JOURNAL OF DIFFERENTIAL EQUATIONS **137**, 302–319 (1997)
ARTICLE NO. DE973269

On Invariant Hypersurfaces of Strongly Monotone Maps

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Received July 9, 1996; revised November 4, 1996

For C^k, $k > 1$, strongly monotone discrete-time dynamical systems, we present simple criteria which ensure that nonmonotone invariant manifolds are C^k.

1. INTRODUCTION

Let $f: \mathbf{R}^m \mapsto \mathbf{R}^m$ be a C^k, $k \geqslant 1$, diffeomorphism to its image. We suppose throughout that the map f has *positive derivative* meaning that the Jacobian matrix $Df(x)$ has positive entries for all $x \in \mathbf{R}^m$.

For $x, y \in \mathbf{R}^m$ we define a partial ordering by $x \geqslant y$ if and only if $x_i \geqslant y_i$ for all i. If $x \geqslant y$ and $x \neq y$ we write $x > y$. If $x_i > y_i$ for all i we write $x \gg y$. The assumption that f has positive derivative has the important consequence that f is *strongly monotone*, that is

$$f(x) \ll f(y) \qquad \text{if} \quad x < y.$$

The dynamics of strongly monotone maps has been the focus of much attention in the recent years through the seminal work of Hirsch and a large amount of work has been devoted to show that strongly monotone systems enjoy special asymptotic properties. Here, we are concerned with certain invariant manifolds naturally associated to f.

A set $A \subset \mathbf{R}^m$ is said *invariant* if $f(A) = A$. We call A *unordered* if no two of its points are related by $>$. In studying the behavior of strongly monotone systems, unordered invariant sets play a fundamental role. For instance, compact omega limit sets of strongly monotone maps [15] or flows [5] are unordered. Similar results hold for compact chain transitive sets [6].

Generalizing earlier results by Hirsch [4], Takáč [15, 16] proved that *any* unordered invariant set for a strongly monotone map (possibly defined on a Banach space) lies in a canonically defined invariant hypersurface. More precisely:

0022-0396/97 $25.00

THEOREM 1.1 (Takáč, Hirsch). *Let $Q \subset \mathbf{R}^m$ be a nonempty unordered invariant set. Then*

(i) *Q lies in an invariant unordered closed hypersurface $V = V(Q) \subset \mathbf{R}^m$.*

(ii) *Let $\pi : \mathbf{R}^m \to E$ denote the orthogonal projection onto the hyperplane E orthogonal to a vector $e \gg 0$. Then $\pi | V$ maps V homeomorphically onto $\pi(V)$, and the maps $\pi | V$ and $(\pi | V)^{-1}$ are (globally) Lipschitz continuous.*

(iii) *If Q is compact, then $\pi(V) = E$.*

Except for the assertion (iii) due to Hirsch [6] this theorem just rephrases (in the finite dimensional setting) Propositions 1.2 and 1.3 of Takáč [16].

Using the terminology introduced in Tereščák an invariant unordered topological hypersurface will be called a *nonmonotone manifold*. Hirsch [4] asked the question of determining conditions under which nonmonotone manifolds are smooth. Brunovský [1] proved that they are C^1 provided that f is Morse–Smale. Mierczyński [10] proved the smoothness of V for a class of competitive system of ODE's. Recently Tereščák [17] proved the C^1 property of nonmonotone manifolds in full generality for C^1 strongly monotone maps taking values in Banach spaces.

The purpose of this paper is to give simple sufficient conditions on f ensuring that V is C^k, $k > 1$. The main results are stated in Section 2 and proved in Section 3. Some applications to cooperative vector fields are considered in Section 4.

2. NOTATION AND MAIN RESULTS

The following notation will be used. A map is $C^{k,r}$, $k \in \mathbf{N}$, $0 \leqslant r < 1$ if it is C^k and its kth derivative is r-Hölder. By convention $C^{k,0}$ means C^k. A $C^{k,r}$ manifold is a manifold whose change of coordinates are $C^{k,r}$ maps.

Let T denote an $m \times m$ matrix with positive entries. The *Birkhoff's contraction coefficient* of T (see [13]) is the number

$$\tau_B(T) = \frac{1 - \sqrt{\Psi(T)}}{1 + \sqrt{\Psi(T)}},$$

where

$$\Psi(T) = \min_{i, j, k, l} \frac{T_{i,k} T_{j,l}}{T_{j,k} T_{i,l}}.$$

Let $K \subset \mathbf{R}^m$ be a compact invariant set. We define the *Birkhoff's contraction coefficient* of f at K as

$$\tau_B(f, K) = \sup_{x \in K} \tau_B(Df(x)).$$

We also use the notation

$$\|Df|K\| = \sup_{x \in K} \|Df(x)\|,$$

where $\|Df(x)\|$ denotes the operator norm associated to the norm $\|\cdot\|$ on $\mathbf{R}^m$.

Let $M(f, K)$ denote the closure of the union of the supports of all f-invariant Borel probability measures with support in K. By the Poincaré recurrence theorem (see, e.g., [9]) $M(f, K)$ is contained in the *Birkhoff center* of $f|K$ (i.e., the closure of recurrent points in K). It can also be characterized as the minimal center of attraction of $f|K$ (see, e.g., [9], Exercise 1.4 page 100)).

Let (X, d) be a metric space with metric d and $g: X \to X$ a continuous map. A nonempty compact set $K \subset X$ is called an *attractor* for g if it is invariant (i.e., $g(K) = K$) and possesses a neighborhood U called a *fundamental neighborhood* such that $g(\bar{U}) \subset U$ and $\bigcap_{n \geq 1} g^n(U) = K$. If K is an attractor for g, the *basin of attraction* of K is the open set defined as

$$B(K, g) = \{x \in X: \lim_{n \to \infty} d(g^n(x), K) = 0\}.$$

We now state the main results of the paper.

THEOREM 2.1. *Let $V \subset \mathbf{R}^m$ be a nonmonotone manifold. Let $K \subset V$ be an attractor for $f|V$ (f restricted to V) and let $M = M(f, K)$. Suppose*

(i) *f is $C^{1+k, r}$ for $k \in \mathbf{N}$ and $0 \leq r < 1$.*
(ii) $\tau_B(f, M)^{1+k+r} \|Df|M\|^{k+r} < 1$.

Then $B(K, f|V)$ is $C^{1+k, r}$.

A more abstract (but also more precise) condition is given by the following theorem from which Theorem 2.1 will be actually deduced.

THEOREM 2.2. *Suppose that assumption* (i) *of Theorem* 2.1 *holds. Suppose also the existence of $\eta > 0$ such that for every Borel invariant and ergodic probability measure μ with support in K*

$$(1 + k + r)\,\lambda_2(\mu) - \lambda_1(\mu) < -\eta,$$

where $\lambda_1(\mu)$ and $\lambda_2(\mu)$ respectively denote the largest and second largest Lyapounov exponents of f with respect to μ. Then $B(K, f|V)$ is $C^{1+k,r}$.

Remarks. • If the map f is dissipative then the maximal compact invariant set in V is an attractor for $f|V$ whose basin is the whole manifold V.

Theorems 2.1 and 2.2 have the following consequences:

- If f is C^2 then $B(K, f|V)$ is always $C^{1,r}$ for some $r>0$.
- If f is C^{1+k}, $k \in \mathbf{N}$ and nonexpansive (i.e. $\|Df(x)\| \leqslant 1$ for all $x \in M$) then $B(K, f|V)$ is C^{1+k}.

As pointed out by the anonymous referee, this later property can actually be deducted from the known theorems on stable manifolds. Indeed, if f is nonexpansive, it follows from Proposition 3.1 in Section 3 that f^n exponentially decreases distances in K, so K consists of finitely many periodic points. Let p be such a point and suppose for simplicity that p is an equilibrium. Eigenvalues of $Df(p)$ are bounded by 1 and eigenvalues of $Df(p)|E_p = Df(p)|T_pV$ are strictly less than 1, so p is pseudo-hyperbolic. Therefore it admits a C^{1+k} $(m-1)$ dimensional strong stable manifold $W^{ss}(p)$ forward invariant by f. Since points near p in V are exponentially attracted to p, $W^{ss}(p) \subset V$, hence $B(p, f|V)$ is C^{1+k}. A similar reasoning applies if p is periodic.

3. PROOF OF THEOREMS 2.1 AND 2.2

In this section we prove Theorems 2.1 and 2.2. The main ingredient is the following proposition known as the *exponential separation property*.

Proposition 3.1. *Let $K \subset \mathbf{R}^m$ be a compact invariant set. The tangent bundle of $\mathbf{R}^m$ restricted to K splits continuously into two subbundles*

$$T_K\mathbf{R}^m = E \oplus L$$

invariant by Df such that

(a) *For each $x \in K$ the fiber L_x is a line spanned by a unit vector $b(x) \geqslant 0$ and the fiber E_x is an unordered hyperplane.*

(b) *There exists $\rho \leqslant \tau_B(f, K) < 1$ and $C > 0$ such that for all $x \in K$, unit vectors $w \in E_x$ we have*

$$\|Df^n(x)\, w\| \leqslant C\rho^n \, \|Df^n(x)\, b(x)\|.$$

This proposition (with the exception of the estimate $\rho \leqslant \tau_B(f, k)$) has been proved by Ruelle [12], Poláčik and Tereščák [11] and Tereščák [17] among others in fairly general settings.

In finite dimension, the use of the Birkhoff's contraction coefficient makes the proof very simple and allows the estimate $\rho \leqslant \tau_B(f, K)$. We will provide such a proof below.

First we need to introduce the *projective distance* on the cone of positive vectors. Let $u \gg 0$ and $v \gg 0$. The projective distance (see [13]) between u and v is the quantity

$$d_B(u, v) = \max_{i, j} \log \left(\frac{u_i v_j}{u_j v_i} \right).$$

It has the properties of a distance with the exception that $d_B(u, v) = 0$ if and only if $v = \lambda u$ for some $\lambda > 0$. A useful property of d_B is that for all vectors $u \gg 0$, $v \gg 0$ and all real numbers $\alpha > 0$, $\beta > 0$

$$d_B(\alpha u, \beta v) = d_B(u, v).$$

Let T be a $n \times n$ matrix with positive entries. By a theorem of Birkhoff whose proof can be found in Seneta [13, Section 3.4] we have the relation

$$d_B(Tu, Tv) \leqslant \tau_B(T)\, d_B(u, v) \tag{1}$$

for all $u \gg 0$, $v \gg 0$.

We begin by a simple inequality which for convenience is isolated in the next lemma.

LEMMA 3.2. *Let $a \in \mathbf{R}^m$ and $b \gg 0$. Suppose that a and 0 are unrelated by $<$ and $a + b \gg 0$. Then*

$$\max(|a|) \leqslant d_B(a + b, b) \frac{\max(a + b) \max(b)^2}{\min(b)^2},$$

where $\max(x) = \max_i x_i$, $\min(x) = \min_i (x_i)$, $|x| = (|x_1|, ..., |x_n|)$.

Proof. Let $i \in \{1, ..., n\}$. Suppose $a_i \geqslant 0$. Since a and 0 are unrelated there exists $j \neq i$ such that $a_j \leqslant 0$. Thus

$$\frac{a_i b_j}{\max(b)^2} \leqslant \frac{a_i b_j - b_i a_j}{b_i b_j} = \frac{a_i + b_i}{b_i} - \frac{a_j + b_j}{b_j} \leqslant \max\left(\frac{a + b}{b}\right) - \min\left(\frac{a + b}{b}\right).$$

Similarly if $a_i \leqslant 0$ we get the same inequality with $|a_i|$ instead of a_i. Thus,

$$\max(|a|)\,\frac{\min(b)}{\max(b)^2} \leqslant \max\left(\frac{a+b}{b}\right) - \min\left(\frac{a+b}{b}\right).$$

On the other hand

$$\log\left(\max\left(\frac{a+b}{b}\right)\right) - \log\left(\min\left(\frac{a+b}{b}\right)\right)$$

$$\geqslant \frac{\max\left(\frac{a+b}{b}\right) - \min\left(\frac{a+b}{b}\right)}{\max\left(\frac{a+b}{b}\right)}$$

$$\geqslant \left(\max\left(\frac{a+b}{b}\right) - \min\left(\frac{a+b}{b}\right)\right)\frac{\min(b)}{\max(a+b)}.$$

By combining this inequality with the preceding one we obtain the desired result. Q.E.D.

Proof of Proposition 3.1. Let $|\cdot|$ denote the norm on $\mathbf{R}^m$ defined by $|x| = \sum_i |x_i|$. For $x \in \mathbf{R}^m \setminus \{0\}$ let $N(x) = x/|x|$. The vector $\mathbf{1} \in \mathbf{R}^m$ is the vector whose all coordinates equal 1. Set

$$\Delta = \{x \in \mathbf{R}^m : x \geqslant 0,\ |x| = 1\}$$

and

$$\Delta_\varepsilon = \{x \in \mathbf{R}^m : x \geqslant \varepsilon \mathbf{1},\ |x| = 1\}.$$

Since f has positive derivative, the compactness of $K \times \Delta$ implies the existence of $\varepsilon > 0$ such that for all $x \in K$

$$N[Df(x)(\Delta)] \subset \Delta_\varepsilon.$$

Let $b_n(x) = N[Df^n(f^{-n}(x))\,\mathbf{1}] \in \Delta_\varepsilon$. We have

$$d_{\mathrm{B}}(b_n(x), b_{n+1}(x)) = d_{\mathrm{B}}(Df^n(f^{-n}(x))\,\mathbf{1}, Df^{n+1}(f^{-n-1}(x))\,\mathbf{1}).$$

By the chain rule and relation (1) we get

$$d_{\mathrm{B}}(b_n(x), b_{n+1}(x)) \leqslant \tau_{\mathrm{B}}(f, K)^n\, d_{\mathrm{B}}(\mathbf{1}, Df(f^{-n-1}(x))\,\mathbf{1}) \leqslant C\tau_{\mathrm{B}}(f, K)^n,$$

where $C = \sup\{d_{\mathrm{B}}(\mathbf{1}, u) : u \in \Delta_\varepsilon\}$. Applying Lemma 3.2 gives

$$|b_n(x) - b_{n+1}(x)| \leqslant \frac{m d_{\mathrm{B}}(b_n(x), b_{n+1}(x))}{\varepsilon^2} \leqslant \frac{mC\tau_{\mathrm{B}}(f, K)^n}{\varepsilon^2}.$$

It follows that $\{b_n\}$ is a Cauchy sequence which converges uniformly toward some function $b\colon K \to \Delta_\varepsilon$. Continuity of b_n implies continuity of b. Also, it is easy to see that

$$N(Df(x)\, b(x)) = b(f(x)).$$

If we apply the same reasoning to the adjoint of Df

$$Df^*\colon K \times \mathbf{R}^m \to K \times \mathbf{R}^m,$$
$$(x, u) \to (f^{-1}(x), Df(f^{-1}(x))^T u),$$

we obtain a continuous function $b^*\colon K \to \Delta_\varepsilon$. We then define the plane E_x as

$$E_x = \{v \in \mathbf{R}^m\colon \langle v, b^*(x) \rangle = 0\},$$

where $\langle\ ,\ \rangle$ is the standard inner product.

We now prove assertion (b). Let $V_x = \{u \in E_x\colon N(u + b(x)) \in \Delta_{\varepsilon/2}\}$. V_x is a compact neighborhood of the origin in E_x. Also, there exists a real number $\eta > 0$ such that $u + b(x) \leqslant \eta \mathbf{1}$ for all $x \in K$, $u \in V_x$. Let $u \in V_x$,

$$d_{\mathrm{B}}(Df^n(x)(u + b(x)), b(f^n(x))) = d_{\mathrm{B}}(Df^n(x)(u + b(x)), Df^n(x)\, b(x)).$$

By the chain rule and relation (1) we get

$$\begin{aligned} d_{\mathrm{B}}(Df^n(x)(u + b(x)), b(f^n(x))) &\leqslant \tau_{\mathrm{B}}(f, K)^n\, d_{\mathrm{B}}(u + b(x), b(x)) \\ &\leqslant \tau_{\mathrm{B}}(f, K)^n\, C', \end{aligned}$$

where $C' = \sup\{d_{\mathrm{B}}(u, v)\colon u, v \in \Delta_{\varepsilon/2}\}$. Then

$$d_{\mathrm{B}}\left(\frac{Df^n(x)\, u}{|Df^n(x)\, b(x)|} + b(f^n(x)), b(f^n(x))\right) \leqslant \tau_{\mathrm{B}}(f, K)^n\, C'. \tag{2}$$

It follows that for n large enough $Df^n(x)\, u/|Df^n(x)\, b(x)| \in V_{b(f^n(x))}$. Now, we apply Lemma 3.2 with $a = Df^n(x)\, u/|Df^n(x)\, b(x)|$ and $b = b(f^n(x))$. This gives

$$\frac{|Df^n(x)\, u|}{|Df^n(x)\, b(x)|} \leqslant \tau_{\mathrm{B}}(f, K)^n\, mC' \frac{\eta}{\varepsilon^2}$$

for all $u \in V_x$. Since V_x is a neighborhood of the origin of E_x, this implies that

$$\frac{\|Df^n(x) \mid E_x\|}{\|Df^n(x)\, b(x)\|} \leqslant \tau_{\mathrm{B}}(f, K)^n\, C''$$

for any norm $\|\cdot\|$ and some constant $C'' > 0$ depending on $\|\cdot\|$. Q.E.D.

Given a compact invariant set $K \subset \mathbf{R}^m$ and a real number $t \geqslant 0$ define the numbers

$$\alpha(f, K, t) = \lim_{n \to \infty} \frac{1}{n} \log \left(\sup_{x \in K} \frac{\|Df^n(x) \mid E_x\|^t}{\|Df^n(x)\, b(x)\|} \right)$$

and

$$\beta(f, K) = \lim_{n \to \infty} \frac{1}{n} \log \left(\sup_{x \in K} \|Df^n(x) \mid E_x\| \right),$$

where the limits exist by a standard subadditivity argument. The following inequalities give some elementary estimates of these numbers:

$$\beta(f, K) \leqslant \log(\|Df \mid K\|), \tag{3}$$

$$\alpha(f, K, t) \leqslant t\alpha(f, K, 1) + (t-1)\, \beta(f, K), \tag{4}$$

and

$$\alpha(f, K, 1) \leqslant \log(\tau_{\mathrm{B}}(f, K)). \tag{5}$$

Inequalities (3) and (4) are straightforward and (5) follows from Proposition 3.1b.

Let μ be a Borel probability measure with support in K for which $f \mid K$ is ergodic. Let $\lambda_1(\mu) > \lambda_2(\mu) > \cdots > \lambda_k(\mu)$ denote the *Lyapounov exponents* of μ. The next result is a straightforward adaptation of a theorem due to Sebastian Schreiber (1995) but for reader's convenience we supply a proof.

Proposition 3.3 (adapted from [14]). *For all $t \geqslant 0$*

$$\alpha(f, K, t) = \sup_{\mu} \, (t\lambda_2(\mu) - \lambda_1(\mu))$$

where the supremum is taken over all ergodic measures with support in K.

Proof. Let $X = \{(x, v) : x \in K, v \in E_x, \|v\| = 1\}$. By compactness of X there exists a sequence $(x_n, v_n) \in X$ such that

$$\alpha(f, K, t) = \lim_{n \to \infty} \frac{1}{n} \log \frac{\|Df^n(x_n)\, v_n\|^t}{\|Df^n(x_n) \cdot b(x_n)\|}.$$

Define a map $G: X \to X$ by

$$G(x, v) = \left(f(x), \frac{Df(x)\, v}{\|Df(x)\, v\|} \right).$$

Let $\{\theta_n\}_{n\geqslant 0}$ be the sequence of probability measures defined on X by

$$\theta_n=\frac{1}{n}\sum_{i=0}^{n-1}\delta_{G^i(x_n,\, v_n)}.$$

By compactness of X we can always suppose (by replacing θ_n by some subsequence if necessary) that the sequence θ_n converges weakly toward some probability measure θ. Continuity of G easily implies that θ is G-invariant.

Let $h\colon X\to\mathbf{R}$ be the map defined by $h(x, v)=\log(\|Df(x)\, v\|^t/\|Df(x)\, b(x)\|)$. By the chain rule we get

$$h(G^i(x, v))=\log\left(\frac{\|Df^{i+1}(x)\, v\|^t}{\|Df^i(x)\, v\|^t\,\|Df(f^i(x))\, b(f^i(x))\|}\right).$$

Since $b(f^i(x))=Df^i(x)\, b(x)/\|Df^i(x)\, b(x)\|$, this implies

$$h(G^i(x, v))=\log\left(\frac{\|Df^{i+1}(x)\, v\|^t}{\|Df^i(x)\, v\|^t}\,\frac{\|Df^i(x)\, b(x)\|}{\|Df^{i+1}(x)\, b(x)\|}\right).$$

Therefore

$$\int_X h\, d\theta=\lim_{n\to\infty}\int_X h\, d\theta_n=\lim_{n\to\infty}\frac{1}{n}\log\frac{\|Df^n(x_n)\, v_n\|^t}{\|Df^n(x_n)\cdot b(x_n)\|}=\alpha(f, K, t).$$

Now, by the ergodic decomposition theorem (see [9], Chapter 6, Theorem 6.4)

$$\int_X h\, d\theta=\int_X\left(\int_X h\, d\theta_{(x,\, v)}\right) d\theta,$$

where $\theta_{(x,\, v)}$ are ergodic G-invariant probability measures. It follows that for each $\varepsilon>0$ there exists an ergodic G-invariant measure $\nu=\theta_{(x,\, v)}$ for some (x, v), such that $\int_X h\, d\nu\geqslant\alpha(f, K, t)-\varepsilon$. Birkhoff's ergodic theorem implies the existence of a Borel set $X'\subset X$ such that $\nu(X')=1$ and

$$\lim_{n\to\infty}\frac{1}{n}\frac{\|Df^n(x)\, v\|^t}{\|Df^n(x)\, b(x)\|}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}h(G^i(x, v))=\int_X h\, d\nu\geqslant\alpha(f, K, t)-\varepsilon$$

for all $(x, v)\in X'$. Let μ be the marginal probability measure defined on K by

$$\mu(B)=\nu\{(x, v)\in X\colon x\in B\}.$$

Clearly μ is f-invariant and ergodic. On the other hand by Oseledec's Theorem (see, e.g., [9], Chapter 11)

$$\lim_{n \to \infty} \frac{1}{n} \log \left(\frac{\|Df^n(x)\, v\|^t}{\|Df^n(x)\, b(x)\|} \right) = t\lambda_2(\mu) - \lambda_1(\mu) \tag{6}$$

for μ almost all $x \in K$ and $v \in E_x$. Therefore

$$\sup_{\mu \text{ ergodic}} \; t\lambda_2(\mu) - \lambda_1(\mu) \geqslant \alpha(f, K, t).$$

To prove the converse inequality observe that, by Oseledec's Theorem, Eq. (6) holds for any ergodic measure μ, for μ almost all $x \in K$ and $v \in E_x$.
Q.E.D.

The following theorem is based on the C^r *section theorem* of Hirsch *et al.* [8]. It mimics the proof of a result due to Hirsch and Pugh [7, Theorem 6.3] concerning the smoothness of invariant splittings of Anosov diffeomorphisms.

THEOREM 3.4. *Let V be a nonmonotone manifold and let $K \subset V$ be an attractor for $f|\,V$. Suppose that f is $C^{1+k,r}$, $k \in \mathbf{N}$, $0 \leqslant r < 1$ and that*

$$\alpha(f, K, 1+k+r) < 0.$$

Then $B(K, f|\,V)$ is $C^{1+k,r}$.

Proof. Let $U \subset V$ be a fundamental neighborhood of K in V having compact closure. There exists an $n \in \mathbf{N}$ so large that $f^n(U) \subset U$. Also, since $\alpha(f, K) \leqslant \log(\rho) < 0$ by Proposition 3.1(b) and $\alpha(f, K, 1+k+r) < 0$ by assumption, it is possible to choose this n such that for all $x \in K$, we have

$$\frac{\|Df^n(x)\,|\,E_x\|}{\|Df^n(x)\, b(x)\|} \leqslant a$$

and

$$\frac{\|Df^n(x)\,|\,E_x\|^{1+k+r}}{\|Df^n(x)\, b(x)\|} \leqslant b$$

for some $a < 1$ and $b < 1$.

Define the map $g = f^{-n}$. From the preceding inequality we get that for all $x \in K$,

$$\frac{\|Dg(x)\, b(x)\|}{m(Dg(x)\,|\,E_x)} \leqslant a$$

and

$$\frac{\|Dg(x)\, b(x)\|}{(m(Dg(x) \mid E_x))^{k+1+r}} \leqslant b,$$

where $m(A) = \inf\{ \|Av\| : \|v\| = 1\}$ denotes the minimal norm. Given $\varepsilon > 0$ there exists a neighborhood $W \subset U$ of K and a C^∞ splitting $E' \oplus L'$ of $T_W \mathbf{R}^m = W \times \mathbf{R}^m$ so that for each $x \in W$ the derivative $Dg(x)$: $E'_x \oplus L'_x \to E'_{g(x)} \oplus L'_{g(x)}$ can be written as a block matrix

$$\begin{pmatrix} A_x & B_x \\ C_x & D_x \end{pmatrix}$$

with

$$\|B_x\| \leqslant \varepsilon, \quad \|C_x\| \leqslant \varepsilon, \quad \|A_x - Dg(\check{x}) \mid E_{\check{x}}\| \leqslant \varepsilon, \quad \|D_x - Dg(\check{x}) \mid L_{\check{x}}\| \leqslant \varepsilon,$$

where $\check{x}$ denotes the closest point to x in K.

Now choose $k \in \mathbf{N}$ large enough so that $f^{nk}(U) \subset W$ and set $X = f^{nk}(U)$. Let $\mathscr{L}(E'_x, L'_x)$ denote the space of linear maps from E'_x to L'_x and let Π: $\mathscr{L} \to X$ be the vector bundle over X whose fiber at x is $\mathscr{L}(E'_x, L'_x)$. Let $\mathscr{D}_x \subset \mathscr{L}(E'_x, L'_x)$ denote the unit ball in $\mathscr{L}(E'_x, L'_x)$ and let $\mathscr{D}$ be the bundle over X whose fiber at x is $\mathscr{D}_x$.

Let $X_0 = f^n(X) = g^{-1}(X)$, $h = g \mid X_0$ and $\mathscr{D}_0 = \mathscr{D} \cap \Pi^{-1}(X_0)$. If ε is small enough we can define a fiber map covering h,

$$\begin{array}{ccc} F\colon \mathscr{D}_0 & \to & \mathscr{D} \\ \downarrow & & \downarrow \\ h\colon X_0 & \to & X, \end{array}$$

given by the formula

$$F(x, P) = (h(x), F_x(P)),$$

where

$$F_x(P) = (C_x + D_x P) \circ (A_x + B_x P)^{-1}.$$

We will proceed by induction on $k \in \mathbf{N}$. We know from Tereščák [17] that V is C^1 and that $T_x V = E_x$ for all $x \in K$. Suppose now that V is C^k, $k \in \mathbf{N}$, $k \geqslant 1$ and f is $C^{1+k,r}$. Let l_x denote the Lipschitz constant of F_x: $D_x \to D_{h(x)}$ and let $\alpha_x = \|Dh^{-1}(h(x))\|$. It easily follows from the definition of F that for all $x \in X$,

$$l_x = \frac{\|Dg(\check{x}) \mid L_{\check{x}}\|}{m(Dg(\check{x}) \mid E_{\check{x}})} + O(\varepsilon)$$

and

$$\alpha_x = \frac{1}{m(Dh(x))} = \frac{1}{m(Dg(\check{x}) \mid E_{\check{x}})} + O(\varepsilon).$$

We then have the following situation.

(i) The map F, h and h^{-1} are C^k and h is overflowing, meaning that $X_0 \subset h(X_0)$.

(ii) For ε small enough $\sup_{x \in X_0} l_x \leqslant a + O(\varepsilon) < 1$ and $\sup_{x \in X_0} (\alpha_x)^{k+r} l_x \leqslant b + O(\varepsilon) < 1$.

Thus according to the C^r and Hölder section theorems [8, Theorems 3.5 and 3.8 and Remark 2, p. 38] there exists a unique section $\sigma: X_0 \to D_0$ whose image is invariant by F. That is

$$F(\text{image}(\sigma)) \cap E_0 = \text{image}(\sigma).$$

Furthermore, σ is $C^{k, r}$.

Given $x \in X_0$, let $P_x \in \mathscr{L}(E'_x, L'_x)$ denote the linear map whose graph is $T_x V$. We claim that $P_x = \sigma(x)$. Since $E_x = T_x V$ for $x \in K$ and $x \in X \to E'_x$ approximates $x \in K \to E_x$ we can always suppose (by choosing the neighborhood U small enough) that $\|P_x\| \leqslant 1$. This gives us a section $P: X_0 \to D_0$ given by $x \to P_x$. By definition of F we see that $\text{graph}(F_x(P_x)) = Dh(x) \cdot T_x V = T_{h(x)} V = \text{graph}(P_{h(x)})$. Hence, $P_{h(x)} = F_x(P_x)$. Then P is an invariant section and by uniqueness $P = \sigma$. This proves that V is $C^{1+k, r}$ in the neighborhood X_0 and K. Since $B(K, f \mid V) = \bigcup_{n \in \mathbf{N}} f^{-n}(X_0)$ and f is $C^{1+k, r}$, $B(K, f \mid V)$ is $C^{1+k, r}$. Q.E.D.

Theorem 2.2 now follows from Proposition 3.3 and Theorem 3.4. To prove Theorem 2.1 remark that $\alpha(f, K, t) = \alpha(f, M(f, K), t)$ by Proposition 3.3. Then use the estimates given by inequalities (3), (4), and (5).

4. APPLICATION TO COOPERATIVE VECTOR FIELDS

We now apply the precedent results to cooperative vector fields. Let $F: \mathbf{R}^m \to \mathbf{R}^m$ denote a smooth vector field on $\mathbf{R}^m$ on generating a local flow $\Phi = \{\Phi_t\}$. We assume that F is *cooperative*,

$$\frac{\partial F_i}{\partial x_j} \geqslant 0 \qquad \text{for} \quad i \neq j,$$

and *irreducible*, the matrices $DF(x) = ((\partial F_i / \partial x_j)(x))_{i, j}$ are irreducible for all $x \in \mathbf{R}^m$.

Under these conditions it is well known (see [3]) that the map $x \to \Phi_t(x)$ has positive derivatives for $t > 0$.

Throughout the remainder of this section we let $V \subset \mathbf{R}^m$ denote a nonmonotone manifold for F and $K \subset V$ an attractor for $\Phi \mid V$ (the flow induced by Φ on V). We further assume that F is C^{1+k} for some $k \in \mathbf{N}$.

Our first corollary concerns three dimensional systems.

COROLLARY 4.1. *Suppose $m = 3$ and that for every equilibrium or periodic point $p \in K$,*

$$(1+k)\,\lambda_2(p) - \lambda_1(p) < -\eta < 0,$$

where $\lambda_1(p) > \lambda_2(p)$ denote the largest and second largest real parts of the eigenvalues (respectively Floquet multipliers) of the equilibrium (respectively periodic point) p. Then $B(K, \Phi \mid V)$ is C^{1+k}.

Proof. Let μ be an ergodic invariant measure for Φ with support in K. By the Poincaré recurrence theorem the set of recurrent points has full measure. Furthermore, since K lies in a manifold homeomorphic to $\mathbf{R}^2$, the Poincaré-Bendixson theorem implies that every recurrent point is either an equilibrium or a periodic point and the result follows from Theorem 2.2. Q.E.D.

Remark. If γ is a nonstationary periodic orbit for Φ then γ is unordered and since 0 is always a Floquet multiplier, Propositions 3.1 and 3.3 imply that $\lambda_1(\gamma) > 0$. Suppose for example, that $K \subset \mathbf{R}^3$ is an irreducible attractor in the sense that $\Phi \mid K$ contains no proper attractor. Then it suffices to verify that $(1+k)\,\lambda_2(p) - \lambda_1(p) < 0$ for all equilibria $p \in K$, to ensure that $B(K, \Phi \mid K)$ is C^{1+k}.

Let $\lambda(x)$ denote the largest eigenvalue of the symmetric matrix

$$\frac{DF(x) + DF(x)^T}{2}$$

and let

$$d^2(x) = \min_{i \neq j} \frac{\partial F_i}{\partial x_j}(x)\,\frac{\partial F_j}{\partial x_i}(x).$$

If $M \subset \mathbf{R}^m$ is a compact invariant set for Φ, set

$$\lambda(M) = \sup_{x \in M} \lambda(x)$$

and

$$d(M) = \inf_{x \in M} d(x).$$

COROLLARY 4.2. *Let $M = M(\Phi, K) \subset V$ denote the closure of the union of the supports of all Φ-invariant Borel probability measures with support in K. Suppose that one of the two following conditions* ((i) *or* (ii)) *hold.*

(i) $k \, \|DF|M\| < 2(k+1)\, d(M)$,

(ii) $k\lambda(M) < 2(k+1)\, d(M)$.

Then $B(K, \Phi \,|\, V)$ is C^{1+k}.

Proof. We will apply Theorem 2.1 to the map Φ_t for some small $t > 0$. If $d(M) = 0$ (i) is impossible and (ii) implies that Φ_t is nonexpansive. As already noticed, this implies that $B(K, \Phi \,|\, V)$ is C^{1+k}.

Suppose that $d(M) \neq 0$. Let $x \in M$ and $t \in \mathbf{R}$. The matrix $A(t) = D\Phi_t(x)$ satisfies the variational equation $dA/dt = DF(\Phi_t(x))\, A$ with initial condition $A(0) = Id$. Therefore $D\Phi_t(x) = Id + tDF(x) + o(t)$. It easily follows that

$$\Psi(D\Phi_t(x)) \sim t^2 \min_{i \neq j} \frac{\partial F_i}{\partial x_j}(x)\, \frac{\partial F_j}{\partial x_i}(x),$$

where $f(t) \sim g(t)$ means $\lim_{t \to 0} f(t)/g(t) = 1$.

Thus, given any $1 > \varepsilon > 0$ there exists $\alpha > 0$ such that $0 < t < \alpha$ implies $\sqrt{\Psi(D\Phi_t(x))} \geqslant td(x)(1-\varepsilon)$. Therefore, for $0 < t < \alpha$,

$$\tau_{\mathrm{B}}(D\Phi_t(x)) \leqslant \frac{1 - td(x)(1-\varepsilon)}{1 + td(x)(1-\varepsilon)} \leqslant 1 - 2td(x)(1-\varepsilon) \leqslant 1 - 2td(M)(1-\varepsilon).$$

Thus,

$$\tau_{\mathrm{B}}(D\Phi_t(x))^{1+k} \leqslant 1 - 2(1+k)\, td(M)(1-\varepsilon) + o(t).$$

On the other hand

$$\begin{aligned} \|D\Phi_t(x)\|^k &= \|Id + tDF(x) + o(t)\|^k \\ &\leqslant (1 + t\, \|DF|M\| + o(t))^k = 1 + kt\, \|DF|M\| + o(t). \end{aligned}$$

Since ε can be chosen arbitrary small, it is now immediate to verify that condition (i) of the corollary implies condition (ii) of Theorem 2.1 for $f = \Phi_t$ and $t > 0$ small enough. Condition (ii) of the corollary is based on the following estimate for $\|D\Phi_t\|$. Let $\|\cdot\|_2$ denote the standard Euclidean norm on $\mathbf{R}^m$ and $\langle\ \rangle$ the associate inner product. Let $u \in \mathbf{R}^m$ with $\|u\|_2 = 1$. The variational equation satisfied by $D\Phi_t(x)$ gives

$$\frac{d\,\|D\Phi_t(x)\,u\|_2^2}{dt} = 2\,\langle D\Phi_t(x)\,u, DF(\Phi_t(x))\,D\Phi_t(x)\,u\rangle$$
$$= \langle (DF(\Phi_t(x)) + DF(\Phi_t(x))^T)\,D\Phi_t(x)\,u, D\Phi_t(x)\,u\rangle$$
$$\leqslant \lambda(x)\,\|D\Phi_t(x)\,u\|_2^2 \leqslant \lambda(M)\,\|D\Phi_t(x)\,u\|_2^2.$$

Thus,

$$\|D\Phi_t(x)\|_2 \leqslant e^{t\lambda(M)} = 1 + \lambda(M)\,t + o(t). \qquad \text{Q.E.D.}$$

Our last corollary is a persistence result.

COROLLARY 4.3. *Assume*

(a) *K is connected.*

(b) *Every equilibrium $p \in K$ (if any) is linearly unstable, that is $\lambda_1(p) > 0$.*

(c) *The condition given in Theorem 2.2 (or the sufficient condition given by Corollary 4.2) holds for $f = \Phi_1$ and $r = 0$.*

Let $U \subset \bar{U} \subset B(K, \Phi \mid V)$ be an open neighborhood of K in V. Then there exists $\delta > 0$ with the following property. Let $\tilde{F}$ denote a C^{1+k} vector field such that

$$\|F - \tilde{F}\|_{\bar{U}, C^1} = \sup_{x \in \bar{U}} \|F(x) - \tilde{F}(x)\| + \|DF(x) - D\tilde{F}(x)\| \leqslant \delta.$$

Then there exists a nonempty compact set $\tilde{K}$ and an immersed C^{1+k} hypersurface $\tilde{B} \subset \mathbf{R}^m$, both invariant by $\tilde{\Phi}$ (the flow induced by $\tilde{F}$), such that $B(\tilde{K}, \tilde{\Phi} \mid \tilde{B}) = \tilde{B}$. Furthermore if $\tilde{F}$ is cooperative, $\tilde{B}$ is a nonmonotone manifold.

Proof. We will obtain this corollary as a consequence of Fenichel persistence theorem [2]. First, notice that hypothesis (c) of the corollary and Proposition 3.3 imply

$$\alpha(f, K, 1 + k) < 0, \qquad (7)$$

where $f = \Phi_1$. We claim that hypothesis (b) implies that

$$\alpha(f, K, 0) < 0 \qquad (8)$$

Before proving (8) let us see how this can be applied to prove the corollary. Inequality (7) and Theorem 3.4 imply that $B(K, \Phi|K)$ is C^{1+k}. Therefore, by standard results on attractors and Lyapounov functions (see, e.g., [18]) there exists a compact connected C^{1+k} manifold D with boundary ∂D such that $K \subset D \subset U$ and $F|V$ points strictly inward ∂D. Using Fenichel's notation, define for $m \in D$, $A^t(m) = D\Phi_t(m)|E_m$, $B^t(m) = D\Phi_t(m)^{-1} b(\Phi_t)(m))$,

$$v(m) = \limsup_{t \to \infty} \|B^t(m)\|^{1/t}$$

and

$$\sigma(m) = \limsup_{t \to \infty} \frac{\log(\|A^t(m)\|)}{-\log(\|B^t(m)\|)}.$$

It is easy to deduce from (7) and (8) that for all $m \in D$, $\sigma(m) < 1/(k+1)$ and $v(m) < 1$. Therefore, by Fenichel persistence theorem [2, Theorem 1], for any C^{1+k} vector field $\tilde{F}$ C^1 close to F on D there is a C^{1+k} manifold $\tilde{D}$, C^{1+k} close to D positively invariant by $\tilde{\Phi}$ (the flow induced by $\tilde{F}$) and such that $\tilde{F}$ points inward $\partial\tilde{D}$. Since the normals to $\tilde{D}$ are close to the normals to D, we can (by choosing $\tilde{F}$ close enough to F) always assume that $\tilde{D}$ is unordered. Also, even if $\tilde{F}$ may not be a cooperative vector field, we can always assume that $\tilde{\Phi}$ has eventually positive derivatives in $\tilde{D}$ [3, Theorem 1.2]. That is there exists $t_0 > 0$ such that $D\tilde{\Phi}_t(x) \gg 0$ for $x \in D$ and $t \geqslant t_0$.

Set $\tilde{B} = \bigcup_{t \geqslant 0} \tilde{\Phi}_{-t}(\tilde{D})$ and $\tilde{K} = \bigcap_{t \geqslant 0} \tilde{\Phi}(\tilde{D})$. By construction $\tilde{B}$ is a C^{1+k} immersed invariant hypersurface and $\tilde{\Phi}|\tilde{B}$ admits $\tilde{K}$ as global attractor. Furthermore, if $\tilde{F}$ is a cooperative, then $\tilde{B}$ is unordered because $\tilde{D}$ is unordered and $\tilde{\Phi}$ has eventually positive derivatives in $\tilde{D}$.

Our last job is to prove inequality (8). Let $\alpha = \inf\{\lambda_1(p) : p \in K \cap F^{-1}(0)\}$ if $K \cap F^{-1}(0) \neq \emptyset$ and $\alpha = \infty$ otherwise. Let $\eta = -\log(\rho)$ where ρ is as in Proposition 3.1.

Suppose to the contrary that $\alpha(f, K, 0) \geqslant 0$. Then, according to Proposition 3.3, there exists an ergodic and invariant measure μ with support in K such that $\lambda_1(\mu) \leqslant \inf\{\eta/2, \alpha\}$.

It follows from Propositions 3.1 and 3.3 that $\lambda_1(\mu) - \lambda_2(\mu) > \eta$. Thus, $\lambda_2(\mu) < -\eta/2$. Now, by the Oseledec's theorem, there exists a set $R \subset K$ with $\mu(R) = 1$ such that for all $x \in R$, $w \in E_x$

$$\limsup_{t \to \infty} \frac{1}{t} \log(\|D\Phi_t(x)\, w\|) \leqslant \lambda_2(\mu) \leqslant -\eta/2,$$

and

$$\lim_{t \to \infty} \frac{1}{t} \log(\| D\Phi_t(x)\, b(x) \|) = \lambda_1(\mu).$$

Since $F(x) \in E_x$,

$$\limsup_{t \to \infty} \frac{1}{t} \log(\| D\Phi_t(x)\, F(x) \|) = \limsup_{t \to \infty} \frac{1}{t} \log(\| F(\Phi_t(x)) \|) \leqslant -\eta/2.$$

This implies that the trajectory of x has finite length and thus converges. On the other hand by Poincaré recurrence theorem we can always choose $x \in R$ such that $x \in \omega(x)$. Such an x is an equilibrium. Thus $\lambda_1(\mu) = \lambda_1(x) \geqslant \alpha$. This is contradictory. Hence (8) holds. Q.E.D.

ACKNOWLEDGMENTS

I am very grateful to Morris Hirsch for inspiring and helpful discussions and to the anonymous referee for careful reading and several suggestions which improved the earlier version of the paper. Part of this work was done during a visit at Berkeley made possible by NATO Grant CRG 950857.

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