} \wedge dA_{1,i} + \frac{2}{3} A_{1,i}^3) - \frac{1}{2} \mathrm{tr}(\widetilde{\Omega}_{1,i} \wedge d\widetilde{\Omega}_{1,i} + \frac{2}{3} \widetilde{\Omega}_{1,i}^3) \right) \tag{4.21}$$

which satisfies $d\widetilde{\omega}_{3,i} = \widetilde{I}_{4,i}$. Under a gauge and local Lorentz transformation with parameters Λ^{g} and Λ^{L} independent of x^{11} one has $\delta\widetilde{\omega}_{3,i} = d\widetilde{\omega}_{2,i}^1$ where the two-forms $\widetilde{\omega}_{2,i}^1$ are

$$\widetilde{\omega}_{2,i}^1 = \frac{1}{(4\pi)^2} \left(\mathrm{tr}(\Lambda^{\mathrm{g}} dA_{1,i}) - \frac{1}{2} \mathrm{tr}(\Lambda^{\mathrm{L}} d\widetilde{\Omega}_{1,i}) \right). \tag{4.22}$$

Then the modified Bianchi identity (4.19) is rewritten as

$$dG_4 = \gamma \sum_{i=1}^{2} \delta_{1,i} \wedge \widetilde{I}_{4,i}, \tag{4.23}$$

[10] This combination of $\mathrm{tr} F_{2,i}^2$ and $\mathrm{tr} R_2^2$ appears when factorizing the twelve-form anomaly polynomial I_{12}, hence the denomination I_4 (see the Appendix C).

[11] By analogy with the Yang–Mills potential (or connection) A_1, the spin connection is the gauge field for local Lorentz transformations.

with $\widetilde{I}_{4,i} = d\widetilde{\omega}_{3,i}$ and $\delta\widetilde{\omega}_{3,i} = d\widetilde{\omega}^1_{2,i}$. For notational convenience, we have introduced the dimensionful quantity

$$\gamma = -(4\pi)^2 \frac{\kappa_{11}^2}{\lambda^2}. \tag{4.24}$$

The expression of the four-form solving equation (4.23) is

$$\begin{aligned} G_4 &= d\Big(C_3 + \frac{b}{2}\gamma \sum_{i=1}^{2} \varepsilon_i \widetilde{\omega}_{3,i}\Big) - \gamma \sum_{i=1}^{2} \delta_{1,i} \wedge \widetilde{\omega}_{3,i} \\ &= dC_3 + (b-1)\gamma \sum_{i=1}^{2} \delta_{1,i} \wedge \widetilde{\omega}_{3,i} + \frac{b}{2}\gamma \sum_{i=1}^{2} \varepsilon_i \widetilde{I}_{4,i} - \frac{b}{2\pi}\gamma dx^{11} \wedge \sum_{i=1}^{2} \widetilde{\omega}_{3,i}. \end{aligned} \tag{4.25}$$

To obtain the last equality, we have used the relation (4.10). In full generality we should allow for a different parameter b for each fixed plane:

$$G_4 = d\Big(C_3 + \frac{1}{2}\gamma \sum_{i=1}^{2} b_i \varepsilon_i \widetilde{\omega}_{3,i}\Big) - \gamma \sum_{i=1}^{2} \delta_{1,i} \wedge \widetilde{\omega}_{3,i}. \tag{4.26}$$

But the conditions of anomaly cancellation are the same for both planes and at the end of the day b_1 and b_2 are determined by the same equation so that $b_1 = b_2 \equiv b$.[12] The presence of the last term in the second equality (4.25) is due to the requirement of the S^1 periodicity of the $\mathbb{Z}_2$-odd "step functions" $\varepsilon_i(x^{11})$.

Five-branes contributions

It is also interesting to consider situations where M-five-branes are present in the theory. Each five-brane (labeled by f) has a six-dimensional world-volume $\mathcal{W}_{6,f}$. Since five-branes wrapping the circle S^1 play no particular role in the discussions of subsections 4.2.3 and 4.3.4 below,[13] we will focus our interest on five-branes that are perpendicular to S^1 (they intersect the circle at x^{11}_f). We consider a configuration with N_5 such five-branes. To each of them one associates a brane current five-form $\delta_5(\mathcal{W}_{6,f})$, analogous to the $\delta_{1,i}$ introduced for the two ten-dimensional fixed planes. This brane current can then be decomposed as $\delta_5(\mathcal{W}_{6,f}) \equiv \delta(x^{11} - x^{11}_f)dx^{11} \wedge \delta_4(\mathcal{W}_{6,f})$, and the Bianchi identity for G_4 gets an extra contribution [229, 233]

$$dG_4|_{\text{5-brane}} = \gamma \sum_{f=1}^{N_5} \delta_5(\mathcal{W}_{6,f}) = \gamma \sum_{f=1}^{N_5} \delta(x^{11} - x^{11}_f) dx^{11} \wedge \delta_4(\mathcal{W}_{6,f}). \tag{4.27}$$

[12] Actually, this equation will turn out to be quadratic in b_i and hence has another solution: $b_1 = -b_2 \equiv \pm b$. However, we will soon see that $b_1 = +b_2 = 1$ is needed as soon as topologically non-trivial configurations exist.

[13] Actually, five-branes wrapping the circle should be eliminated by the $\mathbb{Z}_2$ projection.

The solution G_4 receives then an extra part. When integrating $\delta(x^{11}-x_f^{11})dx^{11}\wedge\delta_4(\mathcal{W}_{6,f})$, one has a choice (just as before with $\delta_{1,i}\wedge\widetilde{I}_{4,i}$) giving rise to more free parameters:

$$\begin{aligned} G_4|_{\text{5-brane}} &= \gamma\sum_{f=1}^{N_5}\Big[\frac{\beta}{2}\Big(\varepsilon_1(x^{11}-x_f^{11})\delta_4(\mathcal{W}_{6,f})-\frac{1}{\pi}dx^{11}\wedge\theta_3(\mathcal{W}_{6,f})\Big) \\ &\qquad -(1-\beta)\delta(x^{11}-x_f^{11})dx^{11}\wedge\theta_3(\mathcal{W}_{6,f})\Big]. \end{aligned} \tag{4.28}$$

Using Eqs. (4.5) and (4.10) it is straightforward to verify that the exterior derivative of the r.h.s. of (4.28) gives the r.h.s. of (4.27) for any choice of the parameter β provided

$$d\theta_3(\mathcal{W}_{6,f})=\delta_4(\mathcal{W}_{6,f}). \tag{4.29}$$

There are many different choices for $\theta_3(\mathcal{W}_{6,f})$, but since $\delta_4(\mathcal{W}_{6,f})$ does not involve the circle coordinate x^{11}, the choice of the primitive $\theta_3(\mathcal{W}_{6,f})$ does not matter here.

4.2.2 Gauge and local Lorentz invariance of G_4

We would like to work with a gauge and local Lorentz invariant four-form G_4: $\delta G_4=0$. While $\widetilde{I}_{4,i}$ is invariant, the Chern–Simons three-form $\widetilde{\omega}_{3,i}$ is not and looking at the solution (4.25) we see that C_3 cannot be invariant either. Note that the five-brane contributions (4.28) to G_4 have no dependence on the fields (the five-brane contributions consist only in delta functions) and thus do not change δC_3. The most general variation δC_3 giving an invariant G_4 is then

$$\delta C_3=dB_2^1-\gamma\sum_{i=1}^{2}\left(\frac{b}{2}\varepsilon_i d\widetilde{\omega}_{2,i}^1+\delta_{1,i}\wedge\widetilde{\omega}_{2,i}^1\right), \tag{4.30}$$

with some two-form B_2^1 linear in the gauge or local Lorentz parameters $\Lambda^{\rm g}$ or $\Lambda^{\rm L}$. To determine B_2^1, we recall that C_{ABC} is odd under $\mathbb{Z}_2$ and hence projected out, while C_{AB11} is even and kept. But $C_{ABC}=0$ only makes sense if it is a gauge invariant statement. In other words, we must have $\delta C_{ABC}=0$, which means, using the ABC component of Eq. (4.30),

$$(dB_2^1)_{ABC}=\gamma\frac{b}{2}\sum_{i=1}^{2}\varepsilon_i(d\widetilde{\omega}_{2,i}^1)_{ABC}=\gamma\frac{b}{2}(d\sum_{i=1}^{2}\varepsilon_i\widetilde{\omega}_{2,i}^1)_{ABC}, \tag{4.31}$$

which is solved by $(B_2^1)_{AB}=\gamma\frac{b}{2}\sum_{i=1}^2\varepsilon_i(\widetilde{\omega}_{2,i}^1)_{AB}$. Therefore we choose $B_2^1=\gamma\frac{b}{2}\sum_{i=1}^2\varepsilon_i\widetilde{\omega}_{2,i}^1$, so that

$$\begin{aligned} \delta C_3 &= \gamma\sum_{i=1}^{2}\left(\frac{b}{2}d\varepsilon_i-\delta_{1,i}\right)\wedge\widetilde{\omega}_{2,i}^1 \\ &= \gamma(b-1)\sum_{i=1}^{2}\delta_{1,i}\wedge\widetilde{\omega}_{2,i}^1-\gamma\frac{b}{2\pi}dx^{11}\wedge\sum_{i=1}^{2}\widetilde{\omega}_{2,i}^1. \end{aligned} \tag{4.32}$$

Without the last term in (4.32), C_3 would be invariant away from the fixed planes. This piece of the variation is again due to enforcing the periodicity of the "step functions" ε_i.

4.2.3 Invariance under large gauge and Lorentz transformations (global definition of G_4)

In the last subsection, we have considered invariance under small transformations that can be continuously deformed to the identity. However, the field strength G_4 must also be invariant under large gauge and large local Lorentz transformations. This will be precisely the case if G_4 is *globally well-defined*, i.e. if dG_4 is exact. There is a simple criterion when this is true [230]: if G_4 is globally well-defined then for any five-cycle $\mathcal{C}_5$ [14] Stokes' theorem tells us that $\int_{\mathcal{C}_5} dG_4 = 0$. For instance, one may take $\mathcal{C}_5 = \mathcal{C}_4 \times S^1$ where $\mathcal{C}_4$ is an arbitrary four-cycle at a fixed value of x^{11} (i.e. at a fixed value in the S^1 direction).[15] Rewriting dG_4 with the help of the Bianchi identity (4.23) completed by the extra five-brane contribution (4.27), G_4 will be invariant under large transformations precisely if the condition $\int_{\mathcal{C}_5} dG_4 = 0$ holds, i.e. precisely if[16]

$$\begin{aligned}\sum_{i=1}^{2}\int_{\mathcal{C}_4}\widetilde{I}_{4,i} &\equiv \frac{1}{(4\pi)^2}\int_{\mathcal{C}_4}\left(\mathrm{tr}F_{2,1}^2-\frac{1}{2}\mathrm{tr}\widetilde{R}_{2,1}^2\right)+\frac{1}{(4\pi)^2}\int_{\mathcal{C}_4}\left(\mathrm{tr}F_{2,2}^2-\frac{1}{2}\mathrm{tr}\widetilde{R}_{2,2}^2\right)\\ &= -N_5,\end{aligned} \tag{4.33}$$

where N_5 is the number of five-branes intersecting S^1 and $\mathcal{C}_4$ at a point. Since we do want G_4 to be invariant also under large transformations, we assume this condition henceforth.

There is one more important piece of information we can obtain from requiring global definition of G_4 and using Stokes' theorem on a five-dimensional manifold $\mathcal{M}_5$ which intersects exactly one of the fixed planes on a four-cycle. For example, we may take the manifold of the form $\mathcal{M}_5 = \mathcal{C}_4 \times I$ where $\mathcal{C}_4$ is a four-cycle and I is the interval $[\underline{x}_1^{11}, \underline{x}_2^{11}]$ with $-\pi < \underline{x}_1^{11} < 0$ and $0 < \underline{x}_2^{11} < \pi$. The boundary is then $\partial\mathcal{M}_5 = \mathcal{C}_4(\underline{x}_2^{11}) - \mathcal{C}_4(\underline{x}_1^{11})$ and Stokes' theorem gives

$$\int_{\mathcal{M}_5} dG_4 = \int_{\mathcal{C}_4(\underline{x}_2^{11})} G_4 - \int_{\mathcal{C}_4(\underline{x}_1^{11})} G_4 . \tag{4.34}$$

The integral on the l.h.s. is evaluated using the modified Bianchi identity. The contribution (4.23) collapses to an integral of $\widetilde{I}_{4,1}$ over the four-cycle $\mathcal{C}_4$ on the fixed plane. The five-brane contribution (4.27) to dG_4 yields an extra term $\gamma N_5(I)$, where $N_5(I)$ is the number of five-branes that intersect the interval I and the four-cycle $\mathcal{C}_4$. On the other hand, the integrals on the r.h.s are evaluated using the solution (4.25) and (4.28) for G_4. Since the components G_{ABC11} do not contribute to the integrations, only the term $\frac{b}{2}\gamma\sum_{i=1}^{2}\varepsilon_i\widetilde{I}_{4,i}$ from (4.25) and the piece $\frac{\beta}{2}\gamma\sum_{f=1}^{N_5}\varepsilon_1(x^{11}-x_f^{11})\delta_4(\mathcal{W}_{6,f})$ from (4.28) are relevant. For the integral at $x^{11} = \underline{x}_2^{11}$ one has $\varepsilon_1(\underline{x}_2^{11}) = 1 - \underline{x}_2^{11}/\pi$ and $\varepsilon_2(\underline{x}_2^{11}) = -\underline{x}_2^{11}/\pi$, while for the one at $x^{11} = \underline{x}_1^{11}$ one has $\varepsilon_1(\underline{x}_1^{11}) = -1 - \underline{x}_1^{11}/\pi$ and $\varepsilon_2(\underline{x}_1^{11}) = -\underline{x}_1^{11}/\pi$. Since $\widetilde{I}_{4,i}$ is independent of x^{11}, we have $\int_{\mathcal{C}_4(\underline{x}_1^{11})}\widetilde{I}_{4,i} = \int_{\mathcal{C}_4(\underline{x}_1^{11})}\widetilde{I}_{4,i} = \int_{\mathcal{C}_4}\widetilde{I}_{4,i}$. Furthermore, the contribution of a five-brane labelled by f, $\frac{\beta}{2}\gamma\int_{\mathcal{C}_4(\underline{x}_2^{11})}\varepsilon_1(x^{11}-x_f^{11})\delta_4(\mathcal{W}_{6,f}) - \frac{\beta}{2}\gamma\int_{\mathcal{C}_4(\underline{x}_1^{11})}\varepsilon_1(x^{11}-x_f^{11})\delta_4(\mathcal{W}_{6,f})$, is given by

[14] Recall that a cycle is closed: a cycle is a compact submanifold without boundary.
[15] One could equally well take any five-cycle homologous to $\mathcal{C}_4 \times S^1$.
[16] The analogous cohomology condition is well-known from the heterotic superstring [226].

$\frac{\beta}{2}\gamma\,(2+(\underline{x}_1^{11}-\underline{x}_2^{11})/\pi)$ if $x_f^{11}\in I$, and reads $\frac{\beta}{2}\gamma(\underline{x}_1^{11}-\underline{x}_2^{11})/\pi$ if $x_f^{11}\notin I$. Using the relation (4.33), Eq. (4.34) then becomes

$$\gamma\int_{\mathcal{C}_4}\widetilde{I}_{4,1}+\gamma N_5(I)=\frac{b}{2}\gamma\left(2\int_{\mathcal{C}_4}\widetilde{I}_{4,1}-\frac{\underline{x}_1^{11}-\underline{x}_2^{11}}{\pi}N_5\right)+\frac{\beta}{2}\gamma\left(2N_5(I)+\frac{\underline{x}_1^{11}-\underline{x}_2^{11}}{\pi}N_5\right). \quad (4.35)$$

It is worth noting that the exact positions of $\underline{x}_1^{11}$ and $\underline{x}_2^{11}$ are arbitrary. Upon slightly varying them such that $N_5(I)$ remains unchanged one concludes that the terms linear in $\underline{x}_1^{11}-\underline{x}_2^{11}$ and the terms independent of $\underline{x}_1^{11}-\underline{x}_2^{11}$ must vanish separately. This produces two equations:[17]

$$(\beta-b)N_5=0 \qquad \text{and} \qquad (1-\beta)N_5(I)+(1-b)\int_{\mathcal{C}_4}\widetilde{I}_{4,1}=0. \quad (4.36)$$

In the absence of five-branes ($N_5=0$), one simply gets $(1-b)\int_{\mathcal{C}_4}\widetilde{I}_{4,1}=0$. If $\int_{\mathcal{C}_4}\widetilde{I}_{4,1}=0$, each of the two source terms in the modified Bianchi identity is cohomologically trivial, so that the $\widetilde{\omega}_{3,i}$ can be globally well-defined. In this case b is unconstrained. On the other hand, if $\int_{\mathcal{C}_4}\widetilde{I}_{4,1}\neq 0$ each of the two source terms is individually cohomologically non-trivial[18], and one must take $b=1$. It is interesting to remark that $b=1$ eliminates the terms in G_4 containing delta functions so that G_4 becomes finite everywhere on S^1.

In presence of five-branes ($N_5\neq 0$), the first equation (4.36) gives $\beta=b$ and the second one then tells us that $(1-b)(N_5(I)+\int_{\mathcal{C}_4}\widetilde{I}_{4,1})=0$. With a variation of the interval I, we may change the number $N_5(I)$ [19] and conclude that $b=1$. The resulting four-form G_4 with $b=\beta=1$ fixed by its global definition then reads:

$$\begin{aligned} G_4|_{b=1,\beta=1} &= dC_3+\frac{\gamma}{2}\left[\sum_{i=1}^{2}\varepsilon_i\widetilde{I}_{4,i}+\sum_{f=1}^{N_5}\varepsilon_1(x^{11}-x_f^{11})\delta_4(\mathcal{W}_{6,f})\right] \\ &\quad -\frac{\gamma}{2\pi}dx^{11}\wedge\left[\sum_{i=1}^{2}\widetilde{\omega}_{3,i}+\sum_{f=1}^{N_5}\theta_3(\mathcal{W}_{6,f})\right]. \end{aligned} \quad (4.37)$$

We can summarize the two last subsections by saying that the invariance of G_4 under (small) gauge and (small) local Lorentz transformations of G_4 corresponds to the transformation (4.32) with the $\mathbb{Z}_2$ projection $C_{ABC}=0$, and that its global definition requires the cohomology condition (4.33) as well as the two equations (4.36).

4.2.4 The case $b=1$

We have just observed that $b=1$ is required whenever $\int_{\mathcal{C}_4}\widetilde{I}_{4,i}\neq 0$ or five-branes not wrapping the circle are present. The case $b=1$ is attractive because it corresponds to a four-form G_4 without delta function singularities. It also presents some other interesting features

[17] With a five-manifold $\mathcal{M}_5$ intersecting the other fixed plane at $x^{11}=\pi$, we would have obtained an analogous set of equations with $\widetilde{I}_{4,2}$ instead of $\widetilde{I}_{4,1}$.

[18] Although their sum is trivial by (4.33).

[19] Unless all five-branes are living on the fixed planes.

when considering the solution (4.25) and the transformation (4.32) from the point of view of the reduction to the perturbative heterotic superstring. To understand this, we consider a situation without five-branes ($N_5 = 0$) and expand G_{ABC11} and C_{AB11} in Fourier modes along S^1:[20]

$$\begin{aligned} G^{(0)}_{ABC11} &= d_{[A}C^{(0)}_{BC]11} + \frac{\gamma}{2\pi}(\widetilde{\omega}_{3,1} + \widetilde{\omega}_{3,2})_{ABC}, \\ G^{(n)}_{ABC11} &= d_{[A}C^{(n)}_{BC]11} - \frac{\gamma}{2\pi}(b-1)(\widetilde{\omega}_{3,1} + (-1)^n\widetilde{\omega}_{3,2})_{ABC}, \quad n > 0. \end{aligned} \tag{4.39}$$

Thanks to the last term in the second equality (4.25), the zero mode does not depend on the parameter b, and we notice that $C^{(0)}_{AB11}$ can be neither gauge nor local Lorentz invariant if we impose the invariance of G_4. The higher modes $C^{(n)}_{AB11}$ are gauge and local Lorentz invariant if and only if $b = 1$. But the truncation of C_{AB11} to its zero-mode is safe only if the higher modes are gauge invariant, and this requirement is crucial when making contact with the ten-dimensional heterotic superstring in the field theory limit.

In the next section we will show that anomaly cancellations relate b^2 and λ^6/κ_{11}^4, but do not fix one or the other. However, the gauge and local Lorentz variation of the topological term will have a contribution which depends on b and which is a variation of a (local) counterterm and hence does not contribute to the formal twelve-form which characterizes the anomaly. The appearance of this term is related to the b-dependent contributions in the variation of C_3, and then to the higher modes in its expansion. On the other hand, a direct calculation of the anomaly-cancelling terms, by first truncating C_3 to $C_3^{(0)}$ and then calculating the resulting ten-dimensional action leads directly to a Green–Schwarz term which corresponds to $b = 1$, the case in which all truncated modes are gauge invariant. This will be done in section 4.5.

The conclusion is then that anomaly cancellation alone does not fix b. The perturbative heterotic limit however (the small S^1 radius limit) selects $b = 1$ because it ensures gauge invariance of the massive modes. This condition is essentially due to compactification on a small space. As we have seen in the preceding subsection, global considerations also impose $b = 1$, provided topologically non-trivial configurations occur.

4.2.5 Reduction to the ten-dimensional heterotic superstring

The field theory (perturbative) limit of the ten-dimensional heterotic superstring corresponds to a situation where the radius of the circle S^1 is very small. The massless heterotic fields are defined (modulo possible rescalings) as the zero-modes of the Fourier expansion on the

[20] For instance, the mth Fourier coefficient for the component G_{ABC11} simply corresponds to the integral

$$G^{(m)}_{ABC11} = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx^{11}\, e^{-imx^{11}} G_{ABC11}(x^{11}), \tag{4.38}$$

where G_{ABC11} is given by the solution (4.25). Recall that we are working here with a circle of unit radius.

circle of the massless M-theory fields. In particular,

$$\begin{aligned} B_{AB} \stackrel{\text{def.}}{=} C^{(0)}_{AB11} &= \frac{1}{2\pi r}\int_{-\pi r}^{\pi r} dx^{11}\, C_{AB11}(x^{11}), \\ \underline{H}_{ABC} \stackrel{\text{def.}}{=} G^{(0)}_{ABC11} &= \frac{1}{2\pi r}\int_{-\pi r}^{\pi r} dx^{11}\, G_{ABC11}(x^{11}), \end{aligned} \tag{4.40}$$

where we have reintroduced an arbitrary radius r. The integration of the $ABC11$ components of the solution (4.25) for G_4 over the circle S^1 leads to

$$\underline{H}_3 = dB_2 + \frac{\gamma}{2\pi r}(\widetilde{\omega}_{3,1} + \widetilde{\omega}_{3,2}). \tag{4.41}$$

In the small radius limit, the Lorentz Chern–Simons forms which enter $\widetilde{\omega}_{3,1}$ and $\widetilde{\omega}_{3,2}$ become the same and we can replace $\widetilde{\omega}_{3,1} + \widetilde{\omega}_{3,2}$ by[21] $\frac{1}{(4\pi)^2}\left(\Omega_{3,\mathrm{YM}} - \Omega_{3,\mathrm{L}}\right)$, so that the expression of $\underline{H}_3$ becomes

$$\underline{H}_3 = dB_2 - \frac{\kappa_{11}^2}{2\pi r\lambda^2}\left(\Omega_{3,\mathrm{YM}} - \Omega_{3,\mathrm{L}}\right). \tag{4.42}$$

To obtain the standard relation of the heterotic superstring [108], we reexpress κ_{11} in terms of the ten-dimensional gravitational coupling κ_{10},

$$\kappa_{11}^2 = (2\pi r)\kappa_{10}^2, \tag{4.43}$$

and rescale the fields $\underline{H}_3$ and B_2 according to

$$\underline{H}_3 = \frac{\kappa_{10}^2}{\lambda^2}\widehat{H}_3, \qquad B_2 = \frac{\kappa_{10}^2}{\lambda^2}\widehat{B}_2. \tag{4.44}$$

Inserting these redefinitions into (4.42), we get the well-known form

$$\widehat{H}_3 = d\widehat{B}_2 - \Omega_{3,\mathrm{YM}} + \Omega_{3,\mathrm{L}}. \tag{4.45}$$

The variation of $\widehat{B}_2$ can be deduced from its definition (4.40) and the expression (4.32) of δC_3:

$$\delta\widehat{B}_2 = \Omega^1_{2,\mathrm{YM}} - \Omega^1_{2,\mathrm{L}}, \tag{4.46}$$

with $\delta\Omega_{3,\mathrm{YM}} = d\Omega^1_{2,\mathrm{YM}}$, $\delta\Omega_{3,\mathrm{L}} = d\Omega^1_{2,\mathrm{L}}$ and $\widetilde{\omega}^1_{2,1} + \widetilde{\omega}^1_{2,2} = \frac{1}{(4\pi)^2}(\Omega^1_{2,\mathrm{YM}} - \Omega^1_{2,\mathrm{L}})$. On the other hand, the invariance of $\widehat{H}_3$ follows from the invariance of G_4.

In presence of an M-five-brane not wrapping S^1, only the zero mode $G^{(0)}_{ABC11}$ is modified. The extra contribution[22] $\frac{\gamma}{2\pi}\theta_3(\mathcal{W}_{6,f})$ appears on the r.h.s. of (4.45) and $d\widehat{H}_3$ gets a piece proportional to $\delta_4(\mathcal{W}_{6,f})$ which is the appropriate source term for a heterotic solitonic five-brane.

[21] $\Omega_{3,\mathrm{YM}}$ and $\Omega_{3,\mathrm{L}}$ are conventionally normalized Chern–Simons three-forms with $d\Omega_{3,\mathrm{YM}} = \mathrm{tr}F^2_{2,1} + \mathrm{tr}F^2_{2,2}$ and $d\Omega_{3,\mathrm{L}} = \mathrm{tr}R^2_2$.

[22] Remember that $\beta = 1$.

To complete the comparison with the bosonic low-energy effective supergravity of the heterotic superstring, we reduce on S^1 (throwing aside the fields odd under $\mathbb{Z}_2$) the bosonic part of the actions (2.51) and (4.3). With the metric ansatz

$$g_{MN} = \begin{pmatrix} \varphi^{-1/4} g_{AB} & 0 \\ 0 & \varphi^2 \end{pmatrix}, \tag{4.47}$$

the ten-dimensional Einstein and Yang–Mills terms, as well as the kinetic Lagrangian for $\widehat{H}_3$ read[23]

$$\mathcal{L}_{10\text{d. partial}} = -\frac{e_{10}}{2\kappa_{10}^2}\left[R + \frac{1}{2}\frac{\kappa_{10}^2}{\lambda^2}\varphi^{-3/4}\text{tr}(F_{AB}F^{AB}) + \frac{1}{12}\left(\frac{\kappa_{10}^2}{\lambda^2}\varphi^{-3/4}\right)^2 \widehat{H}_{ABC}\widehat{H}^{ABC}\right], \tag{4.48}$$

where $e_{10} = \sqrt{-g_{AB}}$. The dimensionless quantity $\lambda^2/\kappa_{10}^{3/2}$ can be absorbed in a redefinition of the field φ to finally obtain

$$\mathcal{L}_{10\text{d. partial}} = -\frac{e_{10}}{2\kappa_{10}^2}\left[R + \frac{1}{2}\kappa_{10}^{1/2}\varphi^{-3/4}\text{tr}(F_{AB}F^{AB}) + \frac{1}{12}\kappa_{10}\varphi^{-3/2}\widehat{H}_{ABC}\widehat{H}^{ABC}\right], \tag{4.49}$$

and the field-dependent gauge coupling constant is

$$g^2 = \kappa_{10}^{3/2}\varphi^{3/4}. \tag{4.50}$$

We conclude that the Lagrangian $\mathcal{L}_{10\text{d. partial}}$ has two parameters: κ_{10} and the expectation value of φ, the latter being related to the radius r of the circle. The gauge coupling constant λ cannot be observed in the heterotic limit.

4.3 Anomaly cancellations

The $\mathbb{Z}_2$ orbifold projection generates a chiral spectrum. As a consequence, the theory presents a chiral gauge, mixed and gravitational quantum anomaly in $\mathcal{M}_{10}$ which can be characterized by a formal twelve-form. In subsection 4.2.2, we have determined the gauge and local Lorentz variation of the three-form field C_3 due to the modification of the Bianchi identity. This variation is at the origin of an anomalous variation of the action, and we will now study the sources for this anomaly inflow and prove anomaly cancellation. The five-brane contributions will enter our discussion separately in subsection 4.3.4.

4.3.1 Anomaly inflow from the topological term

In this subsection, we determine the anomaly inflow coming from the topological interaction term S_{top}. Since $\delta G_4 = 0$, the variation of this part of the action is simply

$$\delta S_{\text{top}} = -\frac{1}{12\kappa_{11}^2}\delta\int_{\mathcal{M}_{11}} C_3 \wedge G_4 \wedge G_4 = -\frac{1}{12\kappa_{11}^2}\int_{\mathcal{M}_{11}} \delta C_3 \wedge G_4 \wedge G_4. \tag{4.51}$$

[23]Notice that G_{ABCD}, as given in Eq. (4.25) with $C_{ABC} = 0$, would contribute to four-derivative terms only.

Since the variation (4.32) of C_3 only contains components with a dx^{11}, the components G_{ABC11} cannot contribute to this integral. We also know that because of the $\mathbb{Z}_2$ symmetry we have $C_{ABC} = 0$. The solution (4.25) then gives

$$G_{ABCD} = \frac{b}{2}\gamma \sum_{j=1}^{2} \varepsilon_j (\widetilde{I}_{4,j})_{ABCD}, \tag{4.52}$$

and it follows that

$$\delta S_{\text{top}} = -\frac{\gamma^3 b^2}{48\kappa_{11}^2} \int_{\mathcal{M}_{11}} \sum_{i=1}^{2} \left[(b-1)\delta_{1,i} - \frac{b}{2\pi} dx^{11}\right] \wedge \widetilde{\omega}_{2,i}^1 \wedge \sum_{j=1}^{2} \varepsilon_j \widetilde{I}_{4,j} \wedge \sum_{k=1}^{2} \varepsilon_k \widetilde{I}_{4,k}. \tag{4.53}$$

Now we can use (4.12) and (4.18) to compute the dx^{11} integrals over S^1. In the integrals not involving $\delta_{1,i}$ it is important that, apart from the $\varepsilon_j \varepsilon_k$, the rest of the integrand, namely $\widetilde{\omega}_{2,i}^1 \wedge \widetilde{I}_{4,j} \wedge \widetilde{I}_{4,k}$, involves only tilde quantities and is therefore independent of x^{11}. The result of the integration over S^1 reads then

$$\begin{aligned} \delta S_{\text{top}} &= -\frac{\gamma^3 b^2}{48\kappa_{11}^2} \int_{\mathcal{M}_{10}} \left[\frac{(b-1)}{3} \sum_{i=1}^{2} \widetilde{\omega}_{2,i}^1 \wedge (\widetilde{I}_{4,i})^2 - \frac{b}{2} \sum_{i,j,k=1}^{2} (\delta_{jk} - \frac{1}{3})\, \widetilde{\omega}_{2,i}^1 \wedge \widetilde{I}_{4,j} \wedge \widetilde{I}_{4,k}\right] \\ &\overset{\text{def.}}{=} \delta S_{\text{top}}^{(1)} + \delta S_{\text{top}}^{(2)}. \end{aligned} \tag{4.54}$$

The second part $\delta S_{\text{top}}^{(2)}$, with its multiple sum over i, j, k, looks quite different from the first one $\delta S_{\text{top}}^{(1)}$. However, we will soon see that both terms correspond to the same anomaly polynomial I_{12}. The calculation of the sums in $\delta S_{\text{top}}^{(2)}$ gives

$$\delta S_{\text{top}}^{(2)} = \frac{\gamma^3 b^3}{144\kappa_{11}^2} \int_{\mathcal{M}_{10}} (\widetilde{\omega}_{2,1}^1 + \widetilde{\omega}_{2,2}^1) \wedge \left((\widetilde{I}_{4,1})^2 + (\widetilde{I}_{4,2})^2 - \widetilde{I}_{4,1} \wedge \widetilde{I}_{4,2}\right), \tag{4.55}$$

and the corresponding invariant formal twelve-form obtained using the descent equations reads[24]

$$\begin{aligned} I_{12}^{\text{top}\,(2)} &= \frac{\gamma^3 b^3}{144\kappa_{11}^2} (\widetilde{I}_{4,1} + \widetilde{I}_{4,2}) \wedge \left((\widetilde{I}_{4,1})^2 + (\widetilde{I}_{4,2})^2 - \widetilde{I}_{4,1} \wedge \widetilde{I}_{4,2}\right) \\ &= \frac{\gamma^3 b^3}{144\kappa_{11}^2} \left((\widetilde{I}_{4,1})^3 + (\widetilde{I}_{4,2})^3\right). \end{aligned} \tag{4.56}$$

This anomaly polynomial has the same structure as the invariant twelve-form corresponding to $\delta S_{\text{top}}^{(1)}$:

$$I_{12}^{\text{top}\,(1)} = -\frac{\gamma^3 b^2 (b-1)}{144\kappa_{11}^2} \left((\widetilde{I}_{4,1})^3 + (\widetilde{I}_{4,2})^3\right). \tag{4.57}$$

[24]The principle of descent equations is explained in Appendix C. This appendix discusses also other important concepts related to anomaly cancellation (e.g. the appearance of local counterterms).

The expression of the complete twelve-form $I_{12}^{\text{top}\,(1)} + I_{12}^{\text{top}\,(2)}$ characterizing anomaly inflow from S_{top} is then[25]

$$I_{12}^{\text{top}} = \frac{\gamma^3 b^2}{144\kappa_{11}^2} \sum_{i=1}^{2} (\widetilde{I}_{4,i})^3 = -\frac{\pi}{3}\left(\frac{(4\pi)^5}{12}\frac{\kappa_{11}^4}{\lambda^6} b^2\right) \sum_{i=1}^{2} (\widetilde{I}_{4,i})^3. \tag{4.58}$$

Since the field G_4 is real, the parameter b is real as well, and b^2 is necessarily positive, as is κ_{11}^4/λ^6. So the sign of the coefficient in I_{12}^{top} is fixed. This reflects the fact that $N=1$ ten-dimensional supergravity is chiral and specific signs appear in the Bianchi identity once the gravitino chirality is chosen. In any case, the above sign is as required to cancel the quantum anomaly generated by chiral fermions.

To make the relation between the two so different looking anomalies $\delta S_{\text{top}}^{(1)}$ and $\delta S_{\text{top}}^{(2)}$ more explicit, we can try to find a local ten-dimensional counterterm Δ_{10} such that

$$\int_{\mathcal{M}_{10}} \sum_{i=1}^{2} \widetilde{\omega}_{2,i}^1 \wedge (\widetilde{I}_{4,i})^2 = \int_{\mathcal{M}_{10}} (\widetilde{\omega}_{2,1}^1 + \widetilde{\omega}_{2,2}^1) \wedge \left((\widetilde{I}_{4,1})^2 + (\widetilde{I}_{4,2})^2 - \widetilde{I}_{4,1} \wedge \widetilde{I}_{4,2}\right) + \delta \int_{\mathcal{M}_{10}} \Delta_{10}. \tag{4.59}$$

Upon applying the descent equations on the two writings (4.56) [first eq.] and (4.57) of the twelve-form anomaly polynomial, we deduce that

$$\begin{aligned} \Delta_{10} &= \frac{2}{3}(\widetilde{\omega}_{3,1} + \widetilde{\omega}_{3,2}) \wedge \left(\widetilde{\omega}_{3,1} \wedge \widetilde{I}_{4,1} + \widetilde{\omega}_{3,2} \wedge \widetilde{I}_{4,2} - \frac{1}{2}\widetilde{\omega}_{3,1} \wedge \widetilde{I}_{4,2} - \frac{1}{2}\widetilde{\omega}_{3,2} \wedge \widetilde{I}_{4,1}\right) \\ &= \widetilde{\omega}_{3,1} \wedge \widetilde{\omega}_{3,2} \wedge (\widetilde{I}_{4,2} - \widetilde{I}_{4,1}), \end{aligned} \tag{4.60}$$

so that δS_{top} may be rewritten as

$$\delta S_{\text{top}} = \frac{\gamma^3 b^2}{144\kappa_{11}^2}\left[\int_{\mathcal{M}_{10}} \left(\widetilde{\omega}_{2,1}^1 \wedge (\widetilde{I}_{4,1})^2 + \widetilde{\omega}_{2,2}^1 \wedge (\widetilde{I}_{4,2})^2\right) - b\,\delta \int_{\mathcal{M}_{10}} \Delta_{10}\right]. \tag{4.61}$$

This ten-form will be useful in the discussion of the heterotic anomaly-cancelling terms (see section 4.5).

4.3.2 Anomaly inflow from the Green–Schwarz term

We must also determine the variation of the Green–Schwarz term. The first question is whether one should take $\int G_4 \wedge X_7$ or $\int C_3 \wedge X_8$ with[26]

$$X_8 = dX_7 \quad \text{and} \quad X_8 = \frac{1}{12(4\pi)^3}\left(\frac{1}{2}\text{tr}R_2^4 - \frac{1}{8}(\text{tr}R_2^2)^2\right). \tag{4.62}$$

Since the modified Bianchi identity implies that $G_4 \neq dC_3$, both choices are inequivalent. We will start with $\int G_4 \wedge X_7$ and it will turn out that the form $\int C_3 \wedge X_8$ does not allow to cancel the one-loop anomalies.

[25] In particular, we note that the terms cubic in b exactly cancel.

[26] Note that X_8 obeys $dX_8 = 0$ and $\delta X_8 = 0$.

Even if the appropriate normalization of the Green–Schwarz term is known independently [222, 77], it is interesting to rederive it from the present anomaly cancellation. Therefore we consider

$$S_{\mathrm{GS}} = \frac{c}{\gamma}\int_{\mathcal{M}_{11}} G_4 \wedge X_7 \tag{4.63}$$

where $\gamma = -(4\pi)^2\kappa_{11}^2/\lambda^2$ as before and c is a dimensionless constant to be determined. Using $\delta G_4 = 0$ and the descent equations $X_8 = dX_7$ and $\delta X_7 = dX_6^1$, the variation of S_{GS} can be written

$$\begin{aligned} \delta S_{\mathrm{GS}} &= \frac{c}{\gamma}\int_{\mathcal{M}_{11}} G_4 \wedge \delta X_7 = \frac{c}{\gamma}\int_{\mathcal{M}_{11}} G_4 \wedge dX_6^1 = -\frac{c}{\gamma}\int_{\mathcal{M}_{11}} dG_4 \wedge X_6^1 \\ &= -c\int_{\mathcal{M}_{11}} \sum_{i=1}^{2} \delta_{1,i} \wedge \widetilde{I}_{4,i} \wedge X_6^1 = -c\int_{\mathcal{M}_{10}} \sum_{i=1}^{2} \widetilde{I}_{4,i} \wedge \widetilde{X}_{6,i}^1. \end{aligned} \tag{4.64}$$

In the last equality, we have replaced X_6^1 by $\widetilde{X}_{6,i}^1$ defined only on the ith fixed plane. Since $\widetilde{I}_{4,i}$ is closed and gauge and local Lorentz invariant, the variation (4.64) corresponds to the twelve-form

$$I_{12}^{\mathrm{GS}} = -c\sum_{i=1}^{2} \widetilde{I}_{4,i} \wedge \widetilde{X}_{8,i}. \tag{4.65}$$

The other choice for the Green–Schwarz term,

$$\widehat{S}_{\mathrm{GS}} = \frac{\hat{c}}{\gamma}\int_{\mathcal{M}_{11}} C_3 \wedge X_8, \tag{4.66}$$

leads to a variation

$$\begin{aligned} \delta\widehat{S}_{\mathrm{GS}} &= \frac{\hat{c}}{\gamma}\int_{\mathcal{M}_{11}} \delta C_3 \wedge X_8 \\ &= \hat{c}(b-1)\int_{\mathcal{M}_{10}} \sum_{i=1}^{2} \widetilde{\omega}_{2,i}^1 \wedge \widetilde{X}_{8,i} - \hat{c}\frac{b}{2\pi}\int_{\mathcal{M}_{11}} \sum_{i=1}^{2} dx^{11} \wedge \widetilde{\omega}_{2,i}^1 \wedge X_8. \end{aligned} \tag{4.67}$$

While in the first term the delta function has the effect of replacing X_8 by $\widetilde{X}_{8,i}$, in the second term X_8 truly depends on x^{11}. The first term corresponds to a twelve-form polynomial $\widehat{I}_{12}^{\mathrm{GS}\,(1)} = \hat{c}(b-1)\sum_{i=1}^{2} \widetilde{I}_{4,i} \wedge \widetilde{X}_{8,i}$ and has the form required for cancelling the one-loop anomaly.[27] Its coefficient however seems to be wrong: if $b = 1$ it vanishes, and if $b \neq 1$ one would be forced to choose a non-standard $\hat{c}$ (except if $b = 0$). In any case, X_8 in the second term depends on the circle coordinate x^{11} and there is no way to make it equal to $\widetilde{X}_{8,i}$ on the ith fixed plane as needed for anomaly cancellation. We conclude that $\widehat{S}_{\mathrm{GS}}$ could be suitable at best only for $b = 0$. But this case is certainly ruled out as it would lead to $\delta S_{\mathrm{top}} = 0$. The possibility to discriminate between S_{GS} and $\widehat{S}_{\mathrm{GS}}$ relies essentially on the presence of the second term in (4.67) and this term is again a consequence of enforcing the periodicity of the "step functions" $\varepsilon_i(x^{11})$.

[27] See Appendix C.

4.3.3 The cancellation of the one-loop anomaly

The sum $I_{12}^{\text{top}} + I_{12}^{\text{GS}}$ must precisely cancel the one-loop anomaly given by[28]

$$I_{12}^{\text{1-loop}} = \sum_{i=1}^{2} \Big(\frac{\pi}{3} (\widetilde{I}_{4,i})^3 + \widetilde{I}_{4,i} \wedge \widetilde{X}_{8,i} \Big). \tag{4.68}$$

The expressions (4.58) and (4.65) of I_{12}^{top} and I_{12}^{GS} cancel this anomaly if and only if

$$b^2 = \frac{12}{(4\pi)^5} \frac{\lambda^6}{\kappa_{11}^4} \tag{4.69}$$

and

$$c = 1. \tag{4.70}$$

The normalization $c = 1$ was known from cancellation of the anomaly due to a five-brane[29], but the quadratic relation for the parameter b is a new result.

Using Eqs. (4.61), (4.64), (4.69) and (4.70), we may now rewrite the total anomalous variation as

$$\begin{aligned} \delta S_{\text{top}} + \delta S_{\text{GS}} &= -\frac{\pi}{3} \int_{\mathcal{M}_{10}} \Big(\widetilde{\omega}_{2,1}^1 \wedge (\widetilde{I}_{4,1})^2 + \widetilde{\omega}_{2,2}^1 \wedge (\widetilde{I}_{4,2})^2 - b\, \delta \Delta_{10} \Big) \\ &\quad - \int_{\mathcal{M}_{10}} \Big(\widetilde{I}_{4,1} \wedge \widetilde{X}_{6,1}^1 + \widetilde{I}_{4,2} \wedge \widetilde{X}_{6,2}^1 \Big). \end{aligned} \tag{4.71}$$

The term proportional to b is the variation of a local ten-dimensional counterterm Δ_{10}, and this is why it does not contribute to the twelve-form of the descent equations. We will see in section 4.5 that in the small radius limit corresponding to the perturbative heterotic superstring, this particular form of the anomalous variation is at the origin of a local counterterm usually unexpected from standard ten-dimensional arguments.

A word on the value of the parameter b

In subsection 4.2.3, we have seen that the parameter b is equal to 1 if $\int_{\mathcal{C}_4} \widetilde{I}_{4,i} \neq 0$. The relation (4.69) then fixes the ratio λ^6/κ_{11}^4. Since this ratio should not depend on the topological sector of the theory, this in turn imply that $b = 1$ even when $\int_{\mathcal{C}_4} \widetilde{I}_{4,i} = 0$, i.e. $b = 1$ always.

4.3.4 The M-five-brane anomaly

The M-five-brane anomaly is a purely gravitational anomaly of the six-dimensional chiral theory living on the world-volume of the five-brane (a chiral tensor multiplet).[30] The invari-

[28] This one-loop anomaly is discussed in Appendix C, where the various expressions refer to ten-dimensional anomalies on a given fixed plane and should be understood as involving only tilde quantities.

[29] See also the following subsection.

[30] There is also the normal bundle anomaly, but it can be treated independently [33, 92] and we will not consider it here. Remark in passing that there is no anomaly associated to the M-2-brane zero-modes (the dimensions of the membrane world-volume is odd).

ant eight-form corresponding to this one-loop anomaly is simply[31]

$$I_8^{\text{5-brane}}(\text{1-loop}) = \hat{X}_8, \tag{4.72}$$

where the sign on the r.h.s actually depends on the choice of chirality. The Bianchi identity for G_4 is modified by the presence of the five-brane as given by (4.27) and according to (4.64) there is an additional five-brane contribution to the variation of the Green–Schwarz term S_{GS}:[32]

$$\begin{aligned} \delta S_{\text{GS}}|_{\text{5-brane}} &= -\frac{1}{\gamma}\int_{\mathcal{M}_{11}} dG_4|_{\text{5-brane}} \wedge X_6^1 = -\int_{\mathcal{M}_{11}} \delta_5(\mathcal{W}_6) \wedge X_6^1 \\ &= -\int_{\mathcal{W}_6} \hat{X}_6^1. \end{aligned} \tag{4.73}$$

This variation is the descendent of $I_8^{\text{GS}} = -\hat{X}_8$, so that it exactly cancels the five-brane one-loop anomaly (4.72).

We should still check that the five-brane contribution to dG_4 does not alter our previous results. Indeed, when solving the Bianchi identity, G_4 has an extra piece:

$$G_4|_{\text{5-brane}} = \gamma\theta_4(\mathcal{W}_6), \tag{4.74}$$

where $d\theta_4(\mathcal{W}_6) = \delta_5(\mathcal{W}_6)$. In subsection 4.2.1, we have explicitly given the expression of $\theta_4(\mathcal{W}_6)$ for a five-brane not wrapping the circle, and in subsection 4.2.3, we have concluded that when five-branes are present in the interval the parameter β should be taken equal to 1. For five-branes wrapping the circle, the brane current $\delta_5(\mathcal{W}_6)$ does not contain x^{11} and any primitive $\theta_4(\mathcal{W}_6)$ is a priori allowed. For the moment, we do not specify which type of five-branes we consider and keep the generic notation $\theta_4(\mathcal{W}_6)$. The important point is that the extra piece in G_4 does not depend on any fields and is thus trivially gauge and local Lorentz invariant. The variation (4.32) of the three-form C_3 is then unchanged and the modification of δS_{top} due to the additional piece in G_4 reads

$$\begin{aligned} \delta S_{\text{top}}|_{\text{5-brane}} &= -\frac{1}{6\kappa_{11}^2}\int_{\mathcal{M}_{11}} \delta C_3 \wedge G_4|_{\text{without 5-brane}} \wedge G_4|_{\text{5-brane}} \\ &= -\frac{\gamma^3 b}{12\kappa_{11}^2}\int_{\mathcal{M}_{11}} \sum_{i=1}^{2}\left((b-1)\delta_{1,i} - \frac{b}{2\pi}dx^{11}\right) \wedge \widetilde{\omega}_{2,i}^1 \wedge \sum_{j=1}^{2} \varepsilon_j \widetilde{I}_{4,j} \wedge \theta_4(\mathcal{W}_6). \end{aligned} \tag{4.75}$$

If the five-brane wraps the circle, $\delta_5(\mathcal{W}_6)$ and $\theta_4(\mathcal{W}_6)$ are independent of x^{11}. The only x^{11}-dependence of the above integrand is then $((b-1)\delta_{1,i} - \frac{b}{2\pi}dx^{11})\varepsilon_j$ which integrates to zero by virtue of the second part of Eq. (4.18) and Eq. (4.11). Hence the anomalous variation of S_{top} is, as expected, not affected by this kind of five-branes.

[31] We put a hat on the eight-form to emphasize the fact that this anomaly concerns a six-dimensional theory (and not a ten-dimensional one).

[32] Recall that the five-brane source $\delta_5(\mathcal{W}_6)$ is such that for any six-form A_6 living in eleven dimensions, one has $\int_{\mathcal{M}_{11}} \delta_5(\mathcal{W}_6) \wedge A_6 = \int_{\mathcal{W}_6} \hat{A}_6$. A possible switch of chirality would produce a minus sign on the r.h.s. of (4.27) and (4.72).

If the five-brane world-volume does not extend in the x^{11}-direction, we have seen that generally $b = 1$ and $\beta = 1$. Using the explicit form of the solution (4.28) with $\beta = 1$ and considering a situation with N_5 such five-branes, the extra contribution (4.75) to $\delta S_{\rm top}$ becomes[33]

$$\begin{aligned} \delta S_{\rm top}|_{\text{5-brane}} &= \frac{\gamma^3}{48\pi\kappa_{11}^2}\int_{\mathcal{M}_{11}}\sum_{i=1}^{2} dx^{11}\wedge\widetilde{\omega}_{2,i}^1\wedge\sum_{j=1}^{2}\varepsilon_j\widetilde{I}_{4,j}\wedge\sum_{f=1}^{N_5}\varepsilon_1(x^{11}-x_f^{11})\delta_4(\mathcal{W}_{6,f}) \\ &= \frac{\gamma^3}{48\pi\kappa_{11}^2}\sum_{f=1}^{N_5}\int_{\mathcal{W}_{6,f}}(\widehat{\omega}_{2,1}^1+\widehat{\omega}_{2,2}^1)\wedge\Big(f_1(x_f^{11})\widehat{I}_{4,1}+f_2(x_f^{11})\widehat{I}_{4,2}\Big), \end{aligned} \tag{4.76}$$

where the integrals

$$f_j(x_f^{11}) = \int dx^{11}\,\varepsilon_j(x^{11})\varepsilon_1(x^{11}-x_f^{11}) \tag{4.77}$$

over the circle are given by

$$f_1(x_f^{11}) = \frac{2\pi}{3} - 2|x_f^{11}| + \frac{(x_f^{11})^2}{\pi} \quad \text{and} \quad f_2(x_f^{11}) = -\frac{\pi}{3} + \frac{(x_f^{11})^2}{\pi}. \tag{4.78}$$

The $f_j(x_f^{11})$ are $\mathbb{Z}_2$ even functions and obey

$$\frac{df_j}{dx_f^{11}}(x_f^{11}) = -2\varepsilon_j(x_f^{11}). \tag{4.79}$$

We have replaced $\widetilde{\omega}_{2,i}^1$ and $\widetilde{I}_{4,i}$ by $\widehat{\omega}_{2,i}^1$ and $\widehat{I}_{4,i}$ to stress that these forms are evaluated on the $\mathcal{W}_{6,f}$ world-volumes.

The extra anomaly inflow from $S_{\rm top}$ into the six-dimensional theory on the five-brane world-volumes is associated to the invariant eight-form

$$I_8^{\text{top (5-brane)}} = \frac{\gamma^3}{48\pi\kappa_{11}^2}\sum_{f=1}^{N_5}\Big[f_1(x_f^{11})\Big((\widehat{I}_{4,1})^2+\widehat{I}_{4,1}\wedge\widehat{I}_{4,2}\Big)+f_2(x_f^{11})\Big((\widehat{I}_{4,2})^2+\widehat{I}_{4,1}\wedge\widehat{I}_{4,2}\Big)\Big]. \tag{4.80}$$

This non-vanishing anomaly localised on the five-brane world-volumes is an interesting new effect which would deserve further study. A somewhat related issue, within the context of Calabi–Yau compactifications with less supersymmetry, appears in Ref. [138] where it is found that gauge fields originate on five-branes. We will come back to this point at the end of Chapter 6.

[33] We omit here possible non-zero contributions of the form

$$\begin{aligned} \delta S_{\rm top}|_{\text{5-brane}} &= -\tfrac{1}{6\kappa_{11}^2}\textstyle\int_{\mathcal{M}_{11}}\delta C_3\wedge G_4|_{\text{5-brane 1}}\wedge G_4|_{\text{5-brane 2}} \\ &= \tfrac{\gamma^3}{48\pi\kappa_{11}^2}\textstyle\int_{\mathcal{M}_{11}}\varepsilon_1(x^{11}-x_{f1}^{11})\varepsilon_1(x^{11}-x_{f2}^{11})dx^{11}\wedge\delta_4(\mathcal{W}_{6,f1})\wedge\delta_4(\mathcal{W}_{6,f2})\wedge\sum_{i=1}^{2}\widetilde{\omega}_{2,i}^1, \end{aligned}$$

which would correspond to exotic configurations where two five-branes intersect on a string.

4.4 The issue of G_4-flux quantization

4.4.1 Does flux quantization hold ?

In the preceding section, we have seen that the cancellation of the one-loop anomaly $I_{12}^{\text{1-loop}}$ only requires the validity of the relation (4.69) between b^2 and λ^6/κ_{11}^4. On the other hand, we have seen in subsection 4.2.3 that if there are topologically non-trivial gauge/gravity configurations such that $\int_{\mathcal{C}_4} \widetilde{I}_{4,i} \neq 0$, one is forced to take $b = 1$ in order to have a globally well-defined four-form field G_4. We will now explore the consequences of the modification of the Bianchi identity for the flux of G_4 and compare with Witten's result on flux quantization [232] obtained in the downstairs approach which in a certain way also corresponds to a fixed value of b. We will firstly evaluate the integral of G_4 on four-cycles $\mathcal{C}_4$ not wrapping the circle S^1 and find the standard flux quantization. The situation for four-cycles of the form $\mathcal{C}_3 \times S^1$ will turn out to be much more unclear. To simplify the discussion, we suppose that there are no five-branes.

Four-cycles not wrapping S^1

We consider a four-cycle $\mathcal{C}_4(x^{11})$ which does not wrap the circle S^1 and is located at a fixed value of $x^{11} \in (0, \pi)$. The integral of G_4 over $\mathcal{C}_4$ can then be written

$$\begin{aligned} \frac{2}{\gamma} \int_{\mathcal{C}_4(x^{11})} G_4 &= b \sum_{i=1}^{2} \varepsilon_i(x^{11}) \int_{\mathcal{C}_4(x^{11})} \widetilde{I}_{4,i} \\ &= b \left[\left(1 - \frac{x^{11}}{\pi}\right) \int_{\mathcal{C}_4(x^{11})} \widetilde{I}_{4,1} - \frac{x^{11}}{\pi} \int_{\mathcal{C}_4(x^{11})} \widetilde{I}_{4,2} \right] \\ &= b \int_{\mathcal{C}_4(x^{11})} \widetilde{I}_{4,1}, \end{aligned} \tag{4.81}$$

where we have used the solution (4.25) to obtain the first equality and the cohomology condition (4.33) to cancel the x^{11}-dependent terms. Now, either $\int_{\mathcal{C}_4(x^{11})} \widetilde{I}_{4,1} = 0$ and b is not fixed, or $\int_{\mathcal{C}_4(x^{11})} \widetilde{I}_{4,1} = n_1 - \frac{1}{2} p_1 \neq 0$ with $n_1, p_1 \in \mathbb{Z}$ [34] and $b = 1$. In both cases the integral of G_4 over $\mathcal{C}_4$ reads

$$\frac{2}{\gamma} \int_{\mathcal{C}_4} G_4 = \left(n_1 - \frac{1}{2} p_1\right) = -\left(n_2 - \frac{1}{2} p_2\right), \tag{4.82}$$

for any four-cycle $\mathcal{C}_4$ not wrapping S^1. This result looks like Witten's flux quantization [232] which was obtained in the downstairs approach and reads

$$\frac{1}{\gamma_{\text{down}}} \int_{\mathcal{C}_4} G_4 = \left(n_1 - \frac{1}{2} p_1\right). \tag{4.83}$$

[34] See Eq. (C.19) in Appendix C.

The slight disagreement comes from the fact that the gravitational couplings κ_{11}^2 in the upstairs and downstairs approach precisely differ by a factor of two [49]:

$$\kappa_{11,\text{down}}^2 = \frac{1}{2}\kappa_{11,\text{up}}^2, \tag{4.84}$$

and since $\gamma = -(4\pi)^2\kappa_{11}^2/\lambda^3$ one has also $\gamma_{\text{down}} = \gamma_{\text{up}}/2$. Of course, in all our formulas $\gamma \equiv \gamma_{\text{up}}$ and one sees that Eq. (4.82) is exactly Witten's flux quantization.

Four-cycles wrapping S^1

Now we consider four-cycles of the form $\mathcal{C}_4 = \mathcal{C}_3 \times S^1$ where $\mathcal{C}_3$ is some three-cycle at a fixed value of x^{11}. Such cycles do not exist in the downstairs approach and we cannot expect the corresponding flux to be related to Witten's quantization condition. Inserting the solution (4.25), the integration of G_4 over $\mathcal{C}_4$ gives

$$\begin{aligned}\textstyle\int_{\mathcal{C}_3\times S^1} G_4 &= \int_{\mathcal{C}_3\times S^1}\left(dC_3 + \gamma\sum_{i=1}^{2}\left((b-1)\delta_{1,i} - \frac{b}{2\pi}dx^{11}\right)\wedge\widetilde{\omega}_{3,i}\right)\\ &= \int_{\mathcal{C}_3\times S^1} dC_3 - \gamma\int_{\mathcal{C}_3}\sum_{i=1}^{2}\widetilde{\omega}_{3,i}.\end{aligned} \tag{4.85}$$

In the small radius limit (as discussed in subsection 4.2.5), this integral exactly reduces to the flux $\int \widehat{H}_3 = \int (d\widehat{B}_2 - \Omega_{3,\text{YM}} + \Omega_{3,\text{L}})$ in the heterotic superstring. The flux of $\widehat{H}_3$ was studied some time ago by Rohm and Witten [180] and these authors concluded that (in an appropriate normalization) it was of the form $n + \eta$ with $n \in \mathbb{Z}, \eta \in \mathbb{R}$. A similar argument holds here and we cannot say anything interesting about the value of (4.85).

4.4.2 A remark on the membrane action

Let us first review the standard argument [232]. In uncompactified M-theory or in M-theory on a smooth manifold one has simply $G_4 = dC_3$. Since this needs not hold globally one typically argues that

$$\int_{\mathcal{C}_4} G_4 = \int_{\mathcal{C}_4^+} dC_3^{(+)} + \int_{\mathcal{C}_4^-} dC_3^{(-)} = \int_{\mathcal{C}_3} (C_3^{(+)} - C_3^{(-)}), \tag{4.86}$$

where $\mathcal{C}_3 = \partial\mathcal{C}_4^+ = -\partial\mathcal{C}_4^-$. But the membrane action $T_2\int_{\mathcal{C}_3} C_3 + \cdots$ (T_2 is the membrane tension) should be well-defined modulo 2π, and one concludes that $T_2\int_{\mathcal{C}_4} G_4 = 2\pi n$ with integer n. Witten has argued [232] that the three-dimensional membrane theory does in certain cases have a so-called parity anomaly [3] which is a sign ambiguity $e^{i\pi p}, p \in \mathbb{Z}$ of the fermion determinant. This implies that actually the well-definedness of the membrane functional integral requires $T_2\int_{\mathcal{C}_4} G_4 = 2\pi(n - \frac{1}{2}p)$. This fits with the flux quantization since $T_2 = -2\pi/\gamma$.[35] However, in our present upstairs discussion, this relation for the G_4-flux holds (with the appropriate redefinition of κ_{11}^2 by a factor of two as discussed above)

[35] This relation can be obtained from the five-brane tension using the Dirac quantization condition.

if $\mathcal{C}_4$ does not wrap the S^1, but is replaced by (4.85) if it does. Of course, this does not mean that the membrane functional integral is no longer well-defined. First, the above argument is spoiled since $G_4 \neq dC_3$ everywhere. Second, a coupling of C_3 to the membrane world-volume $\mathcal{C}_3$ of the type $\int_{\mathcal{C}_3} C_3$ without modification certainly does not lead to a well-defined functional integral since we have observed that C_3 is neither gauge nor local Lorentz invariant. Obviously then there must be corrective terms to restore the invariance. It would be interesting to explicitly construct these terms. A clue is probably provided by Eq. (4.85) which is gauge and local Lorentz invariant.

4.5 The heterotic anomaly-cancelling terms

In this section, we will consider the small-radius limit (srl), $r \to 0$, and compute the anomaly-cancelling terms of the heterotic superstring. We consider only the zero modes along the circle. In particular, we replace C_{AB11} by its zero mode $C^{(0)}_{AB11} = (2\pi r)^{-1} \int_{S^1} dx^{11}\, C_{AB11} \overset{\text{def.}}{=} B_{AB}$ [see Eq. (4.40)]. We must also keep in mind that $C_{ABC} = 0$. Moreover, only the restriction $\widetilde{X}_{7,1} \overset{\text{srl}}{=} \widetilde{X}_{7,2} \overset{\text{srl}}{=} \widetilde{X}_7$ to $\mathcal{M}_{10}$ of the seven-form X_7 in $\mathcal{M}_{11}$ survives, and similarly for its exterior derivative $X_8 = dX_7$: $\widetilde{X}_{8,1} \overset{\text{srl}}{=} \widetilde{X}_{8,2} \overset{\text{srl}}{=} \widetilde{X}_8$ (with $\widetilde{R}_{2,1} \overset{\text{srl}}{=} \widetilde{R}_{2,2} \overset{\text{srl}}{=} R_2$).

Reduction of the topological term

Using Eq. (4.12) to perform the integral over S^1 (with the reintroduction of an arbitrary radius r), and Eq. (4.69) to reexpress b^2 in terms of λ^6/κ_{11}^4, the reduction of the topological term is

$$
\begin{aligned}
S_{\text{top}} &= -\frac{1}{12\kappa_{11}^2} \int_{\mathcal{M}_{11}} C_3 \wedge G_4 \wedge G_4 \\
&= -\frac{b^2\gamma^2}{48\kappa_{11}^2} \int_{\mathcal{M}_{11}} C_3 \wedge \Big(\sum_{i=1}^{2} \varepsilon_i \widetilde{I}_{4,i} \Big) \wedge \Big(\sum_{j=1}^{2} \varepsilon_j \widetilde{I}_{4,j} \Big) \\
&\overset{\text{srl}}{=} -\frac{2\pi r b^2\gamma^2}{144\kappa_{11}^2} \int_{\mathcal{M}_{10}} B_2 \wedge \Big((\widetilde{I}_{4,1})^2 + (\widetilde{I}_{4,2})^2 - \widetilde{I}_{4,1} \wedge \widetilde{I}_{4,2} \Big) \qquad (4.87) \\
&= -\frac{1}{12(4\pi)^5} \int_{\mathcal{M}_{10}} \widehat{B}_2 \wedge \Big[(\mathrm{tr} F_{2,1}^2)^2 + (\mathrm{tr} F_{2,2}^2)^2 - \mathrm{tr} F_{2,1}^2 \wedge \mathrm{tr} F_{2,2}^2 \\
&\qquad + \frac{1}{4} (\mathrm{tr} R_2^2)^2 - \frac{1}{2} \mathrm{tr} R_2^2 \wedge (\mathrm{tr} F_{2,1}^2 + \mathrm{tr} F_{2,2}^2) \Big].
\end{aligned}
$$

To obtain the last equality, we have introduced the ten-dimensional gravitational coupling constant $\kappa_{10}^2 = \kappa_{11}^2/(2\pi r)$ and rescaled B_2 to the heterotic $\widehat{B}_2$ as in Eq. (4.44).

Reduction of the Green–Schwarz term

Inserting the expression (4.25) for G_4 (with the reintroduction of an arbitrary radius r through the replacement of $(2\pi)^{-1}$ by $(2\pi r)^{-1}$), the reduction of the Green–Schwarz term

(4.63) [with $c=1$] gives

$$
\begin{aligned}
S_{\rm GS} &= \frac{1}{\gamma}\int_{\mathcal{M}_{11}} G_{\mathit{4}}\wedge X_{\mathit{7}} \\
&\overset{\rm srl}{=} \frac{2\pi r}{\gamma}\int_{\mathcal{M}_{10}} B_{\mathit{2}}\wedge \widetilde{X}_{\mathit{8}} + \sum_{i=1}^{2}\int_{\mathcal{M}_{11}}\Big((b-1)\delta_{\mathit{1},i}\wedge\widetilde{\omega}_{\mathit{3},i} - \frac{b}{2\pi r}dx^{11}\wedge\widetilde{\omega}_{\mathit{3},i}\Big)\wedge \widetilde{X}_{\mathit{7}} \\
&= -\frac{2\pi r\lambda^2}{(4\pi)^2\kappa_{11}^2}\int_{\mathcal{M}_{10}} B_{\mathit{2}}\wedge\widetilde{X}_{\mathit{8}} - \sum_{i=1}^{2}\int_{\mathcal{M}_{10}}\widetilde{\omega}_{\mathit{3},i}\wedge\widetilde{X}_{\mathit{7}} \\
&= -\frac{1}{12(4\pi)^5}\int_{\mathcal{M}_{10}}\widehat{B}_{\mathit{2}}\wedge\Big(\frac{1}{2}{\rm tr}R_{\mathit{2}}^4 - \frac{1}{8}({\rm tr}R_{\mathit{2}}^2)^2\Big) - \sum_{i=1}^{2}\int_{\mathcal{M}_{10}}\widetilde{\omega}_{\mathit{3},i}\wedge\widetilde{X}_{\mathit{7}}.
\end{aligned}
\tag{4.88}
$$

Now, we define[36] the heterotic eight- and seven-forms $\widehat{X}_{\mathit{8}}$ and $\widehat{X}_{\mathit{7}}$ (with $d\widehat{X}_{\mathit{7}} = \widehat{X}_{\mathit{8}}$):

$$
\begin{aligned}
\widehat{X}_{\mathit{8}} &= ({\rm tr}F_{\mathit{2},1}^2)^2 + ({\rm tr}F_{\mathit{2},2}^2)^2 - {\rm tr}F_{\mathit{2},1}^2\wedge{\rm tr}F_{\mathit{2},2}^2 - \frac{1}{2}{\rm tr}R_{\mathit{2}}^2\wedge({\rm tr}F_{\mathit{2},1}^2 + {\rm tr}F_{\mathit{2},2}^2) \\
&\quad + \frac{1}{2}{\rm tr}R_{\mathit{2}}^4 + \frac{1}{8}({\rm tr}R_{\mathit{2}}^2)^2, \\
\widehat{X}_{\mathit{7}} &= \Omega_{\mathit{3},1}\wedge{\rm tr}F_{\mathit{2},1}^2 + \Omega_{\mathit{3},2}\wedge{\rm tr}F_{\mathit{2},2}^2 - \frac{1}{2}\Omega_{\mathit{3},1}\wedge{\rm tr}F_{\mathit{2},2}^2 - \frac{1}{2}\Omega_{\mathit{3},2}\wedge{\rm tr}F_{\mathit{2},1}^2 \\
&\quad - \frac{1}{4}\Omega_{\mathit{3},\rm L}\wedge({\rm tr}F_{\mathit{2},1}^2 + {\rm tr}F_{\mathit{2},2}^2) - \frac{1}{4}(\Omega_{\mathit{3},1} + \Omega_{\mathit{3},2})\wedge{\rm tr}R_{\mathit{2}}^2 + \frac{1}{2}\Omega_{\mathit{7},\rm L} + \frac{1}{8}\Omega_{\mathit{3},\rm L}\wedge{\rm tr}R_{\mathit{2}}^2.
\end{aligned}
\tag{4.89}
$$

In the expression of $\widehat{X}_{\mathit{7}}$, the gauge and gravitational Chern–Simons forms $\Omega_{\mathit{3},i}$, $\Omega_{\mathit{3},\rm L}$ and $\Omega_{\mathit{7},\rm L}$ are such that $d\Omega_{\mathit{3},i} = {\rm tr}F_{\mathit{2},i}^2$, $d\Omega_{\mathit{3},\rm L} = {\rm tr}R_{\mathit{2}}^2$ and $d\Omega_{\mathit{7},\rm L} = {\rm tr}R_{\mathit{2}}^4$. Contrary to $\widetilde{X}_{\mathit{7}}$ and $\widetilde{X}_{\mathit{8}}$ which contain only gravitational pieces, $\widehat{X}_{\mathit{7}}$ and $\widehat{X}_{\mathit{8}}$ have gravitational, gauge and mixed terms. With these definitions, the sum of the heterotic anomaly-cancelling terms (4.87) and (4.88) can be written

$$
\begin{aligned}
S_{\rm GS,het} &= -\frac{1}{12(4\pi)^5}\int_{\mathcal{M}_{10}}\Big(\widehat{B}_{\mathit{2}}\wedge\widehat{X}_{\mathit{8}} + (\Omega_{\mathit{3},1}+\Omega_{\mathit{3},2}-\Omega_{\mathit{3},\rm L})\wedge\big(\frac{1}{2}\Omega_{\mathit{7},\rm L} - \frac{1}{8}\Omega_{\mathit{3},\rm L}\wedge{\rm tr}R_{\mathit{2}}^2\big)\Big) \\
&= -\frac{1}{12(4\pi)^5}\int_{\mathcal{M}_{10}}\Big(\widehat{B}_{\mathit{2}}\wedge\widehat{X}_{\mathit{8}} + (\Omega_{\mathit{3},\rm YM}-\Omega_{\mathit{3},\rm L})\wedge\widehat{X}_{\mathit{7}}\Big) \\
&\quad + \frac{1}{8(4\pi)^5}\int_{\mathcal{M}_{10}}(\Omega_{\mathit{3},1} - \frac{1}{2}\Omega_{\mathit{3},\rm L})\wedge(\Omega_{\mathit{3},2} - \frac{1}{2}\Omega_{\mathit{3},\rm L})\wedge({\rm tr}F_{\mathit{2},2}^2 - {\rm tr}F_{\mathit{2},1}^2),
\end{aligned}
\tag{4.90}
$$

where $\Omega_{\mathit{3},\rm YM} = \Omega_{\mathit{3},1} + \Omega_{\mathit{3},2}$. In the last equality, the first line displays the standard Green–Schwarz counterterm of the heterotic superstring [108]. The second line is a local counterterm specific to the structure of the anomaly-cancelling term generated by M-theory on $S^1/\mathbb{Z}_2$.

[36] Note that the eight-form used in the textbook by Green, Schwarz and Witten [108] is $X_{\mathit{8}}^{\rm GSW} = \frac{1}{4}\widehat{X}_{\mathit{8}}$.

In terms of the 10-form Δ_{10} introduced in Eqs. (4.59-4.61), this counterterm reads

$$\frac{\pi}{2}\int_{\mathcal{M}_{10}} \Delta_{10} = \frac{3}{2}\left(-\frac{\gamma^3 b^3}{144\kappa_{11}^2}\int_{\mathcal{M}_{10}} \Delta_{10}\right), \tag{4.91}$$

with $b = 1$. As already explained in section 4.2.4, the small-radius limit automatically produces this term with a coefficient corresponding to $b = 1$. The extra factor of 3/2 with respect to Eq. (4.61) is due to the usual choice of including only part of this term, namely $-1/2$ of it, into the standard heterotic counterterm of (4.90), leaving us with a factor of $+1/2 + 1 = 3/2$.

4.6 Summary

In this chapter, we have focussed on the resolution of the Bianchi identity for the four-form invariant field strength of M-theory on the orbifold $S^1/\mathbb{Z}_2$. In the upstairs approach, the Bianchi identity is actually modified by fixed planes (twisted sectors) and five-branes contributions:

$$dG_4 = \gamma \sum_{i=1}^{2} \delta_{1,i} \wedge \widetilde{I}_{4,i} + \gamma \sum_{f=1}^{N_5} \delta_5(\mathcal{W}_{6,f}), \tag{4.92}$$

with $\gamma = -(4\pi)^2\kappa_{11}^2/\lambda^2$. For five-branes perpendicular to the circle S^1,[37] we obtain a solution that depends on two parameters b and β:

$$\begin{aligned} G_4 \;=\; & dC_3 + (b-1)\gamma \sum_{i=1}^{2} \delta_{1,i} \wedge \widetilde{\omega}_{3,i} + \frac{b}{2}\gamma \sum_{i=1}^{2} \varepsilon_i \widetilde{I}_{4,i} - \frac{b}{2\pi}\gamma dx^{11} \wedge \sum_{i=1}^{2} \widetilde{\omega}_{3,i} \\ & + \gamma \sum_{f=1}^{N_5} \Big[\frac{\beta}{2}\Big(\varepsilon_1(x^{11} - x_f^{11})\delta_4(\mathcal{W}_{6,f}) - \frac{1}{\pi} dx^{11} \wedge \theta_3(\mathcal{W}_{6,f})\Big) \\ & \qquad -(1-\beta)\delta(x^{11} - x_f^{11}) dx^{11} \wedge \theta_3(\mathcal{W}_{6,f})\Big]. \end{aligned} \tag{4.93}$$

Imposing that the four-form G_4 should be globally well-defined (i.e. invariant under small and large gauge and local Lorentz transformations), the parameter b appears to be equal to 1 in all topologically non-trivial sectors of the theory (i.e. with non-vanishing gauge/gravitational "instanton numbers" and/or with five-branes). The other parameter, β, must then also be set to 1.

On the other hand, in the absence of five-branes, the cancellation of the one-loop anomaly generated by the $\mathbb{Z}_2$ orbifold projection brings us to a relation between the parameter b and the gauge and gravitational coupling constants: $b^2 = 12(4\pi)^{-5}\lambda^6/\kappa_{11}^4$. The derivation of this equation turns out to be independent of the topological sector considered, and in topologically non-trivial sectors (where $b = 1$) we have

$$\frac{\lambda^6}{\kappa_{11}^4} = \frac{(4\pi)^5}{12}. \tag{4.94}$$

[37] The five-branes which do not wrap the circle are the only ones relevant for M-theory on $S^1/\mathbb{Z}_2$. Their source terms $\delta_5(\mathcal{W}_{6,f})$ can be decomposed as $\delta(x^{11} - x_f^{11})dx^{11} \wedge \delta_4(\mathcal{W}_{6,f})$.

Since this ratio should not depend on the topological sector considered, we are led to admit that $b = 1$ always. Global well-definedness of G_4 and anomaly cancellation thus allow to select a particular solution to the modified Bianchi identity:

$$\begin{aligned} G_4 &= dC_3 + \frac{\gamma}{2}\sum_{i=1}^{2}\varepsilon_i \widetilde{I}_{4,i} - \frac{\gamma}{2\pi} dx^{11} \wedge \sum_{i=1}^{2} \widetilde{\omega}_{3,i} \\ &\quad + \frac{\gamma}{2}\sum_{f=1}^{N_5}\Big[\varepsilon_1(x^{11} - x^{11}_f)\delta_4(\mathcal{W}_{6,f}) - \frac{1}{\pi} dx^{11} \wedge \theta_3(\mathcal{W}_{6,f})\Big]. \end{aligned} \tag{4.95}$$

We have seen that the value $b = 1$ is also very natural when considering the reduction to the heterotic superstring in the small radius limit, since it is only for $b = 1$ that all higher modes of the Fourier expansion on the circle can be consistently decoupled. We explicitly wrote out the anomaly-cancelling terms obtained from this reduction. They differ from what is usually taken in the heterotic superstring by the addition of a well-defined local counterterm. Moreover, we observed that the flux of G_4 automatically obeys Witten's flux quantization for four-cycles not wrapping S^1, while for cycles wrapping the circle the flux is more general. We have also stressed in passing that the naive action for membranes wrapping S^1 is not gauge/local Lorentz invariant and needs to be modified.

Anomaly cancellations in the presence of five-branes are quite subtle. The one-loop anomaly is correctly cancelled by the variation of the Green–Schwarz term, but there is an extra contribution from the topological interaction term signalling that non-trivial things are happening on the five-branes. In the two following chapters, we are going to study further compactifications on Calabi–Yau manifolds of the simple $S^1/\mathbb{Z}_2$ orbifold of M-theory we have considered here. We will then obtain five-brane contributions to the four-dimensional effective action that are very likely related to the extra anomaly found above.

Chapter 5

On the effective $N = 1$ supergravity of M-theory in four dimensions

M-theory compactified on $\mathcal{K}_7 = S^1/\mathbb{Z}_2 \times \mathcal{X}_6$, where $\mathcal{X}_6$ is a Calabi–Yau three-fold, leads to a four-dimensional theory with $N = 1$ local supersymmetry. At present, our knowledge of the resulting effective supergravity is based on the few aspects of M-theory which are quantitatively understood and on the small-orbifold limit[1] which is the perturbative heterotic $E_8 \times E_8$ superstring compactified on the Calabi–Yau space $\mathcal{X}_6$.

In the low-energy limit, M-theory information can be organized as an expansion in powers of the eleven-dimensional gravitational constant κ_{11} [116, 117], the lowest order κ_{11}^{-2} being eleven-dimensional supergravity [51]. As we have seen in the preceding chapter for a compactification on $S^1/\mathbb{Z}_2$ only, the next orders in κ_{11} do include orbifold plane contributions (super Yang–Mills terms) as well as gauge and gravitational anomaly-cancelling terms [116, 230, 117].

Similarly, the effective four-dimensional supergravity (which is a low-energy description) can be formulated as an expansion in the four-dimensional gravitational constant κ, even if a more common choice suggested by string theory is to use the dilaton as expansion parameter. The lowest order κ^{-2} is the $\mathcal{K}_7$ truncation of eleven-dimensional supergravity. The next order includes super Yang–Mills, charged matter kinetic and superpotential contributions. Then come sigma-model anomaly-cancelling terms contributing in particular to gauge threshold corrections. These first corrections to the low-energy limit of compactified M-theory are identical to those obtained from heterotic compactifications on Calabi–Yau. This is certainly expected since the information content is identical. The literature gives a detailed description of these results, with particular attention paid to the "strong-coupling" heterotic limit in which the size of the Calabi–Yau space is smaller than the orbifold length, supersymmetry breaking by gaugino condensation and non-standard embeddings [14, 36, 115, 7, 8, 9, 159, 70, 133, 137, 143, 82, 83, 157, 158, 136, 132, 138].

Our goal in this chapter is to provide a derivation of the effective supergravity which explicitly relates four-dimensional supergravity statements with M-theory aspects like Bianchi identities modified at singularities and anomaly cancellation. We reformulate these basic

[1] Here the small-orbifold limit refers to the limit in which the length of the $S^1/\mathbb{Z}_2$ orbifold direction is small compared to the size of the Calabi–Yau manifold.

facts of M-theory on $\mathcal{K}_7$ directly in terms of four-dimensional supermultiplets and equations. For instance, Bianchi identities from M-theory are promoted to field equations, as constraints on multiplets which are massless modes of M-theory bulk fields. With this formulation, we expect to obtain a clean, direct derivation of the effective supergravity suitable to cases more subtle than the universal modes of M-theory on $\mathcal{K}_7$. A first use of our formalism (the coupling of five-brane moduli supermultiplets) will be presented in the next chapter.

In section 5.1, we establish our basic supergravity formulation, using the bosonic bulk dynamics as a starting point. The resulting Lagrangian is the lowest order in the κ-expansion. It is essentially defined by a dynamical Lagrangian involving tensor fields supplemented by Bianchi identities which are field equations of the theory. We discuss in detail the bosonic component expansion of the supermultiplet action, the question of the gravity frame and the generation of a superpotential. In section 5.2, we introduce the next order corrections, namely gauge multiplets and charged matter contributions. We show that their introduction is controlled by a simple modification of the four-dimensional Bianchi identities, in analogy with the appearance of $\mathbb{Z}_2$ fixed planes contributions in the M-theory Bianchi identities. Section 5.3 discusses anomaly-cancelling terms. We begin by modifying the four-dimensional effective supergravity by adding terms similar to those appearing for gauge threshold corrections in $(2, 2)$ compactifications of the heterotic superstring. These modifications can be formulated in terms of our particular set of multiplets related to M-theory bulk degrees of freedom. We then directly compute these anomaly-cancelling terms by Kaluza–Klein reduction of the ten-dimensional Green–Schwarz counterterms arising from M-theory on $S^1/\mathbb{Z}_2$. Section 5.4 gives some final comments and Appendix D contains some elementary information about the supergravity tensor calculus formalism we employ in the present (and subsequent) chapter.

5.1 The Bulk Lagrangian

Our concern is compactifications of M-theory to four dimensions preserving $N = 1$ supersymmetry. Or compactifications in which supersymmetry would break spontaneously or dynamically at solutions of the effective field equations. As a consequence, the light (massless) modes can be described by a local effective $N = 1$ supergravity Lagrangian, to be understood in the sense of Wilson. In this section, we restrict ourselves to the well-known "bulk dynamics", which follows from $\mathcal{K}_7$ compactification of eleven-dimensional supergravity. The Lagrangian we obtain is the lowest order in the κ-expansion. It describes Kaluza–Klein massless modes of eleven-dimensional supergravity.

We focus on a precise description of two aspects which may be of importance in M-theory compactifications. Firstly, we introduce chiral, linear or vector supermultiplets with constraints in order to obtain a supersymmetric version of the Bianchi identities present in the theory. Secondly, we use the superconformal tensor calculus for $N = 1$ supergravity in which we can easily keep control of a change of gravity frame. This can be a relevant issue since an expansion, perturbative or not, is performed around a gravitational background which selects a gravity frame. Standard Poincaré supergravity is usually written in the Einstein frame, in which the gravitational Lagrangian is $-\frac{1}{2\kappa^2}eR$. Corrections to the lowest order effective action, which includes this gravitational term, induce in general (but not always) corrections to the gravitational Lagrangian which affect the Einstein frame condition.

5.1.1 Superconformal Formalism

We choose to use the superconformal formulation of $N = 1$ supergravity with a chiral compensating multiplet S_0 to generate Poincaré theories by gauge fixing.[2] S_0 has conformal and chiral weights $w = 1$ and $n = 1$. In this formalism, a change of frame[3] corresponds to a different Poincaré gauge condition applied on the modulus of the scalar compensator z_0, which fixes dilatation symmetry. Up to terms with more than two derivatives and up to terms which would contribute to kinetic terms in a fermionic background only [50, 56], the most general supergravity Lagrangian reads[4]

$$\mathcal{L} = \left[S_0\overline{S}_0\Phi\right]_D + \left[S_0^3 W\right]_F + \frac{1}{4}\left[f_{ab}\mathcal{W}^a\mathcal{W}^b\right]_F . \tag{5.1}$$

The symbols $[\ldots]_D$ and $[\ldots]_F$ denote respectively the invariant D- and F-density formulas given by (all fermion contributions are omitted)

$$[V]_D = e(d + \frac{1}{3}cR) \qquad \text{and} \qquad [C]_F = e(f + \overline{f}), \tag{5.2}$$

where V is a vector multiplet with components $(c, \chi, h, k, v_\mu, \lambda, d)$ and C a chiral multiplet with components (z, ψ, f). The real vector multiplet Φ (with zero weights) is a function (in the sense of tensor calculus) of the multiplets present in the theory, including in general the compensating multiplet. The holomorphic function W of the chiral multiplets is the superpotential. The chiral multiplet $\mathcal{W}$ is the gauge field strength for the gauge multiplets and f_{ab} is the holomorphic gauge kinetic function of the chiral multiplets. Besides S_0 and $\mathcal{W}$, we use chiral multiplets with zero weights and neither W nor f_{ab} depends on the compensator.

The chiral $U(1)$ symmetry of the superconformal algebra can be extended to

$$S_0, \quad W, \quad \Phi \quad \longrightarrow \quad \Lambda S_0, \quad \Lambda^{-3}W, \quad (\Lambda\overline{\Lambda})^{-1}\Phi\big|_{S_0\to\Lambda S_0}, \tag{5.3}$$

with an arbitrary chiral multiplet Λ. This symmetry is at the origin of Kähler invariance of Poincaré supergravity. The last transformation suggests that $\log\Phi$ transforms as the corresponding gauge connection. Choosing $\Lambda = W^{1/3}$ eliminates the superpotential (except if it vanishes). It is then possible to use a $U(1)$/Kähler gauge fixing in which the supergravity Lagrangian (5.1) reads

$$\mathcal{L} = \left[S_0\overline{S}_0\Phi\right]_D + c\left[S_0^3\right]_F + \frac{1}{4}\left[f_{ab}\mathcal{W}^a\mathcal{W}^b\right]_F, \tag{5.4}$$

with an arbitrary constant c as superpotential and two arbitrary functions Φ and f_{ab}.

[2] This corresponds to the so-called "old minimal" Poincaré supergravity [208, 90, 87].

[3] Mostly the Einstein or string frames.

[4] Except otherwise mentioned, our notations for superconformal expressions are as in Refs. [130, 131], from where the original literature can be traced back.

The real function Φ depends on matter multiplets, which are either chiral multiplets like the Calabi–Yau universal modulus T, or real linear multiplets (with weights $w=2$, $n=0$) like the dilaton multiplet in the version of the theory with an antisymmetric tensor, or real vector multiplets ($n=0$, w arbitrary) like the multiplet of gauge potentials ($w=0$).

5.1.2 Supermultiplets with constraints

The lowest order (in the κ-expansion) effective four-dimensional supergravity of M-theory compactified on $\mathcal{K}_7$ describes Kaluza–Klein massless modes of eleven-dimensional supergravity. It is the $S^1/\mathbb{Z}_2$ truncation of eleven-dimensional supergravity on a Calabi–Yau three-fold.

The Lagrangian of eleven-dimensional supergravity can be written as [51]

$$\begin{aligned}\mathcal{L}_{\rm CJS} &= \frac{1}{2\kappa_{11}^2}\Big[-e_{11}R-\frac{1}{2\cdot 4!}e_{11}G_{M_1M_2M_3M_4}G^{M_1M_2M_3M_4} \\ &\quad -\frac{1}{6}\frac{1}{3!4!4!}\epsilon^{M_1\ldots M_{11}}C_{M_1M_2M_3}G_{M_4M_5M_6M_7}G_{M_8M_9M_{10}M_{11}}\Big] \\ &\quad + \text{fermionic terms.}\end{aligned} \tag{5.5}$$

Omitting all fields related to the detailed geometry of the Calabi–Yau manifold, the particle content of the four-dimensional theory is the $N=1$ supergravity multiplet, with metric tensor $g_{\mu\nu}$, and matter multiplets including (on-shell) four bosons and four fermions. Two bosons are scalars and correspond to the dilaton and the "universal modulus" of the Calabi–Yau space (the massless volume mode). Two bosons are Kaluza–Klein modes of the four-form field G_4, with Bianchi identity $dG_4=0$. Explicitly, these two last fields and their Bianchi identities read

$$\begin{aligned} &G_{\mu\nu\rho 11}, \qquad \partial_{[\mu}G_{\nu\rho\sigma 11]}=0, \\ &G_{\mu j\bar{k}11}, \qquad \partial_{[\mu}G_{\nu j\bar{k}11]}=0.\end{aligned} \tag{5.6}$$

We will identify these fields with the vector components of two real vector multiplets V (with weights $w=2$ and $n=0$) and V_T ($w=0=n$), and impose the Bianchi identities as field equations using a chiral multiplet S ($w=0=n$) and a real linear multiplet L_T ($w=2$, $n=0$) as Lagrange multipliers. The bulk supergravity Lagrangian takes then the form

$$\mathcal{L}_{\rm B}=\left[-\frac{1}{\sqrt{2}}(S_0\overline{S}_0V_T)^{3/2}V^{-1/2}-(S+\overline{S})V+L_TV_T\right]_D. \tag{5.7}$$

To analyze the Lagrangian (5.7), we will need to define some notations. Since we will only explicitly consider the bosonic sector of the theory, all fermions in the $N=1$ supermultiplets will be omitted. Since also we are concerned with Poincaré supergravity, we will immediately gauge-fix the superconformal symmetries not contained in $N=1$ Poincaré supersymmetry, with one exception, dilatation symmetry: we want to keep the freedom of a frame choice as explicit as possible. These assumptions imply in particular that superconformal covariant derivatives reduce in general to $D^c_\mu\phi=D_\mu\phi-\frac{1}{2}inA_\mu\phi$ for a complex field with chiral weight n and to

$$\Box^c\varphi=\Box\varphi+\frac{1}{6}w\varphi R, \tag{5.8}$$

for a real field φ with Weyl weight w. The gauge boson A_μ of chiral $U(1)$ symmetry is auxiliary and D_μ and $\Box$ would be covariantized with respect to Poincaré symmetries.

The various superconformal multiplets appearing in the Lagrangian (5.7) have the following components expressions[5]

$$\begin{aligned} V &= (C, 0, H, K, v_\mu, 0, d - \Box C - \tfrac{1}{3}CR), \\ V_T &= (C_T, 0, H_T, K_T, T_\mu, 0, d_T - \Box C_T), \\ S &= (s, 0, -f, if, i\partial_\mu s, 0, 0), \\ L_T &= (\ell_T, 0, 0, 0, t_\mu, 0, -\Box\ell_T - \tfrac{1}{3}\ell_T R), \quad t_\mu = \tfrac{1}{2}e\epsilon_{\mu\nu\rho\sigma}\partial^\nu t^{\rho\sigma}, \\ S_0 &= (z_0, 0, -f_0, if_0, iD^c_\mu z_0, 0, 0). \end{aligned} \tag{5.9}$$

The role of the Lagrange multipliers S and L_T follows from the two relations

$$\begin{aligned} e^{-1}[(S+\overline{S})V]_D &= 2(\partial^\mu \operatorname{Im} s)v_\mu - 2\partial^\mu(\operatorname{Re} s\partial_\mu C) + 2d \operatorname{Re} s \\ &\quad -f(H - iK) - \overline{f}(H + iK) \\ &= -2\operatorname{Im} s\partial^\mu v_\mu + 2d\operatorname{Re} s - f(H-iK) - \overline{f}(H+iK) \\ &\quad + \text{derivative}, \\ e^{-1}[L_T V_T]_D &= \ell_T(d_T - \Box C_T) - \tfrac{1}{2}e\epsilon_{\mu\nu\rho\sigma}(\partial^\mu T^\nu)t^{\rho\sigma} + \text{derivative}. \end{aligned} \tag{5.10}$$

Solving for the components of S leads to $\partial^\mu v_\mu = d = H = K = 0$, and V is a linear multiplet L ($w = 2$, $n = 0$). Solving for the components of L_T leads to $d_T - \Box C_T = \partial_{[\mu}T_{\nu]} = 0$, and V_T can be written as $T + \overline{T}$, with a chiral weightless T.[6] Since one can always write $v_\mu = \frac{1}{6}e\epsilon_{\mu\nu\rho\sigma}v^{\nu\rho\sigma}$, we have generated with $\operatorname{Im} s$ and $t_{\mu\nu}$ the Bianchi identities

$$\partial_{[\mu}v_{\nu\rho\sigma]} = \partial_{[\mu}T_{\nu]} = 0. \tag{5.11}$$

A modification of these Bianchi identities, as induced by $S^1/\mathbb{Z}_2$ compactification or by five-brane couplings will then be phrased as a modification of the supermultiplets appearing multiplied by $S + \overline{S}$ or L_T in Eqs. (5.10).

The usefulness of obtaining Bianchi identities via field equations will become apparent with the introduction of higher orders in the κ-expansion. At this stage of the discussion however, it gives a formulation of the familiar duality relating scalars and antisymmetric tensors or, for superfields, chiral and linear multiplets.

Solving in Eq. (5.7) for the Lagrange multipliers S and L_T leads to the "standard form" of the bulk four-dimensional Lagrangian [44, 59]

$$\mathcal{L}_{\text{B},1} = -\frac{1}{\sqrt{2}}\left[\left(S_0\overline{S}_0 e^{-\widehat{K}/3}\right)^{3/2} L^{-1/2}\right]_D, \tag{5.12}$$

[5]Notice that our component expansion of vector multiplets differs in its highest component from Refs. [130, 131].

[6]The complex components T and f_T of the chiral multiplet T are then given by the relations $C_T = 2\operatorname{Re} T$, $T_\mu = -2\partial_\mu \operatorname{Im} T$, $H_T = -2\operatorname{Re} f_T$ and $K_T = -2\operatorname{Im} f_T$.

or, as defined in Eq. (5.1),

$$\Phi = -\left(\frac{2L}{S_0\overline{S}_0}\right)^{-1/2} e^{-\widehat{K}/2}. \tag{5.13}$$

The Kähler potential for the volume modulus T is

$$\widehat{K} = -3\log(T+\overline{T}). \tag{5.14}$$

We will see again below that this standard form is naturally obtained by direct reduction of the CJS version of eleven-dimensional supergravity on $\mathcal{K}_7$. Clearly, theory (5.12) is also the Calabi–Yau truncation of ten-dimensional $N=1$ pure supergravity [44]. Notice that $\widehat{K}$ can be regarded as the Kähler connection for symmetry (5.3), with transformation

$$\widehat{K} \quad \longrightarrow \quad \widehat{K} + 3\log\Lambda + 3\log\overline{\Lambda}, \tag{5.15}$$

such that $S_0\overline{S}_0 e^{-\widehat{K}/3}$ is chiral/Kähler invariant.

A theory with a linear multiplet is in principle dual to an equivalent Lagrangian with the linear multiplet replaced by a chiral one. In our case, solving for V and L_T in expression (5.7) leads to

$$\mathcal{L}_{\rm B,c} = -\frac{3}{2}\left[S_0\overline{S}_0 e^{-K/3}\right]_D, \tag{5.16}$$

with the Kähler potential

$$K = -\log(S+\overline{S}) + \widehat{K} = -\log(S+\overline{S}) - 3\log(T+\overline{T}). \tag{5.17}$$

This familiar chiral form [227] is not the most useful as long as one insists on the four-dimensional translation of eleven-dimensional Bianchi identities.

Notice that one can obtain another equivalent form of the Lagrangian (5.7) by choosing to solve for S and V_T. In this case, the Calabi–Yau modulus is described by a linear multiplet L_T. This form will not be useful since it is known that one-loop string corrections in general break the chiral-linear duality for this modulus: they involve holomorphic functions of T in a F-density which are intrinsically chiral [69]. Finally, there is an obstruction when trying to solve for V and V_T and one cannot write an expression in terms of the chiral S and the linear L_T.

Before turning to explicit component expressions, we should discuss the choice of Poincaré frame, and introduce the expansion in the four-dimensional gravitational coupling κ, which effectively corresponds to the low-energy expansion of M-theory in powers of the eleven-dimensional gravitational constant κ_{11}.

Choice of Poincaré frame

To gauge-fix dilatations, we impose as usual a condition on the Einstein term appearing in the superconformal supergravity Lagrangian. According to the component expression for the D-density and the tensor calculus of superconformal multiplets [130, 131], the Einstein term included in $[S_0\overline{S}_0\Phi]_D$ is [87, 59]

$$\mathcal{L}_{\rm E} = -\frac{1}{2}eR\left[-\frac{2}{3}z_0\overline{z}_0\left(\Phi - \frac{1}{2}\sum_i w_i C_i\frac{\partial\Phi}{\partial C_i}\right)\right], \tag{5.18}$$

where w_i is a Weyl weight, the sum is taken over the linear ($w_i = 2$) and vector (w_i arbitrary) multiplets, and z_0, Φ and C_i are the lowest, scalar components of respectively S_0, Φ and of the vector or linear multiplets.[7]

Applied to the bulk Lagrangian (5.7), expression (5.18) leads to

$$\mathcal{L}_{\mathrm{E}} = -\frac{1}{2}eR\left[(z_0\overline{z}_0C_T)^{3/2}(2C)^{-1/2}\right]. \tag{5.19}$$

As they should, the terms introduced to impose the Bianchi identities do not contribute. We then select the Einstein frame, in which the gravitational Lagrangian is $-\frac{1}{2\kappa^2}eR$, by the dilatation gauge condition

$$\kappa^{-2} = (z_0\overline{z}_0C_T)^{3/2}(2C)^{-1/2}. \tag{5.20}$$

It will be convenient to introduce the (composite) real vector multiplet

$$\Upsilon = (S_0\overline{S}_0V_T)^{3/2}(2V)^{-1/2}, \tag{5.21}$$

with conformal weight two. In the Poincaré theory and in the Einstein frame, its lowest component is precisely equal to κ^{-2}. With this definition, the bulk Lagrangian simply becomes

$$\mathcal{L}_{\mathrm{B}} = \left[-\Upsilon - (S+\overline{S})V + L_TV_T\right]_D, \tag{5.22}$$

and the equation of motion for V (the chiral-linear duality equation)

$$2V(S+\overline{S}) = \Upsilon \tag{5.23}$$

indicates that the Einstein Lagrangian also reads

$$\mathcal{L}_{\mathrm{E}} = -(2C\,\mathrm{Re}\,s)eR. \tag{5.24}$$

In the Einstein frame (Planck units), $\mathrm{Re}\,s = (4\kappa^2C)^{-1}$.

Eq. (5.23) is compatible with the standard relation of heterotic superstrings[8] $2\kappa^2\langle\mathrm{Re}\,s\rangle = \alpha'$ if one identifies $2\langle C\rangle = 1/\alpha'$. This equation defines string units, in which

$$\mathcal{L}_{\mathrm{E}} = -\frac{e^{-2\varphi}}{\alpha'}eR, \tag{5.25}$$

with a dilaton given by $e^{-2\varphi} = \mathrm{Re}\,s$.

Modified Bianchi identities and κ-expansion

Compactification of M-theory on $S^1/\mathbb{Z}_2$ is commonly discussed in an expansion in powers of κ_{11}. Compactification on $\mathcal{K}_7$ can similarly be formulated with κ as expansion parameter. In the upstairs version, Bianchi identities are modified at the ten-dimensional planes fixed by $S^1/\mathbb{Z}_2$. Suppose now that we modify the four-dimensional supersymmetric Bianchi identities of the bulk Lagrangian in the following way:

$$\mathcal{L}_{\mathrm{B}} \quad\longrightarrow\quad \mathcal{L} = \left[-\Upsilon - (S+\overline{S})(V+\Delta_V) + L_T(V_T+\Delta_T)\right]_D, \tag{5.26}$$

[7] We use in general the same notation for the lowest component of Φ and the multiplet itself.

[8] The symbol $\langle\ldots\rangle$ denotes a background value.

with two composite vector multiplets Δ_V ($w = 2$, $n = 0$) and Δ_T ($w = 0 = n$). Solving for the Lagrange multipliers now leads to

$$V = L - \Delta_V, \quad V_T = T + \overline{T} - \Delta_T. \tag{5.27}$$

The Lagrangian to first order in these modifications is then

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\rm B} - \left[\frac{\Upsilon}{2V}\Delta_V - \frac{3}{2}\frac{\Upsilon}{V_T}\Delta_T\right]_D \\ &= \mathcal{L}_{\rm B} - \left[(S+\overline{S})\Delta_V - \frac{3}{2V_T}(\Upsilon\Delta_T)\right]_D, \end{aligned} \tag{5.28}$$

with V and V_T respectively replaced by L and $T + \overline{T}$. The multiplets Δ_V and $\Upsilon\Delta_T$, with "canonical" dimension $w = 2$, appear at order $\Upsilon^0 \sim \kappa^0$, in comparison with bulk terms of order $\Upsilon \sim \kappa^{-2}$. This is the relation with the expansion in powers of κ_{11} of M-theory in the low-energy limit. In M-theory compactification, the multiplets Δ_V and Δ_T can thus be obtained either by considering the modified Bianchi identities on $\mathcal{K}_7$, formulated as in Eq. (5.26), or from corrections to the Lagrangian of eleven-dimensional supergravity on $\mathcal{K}_7$, as in expression (5.28).

Identification of the components

To complete the identification of the four-dimensional supergravity (5.7) with the modes of eleven-dimensional supergravity we need its complete bosonic expansion, which after solving for the chiral $U(1)$ auxiliary field A_μ reads:

$$\begin{aligned} e^{-1}\mathcal{L}_{\rm B} &= -\tfrac{1}{2}\Upsilon R + \tfrac{1}{4}\Upsilon C^{-2}v_\mu v^\mu - \tfrac{3}{4}\Upsilon C_T^{-2}T_\mu T^\mu - t_\mu T^\mu + 2\,{\rm Im}\, s\partial_\mu v^\mu \\ &\quad -\tfrac{1}{4}\Upsilon C^{-2}(\partial_\mu C)(\partial^\mu C) - \tfrac{3}{4}\Upsilon C_T^{-2}(\partial_\mu C_T)(\partial^\mu C_T) \\ &\quad +d(\tfrac{1}{2}\Upsilon C^{-1} - 2\,{\rm Re}\, s) + (d_T - \Box C_T)(\ell_T - \tfrac{3}{2}\Upsilon C_T^{-1}) \\ &\quad +\tfrac{1}{2}(\partial_\mu\Upsilon)[\partial^\mu \log C + \partial^\mu \log\Upsilon] \\ &\quad +e^{-1}\mathcal{L}_{\rm aux.} + \text{derivative}, \end{aligned} \tag{5.29}$$

where $\Upsilon = (z_0\overline{z}_0 C_T)^{3/2}(2C)^{-1/2}$, and

$$\begin{aligned} e^{-1}\mathcal{L}_{\rm aux.} &= -\tfrac{1}{4}\Upsilon C^{-2}(H+iK)(H-iK) + \tfrac{3}{4}\Upsilon C_T^{-2}(H_T+iK_T)(H_T-iK_T) \\ &\quad +f(H-iK) + \overline{f}(H+iK). \end{aligned} \tag{5.30}$$

The last equality is obtained after solving for the f_0 component of S_0.[9]

The above component expansion of the bosonic Lagrangian is useful because it explicitly displays the dependence on the gauge choice for dilatation symmetry. The gravitational constant is the field-dependent quantity Υ. Choosing a Poincaré frame amounts to impose

[9]Note that the result would be different with a superpotential.

the value of this quantity, and to use this condition to eliminate z_0. Notice also that the choice of the phase of z_0 is a gauge condition for the chiral internal $U(1)$ superconformal symmetry. With the exception of z_0, all bosons are $U(1)$-neutral. As a consequence, the bosonic Lagrangian only depends on the modulus of z_0.

Choosing the Einstein frame, $\Upsilon = \kappa^{-2}$, and solving for the components of S and L_T leads to

$$\begin{aligned} e^{-1}\mathcal{L}_{\rm B} &= -\frac{1}{2\kappa^2}R - \frac{1}{4\kappa^2}C^{-2}[(\partial_\mu C)(\partial^\mu C) - v_\mu v^\mu] \\ &\quad -\frac{3}{4\kappa^2}C_T^{-2}[(\partial_\mu C_T)(\partial^\mu C_T) + T_\mu T^\mu], \end{aligned} \tag{5.31}$$

with $v_\mu = \frac{1}{2}e\epsilon_{\mu\nu\rho\sigma}\partial^\nu b^{\rho\sigma}$ since V is a linear multiplet, $C_T = 2\,{\rm Re}\,T$ and $T_\mu = -2\partial_\mu\,{\rm Im}\,T$ since $V_T = T + \overline{T}$. This Lagrangian is to be compared with the reduction of eleven-dimensional supergravity (5.5). The $\mathbb{Z}_2$ orbifold projection eliminates all states which are odd under $x^{11} \to -x^{11}$, and since we disregard massless modes related to the detailed Calabi–Yau geometry, the reduction of the $d = 11$ space-time metric is

$$g_{MN} = \begin{pmatrix} e^{-\gamma}e^{-2\sigma}g_{\mu\nu} & 0 & 0 \\ 0 & e^{2\gamma}e^{-2\sigma} & 0 \\ 0 & 0 & e^{\sigma}\delta_{i\bar{j}} \end{pmatrix}. \tag{5.32}$$

The $SU(3)$-invariant tensor $\delta_{i\bar{j}}$ refers to complex coordinates on the Calabi–Yau space. The surviving components of the four-index tensor G_{MNPQ} are only $G_{\mu\nu\rho 11}$ and $G_{\mu i\bar{j}11}$, with

$$G_{\mu\nu\rho 11} = 3\partial_{[\mu}C_{\nu\rho]11}, \quad G_{\mu i\bar{j}11} = \partial_\mu C_{i\bar{j}11}, \quad C_{i\bar{j}11} = ia(x)\delta_{i\bar{j}}, \tag{5.33}$$

and the four-dimensional Lagrangian for these fields reads

$$\begin{aligned} e^{-1}\mathcal{L}_{\rm CJS} &= -\frac{1}{2\kappa^2}R - \frac{1}{4\kappa^2}\left[9(\partial_\mu\sigma)(\partial^\mu\sigma) + \frac{1}{6}e^{6\sigma}G_{\mu\nu\rho 11}G^{\mu\nu\rho 11}\right] \\ &\quad -\frac{3}{4\kappa^2}\left[(\partial_\mu\gamma)(\partial^\mu\gamma) + e^{-2\gamma}(\partial_\mu a)(\partial^\mu a)\right]. \end{aligned} \tag{5.34}$$

In this expression, κ is the four-dimensional gravitational coupling

$$\kappa^2 = \frac{\kappa_{11}^2}{V_7}, \tag{5.35}$$

$V_7 = V_1 V_6$ being the volume of the compact space $S^1 \times \mathcal{X}_6$.

At this stage, the identification of the bosonic components C, $b_{\mu\nu}$, C_T and T_μ with the bulk fields σ, $C_{\mu\nu 11}$, γ and a can only be determined up to two proportionality constants (one for each "M-theory multiplet" V and V_T). We will define these constants at the end of section 5.2 from the couplings of C and $C_T = 2\,{\rm Re}\,T$ to charged matter and gauge fields, to obtain:

$$\begin{aligned} 4\kappa^2 C &= \frac{\lambda^2}{V_6}e^{-3\sigma}, & 4\kappa^2 b_{\mu\nu} &= \frac{\lambda^2}{V_6}C_{\mu\nu 11}, \\ C_T &= 2\frac{\lambda^2}{V_6}e^{\gamma}, & T_\mu &= 2\frac{\lambda^2}{V_6}\partial_\mu a. \end{aligned} \tag{5.36}$$

The quantity λ is the gauge coupling constant on the $\mathbb{Z}_2$ fixed planes. The dimensionless number λ^2/V_6 will actually never appear in the four-dimensional effective theory.

Bianchi identities and symmetries

In the bulk Lagrangian (5.7), the terms $[-(S+\overline{S})V + L_T V_T]_D$ impose in particular the Bianchi identities (5.6). They are certainly invariant under

$$\begin{array}{lclll} V & \longrightarrow & V+L, & L & \text{linear}, \\ V_T & \longrightarrow & V_T + T + \overline{T}, & T & \text{chiral}. \end{array} \tag{5.37}$$

These symmetries are the supersymmetric extensions of the gauge invariances of Bianchi identities, $\delta G_{\mu\nu\rho 11} = 3\partial_{[\mu}\Lambda_{\nu\rho]}$ and $\delta G_{\mu i\bar{j}11} = i\partial_\mu \Lambda \delta_{i\bar{j}}$. Solving for S and L_T implies then that V and V_T are "pure gauge", $V = L$ and $V_T = T + \overline{T}$. The last equation defines V_T up to a holomorphic redefinition of T, $T \to f(T)$. This redefinition is a symmetry of the bulk Lagrangian if the function Φ simultaneously transforms as in (5.3), with Λ a holomorphic function of T. The equation for the invariance of $S_0\overline{S}_0 V_T$ is

$$f(T) + \overline{f}(\overline{T}) = \frac{T+\overline{T}}{\Lambda(T)\overline{\Lambda}(\overline{T})}, \tag{5.38}$$

and its solution is clearly $Sl(2,\mathbb{R})$ symmetry,

$$T \quad \longrightarrow \quad f(T) = \frac{aT - ib}{icT + d}, \qquad ad - bc = 1, \tag{5.39}$$

the modular invariance of T (T-duality), extended to a continuous symmetry at the lowest order.

This chiral symmetry is generically anomalous: in the presence of a $N = 1$ super Yang–Mills sector, with or without chiral matter, mixed anomalies arise in the triangle diagram for two gauge bosons and one connection $-3\log(T+\overline{T})$ for $Sl(2,\mathbb{R})$ symmetry. This anomaly is cancelled in particular by a Green–Schwarz mechanism as was demonstrated in the effective Lagrangian description of gauge thresholds [56] calculated at one-loop for $(2,2)$ compactifications of the heterotic superstrings [69]. We will see below that this phenomenon is also a useful tool in the construction of effective supergravity theories of M-theory compactifications.

Addition of a superpotential

The standard reduction of eleven-dimensional supergravity with unbroken $N = 1$ supersymmetry does not generate a superpotential. This fact is however not a direct consequence of the eleven-dimensional Bianchi identity or of the Calabi–Yau and $S^1/\mathbb{Z}_2$ symmetries. In principle, the Bianchi identity $\partial_{[M}G_{NPQR]} = 0$ allows a solution

$$G_{ijk11} = 2i\kappa^{-1}h\epsilon_{ijk}, \quad G_{\overline{ijk}11} = -2i\kappa^{-1}h\epsilon_{\overline{ijk}}. \tag{5.40}$$

In these equations, h is a real constant and ϵ_{ijk} is the $SU(3)$-invariant Calabi–Yau tensor. The second term in the Lagrangian (5.5) leads then to a contribution

$$-\frac{e}{\kappa^4}C_T^{-3}(2\kappa^2 C)h^2 \tag{5.41}$$

in the four-dimensional effective supergravity. This contribution corresponds to the addition of a superpotential term

$$[ihS_0^3]_F \tag{5.42}$$

to the bulk Lagrangian, a contribution which however breaks supersymmetry [57]. Since we have insisted in writing Lagrangians in which all Bianchi identities are field equations, we prefer instead to use

$$[U(W+\overline{W})]_D + [S_0^3 W]_F. \tag{5.43}$$

The field equation of the vector multiplet U (with weights $w=2$, $n=0$) implies that the chiral multiplet W ($w=0=n$) is an arbitrary imaginary constant, which can be zero and supersymmetry stays unbroken, or non-zero.

With the addition of a superpotential, the bulk Lagrangian takes its final "off-shell" form

$$\mathcal{L}_{\rm B} = \left[-\Upsilon - (S+\overline{S})V + L_T V_T + U(W+\overline{W})\right]_D + [S_0^3 W]_F, \tag{5.44}$$

in which the Bianchi identities of eleven-dimensional supergravity are translated into field equations of the Lagrange multipliers S, L_T and U. At this stage, the introduction of these multiplets is not fascinating. This approach simply encodes the (Poincaré) dualities relating antisymmetric tensors (in linear multiplets) and scalars (in chiral multiplets), and the Bianchi identity for the superpotential is trivial. But this procedure will prove useful and informative below.

5.2 Gauge and matter contributions from the two $\mathbb{Z}_2$ fixed planes

In this section, we show that the introduction of the next to lowest order corrections (gauge multiplets and charged matter contributions) is controlled by a simple modification of the four-dimensional Bianchi identities, in analogy with the appearance of $\mathbb{Z}_2$ fixed planes contributions in the M-theory Bianchi identities.

We start by considering the dependence on charged matter (in chiral multiplets collectively denoted by M, with $w=0=n$) and gauge multiplets (vector multiplet A, in the adjoint representation, with $w=0=n$) of the effective $N=1$ four-dimensional supergravity for Calabi–Yau compactifications of heterotic superstrings [227, 58, 35]. This dependence is well-known, at least for the "universal" matter multiplets arising from the simplest Calabi–Yau modes of the ten-dimensional super Yang–Mills fields. Information on the non-trivial harmonic modes is more subtle [68], as for generic Calabi–Yau moduli. We then rewrite this theory in a form where explicit Bianchi identities allow a direct comparison with $\mathcal{K}_7$ compactification of M-theory, in the so-called upstairs formulation [116, 117].

The Lagrangian in the chiral formulation (5.16) becomes

$$\mathcal{L}_{\rm c} = -\frac{3}{2}\left[S_0\overline{S}_0 e^{-K/3}\right]_D + [\frac{1}{4}S\mathcal{W}\mathcal{W} + S_0^3 W]_F, \tag{5.45}$$

with the Kähler potential

$$K = -\log(S+\overline{S}) - 3\log(T+\overline{T} - 2\overline{M}e^A M), \tag{5.46}$$

and the superpotential

$$W = \alpha M^3. \tag{5.47}$$

For notational simplicity, we omit traces over the gauge group representation and their normalization factors. The chiral multiplet $\mathcal{W}$ is the gauge field strength for A ($w = 3/2 = n$). The gauge group is in general not simple, and

$$\mathcal{W}\mathcal{W} = \sum_a c^a \mathcal{W}^a \mathcal{W}^a, \tag{5.48}$$

with a real coefficient c^a for each simple or Abelian factor.[10] The superpotential should be understood as a gauge invariant trilinear interaction with coupling constant α defined as an integral over the Calabi–Yau space. In the linear multiplet version, the equivalent expression is [44, 59]

$$\mathcal{L}_l = -\frac{1}{\sqrt{2}}\left[(S_0\overline{S}_0)^{3/2}\hat{L}^{-1/2}e^{-\widehat{K}/2}\right]_D + [\alpha S_0^3 M^3]_F. \tag{5.49}$$

With respect to Eq. (5.12), gauge and matter dependence arises in modifications of the linear multiplet L (to $\hat{L}$) and of $\widehat{K}$: the new modulus and matter Kähler potential is

$$\widehat{K} = -3\log(T + \overline{T} - 2\overline{M}e^A M), \tag{5.50}$$

instead of Eq. (5.14) and

$$\hat{L} = L - 2\Omega, \tag{5.51}$$

where $\Omega(A)$ is the Chern–Simons vector multiplet ($w = 2$, $n = 0$), defined by[11]

$$\Omega = \sum_a c^a \Omega^a, \quad \Sigma(\Omega^a) = \frac{1}{16}\mathcal{W}^a\mathcal{W}^a. \tag{5.52}$$

Insisting as before on Bianchi identities, both forms (5.45) and (5.49) are equivalent to

$$\begin{aligned} \mathcal{L} &= \big[-\Upsilon - (S+\overline{S})(V+2\Omega) \\ &\quad + L_T(V_T + 2\overline{M}e^A M) + (U(W - \alpha M^3) + \text{c.c.})\big]_D + [S_0^3 W]_F \\ &= \big[-\Upsilon - (S+\overline{S})(V+2\Omega) + L_T(V_T + 2\overline{M}e^A M)\big]_D + [S_0^3(ih + \alpha M^3)]_F. \end{aligned} \tag{5.53}$$

Supersymmetric vacua have $h = 0$. As before, solving in the last expression for S and L_T imposes respectively $V = L - 2\Omega = \hat{L}$ and $V_T = T + \overline{T} - 2\overline{M}e^A M$, leading to Eq. (5.49). Alternatively, solving for V and L_T leads back to the chiral form (5.45), with the tensor calculus identity

$$-2[(S+\overline{S})\Omega]_D = \frac{1}{4}\sum_a c^a [S\mathcal{W}^a\mathcal{W}^a]_F + \text{derivative}, \tag{5.54}$$

---

[10] Corresponding to Kac–Moody levels in string theory. All coefficients can be equal to one, as with the "standard embedding", but our discussion is not affected by their presence.

[11] In global Poincaré supersymmetry, $\Sigma(\Omega) = -\frac{1}{4}\overline{DD}\Omega$. A linear multiplet is defined by the condition $\Sigma(L) = 0$.

which follows from Eq. (5.52) and the definition of the F-density, $[\Sigma(\ldots)]_F = -[\ldots]_D$.

This reformulation of the gauge invariant Lagrangian suggests some remarks. Firstly, it enhances the importance of Chern–Simons multiplets in superstring effective actions: gauge fields and matter fields couple to the bulk Lagrangian using a Chern–Simons multiplet. The gauge Chern–Simons multiplet Ω is defined by Eq. (5.52), which indicates that its chiral projection $\Sigma(\Omega)$ is the chiral multiplet for the kinetic super Yang–Mills Lagrangian. Similarly, for chiral matter, the kinetic Wess–Zumino Lagrangian can be written as $[S_0\overline{S}_0\overline{M}e^AM]_D = -[\Sigma(S_0\overline{S}_0\overline{M}e^AM)]_F$, defining

$$\Omega_M = S_0\overline{S}_0\overline{M}e^AM \tag{5.55}$$

as a matter Chern–Simons multiplet ($w = 2$, $n = 0$) which then couples to $S_0\overline{S}_0V_T$ as Ω couples to V.

Secondly, the Chern–Simons vector multiplet $\Omega(A)$ is not gauge invariant: its variation is a linear multiplet. Then, the variation of $[(S+\overline{S})\Omega]_D$ is a derivative and V remains gauge invariant. When solving for S, it simply follows that $\hat{L}$ is gauge invariant and that the linear multiplet transforms as

$$\delta L = 2\delta\Omega. \tag{5.56}$$

Finally, expression (5.53) shows that all gauge and chiral matter contributions can be viewed as the supersymmetrization of modified Bianchi identities imposed by S, L_T and U. This is equally true in the ten-dimensional supergravity-Yang–Mills system: the curl of the antisymmetric tensor field is modified by Chern–Simons contributions which are supersymmetry partners of the super Yang–Mills Lagrangian [46]. This observation provides the link to the approach based on M-theory on $\mathcal{K}_7$, in which the $\mathbb{Z}_2$-fixed planes carrying the Yang–Mills fields induce because of supersymmetry modifications to the Bianchi identity of the four-form field strength of eleven-dimensional supergravity.

In the effective supergravity of M-theory on $\mathcal{K}_7$ ("upstairs formulation"), the various components of the Lagrangian (5.53),

$$\begin{aligned}\mathcal{L} &= \Big[-(S_0\overline{S}_0V_T)^{3/2}(2V)^{-1/2} - (S+\overline{S})(V+2\Omega) \\ &\quad +L_T(V_T + 2\overline{M}e^AM) + [U(W-\alpha M^3) + \text{c.c.}]\Big]_D + [S_0^3W]_F,\end{aligned} \tag{5.57}$$

have the following origin. As already discussed at length, the first term is the bulk supergravity contribution. Then $[(S+\overline{S})(V+2\Omega)]_D$ is the supersymmetrization of the Bianchi identity verified by the component $G_{\mu\nu\rho 11}$ of the four-form field, modified by gauge contributions on the fixed planes. Similarly, $[L_T(V_T+2\overline{M}e^AM)]_D$ and $[U(W-\alpha M^3)+\text{c.c.}]_D$ are respectively the supersymmetric extensions of the Bianchi identities of $G_{\mu j\overline{k}11}$ and G_{ijk11}, when fixed plane contributions are included. Thus, all fixed plane contributions are given at this order by the supersymmetrization of Bianchi identities, as obtained by direct $\mathcal{K}_7$ truncation of the eleven-dimensional identities [116, 117].

At this point, the gauge coupling constant for each simple or Abelian factor a in the gauge group appears to be

$$\frac{1}{g_a^2} = c^a\,\text{Re}\,s = \frac{c^a\Upsilon}{4C}, \tag{5.58}$$

s and C being respectively the lowest scalar component of the chiral S and the vector V (or the linear L). At this order, g_a is the tree-level wilsonnian and physical[12] gauge coupling. The second equality is the lowest component of the equation of motion of the vector multiplet V, Eq. (5.23). In the Einstein frame, $\mathrm{Re}\, s = (4\kappa^2 C)^{-1}$.

It is clear, as already observed [14, 36, 115, 7, 8, 9, 159, 70, 133, 137, 143, 82, 83, 157, 158, 136], that as far as the structure of the four-dimensional effective supergravity is concerned, the same information follows from $\mathcal{K}_7$ compactification of M-theory at the next to lowest order in the κ-expansion and from Calabi–Yau compactifications of the heterotic superstrings, at zero string loop order.

Notice that Eq. (5.58) defines $c^a\, \mathrm{Re}\, s$ as the coefficient of gauge kinetic terms. It defines then this field in terms of the gauge kinetic action on the ten-dimensional $\mathbb{Z}_2$ fixed planes,

$$S_{\text{gauge}} = -\frac{1}{4\lambda^2}\int_{\mathcal{M}_{10}} d^{10}x\, e_{10}^{(11)} \mathrm{tr} F_{AB}F^{AB}, \tag{5.59}$$

reduced on $\mathcal{X}_6$. This action is also at the origin of charged matter kinetic terms, which in the effective supergravity read

$$-\frac{e}{\kappa^2}\frac{\partial^2 \widehat{K}}{\partial \overline{M}\partial M}(D_\mu \overline{M})(D^\mu M) = -\frac{6e}{\kappa^2 C_T}(D_\mu \overline{M})(D^\mu M) + \cdots. \tag{5.60}$$

The Calabi–Yau reduction of the action (5.59) provides then the identification of C and C_T in terms of the bulk fields σ and γ appearing in the metric tensor (5.32). These results have already been displayed in Eq. (5.36).

5.3 Anomaly-cancelling terms

In the ten-dimensional heterotic superstring, cancellation of gauge and gravitational anomalies is a one-loop effect in string or effective supergravity perturbation theory. In the low-energy effective action description, we should then distinguish the Wilson effective supergravity from the standard effective action S_Γ, defined as the generating functional of one-particle irreducible Green's functions. The latter action can be obtained in a diagrammatic expansion built from the Wilson Lagrangian $\mathcal{L}$, itself obtained from string perturbation theory as an expansion

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \cdots, \tag{5.61}$$

the upper index being the string-loop order. The expressions given in the preceding sections were for $\mathcal{L}^{(0)}$, or for the tree-level S_Γ. At the string one-loop level, S_Γ includes tree and one-loop diagrams generated by the Feynman rules of the tree-level Wilson Lagrangian $\mathcal{L}^{(0)}$. These include anomalous loop diagrams. It also includes tree diagrams generated by the Green–Schwarz counterterm [105] introduced in $\mathcal{L}^{(1)}$ to cancel the anomalies generated by $\mathcal{L}^{(0)}$. The mechanism for symmetry restoration implies that $\mathcal{L}^{(1)}$ is not invariant under the restored symmetry.

[12] The coefficient of $-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$ in the generating functional of one-particle irreducible Green's functions.

In four space-time dimensions, the nature of the cancelled anomalies is known from studies of $(2,2)$ compactifications of heterotic superstrings in the Yang–Mills sector [69, 56, 134]: target-space duality of the modulus T has a one-loop anomaly which is cancelled by a counterterm in $\mathcal{L}^{(1)}$, in a generalization to sigma-model anomalies of the Green–Schwarz mechanism [42, 43]. The derivation of the complete counterterm requires a calculation to all orders in the modulus T [69]. However, at the present stage of understanding, the M-theory approach should be regarded as a large-T limit in which T-duality reduces to a shift symmetry in the imaginary part of T.

In this section we would like to obtain some or all counterterms in $\mathcal{L}^{(1)}$ associated with anomaly cancellation in the low-energy description. We are particularly interested in contributions to gauge kinetic terms, the so-called threshold corrections. And we want to formulate these terms using the "M-theory multiplets" V, V_T and W corresponding to the surviving components of G_4, in contrast to the "heterotic multiplets" S (or L) and T. We begin by obtaining the relevant information from the case of heterotic $(2,2)$ symmetric orbifolds.

5.3.1 Information from symmetric (2,2) orbifolds

Retaining only the universal modulus T and a linear dilaton multiplet L, the Wilson one-loop Lagrangian for heterotic symmetric $(2,2)$ orbifolds includes a term[13] [56]

$$\mathcal{L}^{(1)} = -2\delta_{\rm GS}\Big[\hat{L}\log(T+\overline{T})\Big]_D + \sum_a b^a\Big[\log\eta(iT)\mathcal{W}^a\mathcal{W}^a\Big]_F, \tag{5.62}$$

where $\delta_{\rm GS}$ and b^a are numbers depending on the orbifold and $\eta(iT)$ is the Dedekind function. Under $Sl(2,\mathbb{Z})$ T-duality (5.39), the variation of $\mathcal{L}^{(1)}$ is

$$\begin{aligned}\delta\mathcal{L}^{(1)} &= 2\delta_{\rm GS}\Big[\hat{L}\Big(\log\varphi(T)+\log\overline{\varphi}(\overline{T})\Big)\Big]_D + \tfrac{1}{2}b^a\Big[\log\varphi(T)\mathcal{W}^a\mathcal{W}^a\Big]_F \\ &= \tfrac{1}{2}(\delta_{\rm GS}+b^a)\Big[\log\varphi(T)\mathcal{W}^a\mathcal{W}^a\Big]_F,\end{aligned} \tag{5.63}$$

with $\varphi(T) = icT + d$. On the other hand, the triangle one-loop diagram for two gauge fields and one Kähler connection $-3\log(T+\overline{T})$ is anomalous. Its variation is

$$\delta\Delta = \frac{1}{2}A^a\Big[\log\varphi(T)\mathcal{W}^a\mathcal{W}^a\Big]_F, \tag{5.64}$$

where A^a is the chiral-anomaly coefficient, as obtained from the expression of the diagram. The anomaly cancels since one finds that $b^a + \delta_{\rm GS} + A^a = 0$ for all factors in the gauge group (the index a). The one-loop correction to gauge kinetic terms obtained from the component expansion of $\mathcal{L}^{(1)}$ and from the triangle diagram reads

$$-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}\Big[-(\delta_{\rm GS}+A^a)\log(T+\overline{T}) + b^a\log|\eta(iT)|^4\Big]. \tag{5.65}$$

It is modular invariant since the anomaly is cancelled, and its value is controlled by the F-density contribution to $\mathcal{L}^{(1)}$, with coefficients b^a.

[13] In this paragraph, we use $c^a = 1$ and $\hat{L} = L - 2\sum_a \Omega^a$.

The Wilson Lagrangian depends on the coefficients $\delta_{\rm GS}$ and b^a. But the information on gauge thresholds is in the numbers b^a. In general, the parameters of the Wilson Lagrangian computed at a non-trivial loop order are not of direct physical significance and this is here the case of $\delta_{\rm GS}$ in the sector of gauge kinetic terms.

In the large-T limit, T-duality reduces to $\mathrm{Im}\, T \rightarrow \mathrm{Im}\, T +$constant, the Kähler connection $-3\log(T+\overline{T})$ is invariant and, strictly speaking, no anomaly survives to be cancelled. In addition, $\log|\eta(iT)|^4 \sim -\frac{\pi}{3}(T+\overline{T})$ dominates the logarithmic contributions. The threshold correction is then of the simple form

$$-\frac{1}{4}\sum_a \left[-\frac{\pi b^a}{3}(T+\overline{T})\right] F^a_{\mu\nu}F^{a\mu\nu}, \tag{5.66}$$

invariant under the imaginary shift symmetry of T. Its supersymmetrization is

$$-\frac{\pi}{12}\sum_a b^a \Big[T\mathcal{W}^a\mathcal{W}^a\Big]_F = \frac{2\pi}{3}\sum_a b^a \Big[(T+\overline{T})\Omega^a\Big]_D. \tag{5.67}$$

As long as T only is considered, there seems to be no way to identify the D-density contribution to $\mathcal{L}^{(1)}$ in the large-T limit. The introduction of the matter multiplet M changes the picture. In the lowest order Wilson Lagrangian $\mathcal{L}^{(0)}$, the Kähler connection $-3\log(T+\overline{T})$ is modified to $-3\log(T+\overline{T}-2\overline{M}e^AM)$. As a consequence, the contribution with coefficient A^a to the gauge threshold (5.65) involves the quantity $\log(T+\overline{T}-2\overline{M}M)$. Then, either the one-loop term $\mathcal{L}^{(1)}$ is accordingly modified to $-2\delta_{\rm GS}[\hat{L}\log(T+\overline{T}-2\overline{M}e^AM)]_D$ and the parameter $\delta_{\rm GS}$ disappears from gauge thresholds, or, less plausibly, this is not the case and $\delta_{\rm GS}$ acquires a physical significance in M-dependent thresholds. In any case, since the holomorphic Dedekind function cannot depend on $\overline{M}e^AM$, a calculation of the correction $\mathcal{L}^{(1)}$ to the Wilson Lagrangian to first order in $\overline{M}e^AM$ will give access to the D-density Green–Schwarz term and to the parameter $\delta_{\rm GS}$.

From the point of view of M-theory on $\mathcal{K}_7$ or heterotic superstrings on Calabi–Yau threefolds, $\overline{M}e^AM$ arises from the ten-dimensional Chern–Simons terms, at the same order as gauge kinetic terms. This indicates that the D-density contribution to $\mathcal{L}^{(1)}$ should be visible in a reduction of the ten-dimensional Green–Schwarz terms.

5.3.2 The case of M-theory on $\mathcal{K}_7$

Before computing the anomaly-cancelling terms from the low-energy limit of M-theory on $\mathcal{K}_7$, we consider the problem at the level of four-dimensional supergravity only.

In the large-T limit, as discussed in the preceding subsection, the T-dependent corrections to gauge kinetic terms are of the form

$$\frac{1}{4}\sum_a \beta^a \Big[T\mathcal{W}^a\mathcal{W}^a\Big]_F, \tag{5.68}$$

with coefficients which are in principle calculable in heterotic superstrings. To rewrite them in terms of the “M-theory multiplets”, we first note that

$$\frac{1}{4}\sum_a \beta^a \Big[T\mathcal{W}^a\mathcal{W}^a\Big]_F = -2\Big[(T+\overline{T})\sum_a \beta^a\Omega^a\Big]_D. \tag{5.69}$$

Since the field equation of the Lagrange multiplier L_T implies that $V_T = T + \overline{T} - 2\overline{M}e^A M$, this expression can also be written

$$-2\Big[(V_T + 2\overline{M}e^A M)\sum_a \beta^a \Omega^a\Big]_D. \tag{5.70}$$

The right-hand side of Eq. (5.69) is gauge invariant because $\delta\Omega^a$ is a linear multiplet and therefore $\big[(T + \overline{T})\delta\Omega^a\big]_D$ is a derivative. To ensure gauge invariance of expression (5.70), we add the term $\big[L_T(V_T + 2\overline{M}e^A M)\big]_D$ included in Lagrangian (5.53). We obtain

$$\Big[(L_T - 2\sum_a \beta^a \Omega^a)(V_T + 2\overline{M}e^A M)\Big]_D, \tag{5.71}$$

which is gauge invariant if we postulate that

$$\delta L_T = 2\sum_a \beta^a \delta\Omega^a. \tag{5.72}$$

The correction (5.69) to the super Yang–Mills Lagrangian is independent of the matter fields and has a holomorphic character (it can be seen as a correction to the holomorphic gauge kinetic function f_{ab}). To enumerate possible matter-dependent contributions to gauge kinetic terms, we consider for simplicity a single matter multiplet M transforming in some unspecified representation of the gauge group. The first candidate counterterm is a real density:

$$\sum_a \gamma^a \Big[\overline{M}e^A M\Omega^a\Big]_D. \tag{5.73}$$

Gauge invariance requires however its appearance in the combination

$$-2\delta\Big[\overline{M}e^A M(L - 2\sum_a c^a\Omega^a)\Big]_D + 2\gamma\Big[\overline{M}e^A M(L_T - 2\sum_a \beta^a\Omega^a)\Big]_D, \tag{5.74}$$

or

$$-2\delta\Big[\overline{M}e^A M V\Big]_D + 2\gamma\Big[\overline{M}e^A M(L_T - 2\sum_a \beta^a\Omega^a)\Big]_D, \tag{5.75}$$

using "M-theory multiplet" V. In the first counterterm, each gauge group factor contributes with weight c^a, as in lowest order terms. The second contribution can be combined with expression (5.71) into

$$\Big[(L_T - 2\sum_a \beta^a\Omega^a)(V_T + 2(1+\gamma)\overline{M}e^A M)\Big]_D. \tag{5.76}$$

We will also see below that the M-theory anomaly-cancelling terms generate a contribution of the form

$$\epsilon\Big[V|\alpha M^3|^2\Big]_D, \tag{5.77}$$

involving the matter superpotential. Since the factor $1+\gamma$ in expression (5.76) can be eliminated by a rescaling of M, of δ and of the superpotential coupling constant α, we may take $\gamma = 0$ at our level of approximation, and the Wilson Lagrangian up to string one-loop order is expected to become

$$\begin{aligned}\mathcal{L} &= \Big[- \Upsilon - (S+\overline{S})(V+2\Omega) + (U(W-\alpha M^3) + \text{c.c.}) \\ &\quad + (L_T - 2\textstyle\sum_a \beta^a\Omega^a)(V_T + 2\overline{M}e^A M) \\ &\quad + V(\epsilon|\alpha M^3|^2 - 2\delta\overline{M}e^A M)\Big]_D + [S_0^3 W]_F.\end{aligned} \tag{5.78}$$

Notice that the contributions which correspond to string one-loop effects do not include any correction to the Einstein Lagrangian, which remains simply

$$-\frac{1}{2}\Upsilon eR, \qquad \Upsilon = (z_0\overline{z}_0 C_T)^{3/2}(2C)^{-1/2}, \tag{5.79}$$

and the Einstein frame condition remains $\Upsilon = \kappa^{-2}$. This is expected since the gravitational constant in the heterotic superstring is not corrected at string one-loop order.

From the general expression (5.78) in which Bianchi identities are field equations for S, L_T and U, we can derive various equivalent forms. For instance, solving for S, L_T and U leads to the version of the effective supergravity in which the dilaton is described by a linear multiplet:

$$\begin{aligned}\mathcal{L}_l &= \Big[- (S_0\overline{S}_0)^{3/2}\left(T+\overline{T}-2\overline{M}e^A M\right)^{3/2}(2\hat{L})^{-1/2}\Big]_D + \Big[S_0^3(ih+\alpha M^3)\Big]_F \\ &\quad + \Big[\hat{L}\left(\epsilon|\alpha M^3|^2 - 2\delta\overline{M}e^A M\right)\Big]_D + \Big[\tfrac{1}{4}T\textstyle\sum_a \beta^a \mathcal{W}^a\mathcal{W}^a\Big]_F.\end{aligned} \tag{5.80}$$

The second line is a one-loop correction in the perturbative expansion of the heterotic superstring in which $\hat{L}$ is the string loop-counting field [44]. Its T-dependent part corresponds to the Green–Schwarz counterterm found in Ref. [56] for symmetric heterotic orbifolds. Each of these one-loop corrections, with coefficients ϵ, δ and β^a, is related to a well-defined counterterm which can be easily identified in, for instance, the low-energy limit of M-theory on $\mathcal{K}_7$. The Green–Schwarz counterterms controlled by δ and ϵ are intrinsically real D-densities. They will appear as corrections to the Kähler potential, as matter-dependent "wave-function renormalizations", in the dual version with a chiral dilaton multiplet. On the other hand, the holomorphic T-dependent terms are true threshold corrections.

Alternatively, solving for L_T, V and U leads to the version with a chiral dilaton multiplet:

$$\mathcal{L}_c = -\frac{3}{2}\Big[S_0\overline{S}_0 e^{-K/3}\Big]_D + \frac{1}{4}\Big[\sum_a (c^a S + \beta^a T)\mathcal{W}^a\mathcal{W}^a\Big]_F + \Big[S_0^3(ih+\alpha M^3)\Big]_F, \tag{5.81}$$

with the Kähler potential[14]

$$\begin{aligned}K &= -\log\left(S+\overline{S}+2\delta\overline{M}e^A M - \epsilon|\alpha M^3|^2\right) \\ &\quad -3\log\left(T+\overline{T}-2\overline{M}e^A M\right),\end{aligned} \tag{5.82}$$

[14] The superfield Kähler potential includes covariantization contributions e^A which disappear in the bosonic expression used in component expansions.

and the gauge kinetic functions $f^a = c^a S + \beta^a T$. An ambiguity exists however because one can perform a holomorphic redefinition of the two chiral multiplets. For instance,

$$S = \tilde{S} - \delta T, \tag{5.83}$$

leads to the equivalent Kähler potential

$$\begin{aligned} K &= -\log\left(\tilde{S} + \overline{\tilde{S}} - \delta(T + \overline{T} - 2\overline{M}e^A M) - \epsilon|\alpha M^3|^2\right) \\ &\quad -3\log\left(T + \overline{T} - 2\overline{M}e^A M\right), \end{aligned} \tag{5.84}$$

with gauge kinetic functions $f^a = c^a\tilde{S} + (\beta^a - c^a\delta)T$. The origin of this ambiguity at the level of "M-theory multiplets" is interesting. Suppose that we add the counterterm

$$\Delta\mathcal{L} = B\Big[(V + 2\Omega)(V_T + 2\overline{M}e^A M)\Big]_D \tag{5.85}$$

to the fundamental Lagrangian (5.78), with an arbitrary constant B. The theory becomes then

$$\begin{aligned} \mathcal{L}_B &= \Big[-\Upsilon - (S + \overline{S})(V + 2\Omega) + [U(W - \alpha M^3) + \text{c.c.}] \\ &\quad + \left(L_T - 2\textstyle\sum_a(\beta^a - Bc^a)\Omega^a\right)(V_T + 2\overline{M}e^A M) \\ &\quad + V(\epsilon|\alpha M^3|^2 + BV_T + 2(B-\delta)\overline{M}e^A M)\Big]_D + [S_0^3 W]_F. \end{aligned} \tag{5.86}$$

It is gauge invariant provided the appropriate transformation of L_T is postulated. We have apparently obtained a family of four-dimensional supergravity theories, depending on a new parameter B. This is however only true before solving for the Lagrange multiplier multiplets. Firstly, solving for S and L_T leads to the space-time derivative $\Delta\mathcal{L} = B\left[L(T + \overline{T})\right]_D$. The counterterm $\Delta\mathcal{L}$ is then irrelevant in the version of the theory with a linear dilaton. Secondly, if we instead solve for V and L_T, we obtain the Kähler potential

$$\begin{aligned} K &= -\log\left(S + \overline{S} - B(T + \overline{T}) + 2\delta\overline{M}e^A M - \epsilon|\alpha M^3|^2\right) \\ &\quad -3\log\left(T + \overline{T} - 2\overline{M}e^A M\right), \end{aligned} \tag{5.87}$$

and the gauge kinetic functions $f^a = c^a S + (\beta^a - Bc^a)T$. This theory is clearly related to Eq. (5.82) by the holomorphic redefinition $S \to S - BT$, and the choice $B = \delta$ leads to theory (5.84).

This discussion shows that the counterterm $\Delta\mathcal{L}$ is irrelevant in the four-dimensional effective supergravity, that all values of B lead to equivalent Lagrangians, with the same dynamical equations. Further information due, for instance, to compactification of extra dimensions could however appear more natural with a specific value of B, if one insists to use the version of the effective supergravity with a chiral dilaton multiplet. For instance, all corrections linear in T appear as gauge thresholds with the choice $B = 0$. But one could as well use the version with a linear dilaton which is free of ambiguities.

In addition, the holomorphic redefinition (5.83) mixes terms of different orders in the string loop expansion. As a consequence, the distinction between terms in $\mathcal{L}^{(0)}$ and corrections in $\mathcal{L}^{(1)}$ becomes ambiguous in general in the large T limit.

The values of the coefficients δ and ϵ can be inferred from a direct calculation of the M-dependent anomaly-cancelling terms in M-theory on $\mathcal{K}_7$. This is the subject of the next subsection. Notice however that such a calculation only provides the terms of first order in the matter multiplets $\overline{M}e^A M$ and $|\alpha M^3|^2$. To this order, expression (5.82) becomes

$$\begin{aligned} K &= -\log(S+\overline{S}) - 3\log(T+\overline{T}-2\overline{M}e^A M) \\ & \quad -\frac{1}{S+\overline{S}}\Big[2\delta\overline{M}e^A M - \epsilon|\alpha M^3|^2\Big]. \end{aligned} \tag{5.88}$$

The term with coefficient δ has been obtained in direct Calabi–Yau reductions of M-theory on $S^1/\mathbb{Z}_2$ (see, for instance, [157, 158, 136][15]). The charged matter contribution with coefficient ϵ was not included in these analyses.

The gauge contributions appearing in Eq. (5.78) read

$$-2\sum_a \Big[[c^a(S+\overline{S}) + \beta^a(V_T + 2\overline{M}e^A M)]\Omega^a\Big]_D, \tag{5.89}$$

so that the gauge coupling constants are given by

$$\frac{1}{g_a^2} = c^a \,\mathrm{Re}\, s + \frac{1}{2}\beta^a\left(C_T + 2\overline{M}M\right). \tag{5.90}$$

This expression becomes harmonic once the Bianchi identity imposing $C_T + 2\overline{M}M = 2\,\mathrm{Re}\,T$ has been used. Similarly, one obtains from Eq. (5.80)

$$\frac{1}{g_a^2} = \frac{c^a\Upsilon}{4C} + \beta^a\,\mathrm{Re}\,T - c^a\delta\overline{M}M + \frac{c^a\epsilon}{2}|\alpha M^3|^2, \tag{5.91}$$

with $\Upsilon = 2(z_0\overline{z}_0)^{3/2}(\mathrm{Re}\,T - \overline{M}M)^{3/2}C^{-1/2}$. This second form of the gauge couplings is never harmonic since it is obtained from a theory with a linear dilaton multiplet. Both expressions do however agree since the chiral-linear duality relation between $\mathrm{Re}\,s$ and C is

$$\mathrm{Re}\,s = \frac{\Upsilon}{4C} - \delta\overline{M}M + \frac{\epsilon}{2}|\alpha M^3|^2. \tag{5.92}$$

5.3.3 On the eleven-dimensional origin of the anomaly-cancelling terms

In ten dimensions, anomaly-cancelling terms for the $E_8 \times E_8$ heterotic superstring are well-known. There are two terms. The first couples a gauge and Lorentz invariant eight-form $\widehat{X}_8$ to the two-form field $\widehat{B}_2$. The second one is proportional to $\int(\Omega_{3,1} + \Omega_{3,2} - \Omega_{3,\mathrm{L}}) \wedge \widehat{X}_7$, with $d\widehat{X}_7 = \widehat{X}_8$.

In the preceding chapter, we have precisely computed the anomaly-cancelling terms arising from M-theory on $S^1/\mathbb{Z}_2$. These terms arise from the following action terms in eleven dimensions [116, 117, 222, 77, 54]:

$$-\frac{\lambda^2}{(4\pi)^2\kappa_{11}^2}\int_{\mathcal{M}_{11}} G_4 \wedge X_7 - \frac{1}{12\kappa_{11}^2}\int_{\mathcal{M}_{11}} C_3 \wedge G_4 \wedge G_4, \tag{5.93}$$

[15] It has also been obtained, in a quite different context, by Itoyama and Leon [123].

where

$$X_7 = \frac{1}{12(4\pi)^3}\left(\frac{1}{2}\Omega_{7,\mathrm{L}} - \frac{1}{8}(\mathrm{tr}R_2^2)\wedge\Omega_{3,\mathrm{L}}\right), \tag{5.94}$$

κ_{11} is the eleven-dimensional gravitational constant and λ is the gauge coupling constant on both ten-dimensional $\mathbb{Z}_2$-fixed planes, as defined by the gauge action (5.59). The Chern–Simons forms are defined by

$$d\Omega_{7,\mathrm{L}} = \mathrm{tr}R_2^4, \qquad d\Omega_{3,\mathrm{L}} = \mathrm{tr}R_2^2, \qquad d\Omega_{3,i} = \mathrm{tr}F_{2,i}^2, \qquad i=1,2. \tag{5.95}$$

As we have seen in the preceding chapter, cancellation of gauge and gravitational anomalies and coherence of the reduction to ten dimensions impose $\lambda^6 = (4\pi)^5\kappa_{11}^4/12$ [116, 117]. This condition relates the gauge coupling λ and the S^1 radius. Solving the Bianchi identity verified by the four-form field G_4 on $S^1/\mathbb{Z}_2$ and extracting the zero modes leads to the following Green–Schwarz terms

$$\begin{aligned} S_{\mathrm{GS}} \;=\; & -\frac{1}{48\pi}\int_{\mathcal{M}_{10}} \widehat{B}_2\wedge\Big[(I_{4,1})^2+(I_{4,2})^2 - I_{4,1}\wedge I_{4,2} + \frac{1}{(4\pi)^4}\Big(\frac{1}{2}\mathrm{tr}R_2^4 - \frac{1}{8}(\mathrm{tr}R_2^2)^2\Big)\Big] \\ & -\frac{1}{(4\pi)^2}\int_{\mathcal{M}_{10}} (\Omega_{3,1}+\Omega_{3,2}-\Omega_{3,\mathrm{L}})\wedge X_7. \end{aligned} \tag{5.96}$$

In this expression,

$$I_{4,i} = \frac{1}{(4\pi)^2}\Big(\mathrm{tr}F_{2,i}^2 - \frac{1}{2}\mathrm{tr}R_2^2\Big), \qquad i=1,2, \tag{5.97}$$

and

$$\widehat{B}_{AB} = \frac{\lambda^2}{\kappa_{11}^2}\int_{S^1} dx^{11} C_{AB11}. \tag{5.98}$$

As usual, the two-form field $\widehat{B}_2$ couples to $\widehat{X}_8$. But the second term has a particular structure: $\widehat{X}_7$ is replaced by the purely gravitational seven-form (5.94): the anomaly-cancelling terms derived from M-theory differ from the standard expression of the heterotic superstring by a well-defined local counterterm, as permitted by the descent equations.

The Calabi–Yau compactification of $\widehat{B}_{AB}$ leads to two zero modes of the bulk fields,

$$\kappa^2\widehat{B}_{\mu\nu} = \frac{\lambda^2}{V_6}C_{\mu\nu 11}, \qquad \kappa^2\widehat{B}_{i\bar{j}} = \frac{\lambda^2}{V_6}C_{i\bar{j}11}, \tag{5.99}$$

since $\frac{\lambda^2}{\kappa_{11}^2} = \frac{1}{\kappa^2}\frac{\lambda^2}{V_6}\frac{1}{V_1}$. By Eqs. (5.36), these states are related to our four-dimensional bulk multiplets L (or S) and T by

$$\widehat{B}_{\mu\nu} = 4b_{\mu\nu}, \qquad \kappa^2\widehat{B}_{i\bar{j}} = i\,\mathrm{Im}\,T\delta_{i\bar{j}}. \tag{5.100}$$

The number λ^2/V_6 disappears in this identification: it does not play any role in the four-dimensional effective theory.

Our task is then to compute the reduction of S_{GS} on $\mathcal{M}_4\times$(Calabi–Yau). Since we restrict ourselves to contributions with at most two derivatives, we only need the reduction of

$$\Delta = -\frac{1}{48\pi}\int_{\mathcal{M}_{10}} \widehat{B}_2\wedge\Big[(I_{4,1})^2+(I_{4,2})^2 - I_{4,1}\wedge I_{4,2}\Big]. \tag{5.101}$$

The global definition of the four-form field G_4 ("cohomology condition") implies [230]

$$\langle I_{4,1}\rangle = -\langle I_{4,2}\rangle \tag{5.102}$$

for the Calabi–Yau background, which is a $(2,2)$ form. The counterterm Δ becomes then

$$\begin{aligned}\Delta &= -\frac{1}{4(4\pi)^3}\int_{\mathcal{M}_{10}} \widehat{B}_2 \wedge \langle I_{4,1}\rangle \wedge \left(\mathrm{tr}F_{2,1}^2 - \mathrm{tr}F_{2,2}^2\right)\\ &\quad -\frac{1}{12(4\pi)^5}\int_{\mathcal{M}_{10}} \widehat{B}_2 \wedge \left[(\mathrm{tr}F_{2,1}^2)^2 + (\mathrm{tr}F_{2,2}^2)^2 - (\mathrm{tr}F_{2,1}^2)\wedge(\mathrm{tr}F_{2,2}^2)\right] + \cdots,\end{aligned} \tag{5.103}$$

where gravitational contributions with more than two derivatives are omitted. Notice that the two E_8 factors contribute with opposite signs in the background-dependent term.

The standard embedding is defined by $\langle \mathrm{tr}F_{2,2}^2\rangle = 0$, while $\langle \mathrm{tr}F_{2,1}^2\rangle$ is in the $SU(3)$ direction of the maximal embedding $E_6 \times SU(3) \subset E_8$. As a consequence,

$$\langle \mathrm{tr}F_{2,1}^2\rangle = \langle \mathrm{tr}R_2^2\rangle = 2(4\pi)^2\langle I_{4,1}\rangle, \tag{5.104}$$

which in turn leads to

$$\begin{aligned}\Delta &= -\frac{1}{8(4\pi)^5}\int_{\mathcal{M}_{10}} \widehat{B}_2 \wedge \langle \mathrm{tr}R_2^2\rangle \wedge (\mathrm{tr}F_{2,1}^2 - \mathrm{tr}F_{2,2}^2)\\ &\quad -\frac{1}{12(4\pi)^5}\int_{\mathcal{M}_{10}} \widehat{B}_2 \wedge \left[(\mathrm{tr}F_{2,1}^2)^2 + (\mathrm{tr}F_{2,2}^2)^2 - (\mathrm{tr}F_{2,1}^2)\wedge(\mathrm{tr}F_{2,2}^2)\right] + \cdots.\end{aligned} \tag{5.105}$$

To derive the zero modes of $\mathrm{tr}F_{2,1}^2$ and $\mathrm{tr}F_{2,2}^2$, it is simpler to consider the Chern–Simons forms, using the relation

$$(\mathrm{tr}F_{2,i}^2)_{ABCD} = 4\partial_{[A}(\Omega_{3,i})_{BCD]}, \qquad i = 1,2. \tag{5.106}$$

In the standard embedding, the unbroken E_8 group generates a $N = 1$ super Yang–Mills multiplet only. The only massless mode is then

$$(\mathrm{tr}F_{2,2}^2)_{ABCD} \to (\mathrm{tr}F_{2,E_8}^2)_{\mu\nu\rho\sigma}, \quad (\Omega_{3,2})_{ABC} \to (\Omega_{3,E_8})_{\mu\nu\rho}. \tag{5.107}$$

Clearly, these massless modes of $\mathrm{tr}F_{2,2}^2$ and $\Omega_{3,2}$ are respectively components of the four-dimensional $N = 1$ multiplets $\mathcal{W}^2\mathcal{W}^2$ and Ω^2 used earlier. More precisely:

$$\begin{aligned}[\mathcal{W}^2\mathcal{W}^2]_{\text{f-component}} &= -\frac{1}{2}\mathrm{tr}F_{E_8\mu\nu}F_{E_8}^{\mu\nu} - \frac{i}{4e}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}F_{E_8\mu\nu}F_{E_8\rho\sigma} + \cdots,\\ [\Omega^2]_{\text{d-component}} &= \frac{1}{16}\mathrm{tr}F_{E_8\mu\nu}F_{E_8}^{\mu\nu} + \cdots,\\ [\Omega^2]_{\text{vector component}} &= \frac{1}{8e}\epsilon^{\mu\nu\rho\sigma}(\Omega_{3,E_8})_{\nu\rho\sigma} + \cdots.\end{aligned} \tag{5.108}$$

The gauge fields of the E_8 group broken into E_6 generate E_6 gauge fields and the chiral matter multiplet M, transforming in representation **27**. Accordingly, the massless modes of $\mathrm{tr}F^2_{2,1}$ are:

$$\begin{aligned}(\mathrm{tr}F^2_{2,1})_{ABCD} \quad &\longrightarrow \quad (\mathrm{tr}F^2_{2,E_6})_{\mu\nu\rho\sigma},\\ &\longrightarrow \quad (\mathrm{tr}F^2_{2,1})_{\mu\nu i\bar{j}} \;=\; 2\partial_{[\mu}(\Omega_{3,1})_{\nu]i\bar{j}},\\ &\longrightarrow \quad (\mathrm{tr}F^2_{2,1})_{\mu ijk} \;=\; \partial_\mu(\Omega_{3,1})_{ijk},\\ &\longrightarrow \quad (\mathrm{tr}F^2_{2,1})_{\mu\overline{ijk}} \;=\; \partial_\mu(\Omega_{3,1})_{\overline{ijk}}.\end{aligned} \tag{5.109}$$

While as before $(\Omega_{3,1})_{ABC} \rightarrow (\Omega_{3,E_6})_{\mu\nu\rho}$, the other components of $\Omega_{3,1}$ involve the scalar component of the matter multiplet M and require more care since we have already precisely defined the four-dimensional field M by its coupling to the bulk fields. To obtain the correct relations, a detour is helpful.

As already explained, the gauge kinetic action (5.59) also generates the four-dimensional kinetic terms for the matter multiplet M. In the four-dimensional effective Lagrangian, these contributions arise as the highest component of the "matter Chern–Simons multiplet" $\overline{M}e^AM$, which includes $-2(D_\mu\overline{M})(D^\mu M)$. This multiplet also contains in its vector component the matter Chern–Simons form

$$\Omega^M_\mu = i\overline{M}(D_\mu M) - i(D_\mu\overline{M})M. \tag{5.110}$$

This is completely similar to the gauge Chern–Simons multiplet which includes gauge kinetic terms in its d-component and $(\Omega_3)_{\mu\nu\rho}$ in its vector component, as indicated by expressions (5.108). A direct computation of the relation between kinetic terms due to the action (5.59), and the highest component of $\overline{M}e^AM$ delivers then the relation between $(\Omega_{3,1})_{\mu i\bar{j}}$ and this multiplet, by four-dimensional supersymmetry. A similar operation gives the relation between $(\Omega_{3,1})_{ijk}$ and the superpotential multiplet αM^3. The relations are

$$\begin{aligned}(\Omega_{3,1})_{\mu i\bar{j}} &= \frac{1}{6\kappa^2}\left[(D_\mu\overline{M})M - \overline{M}(D_\mu M)\right]\delta_{i\bar{j}} = \frac{i}{6\kappa^2}\delta_{i\bar{j}}[\overline{M}e^AM]_{\text{vector component}},\\ (\Omega_{3,1})_{ijk} &= \frac{1}{3\kappa^3}\alpha\overline{M}^3\epsilon_{ijk} = \frac{1}{3\kappa^3}\epsilon_{ijk}[\alpha\overline{M}^3]_{\text{scalar component}},\end{aligned} \tag{5.111}$$

together with the last equation (5.108) which applies to all gauge Chern–Simons forms.

With the complete identification of the massless modes of the Chern–Simons three-forms, we are equipped for translating the Calabi–Yau reduction of the Green–Schwarz counterterm (5.105) in a four-dimensional supergravity density formula. After straightforward manipulations of Eq. (5.105), we obtain

$$\begin{aligned}\mathcal{L}_{\rm GS} &= -\frac{1}{192(4\pi)^5} I\epsilon^{\mu\nu\rho\sigma}\frac{\lambda^2}{V_6}(\partial_\mu a)(\Omega_{3,E_6}-\Omega_{3,E_8})_{\nu\rho\sigma} \\ &\quad+\frac{i}{384(4\pi)^5\kappa^2} I\epsilon^{\mu\nu\rho\sigma}\frac{\lambda^2}{V_6}(\partial_\mu C_{\nu\rho 11})(MD_\sigma\overline{M}-\overline{M}D_\sigma M) \\ &\quad+\frac{i\alpha^2}{216(4\pi)^5}\frac{V_6}{\kappa^6}\epsilon^{\mu\nu\rho\sigma}\frac{\lambda^2}{\kappa^2 V_6}(\partial_\mu C_{\nu\rho 11})\left[\overline{M}^3(\partial_\sigma M^3)-(\partial_\sigma\overline{M}^3)M^3\right]+\cdots \\ &= \frac{1}{384(4\pi)^5} I\epsilon^{\mu\nu\rho\sigma}T_\mu(\Omega_{3,E_6}-\Omega_{3,E_8})_{\nu\rho\sigma} \\ &\quad-\frac{i}{96(4\pi)^5} I\epsilon^{\mu\nu\rho\sigma}(\partial_\nu b_{\rho\sigma})(MD_\mu\overline{M}-\overline{M}D_\mu M) \\ &\quad-\frac{i\alpha^2}{54(4\pi)^5}\frac{V_6}{\kappa^6}\epsilon^{\mu\nu\rho\sigma}(\partial_\nu b_{\rho\sigma})\left[\overline{M}^3(\partial_\mu M^3)-(\partial_\mu\overline{M}^3)M^3\right]+\cdots,\end{aligned} \tag{5.112}$$

where the dots indicate the terms required by $N=1$ supersymmetry and I is the dimensionless integral

$$I=\kappa^{-2}\int_{\mathcal{X}_6} dV_6\,\delta_{i\bar{i}}\epsilon^{ijk}\epsilon^{\overline{ijk}}\langle{\rm tr}R_2^2\rangle_{jk\bar{j}\bar{k}}, \tag{5.113}$$

in terms of the $(1,1)$ (the metric tensor), $(3,0)$ and $(0,3)$ Calabi–Yau tensors and the background $\langle{\rm tr}R_2^2\rangle$.

Using Eqs. (5.111), we can write the Green–Schwarz counterterm in superfield form as:

$$\begin{aligned}\mathcal{L}_{\rm GS} &= -\frac{I}{48(4\pi)^5}\left[(V_T+2\overline{M}e^A M)(\Omega^1-\Omega^2)\right]_D-\frac{I}{48(4\pi)^5}\left[V\overline{M}e^A M\right]_D \\ &\quad+\frac{1}{27(4\pi)^5}\frac{V_6}{\kappa^6}\left[V|\alpha M^3|^2\right]_D.\end{aligned} \tag{5.114}$$

Comparing the three terms of $\mathcal{L}_{\rm GS}$ with the corresponding parts of (5.78), we can express the coefficients β^a, δ and ϵ in terms of the Calabi–Yau dependent integral I:

$$\delta=\beta^1=-\beta^2=\frac{I}{96(4\pi)^5}\quad,\quad\epsilon=\frac{1}{27(4\pi)^5}\frac{V_6}{\kappa^6}. \tag{5.115}$$

The calculation predicts then $\beta^1=-\beta^2=\delta$, a result already obtained in Refs. [157, 158, 136], for instance. Once again, we stress that we have obtained next-order corrections to the effective Wilson Lagrangian. As argued earlier in subsection 5.3.1, while we expect β^1 and β^2 to have physical significance as coefficients of the modulus-dependent threshold corrections, the parameter δ is not necessarily a physical quantity. To decide of its relevance, a calculation at the same order of threshold corrections in the effective action should be performed, but this computation requires a detailed knowledge of the charged matter spectrum and couplings.

5.4 Summary

In this chapter, we have deduced the structure of the four-dimensional $N = 1$ effective (wilsonnian) supergravity describing the universal massless sector of M-theory compactified on $S^1/\mathbb{Z}_2 \times$ (Calabi–Yau). The model depends on three categories of multiplets: the "M-theory multiplets" V, V_T and W which describe the degrees of freedom of the M-theory four-index tensor, the "source multiplets" M and A in Ω, $\overline{M}e^A M$ and αM^3 which are related to the source terms in M-theory Bianchi identities, and the "Lagrange multiplier multiplets" S, L_T and U which impose by their field equations the Bianchi identities. In addition, the multipliers S and L_T generate the four-dimensional axion-tensor duality which is known to be an important ingredient of the formulation of the string dilaton beyond the lowest order.

An effective supergravity similar to expression (5.78) is in principle valid for generic compactifications of M-theory with unbroken $N = 1$ supersymmetry in four dimensions. One needs to identify the appropriate supermultiplets appearing as sources in the Bianchi identities generated by S, L_T and U. Only the values of coefficients like β^a, δ and ϵ depend on the detailed geometry of the compact space. As we will see in the next chapter, this method is especially useful in deriving contributions to the effective Lagrangian due to non-perturbative states like M-theory five-brane degrees of freedom.

Chapter 6

A five-brane modulus in the effective $N = 1$ supergravity of M-theory

We know that compactification of M-theory on $\mathcal{K}_7 = S^1/\mathbb{Z}_2 \times \mathcal{X}_6$, where $\mathcal{X}_6$ is a Calabi–Yau threefold, leads to $N = 1$ supersymmetry in four space-time dimensions. Five-branes configurations preserve this supersymmetry if their world-volume $\mathcal{W}_6$ enclose four-dimensional Minkowski space $\mathcal{M}_4$ and a holomorphic two-cycle $\mathcal{C}_2$ in $\mathcal{X}_6$ [19, 29, 230]:

$$\mathcal{W}_6 = \mathcal{M}_4 \times \mathcal{C}_2. \tag{6.1}$$

With this embedding, the five-brane massless excitations on the world-volume, which belong to a tensor multiplet of chiral six-dimensional supersymmetry on $\mathcal{W}_6$ [98, 127], produce in the low-energy four-dimensional effective supergravity various $N = 1$ multiplets of massless fields. Some of these modes are deeply related to the Calabi–Yau geometry, and computing their effective theory is a complicated task. There are however universal modes which can be more easily described, the most obvious example being the real scalar associated to the position of the five-brane on the orbifold $S^1/\mathbb{Z}_2$. This "universal five-brane modulus" will be the main subject of this chapter: we will compute its effective supergravity couplings to the modes of M-theory on $\mathcal{K}_7$ which are also perturbative massless states of $E_8 \times E_8$ heterotic superstrings on $\mathcal{X}_6$. In the simplest case of the standard embedding, these modes are the $N = 1$ supergravity and dilaton multiplets, the modulus of the Calabi–Yau volume, $E_6 \times E_8$ gauge fields and chiral matter in representation $(\mathbf{27}, \mathbf{1})$. Lukas, Ovrut and Waldram (LOW) [138] have derived the effective supergravity for these heterotic states in a non-trivial background value of the brane modulus. Our goal here is to obtain a complete[1] effective supergravity for the supermultiplet of the universal brane modulus.

When computing an effective Lagrangian, it is usually important to respect the symmetries of the underlying theory. For instance, the tensor multiplet of five-brane excitations has an antisymmetric tensor with a self-dual field strength. This symmetry has far-reaching implications in four dimensions: the effective theory has a massless antisymmetric tensor dual to a pseudoscalar or, in terms of supermultiplets a chiral multiplet dual to a linear multiplet [204].[2] This observation has immediate implications on the effective supergravity

[1]Including terms with two derivatives or less.
[2]We will call this property "chiral-linear duality".

of the brane modes since only a limited class of chiral multiplets couplings is allowed by chiral-linear duality [87]. Another example is the fact that M-theory on $\mathcal{K}_7$ can be defined by specific Bianchi identities. Their symmetry properties provide information on the supergravity multiplets to be used in their effective description. In the preceding chapter, we have formulated the effective supergravity of M-theory on $\mathcal{K}_7$ without five-branes using Lagrange multiplier superfields to impose by their field equations these Bianchi identities and all their symmetries. This formulation is well adapted to the inclusion of five-brane modes.

The construction reveals some interesting features. It turns out that the contributions of the five-brane universal modulus are closely similar to the perturbative corrections generated by volume moduli [69, 56, 42, 43]. In particular, gauge threshold corrections arise, with a gauge-group independent term linked by supersymmetry to brane kinetic terms. This correction can be regarded as a renormalization of the dilaton field. There is no induced superpotential and the vacuum properties of the scalar potential are not modified. The physics impact of the five-brane fields is in the modification of the M-theory background equation (the "cohomology condition" [230]) and in the gauge-group-dependent threshold corrections, as observed by LOW [138].

In section 6.1, we study the role of the six-form field which couples naturally to the five-brane. We construct a version of the bosonic sector of eleven-dimensional supergravity in which the field equation of the six-form field is the required Bianchi identity. This theory can then easily be coupled to contributions arising from $S^1/\mathbb{Z}_2$ fixed planes or from five-branes. Its reduction on $\mathcal{K}_7$ provides the link with the effective four-dimensional supergravity derived in the preceding chapter and a guiding line for the introduction of five-brane fields. Section 6.2 is devoted to the dynamics of the five-brane massless modes. Our starting point is the self-dual formulation of the bosonic five-brane action, with an auxiliary scalar, as derived by Pasti, Sorokin and Tonin (PST) [167, 13]. The $\mathcal{K}_7$ truncation is performed and supersymmetrized, first in flat space, second in an eleven-dimensional supergravity background. The resulting kinetic Lagrangian for the five-brane modulus multiplet possesses as expected chiral-linear duality: the brane modulus can be either described by a linear supermultiplet $\hat{L}$ or by a chiral $\hat{S}$ with symmetry $\hat{S} \to \hat{S} + ic$ (where c is a real constant). This invariance severely restricts the possible form of the brane contributions in the Lagrangian. We also discuss how the various contributions to the scalar potential cancel each other. The complete effective supergravity coupled to the five-brane Lagrangian is the subject of section 6.3. Following the procedure valid for the Calabi–Yau volume modulus T, we introduce threshold corrections as the most general term allowed by the shift symmetry acting on the brane multiplet $\hat{S}$. We then consider the two dual versions of the effective supergravity, with the dilaton embedded either in a chiral or in a linear multiplet. The analysis of the gauge couplings in the linear version reveals a universal quadratic correction generated by the brane kinetic terms, and a linear dependence in the threshold terms. In the chiral version of the theory, the quadratic correction is moved into the Kähler potential of the chiral dilaton $S + \overline{S}$. This result is strongly similar to standard gauge threshold corrections in the modulus T, which are perturbative one-loop contributions in string theory. We then compare our expressions with the background found by LOW and discuss the modifications of the scalar potential introduced by the brane modulus. The final section 6.4 contains some observations and a few suggestions about possible future developments of our work.

6.1 The six-form field

In this section, we first discuss a formulation of the bosonic sector of eleven-dimensional supergravity in which the Bianchi identity for the four-form G_4 is explicitly given by the field equation of a six-form C_6. This eleven-dimensional field plays the role of a Lagrange multiplier and its Lagrangian can be easily modified to include source contributions arising, for instance, from five-branes.[3] Explicitly, the modified Lagrangian is of the form $C_6 \wedge (dG_4 - \Delta_5)$, where Δ_5 is the five-form source of the Bianchi identity. It also turns out to be at the origin of the four-dimensional "Lagrange multiplets" described in the preceding chapter, where Bianchi identities were field equations. After having introduced C_6 at the level of the bosonic sector of the standard CJS eleven-dimensional supergravity [51], we consider the modifications required by the two ten-dimensional planes fixed under $\mathbb{Z}_2$ and by the presence of five-branes.

6.1.1 Eleven-dimensional supergravity

We begin by considering the standard CJS formulation [51]. In terms of differential forms, the bosonic part of the eleven-dimensional supergravity action is given by

$$2\kappa_{11}^2 S_{\rm CJS} = -\int_{\mathcal{M}_{11}} d^{11}x\, e_{11} R - \frac{1}{2}\int_{\mathcal{M}_{11}} G_4 \wedge *G_4 - \frac{1}{6}\int_{\mathcal{M}_{11}} C_3 \wedge G_4 \wedge G_4, \qquad (6.2)$$

where the two independent fields are the metric (vielbein) and the three-form potential C_3. The four-form field strength G_4 is defined by $G_4 = dC_3$, and $\mathcal{M}_{11}$ is eleven-dimensional Minkowski space. The equation of motion for C_3 that can be computed from the action (6.2) is

$$C_3 \,:\; d *G_4 = -\frac{1}{2} G_4 \wedge G_4, \qquad (6.3)$$

and the Bianchi identity reads

$$dG_4 \equiv d(dC_3) = 0. \qquad (6.4)$$

Note that the action $S_{\rm CJS}$ is invariant under the standard gauge transformation

$$C_3 \rightarrow C_3 + d\Lambda_2, \qquad (6.5)$$

where Λ_2 is a two-form.

Since we would like to incorporate "magnetic" five-branes[4] in our discussion, it is natural to look for an action which contains a seven-form field strength G_7 dual to the usual four-form G_4. The structure of the topological term $C_3 \wedge G_4 \wedge G_4$ in (6.2) does not allow us to completely eliminate the three-form C_3, and an action equivalent to $S_{\rm CJS}$ is

$$\begin{aligned} 2\kappa_{11}^2 S_{\rm 11sd} \;=\; & -\int_{\mathcal{M}_{11}} d^{11}x\, e_{11} R - \frac{1}{2}\int_{\mathcal{M}_{11}} G_4 \wedge *G_4 - \frac{1}{6}\int_{\mathcal{M}_{11}} C_3 \wedge dC_3 \wedge dC_3 \\ & + \int_{\mathcal{M}_{11}} G_7 \wedge (G_4 - dC_3), \end{aligned} \qquad (6.6)$$

[3] Our procedure is similar, but not identical, to the method of de Alwis [55].

[4] As opposed to the "electric" membranes which naturally couple to the CJS action (6.2).

where the four independent fields are now the metric (vielbein), the three-form C_3, the four-form G_4 and the seven-form G_7. The equations of motion for the antisymmetric tensor fields are

$$\begin{aligned} C_3 &: dG_7 = -\frac{1}{2}dC_3 \wedge dC_3, \\ G_4 &: *G_4 = G_7, \\ G_7 &: G_4 = dC_3. \end{aligned} \tag{6.7}$$

Inserting the second and third relations in the first one leads to the original field equation in the CJS version of the theory. The solution of the equation for C_3 is

$$G_7 = -dC_6 - \frac{1}{2}C_3 \wedge dC_3, \tag{6.8}$$

where C_6 is an arbitrary six-form potential. Notice that the invariance of the seven-form G_7 under the gauge transformation (6.5) imposes that

$$C_6 \to C_6 - \frac{1}{2}\Lambda_2 \wedge dC_3 + d\Lambda_5. \tag{6.9}$$

We can now write another equivalent form of the bosonic sector of eleven-dimensional supergravity in which the Bianchi identity is imposed via the six-form field C_6. Substituting the expression (6.8) for G_7 into the action (6.6), we obtain (with a partial integration) a formulation where the four independent fields are the metric, C_3, G_4 and C_6:

$$\begin{aligned} 2\kappa_{11}^2 S_{11\mathrm{sd}'} &= -\int_{\mathcal{M}_{11}} d^{11}x\, e_{11} R - \frac{1}{2}\int_{\mathcal{M}_{11}} G_4 \wedge *G_4 - \frac{1}{2}\int_{\mathcal{M}_{11}} C_3 \wedge dC_3 \wedge (G_4 - \frac{2}{3}dC_3) \\ &\quad + \int_{\mathcal{M}_{11}} C_6 \wedge dG_4. \end{aligned} \tag{6.10}$$

This action is invariant under the gauge symmetries (6.5) and (6.9). The equations of motion for G_4, C_6 and C_3 are now

$$\begin{aligned} G_4 &: & *G_4 &= -dC_6 - \frac{1}{2}C_3 \wedge dC_3, \\ C_6 &: & dG_4 &= 0, \\ C_3 &: & dC_3 \wedge (dC_3 - G_4) &= -\frac{1}{2}C_3 \wedge dG_4. \end{aligned} \tag{6.11}$$

The exterior derivative of the first equation is the CJS equation (6.3) if in addition $G_4 = dC_3$. The second relation is the Bianchi identity (6.4) which says that locally $G_4 = dA_3$. Finally, the third equation implies that C_3 and A_3 can differ by a gauge transformation (6.5) and by irrelevant particular solution to Eq. (6.3).

The truncation of the theory (6.10) on $\mathcal{K}_7$ is as follows. The fields C_3 and G_4 are as usual odd under the symmetry $\mathbb{Z}_2$, while C_6 is even. Since the $\mathbb{Z}_2$ symmetry acts on the S^1

coordinate x^{11}, the universal massless modes of the six-form surviving the truncation will then be $C_{\mu\nu\rho\sigma i\bar{j}}$, $C_{\mu\nu\rho ijk}$,[5] $C_{\mu\nu ij\bar{i}\bar{j}}$, and $C_{ijk\bar{i}\bar{j}\bar{k}}$. Their field equations respectively imply the Bianchi identity for $(dG_4)_{ij\bar{i}\bar{j}11}$ which is a background equation[6], and the four-dimensional Bianchi identities for the massless components $G_{\bar{i}\bar{j}\bar{k}11}$, $G_{\rho k\bar{k}11}$ and $G_{\mu\nu\rho 11}$. Observe also that the modified topological term in the action (6.10) does not survive the truncation.

6.1.2 Defects contributions

If one assumes that the Bianchi identity is not $dG_4 = 0$, but instead $dG_4 = \Delta_5$ with an exact five-form $\Delta_5 = d\Delta_4$, not depending on C_3, G_4 or C_6, the action (6.10) can be consistently modified to become:

$$\begin{aligned} 2\kappa_{11}^2 S &= -\frac{1}{2}\int_{\mathcal{M}_{11}} G_4 \wedge *G_4 - \frac{1}{2}\int_{\mathcal{M}_{11}} C_3 \wedge dC_3 \wedge (G_4 - \Delta_4 - \frac{2}{3}dC_3) \\ &\quad + \int_{\mathcal{M}_{11}} C_6 \wedge (dG_4 - \Delta_5) + \text{Einstein term} + \cdots. \end{aligned} \tag{6.12}$$

The eleven-dimensional independent fields are C_3, G_4 and C_6. The term with C_6 is modified to obtain the new Bianchi identity with the source Δ_5. The additional $\frac{1}{2}C_3 \wedge dC_3 \wedge \Delta_4$ term is a possible addition to cancel the variation under (6.5) of the source contribution $-C_6 \wedge \Delta_5$. We will however see below that if a five-brane is at the origin of the source Δ_5, another modification arises. The dots in action (6.12) denote possible contributions which do not involve the eleven-dimensional bulk fields and are related to the dynamics of the magnetic source Δ_5. An example would be the ten-dimensional kinetic terms for the gauge fields in a compactification of M-theory on $S^1/\mathbb{Z}_2$. The equations of motion for C_3, G_4 and C_6 are

$$\begin{aligned} C_3 \;:\; dC_3 \wedge (dC_3 - G_4 + \Delta_4) &= -\frac{1}{2}C_3 \wedge (dG_4 - \Delta_5), \\ G_4 \;:\; *G_4 &= -dC_6 - \frac{1}{2}C_3 \wedge dC_3, \\ C_6 \;:\; dG_4 &= \Delta_5. \end{aligned} \tag{6.13}$$

A compactification of M-theory on $S^1/\mathbb{Z}_2$ or $\mathcal{K}_7$ has at least two kinds of defects generating sources: M-five-branes and $\mathbb{Z}_2$ fixed planes. A tensor $N = 2$ multiplet of massless excitations lives on the world-volume of each five-brane [98, 127] and an E_8 super Yang–Mills multiplet is located on each fixed plane [116, 230, 117].

Five-brane contributions

A five-brane in an eleven-dimensional background can be described by the following world-volume bosonic action [219, 1, 21]

$$S_{\text{M5}} = \int_{\mathcal{W}_6} \mathcal{L}_{\text{kin.}} - T_5 \int_{\mathcal{W}_6} \hat{C}_6 - \frac{T_5}{2}\int_{\mathcal{W}_6} \hat{C}_3 \wedge d\mathcal{B}_2, \tag{6.14}$$

[5] As well as the conjugate $C_{\mu\nu\rho\bar{i}\bar{j}\bar{k}} = C^*_{\mu\nu\rho ijk}$.

[6] Since it is not associated to any four-dimensional massless mode.

where the hatted fields $\hat{C}_3$ and $\hat{C}_6$ are the eleven-dimensional background fields pulled back onto the six-dimensional world-volume $\mathcal{W}_6$, and the two-form $\mathcal{B}_2$ belongs to the $d = 6$ supermultiplet of the five-brane (see section 6.2 for a more precise description of this multiplet). The first contribution $\mathcal{L}_{\text{kin.}}$ describes the kinematics of the bosonic degrees of freedom. It includes a Born–Infeld term for the induced metric tensor $\hat{g}_{\hat{m}\hat{n}}$ coupled to the three-form $\mathcal{H}_3 \equiv d\mathcal{B}_2 - \hat{C}_3$, which is submitted to a self-duality condition. In the covariant formalism of PST [164, 165, 166], this self-duality condition is generated by an auxiliary scalar field $\mathcal{A}$. Hence, $\mathcal{L}_{\text{kin.}}$ is a functional of $\hat{g}_{\hat{m}\hat{n}}$, $\mathcal{H}_3$ and $\mathcal{A}$, but its precise form is not important for the moment. Notice that invariance under (6.5) of $\mathcal{H}_3$ implies that $\delta\mathcal{B}_2 = \hat{\Lambda}_2$. With this transformation of $\mathcal{B}_2$, the complete action S_{M5} is gauge invariant.

The five-brane action S_{M5} includes the C_6 term

$$-T_5 \int_{\mathcal{W}_6} \hat{C}_6 = -T_5 \int_{\mathcal{M}_{11}} C_6 \wedge \delta_5, \tag{6.15}$$

where the equality for an arbitrary six-form would define the closed delta-function five-form δ_5. Comparison with the C_6 term in action (6.12) indicates that adding a five-brane contribution modifies the source Δ_5 according to $\Delta_5 \to \Delta_5 + 2\kappa_{11}^2 T_5 \delta_5$, without however affecting Δ_4 since gauge invariance is obtained with the new contribution $-\frac{T_5}{2}\int_{\mathcal{M}_{11}} C_3 \wedge d\mathcal{B}_2 \wedge \delta_5$.

Using δ_5 to rewrite the action (6.14) as

$$S_{\text{M5}} = \int_{\mathcal{M}_{11}} \mathcal{L}_{\text{kin.}} \wedge \delta_5 - T_5 \int_{\mathcal{M}_{11}} C_6 \wedge \delta_5 - \frac{T_5}{2} \int_{\mathcal{M}_{11}} C_3 \wedge d\mathcal{B}_2 \wedge \delta_5, \tag{6.16}$$

we obtain a complete action from which the modified Bianchi identity with a five-brane source added can be deduced as an equation of motion for the six-form C_6:

$$\begin{aligned} 2\kappa_{11}^2 S \;=\; & -\frac{1}{2}\int_{\mathcal{M}_{11}} G_4 \wedge *G_4 - \frac{1}{2}\int_{\mathcal{M}_{11}} C_3 \wedge dC_3 \wedge (G_4 - \Delta_4 - \frac{2}{3} dC_3) \\ & + \int_{\mathcal{M}_{11}} C_6 \wedge (dG_4 - \Delta_5 - 2\kappa_{11}^2 T_5 \delta_5) \\ & + 2\kappa_{11}^2 \int_{\mathcal{M}_{11}} \mathcal{L}_{\text{kin.}} \wedge \delta_5 - \kappa_{11}^2 T_5 \int_{\mathcal{M}_{11}} C_3 \wedge d\mathcal{B}_2 \wedge \delta_5 \\ & + \text{Einstein term} + \cdots, \end{aligned} \tag{6.17}$$

where the independent fields are the metric, C_3, G_4, C_6 and $\mathcal{B}_2$.[7] The equations of motion

[7] As well as the translational degrees of freedom of the five-brane world-volume, in the pull-back of $\mathcal{M}_{11}$ onto $\mathcal{W}_6$.

for C_3, G_4, C_6 and $\mathcal{B}_2$ are

$$\begin{aligned}
C_3 &: & dC_3\wedge(dC_3 - G_4+\Delta_4) &= -2\kappa_{11}^2\Big(\frac{\delta\mathcal{L}_{\text{kin.}}}{\delta C_3} - \frac{T_5}{2}d\mathcal{B}_2\Big)\wedge\delta_5 - \frac{1}{2}C_3\wedge(dG_4-\Delta_5),\\
G_4 &: & *G_4 &= -dC_6 - \frac{1}{2}C_3\wedge dC_3,\\
C_6 &: & dG_4 &= \Delta_5 + 2\kappa_{11}^2 T_5\delta_5,\\
\mathcal{B}_2 &: & \frac{\delta\mathcal{L}_{\text{kin.}}}{\delta\mathcal{B}_2}\wedge\delta_5 &= \frac{T_5}{2}dC_3\wedge\delta_5.
\end{aligned}\tag{6.18}$$

Taking the exterior derivative of the first equation and using the equality $d(\frac{\delta\mathcal{L}_{\text{kin.}}}{\delta C_3}) = \frac{\delta\mathcal{L}_{\text{kin.}}}{\delta\mathcal{B}_2}$ which follows from the fact that C_3 and $\mathcal{B}_2$ only appear through $\mathcal{H}_3$ in $\mathcal{L}_{\text{kin.}}$, we recover the last equation when the third relation is taken into account.

This discussion can be easily extended to a configuration with several five-branes. In the case of M-theory on $\mathcal{K}_7$, five-brane world-volumes must be embedded in $\mathcal{M}_{11}$ in a $\mathbb{Z}_2$-invariant way.

$\mathbb{Z}_2$-fixed planes contributions

We now proceed to add the contributions due to $\mathbb{Z}_2$ fixed planes. They will correspond to specific expressions for the source Δ_5 and its primitive Δ_4 in action (6.17). And since these expressions do not depend on the eleven-dimensional or five-brane fields, the equations (6.18) and their significance will remain unchanged.

In the presence of five-branes, M-theory on $S^1/\mathbb{Z}_2\times\mathcal{M}_{10}$ can be defined by the following Bianchi identity [230, 116, 117, 233]:

$$dG_4 = -(4\pi)^2\frac{\kappa_{11}^2}{\lambda^2}\Big[I_{4,1}\wedge\delta_{1,1} + I_{4,2}\wedge\delta_{1,2} + \sum_{f=1}^{N_5}\delta_5(\mathcal{W}_{6,f})\Big],\tag{6.19}$$

where $\mathcal{W}_{6,f}$ is the world-volume of the fth five-brane and $\delta_5(\mathcal{W}_{6,f})$ the corresponding five-form as defined in Eq. (6.15). The $S^1/\mathbb{Z}_2$ direction x^{11} has periodicity 2π, $\mathbb{Z}_2$ acts according to $x^{11}\to -x^{11}$ and the fixed points are at $x^{11}=0$ and π. For each five brane with world-volume $\mathcal{W}_{6,f}$, there exists a five-brane with world-volume given by the image under $\mathbb{Z}_2$ of $\mathcal{W}_{6,f}$. Eq. (6.19) also gives the expression $T_5 = -8\pi^2/\lambda^2$ in terms of the gauge coupling constant λ on the ten-dimensional fixed planes. The Dirac one-forms on S^1 read

$$\delta_{1,1} = \delta(x^{11})\,dx^{11} \quad \text{and} \quad \delta_{1,2} = \delta(x^{11}-\pi)\,dx^{11}.\tag{6.20}$$

Finally, on the ten-dimensional $\mathbb{Z}_2$ fixed planes, at $x^{11}=0$ and $x^{11}=\pi$, live four-forms

$$I_{4,i} = \frac{1}{(4\pi)^2}\Big[\text{tr}F_{2,i}^2 - \frac{1}{2}\text{tr}R_2^2\Big],\quad i=1,2,\tag{6.21}$$

where each $F_{2,i}$ is an E_8 gauge curvature and R_2 is the Riemann curvature. We then conclude that the appropriate bosonic action for M-theory on $S^1/\mathbb{Z}_2$ can be written as:

$$\begin{aligned} S &= \int_{\mathcal{M}_{11}} \mathcal{L}, \\ 2\kappa_{11}^2 \mathcal{L} &= -\frac{1}{2} G_4 \wedge * G_4 - \frac{1}{2} C_3 \wedge dC_3 \wedge (G_4 - \Delta_4 - \frac{2}{3} dC_3) \\ &\quad + C_6 \wedge \Big(dG_4 + (4\pi)^2 \frac{\kappa_{11}^2}{\lambda^2} \Big[I_{4,1} \wedge \delta_{1,1} + I_{4,2} \wedge \delta_{1,2} + \sum_{f=1}^{N_5} \delta_5(\mathcal{W}_{6,f}) \Big] \Big) \\ &\quad + 2\kappa_{11}^2 \sum_{f=1}^{N_5} \Big(\mathcal{L}_{\mathrm{kin.}}(\mathcal{H}_{3(f)}) + \frac{4\pi^2}{\lambda^2} C_3 \wedge d\mathcal{B}_{2(f)} \Big) \wedge \delta_5(\mathcal{W}_{6,f}) \\ &\quad - \frac{\kappa_{11}^2}{\lambda^2} \Big(F_{2,1} \wedge * F_{2,1} \wedge \delta_{1,1} + F_{2,2} \wedge * F_{2,2} \wedge \delta_{1,2} \Big) + \text{Einstein term.} \end{aligned} \tag{6.22}$$

The last line includes the kinetic terms of the E_8 gauge fields living on each fixed plane and the four-form Δ_4 is defined as the solution to the Bianchi identity (6.19) without any five-brane:

$$d\Delta_4 = -(4\pi)^2 \frac{\kappa_{11}^2}{\lambda^2} \Big[I_{4,1} \wedge \delta_{1,1} + I_{4,2} \wedge \delta_{1,2} \Big]. \tag{6.23}$$

Notice that each five-brane has its own tensor $\mathcal{H}_{3(f)} = d\mathcal{B}_{2(f)} - C_3$, up to the identification of a five-brane with its image under $\mathbb{Z}_2$.

It should be remarked that the theory (6.22) is not equivalent to the Hořava–Witten action. It is a generalization of the bosonic sector of eleven-dimensional supergravity and it does not include an anomaly-cancelling term similar to the contribution $\int_{\mathcal{M}_{11}} C_3 \wedge G_4 \wedge G_4$. Cancellation of chiral anomalies requires the addition of appropriate Green–Schwarz counterterms.

6.1.3 The background

The Bianchi identities of M-theory compactified on $\mathcal{K}_7$ are the components of Eq. (6.19) reduced on the Calabi–Yau space. They are also the field equations of the components of C_6 reduced on $\mathcal{K}_7$. Denoting by V_6 the Calabi–Yau volume and using $\kappa_{11}^2 = 2\pi V_6 \kappa^2$, where κ is the four-dimensional gravitational constant, one infers that the dimensionless number λ^2/V_6 can be absorbed in the metric moduli, so that these identities as well as the four-dimensional reduced action depend on a single parameter, the four-dimensional gravitational constant.[8] As mentioned earlier, the field equation of the component $C_{\mu\nu\rho\sigma i\bar{j}}$ is the background equation

$$(dG_4)_{ij\bar{k}\bar{l}11} = -(4\pi)^2 \frac{\kappa_{11}^2}{\lambda^2} \Big[I_{4,1} \wedge \delta_{1,1} + I_{4,2} \wedge \delta_{1,2} + \sum_{f=1}^{N_5} \delta_5(\mathcal{W}_{6,f}) \Big]_{ij\bar{k}\bar{l}11}. \tag{6.24}$$

[8]See the preceding chapter for a detailed discussion of this point. For the same reason, we may choose the S^1 radius to be one.

This equation integrated over a closed five-cycle gives, for a globally well-defined G_4, the standard "cohomology condition" which defines the embedding of the four-dimensional gauge group into $E_8 \times E_8$ [230]. In general, it implies non-zero background values for $(\mathrm{tr}F_{2,1}^2)_{ij\overline{kl}}$ and/or $(\mathrm{tr}F_{2,2}^2)_{ij\overline{kl}}$, and relates these vacuum values to the Calabi–Yau background $(\mathrm{tr}R_2^2)_{ij\overline{kl}}$. Since there are no massless fluctuations associated with this component of dG_4, we may assume that the fluctuation $C_{\mu\nu\rho\sigma i\bar{j}}$ is zero when computing the reduced effective Lagrangian, provided we develop the theory around the appropriate background.

We denote the form degree on $\mathcal{M}_4 \times \mathcal{K}_7$ as (m, n, p, q). The degree on $\mathcal{M}_4$ is m, the holomorphic and anti-holomorphic degrees on the Calabi–Yau space are n and p, and q is the degree on $S^1/\mathbb{Z}_2$. The $SU(3)$ holonomy condition implies that the background $\langle G_4 \rangle$ is a $(0,2,2,0)$ form. The defining equation for the six-form field C_6 is the duality equation $*G_4 = -dC_6 - \frac{1}{2} C_3 \wedge dC_3$. The background $\langle dC_6 \rangle$ is then a $(4,1,1,1)$ form. By $SU(3)$ holonomy and $\mathbb{Z}_2$ symmetry, the background component $\langle C_6 \rangle$ of C_6 is a $(4,1,1,0)$ form and $\langle dC_6 \rangle = \frac{\partial}{\partial x^{11}} \langle C_6 \rangle \, dx^{11}$. The equations defining the background are then[9] $\langle dC_6 \rangle = - * \langle G_4 \rangle$ and

$$\langle d * dC_6 \rangle = -(4\pi)^2 \frac{\kappa_{11}^2}{\lambda^2} \Big\langle I_{4,1} \wedge \delta_{1,1} + I_{4,2} \wedge \delta_{1,2} + \sum_{f=1}^{N_5} \delta_5(\mathcal{W}_{6,f}) \Big\rangle. \tag{6.25}$$

They depend in general on the metric tensor reduced on $\mathcal{M}_4 \times \mathcal{K}_7$ since they use the Hodge dual and Dirac tensorial distributions. This condition has been studied in detail in Refs. [136, 138].

In our approach based on Lagrangian (6.22), however, the background contribution is more involved. Since the six-form field multiplies the Bianchi identity, all C_6 background contributions automatically cancel. But the background values of $G_4 \wedge *G_4$, of the Einstein term and of the gauge and brane (Born–Infeld) kinetic terms are non-zero. We will return to this point when computing the effective four-dimensional scalar potential, which vanishes, in the next section.

Our task now is to obtain the four-dimensional Calabi–Yau reduction of the action (6.22), and to extend it to a Poincaré $N = 1$ supergravity. Without five-branes, the result is well-known either from heterotic superstrings on $\mathcal{X}_6$ [227, 58, 35] or from M-theory on $\mathcal{K}_7$ [157, 158, 136], and the preceding chapter gives a discussion based on Bianchi identities which is also the approach followed here.

6.2 The M-theory five-brane

The action (6.22) includes kinetic terms for the five-brane bosonic degrees of freedom, which in particular are responsible for the propagation of the self-dual three-form $\mathcal{H}_3$. Since we find useful to incorporate in our discussion the largest possible symmetry, we will use for these kinetic terms the formalism developed by PST [164, 165, 166] adapted to the five-brane [167, 13, 47]. In this covariant formulation, the self-duality follows from field equations.

The five-brane has also scalar fields related to the translational modes of its world-volume, and we begin by a brief discussion of the embedding of a world-volume $\mathcal{W}_6$ in $\mathcal{M}_4 \times \mathcal{K}_7$.

[9] When reduced on $\mathcal{M}_4 \times \mathcal{K}_7$, $\langle C_3 \wedge dC_3 \rangle$ vanishes.

6.2.1 Reduction of the M-five-brane bosonic action to four dimensions

In order to preserve $N=1$ four-dimensional supersymmetry, the world-volume of the five-brane, $\mathcal{W}_6$, must include four-dimensional Minkowski space $\mathcal{M}_4$ (to keep Lorentz covariance) and the two additional space-like dimensions have to be a holomorphic cycle $\mathcal{C}_2$ in the Calabi–Yau manifold [19, 29, 230]. We can then choose the five-brane world-volume coordinates as

$$y^{\hat{m}} = (y^\mu, y, \overline{y}), \quad \hat{m} = \hat{1}, \hat{2}, \ldots, \hat{6}, \quad \mu = 1, 2, 3, 4, \tag{6.26}$$

with a complex coordinate y along the Calabi–Yau two-cycle. The embedding of the world-volume in $\mathcal{M}_{11}$ is defined by the functions $x^M(y^{\hat{m}})$, $M = 1, 2, \ldots, 11$, and by the pull-back functions $\partial x^M/\partial y^{\hat{m}}$. In $\mathcal{M}_4 \times \mathcal{K}_7$, we use coordinates $x^M = (x^\mu, x^{11}, z^i, \overline{z}^{\overline{i}})$, $i = 1, 2, 3$, with

$$x^\mu = y^\mu, \quad z^i = z^i(y^\mu, y), \quad \overline{z}^{\overline{i}} = \overline{z}^{\overline{i}}(y^\mu, \overline{y}), \tag{6.27}$$

choosing a parameterization of $\mathcal{M}_4$.

The five-brane excitations are described by a $d=6$ tensor supermultiplet [98, 127]. The fields are a chiral antisymmetric tensor $\mathcal{B}_{\hat{m}\hat{n}}$ (with a self-dual field strength $\mathcal{H}_{\hat{m}\hat{n}\hat{p}}$), five scalar fields $X^{(1)}, X^{(2)}, \ldots, X^{(5)}$ specifying the position of the world-volume $\mathcal{W}_6$ in $\mathcal{M}_{11}$, and their fermionic partners. In our $\mathcal{M}_4 \times \mathcal{K}_7$ reduction, we neglect the detailed structure of the Calabi–Yau manifold. Of the five scalar fields, only one survives as the massless mode of the Calabi–Yau expansion of $x^{11}(y^\mu, y, \overline{y})$,

$$\begin{aligned} x^{11}(y^\mu, y, \overline{y}) &= X(x^\mu) + \text{massive modes}, \\ z^i = z^i(y), \quad \overline{z}^{\overline{i}} &= \overline{z}^{\overline{i}}(\overline{y}), \quad x^\mu = y^\mu. \end{aligned} \tag{6.28}$$

This means that we will only retain the following bosonic five-brane excitations:

$$\mathcal{B}_{\mu\nu}(x^\mu) \equiv \mathcal{B}_{\mu\nu}(y^\mu), \quad \mathcal{B}_{\hat{5}\hat{6}}(x^\mu) \equiv \mathcal{B}_{\hat{5}\hat{6}}(y^\mu), \quad X(x^\mu), \tag{6.29}$$

and the self-duality condition on $\mathcal{H}_{\hat{m}\hat{n}\hat{p}}$ relates $\mathcal{B}_{\mu\nu}$ and $\mathcal{B}_{\hat{5}\hat{6}}$. The background value of the scalar field X is the five-brane position along the $S^1/\mathbb{Z}_2$ orbifold direction x^{11}. Each five-brane generates then in $\mathcal{M}_4$ two bosonic degrees of freedom. By $N=1$ supersymmetry, they will be described by either a chiral or a linear multiplet.

With these choices of embedding and truncation, the world-volume induced metric[10]

$$\hat{g}_{\hat{m}\hat{n}} = \frac{\partial x^M}{\partial y^{\hat{m}}} \frac{\partial x^N}{\partial y^{\hat{n}}} g_{MN} \tag{6.30}$$

reduces in four dimensions to

$$\begin{aligned} \hat{g}_{\mu\nu} &= e^{-\gamma-2\sigma} g_{\mu\nu} + e^{2\gamma-2\sigma}(\partial_\mu X)(\partial_\nu X), \\ \hat{g}_{\hat{5}\hat{6}} &= k^2 e^\sigma, \\ \hat{g}_{\mu\hat{5}} &= \hat{g}_{\mu\hat{6}} = \hat{g}_{\hat{5}\hat{5}} = \hat{g}_{\hat{6}\hat{6}} = 0, \end{aligned} \tag{6.31}$$

[10] The two-index tensor g_{MN} is the eleven-dimensional metric which was defined and used in the preceding chapter [see Eq. (5.32)].

where $k^2 = \delta_{i\bar{i}} \frac{\partial z^i}{\partial y} \frac{\partial \overline{z}^{\bar{i}}}{\partial \overline{y}}$ is a constant (a background value) in our Kaluza–Klein truncation.

To describe the dynamics of the bosonic fields (6.29) and their couplings to four-dimensional supergravity, we need an action for the five-brane coupled to eleven-dimensional supergravity. Using the general formalism developed by PST [164, 165, 166] to write covariant Lagrangians for self-dual (or anti self-dual) tensors, a kappa-symmetric covariant world-volume Lagrangian for the five-brane excitations has been constructed [167, 13, 47], completing earlier work [119, 120, 193, 168, 233, 219, 1, 21]. In a non-trivial eleven-dimensional supergravity background, the action has two parts: a kinetic Lagrangian with a Born–Infeld term involving the three-index tensor $\mathcal{H}_{\hat{m}\hat{n}\hat{p}}$ and a Wess–Zumino term involving both C_3 and its dual C_6. The bosonic action is:[11]

$$
\begin{aligned}
S_{\rm M5} \;&=\; T_5 \int_{\mathcal{W}_6} d^6y \Big(-\sqrt{-\det(\hat{g}_{\hat{m}\hat{n}} + i\mathcal{H}^*{}_{\hat{m}\hat{n}})} - \frac{1}{4}\sqrt{-\hat{g}}\, \mathcal{V}_{\hat{l}} \mathcal{H}^{*\hat{l}\hat{m}\hat{n}} \mathcal{H}_{\hat{m}\hat{n}\hat{p}} \mathcal{V}^{\hat{p}} \Big) \\
&\quad -T_5 \int_{\mathcal{W}_6} \Big(\hat{C}_6 - \frac{1}{2} d\mathcal{B}_2 \wedge \hat{C}_3 \Big).
\end{aligned}
\tag{6.32}
$$

The second line is as in Eq. (6.14) and the first two terms define the kinetic Lagrangian in the PST formalism. In this expression, $\hat{C}_3$ and $\hat{C}_6$ are the eleven-dimensional background fields pulled back onto the world-volume using derivatives $\partial x^M / \partial y^{\hat{m}}$,

$$
\hat{C}_{\hat{m}\hat{n}\hat{p}} = \frac{\partial x^M}{\partial y^{\hat{m}}} \frac{\partial x^N}{\partial y^{\hat{n}}} \frac{\partial x^P}{\partial y^{\hat{p}}} C_{MNP}, \qquad \hat{C}_{\hat{m}_1 \ldots \hat{m}_6} = \frac{\partial x^{M_1}}{\partial y^{\hat{m}_1}} \cdots \frac{\partial x^{M_6}}{\partial y^{\hat{m}_6}} C_{M_1 \ldots M_6}, \tag{6.33}
$$

T_5 is the brane tension and

$$
\begin{aligned}
\mathcal{H}_{\hat{m}\hat{n}\hat{p}} &= 3\, \partial_{[\hat{m}} \mathcal{B}_{\hat{n}\hat{p}]} - \hat{C}_{\hat{m}\hat{n}\hat{p}}, \\
\mathcal{H}^*{}_{\hat{m}\hat{n}} &= \mathcal{H}^*_{\hat{m}\hat{n}\hat{p}} \mathcal{V}^{\hat{p}}, \\
\mathcal{H}^{*\hat{m}\hat{n}\hat{p}} &= \frac{-1}{3!\sqrt{-\hat{g}}}\, \epsilon^{\hat{m}\hat{n}\hat{p}\hat{q}\hat{r}\hat{s}} \mathcal{H}_{\hat{q}\hat{r}\hat{s}}, \\
d\mathcal{B}_2 &= \frac{1}{2}\, \partial_{\hat{m}} \mathcal{B}_{\hat{n}\hat{p}}\, dy^{\hat{m}} \wedge dy^{\hat{n}} \wedge dy^{\hat{p}}.
\end{aligned}
\tag{6.34}
$$

Finally,

$$
\mathcal{V}_{\hat{m}} = \frac{\partial_{\hat{m}} \mathcal{A}}{\sqrt{(\partial_{\hat{n}} \mathcal{A})(\partial^{\hat{n}} \mathcal{A})}}, \qquad (\mathcal{V}_{\hat{m}} \mathcal{V}^{\hat{m}} = 1), \tag{6.35}
$$

where $\mathcal{A}(y^{\hat{m}})$ is the auxiliary scalar field introduced by PST to impose the self-duality of the tensor $\mathcal{H}_{\hat{m}\hat{n}\hat{p}}$ as an equation of motion.

Since we will consider only four-dimensional contributions with up to two derivatives, it will be sufficient to write

$$
\sqrt{-\det(\hat{g}_{\hat{m}\hat{n}} + i\mathcal{H}^*{}_{\hat{m}\hat{n}})} \cong \sqrt{-\hat{g}} \Big(1 - \frac{1}{4} \mathcal{H}^{*\hat{m}\hat{n}} \mathcal{H}^*{}_{\hat{m}\hat{n}} \Big). \tag{6.36}
$$

[11] Our conventions are mostly as in Ref. [47]. We consider here a single five-brane.

The action (6.32) simplifies then to

$$\begin{aligned} S_{\rm M5} &= -T_5 \int_{\mathcal{W}_6} d^6y \sqrt{-\hat{g}} \Big(\frac{1}{4} \mathcal{V}_{\hat{l}} \mathcal{H}^{*\hat{l}\hat{m}\hat{n}} (\mathcal{H}_{\hat{m}\hat{n}\hat{p}} - \mathcal{H}^*{}_{\hat{m}\hat{n}\hat{p}}) \mathcal{V}^{\hat{p}} + 1 \Big) \\ &\quad -T_5 \int_{\mathcal{W}_6} \Big(\hat{C}_6 - \frac{1}{2} d\mathcal{B}_2 \wedge \hat{C}_3 \Big). \end{aligned} \tag{6.37}$$

The PST formalism possesses various local symmetries. One of them allows a gauge choice in which $\mathcal{A}(y^{\hat{m}})$ is a function of y and $\overline{y}$ only, so that

$$\mathcal{V}^\mu = 0, \qquad \mathcal{V}_{\hat{5}} \mathcal{V}^{\hat{5}} + \mathcal{V}_{\hat{6}} \mathcal{V}^{\hat{6}} = 1, \tag{6.38}$$

which preserves four-dimensional Lorentz covariance. With our truncation (6.29) of the five-brane excitations and of the bulk fields, we are led to only retain components

$$\begin{aligned} &\mathcal{H}_{\mu\hat{5}\hat{6}} = \partial_\mu \mathcal{B}_{\hat{5}\hat{6}} - \hat{C}_{\mu\hat{5}\hat{6}}, \qquad \mathcal{H}_{\mu\nu\rho} = 3\partial_{[\mu} \mathcal{B}_{\nu\rho]} - \hat{C}_{\mu\nu\rho}, \\ &\mathcal{B}_{\hat{5}\hat{6}} \equiv k^2 \mathcal{B}, \qquad \hat{C}_{\mu\hat{5}\hat{6}} = k^2 a(x)\, \partial_\mu X, \qquad \hat{C}_{\mu\nu\rho} = 3 C_{[\mu\nu 11}\, \partial_{\rho]} X, \end{aligned} \tag{6.39}$$

where $a(x)$ is defined by $C_{i\bar{j}11} = i a(x) \delta_{i\bar{j}}$. In addition, our reduction of the eleven-dimensional space-time metric (5.32) implies that

$$\sqrt{-\hat{g}} \cong k^2 e e^{-2\gamma - 3\sigma} \Big(1 + \frac{1}{2} e^{3\gamma} (\partial_\mu X)(\partial^\mu X) \Big), \tag{6.40}$$

where $e^2 = -\det(g_{\mu\nu})$ is now the determinant of the four-dimensional space-time metric.

The reduction of the term involving the six-form field follows from two facts. Firstly, with the embedding (6.28) of $\mathcal{W}_6$ into $\mathcal{M}_{11}$, one can write

$$\hat{C}_{\mu\nu\rho\sigma\hat{5}\hat{6}} = -i \frac{\partial z^i}{\partial y} \frac{\partial \overline{z}^{\bar{j}}}{\partial \overline{y}} \left[\langle C \rangle_{\mu\nu\rho\sigma i\bar{j}} + C_{\mu\nu\rho\sigma i\bar{j}} + 4 (\partial_{[\mu} X) C_{11\nu\rho\sigma] i\bar{j}} \right], \tag{6.41}$$

where $\langle C \rangle_{\mu\nu\rho\sigma i\bar{j}}$ is the background contribution discussed in subsection 6.1.3 and $C_{\mu\nu\rho\sigma i\bar{j}}$ is the field fluctuation. Notice that the equations defining this background involve the reduced eleven-dimensional metric[12] and $\langle C \rangle_{\mu\nu\rho\sigma i\bar{j}}$ does depend on the metric moduli σ and γ. Secondly, since C_6 is even under $\mathbb{Z}_2$, $C_{11\nu\rho\sigma i\bar{j}}$ is cancelled by the $\mathcal{K}_7$ reduction and the component $C_{\mu\nu\rho\sigma i\bar{j}}$ generates the background equation and can be omitted in the four-dimensional effective Lagrangian.

The four-dimensional five-brane action reads then

$$\begin{aligned} S_{\rm M5} &= \int_{\mathcal{M}_4} d^4x\, \mathcal{L}_{\rm M5}, \\ \mathcal{L}_{\rm M5} &= -\frac{\tilde{T}}{2} \Big[\frac{1}{3!} e e^{\gamma + 3\sigma} \mathcal{H}_{\mu\nu\rho} \mathcal{H}^{\mu\nu\rho} - \frac{1}{3!} \epsilon^{\mu\nu\rho\sigma} \Big(\partial_\mu \mathcal{B} - (\partial_\mu X) a \Big) \mathcal{H}_{\nu\rho\sigma} \\ &\quad - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\partial_\mu \mathcal{B})(\partial_\nu X) C_{\rho\sigma 11} - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\partial_\mu \mathcal{B}_{\nu\rho})(\partial_\sigma X) a \\ &\quad + e e^{\gamma - 3\sigma} (\partial_\mu X)(\partial^\mu X) + 2e(e^{-2\gamma - 3\sigma} + \langle C \rangle) \Big]. \end{aligned} \tag{6.42}$$

[12] In particular in the Hodge dual.

This derivation uses

$$T_5 \int_{\mathcal{W}_6} d^6y \sqrt{-\hat{g}}\,(\ldots) = \tilde{T} \int_{\mathcal{M}_4} d^4x\, e e^{-2\gamma-3\sigma} \left(1 + \frac{1}{2} e^{3\gamma} (\partial_\mu X)(\partial^\mu X)\right)(\ldots), \tag{6.43}$$

where

$$\frac{\tilde{T}}{T_5} = \int_{\mathcal{C}_2} dy d\overline{y}\, \frac{\partial z^i}{\partial y} \frac{\partial \overline{z}^{\overline{j}}}{\partial \overline{y}} \delta_{i\overline{j}} \tag{6.44}$$

is the volume of the holomorphic two-cycle in the Calabi–Yau manifold, and the definition $\langle C \rangle_{\mu\nu\rho\sigma i\overline{j}} = ie\epsilon_{\mu\nu\rho\sigma} \langle C \rangle \delta_{i\overline{j}}$. The scalar field $\mathcal{B}$ acts as a Lagrange multiplier. It imposes the constraint

$$\epsilon^{\mu\nu\rho\sigma} \partial_\mu \left(\mathcal{H}_{\nu\rho\sigma} + 3(\partial_\nu X) C_{\rho\sigma 11}\right) = \epsilon^{\mu\nu\rho\sigma} \partial_\mu \left(\mathcal{H}_{\nu\rho\sigma} + \hat{C}_{\nu\rho\sigma}\right) = 0. \tag{6.45}$$

Its solution is the second Eq. (6.39). We can then express the Lagrangian (6.42) as a function of the unconstrained fields $\mathcal{H}_{\mu\nu\rho}$, X and $\hat{\mathcal{B}} = \mathcal{B} - Xa$:

$$\begin{aligned} \mathcal{L}_{\mathrm{M5}} = & -\frac{\tilde{T}}{2} \Big[\frac{1}{3!} e e^{\gamma+3\sigma} \mathcal{H}_{\mu\nu\rho} \mathcal{H}^{\mu\nu\rho} + e e^{\gamma-3\sigma} (\partial_\mu X)(\partial^\mu X) \\ & + \frac{1}{3} X \epsilon^{\mu\nu\rho\sigma} \mathcal{H}_{\mu\nu\rho} (\partial_\sigma a) + \frac{1}{2} X^2 \epsilon^{\mu\nu\rho\sigma} (\partial_\mu a)(\partial_\nu C_{\rho\sigma 11}) \\ & - \frac{1}{3!} \epsilon^{\mu\nu\rho\sigma} (\partial_\mu \hat{\mathcal{B}}) \left(\mathcal{H}_{\nu\rho\sigma} - 3X \partial_\nu C_{\rho\sigma 11}\right) + 2e(e^{-2\gamma-3\sigma} + \langle C \rangle) \Big]. \end{aligned} \tag{6.46}$$

The last term seems to indicate the presence of a scalar potential. However, solving the equation defining the six-form background field shows a cancellation: the scalar potential vanishes as expected by the stability of the configuration which is protected by the residual supersymmetry [81, 220]. We will see in subsection 6.2.3 that the supermultiplet structure required to supersymmetrize this bosonic action does not allow the presence of a scalar potential.

The Lagrangians (6.42) and (6.46) are invariant under the residual symmetries:

$$\begin{gathered} \delta C_{\mu\nu 11} = 2\partial_{[\mu} \Lambda_{\nu]}, \\ \delta a = c, \qquad \delta \mathcal{B} = cX, \qquad \delta \hat{\mathcal{B}} = 0, \qquad c = \text{constant}. \end{gathered} \tag{6.47}$$

Note moreover that $\mathcal{B}$ appears in the Lagrangian (6.42) only through its derivatives, so the independent symmetry

$$\delta \mathcal{B} = c', \qquad c' = \text{constant}, \tag{6.48}$$

is also present. Solving for $\hat{\mathcal{B}}$ in Eq. (6.46) leads to a Lagrangian for $\mathcal{B}_{\mu\nu}$ and X, which will be supersymmetrized using a linear multiplet. And solving for $\mathcal{H}_{\mu\nu\rho}$ leads to a theory containing a chiral multiplet with scalar components $\hat{\mathcal{B}}$ and X. This chiral-linear duality is the four-dimensional consequence of the self-duality of the brane three-index tensor $\mathcal{H}_{\hat{m}\hat{n}\hat{p}}$, when expressed in the covariant formalism of PST.

We now consider the supersymmetrization in four space-time dimensions of the reduced five-brane Lagrangian, firstly without supergravity background, secondly with the coupling to the eleven-dimensional background fields.

6.2.2 Supersymmetrization without supergravity background

Our first goal is to identify the supermultiplet content of the effective four-dimensional supergravity expected to arise from our truncation of the five-brane spectrum. The simplest procedure is to consider the flat, zero-background limit of the five-brane Lagrangian (6.46), which becomes

$$\mathcal{L}_{\text{M5, flat}} = -\frac{\tilde{T}}{2}\left[\frac{1}{3!}\mathcal{H}_{\mu\nu\rho}(\mathcal{H}^{\mu\nu\rho} + \epsilon^{\mu\nu\rho\sigma}\,\partial_\sigma\mathcal{B}) + (\partial_\mu X)(\partial^\mu X)\right]. \tag{6.49}$$

Introducing for convenience the four-dimensional vector field

$$v^\mu = \frac{1}{3!}\epsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\nu\rho\sigma}, \tag{6.50}$$

we obtain

$$\mathcal{L}_{\text{M5, flat}} = \frac{\tilde{T}}{2}\Big[v^\mu\left(\partial_\mu\mathcal{B} + v_\mu\right) - (\partial_\mu X)(\partial^\mu X)\Big]. \tag{6.51}$$

Solving for v_μ leads to $v_\mu = -\frac{1}{2}\partial_\mu\mathcal{B}$, so that

$$\mathcal{L}_{\text{M5, flat}} = -\frac{\tilde{T}}{2}\left[\frac{1}{4}(\partial_\mu\mathcal{B})(\partial^\mu\mathcal{B}) + (\partial_\mu X)(\partial^\mu X)\right]. \tag{6.52}$$

Alternatively, solving for $\mathcal{B}$ gives

$$\partial_\mu v^\mu = 0 \quad\longrightarrow\quad \mathcal{H}_{\mu\nu\rho} = 3\partial_{[\mu}\mathcal{B}_{\nu\rho]}, \tag{6.53}$$

and we obtain the equivalent form of the Lagrangian

$$\mathcal{L}_{\text{M5, flat}} = -\frac{\tilde{T}}{2}\Big[\frac{1}{3!}\mathcal{H}^{\mu\nu\rho}\mathcal{H}_{\mu\nu\rho} + (\partial_\mu X)(\partial^\mu X)\Big]. \tag{6.54}$$

This discussion illustrates again how the six-dimensional self-duality condition on $\mathcal{H}_{\hat{m}\hat{n}\hat{p}}$ translates in the truncated four-dimensional theory into a duality equivalence of an antisymmetric tensor $\mathcal{B}_{\mu\nu}$ with a (pseudo)scalar $\mathcal{B}$.

We now observe that expression (6.51) is precisely the bosonic part of the supersymmetric Lagrangian

$$\mathcal{L}_{\text{flat}} = -\tilde{T}\int d^2\theta d^2\overline{\theta}\,\left(\hat{V}^2 - \frac{1}{2}(\hat{S} + \overline{\hat{S}})\hat{V}\right), \tag{6.55}$$

where $\hat{V}$ is a real vector superfield and $\hat{S}$ is a chiral superfield. Using the component expansions

$$\begin{aligned}\hat{V} &= \hat{C} + (\theta\sigma^\mu\overline{\theta})\hat{v}_\mu + \theta\theta(\hat{m} + i\hat{n}) + \overline{\theta\theta}(\hat{m} - i\hat{n})\\ &\quad + \theta\theta\overline{\theta\theta}(\hat{d} - \tfrac{1}{4}\Box\hat{C}) + \cdots,\\ \hat{S} &= \hat{s} - \theta\theta\hat{f}_s - i(\theta\sigma^\mu\overline{\theta})\partial_\mu\hat{s} + \tfrac{1}{4}\theta\theta\overline{\theta\theta}\Box\hat{s} + \cdots,\end{aligned} \tag{6.56}$$

where the dots indicate fermion contributions, the bosonic part of the supersymmetric Lagrangian (6.55) is

$$\begin{aligned}\mathcal{L}_{\text{flat, bos.}} &= \frac{\tilde{T}}{2}\Big[\hat{v}^\mu(\hat{v}_\mu - \partial_\mu \operatorname{Im}\hat{s}) - (\partial_\mu \hat{C})(\partial^\mu \hat{C}) \\ &\qquad -2\hat{d}(2\hat{C} - \operatorname{Re}\hat{s}) - 4(\hat{m}^2+\hat{n}^2) - \Big(\hat{f}_s(\hat{m}-i\hat{n}) + \text{c.c.}\Big)\Big],\end{aligned} \tag{6.57}$$

omitting a space-time derivative. The second line is auxiliary and vanishes when solving for either $\operatorname{Re}\hat{s}$ and $\hat{f}_s$ or $\hat{m}$, $\hat{n}$, $\hat{d}$ and $\hat{f}_s$. The first line is Eq. (6.51).

The chiral-linear duality present in the globally supersymmetric Lagrangian (6.55) is the sequel, in the truncated theory, of the self-duality property of the five-brane antisymmetric tensor. Explicitly, solving for the vector superfield $\hat{V}$ in Eq. (6.55) leads to $\hat{V} = \frac{1}{4}(\hat{S} + \overline{\hat{S}})$. For the bosonic components, this is $\hat{C} = \frac{1}{2}\operatorname{Re}\hat{s}$, $\hat{v}_\mu = \frac{1}{2}\partial_\mu \operatorname{Im}\hat{s}$ and $\hat{m} + i\hat{n} = -\frac{1}{4}\hat{f}_s$. The supersymmetric Lagrangian then becomes

$$\mathcal{L}_{\text{flat}} = \frac{\tilde{T}}{8}\int d^2\theta d^2\overline{\theta}\, \hat{S}\overline{\hat{S}} = -\frac{\tilde{T}}{8}\Big[(\partial_\mu \hat{s})(\partial^\mu \overline{\hat{s}}) - \hat{f}_s\overline{\hat{f}_s}\Big] + \text{fermionic terms.} \tag{6.58}$$

Alternatively, we can rewrite expression (6.55) as

$$\mathcal{L}_{\text{flat}} = -\tilde{T}\int d^2\theta d^2\overline{\theta}\, \hat{V}^2 - \frac{\tilde{T}}{8}\int d^2\theta\, \hat{S}\overline{\mathcal{D}\mathcal{D}}\hat{V} - \frac{\tilde{T}}{8}\int d^2\overline{\theta}\, \overline{\hat{S}}\mathcal{D}\mathcal{D}\hat{V}, \tag{6.59}$$

and solve for the chiral superfield $\hat{S}$, implying that $\hat{V}$ is a real linear multiplet $\hat{L}$, $\overline{\mathcal{D}\mathcal{D}}\hat{L} = \mathcal{D}\mathcal{D}\hat{L} = 0$. For the bosonic components, solving for $\hat{s}$ and $\hat{f}_s$ in expression (6.57) leads to $\hat{d} = \hat{m} = \hat{n} = 0$ and

$$\partial^\mu \hat{v}_\mu = 0 \qquad \longrightarrow \qquad \hat{v}_\mu = -\frac{1}{3!}\epsilon_{\mu\nu\rho\sigma}\mathcal{H}^{\nu\rho\sigma} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu b^{\rho\sigma}. \tag{6.60}$$

The Lagrangian becomes

$$\mathcal{L}_{\text{flat}} = -\tilde{T}\int d^2\theta d^2\overline{\theta}\, \hat{L}^2 = -\frac{\tilde{T}}{2}\Big[(\partial_\mu \hat{C})(\partial^\mu \hat{C}) + \frac{1}{3!}\mathcal{H}^{\mu\nu\rho}\mathcal{H}_{\mu\nu\rho}\Big] + \text{fermionic terms.} \tag{6.61}$$

6.2.3 Supersymmetrization with supergravity background

We now turn on the supergravity background and return to Lagrangian (6.46) to derive its supersymmetric extension.

The description in terms of superconformal multiplets of the supergravity bulk fields has been discussed in detail in Chapter 5. The dilaton and universal modulus are respectively described by two vector multiplets: V with weight $w = 2$, $n = 0$ and V_T with zero weights. Bianchi identities in $\mathcal{M}_{11}$ would constrain V to be linear and V_T to be $T + \overline{T}$ in terms of a chiral multiplet T. Writing the (bosonic) component expansions as

$$\begin{aligned} V &= (C, 0, H, K, v_\mu, 0, d - \Box C - \tfrac{1}{3}CR), \\ V_T &= (C_T, 0, H_T, K_T, T_\mu, 0, d_T - \Box C_T),\end{aligned} \tag{6.62}$$

the identification is

$$\begin{aligned} 4\kappa^2 C &= \frac{\lambda^2}{V_6} e^{-3\sigma}, & 4\kappa^2 v_\mu &= \frac{1}{2}\frac{\lambda^2}{V_6} e\epsilon_{\mu\nu\rho\sigma}\partial^\nu C^{\rho\sigma 11}, \\ C_T &= 2\frac{\lambda^2}{V_6} e^\gamma, & T_\mu &= 2\frac{\lambda^2}{V_6}\partial_\mu a. \end{aligned} \tag{6.63}$$

Since we may redefine the dimensionless quantity λ^2/V_6 by a scaling of the moduli, we take $\lambda^2/V_6 = 1$ from here on. To describe the five-brane degrees of freedom, we introduce as in the preceding subsection two supermultiplets: a vector supermultiplet $\hat{V}$ and a chiral supermultiplet $\hat{S}$. We choose them with zero conformal and chiral weights ($w = 0 = n$). Their bosonic component expansions are

$$\begin{aligned} \hat{V} &= (\hat{C}, 0, \hat{H}, \hat{K}, \hat{v}_\mu, 0, \hat{d} - \Box\hat{C}), \\ \hat{S} &= (\hat{s}, 0, -\hat{f}_s, i\hat{f}_s, i\partial_\mu\hat{s}, 0, 0). \end{aligned} \tag{6.64}$$

To bring the Lagrangian (6.46) in a form appropriate for supersymmetrization in terms of $\hat{V}$, $\hat{S}$, V and V_T, we observe that the dimensions of the brane fields $\hat{\mathcal{B}}$, X and $\mathcal{H}_{\mu\nu\rho}$ do not fit with those of components $\hat{s}$, $\hat{C}$ and $\hat{v}_\mu$.[13] Since the only scale in our four-dimensional Poincaré supergravity should be κ, we first introduce a dimensionless five-brane coupling constant

$$\tilde{T} = \frac{\tau}{\kappa^4}, \tag{6.65}$$

and perform the rescalings

$$\mathcal{H}_{\mu\nu\rho} = \kappa\tilde{\mathcal{H}}_{\mu\nu\rho}, \quad X = \kappa\tilde{X}, \quad \hat{\mathcal{B}} = \kappa\tilde{\mathcal{B}}. \tag{6.66}$$

The action (6.46) rewrites then as

$$\begin{aligned} \mathcal{L}_{\rm M5} = & -\frac{\tau}{2\kappa^2}\Big[\frac{1}{3!}ee^{\gamma+3\sigma}\tilde{\mathcal{H}}_{\mu\nu\rho}\tilde{\mathcal{H}}^{\mu\nu\rho} + ee^{\gamma-3\sigma}(\partial_\mu\tilde{X})(\partial^\mu\tilde{X}) \\ & +\frac{1}{3}\tilde{X}\epsilon^{\mu\nu\rho\sigma}\tilde{\mathcal{H}}_{\mu\nu\rho}(\partial_\sigma a) + \frac{1}{2}\tilde{X}^2\epsilon^{\mu\nu\rho\sigma}(\partial_\mu a)(\partial_\nu C_{\rho\sigma 11}) \\ & -\frac{1}{3!}\epsilon^{\mu\nu\rho\sigma}(\partial_\mu\tilde{\mathcal{B}})\left(\tilde{\mathcal{H}}_{\nu\rho\sigma} - 3\tilde{X}\partial_\nu C_{\rho\sigma 11}\right)\Big] - V_0, \end{aligned} \tag{6.67}$$

with an apparent scalar potential

$$V_0 = \frac{\tau}{\kappa^4}e(e^{-2\gamma-3\sigma} + \langle C\rangle). \tag{6.68}$$

Then, to go to the superconformal formalism, we recall that $1/\kappa^2$ is the Poincaré gauge-fixed value of the multiplet

$$\Upsilon = (S_0\overline{S}_0 V_T)^{3/2}(2V)^{-1/2}. \tag{6.69}$$

[13] Explicitly, one has $[\hat{\mathcal{B}}] = [X] = [\text{mass}]^{-1}$, $[\mathcal{H}_{\mu\nu\rho}] = [\text{mass}]^0$, $[\hat{s}] = [\hat{C}] = [\text{mass}]^0$ and $[\hat{v}_\mu] = [\text{mass}]^1$.

Suppose that we identify the scalar field $\tilde{X}$ with the lowest component $\hat{C}$ of the brane multiplet $\hat{V}$. Identifications (6.63) also indicate that $e^{-3\sigma}$ is the lowest component of $4V\Upsilon^{-1}$, while e^{γ} is the lowest component of $\frac{1}{2}V_T$. We then infer that the first line of the action (6.67) appears in the component expansion of

$$-\tau[VV_T\hat{V}^2]_D, \tag{6.70}$$

which is independent from Υ. Comparison of the $\hat{v}_\mu\hat{v}^\mu$ term with the $\tilde{\mathcal{H}}_{\mu\nu\rho}\tilde{\mathcal{H}}^{\mu\nu\rho}$ term in the actions leads then to the identifications

$$\hat{C} = \tilde{X}, \qquad \hat{v}_\mu = -\frac{e}{3!}\frac{1}{4\kappa^2 C}\epsilon_{\mu\nu\rho\sigma}\tilde{\mathcal{H}}^{\nu\rho\sigma}. \tag{6.71}$$

With these results, the vector component of $V\hat{V}$ is

$$C\hat{v}_\mu + \hat{C}v_\mu = -\frac{1}{4\kappa^2}\frac{e}{3!}\epsilon_{\mu\nu\rho\sigma}(\tilde{\mathcal{H}}^{\nu\rho\sigma} - 3\tilde{X}\partial^\nu C^{\rho\sigma 11}), \tag{6.72}$$

which is the combination appearing in the last line of the Lagrangian (6.67). We finally conclude that

$$\mathcal{L}_{\rm brane} = -\tau\left[VV_T\hat{V}^2 - \frac{1}{2}(\hat{S}+\overline{\hat{S}})V\hat{V}\right]_D \tag{6.73}$$

is the superconformal tensor calculus expression for the five-brane kinetic Lagrangian, with in addition

$$\tilde{\mathcal{B}} = \frac{1}{2}\,{\rm Im}\,\hat{s}. \tag{6.74}$$

Expression (6.73) is independent from the compensating multiplet S_0 and completely frame-independent. Its component expansion does not include any eR term and the Einstein frame condition for dilatation symmetry would not be affected by its addition to bulk (and $S^1/\mathbb{Z}_2$ plane) contributions. The bosonic component expression reads

$$\begin{aligned} e^{-1}\mathcal{L}_{\rm brane} \;=\;& -\tau CC_T\Big((\partial_\mu\hat{C})(\partial^\mu\hat{C}) - \hat{v}_\mu\hat{v}^\mu\Big) + 2\tau C\hat{C}\hat{v}^\mu T_\mu + \tau\hat{C}^2 v^\mu T_\mu \\ & +\tau(\partial_\mu\,{\rm Im}\,\hat{s})(C\hat{v}^\mu + \hat{C}v^\mu) \\ & +\tau\hat{C}^2(C_T d - C d_T) - 2\tau C\hat{C}(\partial_\mu C_T)(\partial^\mu\hat{C}) - \tau\hat{C}^2(\partial_\mu C)(\partial^\mu C_T) \\ & +\tau({\rm Re}\,\hat{s} - 2\hat{C}C_T)\Big(C\hat{d} + \hat{C}d - v^\mu\hat{v}_\mu + (\partial^\mu C)(\partial_\mu\hat{C})\Big) \\ & +e^{-1}\mathcal{L}_{\rm aux.} + \text{total derivative.} \end{aligned} \tag{6.75}$$

The auxiliary Lagrangian vanishes "on-shell": it is a quadratic expression in H, K, H_T, K_T, $\hat{H}$, $\hat{K}$ and $\hat{f}_s$. To compare the above expression with Eq. (6.67), we also need to solve for ${\rm Re}\,\hat{s}$, which is not generated by the reduction of the brane bosonic world-volume action: its presence is required by supersymmetry only. The fourth line is then eliminated. All contributions from the third line are related by supersymmetry to propagation of the background fields and are invisible in Eq. (6.67). And the first two lines with identifications

(6.71) and (6.74) correspond to Eq. (6.67), with the exception of the scalar potential V_0 which cannot arise from the superconformal expression (6.73).

As expected from the self-duality of the brane tensor $\mathcal{H}_{\hat{m}\hat{n}\hat{p}}$, the supergravity Lagrangian (6.73) has chiral-linear duality. Solving for the vector superfield $\hat{V}$ gives $\hat{V} = \frac{1}{4}V_T^{-1}(\hat{S} + \overline{\hat{S}})$ and $\mathcal{L}_{\text{brane}}$ becomes

$$\mathcal{L}_{\text{brane, chiral}} = \frac{\tau}{16}\left[VV_T^{-1}(\hat{S} + \overline{\hat{S}})^2\right]_D . \tag{6.76}$$

Alternatively, solving for the scalar superfield $\hat{S}$ leads to $V\hat{V} = \hat{L}$, where $\hat{L}$ is a real linear superfield. We then obtain the expression

$$\mathcal{L}_{\text{brane, linear}} = -\tau\left[V_T V^{-1}\hat{L}^2\right]_D . \tag{6.77}$$

Chiral-linear duality requires invariance under $\delta\hat{S} =$ an imaginary constant. This symmetry also excludes a superpotential and then the generation of a scalar potential.

The conclusion is that the superconformal Lagrangian (6.73) provides the four-dimensional effective kinetic Lagrangian for the brane modulus multiplet. As in action (6.22), the complete effective four-dimensional supergravity is the known effective theory of orbifold gauge and matter multiplets plus expression (6.73). Most importantly, the brane contributions to the background equations must be taken into account to correctly evaluate the scalar potential. This is the last point we need to discuss before analysing the complete supergravity theory.

6.2.4 Background and scalar potential

Omitting the gauge field contributions on the orbifold planes and considering a single five-brane, the background value of the eleven-dimensional Lagrangian (6.22) reads

$$\langle\mathcal{L}\rangle = \Big\langle -\frac{1}{2\kappa_{11}^2}\left[e_{11}R + \frac{1}{2}G_4 \wedge *G_4\right] + \mathcal{L}_{\text{kin.}}(\mathcal{H}_3) \wedge \delta_5(\mathcal{W}_6)\Big\rangle . \tag{6.78}$$

In our reduction, $\langle\mathcal{L}\rangle$ is in principle a function of the background metric scalar fields $\sigma(x^{11})$ and $\gamma(x^{11})$, and of their first and second derivatives which appear in the components of the Ricci tensor R_{MN}. Using the Einstein background equations, one actually finds that $\langle\mathcal{L}\rangle$ is a derivative,

$$\langle\mathcal{L}\rangle = \frac{1}{2\kappa_{11}^2}\frac{d}{dx^{11}}\left[e^{-3\gamma}\frac{d}{dx^{11}}(\gamma + 2\sigma)\right], \tag{6.79}$$

which disappears after integrating on x^{11}: the four-dimensional effective Lagrangian has zero background value. As a result, the effective four-dimensional scalar potential generated by the brane modulus vanishes.

Taking several branes and the orbifold planes into account leads to the same result: the scalar potential vanishes as long as a superpotential is not generated by charged matter chiral superfields.

6.3 The coupled theory

In compactified M-theory, the presence of the five-brane modulus multiplet does not modify the Bianchi identities verified by the massless components $G_{\mu\nu\rho 11}$, $G_{\mu i\bar{j}11}$ and G_{ijk11}. Its effect on the four-dimensional effective supergravity is simply to add the kinetic Lagrangian (6.73) and to modify the background equation (6.24) by the source terms proportional to $\delta_5(\mathcal{W}_6)$. More changes will occur with gauge thresholds and anomaly-cancelling terms, which can be regarded as "higher-order" corrections.

The complete effective supergravity[14] of M-theory compactified on $(S^1/\mathbb{Z}_2 \times \mathcal{X}_6)$ in presence of a five-brane is then

$$\begin{aligned}\mathcal{L} \;=\; & \Big[-(S_0\overline{S}_0V_T)^{3/2}(2V)^{-1/2} - (S+\overline{S})(V+2\Omega) + (U(W-\alpha M^3) + \text{c.c.}) \\ & +(L_T - 2\textstyle\sum_a \beta^a\Omega^a)(V_T + 2\overline{M}e^AM) + V(\epsilon|\alpha M^3|^2 - 2\delta\overline{M}e^AM)\Big]_D \\ & +\Big[S_0^3 W\Big]_F \\ & -\tau\Big[VV_T\hat{V}^2 - \tfrac{1}{2}(\hat{S}+\overline{\hat{S}})V\hat{V}\Big]_D + \tfrac{1}{4}\tau\Big[\hat{S}\textstyle\sum_a\hat{\beta}^a\mathcal{W}^a\mathcal{W}^a\Big]_F.\end{aligned} \tag{6.80}$$

The first three lines collect all contributions from gauge multiplets[15] and charged matter multiplets[16] M. They also include the contributions of the massless modes of G_4 and of the metric tensor, in the multiplets V, V_T and W. The first term is the bulk Lagrangian [44, 59] produced by the reduction of the CJS theory. The next terms induce by the field equations of the Lagrange multipliers S, U and L_T the Bianchi identities. The solutions are:

$$\begin{array}{lrcll} S\,(\text{chiral}): & V & = & L - 2\Omega & (L \text{ linear}), \\ U\,(\text{vector}): & W & = & \alpha M^3 + ih & (h \text{ real constant}), \\ L_T\,(\text{linear}): & V_T & = & T + \overline{T} - 2\overline{M}e^AM & (T \text{ chiral}). \end{array}$$

Reduction of the action (6.22) shows that the massless components of the six-form field are included in these Lagrange multipliers. The single term in the third line is the superpotential, as defined by the Bianchi identity induced by U. The contributions with coefficients β^a, ϵ and δ are higher-order corrections following from anomaly cancellation. They generate in particular gauge thresholds.

The last line in Eq. (6.80) is the brane Lagrangian (6.73), supplemented by a higher-order correction with coefficients $\tau\hat{\beta}^a$. Its role will be discussed below. The identity

$$\frac{1}{4}\Big[\hat{S}\mathcal{W}^a\mathcal{W}^a\Big]_F = -2\Big[(\hat{S}+\overline{\hat{S}})\Omega^a\Big]_D + \text{ derivative} \tag{6.81}$$

can also be used as a definition of the gauge curvature chiral multiplets $\mathcal{W}^a$.

[14] Up to terms with two derivatives.

[15] In the Chern–Simons superfields Ω^a, $\Omega = \sum_a c^a\Omega^a$, for a gauge group $G = \prod_a G^a$.

[16] The chiral multiplet M denotes a generic charged matter multiplet, for instance a **27** of an E_6 gauge group.

Theory (6.80) has a very simple Einstein term since only the bulk Lagrangian contributes:

$$\mathcal{L}_{\text{Einstein}} = -\frac{1}{2}eR\left[(z_0\overline{z}_0C_T)^{3/2}(2C)^{-1/2}\right], \tag{6.82}$$

where z_0, C_T and C denote the lowest components of S_0, V_T and V. As mentioned already in Eq. (6.69), the Einstein frame is selected by the condition

$$\left(\frac{z_0\overline{z}_0C_T}{2C}\right)^{-3/2} = 2\kappa^2C. \tag{6.83}$$

The Einstein frame will be used below.

Since theory (6.80) contains “auxiliary multiplets” which can be eliminated, we will consider two versions related by chiral-linear duality acting on the dilaton multiplet:

- The *linear version* is obtained by solving for L_T, U and S. The dynamical multiplets are then L, T, M, Ω^a and the brane multiplet $\hat{L}$ or $\hat{S}$. The dilaton is described by the linear multiplet L, which also includes the massless component $G_{\mu\nu\rho 11}$ of the four-form field.
- The *chiral version* is obtained by solving for L_T, U and V, the dynamical multiplets being then S, T, M, Ω^a and the brane multiplet $\hat{L}$ or $\hat{S}$. The dilaton is described by the real part $\text{Re}\, s$ of the scalar component of the chiral multiplet S, while $\text{Im}\, s$ is a component of the six-form field.

For our purposes, it is useful to simplify the theory by solving for the multiplets L_T and U. Their field equations respectively imply that $V_T = T+\overline{T}-2\overline{M}e^AM$, with a chiral modulus multiplet T, and that the superpotential is a cubic gauge invariant function of M which we symbolically write $W(M)=\alpha M^3$, up to a possible constant (which would break supersymmetry). The result is the following effective Lagrangian:

$$\begin{aligned}\mathcal{L} &= \Big[-\Big(S_0\overline{S}_0(T+\overline{T}-2\overline{M}e^AM)\Big)^{3/2}(2V)^{-1/2} - (S+\overline{S})(V+2\Omega) \\ &\quad +V(\epsilon|\alpha M^3|^2 - 2\delta\overline{M}e^AM)\Big]_D - \tau\Big[V(T+\overline{T}-2\overline{M}e^AM)\hat{V}^2 - \tfrac{1}{2}(\hat{S}+\overline{\hat{S}})V\hat{V}\Big]_D \\ &\quad +\Big[S_0^3W(M) + \tfrac{1}{4}\textstyle\sum_a(\beta^aT+\tau\hat{\beta}^a\hat{S})\mathcal{W}^a\mathcal{W}^a\Big]_F.\end{aligned} \tag{6.84}$$

We first omit the higher-order corrections and set $\beta^a=\hat{\beta}^a=\delta=\epsilon=0$. All terms in the Lagrangian are then obtained from the reduction of the higher-dimensional bosonic action (6.22) and of the brane action (6.32), supplemented by $N=1$ supersymmetry. We also choose to describe the brane multiplet by the chiral multiplet $\hat{S}$ by solving for $\hat{V}$. Then, with identity (6.81),

$$\begin{aligned}\mathcal{L} &= \Big[-\Big(S_0\overline{S}_0(T+\overline{T}-2\overline{M}e^AM)\Big)^{3/2}(2V)^{-1/2} \\ &\quad -\Big(S+\overline{S}-\frac{\tau}{16}\frac{(\hat{S}+\overline{\hat{S}})^2}{T+\overline{T}-2\overline{M}e^AM}\Big)V\Big]_D + \Big[S_0^3W(M)+\frac{1}{4}S\sum_a c^a\mathcal{W}^a\mathcal{W}^a\Big]_F,\end{aligned} \tag{6.85}$$

and solving for V leads to the chiral version, in which the (bulk) dilaton is described by S. It is as usual defined by

$$\mathcal{L}_{\rm chiral} = -\frac{3}{2}\Big[S_0\overline{S}_0 e^{-K/3}\Big]_D + \Big[\frac{1}{4}\sum_a f^a \mathcal{W}^a\mathcal{W}^a + S_0^3 W(M)\Big]_F. \qquad (6.86)$$

The real Kähler potential is

$$K = -\log\Big(S+\overline{S} - \frac{\tau}{16}\frac{(\hat{S}+\overline{\hat{S}})^2}{T+\overline{T}-2\overline{M}e^A M}\Big) - 3\log\big(T+\overline{T}-2\overline{M}e^A M\big), \qquad (6.87)$$

and the holomorphic gauge kinetic functions are simply

$$f^a = c^a S. \qquad (6.88)$$

It is important to realize that the brane kinetic terms affect the dilaton Kähler potential, and that this modification cannot be moved into the gauge kinetic function by a holomorphic redefinition of the chiral S: the brane kinetic terms are not harmonic.

Suppose nevertheless that we insist on defining the dilaton as the real quantity

$$\varphi = \mathrm{Re}\, s - \frac{\tau}{32}\frac{(\hat{s}+\overline{\hat{s}})^2}{T+\overline{T}-2\overline{M}M}, \qquad (6.89)$$

as suggested by the Kähler potential (6.87). The coupling for the gauge group factor G^a can then be written as[17]

$$\frac{1}{g_a^2} = \mathrm{Re}\, f^a = c^a\Big(\varphi + \frac{\tau}{32}\frac{(\hat{s}+\overline{\hat{s}})^2}{T+\overline{T}-2\overline{M}M}\Big). \qquad (6.90)$$

In this point of view, the brane contribution appears as a correction to the gauge coupling. However, one cannot find a *holomorphic* function f^a with the field variable φ and this choice of dilaton field is not compatible with the supermultiplet structure required when writing the supergravity Lagrangian in the chiral version.

The addition of the higher-order corrections is straightforward. In the chiral version, the Kähler potential becomes

$$\begin{aligned} K &= -\log\Big(S+\overline{S} - \frac{\tau}{16}\frac{(\hat{S}+\overline{\hat{S}})^2}{T+\overline{T}-2\overline{M}e^A M} + 2\delta\overline{M}e^A M - \epsilon|\alpha M^3|^2\Big) \\ &\quad -3\log\big(T+\overline{T}-2\overline{M}e^A M\big) \end{aligned} \qquad (6.91)$$

while the gauge kinetic functions read

$$f^a = c^a S + \beta^a T + \tau\hat{\beta}^a\hat{S}. \qquad (6.92)$$

The "natural" definition of the dilaton suggested by the Kähler potential is now

$$\varphi = \mathrm{Re}\, s - \frac{\tau}{32}\frac{(\hat{s}+\overline{\hat{s}})^2}{T+\overline{T}-2\overline{M}M} + \delta\overline{M}M - \frac{1}{2}\epsilon|\alpha M^3|^2, \qquad (6.93)$$

[17] It is the "wilsonnian gauge coupling".

and in terms of this dilaton, the gauge couplings become

$$\begin{aligned}\frac{1}{g_a^2} &= c^a\varphi + \beta^a\,\mathrm{Re}\,T - c^a\delta\overline{M}M + \frac{1}{2}c^a\epsilon|\alpha M^3|^2 \\ &+\frac{1}{2}\tau(T+\overline{T}-2\overline{M}M)\Big(\frac{1}{16}c^a\Big(\frac{\hat{s}+\overline{\hat{s}}}{T+\overline{T}-2\overline{M}M}\Big)^2 + \hat{\beta}^a\frac{\hat{s}+\overline{\hat{s}}}{T+\overline{T}-2\overline{M}M}\Big).\end{aligned} \tag{6.94}$$

Returning to the Lagrangian (6.80), the field equation relating $\hat{V}$ and $\hat{S}$ is

$$\hat{V} = \frac{1}{4}\frac{\hat{S}+\overline{\hat{S}}}{T+\overline{T}-2\overline{M}e^AM}, \tag{6.95}$$

and the lowest component $\hat{C}$ of $\hat{V}$ has been identified with $\tilde{X}=\kappa^{-1}X$, which is the brane modulus in the direction x^{11}, in Planck units. The gauge couplings can then finally be expressed as

$$\begin{aligned}\frac{1}{g_a^2} &= c^a\varphi + \beta^a\,\mathrm{Re}\,T - c^a\delta\overline{M}M + \frac{1}{2}c^a\epsilon|\alpha M^3|^2 \\ &+\frac{1}{2}\tau(T+\overline{T}-2\overline{M}M)(c^a\tilde{X}^2+4\hat{\beta}^a\tilde{X}).\end{aligned} \tag{6.96}$$

The linear version is interesting. Solving in Eq. (6.84) for S implies $V=L-2\Omega$, and the resulting effective supergravity reads

$$\begin{aligned}\mathcal{L}_{\text{linear}} &= \Big[-\tfrac{1}{\sqrt{2}}\Big(S_0\overline{S}_0(T+\overline{T}-2\overline{M}e^AM)\Big)^{3/2}(L-2\Omega)^{-1/2} \\ &+(L-2\Omega)\Big(\epsilon|\alpha M^3|^2 - 2\delta\overline{M}e^AM - \tau(T+\overline{T}-2\overline{M}e^AM)\hat{V}^2 \\ &+\tfrac{1}{2}\tau(\hat{S}+\overline{\hat{S}})\hat{V}\Big)\Big]_D + \Big[S_0^3W(M) + \tfrac{1}{4}\textstyle\sum_a(\beta^aT+\tau\hat{\beta}^a\hat{S})\mathcal{W}^a\mathcal{W}^a\Big]_F.\end{aligned} \tag{6.97}$$

In this case, with the identification $\hat{C}=\tilde{X}$ and with the field equation (6.95) relating $\hat{s}$ and $\hat{C}$, computing the gauge couplings leads easily to

$$\begin{aligned}\frac{1}{g_a^2} &= \frac{1}{2}c^a\Big(\frac{z_0\overline{z}_0(T+\overline{T}-2\overline{M}M)}{2C}\Big)^{3/2} + \frac{1}{2}c^a\epsilon|\alpha M^3|^2 - c^a\delta\overline{M}M \\ &+\frac{1}{2}\tau(T+\overline{T}-2\overline{M}M)(c^a\tilde{X}^2+4\hat{\beta}^a\tilde{X}) + \beta^a\,\mathrm{Re}\,T.\end{aligned} \tag{6.98}$$

Comparing with expression (6.96), we find that

$$2\varphi = \Big(\frac{z_0\overline{z}_0(T+\overline{T}-2\overline{M}M)}{2C}\Big)^{3/2} \tag{6.99}$$

or, in the Einstein frame, with condition (6.83) and $C_T = T+\overline{T}-2\overline{M}M$,

$$\varphi = \frac{1}{4\kappa^2C}. \tag{6.100}$$

The compatibility of expressions (6.98) and (6.92) follows then from the field equation of the vector multiplet V in theory (6.84), which is chiral-linear duality:

$$\begin{aligned} S+\overline{S} &= \Big(\frac{S_0\overline{S}_0(T+\overline{T}-2\overline{M}e^AM)}{2V}\Big)^{3/2} + \epsilon|\alpha M^3|^2 - 2\delta\overline{M}e^AM \\ & -\tau(T+\overline{T}-2\overline{M}e^AM)\hat{V}^2 + \frac{1}{2}\tau(\hat{S}+\overline{\hat{S}})\hat{V}. \end{aligned} \tag{6.101}$$

To summarize, in the chiral version of the effective supergravity, the kinetic Lagrangian of the five-brane modulus introduces a quadratic, non-harmonic correction to the dilaton in the Kähler potential. The holomorphic gauge functions and the wilsonnian gauge couplings are not affected by these terms. In the linear version, the kinetic brane Lagrangian generates quadratic, non-harmonic corrections to the field-dependent wilsonnian gauge couplings.

The higher-order brane contribution

$$\Delta_{\text{brane}} = \frac{1}{4}\tau\sum_a \hat{\beta}^a[\hat{S}\mathcal{W}^a\mathcal{W}^a]_F \tag{6.102}$$

is similar to the familiar gauge thresholds in the modulus T, with coefficients β^a. We have seen that the self-duality of the three-index tensor on the brane world-volume leads in four dimensions to a chiral-linear duality. In the effective supergravity, this duality requires invariance under variations of $\hat{S}$ by an imaginary constant. Then, D-terms should depend on $\hat{S}+\overline{\hat{S}}$, and with our set of multiplets, there is a unique F-term compatible with this symmetry: the higher-order correction Δ_{brane}.[18]

This second brane contribution is not generated by reduction of the PST brane action (6.32), as T-dependent threshold corrections do not follow from reduction of the bosonic action (6.22). In that sense, they can be regarded as higher-order terms.

The presence of quadratic and linear brane contributions has been established in the background calculation of Lukas, Ovrut and Waldram [136, 138, 139]. These authors have in particular computed the gauge couplings for a set of branes located at fixed positions along S^1. These positions correspond to constant background values of our scalar field $\tilde{X}$. To compare with our result, it is easier (and sufficient) to consider a single brane, two gauge couplings and a single modulus T, as in our reduction. The variables used by LOW are then the position z along the interval $S^1/\mathbb{Z}_2$ and three charges $\beta^{(0)}$, $\beta^{(2)}$ and $\beta^{(5b.)}$ associated with the two fixed planes and the brane. The variable z is normalized in the interval $[0,1]$ and the charges are quantized: $\beta^{(0)}$ and $\beta^{(2)}$ are half integers, $\beta^{(5b.)}$ is an integer, and the cohomology (or background) condition implies $\beta^{(0)}+\beta^{(2)}+\beta^{(5b.)}=0$. The gauge couplings

[18] The authors of Refs. [139, 45] failed to recognize the importance of self-duality of the world-volume three-form. They attempted to describe the brane modulus with a chiral multiplet and introduced a quadratic holomorphic F-density forbidden by chiral-linear duality and unrelated to brane kinetic terms. The resulting supergravity theory is incorrect.

found by LOW are then

$$\begin{aligned}
\frac{1}{g_1^2} &= \mathrm{Re}\, s + \frac{\epsilon_S}{8\pi}\,\mathrm{Re}\, T[\beta^{(0)} + (1-z)^2\beta^{(5b.)}], \\
\frac{1}{g_2^2} &= \mathrm{Re}\, s + \frac{\epsilon_S}{8\pi}\,\mathrm{Re}\, T[-(\beta^{(0)} + \beta^{(5b.)}) + z^2\beta^{(5b.)}], \\
\frac{1}{g_1^2} - \frac{1}{g_2^2} &= \frac{\epsilon_S}{4\pi}\,\mathrm{Re}\, T[\beta^{(0)} + \beta^{(5b.)} - z\beta^{(5b.)}],
\end{aligned} \tag{6.103}$$

with a dimensionless (arbitrary) parameter ϵ_S related to the Calabi–Yau volume and the S^1 radius. Compare these quantities with our expression (6.96), with $M = 0$ and $c^a = 1$:

$$\begin{aligned}
\frac{1}{g_{1,2}^2} &= \varphi + \mathrm{Re}\, T[\beta^{1,2} + \tau\tilde{X}^2 + 4\tau\hat{\beta}^{1,2}\tilde{X}], \\
\frac{1}{g_1^2} - \frac{1}{g_2^2} &= \mathrm{Re}\, T[\beta^1 - \beta^2 + 4\tau(\hat{\beta}^1 - \hat{\beta}^2)\tilde{X}].
\end{aligned} \tag{6.104}$$

Our parameters are not normalized or quantized. If we merely write $\tilde{X} = \tilde{\lambda} z$, both sets of equations coincide with the trivial statement $\tau = \frac{\epsilon_S}{8\pi\tilde{\lambda}^2}\beta^{(5b.)}$ and the non-trivial relations

$$\beta^1 = -\beta^2 = \frac{\epsilon_S}{8\pi}(\beta^{(0)} + \beta^{(5b.)}), \quad \hat{\beta}_1 = -\frac{\tilde{\lambda}}{2} \quad \text{and} \quad \hat{\beta}_2 = 0. \tag{6.105}$$

These equations are predictions obtained from the solution of the background condition which specify in parts our four arbitrary threshold parameters. They are specific properties of M-theory compactified on $\mathcal{K}_7$. Our effective supergravity reproduces then nicely the background found by LOW. Notice in passing that the dilaton field is incorrectly identified in Eqs. (6.103) as the real part of the chiral s. We have seen that the correct identification is $\varphi = (4\kappa^2 C)^{-1}$, in the linear version of the theory. A background calculation is not sufficient to reach this conclusion: a constant value z of the brane modulus can be the background value of any kind of multiplets (vector, linear, chiral).

6.3.1 The scalar potential

We close this section by a discussion of the impact of the five-brane modulus on the supergravity scalar potential.

We first use the chiral version, defined by the Kähler potential (6.87) and the superpotential $W(M)$. We concentrate on the potential at $M = 0$.[19] We however assume that the superpotential can be nonzero in this limit: this is the case if the component G_{ijk11} of the four-form field is a non-zero constant breaking supersymmetry. As usual, the potential (in the Einstein frame) reads

$$\kappa^4 V(s, T, \hat{s}) = e^K W\overline{W}\left[\sum_{IJ}(K_I + W^{-1}W_I)(K^J + \overline{W}^{-1}\overline{W}^J)(K_J^I)^{-1} - 3\right], \tag{6.106}$$

[19] Since M is a charged field, the potential is always stationary at $M = 0$.

where $K_I = \partial K/\partial z^I$, $K^I = (K_I)^*, \ldots$, and $z_I = (s, T, \hat{s})$. In the absence of the five-brane field $\hat{s}$, the potential takes the simple form $\kappa^4 V(s,T) = e^K W\overline{W}$. An explicit calculation shows that this result is not affected by the contributions of the five-brane modulus. The complete scalar potential at $M=0$ in the chiral version of the theory is then:

$$\kappa^4 V(s,T,\hat{s}) = \frac{W\overline{W}}{(s+\overline{s}-\frac{\tau}{16}\frac{(\hat{s}+\overline{\hat{s}})^2}{T+\overline{T}})(T+\overline{T})^3}. \tag{6.107}$$

This result can be easily understood in the linear version of the theory, or in the original expression (6.84) of the Lagrangian. The five-brane terms do not include any contribution to the scalar potential: we have discussed this point in subsection 6.2.4. In the linear version, the scalar potential is then completely independent from the brane modulus $\hat{C}$. This statement would remain true with several five-branes, since each contributes by adding to Eq. (6.84) a similar term, without any scalar potential.

The appearance of a dependence on $\hat{s}$ of the potential in the chiral version follows from the chiral-linear duality equation (6.101). The relation between s and C is modified by the five-brane to become

$$s+\overline{s}-\frac{\tau}{16}\frac{(\hat{s}+\overline{\hat{s}})^2}{T+\overline{T}} = \frac{1}{2\kappa^2 C} \tag{6.108}$$

with $M=0$ and in the Einstein frame. It is the dependence on C of the scalar potential in the linear version which induces a dependence on $\hat{s}$ in the chiral version. As a consequence, the five-brane modulus does not produce a new minimum equation:

$$\frac{\partial V}{\partial \hat{s}} = -\frac{\tau}{8}\frac{\hat{s}+\overline{\hat{s}}}{T+\overline{T}}\frac{\partial V}{\partial s} \tag{6.109}$$

and $\partial V/\partial s$ is not zero. The impact of the five-brane modulus on the effective scalar potential is then a simple redefinition of the chiral dilaton field s as a function of the (unchanged) C of the linear multiplet formulation.

6.4 Summary

Even if the five-brane is not a perturbative object, it is interesting to consider the brane corrections to the four-dimensional effective supergravity from the perspective of string perturbation theory. The string loop-counting field is our multiplet V with dilaton C, and an n-loop term in the Wilson Lagrangian is characterized by a factor $V^{(3n-1)/2}$ [44]. According to Eq. (6.80), the kinetic Lagrangian of the five-brane modulus multiplet is similar to a one-loop correction, linear in V. The origin of this factor is simple: the kinetic terms are normalized by the world-volume induced metric $\sqrt{-\hat{g}} \sim e^{-3\sigma-2\gamma} \sim CC_T^{-2}$. Compare now with the one-loop corrections in the modulus T, which are completely understood in compactifications of heterotic superstrings. Two kinds of contributions arise [69, 56]. The first is a real gauge-group independent term proportional to the Kähler potential $-3\log(T+\overline{T})$, the "Green–Schwarz" term. The second one is a gauge-group dependent correction which involves a holomorphic function. In the chiral version, the Green–Schwarz term corrects the

Kähler potential of the S field and it can be regarded as a wave-function renormalization of this field. The second term is then a correction to the gauge kinetic functions f^a. The similarity with the five-brane contributions in the Lagrangian (6.80) is obvious. In the case of the volume modulus T, the one-loop corrections can be understood in terms of a cancellation of target-space duality anomalies. The analogy suggests that anomalies could also help to understand the structure of our five-brane contributions. Moreover, the resemblance between the structure of the functions f_1 and f_2 in the variation (4.76) and the expressions (6.104) of $1/g^2_{1,2}$ is certainly not a coincidence. The natural extension of the present thesis would then be to assess the significance of this link. It would also be interesting to consider other non-perturbative states like gaugino condensates since they can be easily included in our formalism.

Appendix A

Notations and conventions

A.1 Coordinates and metrics

Our notation for coordinates is:

$d = 11$ curved space-time:	x^M	$M = 1, \ldots, 11$
$d = 10$ curved space-time:	x^A	$A = 1, \ldots, 10$
$d \leq 9$ curved space-time:	x^μ	$\mu = 1, \ldots, d$
$S^1/\mathbb{Z}_2$ direction:	x^{11}	
Calabi–Yau real coordinates:	x^a	$a = 5, \ldots, 10$
Calabi–Yau complex (Kähler) coordinates:	z^i, $\overline{z}^{\bar{i}}$	$i = 1, 2, 3$
Extended p-brane world-volume coordinates:	$y^{\hat{m}}$	$\hat{m} = \hat{1}, \ldots, \hat{p} + \hat{1}$

Note that $M, N, \ldots$ indices are also used when the space-time dimension d is unspecified. The complex nature of the Calabi–Yau manifold $\mathcal{X}_6$ leads us to define the coordinates

$$z^l = \frac{1}{\sqrt{2}}\left(x^l + ix^{l+3}\right) \quad \text{and} \quad \overline{z}^{\bar{l}} = \frac{1}{\sqrt{2}}\left(x^l - ix^{l+3}\right), \quad \text{with} \quad l = 1, 2, 3, \tag{A.1}$$

as well as the $SU(3)$-invariant Calabi–Yau tensor ϵ_{ijk} which is such that $\epsilon_{123} = \epsilon_{\overline{123}} = 1$.

We consider an $SO(1, 10)$ space-time metric with mostly plus signature $(-, +, +, \ldots, +)$.[1] The reduction of the eleven-dimensional metric to ten dimensions is defined by

$$g_{MN} = \begin{pmatrix} \varphi^{-1/4} g_{AB} & 0 \\ 0 & \varphi^2 \end{pmatrix}, \tag{A.2}$$

while its reduction to four dimensions reads

$$g_{MN} = \begin{pmatrix} e^{-\gamma-2\sigma} g_{\mu\nu} & 0 & 0 \\ 0 & e^{2\gamma-2\sigma} & 0 \\ 0 & 0 & e^{\sigma}\delta_{i\bar{j}} \end{pmatrix}. \tag{A.3}$$

[1]This convention is also the one used by Misner, Thorne and Wheeler [144], Green, Schwarz and Witten [107, 108], Lüst and Theisen [140], and Polchinski [171, 172].

The vielbein determinant in d space-time dimensions is written e_d and one has the equalities

$$e_{11} = \varphi^{-1/4} e_{10} = e^{-\gamma-2\sigma} e_4.$$

The corresponding relations for the gravitational coupling constants are

$$\kappa_{11} = V_1^{1/2} \kappa_{10} = (V_1 V_6)^{1/2} \kappa_4 \tag{A.4}$$

(since $[\kappa_d] = [\text{mass}]^{-\frac{d}{2}+1}$, we can easily verify that these relations make sense from a dimensional point of view).

A.2 Antisymmetric tensors

We define the antisymmetrization of n indices with unit weight:

$$A_{[M_1 \dots M_n]} = \frac{1}{n!} \left(A_{M_1 \dots M_n} \pm (n! - 1) \text{ permutations} \right). \tag{A.5}$$

A.3 Differential forms

The expression of a p-form A_p in terms of its completely antisymmetric components is

$$A_p = \frac{1}{p!} A_{M_1 \dots M_p} dx^{M_1} \wedge \dots \wedge dx^{M_p} = \frac{1}{p!} A_{[M_1 \dots M_p]} dx^{M_1} \wedge \dots \wedge dx^{M_p} \tag{A.6}$$

(the degree of the form is denoted by an italicized subscript). The wedge product of a p-form and a q-form is then defined by

$$A_p \wedge B_q = \frac{1}{p!q!} A_{M_1 \dots M_p} B_{M_{p+1} \dots M_{p+q}} dx^{M_1} \wedge \dots \wedge dx^{M_{p+q}} \stackrel{\text{def.}}{=} C_{p+q}, \tag{A.7}$$

with

$$C_{M_1 \dots M_{p+q}} = \frac{(p+q)!}{p!q!} A_{[M_1 \dots M_p} B_{M_{p+1} \dots M_{p+q}]}. \tag{A.8}$$

Note that

$$A_p \wedge B_q = (-1)^{pq} B_q \wedge A_p. \tag{A.9}$$

The exterior derivative is $d = dx^M \partial_M$ and the curl $F_{p+1} = dA_p$ of a p-form reads then

$$\begin{aligned} dA_p &= \frac{1}{p!} (\partial_M A_{N_1 \dots N_p}) dx^M \wedge dx^{N_1} \wedge \dots \wedge dx^{N_p} \\ &= \frac{1}{(p+1)!} F_{M_1 \dots M_{p+1}} dx^{M_1} \wedge \dots \wedge dx^{M_{p+1}}, \end{aligned} \tag{A.10}$$

with

$$\begin{aligned} F_{M_1 \dots M_{p+1}} &= (p+1) \partial_{[M_1} A_{M_2 \dots M_{p+1}]} \\ &= \partial_{M_1} A_{M_2 \dots M_{p+1}} \pm p \text{ cyclic permutations.} \end{aligned} \tag{A.11}$$

Applying this definition to a wedge product one obtains

$$d(A_p \wedge B_q) = (dA_p) \wedge B_q + (-1)^p A_p \wedge dB_q. \tag{A.12}$$

The Hodge dual form of a p-form A_p on a d-dimensional manifold endowed with a metric of determinant g_d is defined by

$$\begin{aligned} *A_p &= \frac{\sqrt{-g_d}}{p!(d-p)!}\epsilon_{M_1\ldots M_{d-p}N_1\ldots N_p}A^{N_1\ldots N_p}dx^{M_1}\wedge\cdots\wedge dx^{M_{d-p}} \\ &= \frac{1}{(d-p)!}A^*_{M_1\ldots M_{d-p}}dx^{M_1}\wedge\cdots\wedge dx^{M_{d-p}}, \end{aligned} \tag{A.13}$$

where $\epsilon_{M_1\ldots M_d}$ is the completely antisymmetric symbol

$$\epsilon_{M_1\ldots M_d} \overset{\text{def.}}{=} \begin{cases} +1 & \text{if } (M_1,\ldots,M_d) \text{ is an even permutation of } (1,\ldots,d), \\ -1 & \text{if } (M_1,\ldots,M_d) \text{ is an odd permutation of } (1,\ldots,d), \\ 0 & \text{if } (M_1,\ldots,M_d) \text{ are not all different,} \end{cases} \tag{A.14}$$

and

$$A^*_{M_1\ldots M_{d-p}} = \frac{\sqrt{-g_d}}{p!}\epsilon_{M_1\ldots M_{d-p}N_1\ldots N_p}A^{N_1\ldots N_p} \tag{A.15}$$

with $A^{N_1\ldots N_p} = g_d^{N_1O_1}\cdots g_d^{N_pO_p}A_{O_1\ldots O_p}$. The contravariant components are given by

$$A^{*M_1\ldots M_{d-p}} = \frac{-1}{p!\sqrt{-g_d}}\epsilon^{M_1\ldots M_{d-p}N_1\ldots N_p}A_{N_1\ldots N_p}. \tag{A.16}$$

The volume form in p space-time dimensions is $dx^{M_1}\wedge\cdots\wedge dx^{M_p} = \epsilon^{M_1\ldots M_p}d^px$ and the coordinate-invariant integral of a p-form reads

$$\int A_p = \int d^px\, A_{12\ldots p}. \tag{A.17}$$

A.4 Non-Abelian gauge forms

We consider a simple non-Abelian gauge group G[2] with a set of generators (matrices) T^a that obey a Lie algebra

$$[T^a, T^b] = f^{ab}{}_c T^c, \tag{A.18}$$

and are normalized in such a way that $\mathrm{tr}(T^aT^b) = \delta^{ab}$. One gauge field A^a_A is then introduced for each generator (hence the index a). The corresponding gauge derivative (acting for instance on the gauginos χ^a) and gauge field strength (Yang–Mills field strength) are defined in terms of the gauge structure constants f^{abc} by

$$\begin{aligned} D_A\chi^a &= \partial_A\chi^a + f^a{}_{bc}A^b_A\chi^c \\ F^a_{AB} &= \partial_A A^a_B - \partial_B A^a_A + f^a{}_{bc}A^b_A A^c_B. \end{aligned} \tag{A.19}$$

[2]In other words, we assume that G corresponds to a non-Abelian Lie algebra that has no nontrivial invariant subalgebras.

It is however more convenient to make use of the gauge (matrix-valued) one-form

$$A_1 \equiv A^a_A T^a dx^A, \tag{A.20}$$

so that the gauge field strength becomes a two-form

$$F_2 \equiv F^a_2 T^a \equiv \frac{1}{2} F^a_{AB} T^a dx^A \wedge dx^B = dA_1 + A_1^2, \tag{A.21}$$

with A_1^2 standing for the wedge product $A_1 \wedge A_1$. Representing the transformation infinitesimal parameters by the matrix Λ^{g}, a gauge transformation reads

$$\delta_{\mathrm{YM}} A_1 = d\Lambda^{\mathrm{g}} + [A_1, \Lambda^{\mathrm{g}}]. \tag{A.22}$$

The interesting, gauge invariant, object turns out to be[3]

$$\mathrm{tr} F_2^2 \equiv \mathrm{tr}(F_2 \wedge F_2) = d\Omega_{3,\mathrm{YM}}, \tag{A.23}$$

where we have introduced the Chern–Simons three-form

$$\Omega_{3,\mathrm{YM}} = \mathrm{tr}(A_1 \wedge dA_1 + \frac{2}{3} A_1^3). \tag{A.24}$$

The variation of $\Omega_{3,\mathrm{YM}}$ under a gauge transformation is given by

$$\delta_{\mathrm{YM}} \Omega_{3,\mathrm{YM}} = d\Omega^1_{2,\mathrm{YM}}, \tag{A.25}$$

with[4]

$$\Omega^1_{2,\mathrm{YM}} = \mathrm{tr}(\Lambda^{\mathrm{g}} dA_1). \tag{A.26}$$

Note that all this discussion for the Yang–Mills theory can be applied to general relativity with the spin connection replacing the gauge field.

We conclude by giving a few comments on the properties of the exceptional group E_8. E_8 is a 248-dimensional group which has no independent fourth- or sixth-order Casimir invariants. Making use of the fact that E_8 has an $SO(16)$ subgroup, one can actually show that the traces $\mathrm{Tr} F_2^4$ or $\mathrm{Tr} F_2^6$ decompose according to

$$\mathrm{Tr} F_2^4 = \frac{1}{100} (\mathrm{Tr} F_2^2)^2 \quad \text{and} \quad \mathrm{Tr} F_2^6 = \frac{1}{7200} (\mathrm{Tr} F_2^2)^3, \tag{A.27}$$

where the symbol "Tr" stands for a trace in the adjoint representation. One usually prefers using another trace (corresponding to another representation) denoted "tr" and defined by[5]

$$\mathrm{tr} F_2^2 = \frac{1}{30} \mathrm{Tr} F_2^2. \tag{A.28}$$

[3]Using the normalization of the generators, $\mathrm{tr} F_2^2$ can be written $F^a_2 \wedge F^a_2$. Note also that the terms with four gauge potentials contained in $\mathrm{tr} F_2^2$ vanish thanks to Jacobi identity.

[4]The superscript 1 simply indicates that the dependence in the matrix of parameters is linear.

[5]This definition matches the relation

$$\mathrm{Tr} F_2^2 = (n-2) \mathrm{tr} F_2^2$$

between traces in the adjoint and fundamental representations for a general $SO(n)$ group. The coefficient is precisely equal to 30 for $SO(32)$.

A.5 Units

Following a well-established habit in particle physics, we use natural units $\hbar = c = 1$, except for a few formulas where the constants $\hbar$ and c appear explicitly. Denoting the dimension of a quantity A by the symbol $[A]$, we have then in particular the relations

$$[\text{mass}] = [\text{length}]^{-1} = [\text{time}]^{-1} = \text{energy (GeV)}. \tag{A.29}$$

For electromagnetic formulas, we adopt the Heaviside–Lorentz system of units in which the MKS constants ϵ_0 and μ_0 are set to unity, the factors of 4π appearing in the force equations rather than in the field equations (the Heaviside–Lorentz system is a rationalised system). For instance, the fine structure constant, which reads $\alpha = e^2/(4\pi\epsilon_0\hbar c)$ in MKS units, simply becomes $\alpha = e^2/(4\pi)$ in the system usually employed by the high-energy physics community. The value $\alpha \cong 1/137$ (low-energy limit) is the same in the two systems, while the numerical values of e differ.

The Table A.1 gives the mass dimension of some quantities appearing in the text. For example, $[C_3] = [\text{mass}]^0$, where the form notation C_3 actually stands for the components C_{MNO} of the corresponding antisymmetric tensor. Note that the combinations $\lambda^2/\kappa_{10}^{3/2}$, λ^3/κ_{11}^2 and λ^2/V_6 are dimensionless, while $[\kappa_{11}^2/\lambda^2]$ has dimension $[\text{mass}]^{-3}$.

Quantity	Dimension
C_3; G_4	0; 1
A_1; F_2	1; 2
Ω_1; R_2	1; 2
$\widetilde{\omega}^1_{2,i}$; $\widetilde{\omega}_{3,i}$; $\widetilde{I}_{4,i}$	2; 3; 4
X_8	8
$\Omega_{3,\text{YM}}$; $\Omega_{3,\text{L}}$	3; 3
V_1; V_6	-1; -6
κ_d; $\kappa_4 \equiv \kappa$; κ_{10}; κ_{11}	$1 - d/2$; -1; -4; $-9/2$
α'	-2
λ	-3
T_p	$p+1$
$\delta_{1,i}$; δ_4; δ_5	1; 4; 5
θ_3; θ_4	3; 4

Table A.1: Mass dimension of some quantities appearing in the text.

Appendix B

Massless field representations

In this appendix, we have gathered some well-known facts about massless field representations in various dimensions. They prove useful when discussing the massless sector of superstring theories or the supergravity models describing their low-energy effective field theory behaviour.

B.1 Gravitational field

As can be deduced from the linearized Einstein equations of general relativity, the graviton g_{MN} is a traceless symmetric tensor with only $(d-2)$ transverse components (see Ref. [144] for a discussion of the four-dimensional case). Thus, it corresponds to a number of dynamical degrees of freedom given by

$$\frac{1}{2}(d-2)(d-1)-1=\frac{1}{2}(d-3)d. \tag{B.1}$$

The gravitational field has no degrees of freedom below four space-time dimensions.

B.2 Antisymmetric tensor gauge field

Taking into account gauge invariance, a completely antisymmetric tensor of rank n has

$$\begin{pmatrix} d-2 \\ n \end{pmatrix} \overset{\text{def.}}{=} \frac{(d-2)!}{n!(d-2-n)!} \tag{B.2}$$

on-shell components. If the associated field strength tensor is self-dual (or anti-self-dual), this number has to be divided by 2.

B.3 Spinor field

Before discussing the number of on-shell spinor components, we begin with a reminder about the properties of spinors in various dimensions. We restrict ourselves to the reality and chirality features of spinors in Minkowski space-times. A more exhaustive discussion of the properties listed below can be found in Ref. [205].

Spinor representation

A spinor (Dirac) representation of the Lorentz algebra $SO(d-1,1)$ has a complex dimension

$$d_\gamma = \begin{cases} 2^{d/2} & \text{for even values of } d, \\ 2^{(d-1)/2} & \text{for odd values of } d. \end{cases} \tag{B.3}$$

A Dirac spinor in $d = 10$ space-time dimensions has then $2^5 = 32$ complex components.

Majorana condition

A Majorana condition relates a spinor to its complex conjugate. Such a condition is only conceivable for $d = 1, 2, 3, 4$ or 8 (mod 8).

Weyl condition

The matrices allowing the definition of the chirality of a spinor always exist when d is even (Γ_{d+1} is non trivial when d is even). Imposing a Weyl condition amounts to consider only spinors which have one of the two possible chiralities.

Majorana–Weyl condition

It is possible to impose simultaneously Majorana and Weyl conditions on $SO(d-1,1)$ spinors if and only if $d = 2$ (mod 8). A Majorana–Weyl spinor has then 1 real component if $d = 2$, 16 real components if $d = 10$, and $2^{\frac{d}{2}-1}$ real components in the general case.

The allowed conditions in space-time dimensions $d \leq 12$ are summarized in Table B.1, where we have also given the smallest possible dimension d_m of a spinor representation. The total number of supercharges in four dimensions is limited to $N = 4$ for renormalizable supersymmetric models and to $N = 8$ for supergravity theories: the total number of real supercharges must not exceed 16, respectively 32 if one confines oneself to states with helicities less than 1, respectively less than 2. Thus, there exists a maximum dimension beyond which the construction of conventional models including supersymmetry is not possible any more.

On-shell spinor components

The number of pertinent components is reduced by an extra factor of 2 if the spinor satisfies the Dirac equation. So, a Majorana–Weyl spinor in ten space-time dimensions usually describes $\frac{1}{2} \cdot d_m = 8$ propagation modes. For a generic spinor, the number of on-shell real components is given by

$$\frac{1}{k} \cdot \begin{cases} 2^{d/2} & \text{for even values of } d, \\ 2^{(d-1)/2} & \text{for odd values of } d, \end{cases} \tag{B.4}$$

where $k = 2$ for a Weyl or Majorana spinor, $k = 4$ for a Majorana–Weyl spinor.

d	d_γ	Allowed condition(s)	d_m
2	2	M, W, M–W	1
3	2	M	2
4	4	M, W	4
5	4	–	8
6	8	W	8
7	8	–	16
8	16	M, W	16
9	16	M	16
10	32	M, W, M–W	16
11	32	M	32
12	64	M, W	64

Table B.1: Summary of the allowed conditions for $SO(d-1,1)$ spinors in various space-time dimensions d. d_γ symbolizes the (complex) dimension of the Dirac representation, while M, W and M–W denotes respectively Majorana, Weyl and Majorana–Weyl conditions. We have also indicated the smallest possible dimension d_m of a spinor representation (the initial total number of real spinor components is $2d_\gamma$ and one can sometimes reduce this number by imposing any of the allowed conditions).

B.4 Gravitino field

From the point of view of the Lorentz group, a gravitino can be thought of as originating in the product of a vector and a spinor. One may write

$$\text{vector} \otimes \text{spinor} = \text{spinor} \oplus \text{gravitino},$$

so that the number of dynamical degrees of freedom of a gravitino is simply given by

$$\begin{aligned} \text{comp}(\text{vector} \otimes \text{spinor}) - \text{comp}(\text{spinor}) &= (d-2) \cdot \text{comp}(\text{spinor}) - \text{comp}(\text{spinor}) \\ &= (d-3) \cdot \frac{1}{k} \cdot \begin{cases} 2^{d/2} & \text{for even values of } d, \\ 2^{(d-1)/2} & \text{for odd values of } d, \end{cases} \end{aligned} \tag{B.5}$$

where “comp” stands for “on-shell components”. For instance, a Majorana gravitino in eleven space-time dimensions represents $(11-3) \cdot \frac{1}{2} \cdot 32 = 128$ propagating modes.

Appendix C

Anomaly polynomials

This appendix contains a brief introduction to anomalies in ten- and six-dimensional field theories with a particular emphasis on anomaly polynomials [5, 4, 108]. A more comprehensive review can be found in the textbook written by Green, Schwarz and Witten [108] or in the nice Appendix B of Ref. [86]. We adopt the conventions of Ref. [108].

A field theory is called *anomalous* if any of its classical symmetries cannot survive the quantization procedure. The notion of *anomaly* refers to such a "quantum mechanical symmetry breaking".

If the classical symmetry is *global*, the generation of an anomaly by loop quantum corrections is not lethal to the coherence of the model. It simply means that the associated conservation law is not valid in the quantum field theory. For instance, the classical global symmetry $U_L(n) \times U_R(n)$ of QCD with n massless quarks which corresponds to independent unitary transformations of the left-handed and right-handed components of the quarks[1] is broken to a diagonal $SU(n)$ subgroup by quantum effects. In this case, the symmetry breakdown is actually phenomenologically welcome since observations of hadrons show an approximate global symmetry $SU(n)$.

On the other hand, an anomalous *local* (gauge) symmetry does not allow a coherent quantization of the associated gauge field. A common example in the framework of the Standard Model is the anomaly produced by the Feynman diagram depicted in Figure C.1 where three external gauge bosons are coupled through a fermionic triangle.[2] The cancellation of all anomalous currents leads to a relation between the quantum numbers of the quarks and leptons. In a more general setting, the known anomalies are all related to classical chiral symmetries and may be present whenever space-time has an *even* number d of dimensions. They always involve a polygonal diagram with $1 + d/2$ sides and their cancellation restrict the admissible gauge structures.

Mathematically, a chiral anomaly $\mathcal{A}_d$ in d dimensions (corresponding to an anomalous variation $\delta S = \int d^d x\, \mathcal{A}_d$ of an action S under gauge and gravitational local transformations)

[1]The Lagrangian describing the classical dynamics of a free massless fermion contains only two kinetic terms associated respectively to the left and right independent components of the spinor (these components have well-defined separate transformations under the Lorentz group). It obviously has a so-called *chiral symmetry* $U_L(1) \times U_R(1)$.

[2]In a more general context, a similar anomaly is produced by the coupling of an external gauge field with two external gravitons.

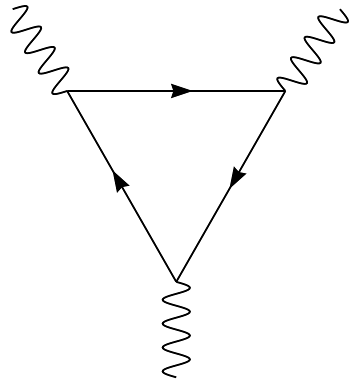

Figure C.1: Triangle diagram at the origin of anomalies in four dimensions.

can be characterized by a formal polynomial I_{d+2} which is unique and gauge invariant.[3] The *anomaly polynomial* is related to $\mathcal{A}_d$ by the following *descent equations*:

$$I_{d+2} = dI_{d+1}, \quad \delta I_{d+1} = d\mathcal{A}_d. \tag{C.1}$$

These equations reflect the fact that the anomaly $\mathcal{A}_d$ is not uniquely defined. One always has the freedom to add a d-dimensional *local counterterm* to the action S:

$$S \to S' = S + \int d^d x\, \Delta_d, \tag{C.2}$$

which from the point of view of $\mathcal{A}_d$ amounts to the redefinition

$$\mathcal{A}_d \to \mathcal{A}'_d = \mathcal{A}_d + \delta\Delta_d + d\Delta_{d-1}, \tag{C.3}$$

with an arbitrary Δ_{d-1}. The descent equations precisely imply that I_{d+1} is defined up to an exterior derivative $d\Delta_d$, so that δI_{d+1} is unique up to a $\delta d\Delta_d = d\delta\Delta_d$. This in turn means that $\mathcal{A}_d$ is only defined up to a variation $\delta\Delta_d$ and an exterior derivative $d\Delta_{d-1}$ as it should. When one applies the descent equations to distinct writings of the same invariant anomaly polynomial I_{d+2},[4] one is then naturally led to different anomalies $\mathcal{A}_d$ and $\mathcal{A}'_d$ which differs by the gauge variation $\delta\Delta_d$ of a local counterterm and possibly by a total derivative $d\Delta_{d-1}$.

C.1 Anomalies in ten dimensions

In ten dimensions, anomalies at one-loop originate from the hexagonal diagram presented in Figure C.2. The six external fields of this graph are gauge fields and/or gravitons (in even number). Three kinds of fields may propagate on the internal loop: chiral spinors and

[3]The symbol δ designates gauge and local Lorentz (infinitesimal) transformations. Similarly, by "gauge invariant", we mean invariant under both types of transformations.

[4]See subsection 4.3.1 for a concrete example.

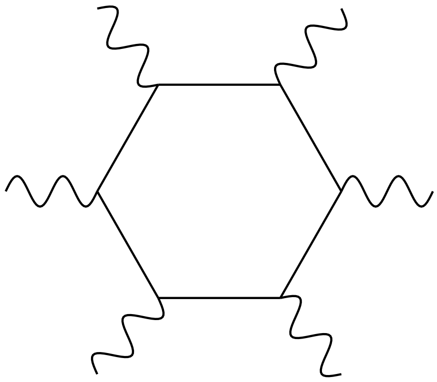

Figure C.2: Hexagonal diagram at the origin of anomalies in ten dimensions.

gravitinos, and self-dual (or anti self-dual) five-forms[5]. The contributions of these fields to the gauge, gravitational and mixed anomalies are given by the following invariant twelve-forms:[6]

$$
\begin{aligned}
I^{(\text{spinor})}_{12,\,\text{gauge}}(F_2) &= \frac{1}{(2\pi)^5 6!}\mathrm{Tr}F_2^6, \\
I^{(\text{spinor})}_{12,\,\text{grav.}}(R_2) &= \frac{1}{(2\pi)^5 6!}\Big(-\frac{1}{504}\mathrm{tr}R_2^6-\frac{1}{384}\mathrm{tr}R_2^4\wedge\mathrm{tr}R_2^2-\frac{5}{4608}\Big(\mathrm{tr}R_2^2\Big)^3\Big), \\
I^{(\text{spinor})}_{12,\,\text{mixed}}(R_2,F_2) &= \frac{1}{(2\pi)^5 6!}\Big(\frac{1}{16}\mathrm{tr}R_2^4\wedge\mathrm{Tr}F_2^2+\frac{5}{64}\Big(\mathrm{tr}R_2^2\Big)^2\wedge\mathrm{Tr}F_2^2-\frac{5}{8}\mathrm{tr}R_2^2\wedge\mathrm{Tr}F_2^4\Big), \\
I^{(\text{gravitino})}_{12,\,\text{grav.}}(R_2) &= \frac{1}{(2\pi)^5 6!}\Big(\frac{55}{56}\mathrm{tr}R_2^6-\frac{75}{128}\mathrm{tr}R_2^4\wedge\mathrm{tr}R_2^2+\frac{35}{512}\Big(\mathrm{tr}R_2^2\Big)^3\Big), \\
I^{(\text{5-form})}_{12,\,\text{grav.}}(R_2) &= \frac{1}{(2\pi)^5 6!}\Big(-\frac{62}{63}\mathrm{tr}R_2^6+\frac{7}{12}\mathrm{tr}R_2^4\wedge\mathrm{tr}R_2^2-\frac{5}{72}\Big(\mathrm{tr}R_2^2\Big)^3\Big),
\end{aligned}
\tag{C.4}
$$

where the symbol "Tr" denotes the trace over generators of the adjoint representation of the gauge group. The anomaly polynomials $I^{(\text{spinor})}_{12}$ and $I^{(\text{gravitino})}_{12}$ are appropriate for chiral Weyl spinors and gravitinos respectively. An additional factor of 1/2 must be included for Majorana–Weyl fermions, and spinors or gravitinos with opposite chiralities contribute with opposite signs. Note also that the chiral spinors are the only one to give a contribution to the gauge and mixed anomalies. The full anomaly polynomial is constructed from a linear combination of the independent field contributions (C.4), and it is not difficult to examine the issue of anomaly cancellation in ten-dimensional chiral supergravity theories.

[5]Although bosonic, these field strengths may contribute to the anomaly and their associated four-forms are often said to be chiral (or antichiral).

[6]Note that the power of (2π) is $-d/2$ (see Ref. [108]). Moreover, one has $I^{(\text{gravitino})}_{12,\,\text{grav.}}-I^{(\text{spinor})}_{12,\,\text{grav.}}=-I^{(\text{5-form})}_{12,\,\text{grav.}}$ (this will have an important consequence below).

C.1.1 Local one-loop anomaly in type IIB supergravity

The ten-dimensional type IIB supergravity has two antichiral Majorana–Weyl spinors, two chiral Majorana–Weyl gravitinos, and a self-dual rank-5 field strength. There are no gauge fields.

The full anomaly is then characterized by the twelve-form

$$\begin{aligned} I_{12} &= -2\cdot\frac{1}{2}I^{(\text{spinor})}_{12,\,\text{grav.}}(R_2)+2\cdot\frac{1}{2}I^{(\text{gravitino})}_{12,\,\text{grav.}}(R_2)+I^{(\text{5-form})}_{12,\,\text{grav.}}(R_2) \\ &\equiv 0, \end{aligned} \tag{C.5}$$

where in the last step we have used the expressions (C.4). One concludes that the ten-dimensional type IIB supergravity is anomaly free.

C.1.2 Local one-loop anomaly in $N=1$ supergravity coupled to vector supermultiplets

The ten-dimensional $N=1$ supergravity multiplet contains an antichiral Majorana–Weyl spinor (dilatino) and a chiral Majorana–Weyl gravitino. There are no five-forms. The Yang–Mills supermultiplets contain chiral Majorana–Weyl spinors (gauginos) in the adjoint representation of the gauge group.

The full anomaly takes then the form

$$\begin{aligned} I_{12} &= \frac{1}{2}\Big(-I^{(\text{spinor})}_{12,\,\text{grav.}}(R_2)+I^{(\text{gravitino})}_{12,\,\text{grav.}}(R_2)\Big) \\ &\quad+\frac{1}{2}\Big(nI^{(\text{spinor})}_{12,\,\text{grav.}}(R_2)+I^{(\text{spinor})}_{12,\,\text{mixed}}(R_2,F_2)+I^{(\text{spinor})}_{12,\,\text{gauge}}(F_2)\Big) \\ &= \frac{1}{2(2\pi)^5 6!}\Big(\frac{496-n}{504}\text{tr}R_2^6-\frac{224+n}{384}\text{tr}R_2^4\wedge\text{tr}R_2^2+\frac{5}{4608}(64-n)(\text{tr}R_2^2)^3 \\ &\qquad+\frac{1}{16}\text{tr}R_2^4\wedge\text{Tr}F_2^2+\frac{5}{64}(\text{tr}R_2^2)^2\wedge\text{Tr}F_2^2-\frac{5}{8}\text{tr}R_2^2\wedge\text{Tr}F_2^4+\text{Tr}F_2^6\Big), \end{aligned} \tag{C.6}$$

where the integer n is the dimension of the gauge group. The expressions (C.4) have been used to obtain the last equality. The cancellation of this anomaly is provided by the so-called *Green–Schwarz mechanism* [105, 106] which rests on the existence in the action of a local counterterm corresponding to the diagram shown in Figure C.3. This procedure necessitates the factorization of I_{12} in a product of a four-form and an eight-form, but such a factorization is impossible for the term $\text{tr}R_2^6$ (the group $SO(9,1)$ has an independent Casimir invariant of order 6). Therefore, one is led to assume that $n=496$. For this particular dimension of the gauge group, the anomaly (C.6) becomes

$$\begin{aligned} I_{12} &= \frac{1}{2(2\pi)^5 6!}\Big(-\frac{15}{8}\text{tr}R_2^4\wedge\text{tr}R_2^2-\frac{15}{32}(\text{tr}R_2^2)^3+\frac{1}{16}\text{tr}R_2^4\wedge\text{Tr}F_2^2 \\ &\qquad+\frac{5}{64}(\text{tr}R_2^2)^2\wedge\text{Tr}F_2^2-\frac{5}{8}\text{tr}R_2^2\wedge\text{Tr}F_2^4+\text{Tr}F_2^6\Big). \end{aligned} \tag{C.7}$$

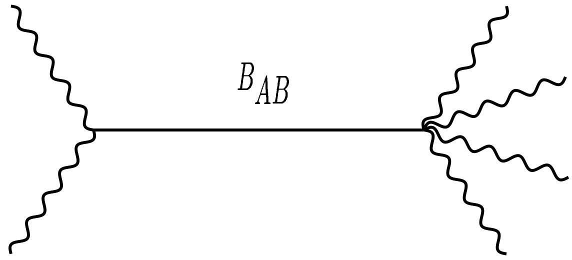

Figure C.3: Tree diagram describing the exchange of a B_{AB} field between gauge bosons and gravitons in ten dimensions. From the point of view of the underlying superstring theory, this graph has the same (one-loop) origin as the hexagonal diagram.

This anomaly can be factorized if and only if the gauge group is such that the term $\mathrm{Tr}F_2^6$ admits the following decomposition:

$$\mathrm{Tr}F_2^6 = \frac{1}{48}\mathrm{Tr}F_2^4 \wedge \mathrm{Tr}F_2^2 - \frac{1}{14400}(\mathrm{Tr}F_2^2)^3. \tag{C.8}$$

There are only two 496-dimensional non-Abelian Lie groups with this property, namely $SO(32)$ and $E_8 \times E_8$. With the decomposition (C.8), the anomaly polynomials finally reads

$$I_{12} = -\frac{15}{2(2\pi)^5 6!}\Big(\mathrm{tr}R_2^2 - \frac{1}{30}\mathrm{Tr}F_2^2\Big) \wedge X_8^{\mathrm{GSW}}, \tag{C.9}$$

where the eight-form X_8^{GSW} is given by

$$\begin{aligned} X_8^{\mathrm{GSW}} &= \frac{1}{8}\mathrm{tr}R_2^4 + \frac{1}{32}(\mathrm{tr}R_2^2)^2 - \frac{1}{240}\mathrm{tr}R_2^2 \wedge \mathrm{Tr}F_2^2 + \frac{1}{24}\mathrm{Tr}F_2^4 - \frac{1}{7200}(\mathrm{Tr}F_2^2)^2 \\ &= \frac{1}{8}\mathrm{tr}R_2^4 + \frac{1}{32}(\mathrm{tr}R_2^2)^2 - \frac{1}{8}\mathrm{tr}R_2^2 \wedge \mathrm{tr}F_2^2 + \frac{5}{4}\mathrm{tr}F_2^4 - \frac{1}{8}(\mathrm{tr}F_2^2)^2, \end{aligned} \tag{C.10}$$

where in the last equality we have used the identity Tr = 30tr (see Appendix A). The anomaly I_{12} has the structure required for a cancellation by a Green–Schwarz counterterm.

C.1.3 Local one-loop anomaly on a $S^1/\mathbb{Z}_2$ fixed plane

The discussion of the one-loop anomaly on the ith fixed plane ($i = 1, 2$) closely mimics the development reproduced in the preceding subsection. For a general gauge group, one has

$$\begin{aligned} I_{12,i}^{\text{1-loop}} &= \frac{1}{4}\Big(- I_{12,\,\mathrm{grav.}}^{(\mathrm{spinor})}(R_{2,i}) + I_{12,\,\mathrm{grav.}}^{(\mathrm{gravitino})}(R_{2,i})\Big) \\ &\quad + \frac{1}{2}\Big(n_i I_{12,\,\mathrm{grav.}}^{(\mathrm{spinor})}(R_{2,i}) + I_{12,\,\mathrm{mixed}}^{(\mathrm{spinor})}(R_{2,i}, F_{2,i}) + I_{12,\,\mathrm{gauge}}^{(\mathrm{spinor})}(F_{2,i})\Big) \\ &= \frac{1}{2(2\pi)^5 6!}\Big(\frac{248 - n_i}{504}\mathrm{tr}R_{2,i}^6 - \frac{112 + n_i}{384}\mathrm{tr}R_{2,i}^4 \wedge \mathrm{tr}R_{2,i}^2 + \frac{5}{4608}(32 - n_i)(\mathrm{tr}R_{2,i}^2)^3 \\ &\quad + \frac{1}{16}\mathrm{tr}R_{2,i}^4 \wedge \mathrm{Tr}F_{2,i}^2 + \frac{5}{64}(\mathrm{tr}R_{2,i}^2)^2 \wedge \mathrm{Tr}F_{2,i}^2 - \frac{5}{8}\mathrm{tr}R_{2,i}^2 \wedge \mathrm{Tr}F_{2,i}^4 + \mathrm{Tr}F_{2,i}^6\Big), \end{aligned} \tag{C.11}$$

where an overall factor of 1/2 is due to the Majorana–Weyl condition. The gravitational contribution $-I^{(\text{spinor})}_{12,\,\text{grav.}} + I^{(\text{gravitino})}_{12,\,\text{grav.}}$ is the half of what would be expected in ten dimensions because of the coupling of eleven-dimensional supergravity to ten-dimensional fields [116, 117]. The dimension of the gauge group is denoted by n_i. Note that we write $R_{2,i}$ instead of R_2 to emphasize the fact that R_2 is meant to be located on the ith plane. The one-loop anomaly has a chance to be cancelled by an appropriate Green–Schwarz mechanism only if the twelve-form $I^{\text{1-loop}}_{12,i}$ factorizes into a four-form and an eight-form. The $\text{tr}R_2^6$ term must then be absent and we must be able to express $\text{Tr}F_2^6$ as a linear combination of $\text{Tr}F_2^2 \wedge \text{Tr}F_2^4$ and $(\text{Tr}F_2^2)^3$. The first condition selects $n_i = 248$, while the only apposite factorization of $\text{Tr}F_2^6$ necessitates the following linear combination:

$$\text{Tr}F^6_{2,i} = \text{Tr}F^2_{2,i} \wedge \left(\frac{1}{24}\text{Tr}F^4_{2,i} - \frac{1}{3600}(\text{Tr}F^2_{2,i})^2\right). \tag{C.12}$$

This condition only holds for the group E_8 for which we have the two identities

$$\text{Tr}F^6_{2,i} = \frac{1}{7200}(\text{Tr}F^2_{2,i})^3 \qquad \text{and} \qquad \text{Tr}F^4_{2,i} = \frac{1}{100}(\text{Tr}F^2_{2,i})^2. \tag{C.13}$$

The group E_8 also has the required dimension $n_i = 248$. With the redefinition Tr = 30tr and the second E_8 identity, the anomaly (C.11) takes the factorized form

$$I^{\text{1-loop}}_{12,i} = \frac{\pi}{3}(I_{4,i})^3 + I_{4,i} \wedge X_{8,i}, \tag{C.14}$$

with the four-form

$$I_{4,i} = \frac{1}{(4\pi)^2}\left(\text{tr}F^2_{2,i} - \frac{1}{2}\text{tr}R^2_{2,i}\right) \tag{C.15}$$

and the eight-form

$$X_{8,i} = \frac{1}{12(4\pi)^3}\left(\frac{1}{2}\text{tr}R^4_{2,i} - \frac{1}{8}\left(\text{tr}R^2_{2,i}\right)^2\right). \tag{C.16}$$

It is interesting to understand the factorization in the case of the $E_8 \times E_8$ weakly coupled heterotic string for which the relevant anomaly polynomial is the sum $I^{\text{1-loop, weak}}_{12} = I^{\text{1-loop}}_{12,1} + I^{\text{1-loop}}_{12,2}$. In the perturbative limit, the two fixed planes coincide and one should take the same value for R_2 in both expressions. Then, the purely gravitational eight-form X_8 is the same and the sum $I^{\text{1-loop, weak}}_{12}$ becomes

$$\begin{aligned} I^{\text{1-loop, weak}}_{12} &= (I_{4,1} + I_{4,2}) \wedge X_8 + \frac{\pi}{3}\left((I_{4,1})^3 + (I_{4,2})^3\right) \\ &= (I_{4,1} + I_{4,2}) \wedge \left[X_8 + \frac{\pi}{3}\left((I_{4,1})^2 + (I_{4,2})^2 - I_{4,1} \wedge I_{4,2}\right)\right]. \end{aligned} \tag{C.17}$$

The second term of the first equality has been factorized thanks to the trivial identity $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$. We can rewrite $I^{\text{1-loop, weak}}_{12}$ in term of traces over the representations of the product group $E_8 \times E_8$ to recover the well-known expression (C.9) of the anomaly for $E_8 \times E_8$ super Yang–Mills coupled to $N = 1$ supergravity:[7]

$$\begin{aligned} I_{12}^{\text{1-loop, weak}} &= \frac{1}{12(4\pi)^5}\left(\mathrm{tr_x} F_2^2 - \mathrm{tr} R_2^2\right) \wedge \left(\frac{1}{2}\mathrm{tr} R_2^4 + \frac{1}{8}(\mathrm{tr} R_2^2)^2 - \frac{1}{2}\mathrm{tr} R_2^2 \wedge \mathrm{tr_x} F_2^2 \right. \\ &\qquad \left. + 5\mathrm{tr_x} F_2^4 - \frac{1}{2}(\mathrm{tr_x} F_2^2)^2\right), \end{aligned} \tag{C.18}$$

where $\mathrm{tr_x}$ is a trace over the $E_8 \times E_8$ group[8].

Note that we have chosen the normalization of the four-forms $I_{4,i}$ such that their integration over a four-cycle equals the integral characteristic class of the E_8 bundle minus a quarter of the even first Pontrjagin class.[9] More explicitly, for any four-cycle $\mathcal{C}_4$ one has

$$\int_{\mathcal{C}_4} I_{4,i} \equiv \frac{1}{(4\pi)^2}\int_{\mathcal{C}_4}\left(\mathrm{tr} F_{2,i}^2 - \frac{1}{2}\mathrm{tr} R_{2,i}^2\right) = m_i - \frac{1}{2}p_i, \quad m_i, p_i \in \mathbb{Z}. \tag{C.19}$$

C.2 Anomalies in six dimensions

In six dimensions, anomalies at one-loop originate from three kinds of fields: chiral spinors and gravitinos, and self-dual (or anti self-dual) three-forms. The contributions of these fields to the gauge, gravitational and mixed anomalies are given by the following invariant eight-forms:[10]

$$\begin{aligned} I_{8,\,\text{gauge}}^{(\text{spinor})}(F_2) &= \frac{1}{(2\pi)^3 4!}\left(-\mathrm{Tr} F_2^4\right), \\ I_{8,\,\text{grav.}}^{(\text{spinor})}(R_2) &= \frac{1}{(2\pi)^3 4!}\left(-\frac{1}{240}\mathrm{tr} R_2^4 - \frac{1}{192}\left(\mathrm{tr} R_2^2\right)^2\right), \\ I_{8,\,\text{mixed}}^{(\text{spinor})}(R_2, F_2) &= \frac{1}{(2\pi)^3 4!}\left(\frac{1}{4}\mathrm{tr} R_2^2 \wedge \mathrm{Tr} F_2^2\right), \\ I_{8,\,\text{grav.}}^{(\text{gravitino})}(R_2) &= \frac{1}{(2\pi)^3 4!}\left(-\frac{49}{48}\mathrm{tr} R_2^4 + \frac{43}{192}\left(\mathrm{tr} R_2^2\right)^2\right), \\ I_{8,\,\text{grav.}}^{(\text{3-form})}(R_2) &= \frac{1}{(2\pi)^3 4!}\left(-\frac{7}{60}\mathrm{tr} R_2^4 + \frac{1}{24}\left(\mathrm{tr} R_2^2\right)^2\right), \end{aligned} \tag{C.20}$$

where Tr denotes the trace over generators of the adjoint representation of the gauge group.

[7] This is the well-known expression for the heterotic string.

[8] This means in particular that $\mathrm{tr_x} F_2^2 = \mathrm{tr} F_{2,1}^2 + \mathrm{tr} F_{2,2}^2$ and $\mathrm{tr_x} F_2^4 = \mathrm{tr} F_{2,1}^4 + \mathrm{tr} F_{2,2}^4$.

[9] The characteristic classes are topological invariants which, in particular, play a role in QCD where they are associated with "instanton numbers".

[10] These anomaly polynomials are of course expressed in terms of the *six*-dimensional gauge and Lorentz curvatures.

C.2.1 Local one-loop anomaly on an M-five-brane

The excitations of the M-five-brane are described by a $d = 6$ $N = 2$ tensor supermultiplet which contains two chiral Weyl spinors, five real scalars and a self-dual rank-3 field strength. There are no gauge fields.

Thus, the five-brane anomaly is purely gravitational and can be encoded in the following eight-form:

$$\begin{aligned} I_8 &= 2I^{(\text{spinor})}_{8,\,\text{grav.}}(R_2) + I^{(\text{3-form})}_{8,\,\text{grav.}}(R_2) \\ &= -\frac{1}{(2\pi)^3 4!}\left(\frac{1}{8}\text{tr}R_2^4 - \frac{1}{32}\left(\text{tr}R_2^2\right)^2\right). \end{aligned} \tag{C.21}$$

Appendix D

$N = 1$ four-dimensional conformal supergravity

In Chapters 5 and 6, we have used the superconformal tensor calculus for $N = 1$ four-dimensional supergravity [125, 130, 131]. In this appendix, we would like to comment on some aspects of this formalism. We also assemble a few useful relations, list the various multiplets appearing in the main text and give an account of the possible Bianchi identities in four dimensions.

D.1 Conformal supergravity as a unified formalism for describing Poincaré supergravity theories

The gauge fields of the pure $N = 1$ Poincaré supergravity theory are the vierbein e_μ^m (gauge field of the space-time translations)[1], the spin connection ω_μ^{mn} (gauge field of the local Lorentz transformations) and the gravitino ψ_μ (gauge field of supersymmetry). The associated Lagrangian contains the usual (non-supersymmetric) Einstein–Hilbert part and the extra Rarita–Schwinger term which describes the gravitino dynamics [93, 62]. The equation of motion for the spin connection is algebraic and the gauge field ω_μ^{mn} can be eliminated. As a consequence, the relevant physical degrees of freedom constituting the pure supergravity multiplet correspond to the vierbein (the graviton) and the gravitino.

However, to obtain an off-shell formulation (i.e. a formulation in which supersymmetry is realized without the help of the equations of motion), one has to introduce *auxiliary fields*. The point is that the choice of the number and nature of these auxiliary fields is not unique. There are thus at least three versions of $N = 1$ Poincaré supergravity: the *old minimal supergravity* [208, 90], the *Sohnius–West new minimal supergravity* [206], and the *Breitenlohner non-minimal supergravity* [34]. These three theories can be regarded as different gauge fixings of conformal supergravity coupled to different multiplets used as compensator. Conformal supergravity is a gauge theory based on the superconformal group with the generators P_m (translations), L_{mn} (Lorentz transformations), K_m (conformal boosts), D

[1]The Latin character m denotes an internal index.

(dilatations), Q (supersymmetry), S (special supersymmetry)[2] and A (chiral $U(1)$ internal symmetry), the corresponding gauge fields (or gauge potentials) being e_μ^m, ω_μ^{mn}, f_μ^m, b_μ, ψ_μ, φ_μ and A_μ. To obtain a super-Poincaré theory, one has to gauge fix the four extraneous local symmetries. The standard procedure is to introduce an extra supermultiplet and impose conditions on some of its components, so that it will not compensate the variances of the other multiplets any more. The set of auxiliary fields present in the Poincaré supergravity multiplet is related to the choice of this *compensating multiplet.* For instance, using a chiral compensating multiplet $S_0 = (z_0, \psi_0, f_0)$ leads to the "old minimal" Poincaré supergravity with six auxiliary degrees of freedom. More precisely, the gauge fixing amounts to take [131]

- $b_\mu = 0$ (conformal boosts gauge fixing),
- ψ_0 with a specific dependence on the fields present in the theory (special supersymmetry gauge fixing),
- $\mathrm{Im}\, z_0 = 0$ (chiral $U(1)$ gauge fixing),
- $|z_0|$ in order to have a particular Poincaré frame, mostly the Einstein or string frames (dilatations gauge fixing).

The surviving component f_0 (a complex scalar) and the gauge potential A_μ can be identified with the six auxiliary fields of the "old minimal" Poincaré supergravity multiplet. The graviton and the gravitino complete the multiplet.

D.2 Some useful relations

Since it is sufficient for our purposes to consider only bosonic quantities, we will systematically omit the fermionic contributions in the following relations. Complete expressions can be found for instance in Refs. [130] and [131].

Conformal and chiral weights

The *conformal* and *chiral weights*[3] are two parameters denoted w and n that precise the behaviour of (the first component of) a supermultiplet under the dilatation and chiral transformations.

General multiplets

- *Complex vector multiplet* $V = (c, 0, h, k, v_\mu, 0, d)$ with complex components and unspecified weights w and n.
- *Real vector multiplet* $V = (c, 0, h, k, v_\mu, 0, d)$ with real components, w arbitrary and $n = 0$. For $w = 0$, it is possible to define a *gauge multiplet.*

[2]These two local supersymmetries correspond respectively to the "square roots" of the translations and conformal boosts.

[3]The conformal weight is sometimes called *Weyl weight.*

- *Chiral multiplet* $C = (z, 0, f)$ with complex components and $w = n$. The conjugate multiplet is $\overline{C} = (\overline{z}, 0, \overline{f})$ with $w_c = w$ and $n_c = -n$. A chiral multiplet C can be embedded in a complex vector multiplet: $V(C) = (z, 0, -f, if, iD^c_\mu z, 0, 0)$ and we usually use the notation $C \equiv V(C)$. After the gauge fixing down to super-Poincaré, the expression of the conformal covariant derivative is $D^c_\mu z = D_\mu z - \frac{1}{2} inA_\mu z$.
- *Complex linear multiplet* $L \equiv V(L) = (c, 0, h, -ih, v_\mu, 0, -\Box^c c - iD^c_\mu v^\mu)$ with complex components and $w - n = 2$. Note that the conformal d'Alembertian is of the form $\Box^c z = \Box z + \frac{1}{6} wRz + \cdots$ [223], where z is a scalar field with conformal weight w.
- *Real linear multiplet* $L \equiv V(L) = (c, 0, 0, 0, v_\mu, 0, -\Box^c c)$ with real components, $w = 2$ and $n = 0$. The component v_μ is subject to the constraint $D^c_\mu v^\mu = 0$.

Component expressions for functions of multiplets

The product of two general vector multiplets, $V = (c, 0, h, k, v_\mu, 0, d)$ with weights w and n, and $\widetilde{V} = (\tilde{c}, 0, \tilde{h}, \tilde{k}, \tilde{v}_\mu, 0, \tilde{d})$ with weights $\tilde{w}$ and $\tilde{n}$, is

$$\begin{aligned} V\widetilde{V} &= \Big(c\tilde{c}, 0, c\tilde{h} + \tilde{c}h, c\tilde{k} + \tilde{c}k, c\tilde{v}_\mu + \tilde{c}v_\mu, 0, \\ &\qquad c\tilde{d} + \tilde{c}d + h\tilde{h} + k\tilde{k} - v_\mu \tilde{v}^\mu - (D^c_\mu c)(D^{c\mu}\tilde{c})\Big), \end{aligned} \tag{D.1}$$

with weights $w + \tilde{w}$ and $n + \tilde{n}$.

The mth power of a vector multiplet $V = (c, 0, h, k, v_\mu, 0, d)$ with weights w and n is given by

$$\begin{aligned} V^m &= \Big(c^m, 0, mc^{m-1}h, mc^{m-1}k, mc^{m-1}v_\mu, 0, \\ &\qquad mc^{m-1}d + \tfrac{1}{2}m(m-1)c^{m-2}[h^2 + k^2 - v_\mu v^\mu - (D^c_\mu c)(D^{c\mu}c)]\Big), \end{aligned} \tag{D.2}$$

with of course weights mw and mn. For a chiral multiplet $C = (z, 0, f)$, this translates into the relation

$$C^m = \Big(z^m, 0, mz^{m-1}f\Big), \tag{D.3}$$

and a general function $g(C)$ (with $w = 0 = n$) has the components

$$g(C) = \Big(g(z), 0, \frac{dg}{dz}(z)f\Big). \tag{D.4}$$

D- and F-density formulas for invariant actions

The D-density formula for a real vector multiplet $V = (c, 0, h, k, v_\mu, 0, d)$ with conformal weight $w = 2$ is

$$[V]_D = e(d + \frac{1}{3}cR), \tag{D.5}$$

while the F-density for a chiral multiplet $C = (z, 0, f)$ with conformal weight $w = 3$ reads

$$[C]_F = e(f + \overline{f}). \tag{D.6}$$

D.3 The various multiplets used in Chapters 5 and 6

In this section, we recall the essential features of the various supermultiplets appearing in the course of our discussion about the effective supergravity of M-theory compactifications. In comparison to Refs. [130, 131] and to section D.2, we have replaced in the expansion of the vector multiplets the highest component d by $d - \Box^c c$. This change was made to facilitate some technical manipulations. It implies that the terms introduced to impose the Bianchi identities do not include any contribution to the Einstein term.

Compensating multiplet

- A chiral multiplet $S_0 = (z_0, 0, -f_0, if_0, iD^c_\mu z_0, 0, 0)$ with $w = 1 = n$.

Lagrange multiplier multiplets

- A chiral multiplet $S = (s, 0, -f, if, i\partial_\mu s, 0, 0)$ with $w = 0 = n$.
- A real linear multiplet $L_T = (\ell_T, 0, 0, 0, t_\mu, 0, -\Box\ell_T - \frac{1}{3}\ell_T R)$ with $w = 2$, $n = 0$ and $t_\mu = \frac{1}{2} e\epsilon_{\mu\nu\rho\sigma}\partial^\nu t^{\rho\sigma}$.
- A real vector multiplet $U = (C_U, 0, H_U, K_U, u_\mu, 0, d_U - \Box U - \frac{1}{3}UR)$ with $w = 2$ and $n = 0$.

M-theory multiplets

- A real vector multiplet $V = (C, 0, H, K, v_\mu, 0, d - \Box C - \frac{1}{3}CR)$ with $w = 2$ and $n = 0$.

  Solving for the Lagrange multiplier S in the D-density $[(S + \overline{S})V]_D$, the multiplet V can be written as a real linear multiplet L with $\partial_\mu v^\mu = d = H = K = 0$.

- A real vector multiplet $V_T = (C_T, 0, H_T, K_T, T_\mu, 0, d_T - \Box C_T)$ with $w = 0 = n$.

  Solving for the Lagrange multiplier L_T in the D-density $[L_T V_T]_D$, the multiplet V_T can be written as $T + \overline{T}$ where the multiplet T (with $w = 0 = n$) is the chiral Calabi–Yau universal modulus multiplet with complex components T and f_T given by the relations $C_T = 2\,\mathrm{Re}\,T$, $T_\mu = -2\partial_\mu\,\mathrm{Im}\,T$, $H_T = -2\,\mathrm{Re}\,f_T$ and $K_T = -2\,\mathrm{Im}\,f_T$.

- A chiral multiplet $W = (w, 0, -f_w, if_w, i\partial_\mu w, 0, 0)$ with $w = 0 = n$.

  Solving for the Lagrange multiplier U in the D-density $[U(W + \overline{W})]_D$, the multiplet W reduces to an arbitrary imaginary constant (the only non-trivial remaining equation is $\partial_\mu\,\mathrm{Im}\,w = 0$).

The relations between the components C, C_T, v_μ, T_μ and $\mathrm{Im}\,w$ of these three M-theory multiplets and the fields σ, γ, $C_{\mu\nu 11}$ and $C_{i\bar{\jmath}11}$ arising from the reduced eleven-dimensional metric and three-form C_3 are given in Eqs. (5.36) and (5.40).

Source multiplets

- A chiral matter multiplet $M = (M, 0, -f_M, if_M, i\partial_\mu M, 0, 0)$ with $w = 0 = n$.

- A real vector gauge multiplet $A \equiv A^a T^a$ where the T^a are the gauge group generators (adjoint representation). In the Wess–Zumino gauge, one has $A^a = (0, 0, 0, 0, A^a_\mu, 0, D^a)$ with $w = 0 = n$.

The gauge field strength for the multiplet A is the chiral multiplet $\mathcal{W}$ with $w = 3/2 = n$. One also defines the Chern–Simons real vector multiplet $\Omega = \Omega(A)$ with $w = 2$ and $n = 0$ through the chiral projection $\Sigma(\Omega) = \frac{1}{16}\mathcal{W}\mathcal{W}$, where

$$\Sigma(V) = \Big(\frac{1}{2}(h - ik), 0, -\frac{1}{2}(d + \Box^c c + iD^c_\mu v^\mu)\Big) \qquad \text{(D.7)}$$

for a general real vector multiplet V with $w = 2$ (in global Poincaré supersymmetry, $\Sigma(V)$ corresponds to $-\frac{1}{4}\overline{DD}V$). The components of Ω can be computed from the F-density of the product $\mathcal{W}\mathcal{W}$ (which gives the usual super Yang–Mills Lagrangian) using the identity $[\Sigma(\ldots)]_F = -[\ldots]_D$.

M-five-brane multiplets

- A chiral multiplet $\hat{S} = (\hat{s}, 0, -\hat{f}_s, i\hat{f}_s, i\partial_\mu\hat{s}, 0, 0)$ with $w = 0 = n$.

- A real vector multiplet $\hat{V} = (\hat{C}, 0, \hat{H}, \hat{K}, \hat{v}_\mu, 0, \hat{d} - \Box\hat{C})$ with $w = 0 = n$.

The relations between the components $\mathrm{Im}\,\hat{s}$, $\hat{C}$ and $\hat{v}_\mu$ of these two M-five-brane multiplets and the fields σ, $C_{\mu\nu 11}$, $C_{i\bar{j}11}$, $\mathcal{B}_{\hat{5}\hat{6}}$, $\mathcal{B}_{\mu\nu}$ and X are given in Eqs. (6.71) and (6.74).

D.4 Bianchi identities in four dimensions

In four space-time dimensions, the Bianchi identities for all antisymmetric tensors are of the form:

$$\begin{array}{llcll}
\text{B1 (for a rank-3 tensor } A_{\mu\nu\rho}\text{):} & 4\partial_{[\mu}(A_{\nu\rho\sigma]} - \Delta_{\nu\rho\sigma]}) & = & 0 & \text{(one eq.),} \\
\text{B2 (for a rank-2 tensor } A_{\mu\nu}\text{):} & 3\partial_{[\mu}(A_{\nu\rho]} - \Delta_{\nu\rho]}) & = & 0 & \text{(four eqs.),} \\
\text{B3 (for a rank-1 tensor } A_{\mu}\text{):} & 2\partial_{[\mu}(A_{\nu]} - \Delta_{\nu]}) & = & 0 & \text{(six eqs.),} \\
\text{B4 (for a rank-0 tensor } A\text{):} & \partial_{\mu}(A - \Delta) & = & 0 & \text{(four eqs.).}
\end{array} \qquad \text{(D.8)}$$

In each identity we have introduced a possible source Δ. These four identities can be inserted in a Lagrangian using the appropriate Lagrange multipliers λ. The Lagrangian terms are respectively:

$$\begin{aligned}
&\text{B1:} \quad \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}(\partial_\mu\lambda)(A_{\nu\rho\sigma} - \Delta_{\nu\rho\sigma}),\\
&\text{B2:} \quad \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\partial_\mu\lambda_\nu)(A_{\rho\sigma} - \Delta_{\rho\sigma}),\\
&\text{B3:} \quad \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\partial_\mu\lambda_{\nu\rho})(A_\sigma - \Delta_\sigma),\\
&\text{B4:} \quad \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}(\partial_\mu\lambda_{\nu\rho\sigma})(A - \Delta).
\end{aligned} \tag{D.9}$$

Each of these expressions possesses a natural supersymmetric extension, with a Lagrange multiplier supermultiplet Λ imposing a constraint on the supermultiplet $\tilde{A} = A - \Delta$. One has:[4]

$$\begin{array}{llll}
\text{B1:} & [(\Lambda + \overline{\Lambda})\tilde{A}]_D, & \Lambda : \text{chiral } (w = 0), & \tilde{A} : \text{real vector } (w = 2),\\
\text{B2:} & [\mathcal{W}(\Lambda)^\alpha \tilde{A}_\alpha]_F, & \Lambda : \text{real vector } (w = 0), & \tilde{A}_\alpha : \text{chiral spinor } (w = 3/2),\\
\text{B3:} & [\Lambda\tilde{A}]_D, & \Lambda : \text{real linear } (w = 2), & \tilde{A} : \text{real vector } (w = 0),\\
\text{B4:} & [\Sigma(\Lambda)\tilde{A}]_F = -\frac{1}{2}[\Lambda(\tilde{A} + \overline{\tilde{A}})]_D, & \Lambda : \text{real vector } (w = 2), & \tilde{A} : \text{chiral } (w = 0).
\end{array} \tag{D.10}$$

In the F-densities corresponding to the cases B2 and B4, $\mathcal{W}(\Lambda)_\alpha$ is the curvature of the real vector supermultiplet Λ and $\Sigma(\Lambda)$ is its chiral embedding.[5] We have never used the identity B2 in the present thesis.

[4]The Greek letter α used in the F-density B2 denotes a spinor index.

[5]In global Poincaré supersymmetry, these objects would be $-\frac{1}{4}\overline{DD}D_\alpha\Lambda$ and $-\frac{1}{4}\overline{DD}\Lambda$ respectively (see Refs. [130, 131]).

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