Isoperimetric inequalities for Laplace and Steklov problems on Riemannian manifolds

Thèse

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par

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Abstract

This thesis is about the spectra of the Laplacian and of the Dirichlet-to-Neumann operators on a compact Riemannian manifold. Both problems have physical interpretations that can be found in [Eva98] and [Ban80]. We focused on finding upper bounds for the eigenvalues, based on the geometry of the manifold. More precisely, we consider if it is possible to obtain upper bounds in which the geometrical term can be separated from the asymptotical term, and if the latter grows optimally regarding the Weyl law.

The first result is dedicated to the construction of a counterexample of a question that follows from a work by B. Colbois, A. El Soufi and A. Girouard [CEG13], where they bound the Laplacian eigenvalues of a hypersurface Σ of dimension $n \geqslant 2$ using its isoperimetric ratio $I(\Sigma)$:

$$\lambda_k(\Sigma) \cdot |\Sigma|^{\frac{2}{n}} \leqslant \gamma(n) I(\Sigma)^{1+\frac{2}{n}} k^{\frac{2}{n}}.$$

I show the bound they obtain is actually optimal, in the sense that one cannot have a bound of the form $\gamma_1(n)I(\Sigma) + \gamma_2(n)k^{\frac{2}{n}}$. In this context, it also means that a "Kröger type" inequality is not possible, even after a certain index.

The second theme is a joint work with S. Kouzayha [KP]. Given a Riemannian manifold (M, g) of dimension n, we are interested in a generalisation of the Laplace problem, that is, we are focused on the variations of the densities σ and ρ of the following problem:

$$\begin{cases} -\mathrm{div}(\sigma \nabla u) & = \lambda \rho u \quad \text{in } M \\ \partial_n u & = 0 \quad \text{on } \partial M. \end{cases}$$

There are *a priori* a lot of parameters, so the metric will be fixed, and the densities will vary following some restrictions. Here σ will be taken as a power of ρ, that is, $\sigma = \rho^{\alpha}$, $\alpha \in (0, 1)$. This choice is motivated by considerations of the conformal spectrum. Indeed, the study of conformal metrics can be linked with a particular case of densities given by $\sigma = \rho^{\frac{n-2}{n}}$. We bring a definitive answer to the boundedness of the eigenvalues when $\alpha \leqslant \frac{n-2}{n}$.

The third topic is about the Steklov problem:

$$\begin{cases} \Delta u & = 0 \quad \text{in } M \\ \partial_\eta u & = \sigma u \quad \text{on } \partial M. \end{cases}$$

That means we search for real values such that there is a corresponding function u which is harmonic in M and proportional to its normal derivative on ∂M.

The interest for Steklov problem has grown only recently, and mathematicians are discovering, with the knowledge of what was done for the Laplace equation, how these two problems

intersect with one another. In the case of domains, Steklov eigenvalues can be seen as the solutions of the weighted Laplacian that is discussed in part 3, where the density is concentrated only on the boundary. In my work, I focused on eigenvalues of a manifold M embedded in the hyperbolic space. The first bound we obtain is in a very general context; the only assumption is that M is embedded in the hyperbolic space of dimension $m \geqslant n$. In the last two results, we impose $m = n + 1$ and the fact that ∂M is the boundary of some connected domain of $\mathbb{R}^n$ or $\mathbb{H}^n$.

Keywords: Riemannian geometry, Laplacian, isoperimetric ratio, hypersurface, density, hyperbolic space

Contents

Chapter 1

Introduction

This introduction will define the tools necessary to understand the problems that will arise in this thesis, and how they are linked together. We will see the motivation behind these spectral problems, some important attempts, techniques and results that have led my research. What is presented in this introduction is inspired by the books of Chavel [Cha84] and Bandle [Ban80], and the reader may refer to them for more details.

1.1 Notation

1.1.1 The Laplace and Steklov problems

In the following, (M, g) will denote a smooth compact connected Riemannian manifold. When M has a boundary ∂M, we will usually impose that it is of class $\mathscr{C}^\infty$. We call $\mathscr{C}^\infty(M)$ the set of functions which are smooth on M, and $\mathscr{C}_c^\infty(M)$ the set of functions which are smooth on M, and having compact support in $M \setminus \partial M$. For $u \in \mathscr{C}^\infty(M)$, we denote by ∇u the *gradient* of u, and the *divergence* of any vector field X will be denoted $\mathrm{div}(X)$. Let us also call $\mathrm{d}V_g$ the volume element associated to the manifold (M, g). We define the norm $\|\cdot\|_1$ as

$$\|u\|_1^2 = \int_M |\nabla u|^2 \mathrm{d}V_g + \int_M u^2 \mathrm{d}V_g.$$

Then, we define respectively the sets $H^1(M)$ and $H_0^1(M)$ as the completion for the norm $\|\cdot\|_1$ of the sets $\mathscr{C}^\infty(M)$ and $\mathscr{C}_c^\infty(M)$. As a consequence $H^1(M)$ and $H_0^1(M)$ are Hilbert spaces. We write $L^2(M)$ for the space of measurable functions for the measure induced by the metric g, that is,

$$L^2(M) = \left\{ u : M \to \mathbb{R} \text{ s.t. } \int_M u^2 \mathrm{d}V_g < \infty \right\}.$$

The space $L^2(M)$ is a Hilbert space when equipped with the inner product

$$(u \cdot v) = \int_M uv \mathrm{d}V_g.$$

Finally, we define the *Laplacian* or *Laplace operator* as

$$\Delta = -\mathrm{div} \circ \nabla.$$

In local coordinates and applied to a function $u \in \mathscr{C}^\infty(M)$, the Laplacian reads:

$$\Delta u = -\frac{1}{\sqrt{\det(g)}} \sum_{i,j=1}^{n} \partial_i \left(g_{ij}^{-1} \sqrt{\det(g)} \partial_j u \right),$$

where g_{ij}^{-1} denotes the coordinate (i, j) of the inverse metric tensor. In this thesis, we consider four equations. Let $u \in \mathscr{C}^\infty(M)$. We separate the cases when M has a boundary or not.
If (M, g) is a manifold without boundary,

- We consider the *closed problem*:

$$\Delta u = \lambda u \text{ in } M. \tag{1.1.1}$$

If (M, g) is a manifold with boundary,

- We define the *Dirichlet problem* as:

$$\begin{cases} \Delta u &= \lambda u \quad \text{in } M \\ u &= 0 \quad \text{on } \partial M. \end{cases} \tag{1.1.2}$$

- We define the *Neumann problem* as:

$$\begin{cases} \Delta u &= \lambda u \quad \text{in } M \\ \partial_\eta u &= 0 \quad \text{on } \partial M. \end{cases} \tag{1.1.3}$$

- When $\dim(M) \geqslant 2$, we define the *Steklov problem* as:

$$\begin{cases} \Delta u &= 0 \quad \text{in } M \\ \partial_\eta u &= \sigma u \quad \text{on } \partial M, \end{cases} \tag{1.1.4}$$

  where $\partial_\eta u$ is the exterior normal derivative of u.

1.1.2 The eigenvalues, a variational characterisation

The following theorem is a classic in spectral geometry (see [Bé86, Chapter III] for the first part and [GP17] for the second part).

Theorem 1.1.1 (Spectral theorem)**.**
Let M be a smooth compact Riemannian manifold, possibly with a smooth boundary. For the closed, Dirichlet, and Neumann problems considered above, the set of eigenvalues constitute a non-negative and discrete spectrum, and the corresponding eigenfunctions can be chosen such that they realise an orthogonal basis of $L^2(M)$.

For the Steklov problem, the set of eigenvalues is also a non-negative and discrete sequence, and the restriction of the eigenfunctions to the boundary realises an orthogonal basis of $L^2(\partial M)$.

Remark 1.1.2.

The number 0 is always an eigenvalue of the closed, the Neumann, and the Steklov problems. Its multiplicity for the closed, the Neumann and the Steklov problems is equal to the number of connected components of M. On the other hand, 0 is never an eigenvalue of the Dirichlet problem. Usually, we will consider a connected manifold. In that case we will use the following notation: first, the notation "Sp" denotes the spectrum of the problem considered, and

- For the closed problem:

$$\mathrm{Sp}(M) = \{0 = \lambda_0(M, \mathrm{Cl}) < \lambda_1(M, \mathrm{Cl}) \leqslant \lambda_2(M, \mathrm{Cl}) \leqslant \cdots \leqslant \lambda_k(M, \mathrm{Cl}) \leqslant \cdots \nearrow +\infty\}.$$

- For the Dirichlet problem:

$$\mathrm{Sp}(M) = \{0 < \lambda_1(M, \mathrm{Di}) \leqslant \lambda_2(M, \mathrm{Di}) \leqslant \cdots \leqslant \lambda_k(M, \mathrm{Di}) \leqslant \cdots \nearrow +\infty\}.$$

- For the Neumann problem:

$$\mathrm{Sp}(M) = \{0 = \lambda_0(M, \mathrm{Ne}) < \lambda_1(M, \mathrm{Ne}) \leqslant \lambda_2(M, \mathrm{Ne}) \leqslant \cdots \leqslant \lambda_k(M, \mathrm{Ne}) \leqslant \cdots \nearrow +\infty\}.$$

- For the Steklov problem:

$$\mathrm{Sp}(M) = \{0 = \sigma_0(M) < \sigma_1(M) \leqslant \sigma_2(M) \leqslant \cdots \leqslant \sigma_k(M) \leqslant \cdots \nearrow +\infty\}.$$

When the context is clear, and in order to avoid heavy notation, we may intentionally omit to write Cl, Di, or Ne for an eigenvalue.

The following theorem is also fundamental:

Theorem 1.1.3 (Green formula)**.**

Let u, v be two real valued functions of class $\mathscr{C}^2$ defined on a smooth compact Riemannian manifold (M, g) with boundary. We call Σ the boundary of M, and $\mathrm{d}\Sigma$ its induced volume element. Then

$$\int_M v\Delta u \,\mathrm{d}V_g = \int_M < \nabla u, \nabla v > \mathrm{d}V_g - \int_\Sigma v\partial_\eta u \,\mathrm{d}\Sigma.$$

The Green formula is especially useful because if we consider any of the four problems above, a Rayleigh quotient will appear, and the eigenvalue can be characterised by a min-max property. This is what the following proposition emphasises:

Proposition 1.1.4 (Min-max principle)**.**

In what follows, the notation E_k designates a linear subspace of dimension k.
If M is a smooth compact manifold with no boundary,

- *The eigenvalues for the closed problem can be written:*

$$\lambda_k(M, \mathrm{Cl}) = \min_{E_{k+1} \subset H^1(M)} \max_{\substack{u \in E_{k+1} \\ u \neq 0}} \frac{\int_M |\nabla u|^2 \mathrm{d}V_g}{\int_M u^2 \mathrm{d}V_g}. \tag{1.1.5}$$

If M is a smooth compact manifold with boundary Σ,

- *The eigenvalues for the Neumann problem can be written:*

$$\lambda_k(M, \mathrm{Ne}) = \min_{E_{k+1}\subset H^1(M)} \max_{\substack{u\in E_{k+1}\\ u\neq 0}} \frac{\int_M |\nabla u|^2 \mathrm{d}V_g}{\int_M u^2 \mathrm{d}V_g}. \tag{1.1.6}$$

- *The eigenvalues for the Dirichlet problem can be written:*

$$\lambda_k(M, \mathrm{Di}) = \min_{E_k\subset H^1_0(M)} \max_{\substack{u\in E_k\\ u\neq 0}} \frac{\int_M |\nabla u|^2 \mathrm{d}V_g}{\int_M u^2 \mathrm{d}V_g}. \tag{1.1.7}$$

- *The eigenvalues for the Steklov problem can be written:*

$$\sigma_k(M) = \min_{E_{k+1}\subset H^1(M)} \max_{\substack{u\in E_{k+1}\\ u\neq 0}} \frac{\int_M |\nabla u|^2 \mathrm{d}V_g}{\int_\Sigma u^2 \mathrm{d}\Sigma}. \tag{1.1.8}$$

In particular, the first non-zero eigenvalues are characterised by:

$$\begin{aligned}
\lambda_1(M) &= \min_{\substack{u\in H^1(M)\\ u\neq 0}} \left\{ \frac{\int_M |\nabla u|^2 \mathrm{d}V_g}{\int_M u^2 \mathrm{d}V_g}, \int_M u \mathrm{d}V_g = 0 \right\} \text{ for the closed and Neumann problems,}\\
\lambda_1(M, \mathrm{Di}) &= \min_{\substack{u\in H^1_0(M)\\ u\neq 0}} \frac{\int_M |\nabla u|^2 \mathrm{d}V_g}{\int_M u^2 \mathrm{d}V_g} \text{ for the Dirichlet problem,}\\
\sigma_1(M) &= \min_{\substack{u\in H^1(M)\\ u\neq 0}} \left\{ \frac{\int_M |\nabla u|^2 \mathrm{d}V_g}{\int_\Sigma u^2 \mathrm{d}\Sigma}, \int_\Sigma u \mathrm{d}\Sigma = 0 \right\} \text{ for the Steklov problem.}
\end{aligned}$$

Remark 1.1.5.

Have in mind that $H^1_0(M) = H^1(M) \cap \{\mathrm{Tr}(u) = 0\}$ (see [Dav95, chapter 6]), where $\mathrm{Tr} : H^1(M) \to L^2(\Sigma)$ is the trace operator. Also, the fact that the appropriate Hilbert space for the problems (1.1.1), (1.1.3), and (1.1.4) is $H^1(M)$, and that the Hilbert space associated to (1.1.2) is $H^1_0(M)$, is not obvious, and a good explanation can be found in [Dav95, chapters 6&7]. It is also worth noticing that the Dirichlet condition is "imposed". Thus some people refer to it as an *essential* boundary condition, while the Neumann condition appears as a *natural* boundary condition, or *free* boundary condition.

Thanks to the min-max characterisation, the following two propositions follow:

Proposition 1.1.6 (Monotonicity of Dirichlet eigenvalues)**.**

Let Ω_1 and Ω_2 be two compact connected domains of a Riemannian manifold such that $\Omega_1 \subset \Omega_2$. Then for all $k \in \mathbb{N}$,

$$\lambda_k(\Omega_2, \mathrm{Di}) \leqslant \lambda_k(\Omega_1, \mathrm{Di}).$$

Proof. We use the min-max characterisation. To every function $u_1 \in H_0^1(\Omega_1)$ we associate a function $u_2 \in H_0^1(\Omega_2)$ as:

$$u_2(x) = \begin{cases} u_1(x) & \text{if } x \in \Omega_1 \\ 0 & \text{if } x \in \Omega_2 \setminus \Omega_1. \end{cases}$$

We can also see $H_0^1(\Omega_1)$ as a more restrictive space. Because both Rayleigh quotients verify $R(u_1) = R(u_2)$, we have the inequality. □

Proposition 1.1.7 (Comparison between the Neumann and the Dirichlet spectra)**.**
Let Ω be a bounded domain of a Riemannian manifold. One has, for every $k \in \mathbb{N}$,

$$\lambda_k(\Omega, \mathrm{Ne}) \leqslant \lambda_{k+1}(\Omega, \mathrm{Di})$$

Proof. The idea is the same as in the proof of the monotonicity of the Dirichlet spectrum: we compare the respective Hilbert spaces associated to the Neumann and Dirichlet problems. As $H_0^1(\Omega) \subset H^1(\Omega)$, the result follows. □

Remark 1.1.8.
Even though we only know how to calculate the spectrum in a few cases, we have some useful methods to bound λ_k from above. Using the min-max formula for the Dirichlet problem (1.1.7), we get that for every subspace $E_k \subset H_0^1(M)$ of dimension k,

$$\lambda_k(M, \mathrm{Di}) \leqslant \max_{\substack{u \in E_k \\ u \neq 0}} R(u).$$

In particular, let $V_1, \ldots, V_k$ be k disjoint subsets of M, and take E_k generated by $u_1, \ldots, u_k$, some disjointly supported functions of $H_0^1(M)$ defined on the V_i's. Now, let u be a function in E_k, that is, we write $u = \sum_{i=1}^{i=k} \alpha_i u_i \, (\alpha_i \in \mathbb{R})$. We also take m defined as $m := \max\limits_{i=1\ldots k} R(u_i)$. In particular, $\int_M |\nabla u_i|^2 \, \mathrm{d}V_g \leqslant m \int_M u_i^2 \mathrm{d}V_g$. We have, for all $u \in E_k$,

$$R(u) = \frac{\int_M |\nabla u|^2 \mathrm{d}V_g}{\int_M u^2 \mathrm{d}V_g} = \frac{\int_M \left|\sum_i \alpha_i \nabla u_i\right|^2 \mathrm{d}V_g}{\int_M \left(\sum_i \alpha_i u_i\right)^2 \mathrm{d}V_g} = \frac{\int_M \sum_i \alpha_i^2 |\nabla u_i|^2 \, \mathrm{d}V_g}{\int_M \sum_i \alpha_i^2 u_i^2 \mathrm{d}V_g} = \frac{\sum_i \alpha_i^2 \int_M |\nabla u_i|^2 \, \mathrm{d}V_g}{\sum_i \alpha_i^2 \int_M u_i^2 \mathrm{d}V_g} \leqslant m.$$

Thus, it is also true for the maximum over all $u \in E_k$, and we get:

$$\lambda_k \leqslant \max_{i=1\ldots k} R(u_i). \tag{1.1.9}$$

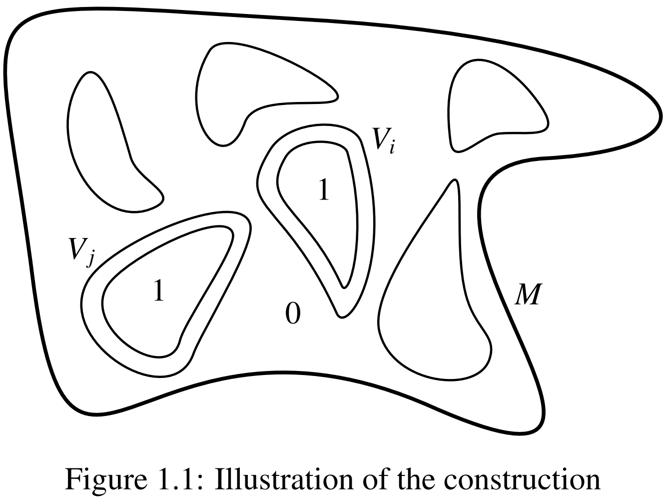

Figure 1.1: Illustration of the construction

The inequality (1.1.9) gives a powerful method to bound any λ_k from above, and it will be used throughout the next chapters. Usually the idea is, for any fixed k, to construct k test functions such that their Rayleigh quotient is easy to bound from above. A classical way of doing this is by defining test functions to be 1 on some disjoint subsets V_i (see Figure 1.1), and let them decrease linearly to 0 on the neighbourhoods of V_i, in such a way that the norm of their gradient is constant.

To that effect we will often consider the distance function, which has a gradient of constant norm 1. More precisely, let K be a closed subset of a smooth and complete Riemannian manifold M without boundary. For $x \in M \setminus K$, define $d^K : x \mapsto \inf_{y \in K} \operatorname{dist}_M(x, y)$. Then d^K is differentiable almost everywhere and

$$\left|\nabla d^K\right| = 1.$$

This result can be found in [MM03, Theorem 3.1], see also [EG92, Section 3.1]. Of course, it is not always easy to find enough "space" that we can use as a support set, as it can also be difficult to bound the Rayleigh quotient of the functions themselves. But some theorems will help us in the next section. Notice that the construction above can obviously be applied for the Neumann, the closed, and the Steklov problems, the only difference is we have to take $k + 1$ disjoint subsets instead of k.

1.2 Discussion around the Bishop-Gromov Theorem

This section is dedicated to explaining in more details what will be used in Chapters 3 and 4. As we have seen at the end of Remark 1.1.8, and given a manifold M, a difficulty is to exhibit disjoint and "heavy" domains of M. To state the lemmas we will use, let us introduce some notation.

For $H \in \mathbb{R}$, let $B^H(r)$ denote a generic ball of radius r in the complete Riemannian and simply connected manifold of constant curvature H. For a Riemannian manifold (M, g), we denote by $B_M(x, r)$ the ball centred in x and of radius r in M.

Theorem 1.2.1 (Bishop-Gromov comparison Theorem)**.**
Let (M, g) be a complete Riemannian manifold of dimension n, $Ric(g) \geqslant (n-1)H$. Then for any $x \in M$, for $0 < r < R$,

$$|B_M(x, r)| \leqslant \left|B^H(r)\right|, \tag{1.2.10}$$

$$\frac{|B_M(x, R)|}{|B_M(x, r)|} \leqslant \frac{\left|B^H(R)\right|}{|B^H(r)|}. \tag{1.2.11}$$

Remark 1.2.2. The volume of any ball of radius r in the hyperbolic space $\mathbb{H}^n$ of constant curvature $H = -1$ can be expressed (see [Cha06, Chapter VI, Section 5]) as follows:

$$\left|B^H(r)\right| = \left|\mathbb{S}^{n-1}\right| \int_0^r \sinh(t)^{n-1} \mathrm{d}t.$$

In particular for $r \leqslant 1$, this implies the existence of $\tilde{\omega}_n$ such that $\left|B^H(r)\right| \leqslant \tilde{\omega}_n r^n$. Thus, using the Bishop-Gromov Theorem, for any complete Riemannian manifold (M, g) of dimension n, $\mathrm{Ric}(g) \geqslant -(n-1)g$, we have that, for $r \leqslant 1$,

$$|B_M(x, r)| \leqslant \tilde{\omega}_n r^n.$$

In other words, as long as we can control the Ricci curvature, that is, bound it from below, we can also control the weight of small balls.

But the main consequence of the Bishop-Gromov comparison Theorem that will be interesting for us is the following result, known as the Packing Lemma (see [Zhu97, Lemma 3.6]). It gives an upper bound on the number of balls of radius ε necessary to cover a ball of radius r, and a bound on the *multiplicity* of the covering, that is the quantity

$$\text{multiplicity} = \max_{y \in B(x,r)} \operatorname{card}\{x_i \text{ s.t. } y \in B(x_i, \varepsilon)\}.$$

Theorem 1.2.3 (Packing Lemma)**.**

Let (M, g) be a complete Riemannian manifold of dimension n such that Ric$(g) \geqslant (n-1)Hg$. Given $\varepsilon > 0$, $r > \frac{\varepsilon}{2}$, and $x \in M$, there is a covering of $B_M(x, r)$ by balls $B_M(x_i, \varepsilon)$, where $x_i \in B_M(x, r)$, such that the number of balls satisfies $N \leqslant \frac{|B^H(2r)|}{|B^H(\varepsilon/4)|}$, and the multiplicity of the covering is bounded by $\frac{|B^H(5\varepsilon)|}{|B^H(\varepsilon/4)|}$.

Proof. The proof can also be found in the article [Zhu97, Lemma 3.6] cited above. First, take a maximal set of points $x_i \in B_M\left(x, r - \frac{\varepsilon}{2}\right)$, such that $\operatorname{dist}(x_i, x_j) \geqslant \frac{\varepsilon}{2}$ for $i \neq j$. The construction implies that the balls $B_M\left(x_i, \frac{\varepsilon}{4}\right)$ are disjoint, but also that the union of the balls $B_M(x_i, \varepsilon)$ covers the ball $B_M(x, r)$. We thus have:

$$\begin{aligned}
N &\leqslant \frac{|B_M(x, r)|}{\min_i \left|B_M\left(x_i, \frac{\varepsilon}{4}\right)\right|} \\
&= \frac{|B_M(x, r)|}{\left|B_M\left(x_0, \frac{\varepsilon}{4}\right)\right|} \text{ for some } x_0 \\
&\leqslant \frac{|B_M(x_0, 2r)|}{\left|B_M\left(x_0, \frac{\varepsilon}{4}\right)\right|} \text{ since } B_M(x, r) \subset B_M(x_0, 2r) \\
&\leqslant \frac{\left|B^H(2r)\right|}{\left|B^H\left(\frac{\varepsilon}{4}\right)\right|}.
\end{aligned}$$

For the bound on the multiplicity, let us take a $y \in B_M(x, r)$ realising the maximum. Now take a ball of the covering containing y. Without loss of generality, we can suppose this ball is $B_M(x_0, \varepsilon)$. But for all x_i such that $B_M(x_0, \varepsilon) \cap B_M(x_i, \varepsilon) \neq \varnothing$, we have $\operatorname{dist}(x_0, x_i) \leqslant 2\varepsilon$. This implies the ball $B_M(x_0, 3\varepsilon)$ contains these (disjoint) balls $B_M\left(x_i, \frac{\varepsilon}{4}\right)$. It follows:

$$\begin{aligned}
\text{multiplicity} &\leqslant \frac{|B_M(x_0, 3\varepsilon)|}{\min\limits_i \left|B_M\left(x_i, \frac{\varepsilon}{4}\right)\right|} \\
&= \frac{|B_M(x_0, 3\varepsilon)|}{\left|B_M\left(x_1, \frac{\varepsilon}{4}\right)\right|} \text{ for some } x_1 \\
&\leqslant \frac{|B_M(x_1, 5\varepsilon)|}{\left|B_M\left(x_1, \frac{\varepsilon}{4}\right)\right|} \text{ since } B_M(x_0, 3\varepsilon) \subset B_M(x_1, 5\varepsilon) \\
&\leqslant \frac{\left|B^H(5\varepsilon)\right|}{\left|B^H\left(\frac{\varepsilon}{4}\right)\right|}.
\end{aligned}$$

□

Remark 1.2.4. We now imitate the idea of [CM08, Section 2], to introduce the function

$$C(n, H, r) = \max_{t \leqslant r} \left(1 + \left\lfloor \frac{\left|B^H(2t)\right|}{\left|B^H(\frac{t}{8})\right|} \right\rfloor \right).$$

This "packing constant" is an upper bound on the number of balls necessary to cover any ball of radius r by balls of radius $\frac{r}{2}$, and only depends on the dimension of M, on a bound on its Ricci curvature, and on the radius r. Moreover, the function $r \mapsto C(n, H, r)$ is **increasing**. The following basic examples will be used in chapter 4.

If $M = \mathbb{R}^n$, then $H = 0$, and a calculation gives us that C does not depend on r. In fact, it is sufficient to take $C(n) = 16^n$.

If $M = \mathbb{H}^n$, the hyperbolic space of constant curvature -1, then C depends on r. But by taking $r \leqslant 1$, and thanks to the increasing character of the function $r \mapsto C(r)$, then it can also be bounded by some constant $C(n)$.

Now, everything is set up for two important lemmas. The critical idea is that the following results are stated in an abstract metric measured space, that is, a triplet (X, d, μ), where X is a set of points, $d : X \times X \to \mathbb{R}_+$ is a distance, and μ is a measure defined on the Borel algebra of X. A first trivial example is to consider $\mathbb{R}^n$ (*resp.* $\mathbb{H}^n$), the distance and the measure being defined with respect to the canonical metric of $\mathbb{R}^n$ (*resp.* $\mathbb{H}^n$). We give richer examples once the lemmas are stated.

The next result can be found in [CEG13, Lemma 2.1] or [CGG18, Lemma 4.1], and is a slight reformulation of a result in [CM08, Corollary 2.3]. It will be used repeatedly in the proofs of Chapter 4. We recall that a measure μ is ***non-atomic*** if for any measurable set $\mathscr{U}_1$ such that $\mu(\mathscr{U}_1) > 0$, there exists a measurable set $\mathscr{U}_2 \subset \mathscr{U}_1$ such that $0 < \mu(\mathscr{U}_2) < \mu(\mathscr{U}_1)$.

Lemma 1.2.5. *Let (X, d, μ) be a metric measured space, complete and locally compact, where μ is a non-atomic measure. Suppose that for all $r > 0$, there exists $C(r) \in \mathbb{N}$ such that each ball of radius r can be covered by $C(r)$ balls of radius $\frac{r}{2}$. Let $K > 0$. If there exists a radius $r_0 > 0$ such that for all $x \in X$,*

$$\mu(B(x, r_0)) \leqslant \frac{\mu(X)}{4C(r_0)^2 K},$$

then there exist K μ-measurable sets $A_1, \dots, A_K$ of X such that for all $i \leqslant K$,

$$\mu(A_i) \geqslant \frac{\mu(X)}{2C(r_0)K} \quad \text{and for } i \neq j,\ d(A_i, A_j) \geqslant 3r_0.$$

The following lemma was first introduced by Asma Hassannezhad in [Has11, Theorem 2.1] and will be used in chapter 3. It is based on two important technical results of Grigor'yan, Netrusov, Yau (see [GNY04, section 3]) and Colbois, Maerten [CM08, Corollary 2.3]. We say that a metric measured space (X, d, μ) satisfies the $(2, N; 1)$-covering property if every ball of radius $r \leqslant 1$ can be covered by N balls of radius $\frac{r}{2}$.

Lemma 1.2.6. *Let (X, d, μ) be a complete and locally compact metric measured space, where μ is a non-atomic measure. Suppose X satisfies the $(2, N; 1)$-covering property. Then for every $k \in \mathbb{N}^*$, there exists a family of $4(k+1)$ measurable sets F_j, G_j with the following properties:*

1. *$F_j \subset G_j$.*
2. *The G_j's are mutually disjoint.*
3. *$\mu(F_j) \geqslant \frac{\mu(X)}{c^2(k+1)}$ with $c = c(N)$ a constant which depends only on N.*
4. *The (F_j, G_j)'s are of the following two types:*
   - *For all j, F_j is an annulus $A = \{r \leqslant d(x,a) < R\}$, and $G_j = 2A = \left\{\frac{r}{2} \leqslant d(x,a) < 2R\right\}$ with $0 \leqslant r \leqslant R$ and $2R < 1$, or:*
   - *For all j, F_j is an open set $F \subset M$ and $G_j = F^{r_0} = \{x \in M; d(x, F) \leqslant r_0\}$ with $r_0 = \frac{1}{1600}$.*

Remark 1.2.7. The $(2, N; 1)$-covering property is only a **metric** notion, that is, the fact for a metric measured space (X, d, μ) to verify the $(2, N; 1)$-covering property does not depend on μ. Let us consider a complete Riemannian manifold $\left(\tilde{M}, \tilde{g}_0\right)$ that contains a bounded domain (M, g_0) with a boundary Σ of class $\mathscr{C}^1$, and g_0 is the restriction of $\tilde{g}_0$ on M. We also assume $\operatorname{Ric}(\tilde{g}_0) \geqslant -(n-1)\tilde{g}_0$, and we consider the metric measured space $\left(\tilde{M}, \tilde{d}_0, \tilde{\mu}_0\right)$ where $\tilde{d}_0$ and $\tilde{\mu}_0$ are defined with respect to $\tilde{g}_0$. Thanks to the Packing Lemma 1.2.3, the metric measured space $\left(\tilde{M}, \tilde{d}_0, \tilde{\mu}_0\right)$ satisfies the $(2, N; 1)$-covering property where N only depends on n. Thus, for any non-atomic measure μ, we can apply Lemma 1.2.5 to the space $\left(\tilde{M}, \tilde{d}_0, \mu\right)$. As we are free to choose any measure μ, this opens multiple interesting options, such as:

- For any Borel set $\mathscr{U}$ of $\tilde{M}$,
  $$\mu(\mathscr{U}) := \int_{\mathscr{U} \cap \Sigma} \mathrm{d}\Sigma,$$
  We will use this in Chapter 4, Sections 2 & 3, by considering $\tilde{M}$ as the Euclidean space, or the hyperbolic space.

- Define a smooth function $\rho : M \to \mathbb{R}_+^*$, such that $\int_M \rho \mathrm{d}V_g = |M|$. We can define a measure $\mu = \rho \mathrm{d}V_g$, or in other words, for any Borel set $\mathscr{U}$ of $\tilde{M}$,
  $$\mu(\mathscr{U}) := \int_{\mathscr{U} \cap M} \rho \mathrm{d}V_g.$$
  This consideration will be used in Chapter 3, Section 1.

1.3 The Weyl law

1.3.1 The Weyl law for the Laplacian

The spectrum of the Laplacian is not known for a general manifold. But in 1912, Weyl [Wey12] proved the following fundamental result, now known as the *Weyl law*, which gives the asymptotic behaviour of the eigenvalues, whether the problem considered is the closed one, or with Dirichlet or Neumann condition if M has a boundary. This result, however, does not give any information on the beginning of the spectrum. On the other hand, one can ask if knowing something on the eigenvalues helps to understand the geometry of the manifold. In 1966, M. Kac [Kac66] popularised the question "*Can one hear the shape of a drum?*", that is, if the spectrum of a manifold M is entirely known, can one recover the "shape" of M from it? In other words, can one find two isospectral manifolds that are not isometric? Actually, an answer to this question was already known. Indeed, in 1964, J. Milnor [Mil64] had already discovered two non-isometric isospectral flat tori of dimension 16. In the following years, several results were found about isospectrality. We shall give some counter-examples, but as the following list is not exhaustive, the reader may refer to [Bro88], [BT90], [Bus88]. In 1980, M.F. Vignéras [Vig80] exhibited compact hyperbolic non isometric surfaces, that have the same spectrum. In 1986, P. Buser [Bus86] also found two non-isometric compact Riemann surfaces that answer negatively Kac's question. It was in 1992 that a counter-example was constructed for planar domains with Dirichlet boundary conditions (see [GWW92]). The authors exhibited two piecewise smooth isospectral domains of $\mathbb{R}^2$, but not isometric. Even so, it is still an open question in the case of smooth domains of $\mathbb{R}^2$.

But if one cannot hear the shape of a drum, then what can we hear? Thanks to the Weyl law, this typical *inverse problem* is already partially answered.

Theorem 1.3.1 (The Weyl law)**.**
Let (M, g) be a compact Riemannian manifold of dimension n as in Theorem 1.1.1. Define the counting function $N(\lambda) = \mathrm{card}\{j \text{ s.t. } \lambda_j \leqslant \lambda\}$, and ω_n as the volume of the n-Euclidean ball of radius 1*. Then*

$$N(\lambda) = \frac{\omega_n}{(2\pi)^n}\,|M|\,\lambda^{\frac{n}{2}} + o\left(\lambda^{\frac{n}{2}}\right).$$

Corollary 1.3.2. *Let us define the constant $W(n) = \frac{(2\pi)^2}{\omega_n^{\frac{2}{n}}}$. Then the following asymptotic behaviour of the eigenvalues holds, and will be the one we use in this thesis.*

$$\lambda_k(M)\,|M|^{\frac{2}{n}} \underset{k\to\infty}{\sim} W(n)k^{\frac{2}{n}}.$$

As a consequence, the Weyl law enables us to recover the volume and the dimension of the manifold. If the reader is interested in the proof on Euclidean domains for Dirichlet boundary conditions, see [Cha84, pp. 31–33], and for the proof in the Neumann case, see [CH89, pp. 436–443]. For a general proof on any Riemanniann manifold, the reader can refer to [Bé86, p. 149]. A more precise approximation of the remainder $R(\lambda) = N(\lambda) - \frac{\omega_n}{(2\pi)^n}\,|M|\,\lambda^{\frac{n}{2}}$ is delicate, and usually leads to look at the geometry of the manifold. For example, in the case of manifolds with non-positive sectional curvature, the remainder can be expressed as $O\left(\frac{\lambda^{\frac{n-1}{2}}}{\ln(\lambda)}\right)$ (see [Bé77]).

1.3.2 The Weyl law for the Steklov problem

This spectrum also has an analogue for the Weyl law (see [GP17]). Let M be a smooth compact Riemannian manifold of dimension n with a smooth boundary. Then

$$\sigma_k(M)\,|\partial M|^{\frac{1}{n-1}} \underset{k\to\infty}{\sim} \sqrt{W(n-1)}k^{\frac{1}{n-1}}. \tag{1.3.12}$$

Notice that the formula (1.3.12) is slightly different from the classical law for the spectrum of the Laplacian. First the normalisation does not depend on the volume of M, but on the volume of its boundary. Moreover, the power of k is $\frac{1}{n-1}$, which can be interpreted as an asymptotically slower growth than the growth of the Laplace spectrum.

1.4 Uniform bounds

As we have seen in the previous section, the Weyl laws show how the spectrum behaves asymptotically, in the closed, Dirichlet and Neumann problems (resp. Steklov problem). Now, take a Riemannian manifold (M, g) of dimension n. It is well-known that if we fix M, and we allow any deformation of the metric g, it is easy to create arbitrarily small or large eigenvalues, simply by applying a homothety to the metric:

$$\lambda_k(M, cg) = \frac{1}{c}\lambda_k(M, g) \quad \text{and} \quad \sigma_k(M, cg) = \frac{1}{\sqrt{c}}\sigma_k(M, g), \tag{1.4.13}$$

where c is a positive constant. In this context, we will often consider the homothetic invariant *normalised eigenvalues* $\lambda_k(M)\,|M|^{\frac{2}{n}}$ (resp. $\sigma_k(M)\,|\partial M|^{\frac{1}{n-1}}$). The above also holds in the case of Euclidean domains.

So a natural question is: for a manifold M, can we uniformly bound the normalised eigenvalues with respect to the geometry of M, preferably with the optimal power in k? In other words, can we find a bound of the type $C(n)k^{\frac{2}{n}}$ for the Laplace problems (resp. $C(n)k^{\frac{1}{n-1}}$ for the Steklov problem) where $C(n)$ depends only on n, and not on M? We investigate this question in the following pages.

1.4.1 Uniform bounds of Laplace eigenvalues

Let us focus on the Laplace problems. We have seen that applying a homothety on M produces arbitrarily small or large eigenvalues, so we consider the homothetic-invariant quantity $\lambda_k(M)\,|M|^{\frac{2}{n}}$. For the Dirichlet spectrum on plane-covering domains, Pólya [Pó61] proved that the eigenvalues are bounded as follows:

$$\lambda_k(\Omega, \mathrm{Di})\,|\Omega| \geqslant W(2)k.$$

He also proved a similar result for the Neumann spectrum on regularly plane-covering domains:

$$\lambda_k(\Omega, \mathrm{Ne})\,|\Omega| \leqslant W(2)k,$$

which has been improved a few years later on plane-covering domains in [Kel66]. In fact, the following conjecture is named after him: given any bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary,

$$\lambda_k(\Omega, \mathrm{Ne})\, |\Omega|^{\frac{2}{n}} \leqslant W(n) k^{\frac{2}{n}} \leqslant \lambda_k(\Omega, \mathrm{Di})\, |\Omega|^{\frac{2}{n}} .$$

The figure 1.2 is an illustration of both the Weyl constant (the dash line) and Pólya's conjecture, that is, the red dots are above the dashed line, and the blue ones are below.

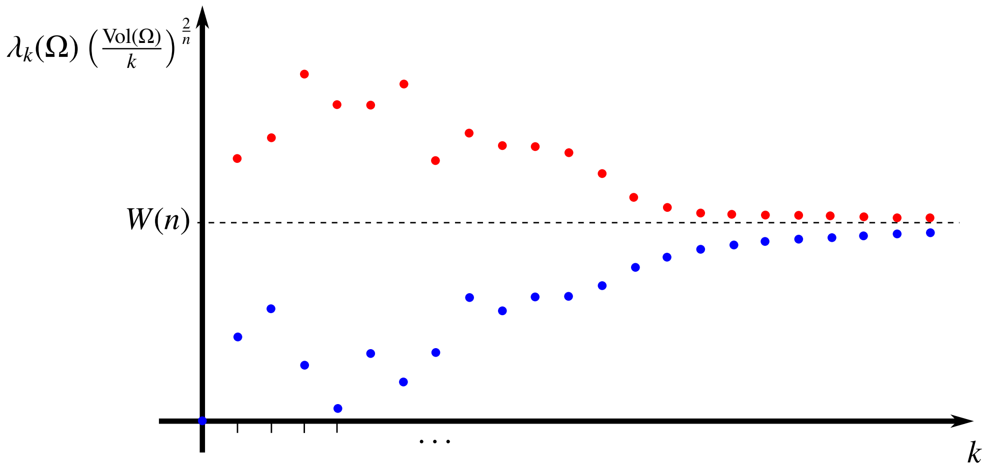

Figure 1.2: In red, conjecture for Dirichlet condition. In blue, for Neumann condition

For the Neumann case, the best result known to date has been answered by Kröger in [Kro92]. He proved the following result:

$$\lambda_k(\Omega, \mathrm{Ne})\, |\Omega|^{\frac{2}{n}} \leqslant \left(\frac{n+2}{n}\right)^{\frac{2}{n}} W(n) k^{\frac{2}{n}}. \tag{1.4.14}$$

The same question could be asked for a general Riemannian manifold M, that is, can one bound the eigenvalues of M in the following way:

$$\lambda_k(M)\, |M|^{\frac{2}{n}} \leqslant C(n) k^{\frac{2}{n}}, \tag{1.4.15}$$

where $C(n)$ is a quantity only depending on n? It was shown in [CD94] that the answer is no. Given a manifold M of dimension $n \geqslant 3$, one can always find a sequence of metrics of volume 1 making the first eigenvalue as large as we want. The idea the authors use is to modify via surgery a small set of M, on which they put a metric having a large λ_1 and a large volume. The union will have its first eigenvalue bounded from below, and a large volume. Using the rescaling (1.4.13), they indeed obtain a manifold of volume 1 having a large first eigenvalue. This result follows the work of Bleecker [Ble83], in which he proved that the sphere $\mathbb{S}^{n\geqslant 3}$ admits a metric with arbitrarily large λ_1 and large volume (see also [Mut80] for an older proof on even-dimensional spheres).

It is important to notice that the result of Colbois and Dodziuk applies in a Riemannian context. This result also holds in the Euclidean space (see [CDE10, Theorem 1.4]), that is, given a compact smooth submanifold M of dimension $n \geqslant 3$ of $\mathbb{R}^{n+1}$, there exists a sequence M_t of smooth submanifolds embedded in $\mathbb{R}^{n+1}$ all diffeomorphic to M such that

$$\lambda_1(M_t)\,|M_t|^{\frac{2}{n}} \underset{t\to\infty}{\longrightarrow} \infty.$$

The proof heavily relies on the fundamental works of Nash and Kuiper on the embedding of manifolds into Euclidean space, and the reader can refer to [CDE10, Section 5] for more details.

One may ask if the same result holds in dimension 2. It was shown in [Kor93] that there exists $C > 0$ such that for any compact orientable surface Σ,

$$\lambda_k(\Sigma)\,|\Sigma| \leqslant C\left(\text{genus}(\Sigma) + 1\right) k, \tag{1.4.16}$$

which means that if the genus of Σ is fixed, the normalised eigenvalues are in fact bounded, and that the result which holds in dimension $n \geqslant 3$ does not hold in dimension 2. However, it is possible to find a sequence of surfaces whose first normalised eigenvalue increases to $+\infty$ (see [BBD88] and [CDE10]). Of course from the last remark, the genus is increasing for such a sequence of surfaces.

In conclusion, given any compact Riemannian manifold M, we know we cannot bound its eigenvalues from above like Kröger did for a Euclidean domain with Neumann boundary conditions. But by imposing geometrical restrictions on M, a lot of authors managed to obtain the following kind of bound for the Laplace eigenvalues:

$$\lambda_k(M)\,|M|^{\frac{2}{n}} \leqslant \text{geom}(M)k^{\frac{2}{n}}. \tag{1.4.17}$$

Here, the term geom(M) is defined in a very general way. What we mean by that is it can designate any relevant geometrical term depending on (M, g), like the Ricci curvature of M or its boundary, the number of connected components of ∂M, the conformal class of g, the isoperimetric ratio, etc. Examples follow in the next pages.

1.4.2 Uniform bounds of Steklov eigenvalues

The last chapter is dedicated to another problem. Over the past few years, this problem has gained a lot of interest and visibility in spectral theory. Take (M, g) a compact Riemannian manifold of dimension n with a $\mathscr{C}^\infty$ boundary ∂M. The equation of interest is the following:

$$\begin{cases} \Delta u &= 0 \quad \text{in } M \\ \partial_n u &= \sigma u \quad \text{on } \partial M. \end{cases}$$

Introduced by Vladimir Steklov in 1902 [Ste02], this equation describes the stationary heat distribution in a domain whose flux passing through the boundary is proportional to the temperature on the boundary. If we add some other boundary conditions, it is useful in hydrodynamics for modeling the waves of a liquid in a closed container, like a glass. In dimension 2, it can also be interpreted as a vibrating membrane having all its mass concentrated on the boundary (see [Ban80, chapter III]). For more detailed interpretations of the problem, the reader can refer to [Pay67].

Similarly to the eigenvalues of the Laplacian, one can ask if we have a Kröger inequality on Euclidean domains, with respect to its associated Weyl law. In [HPS75] it was shown that given a simply connected Euclidean domain $\Omega \subset \mathbb{R}^2$,

$$\sigma_k(\Omega)\,|\partial\Omega| \leqslant 2\pi k. \tag{1.4.18}$$

Notice that this inequality is sharp for $k = 1$; indeed the disk maximises the first non-zero eigenvalue (see the Weinstock inequality in [Wei54]). However, the fact that the ball is a maximiser among the simply connected Euclidean domains, is not true in dimension $n \geqslant 3$ (see [FS17, Theorem 1.1]).

Another important result in the case of domains was found in [CEG11, Theorem 1.2]. The authors proved that there exists a constant $C(n)$ such that for any bounded domain Ω in $\mathbb{R}^n$, $\mathbb{H}^n$, or in an hemisphere of $\mathbb{S}^n$,

$$\sigma_k(\Omega)\,|\partial\Omega|^{\frac{1}{n-1}} \leqslant C(n)k^{\frac{2}{n}}.$$

Notice the power of k, contrary to the inequality (1.4.18), is not optimal with respect to the Weyl law for the Steklov problem when $n \geqslant 3$.

For people trying to find the best asymptotic bounds for the Steklov eigenvalues σ_k, finding bounds whose dependence on k is of the form $k^{\frac{1}{n-1}}$ is perhaps one of the biggest current challenges. Although difficult, a few articles have been published in this context. In 2016, Provenzano and Stubbe [PS17, Theorem 1.7] proved that for a Euclidean domain Ω, $\left|\sigma_k(\Omega) - \sqrt{\lambda_k(\partial\Omega)}\right| \leqslant c_\Omega$, where $\partial\Omega$ is of class $\mathscr{C}^2$, and has only one connected component, and c_Ω depends on several geometric properties of Ω. Notice their result implies that

$$\sqrt{\lambda_k(\partial\Omega)} - c_\Omega \leqslant \sigma_k(\Omega) \leqslant \sqrt{\lambda_k(\partial\Omega)} + c_\Omega.$$

Thanks to the numerous upper bounds obtained on the eigenvalues of the Laplace operator of the form $\lambda_k(\partial\Omega) \leqslant \mathrm{geom}(\partial\Omega)k^{\frac{2}{n-1}}$, their inequality allowed hope in finding more bounds of the following type for a Riemannian manifold M:

$$\sigma_k(M) \leqslant \mathrm{geom}(M)k^{\frac{1}{n-1}}. \tag{1.4.19}$$

The reader may refer to [CGH18] for a more recent result in the same vein.

In the article [CEG11], other links between these spectra were found. The authors showed that if Σ is a closed Riemannian manifold of volume 1, then the Steklov eigenvalues of the cylinder $[-L, L]\times\Sigma$ can be expressed with the help of the Laplace eigenvalues of Σ; the spectrum can be written as:

$$\mathrm{Sp} = \left\{0, \frac{1}{L}, \sqrt{\lambda_k(\Sigma)}\tanh\left(\sqrt{\lambda_k(\Sigma)}L\right), \sqrt{\lambda_k(\Sigma)}\coth\left(\sqrt{\lambda_k(\Sigma)}L\right)\right\}.$$

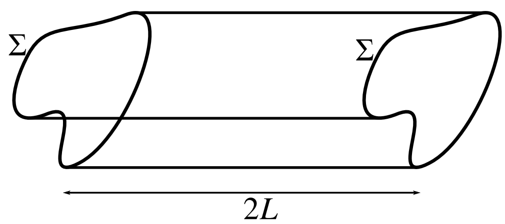

Figure 1.3: The cylinder $[-L, L] \times \Sigma$

This comparison, combined with the results of [CD94], is actually useful to find manifolds with very large σ_1. However, the boundary is not fixed. More recently in [CEG17], the authors take a manifold (M, g) and construct a sequence of metrics g_ε, all conformal to g, and equal to g on the fixed boundary ∂M, such that $\sigma_1(M, g_\varepsilon)$ is as large as desired. These results emphasise the importance of geometrical restrictions; if one wants to find upper bounds, not only are they needed on the boundary, but also on M.

Because it is not possible to bound the eigenvalues of the Steklov problem uniformly relatively to variations of the metric, a geometrical term is often needed. Unsurprisingly, this term brings the same questions as in the Laplace equations.

1.5 Motivation and results for Laplace spectrum

1.5.1 Motivation

The second chapter of this thesis provides an answer to a particular case of the following question. Given a compact manifold of dimension n, can we bound the k-th normalised Laplace eigenvalue of M from above as follows:

$$\lambda_k(M)\,|M|^{\frac{2}{n}} \leqslant \mathrm{geom}(M) + C(n)k^{\frac{2}{n}}? \tag{1.5.20}$$

Before going further, we need to define what we mean by a “Kröger type inequality”, or “inequality of type Kröger”.

Definition 1.5.1.

- We will write $\mathscr{M}(\alpha)$ to designate a family of manifolds with parameter α. In what follows, this set will usually be a family of manifolds verifying geometrical constraints.
- Fix $n \geqslant 1$. An $\mathscr{M}$-inequality of type Kröger is said to be **true** if there exists $C = C(n) > 0$ such that for all α, there exists $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$ and for any $M \in \mathscr{M}(\alpha)$,
$$\lambda_k(M)\,|M|^{\frac{2}{n}} \leqslant Ck^{\frac{2}{n}}.$$
This notion will be important throughout all the chapters of this thesis. Notice that a typical way to get a Kröger type inequality is to obtain an inequality (1.5.20) presented above, and we will see why with some examples presented below.
- On the contrary, an $\mathscr{M}$-inequality of type Kröger is said to be **false** if for all $C > 0$, there exists α such that for any $k_0 \in \mathbb{N}$, there exists $k \geqslant k_0$ and $M \in \mathscr{M}(\alpha)$ such that
$$\lambda_k(M)\,|M|^{\frac{2}{n}} > Ck^{\frac{2}{n}}.$$

To illustrate these definitions, let us study some results that motivated my research. As we have seen on page 13, in 1993, N. Korevaar [Kor93] managed to bound the eigenvalues of a compact orientable surface of $\mathbb{R}^3$ with the help of its genus:

$$\lambda_k(\Sigma)\,|\Sigma| \leqslant C\left(\mathrm{genus}(\Sigma) + 1\right)k. \tag{1.5.21}$$

Eighteen years later, Hassannezhad [Has11] showed the existence of two positive constants C_1 and C_2 such that for all compact surfaces Σ we have:

$$\lambda_k(\Sigma)\,|\Sigma| \leqslant C_1\,\mathrm{genus}(\Sigma) + C_2 k.$$

Let us define $C = C_1 + C_2$, $\mathscr{M}(G) = \left\{\Sigma \text{ compact surface s.t. } \mathrm{genus}(\Sigma) \leqslant G\right\}$, and $k_0 = G$. It is now easy to see that for all $\Sigma \in \mathscr{M}(G)$, and for all $k \geqslant k_0$, we have

$$\lambda_k(\Sigma)\,|\Sigma| \leqslant Ck.$$

Consequently, this "enhanced" result is a Kröger type inequality for the eigenvalues of a compact surface. It is crucial to understand that this last result is really an improvement, because one cannot deduce this inequality from Korevaar's inequality.

In the same paper, A. Hassannezhad found another inequality of this kind, in the context of conformal class of metrics. Given a Riemannian manifold (M, g), we will denote by

$$[g] = \{fg, f \in \mathscr{C}^\infty(M), f > 0\}$$

the conformal class of g. She proved the existence of two constants $A(n), B(n)$ such that for each compact Riemannian manifold (M, g) of dimension n,

$$\lambda_k(M)\,|M|^{\frac{2}{n}} \leqslant A(n)V([g])^{\frac{2}{n}} + B(n)k^{\frac{2}{n}}, \tag{1.5.22}$$

where $V([g])$ is the *min-conformal volume* of (M, g). That is,

$$V([g]) = \inf\left\{|M|_{g_0}, \quad g_0 \in [g], \quad \mathrm{Ricci}(g_0) \geqslant -(n-1)g_0\right\}.$$

This is also an improvement of Korevaar's upper bound on a compact Riemannian manifold (M, g):

$$\lambda_k(M)\,|M|^{\frac{2}{n}} \leqslant c(n, [g])k^{\frac{2}{n}}, \tag{1.5.23}$$

where $c(n, [g])$ only depends on n and the conformal class of g.

The inequality (1.5.22) of A. Hassannezhad gives a Kröger type inequality. Indeed, take $C(n) = A(n) + B(n)$, and define for all W the family

$$\mathscr{M}(W) = \left\{(M, g) \text{ compact Riemannian manifold s.t. } V([g]) \leqslant W\right\},$$

and $k_0 = W$. Then again we obtain, for all $k \geqslant k_0$,

$$\lambda_k(M)\,|M|^{\frac{2}{n}} \leqslant C(n)k^{\frac{2}{n}}.$$

1.5.2 Results

Let Ω be a Euclidean domain of $\mathbb{R}^{n+1}$, and let Σ be the compact hypersurface of dimension n such that $\Sigma = \partial\Omega$. Another natural and interesting quantity related to Σ is its isoperimetric ratio

$$I(\Sigma) = \frac{|\Sigma|}{|\Omega|^{\frac{n}{n+1}}}.$$

Similar to the result of Korevaar on conformal class (1.5.23), the result in [CEG13, Theorem 1.1] is an inequality involving the isoperimetric ratio of Σ.

Theorem 1.5.2. *There exists an explicit constant $\gamma(n)$ such that, for all bounded domains Ω in $\mathbb{R}^{n+1}$ with a $\mathscr{C}^2$-boundary $\Sigma = \partial\Omega$, and for all $k \geqslant 1$,*

$$\lambda_k(\Sigma)\,|\Sigma|^{\frac{2}{n}} \leqslant \gamma(n) I(\Sigma)^{1+\frac{2}{n}} k^{\frac{2}{n}} \tag{1.5.24}$$

with $\gamma(n) = \frac{2^{10n+18+8/n}}{n+1}\omega_{n+1}^{\frac{1}{n+1}}$.

Notice the term $I(\Sigma)$ is multiplied by the asymptotic term $k^{\frac{2}{n}}$. Again, the goal was to find an inequality where the geometric and asymptotic terms were separated, that is, does there exist a constant $A = A(n) > 0$ and a continuous function $f : [0, +\infty) \to \mathbb{R}$ such that for any connected hypersurface Σ and for all k,

$$\lambda_k(\Sigma)\,|\Sigma|^{\frac{2}{n}} \leqslant f(I(\Sigma)) + Ak^{\frac{2}{n}}? \tag{1.5.25}$$

If true, this would mean that an $\mathscr{M}$-inequality of type Kröger is true, where

$$\mathscr{M}(J) = \left\{\Sigma \text{ connected hypersurface of } \mathbb{R}^{n+1} \text{ s.t. } I(\Sigma) \leqslant J\right\}.$$

Indeed, it would be sufficient to take $C = A + 1$ and for all J, to define $k_0 = \left(\max\limits_{t\in[0,J]} f(t)\right)^{\frac{n}{2}}$. Then, inequality (1.5.25) would become, for all Σ in $\mathscr{M}(J)$:

$$\lambda_k(\Sigma)\,|\Sigma|^{\frac{2}{n}} \leqslant \max_{t\in[0,J]} f(t) + Ak^{\frac{2}{n}} = k_0^{\frac{2}{n}} + Ak^{\frac{2}{n}} \leqslant k^{\frac{2}{n}} + Ak^{\frac{2}{n}} = (A+1)k^{\frac{2}{n}}.$$

In this case however, we actually proved that this $\mathscr{M}$-inequality of type Kröger is **false**.

Theorem 1.5.3 (see Theorem 2.1.2)**.**

Let $n \geqslant 2$. For all $A > 0$, there exists $J > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$, there exists $\Sigma \in \mathscr{M}(J)$ verifying:

$$\lambda_k(\Sigma)\,|\Sigma|^{\frac{2}{n}} > 2Ak^{\frac{2}{n}}.$$

Moreover in dimension $n \geqslant 3$, the hypersurface Σ can be chosen to be diffeomorphic to a given hypersurface.

Let us remark that it is not immediate whether or not one can "separate" the geometric term from the asymptotic one. For example, let us call $i(\Sigma)$ the **intersection index**, that is, for a hypersurface, the maximal number of points you can get on a line intersecting Σ. Let us consider the following result [CDE10]:

$$\lambda_k(\Sigma)\,|\Sigma|^{\frac{2}{n}} \leqslant B(n) i(\Sigma)^{1+\frac{2}{n}} k^{\frac{2}{n}}, \tag{1.5.26}$$

where $B(n)$ is an explicit constant depending only on the dimension n. Then it is still an open problem to know whether or not we can bound the eigenvalues in the following way:

$$\lambda_k(\Sigma)\,|\Sigma|^{\frac{2}{n}} \leqslant F(i(\Sigma)) + B_2(n)k^{\frac{2}{n}},$$

where $B_2(n)$ is a constant depending only on n and F is a continuous function.

This problem does not have an obvious solution, and conversely, for the moment it seems that obtaining a result of the same type as in Theorem 1.5.3 is not easy. Indeed the hope would be to find an inequality linking the intersection index and the isoperimetric ratio of a hypersurface in the following way:

$$i(\Sigma) \leqslant CI(\Sigma), \tag{1.5.27}$$

where C is a positive constant. This would in particular imply that a Kröger type inequality in the context of hypersurfaces having their intersection index bounded, is false. However, inequality (1.5.27) is not possible. To see this impossibility, the reader can observe the following figure:

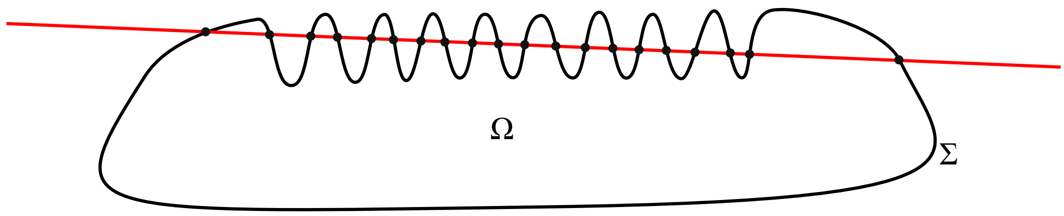

Figure 1.4: A very large intersection index

This illustrates a hypersurface Σ having a bounded isoperimetric ratio but a very large intersection index, thus contradicting inequality (1.5.27).

1.6 Motivation and results for the Laplacian with density

1.6.1 Motivation

In chapter 3, we study a very general equation (see [Hen06, chapters 9&10]):

$$\begin{cases} -\mathrm{div}(\sigma\nabla u) &= \lambda\rho u \quad \text{in } M, \\ \partial_\eta u &= 0 \quad \text{on } \partial M, \end{cases} \tag{1.6.28}$$

where M is a smooth Riemannian manifold and σ and ρ are two positive $\mathscr{C}^1$ functions defined on M. The spectrum of (1.6.28) is discrete, and can be ordered in a positive nondecreasing sequence that tends to infinity. We denote the k-th eigenvalue of this problem as $\lambda_k(\rho,\sigma)$. In [CES15] they study the particular case $\sigma = \rho$. The authors proved that on a compact orientable surface Σ of dimension 2,

$$\lambda_k(\sigma,\sigma) \leqslant C\frac{\|\sigma\|_\infty}{\|\sigma\|_1}\left(\mathrm{genus}(\Sigma)+1\right)k,$$

where C is a universal constant, and $\|\cdot\|_p$ designates the usual p-norm. The authors also proved that on a compact Riemannian manifold (M,g) of dimension $n \geqslant 3$,

$$\lambda_k(\sigma,\sigma) \leqslant C([g])\frac{\|\sigma\|_{\frac{n}{n-2}}}{\|\sigma\|_1}k^{\frac{2}{n}},$$

where $C([g])$ is a constant depending only on the conformal class of g.

Note that the problem (1.6.28) contains more parameters than the classical Neumann equation, so in this thesis, we will simply fix the metric and look at variations of the densities σ and ρ. One can ask the same questions for the Laplace equations in general. That is, can one uniformly bound the eigenvalues relatively to the variation of the densities? If so, can we get a Kröger type bound? As we will explain in chapter 3, these questions have already been studied in some particular cases by B. Colbois, A. El Soufi, A. Savo and A. Hassannezhad. Namely, when $\sigma = 1$ (see [CE18, Corollary 4.1]), or $\sigma = \rho^{\frac{n-2}{n}}$ (see [Has11, Theorem 1.1]), the eigenvalues are indeed bounded for $n \geqslant 2$. But when $\sigma = \rho$, we have a counter-example of manifold where the first non-zero eigenvalue can become very large (see [CES15, Theorem 5.2]).

It is worth noticing that B. Colbois and A. El Soufi managed in [CE18, Theorem 4.2], to obtain an inequality with geometric and asymptotic terms separated. Let (M, g) be a bounded domain with a smooth boundary of a complete Riemannian manifold $(\tilde{M}, \tilde{g})$ of dimension $n \geqslant 2$ such that $g = \tilde{g}_{|M}$ and $\mathrm{Ricci}(\tilde{g}) \geqslant -(n-1)\mathrm{ric}_0\, \tilde{g}$ for any $\mathrm{ric}_0 > 0$. Under the condition that $\int_M \sigma \mathrm{d}V_g = |M|_g$, then:

$$\lambda_k(1, \sigma)|M|_g^{\frac{2}{n}} \leqslant A_n k^{\frac{2}{n}} + B_n \mathrm{ric}_0 |M|_g^{\frac{2}{n}}.$$

1.6.2 Results

We were interested in the particular case $\sigma = \rho^\alpha$. The resulting variational characterisation is:

$$\lambda_k(\rho, \rho^\alpha) = \min_{E_{k+1} \subset H^1(M)} \max_{\substack{u \in E_{k+1} \\ u \neq 0}} \frac{\int_M |\nabla u|^2 \rho^\alpha \mathrm{d}V_g}{\int_M u^2 \rho \mathrm{d}V_g}. \tag{1.6.29}$$

The two following results were found in collaboration with S. Kouzayha. The first one is an inequality where the bound does not depend on the density ρ, thus bringing a definitive and positive answer to the boundedness of the eigenvalues when $\alpha \in \left(0, \frac{n-2}{n}\right)$. The second theorem is a counter-example to this boundedness when M is a manifold of revolution.

Theorem 1.6.1 (see Theorem 3.1.1)**.**
Let $n \geqslant 2$. There exist A_n, B_n two positive constants such that, for all bounded domains M with boundary of class $\mathscr{C}^1$ of a complete Riemannian manifold $(\tilde{M}, \tilde{g}_0)$ of dimension n, verifying Ricci$(\tilde{g}_0) \geqslant -(n-1)\tilde{g}_0$, we have, for every metric g conformal to $g_0 := \tilde{g}_{0|_M}$, for all $\alpha \in \left(0, \frac{n-2}{n}\right)$, and every density ρ such that $\int_M \rho \mathrm{d}V_g = |M|_g$:

$$\lambda_k(\rho, \rho^\alpha)|M|_g^{\frac{2}{n}} \leqslant A_n k^{\frac{2}{n}} + B_n |M|_{g_0}^{\frac{2}{n}}.$$

Corollary 1.6.2 (see Corollary 3.1.3)**.**
Let M be a bounded domain with boundary of class $\mathscr{C}^1$ of a complete Riemannian manifold $(\tilde{M}, \tilde{g})$ of dimension $n \geqslant 2$ such that Ricci$(\tilde{g}) \geqslant -(n-1)ric_0\tilde{g}$, and let $g = \tilde{g}_{|_M}$. For every density ρ such that $\int_M \rho \mathrm{d}V_g = |M|_g$, we have:

$$\lambda_k(\rho, \rho^\alpha)|M|_g^{\frac{2}{n}} \leqslant A_n k^{\frac{2}{n}} + B_n ric_0 |M|_g^{\frac{2}{n}},$$

where $\alpha \in \left(0, \frac{n-2}{n}\right)$ and $ric_0 > 0$.

Definition 1.6.3. We take the same definition of a *manifold of revolution with boundary* as in [CES15, Section 5], where the authors define it as a Riemannian manifold with boundary, having a distinct point N such that $(M \setminus \{N\}, g)$ is isometric to $\left((0,R] \times \mathbb{S}^{n-1}, dr^2 + \theta(r)^2 g_{\mathbb{S}^{n-1}}\right)$. Here $g_{\mathbb{S}^{n-1}}$ designates the standard metric on the sphere, and $r \mapsto \theta(r)$ is a smooth positive function on $(0,R]$ that vanishes at 0. We also impose $\theta'(0) = 1$ and $\theta''(0) = 0$. Notice that M is compact, connected, and has a smooth boundary $\partial M = \{R\}\times\mathbb{S}^{n-1}$, isometric to the $(n-1)$-sphere of radius $\theta(R)$.

Theorem 1.6.4 (see Theorem 3.3.1)**.**
Let Ω be a manifold of revolution with boundary of dimension $n \geqslant 2$. If $\alpha \in \left(\frac{n-2}{n}, 1\right)$, then

$$\sup_{\rho} \lambda_1(\rho, \rho^{\alpha})|\Omega|_g^{\frac{2}{n}} = +\infty.$$

Remark 1.6.5. A natural question emerges: what happens when $\alpha > 1$? We believe the supremum is not uniformly bounded and can be infinite for some manifolds. However, our attempts in that direction remained fruitless. For example, the method used in proposition 3.3.3 page 39 gives, after some calculations, a negative density. Another difficulty lies in the fact that when $\alpha > 1$, and with the usual densities we consider, the minoration of the first eigenvalue is not trivial.

1.7 Motivation and results for Steklov spectrum

1.7.1 Motivation

As described previously, we are interested in works that separate geometrical and asymptotical terms. First of all, let us present what happens in the case of a smooth and compact orientable surface M. In [GP12], the authors proved that

$$\sigma_k(M)\,|\partial M| \leqslant 2\pi\left(\text{genus}(M) + b\right)k, \tag{1.7.30}$$

where b is the number of boundary components of M.

An analog of this inequality can be found in [CEG11, Theorem 1.5]: there exists C such that for any compact orientable surface,

$$\sigma_k(M)\,|\partial M| \leqslant C\left\lfloor\frac{\text{genus}(M)+3}{2}\right\rfloor k.$$

The reader will notice that, because the right-hand side does not depend on the number of components b of the boundary, this inequality is actually an improvement of (1.7.30) for b large enough. In this case, A. Hassannezhad [Has11] also found a Kröger type bound. She proved that given a compact surface M and Ω a domain of M, then the eigenvalues of Ω are bounded in the following way:

$$\sigma_k(\Omega)\,|\partial\Omega| \leqslant A\,\text{genus}(M) + Bk.$$

The geometrical restrictions such as the genus, the isoperimetric ratio, or the intersection index are not the only useful quantities to bound the eigenvalues. In the article [IM11, Theorem

1.2], the authors managed to bound the first Steklov eigenvalue of a domain relatively to the mean curvatures of its boundary. Let Ω be a compact domain of $\mathbb{R}^n$, and let H_r denote the r-th mean curvature of $\partial\Omega$ in $\mathbb{R}^n$ (see [Rei73]). Then their theorem states that for all $r \in \{0 \dots n-1\}$,

$$\sigma_1(\Omega) \left(\int_{\partial\Omega} H_{r-1} \mathrm{d}V_g \right)^2 \leqslant n\,|\Omega| \left(\int_{\partial\Omega} H_r^2 \mathrm{d}V_g \right),$$

and equality holds if and only if Ω is the Euclidean ball. They also obtain a very similar bound in the case of submanifolds of dimension n immersed in $\mathbb{R}^m$, for some $m > n$.

This type of bound on the first eigenvalue of the Laplacian had already been obtained by Reilly in [Rei77]. He proved that, given a compact hypersurface (Σ, g) without boundary of dimension n in $\mathbb{R}^{n+1}$, then for all $r \in \{0, \dots, n\}$,

$$\lambda_1(M) \left(\int_M H_{r-1} \mathrm{d}V_g \right)^2 \leqslant n\,|M| \left(\int_M H_r^2 \mathrm{d}V_g \right).$$

The equality holds if and only if M is isometric to $\mathbb{S}^n$. He also obtained a similar bound in the case of submanifolds of dimension n immersed in $\mathbb{R}^m$, for $m > n+1$. It is important to know that this result for λ_1 has been generalised in the hyperbolic space in [EI92] (see also [EI00] and the work of Grosjean [Gro02], [Gro04]). However it is not known if the same generalisation in the hyperbolic space can be made for the Steklov eigenvalue σ_1.

1.7.2 Results

In chapter 4, we will generalise results from Euclidean space, obtained in [CGG18], to hyperbolic space. We don't get a "clean" separation of the asymptotical and geometrical terms in the first result, but the latter two are simpler, as we make more assumptions on the boundary of M. Let $\mathbb{H}^m$ be the hyperbolic space of dimension m. We use the representation

$$\mathbb{H}^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m, x_m > 0\} \text{ where the metric reads } \mathrm{d}s^2 = \frac{\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_m^2}{x_m^2}.$$

The first inequality is presented in a general context. We have the following bound:

Theorem 1.7.1 (see Theorem 4.2.1)**.**
Let Σ be a smooth and compact submanifold of dimension $n-1$ in $\mathbb{H}^m$, and let $M \subset \mathbb{H}^m$ be a smooth manifold with boundary Σ. Then for all $k \geqslant 1$,

$$\sigma_k(M) \leqslant \frac{|M|}{|\Sigma|} \left(A_0(n,m) + A(\Sigma)\, k^{\frac{2}{n-1}} \right),$$

where $A_0(n,m)$ only depends on the dimensions n and m, and $A(\Sigma)$ depends on n, on m, on a lower bound for the Ricci curvature of Σ, on its volume $|\Sigma|$, on its maximal distortion γ (see definition 4.2.2), and on its number b of connected components.

In the following, the notation $\mathbb{R}^n \times \{1\} \subset \mathbb{H}^{n+1}$ means we consider the set $\{(x_1, \dots, x_n, 1)\} \subset \mathbb{H}^{n+1}$. The induced metric on this space is the Euclidean one, hence we refer to it as $\mathbb{R}^n \times \{1\}$, or

$\mathbb{R}^n$ for convenience. In a similar way, the notation $\{0\} \times \mathbb{H}^n \subset \mathbb{H}^{n+1}$ means we consider the set $\{(0, x_2, \dots, x_{n+1})\}$ with induced metric $\mathrm{d}s^2 = \left(\mathrm{d}x_2^2 + \cdots + \mathrm{d}x_{n+1}^2\right)/x_{n+1}^2$. As it is isometric to the hyperbolic space of dimension n, we refer to it as $\{0\} \times \mathbb{H}^n$, or $\mathbb{H}^n$ for convenience. The last two theorems impose more restrictions on the boundary ∂M.

Theorem 1.7.2 (see Theorem 4.3.1)**.**
Let Σ be an $(n-1)$-dimensional, connected, smooth hypersurface in $\mathbb{R}^n \times \{1\} \subset \mathbb{H}^{n+1}$ and let $\Omega \subset \mathbb{R}^n$ denote a domain with boundary $\partial\Omega = \Sigma$. Then, for every manifold M such that $\partial M = \Sigma$, for each $k \geqslant 1$,

$$\sigma_k(M) \leqslant |M|\nu(n)\left(\frac{1}{|\Omega|^{\frac{n-1}{n}}} + \frac{k^{\frac{2}{n-1}}}{|\Omega|^{\frac{n+1}{n}}}\right),$$

where $\nu(n)$ is a constant that only depends on n.

Theorem 1.7.3 (see Theorem 4.4.1)**.**
Let Σ be an $(n-1)$-dimensional, connected, smooth hypersurface in $\{0\} \times \mathbb{H}^n \subset \mathbb{H}^{n+1}$ and let $\Omega \subset \{0\} \times \mathbb{H}^n$ denote a domain with boundary $\partial\Omega = \Sigma$. Then, for every manifold M such that $\partial M = \Sigma$, for each $k \geqslant 1$,

$$\sigma_k(M) \leqslant |M|\tilde{\nu}(n)\left(\frac{1}{|\Omega|^{\frac{n-1}{n}}} + \frac{k^{\frac{2}{n-1}}}{|\Omega|^{\frac{n+1}{n}}}\right),$$

where $\tilde{\nu}(n)$ is a constant that only depends on n.

Chapter 2

Counter-example to a Kröger type spectral inequality

2.1 Introduction

In this chapter, we will consider smooth and compact hypersurfaces of $\mathbb{R}^{n+1}$, namely submanifolds of dimension n in $\mathbb{R}^{n+1}$ equipped with the induced metric. The associated spectrum of the Laplace operator $\Delta = -\mathrm{div} \circ \nabla$, is discrete, positive, and denoted

$$0 = \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \nearrow +\infty.$$

As we have seen in the introduction, B. Colbois, E. Dryden et A. El Soufi showed that an inequality of Kröger type is not possible in the context of compact hypersurfaces without boundary. More precisely they showed in [CDE10, Theorem 1.4] that if Σ is a hypersurface of $\mathbb{R}^{n+1}$ with $n \geqslant 3$,

$$\sup_X \lambda_1(X(\Sigma))\,|X(\Sigma)|^{\frac{2}{n}} = \infty, \tag{2.1.1}$$

where the supremum is taken over the set of embeddings of Σ into $\mathbb{R}^{n+1}$.

In dimension 2 we have the result:

$$\sup_\Sigma \lambda_1(\Sigma)\,|\Sigma|^{\frac{2}{n}} = \infty, \tag{2.1.2}$$

where the supremum is taken over the set of compact orientable surfaces $\Sigma \subset \mathbb{R}^3$.

Therefore we have to impose geometric restrictions on Σ in order to bound the eigenvalues from above. For example in the same paper the authors proved that if Σ is a convex hypersurface of dimension n, then

$$\lambda_k(\Sigma)\,|\Sigma|^{\frac{2}{n}} \leqslant c(n)k^{\frac{2}{n}}$$

where $c(n)$ is a constant depending only on the dimension.

For compact orientable surfaces Σ of $\mathbb{R}^3$, a natural restriction is the genus. A. Hassannezhad [Has11] showed the existence of two constants C_1 and C_2 such that for all compact orientable surfaces Σ of $\mathbb{R}^3$ we have:

$$\lambda_k(\Sigma)\,|\Sigma| \leqslant C_1\,\mathrm{genus}(\Sigma) + C_2 k.$$

Let us now focus on the main result of the article [CEG13] from B. Colbois, A. El Soufi and A. Girouard in which they link the eigenvalues of a compact hypersurface to the isoperimetric ratio:

Theorem 2.1.1. *For all bounded domains Ω of $\mathbb{R}^{n+1}$ with a $\mathscr{C}^2$-boundary $\Sigma = \partial\Omega$, and for all $k \geqslant 1$,*

$$\lambda_k(\Sigma)\cdot|\Sigma|^{\frac{2}{n}} \leqslant \gamma(n) I(\Sigma)^{1+\frac{2}{n}} k^{\frac{2}{n}}$$

with $I(\Sigma) = \dfrac{|\Sigma|}{|\Omega|^{\frac{n}{n+1}}}$ the isoperimetric ratio and $\gamma(n) = \frac{2^{10n+18+8/n}}{n+1}\omega_{n+1}^{\frac{1}{n+1}}$.

Thus the question is whether there is a Kröger type inequality for large k in dimension $n \geqslant 2$. If there exists a constant $A > 0$ and a continuous function $f : (0, +\infty) \to \mathbb{R}$ such that for all hypersurfaces Σ and for all k,

$$\lambda_k(\Sigma)\,|\Sigma|^{\frac{2}{n}} \leqslant f(I(\Sigma)) + Ak^{\frac{2}{n}}, \tag{2.1.3}$$

we have seen in the introduction that it would imply an inequality of type Kröger. We will show that this is actually not possible by constructing a family of counter-examples. To that effect, we first define the following set:

$$\mathscr{M}(J) = \left\{\Sigma \text{ connected hypersurface of } \mathbb{R}^{n+1} \text{ s.t. } I(\Sigma) \leqslant J\right\}$$

Theorem 2.1.2.
Let $n \geqslant 2$. For all $A > 0$, there exists $J > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$, there exists $\Sigma \in \mathscr{M}(J)$ verifying:

$$\lambda_k(\Sigma)\,|\Sigma|^{\frac{2}{n}} > 2Ak^{\frac{2}{n}}.$$

Moreover in dimension $n \geqslant 3$, the hypersurface Σ can be chosen to be diffeomorphic to a given hypersurface.

The idea of the construction is as follows and will be done in sections 2.2 and 2.3. First we take a submanifold of $\mathbb{R}^{n+1}$ and modify it to get its first eigenvalue as large as desired. Then we consider the disjoint union of this hypersurface with the sphere $\mathbb{S}^n$, whose spectrum is well-known. This way we can compare the spectrum of the union with the one of the sphere.

We then connect them by gluing a tube, thin enough not to alter the behaviour of the spectrum, and making the union diffeomorphic to the first submanifold. To that effect we use the results of C. Anné (see [Ann87], and also [Ann84, Sections A and C]). Her work mainly consisted in defining the correct settings for the description of the spectrum of a limit of manifolds. The following theorem shows how to "glue" the two connected components of a disconnected manifold, and how its spectrum behaves. Let us be more precise.

Theorem 2.1.3 (Colette Anné [Ann87]).

Let X_1 and X_2 be two submanifolds of $\mathbb{R}^{n+1}$, and X_1 is compact, without boundary of dimension n. We also impose that dim$(X_2) = q < n$, $\partial X_2 \subset X_1$, X_2 is transverse to X_1 and $X_1 \cap X_2 = \partial X_2 = Y$. There exist piecewise $\mathscr{C}^\infty$ manifolds M_ε, that converge to $X_1 \sqcup X_2$ for the Hausdorff distance. There exists a natural way to define a Laplacian on M_ε and its spectrum converges to the union of the spectrum of X_1 and the Dirichlet spectrum of X_2, which also allow to define a natural Laplacian on $X_1 \cup X_2$.

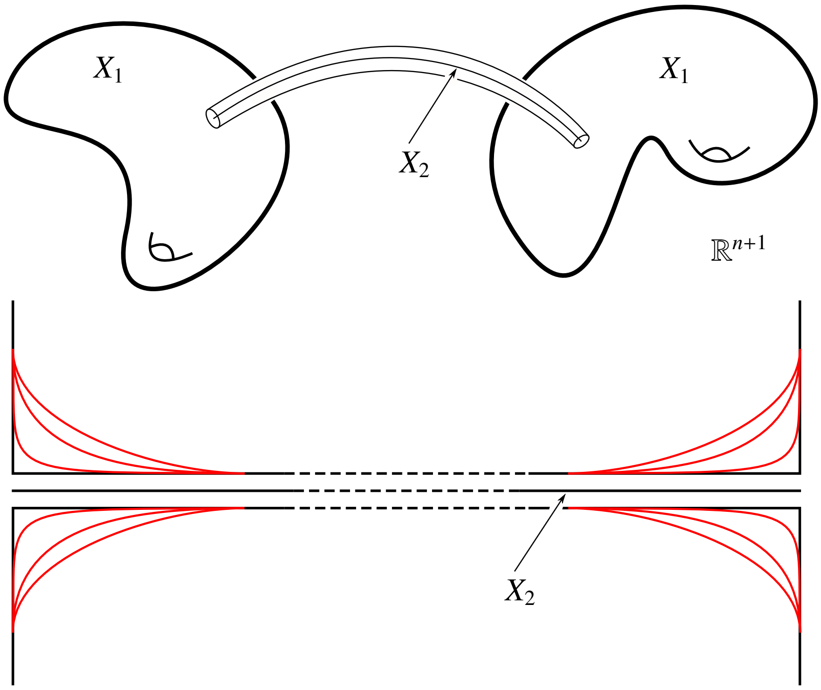

Figure 2.1: On top, illustration of M_ε. On the bottom, smoothing of the manifold

In our case, we wanted to work with smooth hypersurfaces. To that effect, we referred to [Ann84, Section A], where the author presents *quasi-isometries*. In particular, she uses the following lemma:

Lemma 2.1.4. *Let g_1 and g_2 be two metrics defined on the same compact manifold X of dimension n, and let $A, B > 0$ such that (X, g_2) is (A, B)-quasi-isometric to (X, g_1), that is, $Ag_1 \leqslant g_2 \leqslant Bg_1$. Then for all $k \in \mathbb{N}$,*

$$\frac{A^{\frac{n}{2}}}{B^{1+\frac{n}{2}}}\lambda_k(X, g_1) \leqslant \lambda_k(X, g_2) \leqslant \frac{B^{\frac{n}{2}}}{A^{1+\frac{n}{2}}}\lambda_k(X, g_1).$$

This result is interesting for us because we can approach M_ε by a sequence $\left(X_\beta\right)_{\beta>0}$ of smooth hypersurfaces, such that X_β is $(1-\beta, 1+\beta)$-quasi-isometric to M_ε (see figure 2.1 for an illustration). The lemma makes sure their spectra are arbitrarily close to each other. It will be used at the end of Section 2.3.

In the end we obtain a connected hypersurface having an important property; the beginning of its spectrum is the same as that of the sphere, but its area is sufficiently large. We thus show that even with a controlled isoperimetric ratio, it is possible to find a hypersurface with a sufficiently large spectrum to contradict the inequality (2.1.3). The next section covers the tools required for this construction.

2.2 Non-connected case

Let $\mathscr{M}_{disc}(J) = \left\{\Sigma \text{ disconnected hypersurface of } \mathbb{R}^{n+1} \text{ s.t. } I(\Sigma) \leqslant J\right\}$. In this first part we will prove the following result:

Theorem 2.2.1.
Let $n \geqslant 2$. For all $A > 0$, there exists $J > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$, there exists $\tilde{\Sigma} = \tilde{\Sigma}_k \in \mathscr{M}_{disc}(J)$ verifying:

$$\lambda_k(\tilde{\Sigma}_k)\left|\tilde{\Sigma}_k\right|^{\frac{2}{n}} > 2Ak^{\frac{2}{n}}.$$

First of all, let us take a look at these two cases:

- Recall result (2.1.1) and let M be a manifold of dimension $n \geqslant 3$ which can be embedded in $\mathbb{R}^{n+1}$. Then there exists a sequence of embeddings X such that $\lambda_1(X(M))\,|X(M)|^{\frac{2}{n}}$ is as large as desired. If we call $\Sigma = X(M)$ the image of this embedding, it is equivalent to say:

$$\forall L > 0, \exists \Sigma \text{ s.t. } \lambda_1(\Sigma)\,|\Sigma|^{\frac{2}{n}} \geqslant L. \tag{2.2.4}$$

- Recall result (2.1.2). We know that for all $L > 0$, there exists a compact orientable surface $\Sigma \subset \mathbb{R}^3$ such that

$$\lambda_1(\Sigma)\,|\Sigma|^{\frac{2}{n}} \geqslant L. \tag{2.2.5}$$

Then for any dimension $n \geqslant 2$, for all k, there exists Σ_k such that

$$\lambda_1(\Sigma_k)\,|\Sigma_k|^{\frac{2}{n}} \geqslant 4A\,(\lambda_k(\mathbb{S}^n) + 1)\,\frac{|\mathbb{S}^n|^{\frac{2}{n}}}{W(n)}.$$

Moreover in dimension $n \geqslant 3$, the hypersurface Σ_k can be chosen diffeomorphic to a given hypersurface. As $\lambda_1(\Sigma_k)\,|\Sigma_k|^{\frac{2}{n}}$ is a homothetic invariant quantity, we can assume that

$$|\Sigma_k|^{\frac{2}{n}} = 4A\frac{|\mathbb{S}^n|^{\frac{2}{n}}}{W(n)}, \qquad \lambda_1(\Sigma_k) \geqslant \lambda_k(\mathbb{S}^n) + 1.$$

We will denote by Ω_k the domain of dimension $n+1$ such that $\partial\Omega_k = \Sigma_k$. The notation "$M \sqcup N$" will designate the disjoint union of two manifolds M and N. To this union we can associate its spectrum, which is the union of the two spectra of M and N. The notation $\tilde{\Sigma}_k = \Sigma_k \sqcup \mathbb{S}^n$ will refer to the union of Σ_k with the sphere $\mathbb{S}^n$.

Lemma 2.2.2.
We have:

$$\begin{cases} \lambda_0(\tilde{\Sigma}_k) = \lambda_1(\tilde{\Sigma}_k) = 0 \\ \lambda_j(\tilde{\Sigma}_k) = \lambda_{j-1}(\mathbb{S}^n)\ \forall j = 2\dots k+1 \end{cases} \tag{2.2.6}$$

Proof. We know that $\mathrm{Sp}(\Sigma_k \sqcup \mathbb{S}^n) = \mathrm{Sp}(\Sigma_k) \cup \mathrm{Sp}(\mathbb{S}^n)$ and also that $\lambda_1(\Sigma_k) \geqslant \lambda_k(\mathbb{S}^n) + 1$. As a consequence the beginning of the spectrum of the union of $\mathbb{S}^n$ with Σ_k can be represented as follows:

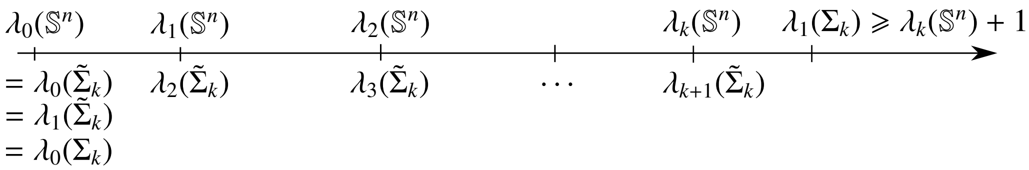

Figure 2.2: The eigenvalues are shifted

We can now prove the following lemma:

Lemma 2.2.3. *There exists $k_0 > 0$ such that $\forall k \geqslant k_0$ we have*

$$\lambda_k(\tilde{\Sigma}_k) \left|\tilde{\Sigma}_k\right|^{\frac{2}{n}} > 2Ak^{\frac{2}{n}}.$$

Proof. By the choice of the volume of Σ_k as above, we get:

$$\left|\tilde{\Sigma}_k\right|^{\frac{2}{n}} \geqslant |\Sigma_k|^{\frac{2}{n}} = 4A\frac{|\mathbb{S}^n|^{\frac{2}{n}}}{W(n)}.$$

Therefore

$$\frac{\lambda_k(\tilde{\Sigma}_k) \left|\tilde{\Sigma}_k\right|^{\frac{2}{n}}}{k^{\frac{2}{n}}} = \frac{\lambda_{k-1}(\mathbb{S}^n) \left|\tilde{\Sigma}_k\right|^{\frac{2}{n}}}{k^{\frac{2}{n}}} \geqslant 4A\frac{\lambda_{k-1}(\mathbb{S}^n)\, |\mathbb{S}^n|^{\frac{2}{n}}}{W(n)k^{\frac{2}{n}}}.$$

But we know Weyl's asymptotic law for the spectrum of $\mathbb{S}^n$ gives:

$$\lim_{k\to\infty} \frac{\lambda_{k-1}(\mathbb{S}^n)\, |\mathbb{S}^n|^{\frac{2}{n}}}{W(n)k^{\frac{2}{n}}} = 1.$$

So there exists $k_0 > 0$ such that $\forall k \geqslant k_0$,

$$\frac{\lambda_{k-1}(\mathbb{S}^n)\, |\mathbb{S}^n|^{\frac{2}{n}}}{W(n)k^{\frac{2}{n}}} > \frac{1}{2}$$

so it is quite clear that for $k \geqslant k_0$,

$$\lambda_k(\tilde{\Sigma}_k) \left|\tilde{\Sigma}_k\right|^{\frac{2}{n}} > \frac{1}{2} 4Ak^{\frac{2}{n}} = 2Ak^{\frac{2}{n}}.$$

$\square$

Lemma 2.2.4. *The isoperimetric ratio of $\tilde{\Sigma}_k$ is bounded as follows:*

$$I(\tilde{\Sigma}_k) \leqslant (n+1)^{\frac{n}{n+1}} \left(\left(\frac{4A}{W(n)} \right)^{\frac{n}{2}} + 1 \right) |\mathbb{S}^n|^{\frac{1}{n+1}}.$$

Note that the bounds do not depend on k.

Proof. Let us call Ω_k the interior of Σ_k. Then the interior of $\tilde{\Sigma}_k$ is $\Omega_k \sqcup \mathbb{B}^{n+1}$.
Consequently $\left|\Omega_k \sqcup \mathbb{B}^{n+1}\right| \geqslant \left|\mathbb{B}^{n+1}\right| = \frac{1}{n+1}|\mathbb{S}^n|$,
and $\left|\tilde{\Sigma}_k\right| = |\Sigma_k| + |\mathbb{S}^n| = \left(\frac{4A}{W(n)}\right)^{\frac{n}{2}}|\mathbb{S}^n| + |\mathbb{S}^n| = \left(\left(\frac{4A}{W(n)}\right)^{\frac{n}{2}} + 1\right)|\mathbb{S}^n|$.
Then

$$
\begin{aligned}
I(\tilde{\Sigma}_k) &= \frac{\left|\tilde{\Sigma}_k\right|}{\left|\Omega_k \sqcup \mathbb{B}^{n+1}\right|^{\frac{n}{n+1}}} = \frac{\left(\left(\frac{4A}{W(n)}\right)^{\frac{n}{2}} + 1\right)|\mathbb{S}^n|}{\left|\Omega_k \sqcup \mathbb{B}^{n+1}\right|^{\frac{n}{n+1}}} \\
&\leqslant \frac{\left(\left(\frac{4A}{W(n)}\right)^{\frac{n}{2}} + 1\right)|\mathbb{S}^n|}{\left(\frac{1}{n+1}|\mathbb{S}^n|\right)^{\frac{n}{n+1}}} = (n+1)^{\frac{n}{n+1}}\left(\left(\frac{4A}{W(n)}\right) + 1\right)^{\frac{n}{2}}|\mathbb{S}^n|^{\frac{1}{n+1}}.
\end{aligned}
$$

□

We call J the quantity in the right-hand side of Lemma 2.2.4. This implies $I(\tilde{\Sigma}_k) \leqslant J$, that is, $\tilde{\Sigma}_k \in \mathscr{M}_{disc}(J)$. Now we can prove Theorem 2.2.1 for non-connected hypersurfaces.

Proof. For all $A > 0$, Lemmas 2.2.3 and 2.2.4 give the existence of J and k_0 such that for all $k \geqslant k_0$, the hypersurface $\tilde{\Sigma}_k \in \mathscr{M}_{disc}(J)$ verifies

$$\lambda_k(\tilde{\Sigma}_k)\left|\tilde{\Sigma}_k\right|^{\frac{2}{n}} > 2Ak^{\frac{2}{n}}.$$

□

2.3 Tubular attachment

We just discussed an example where the hypersurface was disconnected. We can obtain the same result for a connected hypersurface, simply by joining the two components of $\tilde{\Sigma}_k$. This is possible while keeping control on the spectrum and on the isoperimetric ratio, and we are going to show that in the following proposition. The construction of Σ being simple but technical, the reader may refer to the figure 2.3 for a visual intuition.

Proposition 2.3.1.
Let Σ_1 and Σ_2 be two compact and connected hypersurfaces of $\mathbb{R}^{n+1}$. For all $\varepsilon > 0$ and for all $N \in \mathbb{N}^*$, there exists a hypersurface $\Sigma = \Sigma_{\varepsilon,N}$ such that

$$\left|I(\Sigma_{\varepsilon,N}) - I(\Sigma_1 \sqcup \Sigma_2)\right| < \varepsilon$$

and $\forall k \leqslant N$

$$\left|\lambda_k(\Sigma_{\varepsilon,N}) - \lambda_k(\Sigma_1 \sqcup \Sigma_2)\right| < \varepsilon.$$

Proof.

Let $h > 0$. Without loss of generality we can assume that $\Sigma_1 \subset \{x_{n+1} \geqslant h\}$ and $\Sigma_2 \subset \{x_{n+1} \leqslant -h\}$ and that the points $(0, \dots, 0, h)$ and $(0, \dots, 0, -h)$ belong to Σ_1 and Σ_2 respectively. We denote $p = (0, \dots, 0, h)$ and $q = (0, \dots, 0, -h)$.

A hypersurface can be locally represented by a graph. As a consequence, for all $\delta > 0$ small enough, there exists a function $f : \mathbb{B}^n(0, 4\delta) \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ such that the set $V_1 = \{(x, f(x)), x \in \mathbb{B}^n(0, 4\delta)\}$ is an open set containing p in Σ_1.

Note that it naturally implies $f(0, \dots, 0) = h$ and $\forall x \in \mathbb{B}^n(0, 4\delta), f(x) \geqslant h$.

Now let $f_1 \in \mathscr{C}^\infty(\mathbb{B}^n(0, 4\delta), \mathbb{R})$ such that $f_1 \geqslant h$ and

$$f_1(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{B}^n(0, 4\delta) \setminus \mathbb{B}^n(0, 2\delta) \\ h & \text{if } x \in \mathbb{B}^n(0, \delta). \end{cases}$$

Let us call $V_1^\delta = \{(x, f_1(x)), x \in \mathbb{B}^n(0, 4\delta)\}$ the graph of this new function f_1. It will generate a new hypersurface Σ_1^δ equal to $(\Sigma_1 \setminus V_1) \cup V_1^\delta$, flat on the open set $\mathbb{B}^n(0, \delta) \times \{h\}$.

We modify Σ_2 around q on an open set V_2 in the same way, in order to create an open set V_2^δ. Thus we form a hypersurface $\Sigma_2^\delta = (\Sigma_2 \setminus V_2) \cup V_2^\delta$, flat on the open set $\mathbb{B}^n(0, \delta) \times \{-h\}$.

For the following we will use the notation $B_1^\delta = \mathbb{B}^n(0, \delta) \times \{h\} \subset \Sigma_1^\delta$ and $B_2^\delta = \mathbb{B}^n(0, \delta) \times \{-h\} \subset \Sigma_2^\delta$. We now just have to glue the two parts together.

For this purpose we call $R^{\delta,h}$ the surface of revolution defined by

$$R^{\delta,h} = \left\{(x_1, \dots, x_{n+1}) : x_1^2 + \dots + x_n^2 = \varphi^2(x_{n+1}) \text{ and } -h \leqslant x_{n+1} \leqslant h\right\}$$

where $\varphi : (-h, h) \longrightarrow \mathbb{R}$ is an even function that is increasing and bounded from above by δ on $[0, h]$ and such that $\varphi\left(\left[-\frac{h}{2}, \frac{h}{2}\right]\right) = \frac{\delta}{2}$. We also choose φ such that the connected hypersurface

$$\Sigma^{\delta,h} = \Sigma_1^\delta \cup R^{\delta,h} \cup \Sigma_2^\delta$$

is $\mathscr{C}^\infty$ in the neighbourhood of $\partial R^{\delta,h}$. The results of Colette Anné will allow us to conclude. When δ tends to 0, the spectrum of $\Sigma^{\delta,h}$ tends to the spectrum of the disjoint union of Σ_1, Σ_2, and the segment $[-h, h]$ with Dirichlet boundary conditions.

The first eigenvalue $\lambda_1(h)$ of the segment $[-h, h]$ for these conditions is equal to $\frac{\pi^2}{4h^2}$. As a consequence, if we take an integer $N > 0$, there exists $h > 0$ small enough such that $\lambda_1(h) > \lambda_N(\Sigma_1 \sqcup \Sigma_2)$. Finally, for all $\varepsilon, N > 0$, there exist $\delta > 0$ and $h > 0$ small enough such that

$$\left|\lambda_k(\Sigma^{\delta,h}) - \lambda_k(\Sigma_1 \sqcup \Sigma_2)\right| < \varepsilon \qquad k = 0, \dots, N.$$

It is also clear that the volume of $\Sigma^{\delta,h}$ converges, when δ tends to 0, to the volume of $\Sigma_1 \sqcup \Sigma_2$. Likewise the volume of the interior of $\Sigma^{\delta,h}$ converges to the volume of the interior of $\Sigma_1 \sqcup \Sigma_2$ when δ tends to 0. This shows that $I\left(\Sigma^{\delta,h}\right)$ tends to $I\left(\Sigma_1 \sqcup \Sigma_2\right)$ when δ tends to 0. □

Theorem 2.1.2 in the connected case follows immediately.

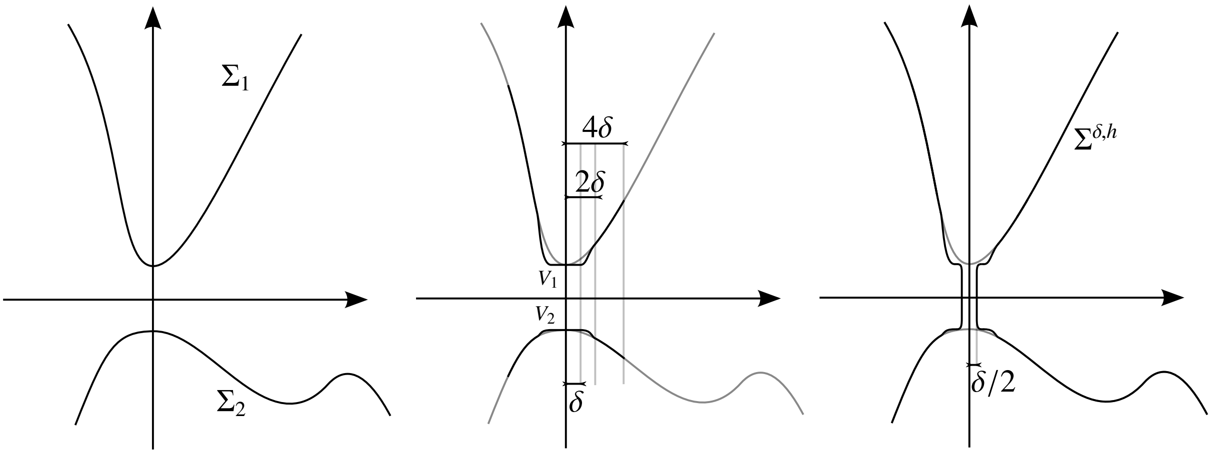

Figure 2.3: Gluing the hypersurfaces

□

Chapter 3

Laplacian with density

Let (M, g) be a compact Riemannian manifold with a boundary of class $\mathscr{C}^1$. We are interested in the spectrum of the weighted Laplacian on M with Neumann boundary conditions. More precisely, given ρ and σ two positive functions on M, we study the eigenvalues of the equation $-\mathrm{div}(\sigma\nabla u) = \lambda\rho u$. Inspired by a recent work of B. Colbois and A. El Soufi [CE18], we investigate upper bounds for the eigenvalues in the case where $\sigma = \rho^\alpha$, $\alpha > 0$. We show that $\alpha = \frac{n-2}{n}$ plays a critical role in the estimation of the spectrum when the total mass of ρ is fixed.

3.1 Introduction

Let (M, g) be a compact Riemannian manifold of dimension $n \geqslant 2$ with a boundary of class $\mathscr{C}^1$. Let ρ and σ be two positive continuous functions defined on M. For all $u \in H^1(M)$, we denote by ∇u the gradient of u with respect to the metric g and we consider the Rayleigh quotient

$$R_{(g,\sigma,\rho)}(u) = \frac{\int_M |\nabla u|^2 \sigma \mathrm{d}V_g}{\int_M u^2 \rho \mathrm{d}V_g}. \tag{3.1.1}$$

Its corresponding eigenvalues are given, for $k \in \mathbb{N}$, by

$$\lambda_k^g(\rho, \sigma) = \inf_{E_{k+1} \subset H^1(M)} \sup_{u \in E_{k+1}\setminus\{0\}} R_{(g,\sigma,\rho)}(u),$$

where E_{k+1} runs through the $(k+1)$-dimensional vector subspaces of $H^1(M)$ and $\mathrm{d}V_g$ is the volume element induced by the metric g.
Under some regularity conditions on ρ and σ, $\lambda_k^g(\rho, \sigma)$ is the k-th eigenvalue of the problem

$$-\mathrm{div}(\sigma\nabla u) = \lambda\rho u \quad \text{in } M, \tag{3.1.2}$$

with Neumann conditions on the boundary. When there is no risk of confusion, we use $\lambda_k(\rho, \sigma)$ instead of $\lambda_k^g(\rho, \sigma)$. We said in the introduction that the spectrum of (3.1.2) is discrete, and can be ordered in a positive nondecreasing sequence that tends to infinity. Also, $\lambda_0(\rho, \sigma) = 0$, the constant functions being eigenfunctions for λ_0. Although we can't find the eigenvalues explicitly in general, we can estimate them when we fix the total mass of ρ to get some interesting inequalities.

What we do here is a continuation of several works that aimed to find a good choice of geometric restriction such that the supremum of the eigenvalues is bounded from above. The conformal spectrum is an important one and has been widely studied. Indeed, let us introduce the quantity

$$\lambda_k^c(M,[g]) = \sup_{g'\in[g]} \lambda_k(M,g'),$$

then an upper bound on this quantity was found by Korevaar [Kor93, Theorem 0.4]. A lower bound was later found by B. Colbois, A. El Soufi in [CE03, Corollary 1]. Together, these results read:

$$n\omega_n^{\frac{2}{n}} k^{\frac{2}{n}} \leqslant \lambda_k^c(M,[g]) \leqslant C([g]) k^{\frac{2}{n}} \tag{3.1.3}$$

where ω_n is the volume of the unit ball in dimension n and $C([g])$ is a constant depending only on n and on the conformal class of g. The lower bound is actually a corollary of an interesting result concerning the gap between two extremal eigenvalues, which states that

$$\lambda_{k+1}^c(M,[g]) - \lambda_k^c(M,[g]) \geqslant n^{\frac{n}{2}}\omega_n.$$

As the problem of the Laplacian with densities is very general, we chose to restrict ourselves to some particular class of densities. In this chapter, we are interested in the problem (3.1.2) when $\sigma = \rho^\alpha$, and $\alpha \in [0,1]$. The results we obtain on the spectrum $\lambda_k^g(\rho,\rho^\alpha)$ are supported by the three following important theorems, that led our motivation and intuition:

- When $\alpha = 0$, A. El Soufi and B. Colbois proved in [CE18, Corollary 4.1] that, for any compact Riemannian manifold (M, g_0) with density ρ, and for any metric g conformal to g_0 such that $\int_M \rho \mathrm{d}V_g = |M|_g$, one has

$$\lambda_k(\rho,1)|M|_g^{\frac{2}{n}} \leqslant C_n k^{\frac{2}{n}} + D_n |M|_{g_0}^{\frac{2}{n}},$$

  where C_n and D_n are constants depending only on n.

- Another important problem is when $\alpha = 1$, that is, $\sigma = \rho$. The associated operator is called the Witten Laplacian and was treated by B. Colbois, A. El Soufi and A. Savo. In their work [CES15, Theorem 5.2], they proved that, contrary to the previous cases, one cannot bound the eigenvalues from above on all manifolds. Indeed, on a compact manifold of revolution endowed with the Gaussian radial density $\rho_m = e^{-m|x|^2}$, one has for m large enough,

$$\lambda_1(\rho_m,\rho_m) \geqslant m.$$

- A further important case is when $\alpha = \frac{n-2}{n}$, and actually comes from several works on the conformal spectrum. We get an upper bound thanks to the work of A. Hassannezhad [Has11, Theorem 1.1], where she finds an inequality for the classical Laplace equation:

$$\lambda_k^g(1,1)|M|_g^{\frac{2}{n}} \leqslant A_n k^{\frac{2}{n}} + B_n V([g])^{\frac{2}{n}}$$

  where A_n and B_n are two constants which only depend on n, and $V([g])$ is the geometric quantity defined by:

$$V([g]) = \inf\left\{|M|_{g'} : g' \text{ is conformal to } g \text{ and } \mathrm{Ricci}(g') \geqslant -(n-1)g'\right\}.$$

If we now take ρ a positive continuous function on M satisfying $\int_M \rho \mathrm{d}V_g = |M|_g$, one can easily check that $|M|_{\rho^{\frac{2}{n}}g} = \int_M \rho \mathrm{d}V_g = |M|_g$ and $\lambda_k^{\rho^{\frac{2}{n}}g}(1,1) = \lambda_k^g(\rho, \rho^{\frac{n-2}{n}})$. Since g and $\rho^{\frac{2}{n}}g$ are conformal, then $V([\rho^{\frac{2}{n}}g]) = V([g])$. Thus we obtain

$$\lambda_k^g\left(\rho, \rho^{\frac{n-2}{n}}\right)|M|_g^{\frac{2}{n}} = \lambda_k^{\rho^{\frac{2}{n}}g}(1,1)|M|^{\frac{2}{n}}_{\rho^{\frac{2}{n}}g} \leqslant A_n k^{\frac{2}{n}} + B_n V([g])^{\frac{2}{n}},$$

which gives us an upper bound for $\lambda_k^g(\rho, \rho^\alpha)$ in the case where α is equal to $\frac{n-2}{n}$.

For the following we will denote by $\lambda^*_{k,\alpha}(M, g)$ the supremum of $\lambda_k^g(\rho, \rho^\alpha)$ on the set of all densities ρ satisfying $\int_M \rho \mathrm{d}V_g = |M|_g$, that is,

$$\lambda^*_{k,\alpha}(M, g) = \sup_\rho \left\{ \lambda_k^g(\rho, \rho^\alpha), \int_M \rho \mathrm{d}V_g = |M|_g \right\}.$$

We believe that a uniform lower bound of the same type as in equation (3.1.3) does not exist for this supremum if $\alpha \in \left[0, \frac{n-2}{n}\right]$. Indeed, when $\alpha = 0$, we can always find a 1-parameter family of metrics of volume 1 making it very small. The construction of these metrics can be found in [CE18, Theorem 5.1], and should be generalised for $\alpha \in \left(0, \frac{n-2}{n}\right]$. More generally, it is still an open question to know if there exists a non-negative bound for the spectral gap with densities, that is:

$$\lambda^*_{k+1,\alpha}(M, g) - \lambda^*_{k,\alpha}(M, g) > K(g)$$

where $K(g) \geqslant 0$ is a constant depending only on the metric.

An interesting question is to study the behaviour of the spectrum for different values of α. In fact, through the theorems 3.1.1 and 3.3.1, we will see that $\alpha = \frac{n-2}{n}$ plays a significant role. To highlight this, we first show $\lambda^*_{k,\alpha}(M, g)$ is bounded when α runs over the interval $\left(0, \frac{n-2}{n}\right)$:

Theorem 3.1.1. *Let $n \geqslant 2$. Then there exist A_n, B_n two positive constants such that, for any bounded domain M with boundary of class $\mathscr{C}^1$ of a complete Riemannian manifold $(\tilde{M}, \tilde{g}_0)$ of dimension n, verifying Ricci$(\tilde{g}_0) \geqslant -(n-1)\tilde{g}_0$, we have, for every metric g conformal to $g_0 := (\tilde{g}_0)_{|M}$, for all $\alpha \in \left(0, \frac{n-2}{n}\right)$, and every density ρ such that $\int_M \rho \mathrm{d}V_g = |M|_g$:*

$$\lambda_k(\rho, \rho^\alpha)|M|_g^{\frac{2}{n}} \leqslant A_n k^{\frac{2}{n}} + B_n |M|_{g_0}^{\frac{2}{n}}.$$

Remark 3.1.2.

Let g be a metric defined on M such that $g_0 = \mathrm{ric}_0 g$, for some $\mathrm{ric}_0 > 0$. Then if Ricci $(g_0) \geqslant -(n-1)g_0$, we have Ricci$(g) \geqslant -(n-1)\mathrm{ric}_0 g$ and $|M|_{g_0} = \mathrm{ric}_0^{\frac{n}{2}}|M|_g$. The following corollary follows:

Corollary 3.1.3.

Let M be a bounded domain with boundary of class $\mathscr{C}^1$ of a complete Riemannian manifold $(\tilde{M}, \tilde{g})$ of dimension $n \geqslant 2$ such that Ricci$(\tilde{g}) \geqslant -(n-1)ric_0\tilde{g}$ and let $g = \tilde{g}_{|M}$. For every density ρ such that $\int_M \rho \mathrm{d}V_g = |M|_g$, we have:

$$\lambda_k(\rho, \rho^\alpha)|M|_g^{\frac{2}{n}} \leqslant A_n k^{\frac{2}{n}} + B_n ric_0 |M|_g^{\frac{2}{n}}$$

where $\alpha \in \left(0, \frac{n-2}{n}\right)$ and $ric_0 > 0$.

In the last section we show that $\lambda^*_{k,\alpha}(M, g)$ is infinite when α belongs to $\left(\frac{n-2}{n}, 1\right)$.

3.2 Bounding the eigenvalues from above

In this section, we suppose that $\alpha \in \left(0, \frac{n-2}{n}\right)$. We define M as a bounded submanifold of dimension n of a complete Riemannian manifold $(\tilde{M}, \tilde{g}_0)$ with Ricci curvature bounded from below and ρ as a positive continuous function on M. Using the same argument of A. El Soufi and B. Colbois [CE18, Theorem 4.1], we are able to maximise the eigenvalues $\lambda_k(\rho, \rho^\alpha)$ on the class of metrics conformal to $g_0 = \tilde{g}_{0|_M}$ under the preservation of the total mass of ρ.

Proof of Theorem 3.1.1. To prove Theorem 3.1.1, we construct $k+1$ test functions on M with disjoint supports and controlled Rayleigh quotients. The idea is to use the covering property that was applied by A. Hassannezhad in [Has11, Theorem 2.1] to find $k+1$ functions defined on M with disjoint supports and controlled Rayleigh quotients. Let μ be the measure defined by its volume element $\mu = \rho \mathrm{d}V_g$. Since $\mathrm{Ric}(\tilde{g}_0) \geqslant -(n-1)\tilde{g}_0$, the metric measured space $(\tilde{M}, \tilde{d}_0, \mu)$ satisfies the (2;N;1)-covering property for some fixed N (see [Has11]), and we can apply Lemma 1.2.6. Define the distance d_0 as the restriction on M of the distance $\tilde{d}_0$ induced by $\tilde{g}_0$. We are going to treat two cases separately:

First case: F_j is a generic annulus A and $G_j = 2A$.

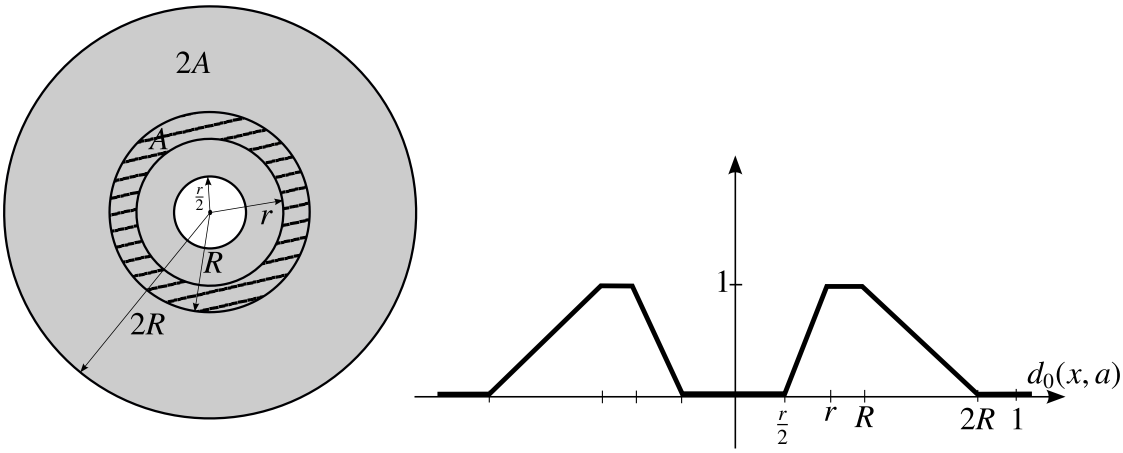

Figure 3.1: Behaviour of u_A

Define the function u_A supported in $G_j = 2A$ by:

$$u_A(x) = \begin{cases} \frac{2}{r} d_0(x,a) - 1 & \text{if} \quad \frac{r}{2} < d_0(x,a) < r \\ 1 & \text{if} \quad r < d_0(x,a) < R \\ 2 - \frac{1}{R} d_0(x,a) & \text{if} \quad R < d_0(x,a) < 2R. \end{cases}$$

Since $u = 1$ on A, then we have

$$\int_M u_A^2 \rho \mathrm{d}V_g \geqslant \int_A u_A^2 \rho \mathrm{d}V_g = \int_A \rho \mathrm{d}V_g = \mu(A) \geqslant \frac{\mu(M)}{c^2(k+1)}. \tag{3.2.4}$$

On the other hand, using Hölder's inequality repeatedly on the integral $\int_M |\nabla^g u_A|^2 \rho^\alpha \mathrm{d}V_g$ and the fact that the generalised Dirichlet energy $\int_{2A} |\nabla^g u_A|^n \mathrm{d}V_g$ is a conformal invariant, we get

$$\begin{aligned}\int_M |\nabla^g u_A|^2 \rho^\alpha \mathrm{d}V_g = \int_{2A} |\nabla^g u_A|^2 \rho^\alpha \mathrm{d}V_g &\leqslant \left(\int_{2A} |\nabla^g u_A|^n \mathrm{d}V_g\right)^{\frac{2}{n}} \left(\int_{2A} \rho^{\frac{n\alpha}{n-2}} \mathrm{d}V_g\right)^{\frac{n-2}{n}} \\ &\leqslant \left(\int_{2A} |\nabla^g u_A|^n \mathrm{d}V_g\right)^{\frac{2}{n}} \left(\int_{2A} \rho \mathrm{d}V_g\right)^{\alpha} \left(\int_{2A} 1 \mathrm{d}V_g\right)^{\frac{n-2}{n}-\alpha} \\ &= \left(\int_{2A} |\nabla^{g_0} u_A|^n \mathrm{d}V_{g_0}\right)^{\frac{2}{n}} \mu(2A)^\alpha |2A|_g^{\frac{n-2}{n}-\alpha}.\end{aligned}$$

Since

$$|\nabla^{g_0} u_A| = \begin{cases} \frac{2}{r} & \text{if} \quad \frac{r}{2} < d_0(x,a) < r \\ 0 & \text{if} \quad r < d_0(x,a) < R \\ \frac{1}{R} & \text{if} \quad R < d_0(x,a) < 2R, \end{cases}$$

we obtain:

$$\int_{2A} |\nabla^{g_0} u_A|^n \mathrm{d}V_{g_0} \leqslant \left(\frac{2}{r}\right)^n |B(a,r)|_{g_0} + \left(\frac{1}{R}\right)^n |B(a,2R)|_{g_0}. \tag{3.2.5}$$

But as $r \leqslant 2R \leqslant 1$ and $\mathrm{Ric}(g_0) \geqslant -(n-1)g_0$, we know thanks to the remark 1.2.2 presented page 6, that the right-hand side of inequality (3.2.5) is bounded from above by a quantity $\tilde{A}_n$ depending only on n.

Hence

$$\int_M |\nabla^g u_A|^2 \rho^\alpha \mathrm{d}V_g \leqslant \left(\tilde{A}_n\right)^{\frac{2}{n}} \mu(2A)^\alpha |2A|_g^{\frac{n-2}{n}-\alpha}. \tag{3.2.6}$$

From (3.2.4) and (3.2.6), we deduce that the Rayleigh quotient is bounded as follows:

$$R_{(g,\rho,\rho^\alpha)}(u_A) = \frac{\int_M |\nabla^g u_A|^2 \rho^\alpha \mathrm{d}V_g}{\int_M u_A^2 \rho \mathrm{d}V_g} \leqslant \left(\tilde{A}_n\right)^{\frac{2}{n}} c^2 \frac{\mu(2A)^\alpha |2A|_g^{\frac{n-2}{n}-\alpha}}{\mu(M)} (k+1). \tag{3.2.7}$$

Second case: F_j is a generic subset V of M and $G_j = V^{r_0}$, the set at distance r_0 from V.

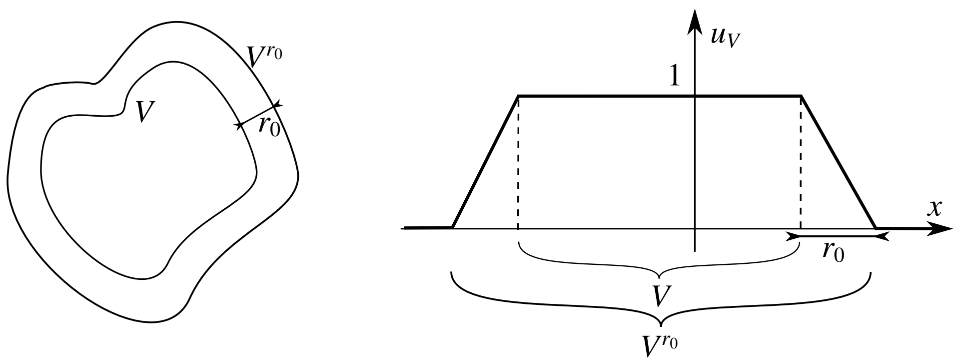

Figure 3.2: Behavior of u_V

We define the function u_V supported in V^{r_0} by:

$$u_V(x) = \begin{cases} 1 & \text{if} \quad x \in V \\ 1 - \frac{1}{r_0} d_0(x, V) & \text{if} \quad x \in V^{r_0} \backslash V \\ 0 & \text{if} \quad x \in M \setminus V^{r_0} \end{cases}$$

Here we have:

$$\int_M u_V^2 \rho \mathrm{d}V_g \geqslant \int_V \rho \mathrm{d}V_g = \mu(V) \geqslant \frac{\mu(M)}{c^2(k+1)}$$

and we also use Hölder as in the previous case:

$$\begin{aligned} \int_M |\nabla^g u_V|^2 \rho^\alpha \mathrm{d}V_g = \int_{V^{r_0}} |\nabla^g u_V|^2 \rho^\alpha \mathrm{d}V_g &\leqslant \left(\int_{V^{r_0}} |\nabla^g u_V|^n \mathrm{d}V_g \right)^{\frac{2}{n}} \left(\int_{V^{r_0}} \rho^{\frac{n\alpha}{n-2}} \mathrm{d}V_g \right)^{\frac{n-2}{n}} \\ &\leqslant \left(\int_{V^{r_0}} |\nabla^g u_V|^n \mathrm{d}V_g \right)^{\frac{2}{n}} \left(\int_{V^{r_0}} \rho \mathrm{d}V_g \right)^{\alpha} \left(\int_{V^{r_0}} 1 \mathrm{d}V_g \right)^{\frac{n-2}{n} - \alpha} \\ &= \left(\int_{V^{r_0}} |\nabla^{g_0} u_V|^n \mathrm{d}V_{g_0} \right)^{\frac{2}{n}} \mu(V^{r_0})^\alpha |V^{r_0}|_g^{\frac{n-2}{n} - \alpha}. \end{aligned}$$

Since

$$|\nabla^{g_0} u_V(x)| = \begin{cases} 0 & \text{if} \quad x \in V \\ \frac{1}{r_0} & \text{if} \quad x \in V^{r_0} \backslash V \end{cases}$$

we get

$$\int_{V^{r_0}} |\nabla^{g_0} u_V|^n \mathrm{d}V_{g_0} = \frac{1}{r_0^n} |V^{r_0} \backslash V|_{g_0} \leqslant \frac{1}{r_0^n} |V^{r_0}|_{g_0},$$

and then

$$\int_M |\nabla^g u_V|^2 \rho^\alpha \mathrm{d}V_g \leqslant \frac{|V^{r_0}|_{g_0}^{\frac{2}{n}} \mu(V^{r_0})^\alpha \, |V^{r_0}|_g^{\frac{n-2}{n} - \alpha}}{r_0^2}.$$

We conclude that

$$R_{(g,\rho,\rho^\alpha)}(u_V) \leqslant B_n \frac{|V^{r_0}|_{g_0}^{\frac{2}{n}} \mu(V^{r_0})^\alpha |V^{r_0}|_g^{\frac{n-2}{n} - \alpha}}{\mu(M)} (k+1), \tag{3.2.8}$$

where $B_n = \frac{c^2}{r_0^2}$ depends only on n. Thus we were able to bound the two Rayleigh quotients by some quantities. We are going to show that these quantities can actually be bounded in the following way.

We use the next lemma, whose proof is given at the end of this section.

Lemma 3.2.1. *Let M be a Riemannian manifold and let ν_1, ν_2, ν_3 be any measures on M. Take a collection of K disjoint open subsets $(U_i)_i$ in M. If $K \geqslant 4k+1$ for some $k \in \mathbb{N}^*$, then there exist $k+1$ open subsets in this collection such that they satisfy the three conditions below:*

$$\nu_1(U_i) \leqslant \frac{\nu_1(M)}{k+1}, \qquad \nu_2(U_i) \leqslant \frac{\nu_2(M)}{k+1} \quad \text{and} \quad \nu_3(U_i) \leqslant \frac{\nu_3(M)}{k+1}.$$

As the $4k+4$ sets G_j are disjoint, we deduce there exist $k+1$ sets among them satisfying the following three inequalities:

$$|G_j|_{g_0} \leqslant \frac{|M|_{g_0}}{k+1}, \qquad |G_j|_g \leqslant \frac{|M|_g}{k+1} \quad \text{and} \quad \mu(G_j) \leqslant \frac{\mu(M)}{k+1}.$$

Using the previous estimates of the Rayleigh quotients (3.2.7) and (3.2.8), we obtain $k+1$ disjointly supported functions u_j, $j = 1, \dots, k+1$, with Rayleigh quotients satisfying either

$$\begin{aligned}
R_{(g,\rho,\rho^\alpha)}(u_j) &\leqslant \left(\tilde{A}_n\right)^{\frac{2}{n}} c^2 \frac{\mu(G_j)^\alpha |G_j|_g^{\frac{n-2}{n}-\alpha}}{\mu(M)}(k+1) \\
&\leqslant \left(\tilde{A}_n\right)^{\frac{2}{n}} c^2 \frac{\mu(M)^\alpha |M|_g^{\frac{n-2}{n}-\alpha}}{(k+1)^\alpha (k+1)^{\frac{n-2}{n}-\alpha}\mu(M)}(k+1) \\
&= \left(\tilde{A}_n\right)^{\frac{2}{n}} c^2 \left(\frac{k+1}{|M|_g}\right)^{\frac{2}{n}} \\
&\leqslant A_n \left(\frac{k}{|M|_g}\right)^{\frac{2}{n}},
\end{aligned}$$

or

$$\begin{aligned}
R_{(g,\rho,\rho^\alpha)}(u_j) &\leqslant B_n \frac{|G_j|_{g_0}^{\frac{2}{n}} \mu(G_j)^\alpha |G_j|_g^{\frac{n-2}{n}-\alpha}}{\mu(M)}(k+1) \\
&\leqslant B_n \frac{|M|_{g_0}^{\frac{2}{n}} \mu(M)^\alpha |M|_g^{\frac{n-2}{n}-\alpha}}{(k+1)^{\frac{2}{n}}(k+1)^\alpha (k+1)^{\frac{n-2}{n}-\alpha}\mu(M)}(k+1) \\
&= B_n \left(\frac{|M|_{g_0}}{|M|_g}\right)^{\frac{2}{n}}.
\end{aligned}$$

Consequently $\lambda_k^g(\rho,\rho^\alpha)$ is bounded above and

$$\lambda_k^g(\rho,\rho^\alpha)|M|_g^{\frac{2}{n}} \leqslant A_n k^{\frac{2}{n}} + B_n |M|_{g_0}^{\frac{2}{n}}.$$

□

Proof. of Lemma 3.2.1

First, notice that at most k subsets are such that $\nu_1(U_i) > \frac{\nu_1(M)}{k+1}$. Indeed, assume there exist $k+1$ subsets verifying $\nu_1(U_i) > \frac{\nu_1(M)}{k+1}$. Then the volume for ν_1 of these (disjoint) subsets would be greater than the volume of M, which is a contradiction.

Now we know we can work on a collection of $K-k$ sets U_i satisfying $\nu_1(U_i) \leqslant \frac{\nu_1(M)}{k+1}$. We repeat the idea to take $K-2k$ sets from this collection that satisfy $\nu_1(U_i) \leqslant \frac{\nu_1(M)}{k+1}$ and $\nu_2(U_i) \leqslant \frac{\nu_2(M)}{k+1}$. We repeat again, and finally extract from these the $K-3k$ smallest sets for the measure ν_3. As $K \geqslant 4k+1$, we have $K-3k \geqslant k+1$, which finishes the proof.

□

3.3 Construction of densities with large λ_1

Now that we have seen we can bound $\lambda_k^g(\rho,\rho^\alpha)$ for $\alpha \in \left(0, \frac{n-2}{n}\right)$, we are going to show that for $\alpha \in \left(\frac{n-2}{n}, 1\right)$, the supremum $\lambda_{1,\alpha}^*(M,g)$ can be equal to $+\infty$ for a certain type of manifold. The reader can refer to the definition 1.6.3 of a manifold of revolution.

Theorem 3.3.1. *Let Ω be a manifold of revolution of dimension $n \geqslant 2$. If $\alpha \in \left(\frac{n-2}{n}, 1\right)$, then*

$$\lambda_{k,\alpha}^*(\Omega) = +\infty.$$

Proof. Without loss of generality, we assume $0 \in \Omega$. For all $m \geqslant 1$, we define the radial density function ρ_m by

$$\rho_m(x) = e^{-m|x|^2}.$$

As $\rho_m \leqslant 1$ and $\alpha < 1$, we get $\rho_m^\alpha \geqslant \rho_m$. Then

$$\lambda_1(\rho_m, \rho_m^\alpha) \geqslant \lambda_1(\rho_m, \rho_m).$$

According to A. Savo, A. El Soufi, and B. Colbois (see [CES15, Theorem 5.2]), we know that in dimension larger than 2, there exists an m_0 such that for all $m \geqslant m_0$, $\lambda_1(\rho_m, \rho_m) \geqslant m$. Thus for $m \geqslant m_0$,

$$\lambda_1(\rho_m, \rho_m^\alpha) \geqslant m. \tag{3.3.9}$$

Let $\tilde{\rho}_m = \frac{\rho_m |\Omega|}{\int_\Omega \rho_m \mathrm{d}x}$. It is clear that $\tilde{\rho}_m$ is a continuous bounded function on Ω with $\int_\Omega \tilde{\rho}_m \mathrm{d}x = |\Omega|$. Thanks to the variational characterisation (3.1.1), we get

$$\lambda_1(\tilde{\rho}_m, \tilde{\rho}_m^\alpha) = \inf_{u\in E_1} \frac{\int_\Omega |\nabla u|^2 \left(\frac{\rho_m|\Omega|}{\int_\Omega \rho_m \mathrm{d}x}\right)^\alpha \mathrm{d}x}{\int_\Omega u^2 \left(\frac{\rho_m|\Omega|}{\int_\Omega \rho_m \mathrm{d}x}\right) \mathrm{d}x} = \lambda_1(\rho_m, \rho_m^\alpha) \left(\frac{1}{|\Omega|}\int_\Omega \rho_m \mathrm{d}x\right)^{1-\alpha}. \tag{3.3.10}$$

Using (3.3.9) and (3.3.10), we get

$$\lambda_1(\tilde{\rho}_m, \tilde{\rho}_m^\alpha) \geqslant m \left(\frac{1}{|\Omega|}\int_\Omega \rho_m \mathrm{d}x\right)^{1-\alpha}.$$

It remains to estimate $\left(\int_\Omega \rho_m \mathrm{d}x\right)^{1-\alpha}$.

Lemma 3.3.2. *For m large enough,*

$$\int_\Omega e^{-m|x|^2} \mathrm{d}x \geqslant e^{-n} m^{-\frac{n}{2}}.$$

Proof. As 0 is in Ω, there exists $L > 0$ such that Ω contains the n-square $(-L, L)^n$.

Therefore,

$$\int_\Omega e^{-m|x|^2} \mathrm{d}x \geqslant \left(\int_{-L}^{L} e^{-mt^2} \mathrm{d}t\right)^n.$$

Notice that $t \mapsto e^{-mt^2}$ is a decreasing function on $(0, L)$. Moreover this function takes the value e^{-1} if $t = m^{-\frac{1}{2}}$. We deduce that

$$\int_{-L}^{L} e^{-mt^2} \mathrm{d}t \geqslant e^{-1} m^{-\frac{1}{2}},$$

and the lemma is proved. One can refer to the figure 3.3 for a visual intuition.

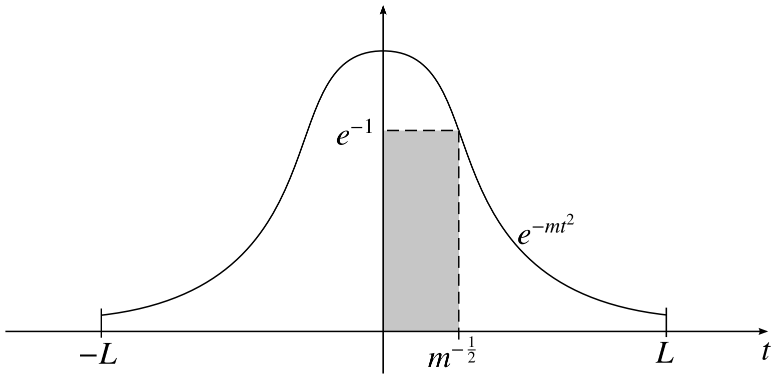

Figure 3.3: Minoration of the integral by the area of the rectangle

□

Thanks to this lemma, we finally obtain

$$\lambda_1(\tilde{\rho}_m, \tilde{\rho}_m^\alpha) \geqslant \frac{e^{-n(1-\alpha)}}{|\Omega|^{1-\alpha}} m^{1-\frac{n}{2}(1-\alpha)}.$$

Since $\alpha > \frac{n-2}{n}$, then $1 - \frac{n}{2}(1-\alpha) > 0$ which means that $\lambda_1(\tilde{\rho}_m, \tilde{\rho}_m^\alpha) \underset{m\to\infty}{\longrightarrow} \infty$ and the proof is complete. □

In the proposition below, we will show that in dimension 1, the previous result (Theorem 3.3.1) holds true for $\alpha \in (0, 1)$.

Proposition 3.3.3. *Let us take $M = (-1, 1)$. Then for all $\alpha \in (0, 1)$,*

$$\lambda_{1,\alpha}^*(M) = +\infty.$$

Proof. The above equation

$$-\mathrm{div}(\rho^\alpha \nabla u) = \lambda \rho u$$

with Neumann boundary conditions on $(-1, 1)$ becomes

$$\rho^{\alpha-1} u'' + \alpha \rho^{\alpha-2} \rho' u' + \lambda u = 0$$

$$\rho^{\alpha-1} u'' + \frac{\alpha}{\alpha - 1} \left(\rho^{\alpha-1}\right)' u' + \lambda u = 0.$$

We can differentiate to obtain

$$\rho^{\alpha-1} u''' + \left(\rho^{\alpha-1}\right)' u'' + \frac{\alpha}{\alpha - 1} \left(\rho^{\alpha-1}\right)' u'' + \frac{\alpha}{\alpha - 1} \left(\rho^{\alpha-1}\right)'' u' + \lambda u' = 0.$$

Now we define $y = u'$ and the equation (3.1.2) becomes

$$\begin{cases} \rho^{\alpha-1}y'' + \dfrac{2\alpha-1}{\alpha-1}\left(\rho^{\alpha-1}\right)' y' + \left(\dfrac{\alpha}{\alpha-1}\left(\rho^{\alpha-1}\right)'' + \lambda\right) y = 0 \quad \text{in} \quad (-1,1) \\ y(1) = y(-1) = 0 \end{cases}$$

Remark that if we multiply by ρ^α we get

$$\rho^{2\alpha-1}y'' + (2\alpha-1)\rho^{2\alpha-2}\rho' y' + \left(\frac{\alpha}{\alpha-1}\left(\rho^{\alpha-1}\right)'' + \lambda\right)\rho^\alpha y = 0,$$

i.e.

$$\left(\rho^{2\alpha-1}y'\right)' + \left(\lambda - \frac{\alpha}{1-\alpha}\left(\rho^{\alpha-1}\right)''\right)\rho^\alpha y = 0. \tag{3.3.11}$$

Now let m be a positive integer. We want to choose $\rho = \rho_m$ such that $\lambda_1\left(\rho_m, \rho_m^\alpha\right) \geqslant m$. To do this we first solve the equation $\frac{\alpha}{1-\alpha}\left(\rho^{\alpha-1}\right)'' = m$ which admits (at least) one positive solution on $(-1,1)$. We choose the density such that $\rho_m(x)^{\alpha-1} = \frac{1-\alpha}{2\alpha}\left(1 + mx^2\right)$, that is,

$$\rho_m(x) = \left(\frac{2\alpha}{1-\alpha}\right)^{\frac{1}{1-\alpha}} \frac{1}{\left(1+mx^2\right)^{\frac{1}{1-\alpha}}}.$$

Now we multiply the equation (3.3.11) by y and integrate it by parts to get

$$\int_{-1}^{1}\left(\rho_m^{2\alpha-1}y'\right)' y\mathrm{d}x + (\lambda - m)\int_{-1}^{1}\rho_m^\alpha y^2\mathrm{d}x = 0,$$

$$\left[\rho_m^{2\alpha-1}y'y\right]_{-1}^{1} - \int_{-1}^{1}\rho_m^{2\alpha-1}(y')^2\mathrm{d}x + (\lambda - m)\int_{-1}^{1}\rho_m^\alpha y^2\mathrm{d}x = 0.$$

But we know that $y = u'$ vanishes at -1 and 1. We obtain the following

$$(\lambda - m)\int_{-1}^{1}\rho_m^\alpha y^2\mathrm{d}x = \int_{-1}^{1}\rho_m^{2\alpha-1}(y')^2\mathrm{d}x \geqslant 0.$$

$$\text{So } \lambda = \lambda\left(\rho_m, \rho_m^\alpha\right) \geqslant m.$$

Again, we use the idea of Lemma 3.3.2 to see that the normalised eigenvalue is not bounded:

$$\lambda_1(\tilde{\rho}_m, \tilde{\rho}_m^\alpha) \geqslant m \cdot m^{-\frac{1}{2}} = m^{\frac{1}{2}}.$$

The number m being arbitrarily large, this concludes the proof. □

Chapter 4

Bounding the Steklov eigenvalues in $\mathbb{H}^{n+1}$

4.1 Introduction

Let M be a compact connected Riemannian manifold with a smooth boundary, and consider the following Steklov problem:

$$\begin{cases} \Delta u &= 0 \quad \text{in } M \\ \partial_\eta u &= \sigma u \quad \text{on } \partial M. \end{cases} \tag{4.1.1}$$

As we have seen in the introduction, the spectrum of M is non-negative, discrete, and tends to infinity as k goes to infinity.

$$0 = \sigma_0(M) < \sigma_1(M) \leqslant \cdots \leqslant \sigma_k(M) \leqslant \cdots \nearrow +\infty.$$

This spectrum matches with the set of eigenvalues of a pseudo-differential operator, called the *Dirichlet-to-Neumann map*. For any function u defined on ∂M, this map associates the normal derivative of its harmonic extension. See [GP17] or [Ban80, p. 95] for details.

In the preliminaries, we saw that obtaining bounds of the form $\sigma_k(M) \leqslant \mathrm{geom}(M)k^{\frac{1}{n-1}}$, is difficult, and only a few results have been found in this direction. This chapter is more modest, as we will find bounds of the form $\sigma_k(M) \leqslant \mathrm{geom}(M)k^{\frac{2}{n-1}}$. We will focus on generalising some results of an article by Colbois, Girouard and Gittins [CGG18], but instead of embedding M into $\mathbb{R}^{n+1}$, we embed it into $\mathbb{H}^{n+1}$. We will use three different techniques, although the main ingredient is the use of Lemma 1.2.5 page 8, that enables us to find sets "heavy enough" but also "far enough" from each other.

The work we present here carries some inherent difficulties coming from the hyperbolic space. Namely, problems of volume growth, and distances. Indeed, because we want to build domains in which we define test functions, we will have to make sure the balls are not "too big". We will also have to check that these domains are disjoint. This can be achieved by controlling the distortion (see first section), or by using the properties of the space in which the boundary is embedded (see second and third sections).

In the last two sections, and contrary to the result cited above, we will make an assumption on the boundary of M, that also has to be the boundary of a domain of $\mathbb{H}^n$ or $\mathbb{R}^n$. This will enable us to use isoperimetric inequalities, that will automatically ensure we have enough "space" on the boundary to work with.

The section 4.2 is dedicated to a general result, that adapts the proof of a theorem in the article cited above in the hyperbolic space. The proof can be easily adapted to a manifold embedded in a complete space whose Ricci curvature is non-positive and bounded from below. If the curvature of the space is positive, and as the proof needs a control on the isoperimetric ratio of the space, the result will be true, for instance, when M is embedded in the hemisphere.

The sections 4.3 and 4.4 concentrate on more specific problems. We will suppose that the boundary of M is a connected submanifold of $\mathbb{H}^{n+1}$, which also realises the boundary of a domain of $\{0\} \times \mathbb{H}^n$ or $\mathbb{R}^n \times \{1\}$.

Remark 4.1.1. The same questions could be asked in a more general context, for example in the case of minimal submanifolds: given Σ, a submanifold of dimension $m-1$ of $\mathbb{R}^n$, and a submanifold Ω of dimension m of $\mathbb{R}^n$, realising the minimal volume inside Σ. Consider a submanifold manifold M of $\mathbb{R}^n$ such that $\partial M = \Sigma$. Can we bound the eigenvalues of M with the help of Ω? If so, can we obtain a Kröger type result, and what geometrical quantities are involved in the inequality? However, this question is still open and will not be treated.

4.2 A general case

In this section we embed Σ in $\mathbb{H}^m$. Notice that the proof of this theorem gives the main tools that will be important in the last two sections. The theorem can be formulated as follows:

Theorem 4.2.1.
Let Σ be a smooth and compact submanifold of dimension $n-1$ in $\mathbb{H}^m$, and let $M \subset \mathbb{H}^m$ be a smooth manifold with boundary Σ. Then for all $k \geqslant 1$,

$$\sigma_k(M) \leqslant \frac{|M|}{|\Sigma|} \left(A_0(n,m) + A(\Sigma)\, k^{\frac{2}{n-1}} \right),$$

where $A_0(n,m)$ only depends on the dimensions n and m, and $A(\Sigma)$ depends on n, on m, on a lower bound for the Ricci curvature of Σ, on its volume $|\Sigma|$, on its maximal distortion γ (see definition 4.2.2), and on its number b of connected components.

Definition 4.2.2. The distortion of a connected submanifold $\Sigma \subset \mathbb{H}^m$ is defined by:

$$\mathrm{disto}(\Sigma) := \sup_{x \neq y \in \Sigma} \frac{\mathrm{dist}_\Sigma(x,y)}{\mathrm{dist}_{\mathbb{H}^m}(x,y)}.$$

Proof of Theorem 4.2.1. The beginning of this proof is very similar to the one of [CGG18, Theorem 1.1]. Let $\Sigma_1, \ldots, \Sigma_b$ be the connected components of Σ. As Σ is compact, the number

$$\gamma := \max_{1 \leqslant i \leqslant b} \mathrm{disto}(\Sigma_i)$$

is well defined. Now let us define a measure μ supported on Σ: for any Borel set $\mathscr{U} \subset \mathbb{H}^m$, we define μ as:

$$\mu(\mathscr{U}) = \int_{\mathscr{U} \cap \Sigma} \mathrm{d}V_\Sigma.$$

We now claim the following lemma:

Lemma 4.2.3. *There exists $\delta > 0$ such that for all $r \leqslant 1$,*

$$\mu(B_{\mathbb{H}^m}(x,r)) \leqslant \delta r^{n-1},$$

and δ depends on n, b, γ, and on a lower bound for the Ricci curvature of Σ.

Proof. For all $x \in \mathbb{H}^m$, there exists $x_i \in \Sigma_i$ such that for all $r > 0$,

$$B_{\mathbb{H}^m}(x,r) \cap \Sigma \subset \bigcup_{i=1}^{b} B_{\Sigma_i}(x_i, 2\gamma r). \tag{4.2.2}$$

To see that, we first define for all i, a point x_i in Σ_i realising the distance from x to Σ_i:

$$\mathrm{dist}_{\mathbb{H}^m}(x, x_i) = \mathrm{dist}_{\mathbb{H}^m}(x, \Sigma_i).$$

Remark that if $B_{\mathbb{H}^m}(x,r) \cap \Sigma = \emptyset$, inclusion (4.2.2) is trivially true. If $B_{\mathbb{H}^m}(x,r) \cap \Sigma \neq \emptyset$, let us take $y \in B_{\mathbb{H}^m}(x,r) \cap \Sigma$. By definition, there exists j such that $y \in \Sigma_j$. We have:

$$\mathrm{dist}_{\Sigma_j}(y, x_j) \leqslant \gamma\, \mathrm{dist}_{\mathbb{H}^m}(y, x_j) \leqslant \gamma \left(\mathrm{dist}_{\mathbb{H}^m}(y,x) + \mathrm{dist}_{\mathbb{H}^m}(x, x_j)\right) \leqslant 2\gamma r,$$

from which we deduce $y \in B_{\Sigma_j}(x_j, 2\gamma r) \subset \bigcup_{i=1}^{b} B_{\Sigma_i}(x_i, 2\gamma r)$, and inclusion (4.2.2) is true for all $r > 0$. The volume of the balls $B_{\mathbb{H}^m}(x,r)$ for the measure μ is thus bounded in the following way:

$$\mu\left(B_{\mathbb{H}^m}(x,r)\right) = \mu\left(B_{\mathbb{H}^m}(x,r) \cap \Sigma\right) \leqslant b \max_i \mu\left(B_{\Sigma_i}(x_i, 2\gamma r)\right) = b \max_i \left|B_{\Sigma_i}(x_i, 2\gamma r)\right|.$$

As Σ is compact, it has its Ricci curvature bounded, so there exists some $\eta > 0$ such that each ball of Σ of radius $R \leqslant 2\gamma$ is bounded from below in the following way:

$$|B_\Sigma(R)| \leqslant \eta R^{n-1}, \tag{4.2.3}$$

where η depends on n, on the maximal distortion γ, and on a lower bound for the Ricci curvature of Σ.

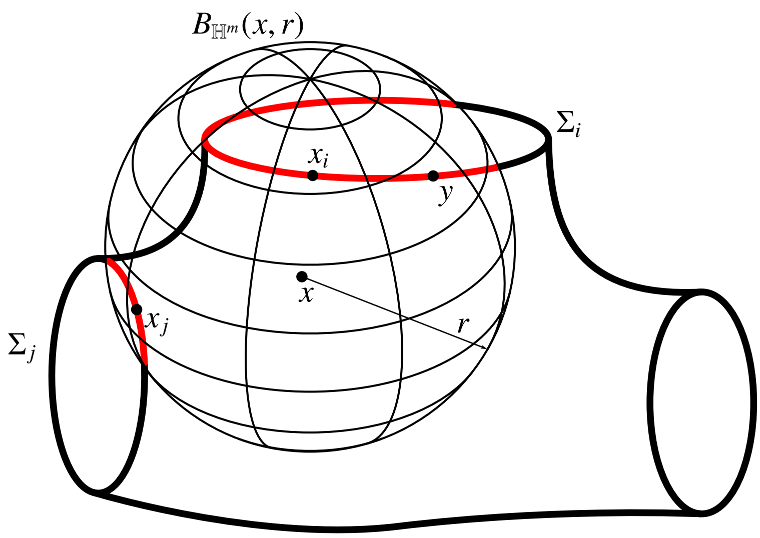

Figure 4.1: In red: intersection of $B_{\mathbb{H}^m}(x,r)$ with Σ

From that we deduce that the volume of the balls of $\mathbb{H}^m$ for the measure μ is also bounded. More precisely we say there exists $\delta > 0$ such that for all $r > 0$,

$$\mu(B_{\mathbb{H}^m}(x, r)) \leqslant \delta r^{n-1},$$

where δ depends on n, b, γ, and on a lower bound for the Ricci curvature of Σ. □

The hyperbolic space $\mathbb{H}^m$ has constant sectional curvature -1. We know, thanks to the Packing Lemma 1.2.3, that there exists a quantity $C(m, r)$ such that any ball of radius r can be covered by $C(m, r)$ balls of radius $\frac{r}{2}$, and $r \mapsto C(m, r)$ is increasing. For simplicity, we will just designate this quantity by $C(r)$ in the following steps.

Let us define the quantity

$$r_k = \left(\frac{|\Sigma|}{8C(1)^2(k+1)\delta}\right)^{\frac{1}{n-1}},$$

and $k_0 = \left\lfloor \frac{|\Sigma|}{8C(1)^2\delta} \right\rfloor$. Consequently we have $k_0 + 1 \geqslant \frac{|\Sigma|}{8C(1)^2\delta}$, thus for $k \geqslant k_0$,

$$r_k \leqslant \left(\frac{k_0+1}{k+1}\right)^{\frac{1}{n-1}} \leqslant 1.$$

By definition, $|\Sigma| = \mu(\mathbb{H}^m)$. Moreover, $r \mapsto C(r)$ is an increasing function, so for all $x \in \mathbb{H}^m$,

$$\mu(B_{\mathbb{H}^m}(x, r_k)) \leqslant \delta r_k^{n-1} = \frac{\mu(\mathbb{H}^m)}{4C(1)^2(2k+2)} \leqslant \frac{\mu(\mathbb{H}^m)}{4C(r_k)^2(2k+2)}.$$

All the ingredients are there to apply Lemma 1.2.5 , which ensures the existence of $2k + 2$ sets $A_1, \dots, A_{2k+2}$ of $\mathbb{H}^m$ verifying

$$\mu(A_i) \geqslant \frac{|\Sigma|}{4C(r_k)(k+1)} \quad \text{and for } i \neq j,\ \mathrm{dist}_{\mathbb{H}^m}(A_i, A_j) \geqslant 3r_k.$$

In particular, let us remark that by definition of the measure μ, the A_i's intersect Σ and M.

Now, we define the "r_k-neighbourhood" of A_i as

$$A_i^{r_k} = \{x \in \mathbb{H}^m \text{ s.t. } \mathrm{dist}_{\mathbb{H}^m}(x, A_i) < r_k\} \subset \mathbb{H}^m.$$

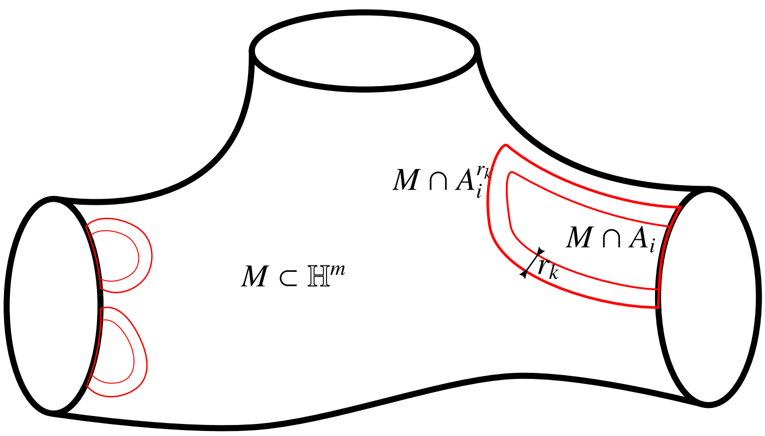

Figure 4.2: Construction of the domains on M

The $\left(M \cap A_i^{r_k}\right)_{i=1}^{2k+2}$ being disjoint, we pick the $k+1$ smallest, which verify for all $i \leqslant k+1$,

$$|M \cap A_i^{r_k}| \leqslant \frac{|M|}{k+1}.$$

Now we build the test functions g_i with support in $A_i^{r_k}$, defined for all $x \in A_i^{r_k}$ by

$$g_i(x) = 1 - \frac{\mathrm{dist}_{\mathbb{H}^m}(x, A_i)}{r_k}.$$

In $\mathbb{H}^m$, we have $|\nabla \mathrm{dist}_{\mathbb{H}^m}|_{\mathbb{H}^m} = 1$. Thus in M we have $|\nabla g_i|_M = \frac{1}{r_k}|\nabla \mathrm{dist}_{\mathbb{H}^m}|_M \leqslant \frac{1}{r_k}$. The energy will then be bounded in the following way:

$$\int_M |\nabla g_i|_M^2 \mathrm{d}V_M = \int_{M \cap A_i^{r_k}} |\nabla g_i|_M^2 \mathrm{d}V_M \leqslant \frac{|M \cap A_i^{r_k}|}{r_k^2} \leqslant \frac{|M|}{(k+1)r_k^2}.$$

Finally, the Rayleigh quotient is bounded as follows:

$$\begin{aligned} R(g_i) = \frac{\int_M |\nabla g_i|_M^2 \mathrm{d}V_M}{\int_\Sigma g_i^2 \mathrm{d}V_\Sigma} &\leqslant \frac{|M|}{(k+1)r_k^2}\frac{1}{\mu(A_i)} \\ &\leqslant \frac{|M|}{(k+1)} \left(\frac{8C(1)^2(k+1)\delta}{|\Sigma|}\right)^{\frac{2}{n-1}} \frac{4C(r_k)(k+1)}{|\Sigma|} \\ &\leqslant \frac{|M|}{|\Sigma|} \left(\frac{8C(1)^2(k+1)\delta}{|\Sigma|}\right)^{\frac{2}{n-1}} 4C(1) \\ &\leqslant \frac{|M|}{|\Sigma|} \left(\frac{16C(1)^2\delta k}{|\Sigma|}\right)^{\frac{2}{n-1}} 4C(1). \end{aligned}$$

The previous estimate is true for $k \geqslant k_0$, and in particular for $k = k_0$. But we know that by definition, $k_0 \leqslant \frac{|\Sigma|}{8C(1)^2\delta}$. We deduce

$$\sigma_{k_0}(M) \leqslant \frac{|M|}{|\Sigma|} \left(\frac{16C(1)^2\delta k_0}{|\Sigma|}\right)^{\frac{2}{n-1}} 4C(1) \leqslant \frac{|M|}{|\Sigma|} \left(\frac{16C(1)^2\delta}{|\Sigma|} \frac{|\Sigma|}{8C(1)^2\delta}\right)^{\frac{2}{n-1}} 4C(1) = A_0(n,m)\frac{|M|}{|\Sigma|}.$$

For all $k \geqslant 1$, we thus obtain:

$$\sigma_k(M) \leqslant \frac{|M|}{|\Sigma|} \left(A_0(n,m) + A(\Sigma)\, k^{\frac{2}{n-1}}\right),$$

where $A_0(n,m)$ only depends on the dimensions n and m, and

$$A(\Sigma) = \left(\frac{16C(1)^2 b\eta(2\gamma)^{n-1}}{|\Sigma|}\right)^{\frac{2}{n-1}} 4C(1)$$

depends on n, on m, on a lower bound for the Ricci curvature of Σ, on its volume $|\Sigma|$, on its maximal distortion γ, and on its number b of connected components.

$\square$

4.3 Embedding the Euclidean space $\mathbb{R}^n$ horizontally into $\mathbb{H}^{n+1}$

We generalise Theorem 1.5 of [CGG18] to the following setting. We first study a case when the domain Ω is embedded "horizontally" in $\mathbb{H}^{n+1}$, that is, $\Omega \subset \mathbb{R}^n \times \{1\}$. We prove the following result:

Theorem 4.3.1.
Let Σ be an $(n-1)$-dimensional, connected, smooth hypersurface in $\mathbb{R}^n \times \{1\} \subset \mathbb{H}^{n+1}$ and let $\Omega \subset \mathbb{R}^n$ denote a domain with boundary $\partial\Omega = \Sigma$. Then, for every manifold M such that $\partial M = \Sigma$, for each $k \geqslant 1$,

$$\sigma_k(M) \leqslant |M|\nu(n) \left(\frac{1}{|\Omega|^{\frac{n-1}{n}}} + \frac{k^{\frac{2}{n-1}}}{|\Omega|^{\frac{n+1}{n}}} \right),$$

where $\nu(n)$ is a constant that only depends on n.

The next propositions will be useful:

Proposition 4.3.2.
For every ball of $\mathbb{R}^n$ of center x and radius r,

$$|B(x,r)| = \omega_n r^n \qquad |\partial B(x,r)| = \rho_n r^{n-1}$$

with

$$\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \qquad \rho_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

The well known isoperimetric inequality in $\mathbb{R}^n$ can be found in [Bar03]:

Proposition 4.3.3. *Isoperimetric inequality in $\mathbb{R}^n$.*
For any domain Ω of $\mathbb{R}^n$,

$$|\partial\Omega| \geqslant \frac{|\Omega|^{\frac{n-1}{n}}}{I(n)},$$

where $I(n) = \left(n\omega_n^{\frac{1}{n}}\right)^{-1}$.

Proposition 4.3.4.
For all points $x, y \in \mathbb{R}^n \subset \mathbb{H}^{n+1}$,

$$\mathrm{dist}_{\mathbb{H}^{n+1}}(x,y) = 2\sinh^{-1}\left(\frac{1}{2}\mathrm{dist}_{\mathbb{R}^n}(x,y)\right).$$

Proof.
The geodesic joining two points x and y in the horizontal hyperplane $\mathbb{R}^n \times \{1\} \subset \mathbb{H}^{n+1}$ is a circular arc. Without loss of generality, we can assume that this geodesic is in the plane $x_1 = \cdots = x_{n-1} = 0$, and that the circle has its centre situated at 0. We will denote the points as $x = (0, \ldots, -e, 1)$ and $y = (0, \ldots, e, 1)$ (the number e is equal to $\frac{1}{2}\mathrm{dist}_{\mathbb{R}^n}(x,y)$). Consequently

the equation of the geodesic is $x_{n+1} = \sqrt{1+e^2-x_n^2}$, thus $\mathrm{d}x_{n+1} = -\frac{x_n}{\sqrt{1+e^2-x_n^2}}\mathrm{d}x_n$ and $\mathrm{d}x_{n+1}^2 = \frac{x_n^2}{x_{n+1}^2}\mathrm{d}x_n^2$. The integration over the geodesic gives:

$$\begin{aligned}
\mathrm{dist}_{\mathbb{H}^{n+1}}(x,y) &= \int \frac{\sqrt{\mathrm{d}x_n^2+\mathrm{d}x_{n+1}^2}}{x_{n+1}} = \int_{-e}^{e} \frac{\sqrt{1+\frac{x_n^2}{x_{n+1}^2}}}{x_{n+1}}\mathrm{d}x_n \\
&= \int_{-e}^{e} \frac{\sqrt{x_n^2+x_{n+1}^2}}{x_{n+1}^2}\mathrm{d}x_n = \int_{-e}^{e} \frac{\sqrt{1+e^2}}{1+e^2-x_n^2}\mathrm{d}x_n \\
&= 2\tanh^{-1}\left(\frac{e}{\sqrt{1+e^2}}\right) = 2\sinh^{-1}(e) = 2\sinh^{-1}\left(\frac{1}{2}\mathrm{dist}_{\mathbb{R}^n}(x,y)\right).
\end{aligned}$$

□

Proof of Theorem 4.3.1.

First step.

We first want to build domains where the test functions will be defined. To that effect we define the following quantities:

$$P(n) = \rho_n I(n) + (4\omega_n)^{\frac{n-1}{n}} \qquad D = \frac{\sinh(5)}{5} > 1 \qquad k_0 = \left\lfloor \frac{|\Omega|^{\frac{n-1}{n}}}{4P(n)} \right\rfloor$$

and the radii:

$$r_k = \frac{1}{4D}\left(\frac{|\Omega|^{\frac{n-1}{n}}}{4(2k+2)P(n)}\right)^{\frac{1}{n-1}} \quad \text{and} \quad \overline{r_k} = \frac{1}{4D}\left(\frac{|\Omega|}{2(2k+2)\omega_n}\right)^{\frac{1}{n}}.$$

Remark 4.3.5.

- We have $k_0 + 1 \geqslant \dfrac{|\Omega|^{\frac{n-1}{n}}}{4P(n)}$ and as $D > 1$, then for all $k \geqslant k_0$,

$$r_k \leqslant 1.$$

- Notice also that for all $k \in \mathbb{N}$,

$$\frac{r_k}{\overline{r_k}} = \frac{[2(2k+2)\omega_n]^{\frac{1}{n}}}{[4(2k+2)P(n)]^{\frac{1}{n-1}}} \leqslant \frac{\omega_n^{\frac{1}{n}}}{P(n)^{\frac{1}{n-1}}} \leqslant \frac{\omega_n^{\frac{1}{n}}}{[4\omega_n]^{\frac{1}{n}}} \leqslant 1.$$

  Thus for all $k \geqslant 0$,

$$r_k \leqslant \overline{r_k}.$$

In what follows, we consider $k \geqslant k_0$.

The idea is now to make sure we have enough "space" in Ω to define a sufficient number of disjoint domains on which we can define suitable test functions. Once this is done, the work is not over yet, because we will need to "transport" these functions on the corresponding sets in M. As we will see, the construction of this transportation is not as canonical as in [CGG18].

Let $\left(x^j\right)_{j=1}^{2k+2}$ be some collection of $2k+2$ points of Ω. Let Ω_0 and Σ_0 be the following subsets of $\mathbb{R}^n$:

$$\Omega_0 = \Omega \setminus \bigcup_{j=1}^{2k+2} B(x^j, 4Dr_k) \qquad \Sigma_0 = \Sigma \setminus \bigcup_{j=1}^{2k+2} B(x^j, 4Dr_k)$$

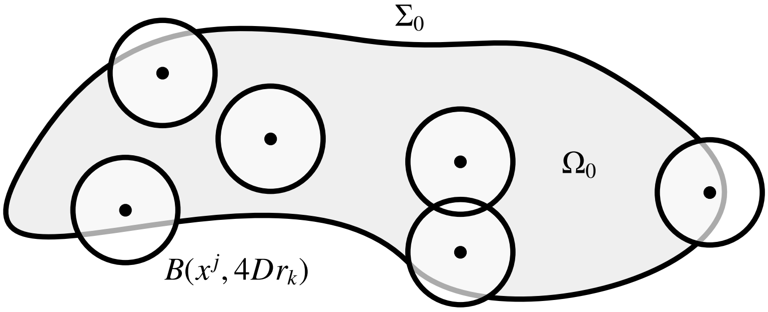

Figure 4.3: Illustration of the balls of radius $4Dr_k$

The purpose of what follows is to show that, due to volume considerations, there is no union of $(2k+2)$ balls of radius $4Dr_k$ that can either cover Ω or Σ. Because of proposition 4.3.2, we have

$$\begin{aligned}\left|\bigcup_{j=1}^{2k+2} B(x^j, 4Dr_k)\right| &\leqslant (2k+2)\left|B(x^j, 4Dr_k)\right| = (2k+2)\omega_n\,(4Dr_k)^n \\ &\leqslant (2k+2)\omega_n\,(4D\overline{r_k})^n = \frac{|\Omega|}{2}.\end{aligned}$$

That is, $|\Omega_0| \geqslant \frac{1}{2}|\Omega|$: the union of the balls does not cover Ω. We now want to show it cannot cover Σ. Again, thanks to proposition 4.3.2, and because $\frac{\rho_n}{P(n)} \leqslant \frac{1}{I(n)}$,

$$\begin{aligned}\left|\bigcup_{j=1}^{2k+2} \partial B(x^j, 4Dr_k)\right| &\leqslant (2k+2)\left|\partial B(x^j, 4Dr_k)\right| = (2k+2)\rho_n\,(4Dr_k)^{n-1} \\ &= (2k+2)\rho_n\left(\frac{|\Omega|^{\frac{n-1}{n}}}{4(2k+2)P(n)}\right) = \frac{\rho_n|\Omega|^{\frac{n-1}{n}}}{4P(n)} \leqslant \frac{|\Omega|^{\frac{n-1}{n}}}{4I(n)}.\end{aligned}$$

As the inclusion $\partial\Omega_0 \subset \Sigma_0 \cup \left(\bigcup_{j=1}^{2k+2} \partial B(x^j, 4Dr_k)\right)$ holds, $|\partial\Omega_0| \leqslant |\Sigma_0| + \left|\bigcup_{j=1}^{2k+2} \partial B(x^j, 4Dr_k)\right|$. Thus, using the isoperimetric inequality 4.3.3,

$$\begin{aligned}|\Sigma_0| &\geqslant |\partial\Omega_0| - \left|\bigcup_{j=1}^{2k+2} \partial B(x^j, 4Dr_k)\right| \geqslant \frac{|\Omega_0|^{\frac{n-1}{n}}}{I(n)} - \frac{|\Omega|^{\frac{n-1}{n}}}{4I(n)} \\ &\geqslant \frac{|\Omega|^{\frac{n-1}{n}}}{2^{\frac{n-1}{n}} I(n)} - \frac{|\Omega|^{\frac{n-1}{n}}}{4I(n)} = \frac{|\Omega|^{\frac{n-1}{n}}}{I(n)}\left(\frac{1}{2^{\frac{n-1}{n}}} - \frac{1}{4}\right) \geqslant \frac{|\Omega|^{\frac{n-1}{n}}}{4I(n)} > 0.\end{aligned}$$

Second step.
Let us now define the measure μ as, for any Borelian set $\mathscr{U} \subset \mathbb{R}^n$,

$$\mu(\mathscr{U}) = \mu(\mathscr{U} \cap \Sigma) = \int_{\mathscr{U} \cap \Sigma} \mathrm{d}\Sigma.$$

We pick the $2k+2$ biggest balls of radius Dr_k for the measure μ, taken in a decreasing order. The construction of these disjoint balls is of course possible because of the previous step. More precisely, let us define the points $(x_i)_{i=1}^{2k+2}$ such that:

$$\mu(B(x_1, Dr_k)) = \sup_{x \in \Omega} \mu(B(x, Dr_k)),$$

and for all $j = 2, \ldots, 2k+2$,

$$\mu(B(x_j, Dr_k)) = \sup \left(\mu(B(x, Dr_k)); x \in \Omega \setminus \bigcup_{i=1}^{j-1} B(x_i, 4Dr_k) \right)$$

With this construction, the properties below are satisfied:

- The balls of radius $2Dr_k$ are all disjoint,
- $\mu(B(x_1, Dr_k)) \geqslant \cdots \geqslant \mu(B(x_{2k+2}, Dr_k))$,
- For all $x \in \Omega_0 = \Omega \setminus \bigcup_{j=1}^{2k+2} B(x_j, 4Dr_k)$,

$$\mu(B(x, Dr_k)) \leqslant \mu(B(x_{2k+2}, Dr_k)).$$

In the last two steps we separate two cases, as shown qualitatively in figure 4.4, where we say that $\mu(B(x_{2k+2}, Dr_k))$ is "large" or "small" if it is larger or smaller than some value given below.

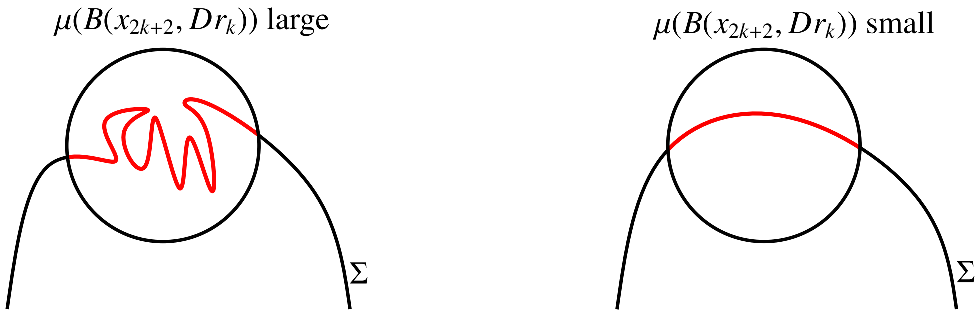

Figure 4.4: The two cases

Third step. We first need the following lemma, that ensures the fact that if two points are "far enough" from one another in $\mathbb{R}^n$, then they also are far from each other in $\mathbb{H}^{n+1}$.

Lemma 4.3.6.
For all $x, y \in \mathbb{R}^n \subset \mathbb{H}^{n+1}$,

$$\textit{if } \operatorname{dist}_{\mathbb{R}^n}(x, y) \geqslant 2Dr_k, \textit{ then } \operatorname{dist}_{\mathbb{H}^{n+1}}(x, y) \geqslant 2r_k.$$

This lemma is proved on page 53.

Let us first assume the volume of the ball $B(x_{2k+2}, Dr_k)$ is "large" for the measure μ; more precisely

$$\mu(B(x_{2k+2}, Dr_k)) \geqslant \frac{|\Omega|^{\frac{n-1}{n}}}{4I(n)} \cdot \frac{1}{4C(n)^2(2k+2)}. \tag{4.3.4}$$

Then we build our test functions on M. To do so, we first need adequate support sets on M. We define the following $2k+2$ sets E_j:

$$E_j = \left\{ x \in M \text{ s.t. } \operatorname{dist}_M\left(x, B(x_j, Dr_k) \cap \Sigma\right) \leqslant r_k \right\}.$$

It is important to notice that, as the balls in $\mathbb{R}^n$ are sufficiently far apart, then

$$\forall i \neq j, \qquad E_i \cap E_j = \varnothing.$$

Indeed, take $x \in B(x_i, Dr_k) \cap \Sigma$ and $y \in B(x_j, Dr_k) \cap \Sigma$, $i \neq j$. Then by construction of the balls in Ω, we have $\operatorname{dist}_{\mathbb{R}^n}(x, y) \geqslant 2Dr_k$. Thanks to Lemma 4.3.6, we get:

$$\operatorname{dist}_M(x, y) \underset{M \subset \mathbb{H}^{n+1}}{\geqslant} \operatorname{dist}_{\mathbb{H}^{n+1}}(x, y) \geqslant 2r_k.$$

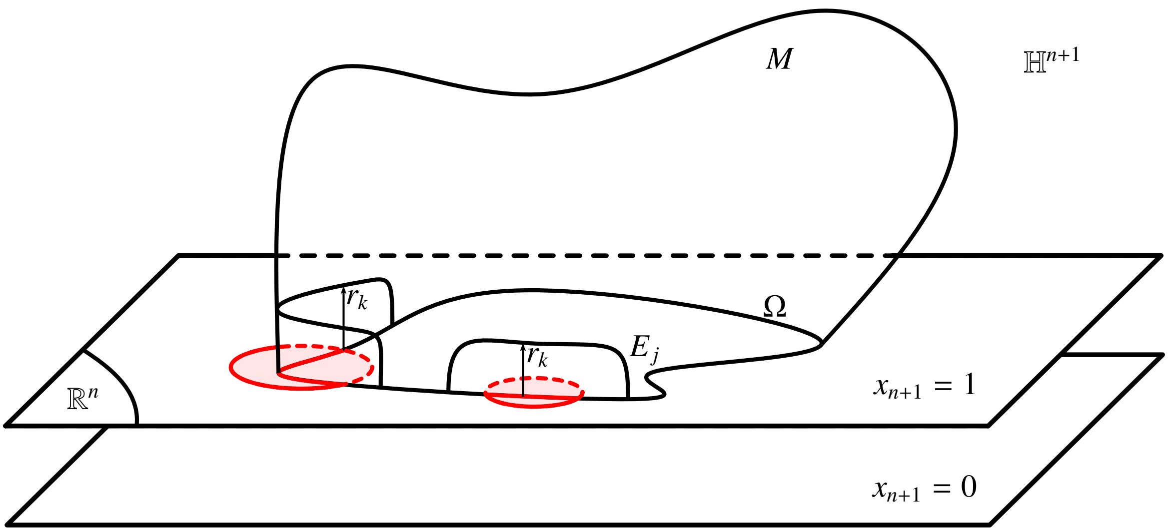

Figure 4.5: Illustration of the balls of $\Omega \subset \mathbb{R}^n$ and the sets $E_j \subset M$

The figure 4.5 helps to understand the construction. The balls of $\mathbb{R}^n$ are coloured in red, and have a red intersection with $\partial\Omega$. The sets $E_j \subset M$ are the points at distance r_k from this intersection. Now it is easy to see that $\operatorname{dist}_M(E_i, E_j) > 0$: the $2k+2$ sets E_j are **disjoint** in M. Consequently, the $k+1$ smallest of them verify

$$|E_j| \leqslant \frac{|M|}{k+1}.$$

On these $k+1$ sets we now define the functions f_j as follows:

$$f_j(x) = \min \left\{ 1, 2 - \frac{1}{r_k} \operatorname{dist}_M\left(x, B(x_j, r_k) \cap \Sigma\right) \right\},$$

so that on each E_j, we have $|\nabla f_j|_M^2 = \dfrac{1}{r_k^2}$. The numerator of the Rayleigh quotient is bounded:

$$\int_M |\nabla f_j|_M^2 = \int_{E_j} |\nabla f_j|_M^2 = \frac{|E_j|}{r_k^2} \leqslant \frac{|M|}{(k+1)r_k^2}.$$

From our assumption 4.3.4 we have, for all $j = 1, \dots, k+1$,

$$\mu(B(x_j, Dr_k)) \geqslant \frac{|\Omega|^{\frac{n-1}{n}}}{4I(n)} \cdot \frac{1}{4C(n)^2(2k+2)},$$

which leads to

$$\int_\Sigma f_j^2 \mathrm{d}\Sigma \geqslant \int_{B(x_j, Dr_k)\cap\Sigma} \mathrm{d}\Sigma = \mu(B(x_j, Dr_k)) \geqslant \frac{|\Omega|^{\frac{n-1}{n}}}{4I(n)} \cdot \frac{1}{4C(n)^2(2k+2)}.$$

The Rayleigh quotient can thus be bounded as follows:

$$\begin{aligned}
R(f_j) = \frac{\int_M |\nabla f_j|_M^2 \mathrm{d}V_g}{\int_\Sigma f_j^2 \mathrm{d}\Sigma} &\leqslant \frac{|M|}{(k+1)r_k^2 \left(\frac{|\Omega|^{\frac{n-1}{n}}}{4I(n)} \cdot \frac{1}{4C(n)^2(2k+2)} \right)} \\
&= \frac{|M|}{\frac{k+1}{16D^2} \left(\frac{|\Omega|^{\frac{n-1}{n}}}{4(2k+2)P(n)} \right)^{\frac{2}{n-1}} \left(\frac{|\Omega|^{\frac{n-1}{n}}}{4I(n)} \cdot \frac{1}{4C(n)^2(2k+2)} \right)} \\
&\leqslant \alpha(n) \frac{|M|}{|\Omega|^{\frac{n+1}{n}}} k^{\frac{2}{n-1}},
\end{aligned}$$

with $\alpha(n)$ a constant depending only on n.

Fourth step. For this last step we suppose that

$$\mu(B(x_{2k+2}, Dr_k)) \leqslant \frac{|\Omega|^{\frac{n-1}{n}}}{4I(n)} \cdot \frac{1}{4C(n)^2(2k+2)}. \tag{4.3.5}$$

This time, the ball $B(x_{2k+2}, Dr_k)$ is not "heavy" enough for the measure μ. By construction, for all $x \in \Omega_0$,

$$\mu(B(x, Dr_k)) \leqslant \mu(B(x_{2k+2}, Dr_k)) \leqslant \frac{|\Omega|^{\frac{n-1}{n}}}{4I(n)} \cdot \frac{1}{4C(n)^2(2k+2)} \leqslant \frac{|\Sigma_0|}{4C(n)^2(2k+2)}.$$

We now consider $\left(\mathbb{R}^n, \mathrm{dist}_{\mathbb{R}^n}, \mu_0\right)$ where μ_0 is the measure such that for any Borel set $\mathscr{U}$, we have $\mu_0(\mathscr{U}) = \mu_0(\mathscr{U} \cap \Sigma_0) = \int_{\mathscr{U}\cap\Sigma_0} \mathrm{d}\Sigma_0$. In particular, $\mu_0(\mathbb{R}^n) = |\Sigma_0|$. The last inequality thus implies:

$$\mu_0(B(x, Dr_k)) \leqslant \mu(B(x, Dr_k)) \leqslant \frac{\mu_0(\mathbb{R}^n)}{4C(n)^2(2k+2)}.$$

We use Lemma 1.2.5 in order to have the existence of domains of $\mathbb{R}^n$ verifying:

- $\mu_0(A_i) \geqslant \dfrac{\mu_0(\mathbb{R}^n)}{2C(n)(2k+2)}$,
- $\mathrm{dist}_{\mathbb{R}^n}(A_i, A_j) \geqslant 3Dr_k$ for $i \neq j$.

Then we define the domains of M as follows:

$$F_j = \left\{x \in M \text{ s.t. } \mathrm{dist}_M\left(x, A_j \cap \Sigma\right) \leqslant r_k\right\}.$$

Comparably to the third step, we have to check that the sets F_j are all disjoint in M. Indeed, take $x \in A_i \cap \Sigma$ and $y \in A_j \cap \Sigma,\ i \neq j$. Thanks to Lemma 4.3.6, we get:

$$\mathrm{dist}_M(x, y) \underset{M \subset \mathbb{H}^{n+1}}{\geqslant} \mathrm{dist}_{\mathbb{H}^{n+1}}(x, y) \geqslant 2r_k,$$

which implies $\mathrm{dist}_M(F_i, F_j) > 0$, and the $2k+2$ sets F_j are disjoint in M. Thus there exist $k+1$ of them such that:

$$|F_j| \leqslant \frac{|M|}{k+1}.$$

On these $k+1$ sets we define the functions f_j as follows:

$$f_j(x) = \min\left\{1, 2 - \frac{1}{r_k}\mathrm{dist}_M\left(x, A_j \cap \Sigma\right)\right\}.$$

On each F_j, we have that $|\nabla f_j|_M^2 = \dfrac{1}{r_k^2}$. Integrating over M gives

$$\int_M |\nabla f_j|_M^2 = \int_{F_j} |\nabla f_j|_M^2 = \frac{|F_j|}{r_k^2} \leqslant \frac{|M|}{(k+1)r_k^2}.$$

On Σ we also have

$$\begin{aligned}\int_\Sigma f_j^2 \mathrm{d}\Sigma &\geqslant \int_{A_j \cap \Sigma} \mathrm{d}\Sigma = \int_{A_j \cap \Sigma_0} \mathrm{d}\Sigma_0 = \mu_0\left(A_j\right) \\ &\geqslant \frac{\mu_0(\mathbb{R}^n)}{2C(n)(2k+2)} = \frac{|\Sigma_0|}{2C(n)(2k+2)} \geqslant \frac{|\Omega|^{\frac{n-1}{n}}}{4I(n)} \cdot \frac{1}{2C(n)(2k+2)}.\end{aligned}$$

Finally, the Rayleigh quotient is bounded as follows:

$$R(f_j) \leqslant \beta(n) \frac{|M|}{|\Omega|^{\frac{n+1}{n}}} k^{\frac{2}{n-1}},$$

and $\beta(n)$ is a constant depending only on the dimension.

Thus we have $\forall k \geqslant k_0$,

$$\sigma_k(M) \leqslant \delta(n) \frac{|M|}{|\Omega|^{\frac{n+1}{n}}} k^{\frac{2}{n-1}},$$

with $\delta(n) = \max\left(\alpha(n), \beta(n)\right)$. Now all we have to do is to bound $\sigma_k(M)$ for $k \leqslant k_0$.

As $k_0 = \left\lfloor \frac{|\Omega|^{\frac{n-1}{n}}}{4P(n)} \right\rfloor \leqslant \frac{|\Omega|^{\frac{n-1}{n}}}{4P(n)}$, we have

$$\sigma_{k_0}(M) \leqslant \delta(n)\frac{|M|}{|\Omega|^{\frac{n+1}{n}}}k_0^{\frac{2}{n-1}} \leqslant \delta(n)\frac{|M|}{|\Omega|^{\frac{n+1}{n}}}\left(\frac{|\Omega|^{\frac{n-1}{n}}}{4P(n)}\right)^{\frac{2}{n-1}} = \varepsilon(n)\frac{|M|}{|\Omega|^{\frac{n+1}{n}-\frac{2}{n}}} = \varepsilon(n)\frac{|M|}{|\Omega|^{\frac{n-1}{n}}}.$$

Finally, define $\nu(n) = \max(\varepsilon(n), \delta(n))$, so that the eigenvalues are bounded as follows, for all $k \geqslant 0$:

$$\sigma_k(M) \leqslant \varepsilon(n)\frac{|M|}{|\Omega|^{\frac{n-1}{n}}} + \delta(n)\frac{|M|}{|\Omega|^{\frac{n+1}{n}}}k^{\frac{2}{n-1}} = |M|\nu(n)\left(\frac{1}{|\Omega|^{\frac{n-1}{n}}} + \frac{k^{\frac{2}{n-1}}}{|\Omega|^{\frac{n+1}{n}}}\right).$$

□

Proof of Lemma 4.3.6.

The idea is to look at two cases: if two points are really far apart in $\mathbb{R}^n$, they can't be close in the hyperbolic space, and if they are close in $\mathbb{R}^n$, then the distances in $\mathbb{R}^n$ and $\mathbb{H}^{n+1}$ are comparable.

- If $\mathrm{dist}_{\mathbb{R}^n}(x,y) \leqslant 2\sinh(5)$, notice the function $t \mapsto \frac{\sinh^{-1}(t)}{t}$ is decreasing on $\mathbb{R}_+^*$.

  As a consequence we can write

$$\begin{aligned}\frac{\mathrm{dist}_{\mathbb{H}^{n+1}}(x,y)}{\mathrm{dist}_{\mathbb{R}^n}(x,y)} &= \frac{2}{\mathrm{dist}_{\mathbb{R}^n}(x,y)}\sinh^{-1}\left(\frac{\mathrm{dist}_{\mathbb{R}^n}(x,y)}{2}\right)\\ &\geqslant \frac{2}{2\sinh(5)}\sinh^{-1}\left(\frac{2\sinh(5)}{2}\right) = \frac{5}{\sinh(5)} = \frac{1}{D},\end{aligned}$$

  Thus we can bound the distance in $\mathbb{H}^{n+1}$ like this:

$$\mathrm{dist}_{\mathbb{H}^{n+1}}(x,y) \geqslant \frac{\mathrm{dist}_{\mathbb{R}^n}(x,y)}{D} \geqslant \frac{2Dr_k}{D} = 2r_k.$$

- If $\mathrm{dist}_{\mathbb{R}^n}(x,y) \geqslant 2\sinh(5)$, then as the function $t \mapsto \sinh^{-1}(t)$ increases, and because of the remark 4.3.5,

$$\begin{aligned}\mathrm{dist}_{\mathbb{H}^{n+1}}(x,y) &= 2\sinh^{-1}\left(\frac{\mathrm{dist}_{\mathbb{R}^n}(x,y)}{2}\right) \geqslant 2\sinh^{-1}\left(\frac{2\sinh(5)}{2}\right)\\ &= 10 \geqslant 10r_k > 2r_k.\end{aligned}$$

□

4.4 Embedding the hyperbolic space $\mathbb{H}^n$ vertically into $\mathbb{H}^{n+1}$

Let Ω be a domain in $\{0\} \times \mathbb{H}^n \subset \mathbb{H}^{n+1}$. This result is very similar, and the main difference will be on how we "transfer" the test functions from Ω to M. We will obtain the following result:

Theorem 4.4.1.
Let Σ be an $(n-1)$-dimensional, connected, smooth hypersurface in $\{0\} \times \mathbb{H}^n \subset \mathbb{H}^{n+1}$ and $\Omega \subset \{0\} \times \mathbb{H}^n$ denote a domain with boundary $\partial\Omega = \Sigma$. Then, for every manifold M such that $\partial M = \Sigma$, for each $k \geqslant 1$,

$$\sigma_k(M) \leqslant |M|\tilde{\nu}(n)\left(\frac{1}{|\Omega|^{\frac{n-1}{n}}} + \frac{k^{\frac{2}{n-1}}}{|\Omega|^{\frac{n+1}{n}}}\right),$$

where $\tilde{\nu}(n)$ is a constant that only depends on n.

To prove the theorem, we need the following statement:

Proposition 4.4.2. *Isoperimetric inequality in $\mathbb{H}^n$ (see [Sch48] and [Cro84]);*
There exists a constant $J(n)$ such that for any domain Ω of $\mathbb{H}^n$,

$$|\partial\Omega| \geqslant \frac{|\Omega|^{\frac{n-1}{n}}}{J(n)}. \tag{4.4.6}$$

We define the measure μ similarly as in the previous sections. For any Borelian set $\mathscr{U} \subset \mathbb{H}^n$, $\mu(\mathscr{U}) = \mu(\mathscr{U} \cap \Sigma) = \int_{\mathscr{U}\cap\Sigma} \mathrm{d}\Sigma$.

Proof of Theorem 4.4.1. We separate the proof into four steps.

First step.

Like in the last section the first step is to build domains that will support the test functions. We define the following quantities:

$$Q(n) = \rho_n J_n + (4\omega_n)^{\frac{n-1}{n}} \qquad k_0 = \left\lfloor \frac{|\Omega|^{\frac{n-1}{n}}}{4Q(n)} \right\rfloor$$

And the radii:

$$r_k := \frac{1}{4}\left(\frac{|\Omega|^{\frac{n-1}{n}}}{4(2k+2)Q(n)}\right)^{\frac{1}{n-1}} \qquad \text{and} \qquad \overline{r_k} := \frac{1}{4}\left(\frac{|\Omega|}{2(2k+2)\omega_n}\right)^{\frac{1}{n}}$$

Remark 4.4.3. For the same reasons as in the last section, we have, for all $k \geqslant k_0$,

$$r_k \leqslant 1 \qquad \text{and} \qquad r_k \leqslant \overline{r_k}.$$

Proposition 4.4.4. *We recall the volume of the balls and spheres of radius r in the hyperbolic space [Cha06, Chapter VI, Section 5]:*

$$|B_{\mathbb{H}^n}(r)| = \left|\mathbb{S}^{n-1}\right| \int_0^r \sinh(t)^{n-1}\mathrm{d}t, \qquad |\partial B_{\mathbb{H}^n}(r)| = \left|\mathbb{S}^{n-1}\right| \sinh(r)^{n-1}.$$

In particular for $r \leqslant 1$, this implies the existence of constants $\tilde{\omega}_n, \tilde{\rho}_n$ such that

$$|B_{\mathbb{H}^n}(r)| \leqslant \tilde{\omega}_n r^n, \qquad\qquad |\partial B_{\mathbb{H}^n}(r)| \leqslant \tilde{\rho}_n r^{n-1}.$$

Again, we want to verify that no union of $2k+2$ balls of radius $4r_k$ can cover either Ω or Σ. Thanks to Propositions 4.4.2 and 4.4.4, we already have the material necessary to prove this, thus we spare the reader from the same calculations as in the last section. Let us define the following sets:

$$\Omega_0 = \Omega \setminus \bigcup_{j=1}^{2k+2} B(x_j, 4r_k) \qquad \text{and} \qquad \Sigma_0 = \Sigma \setminus \bigcup_{j=1}^{2k+2} B(x_j, 4r_k).$$

Then, because the radius r_k is well chosen, the balls of radius $4r_k$ are not too big, and cannot cover all Ω:

$$|\Omega_0| \geqslant \frac{1}{2}|\Omega|.$$

Moreover, because of the isoperimetric inequality in $\mathbb{H}^n$ 4.4.2, these balls cannot cover all of Σ, and

$$|\Sigma_0| \geqslant \frac{|\Omega|^{\frac{n-1}{n}}}{4J(n)}.$$

Second step.

This step is also very similar as we pick the $2k+2$ biggest balls of radius r_k for the measure μ, taken in a decreasing order. The points $(x_i)_{i=1}^{2k+2}$ are chosen such that:

$$\mu(B(x_1, r_k)) = \sup_{x \in \Omega} \mu(B(x, r_k)),$$

and for all $j = 2 \dots 2k+2$,

$$\mu(B(x_j, r_k)) = \sup \left(\mu(B(x, r_k)); x \in \Omega \setminus \bigcup_{i=1}^{j-1} B(x_i, 4r_k) \right)$$

Third step.

Let us first assume the volume of the ball $B(x_{2k+2}, r_k)$ is “big” for the measure μ; more precisely

$$\mu(B(x_{2k+2}, r_k)) \geqslant \frac{|\Omega|^{\frac{n-1}{n}}}{4J(n)} \cdot \frac{1}{4C(1)^2(2k+2)}. \tag{4.4.7}$$

We have to build the domains on M. To that effect, let us define the geodesic $\mathscr{C}_j$ passing through x_j, and following the unit vector normal to $\mathbb{H}^n$ in $\mathbb{H}^{n+1}$. Then we build our test functions on M. First we need adequate support sets on M. We define the following $2k+2$ sets E_j:

$$E_j = M \cap \left\{ x \text{ s.t. } \operatorname{dist}_{\mathbb{H}^{n+1}}(x, \mathscr{C}_j) \leqslant 4r_k \right\}$$

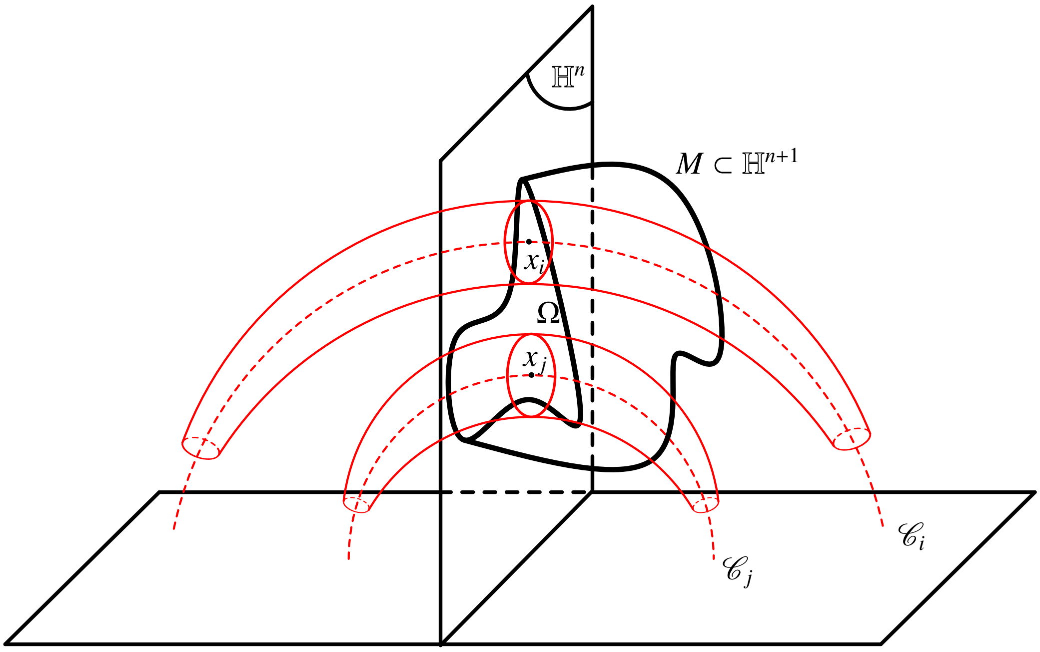

Figure 4.6: Construction of the transfer from Ω to M

Let us observe Figure 4.6, and the dashed lines $\mathscr{C}_i$ and $\mathscr{C}_j$. In the hyperbolic space, we know that two geodesics passing perpendicularly through $\mathbb{H}^n$ follow circular arcs whose centers are in $\{x_1 = x_{n+1} = 0\}$. In particular, the tubes E_j at distance $4r_k$ from their respective geodesic cannot intersect each other. Thus, the $2k+2$ sets E_j are disjoint in M. Consequently, the $k+1$ smallest verify

$$|E_j| \leqslant \frac{|M|}{k+1}.$$

Let us now define the functions $f_j : \mathbb{H}^{n+1} \to \mathbb{R}_+$ as follows:

$$f_j(x) = \min\left\{1, 2 - \frac{1}{r_k}\operatorname{dist}_{\mathbb{H}^{n+1}}\left(x, \mathscr{C}_j\right)\right\}.$$

In $\mathbb{H}^{n+1}$, we have $|\nabla f_j|^2_{\mathbb{H}^{n+1}} \leqslant \dfrac{1}{r_k^2}$, so $|\nabla f_j|^2_M$ is also bounded by $\dfrac{1}{r_k^2}$. As a consequence we have:

$$\int_M |\nabla f_j|^2_M \mathrm{d}V_g = \int_{E_j} |\nabla f_j|^2_M \mathrm{d}V_g \leqslant \frac{|E_j|}{r_k^2} \leqslant \frac{|M|}{(k+1)r_k^2}.$$

From our assumption (4.4.7) we have, for all $j = 1, \dots, k+1$,

$$\mu(B(x_j, r_k)) \geqslant \mu(B(x_{2k+2}, r_k)) \geqslant \frac{|\Omega|^{\frac{n-1}{n}}}{4J(n)} \cdot \frac{1}{4C(1)^2(2k+2)},$$

which leads to

$$\int_\Sigma f_j^2 \mathrm{d}\Sigma \geqslant \int_{B(x_j,r_k)\cap\Sigma} \mathrm{d}\Sigma = \mu(B(x_j, r_k)) \geqslant \mu(B(x_{2k+2}, r_k)) \geqslant \frac{|\Omega|^{\frac{n-1}{n}}}{4J(n)} \cdot \frac{1}{4C(1)^2(2k+2)}.$$

The Rayleigh quotient can thus be bounded as follows:

$$\begin{aligned}
R(f_j) = \frac{\int_M |\nabla f_j|_M^2 \mathrm{d}V_g}{\int_\Sigma f_j^2 \mathrm{d}\Sigma} &\leqslant \frac{|M|}{(k+1)r_k^2\left(\frac{|\Omega|^{\frac{n-1}{n}}}{4J(n)} \cdot \frac{1}{4C(1)^2(2k+2)}\right)} \\
&= \frac{|M|}{\frac{k+1}{16D^2}\left(\frac{|\Omega|^{\frac{n-1}{n}}}{4(2k+2)Q(n)}\right)^{\frac{2}{n-1}}\left(\frac{|\Omega|^{\frac{n-1}{n}}}{4J(n)} \cdot \frac{1}{4C(1)^2(2k+2)}\right)} \\
&\leqslant \tilde{\alpha}(n)\frac{|M|}{|\Omega|^{\frac{n+1}{n}}}k^{\frac{2}{n-1}},
\end{aligned}$$

with $\tilde{\alpha}(n)$ a constant depending only on n.

Fourth step.

If on the contrary, the volume of the ball of center x_{2k+2} is too small, that is,

$$\mu(B(x_{2k+2}, r_k)) \leqslant \frac{|\Omega|^{\frac{n-1}{n}}}{4J(n)} \cdot \frac{1}{4C(1)^2(2k+2)}, \tag{4.4.8}$$

which implies that for all $x \in \Omega_0$,

$$\mu(B(x, r_k)) \leqslant \frac{|\Omega|^{\frac{n-1}{n}}}{4J(n)} \cdot \frac{1}{4C(1)^2(2k+2)} \leqslant \frac{|\Sigma_0|}{4C(1)^2(2k+2)}. \tag{4.4.9}$$

We then apply Lemma 1.2.5 to the space $(\mathbb{H}^n, \mathrm{dist}_{\mathbb{H}^n}, \mu_0)$, where μ_0 is the restriction of μ to Σ_0. By using the same arguments as in the last section, we get the existence of sets that are big enough for the measure μ_0, but also sufficiently far apart in $\mathbb{H}^n$. Similarly to the last step, we define $\mathscr{D}_j$ as the union of geodesics passing through these sets, following the unit vector normal to $\mathbb{H}^n$. We then define for each j, the set at distance smaller than r_k from $\mathscr{D}_j$, and take the intersection with M:

$$F_j = M \cap \left\{x \text{ s.t. } \mathrm{dist}_{\mathbb{H}^{n+1}}(x, \mathscr{D}_j) \leqslant r_k\right\}.$$

The sets F_j are disjoint, and we can now imitate what we did in the third step by picking the $k+1$ smallest. Then by constructing test functions, equal to 1 on $\mathscr{D}_j$, and decreasing linearly to 0 on F_j. From all that, we can finally bound $\sigma_k(M)$ for $k \geqslant k_0$; there exists $\tilde{\delta}(n)$ such that

$$\sigma_k(M) \leqslant \tilde{\delta}(n)\frac{|M|}{|\Omega|^{\frac{n+1}{n}}}k^{\frac{2}{n-1}}.$$

Now what we want to do is to bound $\sigma_k(M)$ for $k \leqslant k_0$.

As $k_0 = \left\lfloor \frac{|\Omega|^{\frac{n-1}{n}}}{4Q(n)} \right\rfloor \leqslant \frac{|\Omega|^{\frac{n-1}{n}}}{4Q(n)}$, we have

$$\sigma_{k_0}(M) \leqslant \tilde{\delta}(n)\frac{|M|}{|\Omega|^{\frac{n+1}{n}}}k_0^{\frac{2}{n-1}} \leqslant \tilde{\delta}(n)\frac{|M|}{|\Omega|^{\frac{n+1}{n}}}\left(\frac{|\Omega|^{\frac{n-1}{n}}}{4Q(n)}\right)^{\frac{2}{n-1}} = \tilde{\varepsilon}(n)\frac{|M|}{|\Omega|^{\frac{n+1}{n}-\frac{2}{n}}} = \tilde{\varepsilon}(n)\frac{|M|}{|\Omega|^{\frac{n-1}{n}}}.$$

Finally, define $\tilde{\nu}(n) = \max(\tilde{\varepsilon}(n), \tilde{\delta}(n))$, so that the eigenvalues are bounded as follows, for all $k \geqslant 0$:

$$\sigma_k(M) \leqslant \tilde{\varepsilon}(n)\frac{|M|}{|\Omega|^{\frac{n-1}{n}}} + \tilde{\delta}(n)\frac{|M|}{|\Omega|^{\frac{n+1}{n}}}k^{\frac{2}{n-1}} = |M|\tilde{\nu}(n)\left(\frac{1}{|\Omega|^{\frac{n-1}{n}}} + \frac{k^{\frac{2}{n-1}}}{|\Omega|^{\frac{n+1}{n}}}\right).$$

□

List of Figures

Bibliography

[Ann84] Colette Anné. *Exemples de convergence de valeurs propres sur des surfaces ayant une anse très fine*. PhD thesis, Institut Fourier, 1984.

[Ann87] Colette Anné. Spectre du laplacien et écrasement d'anses. *Annales scientifiques de l'École Normale Supérieure*, 20(2):271–280, 1987.

[Bé77] Pierre H. Bérard. On the wave equation on a compact Riemannian manifold without conjugate points. *Mathematische Zeitschrift*, 155:249–276, 1977.

[Bé86] Pierre H. Bérard. *Spectral geometry: direct and inverse problems*, volume 1207 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[Ban80] Catherine Bandle. *Isoperimetric inequalities and applications*, volume 7 of *Monographs and Studies in Mathematics*. Pitman, Boston, Mass.-London, 1980.

[Bar03] Franck Barthe. Around the Brunn-Minkowski inequality. *Annales de la Faculté des Sciences de Toulouse. Mathématiques. Série VI*, 12(2):127–178, 2003.

[BBD88] Peter Buser, Marc Burger, and Józef Dodziuk. Riemann surfaces of large genus and large λ_1. In *Geometry and Analysis on Manifolds*, pages 54–63, Berlin, Heidelberg, 1988. Springer Berlin Heidelberg.

[Ble83] David D. Bleecker. The spectrum of a Riemannian manifold with a unit Killing vector field. *Transactions of the American Mathematical Society*, 275(1):409–416, 1983.

[Bro88] Robert Brooks. Constructing isospectral manifolds. *American Mathematical Monthly*, 95(9):823–839, 1988.

[BT90] Robert Brooks and Richard Tse. Correction to: "Isospectral surfaces of small genus" [Nagoya Math. J. **107** (1987), 13–24; MR0909246 (88m:58182)]. *Nagoya Mathematical Journal*, 117:227, 1990.

[Bus86] Peter Buser. Isospectral Riemann surfaces. *Université de Grenoble. Annales de l'Institut Fourier*, 36(2):167–192, 1986.

[Bus88] Peter Buser. Cayley graphs and planar isospectral domains. In *Geometry and analysis on manifolds (Katata/Kyoto, 1987)*, volume 1339 of *Lecture Notes in Math.*, pages 64–77. Springer, Berlin, 1988.

[CD94] Bruno Colbois and Józef Dodziuk. Riemannian metrics with large λ_1. *Proceedings of the American Mathematical Society*, 122(3):905–906, 1994.

[CDE10] Bruno Colbois, Emily B. Dryden, and Ahmad El Soufi. Bounding the eigenvalues of the Laplace–Beltrami operator on compact submanifolds. *Bulletin of the London Mathematical Society*, 42(1):96–108, 2010.

[CE03] Bruno Colbois and Ahmad El Soufi. Extremal eigenvalues of the Laplacian in a conformal class of metrics: The 'conformal spectrum'. *Annals of Global Analysis and Geometry*, 24(4):337–349, 2003.

[CE18] Bruno Colbois and Ahmad El Soufi. Spectrum of the Laplacian with weights. *Annals of Global Analysis and Geometry; to appear*, Jun 2018.

[CEG11] Bruno Colbois, Ahmad El Soufi, and Alexandre Girouard. Isoperimetric control of the Steklov spectrum. *Journal of Functional Analysis*, 261(5):1384–1399, 2011.

[CEG13] Bruno Colbois, Ahmad El Soufi, and Alexandre Girouard. Isoperimetric control of the spectrum of a compact hypersurface. *Journal für die Reine und Angewandte Mathematik. [Crelle's Journal]*, 683:49–65, 2013.

[CEG17] Bruno Colbois, Ahmad El Soufi, and Alexandre Girouard. Compact manifolds with fixed boundary and large Steklov eigenvalues. *Proceedings of the American Mathematical Society; to appear*, January 2017.

[CES15] Bruno Colbois, Ahmad El Soufi, and Alessandro Savo. Eigenvalues of the Laplacian on a compact manifold with density. *Communications in Analysis and Geometry*, 23(3):639–670, 2015.

[CGG18] Bruno Colbois, Alexandre Girouard, and Katie Gittins. Steklov eigenvalues of submanifolds with prescribed boundary in Euclidean space. *The Journal of Geometric Analysis; to appear*, 2018.

[CGH18] Bruno Colbois, Alexandre Girouard, and Asma Hassannezhad. The Steklov and Laplacian spectra of Riemannian manifolds with boundary. *ArXiv e-prints*, 2018.

[CH89] Richard Courant and David Hilbert. *Methods of mathematical physics. Vol. I*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1989. Partial differential equations, Reprint of the 1962 original, A Wiley-Interscience Publication.

[Cha84] Isaac Chavel. *Eigenvalues in Riemannian geometry*, volume 115 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1984.

[Cha06] Isaac Chavel. *Riemannian geometry*, volume 98 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2006. A modern introduction.

[CM08] Bruno Colbois and Daniel Maerten. Eigenvalues estimate for the Neumann problem of a bounded domain. *Journal of Geometric Analysis*, 18(4):1022–1032, 2008.

[Cro84] Christopher B. Croke. A sharp four dimensional isoperimetric inequality. *Commentarii Mathematici Helvetici*, 59(1):187–192, Dec 1984.

[Dav95] E. Brian Davies. *Spectral theory and differential operators*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1995.

[EG92] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[EI92] Ahmad El Soufi and Saïd Ilias. Une inégalité du type "Reilly" pour les sous-variétés de l'espace hyperbolique. *Commentarii Mathematici Helvetici*, 67(2):167–181, 1992.

[EI00] Ahmad El Soufi and Saïd Ilias. Second eigenvalue of schrödinger operators and mean curvature. *Communications in Mathematical Physics*, 208(3):761–770, Jan 2000.

[Eva98] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[FS17] Ailana Fraser and Richard Schoen. Shape optimization for the Steklov problem in higher dimensions. *ArXiv e-prints*, 2017.

[GNY04] Alexander Grigor'yan, Yuri Netrusov, and Shing-Tung Yau. Eigenvalues of elliptic operators and geometric applications. *Surveys in Differential Geometry, IX*, pages 147–218, 2004.

[GP12] Alexandre Girouard and Iosif Polterovich. Upper bounds for Steklov eigenvalues on surfaces. *Electronic Research Announcements in Mathematical Sciences*, 19:77–85, 2012.

[GP17] Alexandre Girouard and Iosif Polterovich. Spectral geometry of the Steklov problem. In *Shape optimization and spectral theory*, pages 120–148. De Gruyter Open, Warsaw, 2017.

[Gro02] Jean-François Grosjean. Upper bounds for the first eigenvalue of the Laplacian on compact submanifolds. *Pacific Journal of Mathematics*, 206(1):93–112, 2002.

[Gro04] Jean-François Grosjean. Extrinsic upper bounds for the first eigenvalue of elliptic operators. *Hokkaido Mathematical Journal*, 33(2):319–339, 2004.

[GWW92] Carolyn Gordon, David L. Webb, and Scott Wolpert. One cannot hear the shape of a drum. *American Mathematical Society. Bulletin. New Series*, 27(1):134–138, 1992.

[Has11] Asma Hassannezhad. Conformal upper bounds for the eigenvalues of the Laplacian and Steklov problem. *Journal of Functional Analysis*, 261(12):3419 – 3436, 2011.

[Hen06] Antoine Henrot. *Extremum problems for eigenvalues of elliptic operators*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[HPS75] Joseph Hersch, Lawrence E. Payne, and Menahem M. Schiffer. Some inequalities for Stekloff eigenvalues. *Archive for Rational Mechanics and Analysis*, 57:99–114, 1975.

[IM11] Saïd Ilias and Ola Makhoul. A Reilly inequality for the first Steklov eigenvalue. *Differential Geometry and its Applications*, 29(5):699 – 708, 2011.

[Kac66] Marc Kac. Can one hear the shape of a drum? *The American Mathematical Monthly*, 73(4):1–23, 1966.

[Kel66] Richard Kellner. On a theorem of Polya. *Amer. Math. Monthly*, 73:856–858, 1966.

[Kor93] Nicholas Korevaar. Upper bounds for eigenvalues of conformal metrics. *Journal of Differential Geometry*, 37(1):73–93, 1993.

[KP] Salam Kouzayha and Luc Pétiard. Boundedness of the eigenvalues for the Laplacian with density. In preparation.

[Kro92] Pawel Kröger. Upper bounds for the Neumann eigenvalues on a bounded domain in Euclidean space. *Journal of Functional Analysis*, 106(2):353–357, 1992.

[Mil64] John Milnor. Eigenvalues of the Laplace operator on certain manifolds. *Proceedings of the National Academy of Sciences of the United States of America*, 51:542, 1964.

[MM03] Carlo Mantegazza and Andrea Carlo Mennucci. Hamilton-Jacobi equations and distance functions on Riemannian manifolds. *Appl. Math. Optim.*, 47(1):1–25, 2003.

[Mut80] Hideo Mutô. The first eigenvalue of the Laplacian on even-dimensional spheres. *The Tôhoku Mathematical Journal. Second Series*, 32(3):427–432, 1980.

[Pó61] George Pólya. On the eigenvalues of vibrating membranes. *Proceedings of the London Mathematical Society. Third Series*, 11:419–433, 1961.

[Pay67] Lawrence E. Payne. Isoperimetric inequalities and their applications. *SIAM Review*, 9(3):453–488, 1967.

[PS17] Luigi Provenzano and Joachim Stubbe. Weyl-type bounds for Steklov eigenvalues. *Journal of Spectral Theory; to appear*, 2017. arXiv:1711.06458v2.

[Rei73] Robert C. Reilly. Variational properties of functions of the mean curvatures for hypersurfaces in space forms. *Journal of Differential Geometry*, 8:465–477, 1973.

[Rei77] Robert C. Reilly. On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space. *Commentarii Mathematici Helvetici*, 52(1):525–533, Dec 1977.

[Sch48] Erhard Schmidt. Die Brunn-Minkowskische Ungleichung und ihr Spiegelbild sowie die isoperimetrische Eigenschaft der Kugel in der euklidischen und nichteuklidischen Geometrie. I. *Mathematische Nachrichten*, 1:81–157, 1948.

[Ste02] Vladimir Steklov. Sur les problèmes fondamentaux de la physique mathématique. *Annales scientifiques de l'École Normale Supérieure*, 19:191–259, 1902.

[Vig80] Marie-France Vignéras. Variétés riemanniennes isospectrales et non isométriques. *Annals of Mathematics. Second Series*, 112(1):21–32, 1980.

[Wei54] Robert Weinstock. Inequalities for a classical eigenvalue problem. *J. Rational Mech. Anal.*, 3:745–753, 1954.

[Wey12] Hermann Weyl. Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen. *Mathematische Annalen*, 71:441–479, 1912.

[Zhu97] Shunhui Zhu. The comparison geometry of Ricci curvature. In *Comparison geometry (Berkeley, CA, 1993–94)*, volume 30 of *Math. Sci. Res. Inst. Publ.*, pages 221–262. Cambridge Univ. Press, Cambridge, 1997.