Amenable actions of discrete groups

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Résumé

Cette thèse porte sur la question de la moyennabilité des actions de groupes discrets. L'objectif de cette thèse est l'étude de la classe $\mathcal{A}$ des groupes dénombrables admettant une action moyennable, fidèle et transitive sur un ensemble dénombrable infini. Un des résultats principaux de ce travail est de démontrer que tout sous-groupe d'indice fini d'un produit amalgamé de deux groupes libres au dessus d'un sous-groupe cyclique (en particulier les groupes de surfaces) est contenue dans $\mathcal{A}$. En outre, des propriétés héréditaires de la classe $\mathcal{A}$ sont établies, et de plus des résultats sur les groupes de Coxeter à angle droit sont présentés.

Mots clés. Actions moyennables de groupes, groupes à 1-relateur cycliquement pincé, groupes de Coxeter à angle droit, automorphismes de graphes avec infinité de bouts.

Keywords. amenable actions of groups, cyclically pinched 1-relator groups, right-angled Coxeter groups, automorphisms of infinitely many ended graphs.

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Contents

Chapter 1

Introduction

The theory of amenability was launched when the so-called Hausdorff-Banach-Tarski paradox, a striking counter-intuitive theorem, was announced in 1924 [28]. By generalizing the earlier work of Hausdorff, Banach and Tarski proved that, in n-dimensional Euclidian space $\mathbb{R}^n$ with $n \geq 3$, every two bounded subsets A and B with non-empty interiors can be decomposed as $A = \cup_{i=1}^k A_i$, $B = \cup_{i=1}^k B_i$ such that the set A_i can be transformed to B_i by some isometry ϕ_i of $\mathbb{R}^n$ for each i; however, such a *paradoxical* decomposition does not exist in $\mathbb{R}$ nor in $\mathbb{R}^2$. It was von Neumann who first noticed that the whole problem is essentially due to the algebraic property of the isometry group $SO(n)$ rather than the geometry of the space $\mathbb{R}^n$ [82]. The natural generalization of such a problem is to replace $\mathbb{R}^n$ by an arbitrary set X and $SO(n)$ by a group G acting on X. The next definition is due to von Neumann [82]:

Definition. An action of G on X is *amenable* if there exists a G-invariant mean on X, i.e. a map $\mu : 2^X = \mathcal{P}(X) \to [0, 1]$ such that $\mu(X) = 1$, $\mu(A \cup B) = \mu(A) + \mu(B)$ for every pair of disjoint subsets A, B of X, and $\mu(gA) = \mu(A)$, $\forall g \in G$, $\forall A \subseteq X$.

In his article [82], von Neumann introduced the class of amenable groups (called "messbar" in German) as groups whose action on themselves by left multiplication is amenable. He noticed that on the one hand (i) a free group on two generators in the group $SO(n)$ appears exactly when $n \geq 3$ causing such a paradoxical decomposition, and on the other hand (ii) any action of an amenable group is amenable.

Regarding (i), clearly the existence of a paradoxical decomposition of G on itself obstructs for G to be amenable (and the converse is also true by Tarski's work [76]), so that if G contains non-abelian free subgroups then G is non-amenable. The converse problem became known as the "von Neumann conjecture" in the 1950's, which was first formulated by Day in [22] where he also invented the term "amenable". Tits [79] gave a positive answer to this conjecture for the linear groups. For the general case, the first counterexample for a finitely generated group was given by Ol'shanskii [63] in 1980. A counterexample for a finitely presented group was given by Ol'shanskii and Sapir [64] in 2002.

About (ii), it is not hard to see that the converse holds for free actions. The question whether the converse holds for more general amenable actions was first formulated by Greenleaf in 1969 in [35, p.18]:

Question. If there is a G-invariant mean on X where G acts reasonably, is G amenable?

The reasonable action should be in the first place *faithful*, otherwise it is sufficient to examine the quotient group; and the action should furthermore be *transitive* since any G-action with finite orbit is amenable. The question remained longtime unanswered until van Douwen gave a counterexample in [81] which was published after his death:

Theorem 1. (van Douwen, 1990) *The free group on two generators F_2 admits a transitive, almost free (i.e. each non-trivial element of F_2 fixes only finitely many points) and amenable action on an infinite countable set.*

For one reason or another, the result of van Douwen did not appeal to a large number of mathematicians and it seemed that his work faded away next to the numerous results published these days. It was in 2004 when a conversation between Grigorchuk and Monod resurrected the interest of the question of Greenleaf, both not aware of the work of van Douwen at that time (but found later on). On the one side, Grigorchuk and Nekrashevych showed in [36] that the free groups admit faithful, transitive and amenable actions; and on the other side, Monod and Glasner obtained in [33] the same results independently and simultaneously. More generally, the second authors defined the following class of countable groups

$$\mathcal{A} = \{G \text{ countable} \mid G \text{ admits a faithful, transitive and amenable action}\}$$

and they showed numerous results concerning free products, whose main theorem states as follows:

Theorem 2. (Glasner-Monod, 2007) *The free product of any two countable groups is in $\mathcal{A}$ unless one factor has the fixed point property and the other has the virtual fixed point property.*

The main purpose of this thesis is the study of the class $\mathcal{A}$ of Glasner-Monod. When investigating the class $\mathcal{A}$, once the work is done for the free products, the natural next step is to investigate the amalgamated free products. Whilst the class $\mathcal{A}$ is closed under direct products, free products and extension of co-amenable subgroups, it is not closed under amalgamated free products even if two factors are amenable groups. Moreover, the class $\mathcal{A}$ is not closed under passing to (co-amenable) subgroups, and it is not known whether $\mathcal{A}$ is closed under passing to finite index subgroups or not.

The first major result of this thesis is the following theorem:

Theorem 3. *Any finite index subgroup of a double $\mathbb{F}_2 *_{\mathbb{Z}} \mathbb{F}_2$ of a free group on two generators over a cyclic subgroup is in $\mathcal{A}$.*

The starting point of the work on this theorem was to prove that the surface groups are in $\mathcal{A}$. The surface group Γ_2 of genus 2 is a particular case of a double of $\mathbb{F}_2$ over the cyclic subgroup generated by the commutator of two generators of $\mathbb{F}_2$, and the surface groups Γ_n with $n \geq 3$ are finite index subgroups of Γ_2. While contemplating the surface groups, we observed that a similar method could be realized for a certain type of reduced words (called "special words") including the commutators, and that every reduced word can be transformed to a special word by some automorphisms of $\mathbb{F}_2$.

The success for surface groups led us to examine the class of cyclically pinched one-relator groups, i.e. free products of two free groups $\mathbb{F}_n$ and $\mathbb{F}_m$ with $n,\ m \geq 2$ over a cyclic subgroup $\mathbb{Z} \simeq \langle c = d \rangle$ generated by a cyclically reduced non-primitive word c in $\mathbb{F}_n$ and d in $\mathbb{F}_m$. Being a natural generalization of surface groups, the class of cyclically pinched one-relator groups is one of the most studied classes of one-relator groups. It has been shown that in general, such a group shares many of the algebraic properties of surface groups [10].

The second central result of this thesis is the following theorem:

Theorem 4. *Any finite index subgroup of cyclically pinched one-relator group $\mathbb{F}_n *_{\mathbb{Z}} \mathbb{F}_m$ is in $\mathcal{A}$.*

As we are investigating a class of groups, stable properties should be considered. The next result gives some hereditary properties for the case of amalgamated free products:

Theorem 5.

1. *Let G, H be amenable groups and let A be a common finite subgroup of G and H. Then the amalgamated free product $G *_A H$ is in $\mathcal{A}$.*
2. *Let H be an amenable group and let $\pi : G \twoheadrightarrow H$ be a group epimorphism and let $A < G$ be a subgroup such that $\pi|_A$ is injective and $[H : \pi(A)] \geq 2$. Then the amalgamated free product $G *_A H = \langle G, H | \pi(a) = a, \forall a \in A \rangle$ is in $\mathcal{A}$. In particular, if G is amenable, the double of G over A is in $\mathcal{A}$.*

Exploring further the structure of the class $\mathcal{A}$, we focus on right-angled Coxeter groups. The right-angled Coxeter group is a group defined by a finite graph $\Gamma = (E(\Gamma), V(\Gamma))$ up to isomorphism and admits a presentation

$$W(\Gamma) = \langle s_1, s_2, \ldots, s_m | s_i s_j = s_j s_i \text{ if } (s_i, s_j) \in E(\Gamma); s_i^2 = 1, \forall 1 \leq i \leq m \rangle.$$

Except when the graph Γ is complete or disconnected, the associated group $W(\Gamma)$ is a non-trivial amalgamated free product. We show that a large class of right-angled Coxeter groups is contained in $\mathcal{A}$ (Corollary 5.13).

This manuscript consists of four chapters and three appendices. The first chapter introduces the essential notions for the understanding of the topic. Besides general facts concerning amalgamated free products, this chapter also reviews some fundamental results relevant to the amenability of groups.

In chapter 2, we give the solution to a problem concerning the theory of ends proposed to the author by Tullio Ceccherini-Silberstein at the beginning of her thesis. This chapter is not directly related to the study of class $\mathcal{A}$. Background information on the theory of ends on graphs and groups are discussed here. We then present the proof of the non-amenability of groups with infinitely many ends *without* appealing to the difficult theorem of Stallings. This work in collaboration with Alain Valette is published in [58] and the complete text may be found in Appendix A.

Chapter 3 is the core of this thesis. It contains a survey of class $\mathcal{A}$ and the principal results presented above. The result concerning Theorem 3 is accepted for publication in [56], and the result concerning Theorem 4 is submitted and available as preprint in [57]. Both papers are reproduced in Appendices B and C.

The final chapter focuses mainly on right-angled Coxeter groups. It also includes short discussion on right-angled Artin groups and braid groups.

Chapter 2

Preliminaries

This chapter is devoted to present the basic definitions and fundamental results necessary to the understanding of the following chapters. In the first section, we recall classical notions on free products with amalgamation. In the next section, we expound some well-known results around the amenability of groups.

2.1 Free products and amalgamated free products

2.1.1 Free groups and group presentations

Let $A = \{a_1, a_2, \ldots, a_n\}$ be an alphabet and let $S = A \cup A^{-1} = \{a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}\}$. A *word* on S is a finite sequence of elements of S. A word w on S is *reduced* if there is no subsequence of the form $a_i a_i^{-1}$ or $a_i^{-1} a_i$.

Definition 2.1.1. The *free group on n generators*, denoted by $\mathbb{F}_n$, is the group of all reduced words on $S = A \cup A^{-1}$, where the unit element is the empty word and the operation is given by juxtaposition of words followed by reductions (i.e. by deleting any pair of the form $a_i a_i^{-1}$ or $a_i^{-1} a_i$). The integer n is called the *rank* of $\mathbb{F}_n$.

Theorem 2.1. (Universal property of free groups) *Let $\mathbb{F}_n$ be a free group on S. Given any application f from A to a group G, there exists a unique homomorphism $\widetilde{f} : \mathbb{F}_n \to G$ such that $\widetilde{f}|_A = f$.*

Proof. If $w = a_{i_1}^{\pm 1} a_{i_2}^{\pm 1} \cdots a_{i_k}^{\pm 1}$ is a reduced word on S, we define

$$\widetilde{f}(w) = f(a_{i_1})^{\pm 1} f(a_{i_2})^{\pm 1} \cdots f(a_{i_k})^{\pm 1}.$$

It is easy to verify that $\widetilde{f}$ is the unique homomorphism. □

Corollary 2.2. *Every group generated by n elements is a quotient of $\mathbb{F}_n$.*

Example 2.1.2. The abelianization of $\mathbb{F}_n$ is $\mathbb{Z}^n$, $\forall n \geq 1$.

The most important theorem on subgroups of free groups is due to J. Nielsen (1921) for finitely generated groups and to O. Schreier (1927) in general case:

Theorem 2.3. (Nielsen-Schreier Theorem) *Every subgroup of a free group is a free group.*

Definition 2.1.3. Let $\mathbb{F}_n$ be a free group on S and let R be a set of some reduced words on S. A group G admits a *presentation*

$$G = \langle\, S \,|\, R \,\rangle$$

if it is isomorphic to the quotient of $\mathbb{F}_n$ by the normal subgroup generated by R. We call R the *set of defining relations* and the elements $r \in R$ the *relators* of the presentation.

Instead of writing r_i in the defining relation, it is customary to write $r_i = 1$.

Example 2.1.4. 1. $\mathbb{F}_n = \langle a_1, \ldots, a_n | \emptyset \rangle = \langle a_1, \ldots, a_n \rangle$.

2. $\mathbb{Z}^n = \langle a_1, \ldots, a_n | a_i a_j a_i^{-1} a_j^{-1}, \forall i, j \rangle = \langle a_1, \ldots, a_n | a_i a_j = a_j a_i, \forall i, j \rangle$.

3. The infinite dihedral group D_∞ admits a presentation

$$D_\infty = \langle x, y | x^2, y^2 \rangle = \langle x, y | x^2 = y^2 = 1 \rangle.$$

A *one-relator group* is the one that admits a presentation with one defining relator (as $\mathbb{Z}^2$ in Example 2.1.4). The theory of one-relator groups is developed from the work of Dehn in [24], where he solved some important problems in combinatorial group theory known as the *word problem, conjugacy problem* and *isomorphism problem* for the fundamental groups of closed orientable surfaces of genus greater than or equal to 2. One of the basic theorems concerning one-relator groups is due to Magnus [54]:

Theorem 2.4. (The Freiheitssatz, 1930) *Let $G = \langle a_1, \ldots, a_n | r \rangle$ be a one-relator group with $r \in \mathbb{F}_n$ a cyclically reduced word (i.e. r is reduced and if $r = a_{i_1} \cdots a_{i_k}$ then $a_{i_1}^{-1} \neq a_{i_k}$) where a generator a_j appears in r. Then the subgroup H generated by $\{a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n\}$ is free on $n-1$ generators.*

2.1.2 Free products

Definition 2.1.5. Let G, H be groups with presentations $G = \langle S_G | R_G \rangle$ and $H = \langle S_H | R_H \rangle$ such that S_G and S_H are disjoint. The *free product* of G and H is the group:

$$G * H = \langle S_G, S_H | R_G, R_H \rangle.$$

The groups G and H are called the *factors* of $G * H$.

Example 2.1.6. 1. $\mathbb{F}_2 \simeq \mathbb{Z} * \mathbb{Z}$.

2. $\mathbb{F}_{n+m} \simeq \mathbb{F}_n * \mathbb{F}_m$, $\forall n, m \geq 1$.

3. $D_\infty \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

4. The modular group $PSL_2(\mathbb{Z})$ is a free product $PSL_2(\mathbb{Z}) \simeq \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

A *normal form*, or equivalently *reduced sequence* of $G * H$ is a sequence g_1, $\dots$, g_n, $n \geq 1$, of elements of $G * H$ such that each $g_i \neq 1$ is in one of the factors, and successive g_i, g_{i+1} are not in the same factor. A sequence $g_1, \cdots, g_n$ is *cyclically reduced* if all cyclic permutations of $g_1, \cdots, g_n$ are reduced. The sequence is called *weakly cyclically reduced* if it is reduced and $g_1 \neq g_n^{-1}$. "Cyclically reduced" requires that there is neither cancellation nor consolidation between g_1 and g_n, while "weakly cyclically reduced" allows g_1 and g_n to be in the same factor.

The next theorem shows that one can write every word of $G * H$ in a unique way given by the normal form. One may refer for example to Chapter IV, Theorem 1.2. in [53] for the proof.

Theorem 2.5. *In the free product $G*H$, the following two equivalent properties hold:*

1. *If $g_1, \dots, g_n$ is a reduced sequence, then $w = g_1 \cdots g_n \neq 1$ in $G * H$.*

2. *Every element w of $G * H$ can be uniquely written as $w = g_1 \cdots g_n$, where $g_1, \dots, g_n$ is a reduced sequence.*

Clearly, every element of $G * H$ is conjugate to a cyclically reduced element. To test the conjugacy of two words, there is a well-known theorem (Chapter IV, Theorem 1.4. in [53]):

Theorem 2.6. *Two cyclically reduced elements $w = g_1 \cdots g_n$ and $u = h_1 \cdots h_m$ are conjugate in a free product $G * H$ if and only if the sequences $g_1, \dots, g_n$ and $h_1, \dots, h_m$ are cyclic permutations of each other.*

It is straightforward to prove that the free product of two groups is unique up to isomorphism:

Theorem 2.7. (Universal property of free products) *Let G_1, G_2 be two groups. There is a unique group $G = G_1 * G_2$ and injections $i_1 : G_1 \hookrightarrow G$ and $i_2 : G_2 \hookrightarrow G$ such that for every group H and for every two homomorphisms $f_1 : G_1 \to H$ and $f_2 : G_2 \to H$, there exists a unique homomorphism $f : G \to H$ such that $f \circ i_1 = f_1$ and $f \circ i_2 = f_2$.*

About the structure of subgroups of free products, we have:

Theorem 2.8. (Kurosh subgroup theorem, 1934) *Let $G = G_1 * G_2$ be a free product, and let $H < G$ be a subgroup of G. Then H is isomorphic to the free product*

$$H = F * (*_i H_i)$$

where F is a free group and H_i is the intersection of H with a conjugate of some factor of G.

For example, if $G = \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$ with prime numbers p and q, then a finitely generated subgroup H of G is isomorphic to a free product of a free group, some copies of $\mathbb{Z}/p\mathbb{Z}$, and some copies of $\mathbb{Z}/q\mathbb{Z}$. Let us mention that the subgroups of finite index in such groups are classified in [50].

By the Kurosh subgroup Theorem, immediately we have:

Corollary 2.9. *If a subgroup H of a free product $G_1 * G_2$ intersects trivially with every conjugate of the factors, then H is free.*

One way to construct free subgroups in a given group is attributed to F. Klein:

Theorem 2.10. (Ping-pong Lemma) *Let Γ be a group acting on a set X. Let G_1, G_2 be two subgroups of Γ such that G_1 contains at least 3 elements and G_2 contains at least 2 elements. Suppose that there exist disjoint non-empty subsets Y_1, Y_2 of X such that $g_1(Y_1) \subseteq Y_2$, $\forall g_1 \in G_1 \setminus \{1\}$ and $g_2(Y_2) \subseteq Y_1$, $\forall g_2 \in G_2 \setminus \{1\}$. Then the subgroup of Γ generated by $G_1 \cup G_2$ is isomorphic to the free product $G_1 * G_2$.*

As an example, it is easy to see that the two matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ generate a free subgroup on 2 generators of $SL_2(\mathbb{Z})$. In general, if G is a torsion-free hyperbolic group and g, $h \in G$ do not commute, then there exists $M \geq 1$ such that for any integers $n \geq M$, $m \geq M$, the subgroup $H = \langle g^n, h^m \rangle$ of G is free of rank two (see [38] for the proof).

2.1.3 Free products with amalgamation

Definition 2.1.7. Let G, H be groups with presentations $G = \langle S_G | R_G \rangle$ and $H = \langle S_H | R_H \rangle$. Let A be a group such that there exist two injective homomorphisms $i : A \hookrightarrow G$ and $j : A \hookrightarrow H$. The *free product of G and H with amalgamation over A* is the group:

$$G *_A H = \langle S_G, S_H | R_G, R_H, i(a) = j(a), \forall a \in A \rangle.$$

The groups G and H are called the $factors$ of the free product with amalgamation, and the group A is called the *amalgamated subgroup*. We can view the group $G *_A H$ as the quotient of the free product $G * H$ by the normal subgroup generated by $\{i(a)^{-1} j(a), \forall a \in A\}$.

Often we take $A < G$ a subgroup of G, $B < H$ a subgroup of H and $\phi : A \to B$ an isomorphism. So the corresponding amalgamated free product of G and H is

$$\langle G, H | a = \phi(a), \forall a \in A \rangle.$$

Example 2.1.8.

1. The free product $G * H$, with $A = \{1\}$.
2. The fundamental group $\pi_1(\Sigma_g)$ of the closed orientable surface Σ_g of genus $g \geq 2$ can be written as $\pi_1(\Sigma_g) = \mathbb{F}_{2(g-1)} *_{\mathbb{Z}} \mathbb{F}_2$ (see Section 2.1.5 for details).

3. $SL_2(\mathbb{Z}) \simeq \mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$, where

$$\mathbb{Z}/6\mathbb{Z} \simeq \left(\begin{array}{cc} 1 & -1 \\ 1 & 0 \end{array} \right), \ \mathbb{Z}/4\mathbb{Z} \simeq \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \text{ and } \mathbb{Z}/2\mathbb{Z} \simeq \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right).$$

Definition 2.1.9. An amalgamated free product $G *_A H$ is called *double of G over A* if H is isomorphic to G and the amalgamated subgroup of H is given by the isomorphism. Such a group has a presentation $\langle G, \phi(G) | a = \phi(a), \forall a \in A \rangle$ where $\phi : G \to H$ is an isomorphism.

Example 2.1.10. The fundamental group $\pi_1(\Sigma_2)$ of the closed orientable surface of genus 2 is double of $\mathbb{F}_2$ over $\mathbb{Z}$:

$$\langle a_1, a_2, b_1, b_2 | [a_1, a_2] = [b_1, b_2] \rangle \simeq \mathbb{F}_2 *_{\mathbb{Z}} \mathbb{F}_2,$$

where the first factor $\mathbb{F}_2$ is generated by a_1, a_2, the second factor $\mathbb{F}_2$ is generated by b_1, b_2, and the isomorphism $\phi : \langle a_1, a_2 \rangle \to \langle b_1, b_2 \rangle$ is given by $\phi(a_1) = b_1$ and $\phi(a_2) = b_2$.

Let $\Gamma = G *_A H$ be an amalgamated free product. Choose a set of representatives S_1 (respectively S_2) of left cosets of A in G (respectively in H).

Theorem 2.11. *(Theorem 1 in [72]) Every g in Γ can be uniquely written as $g = as_1s_2 \cdots s_n$, $n \geq 0$, with $a \in A$ and $s_i \in S_{i_k} \setminus \{1\}$ such that $s_i^{-1} \neq s_{i+1}$, $\forall 1 \leq i \leq n-1$.*

So every $\gamma \in \Gamma \setminus A$ can be written as one of the following four forms:

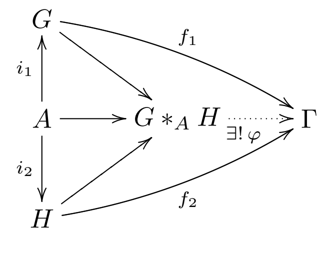
$$(i)\, ag_1h_1 \cdots g_nh_n;\ (ii)\, ah_1g_1 \cdots g_nh_n;\ (iii)\, ag_1h_1 \cdots h_ng_n;\ (iv)\, ah_1g_1 \cdots h_ng_n$$

where $a \in A$, $g_k \in G \setminus A$ and $h_k \in H \setminus A$ for all k, and $n \geq 1$.

Remark 2.1.11. It is possible to state the above theorem without using the sets of representatives S_i (cf. Theorem 2 in [72]).

The free product with amalgamation satisfies the universal property which can be proved in a similar way as for free products:

Theorem 2.12. (Universal property of free products with amalgamation) *Let $i_1 : A \hookrightarrow G$, $i_2 : A \hookrightarrow H$ be injective homomorphisms. For every group Γ and every homomorphisms $f_1 : G \to \Gamma$, $f_2 : H \to \Gamma$ such that $f_1 \circ i_1 = f_2 \circ i_2$, there exists a unique homomorphism $\varphi : G *_A H \to \Gamma$ such that the following diagram commutes:*



There is an amalgamated free product version of the Kurosh subgroup theorem due to H. Neumann (1948):

Theorem 2.13. *(Chapter IV. Theorem 6.6. in [53]) Let* $G = \langle G_1, G_2 | a = \phi(a), \forall a \in A\rangle$ *be a non-trivial free product with amalgamation. Let* H *be a subgroup of* G *such that all conjugates of* H *intersect* A *trivially. Then* H *is isomorphic to the free product:*

$$H \simeq K * (*_i H_i)$$

where K *is a free group and* H_i *is the intersection of* H *with a conjugate of some factor of* G*.*

As a corollary, if a subgroup H of an amalgamated free product G intersects trivially with all conjugates of the factors of G, then H is free.

Remark 2.1.12. Aside from amalgamated free products, there is another basic group construction; that is the *HNN extension* which is defined as follows. Let G be a group and let A and B be subgroups of G with $\phi : A \to B$ an isomorphism. The *HNN extension of* G *relative to* ϕ is the group:

$$HNN(G, A, B, \phi) = \langle G, t | t^{-1}at = \phi(a), \forall a \in A\rangle.$$

The group G is called the *base* of $HNN(G, A, B, \phi)$, the letter t is called the *stable letter*, and the subgroups A and B are called the *associated subgroups*. For example, for n, $m \in \mathbb{Z}$, the group

$$HNN(\mathbb{Z}, n\mathbb{Z}, m\mathbb{Z}, \phi) = \langle a, b | b^{-1}a^n b = a^m\rangle =: BS(m, n)$$

is an HNN extension of $\mathbb{Z} = \langle a\rangle$ with subgroups $n\mathbb{Z} = \langle a^n\rangle$ and $m\mathbb{Z} = \langle a^m\rangle$ with $\phi(a^n) = a^m$, and b is the stable letter. This group is called the *Baumslag-Solitar group*, introduced by these authors as examples of two-generator one-relator non-Hopfian groups for a large number of pairs m and n [14].

2.1.4 Bass-Serre theory

In this section, we shall give a brief description of Bass-Serre theory. To fix the notations, let us recall some definitions on graphs given in [72]. A *graph* X consists of the set of vertices $V(X)$ and the set of edges $E(X)$, and two applications $E(X) \to E(X)$; $e \mapsto \bar{e}$ such that $\bar{\bar{e}} = e$ and $\bar{e} \neq e$, and $E(X) \to V(X) \times V(X)$; $e \mapsto (i(e), t(e))$ such that $i(e) = t(\bar{e})$. An element $e \in E(X)$ is a *directed edge* of X and $\bar{e}$ is the *inverse edge* of e. For all $e \in E(X)$, $i(e)$ is the *initial vertex* of e and $t(e)$ is the *terminal vertex* of e.

Let G be a group acting on a tree X. Throughout this section, the action will be without inversion (i.e. $ge \neq \bar{e}$, for all $g \in G$ and for every edge $e \in E(X)$). An action on X is $free$ if it acts without inversion and it does not fix any vertex.

An important connection between free actions on a tree and free groups is given by the following theorem:

Theorem 2.14. *(Theorem 4 in [72]) A group G acts freely on a tree if and only if G is a free group.*

This gives a very short proof of Nielsen-Schreier Theorem. Indeed, if H is a subgroup of a free group G, then there is a tree X on which G acts freely, so the subgroup H acts also on X freely, thus H is a free group by Theorem 2.14.

Now we shall give the connection between other actions and amalgamated free products.

A *fundamental domain of X mod G* is a subgraph T of X such that $T \simeq G \setminus X$, where $G \setminus X$ is the quotient graph.

Proposition 2.15. *(Proposition 17 in [72]) Let G be a group acting on a tree X. There exists a fundamental domain of X mod G if and only if $G \setminus X$ is a tree.*

Example 2.1.13. If T has one edge and two vertices, we have the graph:

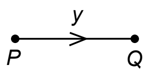


Figure 2.1: Segment T

A graph isomorphic to Figure 2.1 is called *segment*.

The crucial result is that there is an equivalence between groups acting on a tree with a segment as a fundamental domain and amalgamated free products. Precisely, we have:

Theorem 2.16. *(Theorem 6 in [72]) Let G be a group acting on a tree X. Let T be a segment as in Figure 2.1 of X. Suppose that this segment is a fundamental domain of X mod G. Then*

$$G \simeq Stab_G(P) *_{Stab_G(y)} Stab_G(Q),$$

where $Stab_G(P)$, $Stab_G(Q)$ and $Stab_G(y)$ are the stabilizers of P, Q and y respectively.

Conversely,

Theorem 2.17. *(Theorem 7 in [72]) Let $G = H *_A K$ be an amalgamated free product. Then there is a tree X (unique up to isomorphism) such that*

(i) G acts on X with a fundamental domain a segment T as in Figure 2.1;

(ii) $Stab_G(P) = H$, $Stab_G(Q) = K$ and $Stab_G(y) = A$.

In fact, this tree X is defined as follows:

· the vertex set $V(X) = G/H \sqcup G/K$,

· the edge set $E(X) = G/A$,

· $i(gA) = gH$ and $t(gA) = gK$, $\forall gA \in E(X)$.

Remark 2.1.14. There is an equivalence between groups acting on a tree with a *loop* (Figure 2.2) as the quotient graph and HNN-extensions:

1. Let G be a group acting on a tree X with a loop as the quotient graph of X mod G. If x, $y \in E(X)$ and $t \in G$ verify $P = t(x) = i(y)$ and $t \cdot x = y$, then $G \simeq HNN(Stab_G(P), Stab_G(y), Stab_G(x), \phi)$, where $\phi : Stab_G(y) \to Stab_G(x)$ is defined by $\phi(g) = t^{-1}gt$.

2. Conversely, if $G = HNN(H, A, B, \phi)$, then there is a tree X on which G acts with a quotient graph a loop and the tree X is given by:

   · the vertex set $V(X) = G/H$,

   · the edge set $E(X) = G/A \sqcup G/B$,

   · $i(gA) = gH$, $t(gA) = gtH$ and $\overline{gA} = gtB$, $\forall gA \in E(X)$,

   · $i(gB) = gH$, $t(gB) = gt^{-1}H$ and $\overline{gB} = gt^{-1}A$, $\forall gB \in E(X)$.

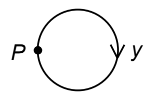


Figure 2.2: Loop

Remark 2.1.15. More generally, the fundamental theorem of Bass-Serre theory states that if G acts on a tree X without inversion, then G is isomorphic to the fundamental group of the quotient graph of groups $G \setminus X$ and the tree X is isomorphic to the Bass-Serre covering tree $\widetilde{G \setminus X}$ (Theorem 13 in [72]). This applies for example to Kurosh subgroup theorem. Indeed, let $G = H * K$ be a free product. There is a G-set tree X such that G is isomorphic to the fundamental group of the quotient graph of groups $G \setminus X$, which is a segment, and the vertex groups are H and K, and the edge groups are trivial. If $Q < G$ is a subgroup, then Q acts also on X, and the vertex groups of the quotient graph of groups $Q \setminus X$ are $Q \cap gStab_G(H)g^{-1}$ and $Q \cap gStab_G(K)g^{-1}$ for $g \in G$ (since $gStab_G(H)g^{-1} = Stab_G(gH)$), and the edge groups are trivial. By the theorem of Bass-Serre, Q is isomorphic to the fundamental group $\pi_1(Q \setminus X)$ of the quotient graph of groups $Q \setminus X$ and $\pi_1(Q \setminus X)$ is generated by the vertex groups and the fundamental group F (in the ordinary sense) of the underlying graph $Q \setminus X$ which is free. Therefore Q is isomorphic to a free product $F * (*_i Q_i)$ where F is free and Q_i is the intersection of Q with a conjugate of some factor of G. See the Section 5.5 in [72] for omitted definitions and more detail.

2.1.5 Free product of free groups with cyclic amalgamation

A *surface group* Γ_g is the fundamental group of a closed orientable surface Σ_g of genus $g \geq 2$. By van Kampen's theorem, Γ_g has the presentation

$$\langle a_1, b_1, \dots, a_g, b_g | [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle.$$

By letting $c = [a_1, b_1] \cdots [a_{g-1}, b_{g-1}]$ and $d = [a_g, b_g]^{-1}$, the group decomposes as the free product of the free group $\mathbb{F}_{2(g-1)}$ on $a_1, b_1, \dots, a_{g-1}, b_{g-1}$ and the free group $\mathbb{F}_2$ on a_g, b_g amalgamated over the cyclic subgroup generated by c in $\mathbb{F}_{2(g-1)}$ and d in $\mathbb{F}_2$, hence $\Gamma \simeq \mathbb{F}_{2(g-1)} *_{\mathbb{Z}} \mathbb{F}_2$.

About subgroups of surface groups, Karrass, Hoare and Solitar showed in [1] and [2] the following theorem:

Theorem 2.18. *Any subgroup of finite index in a surface group is again a surface group, and any subgroup of infinite index is a free group.*

The converse of the finite index case of this theorem is proved in [88] and [89]:

Theorem 2.19. *A torsion-free finite extension of surface group is again a surface group.*

So one sees for example that any $2g-1$ elements of a surface group Γ_g generate a free subgroup: indeed, any finite index subgroup of Γ_g is the fundamental group of a covering space $\Sigma_{g'}$ of Σ_g with $g' \geq g$ and the rank of $\Gamma_{g'}$ is $2g'$, so a subgroup generated by $g' \leq 2g - 1$ elements must be of infinite index.

A generalization of surface groups leads to the concept of the class of cyclically pinched one-relator groups. A group G is *cyclically pinched one-relator group* if it admits a presentation

$$G = \langle a_1, \dots, a_n, b_1, \dots, b_m | c = d \rangle$$

where $1 \neq c = c(a_1, \dots, a_n)$ is a cyclically reduced non-primitive (not part of a basis) word in the free group $A = \langle a_1, \dots, a_n \rangle$, and $1 \neq d = d(b_1, \dots, b_m)$ is a cyclically reduced non-primitive word in the free group $B = \langle b_1, \dots, b_m \rangle$.

Such a group is the amalgamated free product $A *_{\mathbb{Z}} B$ of two free groups A and B over the cyclic subgroup generated by $c = d$.

Let us cite some results on cyclically pinched one-relator groups.

Definition 2.1.16. A group G is *residually finite* if the following equivalent properties hold:

- for each non-identity element in G, there is a normal subgroup of finite index not containing that element;
- the intersection of all its (normal) subgroups of finite index is trivial;
- G can be embedded inside the direct product of a family of finite groups;

Example 2.1.17. $SL_n(\mathbb{Z})$ is residually finite for every $n \geq 1$. Indeed, the family of reductions mod p, where p is a prime number, separates points of $SL_n(\mathbb{Z})$.

Theorem 2.20. (G. Baumslag, [11]) *Cyclically pinched one-relator groups are residually finite.*

Notice that the above theorem is not true for one-relator groups in general: the Baumslag-Solitar groups $B(p, q)$ with distinct prime numbers p, q, are non-Hopfian, so they are not residually finite [14]. Other counterexamples are recently given by Baumslag, Miller and Troeger in [32].

The *conjugacy separability* of cyclically pinched one-relator groups has been shown by J. Dyer [26], that is, for any two non-conjugate elements g and h of a cyclically pinched one-relator group G, there is a finite quotient of G such that the images of g and h are not conjugate. Brunner, Burns and Solitar [3] proved that a cyclically pinched one-relator group G is *subgroup separable* or *LERF*, that is, if H is any finitely generated subgroup of G and $g \in G \setminus H$, then there exists a finite quotient of G such that the image of g lies outside of the image of H. Equivalently, H is the intersection of finite index subgroups containing it.

Concerning the decision problem, the three problems are solvable for cyclically pinched one-relator groups: the word problem is obtained by the Freiheitssatz, the conjugacy problem was solved by Lipschutz [51] (and Juhasz [44] for one-relator groups in general), and the isomorphism problem was answered by Rosenberger [71] (and the isomorphism problem for one-relator groups in general remains unsolved).

From a result of G. Baumslag and P. Shalen in [13], a one-relator group with at least three generators admits an amalgamated free product decomposition $A *_C B$ such that the three factors A, B and C are finitely generated. Such a decomposition is called the *Baumslag-Shalen decomposition.* Except cyclically pinched one-relator groups, very little is known for the factors in general. Some partial results are given in [29]:

Theorem 2.21. *Let G be a torsion-free one-relator group with Baumslag-Shalen decomposition $A *_C B$ such that A and B are free groups. If either C has finite index in both factors or C is in the derived subgroup in both factors, then G is a cyclically pinched one-relator group.*

For more results and details on cyclically pinched one-relator groups, one may refer to [30].

2.2 Amenability

2.2.1 Equivalent definitions

For a set X, a *mean* on X is a map $\mu : 2^X = \mathcal{P}(X) \to [0, 1]$ such that

(i) $\mu(X) = 1$;

(ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ for every pair of disjoint subsets A, B of X.

The set of all means on X is denoted by $\mathcal{M}(X)$. In other words, the set $\mathcal{M}(X)$ is the set of finitely additive probability measures defined on all subsets of X. If G acts on X, a mean $\mu \in \mathcal{M}(X)$ is said to be *G-invariant* if

(iii) $\mu(gA) = \mu(A)$, for all $g \in G$ and $A \subseteq X$.

The set $\mathcal{M}(X) \subset [0,1]^{\mathcal{P}(X)}$ endowed by product topology, is a convex compact topological space.

Definition 2.2.1. An action of G on X is *amenable* if there exists a G-invariant mean μ in $\mathcal{M}(X)$. A group G is *amenable* if the left multiplication of G on itself is amenable.

Example 2.2.2.
1. Finite groups are amenable by taking the normalized counting measure.
2. $\mathbb{Z}$ is amenable; the limit point of the sequence of means $\mu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_i \in \mathcal{M}(\mathbb{Z})$ is $\mathbb{Z}$-invariant.

There are many equivalent formulations for the definition of the amenability. For the proof, one may refer for example to [9]:

Theorem 2.22. *The following are equivalent:*

1. *The group G is amenable.*
2. *The group G is not paradoxical.*
3. *(Fixed point property) Any continuous affine action of G on a non-empty convex compact subset X of a locally convex topological vector space has a fixed point.*
4. *(Reiter's condition) For every finite subset $F \subset G$ and every $\varepsilon > 0$, there exists $\phi \in Proba(G) = \{\phi \in \ell^1(G) | \phi \geq 0, ||\phi||_1 = 1\}$ such that $||g\phi - \phi||_1 < \varepsilon$, $\forall g \in F$.*
5. *(Følner's condition) For every finite subset $F \subset G$ and every $\varepsilon > 0$, there exists a finite subset $A \subset G$ such that $|gA \triangle A| < \varepsilon |A|$, $\forall g \in F$.*
6. *$1_G \prec \lambda_G$, i.e. the left unitary representation $(\lambda_G, \ell^2(G))$ of G almost has invariant vectors.*

Remark 2.2.3.
1. An action of G on X is *paradoxical* if for some positive integers m and n, there exist pairwise disjoint subsets $A_1, \ldots, A_m, B_1, \ldots, B_n$ of X and $g_1, \ldots, g_m, h_1, \ldots, h_n$ of G such that $X = \cup_{i=1}^{m} g_i A_i = \cup_{i=1}^{n} h_i B_i$. A group is *paradoxical* if the G-action on itself by left multiplication is paradoxical. The equivalence between 1 and 2 in Theorem 2.22 follows from Tarski's Theorem (Corollary 9.2. in [84]) which states the equivalence between the presence of invariant measures and the absence of paradoxical decompositions.

2. In the case where G is finitely generated (i.e. there is a finite subset Q of G such that $G = \cup_{n\in\mathbb{N}} Q^n$), the Følner's condition is equivalent to the existence of a sequence $\{A_n\}_{n\geq 1}$ of finite non-empty sets A_n of G such that for every g in some finite generating set Q of G, one has
$$\lim_{n\to\infty} \frac{|A_n \triangle g \cdot A_n|}{|A_n|} = 0.$$
Such a sequence is called *Følner's sequence* of G.

3. It is straightforward to prove that the Følner's sequence verifies:
$$\lim_{n\to\infty} \frac{|\partial A_n|}{|A_n|} = 0,$$
where ∂A_n is the boundary of A_n in the Cayley graph $\mathcal{G}(G, S)$.

Example 2.2.4. Any sequence of intervals $\{I_n\}_{n\geq 1}$ with $|I_n| \to \infty$ is a Følner sequence of $\mathbb{Z}$.

2.2.2 Stability properties

For the hereditary properties, one has the following theorem which can be easily proved by using the fixed point property of Theorem 2.22:

Theorem 2.23. *1. Every subgroup of an amenable group is amenable.*

2. If G is amenable and N is a normal subgroup of G, then the quotient G/N is amenable.

3. Let N be a normal subgroup of G. If N and G/N are amenable, then G is amenable.

4. The direct union of amenable groups is amenable.

Thus we see that G is amenable if and only if it is locally amenable, that is, for every finite subset F of G, the subgroup $\langle F \rangle$ of G generated by F is amenable. It is immediate that the finite direct product of amenable groups is amenable. Thus by the fundamental theorem of finitely generated abelian groups[1], we have:

Corollary 2.24. *Abelian groups are amenable. Therefore, virtually nilpotent groups and virtually solvable groups are amenable.*

Remark 2.2.5. 1. The infinite direct sum of amenable groups is amenable since $\bigoplus_{i\geq 1} G_i$ is isomorphic to the group $\bigcup_{n\geq 1} \prod_{i=1}^{n} G_i$.

2. The infinite direct product of amenable groups is not necessarily amenable; take $G_n = SL_2(\mathbb{Z}/n\mathbb{Z})$, for $n \geq 2$ and notice that the groups G_n are amenable but the group $\prod_{n\geq 2} G_n$ contains $SL_2(\mathbb{Z})$, a non-amenable subgroup (cf. Section 2.2.3).

[1]It states that a finitely generated abelian group is the direct sum of a free abelian group of finite rank and a finite abelian group.

A group is called *elementary amenable* if it can be constructed from finite groups and abelian groups by the operations of taking subgroups, quotients, extensions, and direct unions. By Theorem 2.23, the class EM of elementary amenable groups is contained in the class of amenable groups. The inclusion is strict since Grigorchuk's group of intermediate growth is not elementary amenable (see [37]).

Definition 2.2.6. A subgroup H of G is *co-amenable* in G if the action of G on G/H is amenable.

Example 2.2.7. Any finite index subgroup of G is co-amenable in G.

It is clear that if H is a normal subgroup of G, then the subgroup H is co-amenable if and only if G/H is amenable. By using an argument of the barycentre, one can show that

Theorem 2.25. *A subgroup H of G is co-amenable if and only if for every action of G on a convex compact non-empty space K, $K^H \neq \emptyset$ implies $K^G \neq \emptyset$, where the set K^G is the set of G-fixed points.*

Let $K < H < G$. As a consequence of the theorem, if K is co-amenable in H and H is co-amenable in G, then K is co-amenable in G; and if K is co-amenable in G, then H is co-amenable in G.

2.2.3 Non-amenable groups

As mentioned in the introduction, one of the fundamental works of von Neumann in [82] was to show that the presence of the free subgroup on two generators in the group of isometries of $\mathbb{R}^3$ causes a paradoxical decomposition, responsible of the Banach-Tarski paradox [28].

Theorem 2.26. *Free groups on $n \geq 2$ generators are non-amenable.*

Proof. It is enough to see that $\mathbb{F}_2 = \langle a, b\rangle$ is non-amenable. Indeed, if A (respectively A^{-1}, B, and B^{-1}) is the set of all reduced words starting with a (respectively with a^{-1}, b and b^{-1}) of $\mathbb{F}_2$, then $\mathbb{F}_2$ can be written as $\mathbb{F}_2 = A \sqcup aA^{-1} = B \sqcup bB^{-1}$ which gives a paradoxical decomposition. □

Regarding free products, we have:

Theorem 2.27. *(Proposition 4 in [72]) Let G, H be groups and let $\phi : G*H \to G\times H$ be the canonical homomorphism. Let $X = \{[g, h] | g \in G\setminus\{1\}, h \in H\setminus\{1\}\}$ be viewed as a subset of $G * H$. Then the kernel of ϕ is freely generated by X.*

Thus for non-trivial groups G and H, if at least one factor has more than two elements then the set X contains at least two elements so the kernel is free on $n \geq 2$ generators, therefore the free product $G * H$ is non-amenable. In the case where G and H are finite, we see that the free product $G * H$ has a finite index free subgroup.

Lemma 2.28. *If $G = G_1 *_A G_2$ is an amalgamated free product such that $[G_1 : A] \geq 2$ and $[G_2 : A] \geq 3$, then G contains a free subgroup on 2 generators.*

Proof. Let X be the Bass-Serre tree given as in Theorem 2.17. By assumption, there exist $g_1 \in G_1 \setminus A$ and $g_2, g_2' \in G_2 \setminus A$. Let $x := (g_1 g_2)^2$, $y := (g_1 g_2')^2$ and let

$$A_1 = \{hG_i \in G/G_1 \sqcup G/G_2 \mid h \text{ is a reduced word starting by } g_1 g_2'\}$$

$$A_2 = \{hG_i \in G/G_1 \sqcup G/G_2 \mid h \text{ is a reduced word starting by } g_2'^{-1} g_1^{-1}\}$$

$$B_1 = \{hG_i \in G/G_1 \sqcup G/G_2 \mid h \text{ is a reduced word starting by } g_1 g_2\}$$

$$B_2 = \{hG_i \in G/G_1 \sqcup G/G_2 \mid h \text{ is a reduced word starting by } g_2^{-1} g_1^{-1}\}$$

be pairwise disjoint subsets of vertices of X. Then for every $k \geq 1$, we have $x^k(A_1) \subset B_1$, $x^{-k}(A_1) \subset B_2$, $x^k(A_2) \subset B_1$ and $x^{-k}(A_2) \subset B_2$. So

$$x^k(A_1 \cup A_2) \subset B_1 \cup B_2$$

for every $k \neq 0$. Similarly we have

$$y^k(B_1 \cup B_2) \subset A_1 \cup A_2$$

for every $k \neq 0$. Thus by the Ping-Pong lemma, the subgroup $\langle x, y \rangle$ generated by x and y is free. □

As a consequence, if $G = G_1 *_A G_2$ is an amalgamated free product such that $[G_1 : A] \geq 2$ and $[G_2 : A] \geq 3$, it is non-amenable.

Concerning one-relator groups, every one-relator group with more than two generators is non-amenable by the Freiheitssatz. Otherwise, one-relator groups with two generators are non-amenable except the following list [18] which follows from the results in [47].

1. $\langle a | a^n = 1 \rangle \simeq \mathbb{Z}_n$, for all $n \geq 1$;
2. $\langle a, b | b = 1 \rangle \simeq \mathbb{Z}$;
3. $\langle a, b | bab^{-1} = a^n \rangle$, $n \neq 0$.
   (i) $\langle a, b | bab^{-1} = a \rangle \simeq \mathbb{Z}^2$;
   (ii) $\langle a, b | bab^{-1} = a^{-1} \rangle$ contains a subgroup isomorphic to $\mathbb{Z}^2$ of index two;
   (iii) if $n \neq 0, \pm 1$, the group is 2 step solvable.

Another important class of non-amenable groups is the class of infinite Kazhdan groups which is presented very briefly in the next section. Non-amenability of infinite Kazhdan groups is implied by Theorem 2.29.

Remark 2.2.8. Day asked in [22] whether the lack of free subgroups of rank ≥ 2 implies the amenability of the group. The answer is given negatively by Ol'shanskii [63] where he constructed the "Tarski monster groups", i.e. infinite non-commutative groups such that every proper subgroup is a cyclic group, by showing that they are non-amenable in both torsion-free and torsion cases ([61], [62]). Such a group is necessarily simple and clearly does not contain $\mathbb{F}_2$. Another counter-example is given by Adyan [5]: the Burnside group of odd exponent $n > 665$ with at least two generators $B(m,n) = \langle a_1, \ldots, a_m | w^n = 1, \forall w \in \mathbb{F}_m \rangle$ is not amenable. A counterexample for finitely presented group is given by Ol'shanskii and Sapir in [64].

Remark 2.2.9. *Thompson's group F* is the group of piecewise linear homeomorphisms of the interval $[0,1]$ with dyadic rational fracture points and the slopes are powers of 2. The group F is not elementary amenable, and does not contain a free subgroup on 2 generators [43]. The amenability of F has been unknown for several decades. Recently, E. Shavgulidze posted a preprint claiming to prove that the group F is amenable [73]. There is an active discussion on websites concerning on this paper and many specialists such as M. Sapir, V. Guba and M. Brin seem to be convinced that the result is correct.

2.2.4 Kazhdan's property (T)

In this section, we very briefly discuss the Property (T) of Kazhdan and some well-known facts related to our topic. All about Kazhdan's Property (T) can be found in [9].

Let G be a topological group. Let $\mathcal{H}$ be a complex Hilbert space. Denote by $\mathcal{U}(\mathcal{H})$ the group of all bounded unitary operators $U : \mathcal{H} \to \mathcal{H}$.

A *unitary representation* of G in $\mathcal{H}$ is a group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ such that the application $G \to \mathcal{H}$; $g \mapsto \pi(g)v$ is continuous for every $v \in \mathcal{H}$. We write $(\pi, \mathcal{H})$ for such a representation.

Definition 2.2.10. Let $\mathcal{H}$ be a complex Hilbert space and let $(\pi, \mathcal{H})$ be a unitary representation of a topological group G.

(1) The representation $(\pi, \mathcal{H})$ *almost has invariant vectors* if for every $\varepsilon > 0$, and for every compact subset $K \subset G$, there exists $v \in \mathcal{H}$, $||v|| = 1$ such that $||\pi(g)v - v|| < \varepsilon$, $\forall g \in K$.

(2) The representation $(\pi, \mathcal{H})$ has non-zero *invariant vectors* if there is $v \in \mathcal{H}$, $v \neq 0$ such that $\pi(g)v = v$, $\forall g \in G$.

Definition 2.2.11. A group G has *Kazhdan's Property (T)* if every unitary representation $(\pi, \mathcal{H})$ of G having almost invariant vectors has non-zero invariant vectors.

Such a group is also called a *Kazhdan group*.

For our interest on the relation with amenability, we have:

Theorem 2.29. *(Theorem 1.1.5 in [9]) A locally compact Kazhdan group is amenable if and only if it is compact.*

So infinite discrete Kazhdan groups are never amenable.

Closely related to Kazhdan's property (T) is the *relative Property (T)*. For $H < G$ a closed subgroup, the pair (G, H) has *relative Property (T)* if, whenever a unitary representation $(\pi, \mathcal{H})$ of G almost has invariant vectors, it has a non-zero H-invariant vector. By definition, G has Property (T) if and only if the pair (G, G) has relative Property (T); if H is closed subgroup of G having Kazhdan's Property (T), then (G, H) has relative Property (T).

Similar to Theorem 2.29, we have:

Proposition 2.30. *The pair (G, H) has relative Property (T) and G is amenable if and only if H is compact.*

Example 2.2.12.

1. $SL_n(\mathbb{R})$ has Property (T), for every $n \geq 3$ (Theorem 1.4.15 in [9]).
2. $SL_n(\mathbb{R}) \ltimes \mathbb{R}^n$ has Property (T), for every $n \geq 3$ (Corollary 1.4.16 in [9]).
3. $SL_2(\mathbb{R})$ and $SL_2(\mathbb{R}) \ltimes \mathbb{R}^2$ do not have Property (T) (Remark 1.4.14 in [9]).
4. $(SL_2(\mathbb{R}) \ltimes \mathbb{R}^2, \mathbb{R}^2)$ has relative Property (T) (Corollary 1.4.13 in [9]).

Property (T) is inherited by quotients (Theorem 1.3.4 in [9]), by lattices (Theorem 1.7.1 in [9]) and by extensions (Proposition 1.7.6 in [9]).

In particular, the infinite discrete groups $SL_n(\mathbb{Z})$ and $SL_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$ have Property (T) for every $n \geq 3$. As the prototype example for the pair of discrete groups with relative Property (T), we have

Theorem 2.31. *(Theorem 4.2.2 in [9]) The pair $(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ has relative Property (T).*

Notice that the Theorem 2.31 follows also from Theorem 1.4.5 in [9].

Chapter 3

Amenable actions and ends of graphs

In this chapter, we briefly discuss the notion of ends of graphs. We first recall definitions and fundamental results on ends of graphs and groups, and then we present our result on amenable actions on graphs published in [58] which is taken up in the Appendix A.

3.1 Ends of graphs

Throughout this chapter, a graph $X = (V, E)$ will be a non-oriented graph without multiple edges. By abuse of notation, we will identify the vertex set V and X.

Two vertices x and y are *adjacent* or *neighbors* if (x, y) is an edge. A graph is *locally finite* if every vertex has finitely many neighbors. A *path* in X is a sequence of vertices $x_1, x_2, \ldots,$ such that (x_i, x_{i+1}) is an edge, for all $i \geq 1$ and $x_i \neq x_j$ for all $i \neq j$. A *circuit* is a sequence of adjacent vertices $x_1, \ldots, x_n$ such that $x_1 = x_n$.

A graph is *connected* if for every pair of vertices v and u, there is a path from v to u. A connected graph is a *tree* if it does not contain any circuit. The *distance* of v and u denoted by $d_X(v, u)$ is the number of edges of the shortest path from v to u. It is easy to see that (X, d_X) is a metric space.

Definition 3.1.1. Let $X = (V, E)$ be a locally finite, infinite connected graph. A *ray* is an infinite path $x = \{x_1, x_2, \ldots\}$ of distinct vertices. Two rays x and y are *equivalent* if for every finite subset $F \subset X$, there is a path z outside of F which connects some vertices of x and some vertices of y.

In other words, two rays x and y are equivalent if there is a third ray which meets each of x and y infinitely often.

Definition 3.1.2. *Ends of* X, denoted by ∂X is the equivalent classes of rays:

$$\partial X = \{\text{ray } x \in X\}/\sim,$$

where $\sim$ is the equivalence of rays. A ray x *ends to* (or *converges to*) $w \in \partial X$ if $[x]_\sim = w$.

Example 3.1.3. 1. If X is finite, then $\partial X = \emptyset$;

2. If X is as in Figure 3.1, then $|\partial X| = 2$;
3. If X is as in Figure 3.2, then $|\partial X| = 1$;
4. If X is a regular tree of valence $n \geq 3$, then $|\partial X| = \infty$.

Figure 3.1: A graph with two ends

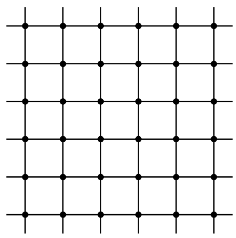


Figure 3.2: A graph with one end

Remark 3.1.4. The above definition corresponds to Halin [39] and Woess [85]. In general, in a locally compact connected topological space X, the set of ends is represented as the projective limit as in [69]: For $K \subset X$ a finite subset, let $\mathcal{C}_K$ be the set of connected components of $X \setminus K$. If $(K_n)_{n\geq 1}$ is a sequence of ascending finite subsets of X, we have a natural projection $\varphi_n : \mathcal{C}_{K_{n+1}} \to \mathcal{C}_{K_n}$; the projective limit of the system

$$\varprojlim(\mathcal{C}_{K_n}, \varphi_n)$$

is the *set of ends* of X.

If $F \subset X$ is a finite subset, then $X \setminus F$ has finitely many pairwise disjoint connected components $C_1, \ldots, C_k$ since X is locally finite. For such a connected component C, We denote

$$\partial C = \{w \in \partial X |\text{ there exists a ray } x \in C \text{ which ends to } w\}.$$

Let w be an end. For $F \subset X$ a finite subset of X, there is exactly one connected component of $X \setminus F$ containing a ray which ends to w. Let $C(F, w)$

be this connected component of $X \setminus F$. It is clear that if $F' \subset X$ is another finite subset, the intersection $C(F, w) \cap C(F', w)$ contains a ray which ends to w. i.e. $w \in \partial(C(F, w) \cap C(F', w))$. So varying the finite set F and the end w, the family of all $C(F, w)$ forms a basis of the topology.

Lemma 3.1. *Endowed with the topology generated by the above basis, the space* $\overline{X} := X \cup \partial X$ *is totally disconnected, compact, Hausdorff space in which* X *is an open dense subset and* ∂X *is compact.*

Proof. Let $\{x_m\}_{m\geq 1}$ be a sequence of distinct elements of X. If $(K_n)_{n\geq 1}$ is a sequence of ascending finite subsets of X and $\mathcal{C}_{K_n}$ is the set of connected components of $X\setminus K_n$, then for each $n \geq 1$, there is a connected component $C_n \in \mathcal{C}_{K_n}$ containing infinitely many elements of $\{x_m\}$. So there is a subsequence $\{x_{m_k}\}$ such that x_{m_k} lies in $\cap_n C_n$ for all k, and it converges to an end. □

The space $\overline{X} := X \cup \partial X$ is called the *end-compactification* of X.

3.2 Ends of groups

Definition 3.2.1. Let G be a finitely generated group with a finite generating subset S. The *set of ends* of G is defined to be the set of ends of the Cayley graph $\mathcal{G}(G, S)$.

Remark 3.2.2. The cardinality of the set of ends of G does not depend on the generating set. Indeed, if S and S' are two generating sets of G, the two Cayley graphs $\mathcal{G}(G, S)$ and $\mathcal{G}(G, S')$ are *quasi-isometric.*

Recall that two metric space X and Y are quasi-isometric if there exists a map (called a quasi-isometry) $f : X \to Y$ such that

(i) $\frac{1}{\lambda} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + C$, for every x_1, $x_2 \in X$;

(ii) for every $y \in Y$, there exists $x \in X$ such that $d_Y(f(x), y) \leq C$,

for some constants $\lambda \geq 1$ and $C \geq 0$.

So if $\{x_n\}$ is a ray in X and $f : X \to Y$ is a quasi-isometry, then there is a subsequence of $\{f(x_n)\}$ which converges to an end in ∂Y; conversely, if $\{y_n\}$ is a ray in Y, then there is a sequence $\{x_n\}$ in X such that $d_Y(f(x_n), y_n) \leq C$, and it contains a subsequence converging to an end in ∂X; if $\{x_n\}$ and $\{y_n\}$ are two rays which end to $w \neq w' \in \partial X$ respectively, then there exist subsequences of $\{f(x_n)\}$ and $\{f(y_n)\}$ which end to $\xi \neq \xi' \in \partial Y$ respectively. Therefore $|\partial X| = |\partial Y|$ for two quasi-isometric spaces X and Y. But the converse is false: for $n \neq m \geq 2$, the groups $\mathbb{Z}^n$ and $\mathbb{Z}^m$ have one end, but they are not quasi-isometric since they do not have the same growth type (cf. Proposition 27 in [23]).

We therefore denote the set of ends $B(\mathcal{G}(G, S)) =: B(G)$ without specifying the generating set.

Example 3.2.3. 1. $|B(\mathbb{Z})| = 2$;

2. $|B(\mathbb{Z}^n)| = 1$, for all $n \geq 2$;

3. $|B(\mathbb{F}_n)| = \infty$, for all $n \geq 2$.

One of the basic result on ends of groups is due to Hopf [41]:

Theorem 3.2. (Hopf, 1943) *A group has 0, 1, 2, or infinitely many ends.*

Proof. Let X be the Cayley graph of a finitely generated group G with respect to some finite generating subset S. Let us show that if G has more than two ends, then $B(G) = \infty$. By contradiction, suppose that G has $n \geq 3$ ends, and let $K \subset X$ be a connected finite subset of X such that $X \setminus K$ has n connected components $C_1, \ldots, C_n$. Since G acts transitively on X, we can suppose that $1 \in K$. Since G acts properly on X, the set $F_K = \{g \in G \,|\, gK \cap K \neq \emptyset\}$ is finite. For $h \in C_1 \setminus F_K$, hK is a connected finite subset of $X \setminus K$, intersecting C_1 since $h \in hK \cap C_1$. Thus we have $hK \subset C_1$. Since $X \setminus hK$ has n connected components, $X \setminus (K \cup hK)$ has at least $2(n-1)$ connected components which contradicts to $B(G) = n$. □

It is clear that $B(G) = 0$ if and only if G is finite; moreover $B(G) = 2$ if and only if G is virtually infinite cyclic (Theorem 5 in [69]; notice that the sufficient condition is obvious since a group is quasi-isometric to its finite index subgroup so they have the same number of ends). A much more difficult result of Stallings says that if $B(G)$ is infinite, then G splits over a finite subgroup:

Theorem 3.3. (Stallings, 1968, [74]) *G has infinitely many ends if and only if G is either*

(i) an amalgamated free product $\Gamma_1 *_A \Gamma_2$ *with A finite, such that* $\min\{[\Gamma_1 : A], [\Gamma_2 : A]\} \geq 2$*, not both 2;*

or

(ii) an HNN-extension $HNN(\Gamma, A, \varphi)$ *with A finite, such that* $\min\{[\Gamma : A], [\Gamma : \varphi(A)]\} \geq 2$*, not both 2.*

In particular, a group having infinitely many ends is non-amenable since it contains a free subgroup on two generators (cf. Lemma 2.28 in Chapter 2).

Remark 3.2.4. If G is a discrete infinite group having Kazhdan's property (T), then G has only one end. Indeed, it is known that (Swan, [75]) for an infinite group G, we have

$$|B(G)| = 1 + \dim_{\mathbb{C}} H^1(G, \mathbb{C}G),$$

where $H^1(G, \mathbb{C}G)$ is the first cohomology space with coefficients in $\mathbb{C}G$ where $\mathbb{C}G$ is viewed as a submodule of $\ell^2(G)$. Bekka and Valette showed in [15] that the G-module embedding $\mathbb{C}G \hookrightarrow \ell^2(G)$ induces an embedding $H^1(G, \mathbb{C}G) \hookrightarrow H^1(G, \ell^2(G))$. Thus

$$|B(G)| \leq 1 + \dim_{\mathbb{C}} H^1(G, \ell^2(G)).$$

Now let G be a locally compact infinite group having Kazhdan's property (T). By the Delorme-Guichardet theorem (see [9]), the first cohomology group $H^1(G, \pi)$ vanishes for any unitary representation π of G. In particular,

$$\dim_{\mathbb{C}} H^1(G, \ell^2(G)) = 0,$$

so $|B(G)| = 1$.

3.3 Amenable actions on graphs with infinitely many ends

Let X be a locally finite graph with infinitely many ends. Let G be a group of automorphisms of X. Recall that an automorphism of G is an isometry of X onto itself with respect to the metric, i.e. $d_X(x, y) = d_X(gx, gy)$, for every x, $y \in X$ and $g \in G$. The action of G on X extends to an action of G on $\overline{X}$ by homeomorphism.

Lemma 3.4. *Let G be a group of automorphisms of X. If there is $x_0 \in X$ such that $\overline{Gx_0} = \partial X$, then $\overline{Gx} = \partial X$, for every $x \in X$.*

Proof. We show that for $w \in \partial X$ an end and $x \in X$, there is a sequence $\{g_n\}_{n\geq 1}$ in G such that $g_n x \xrightarrow[n\to\infty]{} w$. Let F be a finite path from x_0 to x and let C be a connected component of $X \setminus F$ such that $w \in \partial C$. By assumption, $g_n x_0 \xrightarrow[n\to\infty]{} w$ for some $g_n \in G$, so there is n_0 such that $g_n x_0 \in C$ and $d(g_n x_0, F) > d(x, x_0)$, $\forall n \geq n_0$ so that $d(g_n F, F) > 0$ since G acts by isometries. Thus $g_n F \cap F = \emptyset$, $\forall n \geq n_0$. Since $g_n F$ is connected and $g_n x_0 \in C$, it follows that $g_n F \subset C$, in particular $g_n x \in C$, $\forall n \geq n_0$ which means that $g_n x \xrightarrow[n\to\infty]{} w$. □

Every automorphism of X extends continuously to ∂X. There are three types of automorphisms of X, classified by Halin [39]:

Theorem 3.5. *An automorphism $g \in Aut(X)$ is exactly one of the three types:*

1. *elliptic, if g stabilizes some non-empty finite subset of X;*
2. *parabolic, if g is non-elliptic and fixes exactly one end;*
3. *hyperbolic, if g is non-elliptic and fixes exactly two ends.*

Equivalently, an automorphism g is hyperbolic if there are two ends w, w' fixed by g such that $\lim_{n\to\infty} g^n x = w$ and $\lim_{n\to\infty} g^{-n} x = w'$, $\forall x \in X$; and g is parabolic if there is a unique end w fixed by g such that $\lim_{n\to\infty} g^n x = w = \lim_{n\to\infty} g^{-n} x$, $\forall x \in X$. If g is non-elliptic, then g leaves invariant a unique line, called the axis of g, on which g acts by translation.

When X is a locally finite infinite tree, Tits [78,80], Nebbia [59,60] and Pays and Valette [68] showed that a group of automorphisms of a locally finite tree either (i) contains a free group on two generators acting freely, or (ii) fixes a vertex, an edge, an end or a pair of ends. For an arbitrary infinite graph, similar results are shown for example in [39], [86] and [85].

In [58], we are interested in amenable actions of group of automorphisms of a locally finite infinite graph. We showed:

Theorem 3.6. (Theorem A.1 in Appendix A) *Let X be a locally finite graph with infinitely many ends. Let G be a group of automorphisms of X. Assume that the action of G on X is amenable and there exists $x_0 \in X$ such that $\overline{Gx_0} = \overline{X}$. Then there is a unique G-fixed end in ∂X, and the action of G on X is not proper.*

Recall that an action is *proper* if $\forall R > 0$, the set $\{g \in G \mid d(x_0, gx_0) \leq R\}$ is finite.

By Lemma 3.4, the assumption implies that Gx is dense in $\overline{X}$ for every $x \in X$. For a fixed vertex $x_0 \in X$, let B_n be the ball of radius n centered at x_0. Upon replacing B_n by $\widehat{B_n} = B_n \cup F$, where F is the union of finite connected components of $X \setminus B_n$, we can suppose that every connected component of $X \setminus B_n$ is infinite. There is n_0 such that $X \setminus B_n$ has at least three connected components, for all $n \geq n_0$. Then if μ is a G-invariant mean, there exists a connected component C_{B_n} of $X \setminus B_n$ such that $\mu(C_{B_n}) = 1$ (Claim 2. in Appendix A). Then the set

$$D_n = \Big(\bigcap_{g \in G} \overline{gC_{B_n}} \Big) \cap \partial X$$

is non-empty (Lemma A.2 in Appendix A) and the set $\bigcap_{n \geq n_0} D_n$ is reduced to the unique fixed point of G (Claim 3. in Appendix A).

About the properness of the action, for two hyperbolic elements h and g which fix pairwise different ends apart from the common fixed end, we show that the set $\{h^n g h^{-n} x_0 | n \in \mathbb{N}\}$ is bounded (Claim 4 in Appendix A).

Corollary 3.7. *A finitely generated group having infinitely many ends is not amenable.*

Proof. It follows from the Theorem 3.6 since a group acts properly and transitively on its Cayley graph. □

Chapter 4

Amenable actions and the class $\mathcal{A}$

In this chapter, we introduce the class $\mathcal{A}$ of countable groups admitting amenable, transitive and faithful actions on an infinite countable set. This chapter is divided into 3 parts: we first give an overview on the class $\mathcal{A}$. Next, we present the results on actions of amalgamated free products published in [56] and [57] which are taken up in Appendices B and C. Finally we give further results on stability properties of the class $\mathcal{A}$.

4.1 Definitions and generalities

Let X be an infinite countable set and G be a countable group acting on X.

Recall that an action of G on X is *amenable* if there exists a G-invariant mean on X, i.e. a map $\mu : 2^X = \mathcal{P}(X) \to [0,1]$ such that $\mu(X) = 1$, $\mu(A \cup B) = \mu(A) + \mu(B)$ for every pair of disjoint subsets A, B of X, and $\mu(gA) = \mu(A)$, $\forall g \in G$, $\forall A \subseteq X$.

Remark 4.1.1. The above definition is due to Greenleaf [35] (and also to von Neumann [82]). The notion of amenability for group actions introduced by Zimmer [90] is different from ours; an action by homeomorphisms of a countable discrete group G on a locally compact Hausdorff space X is *Zimmer amenable* if there exists a sequence of continuous maps $m^n : X \to Proba(G)$ such that $\lim_{n\to\infty} \sup_{x\in X} \|gm^n_x - m^n_{gx}\|_1 = 0$, for every $g \in G$ (cf. [90], [65], [40], [8]). With this definition, a group is amenable if and only if the action on an one-point space is Zimmer amenable, while such an action is always Greenleaf amenable (cf. Reiter's condition for amenability in Theorem 4.1). On the other hand, the action of G on itself by left multiplication is always Zimmer amenable since $m^n : G \to Proba(G)$ defined by $m^n_g = \delta_g$, $\forall n$, where $\delta_g(x) = 1$ if $x = g$ and zero otherwise, verifies $h \cdot m^n_g = m^n_{hg}$, $\forall g$, $h \in G$. More generally, the action of G on a homogenous space G/H is Zimmer amenable if and only if the subgroup H is amenable (cf. Theorem 1.9. in [90]). It is Greenleaf amenable if and only if H

is co-amenable in the sense of Definition 2.2.6. From now on, we will use the definition of an amenable action in the meaning of Greenleaf amenable action.

As for the amenability of groups, there are well-known equivalent definitions for the amenability of group actions:

Theorem 4.1. *The following are equivalent:*

(1) The action of G on X is amenable.

(2) The action of G on X does not admit a paradoxical decomposition.

(3) (Reiter's condition) For every finite subset $F \subset G$, there exists a sequence $\phi_n \in Proba(X)$ such that $||g\phi_n - \phi_n||_1 \xrightarrow[n\to\infty]{} 0$, $\forall g \in F$.

(4) (Følner's condition) For every finite subset $F \subset G$ and every $\varepsilon > 0$, there exists a finite subset $A \subset X$ such that $|gA \vartriangle A| < \varepsilon|A|$, $\forall g \in F$.

(5) $1_G \prec \lambda_X$, i.e. the left unitary representation $(\lambda_X, \ell^2(X))$ of X almost has G-invariant vectors.

One may prove this in the same manner as for amenable groups (see for example [9]).

Remark 4.1.2. Since G is countable, the action of G on X is amenable if and only if there exists a sequence $\{A_n\}_{n\geq 1}$ of finite non-empty subsets of X such that for every $g \in G$, one has

$$\lim_{n\to\infty} \frac{|A_n \vartriangle g \cdot A_n|}{|A_n|} = 0.$$

Such a sequence is called a *Følner sequence* for the action of G on X.

The amenability of group actions is functorial, in the following sense:

Lemma 4.2. *Let X, Y be G-sets and $f : X \to Y$ be a G-equivariant map (i.e. $f(g \cdot x) = g \cdot f(x)$, $\forall g \in G$, $\forall x \in X$). If the G-action on X is amenable, then the G-action on Y is also amenable.*

Proof. Let $\mathcal{M}(X)$ be the set of all means on X. The co-variant map $f_* : \mathcal{M}(X) \to \mathcal{M}(Y)$ defined by $f_*\mu(B) = \mu(f^{-1}(B))$, $\forall B \subset Y$, is G-equivariant. Indeed, for every $g \in G$ and $\mu \in \mathcal{M}(X)$, we have $g \cdot (f_*\mu)(B) = (f_*\mu)(g^{-1}B) = \mu(f^{-1}(g^{-1}B)) = \mu(g^{-1}(f^{-1}B)) = (g \cdot \mu)(f^{-1}B) = f_*(g \cdot \mu)(B)$, for every subset $B \subset Y$ since f is G-equivariant. So if μ is a G-invariant mean on X, we have

$$g \cdot (f_*\mu)(B) = (g \cdot \mu)(f^{-1}(B)) = \mu(f^{-1}B) = (f_*\mu)(B),$$

for every $g \in G$ and $B \subset Y$, i.e. $f_*\mu$ is a G-invariant mean on Y. □

Corollary 4.3. *The followings are equivalent:*

(i) G is an amenable group.

(ii) Every G-action is amenable.

(iii) There is a free amenable G-action.

Proof. $(i) \Leftrightarrow (ii) \Rightarrow (iii)$ is obvious. For $(iii) \Rightarrow (i)$, let X be a G-set with free amenable action. If A is a family of representatives of orbits then the G-equivariant map $f : G \times A \to X$ defined by $f(g, a) = ga$ is bijective since the G-action on X is free. So f^{-1} is well-defined and the action of G on $G \times A$ defined by $h \cdot (g, a) = (hg, a)$ is amenable. Thus the G-action on itself is amenable since there is an obvious G-equivariant map from $G \times A$ to G. □

Corollary 4.4. *Any subgroup H of an amenable group G is amenable.*

Proof. The action of H on G is amenable and free. □

Remark 4.1.3. If G has Property (T), then every amenable G-action has some non-empty finite orbit.

Proof. We use the definition of amenability in terms of left unitary representation as mentioned in Theorem 4.1: the action of G on X is amenable if and only if the left unitary representation λ_X on $\ell^2(X)$ almost has G-invariant vectors.

So if a Kazhdan group G acts on X amenably, there is a non-zero vector $v \in \ell^2(X)$ such that $\lambda_X(g)v(x) = v(g^{-1}x) = v(x)$, $\forall g \in G$ and $\forall x \in X$. This means that v is constant on each orbit, so there must be finite orbits since $v \in \ell^2(X)$. □

4.2 The class $\mathcal{A}$

For the study of amenable actions of a group G, some restrictions on the G-action are needed in order to avoid trivial cases:

Example 4.2.1.

1. Let $\mathbb{F}_n$ be a free group of rank $n \geq 2$ and $N \triangleleft \mathbb{F}_n$ be a non-trivial normal subgroup such that the quotient group $\mathbb{F}_n/N$ is amenable (for example $N = \mathbb{F}'_n$ the commutator subgroup). Then the action of $\mathbb{F}_n$ on $\mathbb{F}_n/N$ is amenable, but the action factors through $\mathbb{F}_n/N$.
2. Any G-action having a finite orbit is amenable; indeed, if X is a G-set and Gx_0 is a finite orbit, there is a G-equivariant map from Gx_0 to X, so by Lemma 4.2 the G-action on X is amenable.

These examples show that we need the action to be faithful and transitive in addition to the amenability of the action. In this direction, Y. Glasner and N. Monod [33] proposed to study the class $\mathcal{A}$ of all countable groups which admit a faithful, transitive and amenable action on an infinite countable set:

$$\mathcal{A} = \{G \text{ countable} \mid G \text{ admits a faithful, transitive and amenable action}\}.$$

Example 4.2.2. Amenable groups are in $\mathcal{A}$ since the G-action on itself is free and transitive.

The first non-trivial example was given by van Douwen [81]:

Theorem 4.5. (van Douwen, 1990) *The free groups are in $\mathcal{A}$.*

In [81], he showed that there is an action of a free group of rank two F_2 on an infinite countable set X such that the action is transitive, each non-trivial element of F_2 fixes only finitely many points, and admits a F_2-invariant mean on X.

More generally, Y, Glasner and N. Monod showed the following theorem:

Theorem 4.6. (Glasner-Monod, 2007 [33]) *The free product of any two countable groups is in $\mathcal{A}$ unless one factor has the fixed point property and the other has the virtual fixed point property.*

A group G has the *fixed point property* if any amenable G-action has fixed points, and G has the *virtual fixed point property* if it has a finite index subgroup that has the fixed point property.

Remark 4.2.3.

1. If G is infinite with Kazhdan's property (T), then $G \notin \mathcal{A}$ since any amenable action has finite orbits (Remark 4.1.3).

2. Let H be a normal subgroup of $G \in \mathcal{A}$. If the pair (G, H) has relative Property (T), then H has finite exponent, i.e. there is an integer n such that for every $h \in H$, $h^n = 1$ (Lemma 4.3. in [33]). In fact, if the pair (G, H) has relative Property (T), for every amenable G-action on X, there is a finite non-empty H-orbit $F \subset X$. So if the G-action is moreover transitive, one can show that $h^{|F|!}$ acts trivially on X, for all $h \in H$.

3. If G has the fixed point property then it cannot have non-trivial finite index subgroup H since otherwise the natural G-action on the quotient G/H would be amenable without fixed point. So if G has the virtual fixed point property then G has a *minimal finite index subgroup*, in the sense that the subgroup does not have non-trivial finite index subgroup (Remark 1.4 in [33]). So for example residually finite groups do not have the virtual fixed point property.

4. If G has the virtual fixed point property, then every amenable G-action has finite orbits. Indeed, let $H < G$ be a finite index subgroup having the fixed point property and let $G \curvearrowright X$ be an amenable action. If $x \in X$ is a H-fixed point, then H is of finite index in the stabilizer $Stab_G(x)$ of x in G. Thus $[G : Stab_G(x)]$ is finite.

   However, the converse is false; for example the group $SL_3(\mathbb{Z})$ has Property (T) of Kazhdan and is residually finite.

By Theorem 4.6, one can see that for any countable groups G and H, the free product $G * H$ is in $\mathcal{A}$ as soon as one factor does not have the virtual fixed point property. Denote $\mathcal{NVF}$ the class of groups not having the virtual fixed point property. While $\mathcal{A}$ is contained in $\mathcal{NVF}$ clearly, the class $\mathcal{NVF}$ is much larger. For example, $\mathcal{NVF}$ contains the class $\mathcal{B}$ of all countable groups admitting an amenable action without finite orbits and the class of residually finite groups as mentioned in Remark 4.2.3. Moreover, an observation by Kazhdan [48] shows that any countable group that is not finitely generated is in $\mathcal{B}$. For more information on this fixed point property, see [33].

Let us mention that another construction of amenable actions of non-abelian free groups is obtained by R. Grigorchuk and V. Nekrashevych in [36].

The class $\mathcal{A}$ is closed under direct products and free products, and extension of co-amenable subgroups (Proposition 1.7 in [33]).

Proposition 4.7. *For any countable groups G and H, we have:*

(1) G, $H \in \mathcal{A} \Leftrightarrow G \times H \in \mathcal{A}$;

(2) G, $H \in \mathcal{A} \Rightarrow G * H \in \mathcal{A}$;

(3) If $H < G$ is co-amenable, then $H \in \mathcal{A} \Rightarrow G \in \mathcal{A}$.

Recall that a subgroup $H < G$ is *co-amenable* if the action of G on the homogeneous space G/H is amenable. If the action $G \curvearrowright X$ is transitive, then for a fixed $x \in X$, the set X identifies to the quotient $G/Stab_G(x)$. So in terms of co-amenability, we have $G \in \mathcal{A}$ if and only if there exists a co-amenable subgroup $H < G$ such that

$$\bigcap_{g \in G} gHg^{-1} = \{1\}.$$

On the other hand, in general the class is not closed under passing to (co-amenable) subgroups. As an example, take $G = (\bigoplus_{\mathbb{Z}} Q) \rtimes \mathbb{Z}$, with $Q \notin \mathcal{A}$ as in [55]. The subgroup $K = \bigoplus_{\mathbb{N}} Q$ is not in $\mathcal{A}$, but it was shown to be co-amenable in G while $G \in \mathcal{A}$ via the G-action on G/K. Let us mention that the stability of the class $\mathcal{A}$ under passing to finite index subgroups, is an open question.

Moreover, the class $\mathcal{A}$ is not closed under semidirect products; for example one may take the group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. While $SL_2(\mathbb{Z})$ is in $\mathcal{A}$ since it contains a free group of finite index and $\mathbb{Z}^2 \in \mathcal{A}$ clearly, the semidirect product $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ is not in $\mathcal{A}$ since the pair $(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ has relative property (T) and $\mathbb{Z}^2$ does not have finite exponent (Remark 4.2.3).

Likewise, this group is another example which shows that the class $\mathcal{A}$ is not closed under amalgamated free products in general. In fact, the group $SL_2(\mathbb{Z})$ can be viewed as an amalgamated free product:

$$SL_2(\mathbb{Z}) \simeq (\mathbb{Z}/6\mathbb{Z}) *_{(\mathbb{Z}/2\mathbb{Z})} (\mathbb{Z}/4\mathbb{Z}).$$

So one may see the group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ as the amalgamated free product $G *_A H$ of $G = \mathbb{Z}/4\mathbb{Z} \ltimes \mathbb{Z}^2$ and $H = \mathbb{Z}/6\mathbb{Z} \ltimes \mathbb{Z}^2$ along $A = \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}^2$ and notice that the three groups G, H and A are in $\mathcal{A}$ since they are amenable.

Remark 4.2.4. A well-known strong negation to Kazhdan's Property (T) is the *Haagerup Property* or *a-T-menability* (see [66]). At first sight, the class $\mathcal{A}$ shares many similar properties with groups having the Haagerup Property, but notice that a group in $\mathcal{A}$ does not necessarily have the Haagerup Property. For example, the group $SL_3(\mathbb{Z}) * SL_3(\mathbb{Z})$ is in $\mathcal{A}$ by Theorem 4.6 (since $SL_3(\mathbb{Z})$ is residually finite) but does not have the Haagerup Property since $SL_3(\mathbb{Z})$ has Property (T).

Question. Is every group having the Haagerup Property in $\mathcal{A}$?

4.3 Amalgamated free products case

4.3.1 Double of $\mathbb{F}_2$ over $\mathbb{Z}$

The first result concerning amalgamated free products is given in [56]:

Theorem 4.8. *(1) The free product of two free groups of rank two with cyclic amalgamation $\mathbb{F}_2 *_{\langle c\rangle} \mathbb{F}_2$, where the cyclic subgroup $\langle c\rangle$ embeds in each factor as subgroup generated by some common word c on the generating sets, is in $\mathcal{A}$.*

(2) For any finite index subgroup H of $\mathbb{F}_2 *_{\langle c\rangle} \mathbb{F}_2$ as above, H belongs to $\mathcal{A}$.

The basic idea of the proof is the use of the Baire category theorem. For X an infinite countable set, the symmetric group $Sym(X)$ of X endowed with the topology of pointwise convergence (i.e. α_n converges to α if for every finite subset F of X, there exists n_0 such that $\alpha_n|_F = \alpha|_F$, for all $n \geq n_0$) is a Baire space (cf. Section B.2 in Appendix B). Recall that a subset $Y \subset Sym(X)$ is *meagre* if it is a union of countably many closed subsets with empty interior; and *generic* or *dense* G_δ if its complement $Sym(X) \setminus Y$ is meagre. Baire's theorem states that in a complete metric space, the intersection of countably many dense open subsets is dense, in particular not empty. Thus in order to find a permutation α of X having the properties $\{P_i\}_{i\geq 1}$, it is enough to prove that the set

$$\mathcal{U}_i = \{\alpha \in Sym(X) | \alpha \text{ satisfies the property } P_i\}$$

is generic in $Sym(X)$ and take $\alpha \in \cap_{i\geq 1}\mathcal{U}_i$. Such a approach has been used for example in [33], [25], [20] and [27].

About Theorem 4.8, it is enough to prove for the case where the word c is a special word on two generators α, β[1]; indeed, we have

Lemma 4.9. *(Lemma B.18 in Appendix B) For any reduced word $c = c(\alpha, \beta)$ on $\{\alpha^{\pm 1}, \beta^{\pm 1}\}$, there exists an automorphism a of $\mathbb{F}_2$ such that $a(c)$ is a special word.*

[1] c is *special* if it is a cyclically reduced word on $\alpha^{\pm 1}$ and $\beta^{\pm 1}$ such that either (i) the sum $S_c(\alpha)$ of the exponents of α in the word c and the sum $S_c(\beta)$ are both zero, or (ii) $S_c(\alpha)$ divides $S_c(\beta)$; for example the commutator $[\alpha, \beta]$ is special)

By fixing a transitive action of a permutation β on X, we showed that for a special word c on $\alpha^{\pm1}$ and $\beta^{\pm1}$, there exists an action of a free group $\mathbb{F}_2 = \langle\alpha, \beta\rangle$ of rank 2 such that:

(1) the action of $\mathbb{F}_2$ on X is transitive and faithful;

(2) for all $w \in \mathbb{F}_2 \setminus \langle c\rangle$, there exist infinitely many $x \in X$ such that $cx = x$, $cwx = wx$ and $wx \neq x$. In particular, there are infinitely many fixed points of c in X;

(3) there exists a pairwise disjoint Følner sequence for $\mathbb{F}_2$ which is fixed by c;

(4) for every finite index subgroup H of $\mathbb{F}_2$, the H-action on X is transitive,

by taking appropriate generic sets in $Sym(X)$ (Proposition B.1, B.2, B.3 and B.4 in Appendix B).

Once we have the $\mathbb{F}_2$-action on X above with $\{A_n\}_{n\geq1}$ a pairwise disjoint Følner sequence, the construction of the action of the double of $\mathbb{F}_2$ is the following. Let $Z_c = \{\sigma \in Sym(X) \mid \sigma c = c\sigma\}$ be the centralizer of c. The set Z_c is closed subset of $Sym(X)$ so it is a Baire space. For $\sigma \in Z_c$, let $G = \mathbb{F}_2$ and $H = \mathbb{F}_2^\sigma$, where $\mathbb{F}_2^\sigma = \sigma^{-1}\mathbb{F}_2\sigma$ is the group isomorphic to $\mathbb{F}_2$ obtained by conjugating σ. By the universal property, the amalgamated free product of $G *_{\langle c\rangle} H$ acts on X by $g \cdot x = gx$ and $h \cdot x = \sigma^{-1}h\sigma x$, for all $g \in G$ and $h \in H$.

The action of the amalgam $G *_{\langle c\rangle} H$ on X is clearly transitive. In order that the action is also faithful and amenable, we show:

Proposition 4.10. *(Proposition B.4 and B.16 in Appendix B) The sets*

$$\mathcal{O}_1 = \{\sigma \in Z_c \mid \textit{the action of } G *_{\langle c\rangle} H \textit{ on } X \textit{ is faithful}\}$$

and

$$\mathcal{O}_2 = \{\sigma \in Z_c \mid \textit{there exists } \{A_{n_k}\}_{k\geq1} \textit{ a subsequence of } \{A_n\}_{n\geq1} \textit{ such that } \sigma(A_{n_k}) = A_{n_k},\ \forall k \geq 1\}$$

are generic in Z_c.

To prove the genericity of the sets $\mathcal{O}_1$ and $\mathcal{O}_2$ in Z_c, we need to show that in each $i = 1, 2$, for every $\sigma \in Z_c \setminus \mathcal{O}_i$ and for every finite subset F in X, there is a permutation $\sigma' \in Z_c$ defined as $\sigma|_F = \sigma'|_F$ such that we can extend the definition of σ' outside of the finite subset F in a way that $\sigma' \in \mathcal{O}_i$. It is not hard to extend a permutation γ outside of a finite subset so as to move an element $x \in X$ by a word w^γ (in case of $\mathcal{O}_1$), or to fix the elements of a subsequence of A_n (in case of $\mathcal{O}_2$); the difficult part is to make sure that the extended permutation is in the centralizer of c.

One answer to solve this difficulty is to extend the definition of σ' only on the set of the fixed points of c (which is infinite by construction) since $\sigma\mathrm{Fix}(c) = \mathrm{Fix}(c)$ for every $\sigma \in Z_c$. That explains the choice of the previous $\mathbb{F}_2$-action on X.

Theorem 4.8 applies for example to surface groups Γ_g, the fundamental group of a closed orientable surface of genus $g \geq 2$, by viewing Γ_g as a finite index subgroup of Γ_2 which is in $\mathcal{A}$ being a double of $\mathbb{F}_2$ over $\mathbb{Z}$:

Theorem 4.11. *(Theorem B.21 in Appendix B) The surface group Γ_g is in $\mathcal{A}$ for every $g \geq 1$.*

As an easy corollary, we see that

Corollary 4.12. *(Corollary B.22 in Appendix B) For any compact surface S, the fundamental group $\pi_1(S)$ of S is in $\mathcal{A}$.*

For more interesting application, we have

Corollary 4.13. *(Example B.5.1 in Appendix B) For any 3-manifold M which virtually fibers over the circle, the fundamental group $\pi_1(M)$ of M is in $\mathcal{A}$.*

One can find some examples of the fundamental group of such manifolds in [6], which includes the Bianchi groups $\mathrm{PSL}(2, \mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ with d a positive integer.

4.3.2 Cyclically pinched one-relator groups

Doubles of $\mathbb{F}_2$ over $\mathbb{Z}$ and surface groups are special cases of the class of *cyclically pinched one-relator groups*. Recall that these are the groups admitting a presentation

$$G = \langle a_1, \dots, a_n, b_1, \dots, b_k | c = d \rangle$$

where $1 \neq c = c(a_1, \dots, a_n)$ is a cyclically reduced non-primitive word in the free group $\mathbb{F}_n = \langle a_1, \dots, a_n \rangle$, and $1 \neq d = d(b_1, \dots, b_k)$ is a cyclically reduced non-primitive word in the free group $\mathbb{F}_k = \langle b_1, \dots, b_k \rangle$. Such a group is the amalgamated free product of two free groups $\mathbb{F}_n$ and $\mathbb{F}_k$ over the cyclic subgroup generated by $c = d$.

By generalizing the method of the previous subsection, we show in [57] the same results for cyclically pinched one-relator groups:

Theorem 4.14. *1. For every two integers n, $m \geq 1$, the free product with cyclic amalgamation $\mathbb{F}_{n+1} *_{\langle c=d \rangle} \mathbb{F}_{m+1}$, where the exponent sum of some generator occurring in $c \in \mathbb{F}_{n+1}$ (respectively $d \in \mathbb{F}_{m+1}$) is zero, is in $\mathcal{A}$.*

2. Every finite index subgroup H of such a group is in $\mathcal{A}$.

This theorem shows that every cyclically pinched one-relator group is in $\mathcal{A}$ since there is an automorphism ϕ of $\mathbb{F}_{n+1}$ such that the exponent sum of some generator occurring in $\phi(c)$ is zero (Lemma C.9 in Appendix C).

The idea in this case is the application of Baire's theorem on the product $(Sym(X))^n$ of the permutation group $Sym(X)$. The product $(Sym(X))^n$ is a Baire space, so in order to search for n-tuples of permutations $\alpha = (\alpha_1, \dots, \alpha_n) \in (Sym(X))^n$ satisfying the properties $\{P_i\}_{i \geq 1}$, we show that the set

$$\mathcal{U}_i = \{\alpha = (\alpha_1, \dots, \alpha_n) \in (Sym(X))^n | \alpha \text{ satisfies } P_i\}$$

is generic in $(Sym(X))^n$ and take $\alpha \in \cap_{i\geq 1}\mathcal{U}_i$.

As in the case of $\mathbb{F}_2$, we need to first construct an $\mathbb{F}_{n+1}$-action: for a fixed integer $n \geq 1$, a fixed transitive permutation β, and a cyclically reduced word c on $n+1$ generators such that the exponent sum S_c of some generator occurring in c is zero, we construct an action of $\mathbb{F}_{n+1}$ satisfying the following properties (Proposition C.5 in Appendix C):

(1) the action of $\mathbb{F}_{n+1}$ on X is transitive and faithful;

(2) for every non trivial word $w \in \mathbb{F}_{n+1} \setminus \langle c \rangle$, there exist infinitely many $x \in X$ such that $cx = x$, $cwx = wx$ and $wx \neq x$;

(3) there exists a pairwise disjoint Følner sequence $\{A_m\}_{m\geq 1}$ for $\mathbb{F}_{n+1}$ which is fixed by c, and $|A_m| = m$, $\forall m \geq 1$;

(4) for all $k \geq 1$, there are infinitely many $\langle c \rangle$-orbits of size k;

(5) every $\langle c \rangle$-orbit is finite;

(6) for every finite index subgroup H in $\mathbb{F}_{n+1}$, the H-action is transitive.

The last step is the construction of the amalgam. When we have the action $\mathbb{F}_{n+1} \curvearrowright X$ with $c \in \mathbb{F}_{n+1}$ and a Følner sequence $\{A_k\}_{k\geq 1}$ as above (respectively the action $\mathbb{F}_{m+1} \curvearrowright X$ with $d \in \mathbb{F}_{m+1}$ and a Følner sequence $\{B_k\}_{k\geq 1}$), we work in the Baire space $Z = \{\sigma \in Sym(X) \mid \sigma c = d\sigma\}$; it is a Baire space since clearly it is closed and it is not empty since two words c and d have the same permutation type $(\infty, \infty, \ldots, ; 0)$. For $\sigma \in Z$, let $G = \mathbb{F}_{n+1}$, $H = \mathbb{F}^{\sigma}_{m+1}$ and we consider the amalgamated free product of $G *_{\langle c=d \rangle} H$. The action of the amalgamated free product on X is clearly transitive; and moreover it is faithful and amenable thanks to the following proposition:

Proposition 4.15. *(Propositions C.6 and C.7 in Appendix C) The sets*

$$\mathcal{O}_1 = \{\sigma \in Z \mid \text{ the action of } G *_{\langle c=d \rangle} H \text{ on } X \text{ is faithful } \}$$

and

$$\mathcal{O}_2 = \{\sigma \in Z \mid \exists\ \{k_l\}_{l\geq 1} \text{ a subsequence of } k \text{ such that } \sigma(A_{k_l}) = B_{k_l},\ \forall l \geq 1\ \}$$

are generic in Z.

Again as in the case of the double, we have to be careful when we extend the definition of a permutation σ outside of a finite set: but the problem is solved if we work in the infinite sets $\mathrm{Fix}(c)$ and $\mathrm{Fix}(d)$ since $\sigma\mathrm{Fix}(c) = \mathrm{Fix}(d)$ for every $\sigma \in Z$.

Remark 4.3.1. For the case where $n = 1$ and $k \geq 2$ in the definition of cyclically pinched one-relator group, i.e.

$$G = \langle a, b_1, \ldots, b_k | a^N = d \rangle \simeq \mathbb{Z} *_{\langle a^N = d \rangle} \mathbb{F}_k$$

with $N \neq 0$ and $d = d(b_1, \ldots, b_k)$ is a cyclically reduced non-primitive word in $\mathbb{F}_k = \langle b_1, \ldots, b_k \rangle$, we can construct a faithful $\mathbb{Z}$-action satisfying (2), (3), (4) and (5) as above and $\mathbb{F}_k$-action as before. Then $\mathbb{Z} *_{\langle a^N = d \rangle} \mathbb{F}_k$ is also in $\mathcal{A}$ since in order that an amalgamated free product acts transitively, it suffices that only one factor acts transitively.

For the case where $n = k = 1$, then the cyclically pinched one-relator group is a Torus knot group so that it is also in $\mathcal{A}$ (cf. Example 4.4.5).

4.4 Hereditary properties

4.4.1 Double of amenable groups

As we have seen in Section 4.2, in general the amalgamated free product is not in $\mathcal{A}$ even if the factors are amenable groups. But it is true if the amalgam is a double; or more generally, if one factor surjects onto the other factor which is amenable with some extra condition. To see this, we first need a lemma:

Lemma 4.16. *Let G_1, G_2 be groups and $\pi : G_1 \twoheadrightarrow G_2$ be an epimorphism. Let $A < G_1$ be a subgroup such that $\pi|_A$ is injective and let*

$$G = G_1 *_A G_2 = \langle G_1, G_2 | a = \pi(a), \forall a \in A \rangle$$

be the amalgamated free product given by π. Then the kernel of the homomorphism $\psi : G \to G_2$ defined by $\psi(g) = \pi(g)$, and $\psi(h) = h$ for all $g \in G_1$ and $h \in G_2$, is a free product.

Proof. It is clear that $\mathrm{Ker}(\psi) \cap A = 1$. So by Theorem 2.13 in Chapter 2, $\mathrm{Ker}(\psi)$ is a free product $K * (*_i H_i)$ where K is a free group and H_i is the intersection of $\mathrm{Ker}(\psi)$ with a conjugate of some factor. □

Corollary 4.17. *Let H be an amenable group and let $\pi : G \twoheadrightarrow H$ be a group epimorphism and let $A < G$ be a subgroup such that $\pi|_A$ is injective and $[H : \pi(A)] \geq 2$. Then the amalgamated free product $G *_A H$ given by π is in $\mathcal{A}$. In particular, if G is amenable, the double of G over A is in $\mathcal{A}$.*

Proof. As mentioned in Lemma 4.16, the normal subgroup $\mathrm{Ker}(\psi) = N$ is a free product $K * (*_i H_i)$ where K is a free group and H_i is the intersection of N with a conjugate of some factor. By the theory of Bass-Serre, if X is the Bass-Serre tree of $G *_A H$ and $Y = N \setminus X$ is the quotient graph, then the free group K is isomorphic to the fundamental group $\pi_1(Y, T)$ of the graph Y relatively to the maximal tree T, which is generated by g_y with $y \in O - T$ where O is an orientation of Y (cf. Remarque in p. 61 in [72]). Thus in order to have $K \neq \{1\}$, it suffices that the quotient graph $Y = N \setminus X$ is not a tree. If there exist $x \in G$ and $h \in N$ such that $hH = xH$ and $hA \neq xA$ (see Figure 4.1), then the quotient graph Y will have a circuit of length 2.

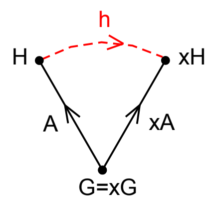


Figure 4.1:

Thus, it is sufficient to find $x \in G$ and $h \in N$ such that $x \in G \setminus A$ and $x^{-1}h \in H \setminus A$.

By assumption on the index of A, there exist $z \in H \setminus \pi(A)$ so that $z^{-1} \in H \setminus \pi(A)$. Let $x \in G \setminus A$ such that $\pi(x) = z$. Let $h = x\pi(x^{-1})$. Then $\psi(h) = \psi(x)\psi(\pi(x^{-1})) = \pi(x)\pi(x^{-1}) = 1$, so $h \in N$, and we have

$$x^{-1}h = x^{-1}x\pi(x^{-1}) = \pi(x^{-1}) = z^{-1} \in H \setminus A.$$

Therefore, the normal subgroup N is a free product $K*(*_i H_i)$ with $K \neq \{1\}$, and the quotient is the amenable group H. So N is co-amenable in $G *_A H$ and is in $\mathcal{A}$, thus $G *_A H$ is in $\mathcal{A}$ by (3) of Proposition 4.7. For the second statement, take $\pi = \mathrm{Id}_G$. $\square$

Example 4.4.1. (see Chapter 5) The right-angled Coxeter group of handled graph (including the n-gons) is in $\mathcal{A}$.

4.4.2 Amalgamated free products over a finite subgroup

Definition 4.4.2. Let G, H be two countable groups and let A be a common finite subgroup of G and H. We say that the triple (G, H, A) is in the class $\mathcal{A}'$ if there exist a G-action on X and a H-action on Y such that

(i) the action $G \curvearrowright X$ is transitive;

(ii) for every element g of $G \setminus A$, and h of $H \setminus A$, the sets

$$\mathrm{supp}_A(g) = \{x \in X | Ax \cap gAx = \emptyset\}$$

$$\mathrm{supp}_A(h) = \{x \in Y | Ax \cap hAx = \emptyset\}$$

are infinite;

(iii) there exist Følner sequences $\{C_n\}_{n\geq 1}$ of $G \curvearrowright X$ and $\{D_n\}_{n\geq 1}$ of $H \curvearrowright Y$ such that

(iii)-1. $|C_n| = |D_n|$, $\forall n \geq 1$;

(iii)-2. the sets $\{A \cdot C_n\}_{n\geq 1}$, $\{A \cdot D_n\}_{n\geq 1}$ are pairwise disjoint;

(iv) the action of A on X and Y are free.

Note that if $(G, H, A) \in \mathcal{A}'$ then $G \in \mathcal{A}$.

Proposition 4.18. *If* $(G,H,A) \in \mathcal{A}'$ *then* $G *_A H \in \mathcal{A}$.

Proof. Let X be a countable set carrying actions of G and H as in Definition 4.4.2. Since A acts freely in the two actions, conjugating we may assume that the G-action and the H-action coincide on H.

Now let $\{C_n\}_{n\geq 1}$ (respectively $\{D_n\}_{n\geq 1}$) be the Følner sequence for G (respectively for H) satisfying the condition (iii) as in Definition 4.4.2. Set $Z = \{\sigma \in Sym(X) \,|\sigma a = a\sigma, \forall a \in A\}$; this is a closed subset in $Sym(X)$ so it is a Baire space. For $\sigma \in Z$, let $H^\sigma = \sigma^{-1}H\sigma$. By universality, the amalgamated free product $G *_A H^\sigma$ acts on X by $g \cdot x = gx$ and $h \cdot x = \sigma^{-1}h\sigma x$ for all $g \in G$ and $h \in H$. We shall prove that the sets

$$\mathcal{O}_1 = \{\sigma \in Z | \text{ the action } G *_A H^\sigma \text{ on } X \text{ is faithful } \},$$

and

$$\mathcal{O}_2 = \{\sigma \in Z | \text{ there is a subsequence } \{n_k\} \text{ of } n \text{ such that } \sigma(C_{n_k}) = D_{n_k}, \forall k\}$$

are generic in Z.

Indeed, for the genericity of $\mathcal{O}_1$, we shall prove that for every non-trivial word $w \in G *_A H$, the set

$$\mathcal{V}_w = \{\sigma \in Z | w^\sigma = \mathrm{Id}_X\}$$

is closed and of empty interior. It is clear that the set $\mathcal{V}_w$ is closed. To prove that the set $\mathcal{V}_w$ is of empty interior, let us consider the case where $w = ag_nh_n \cdots g_1h_1$ with $a \in A$, $g_i \in G \setminus A$ and $h_i \in H \setminus A$ (the other three cases are similar). The corresponding element of $Sym(X)$ given by the action is $w^\sigma = ag_n\sigma^{-1}h_n\sigma \cdots g_1\sigma^{-1}h_1\sigma$. Let $\sigma \in \mathcal{V}_w$. Let $F \subset X$ be a finite subset. Choose $x_0 \notin F \cup \sigma(F)$ such that $Ax_0 \cap (F \cup \sigma(F)) = \emptyset$. Inductively on $1 \leq i \leq n$, we choose a new point $x_{4i-3} \in \mathrm{supp}_A(h_i)$ such that Ax_{4i-3} and h_iAx_{4i-3} are outside of the finite set of all points defined before (this is possible by (ii) in Definition 4.4.2). Then we define

$$\sigma'(ax_{4i-4}) := ax_{4i-3} \text{ and } \sigma'(a\sigma^{-1}(x_{4i-3})) := a\sigma(x_{4i-4}),$$

for all $a \in A$. Then set $x_{4i-2} := h_ix_{4i-3}$. We choose again a new point $x_{4i-1} \in \mathrm{supp}_A(g_i)$ such that Ax_{4i-1} and g_iAx_{4i-1} are outside of the finite set of all points considered so far. We then define

$$\sigma'(ax_{4i-1}) := ax_{4i-2} \text{ and } \sigma'(a\sigma^{-1}(x_{4i-2})) := a\sigma(x_{4i-1}),$$

for all $a \in A$. Then set $x_{4i} := g_ix_{4i-1}$.

Every point v on which σ' is defined verifies $\sigma'a(v) = a\sigma'(v)$, $\forall a \in A$. Indeed, let a, $a' \in A$. Then,

- $\sigma'a(a'x_{4i-4}) = \sigma'(aa'x_{4i-4}) = aa'x_{4i-3} = a(a'x_{4i-3}) = a\sigma'(a'x_{4i-4})$;
- $\sigma'a(a'\sigma^{-1}(x_{4i-3})) = \sigma'(aa'\sigma^{-1}(x_{4i-3})) = aa'\sigma(x_{4i-4}) = a(a'\sigma(x_{4i-4})) = a\sigma'(a'\sigma^{-1}(x_{4i-3}))$;

· $\sigma' a(a' x_{4i-1}) = \sigma'(aa' x_{4i-1}) = aa' x_{4i-2} = a(a' x_{4i-2}) = a\sigma'(a' x_{4i-1})$;

· $\sigma' a(a'\sigma^{-1}(x_{4i-2})) = \sigma'(aa'\sigma^{-1}(x_{4i-2})) = aa'\sigma(x_{4i-1}) = a(a'\sigma(x_{4i-1})) = a\sigma'(a'\sigma^{-1}(x_{4i-2}))$.

By construction, the $4n+1$ points obtained by the right subwords of $w^{\sigma'}$ are all distinct and in particular $w^{\sigma'} x_0 \neq x_0$. We then define σ' to be σ on every points except these finite points, so that $\sigma' \in Z \setminus \mathcal{V}_w$ satisfies $\sigma'|_F = \sigma|_F$. Furthermore, if $w = g \in G \setminus \{\mathrm{Id}\}$, then there exists $x \in X$ such that $gx \neq x$ since G acts faithfully on X. This concludes the genericity of $\mathcal{O}_1$.

About the genericity of $\mathcal{O}_2$, let us write $\mathcal{O}_2 = \bigcap_{N\in\mathbb{N}} \{\sigma \in Sym(X)|$ there exists $m \geq N$ such that $\sigma(C_m) = D_m\}$. We shall show that for every $N \in \mathbb{N}$, the set $\mathcal{V}_N = \{\sigma \in Sym(X) | \forall m \geq N, \sigma(C_m) \neq D_m\}$ is of empty interior (the closedness is clear). Let $F \subset X$ be a finite subset and $\sigma \in \mathcal{V}_N$. Let $m \geq N$ large enough such that $A \cdot x_i \cap (F \cup \sigma(F)) = \emptyset$ and $A \cdot y_i \cap (F \cup \sigma(F)) = \emptyset$, for every $x_i \in C_m$ and $y_i \in D_m$, $\forall 1 \leq i \leq |C_m| = |D_m|$. This is possible by (iii) in Definition 4.4.2. By (iv) in Definition 4.4.2, we have $|A \cdot x_i| = |A \cdot y_i|$, $\forall i$. We then define

$$\sigma'(ax_i) := ay_i \text{ and } \sigma'(a\sigma^{-1}(y_i)) := a\sigma(x_i),$$

for every $1 \leq i \leq |C_m|$ and $a \in A$. For all other points, we define σ' to be equal to σ so that $\sigma' \in Z \setminus \mathcal{V}_N$ and $\sigma'|_F = \sigma|_F$.

Let $\sigma \in \mathcal{O}_1 \cap \mathcal{O}_2$. Clearly the action of $G *_A H^\sigma$ is transitive and faithful. Let $\{C_{n_k}\}_{k\geq 1}$ be the subsequence of $\{C_n\}_{n\geq 1}$ such that $\sigma(C_{n_k}) = D_{n_k}$, $\forall k \geq 1$. The sequence $\{C_{n_k}\}_{k\geq 1}$ is a Følner sequence for G; and for every $h \in H$, we have

$$\begin{aligned} \lim_{k\to\infty} \frac{|C_{n_k} \triangle h \cdot C_{n_k}|}{|C_{n_k}|} &= \lim_{k\to\infty} \frac{|C_{n_k} \triangle \sigma^{-1} h \sigma C_{n_k}|}{|C_{n_k}|} = \lim_{k\to\infty} \frac{|\sigma C_{n_k} \triangle h\sigma C_{n_k}|}{|C_{n_k}|} \\ &= \lim_{k\to\infty} \frac{|D_{n_k} \triangle h D_{n_k}|}{|D_{n_k}|} = 0 \end{aligned}$$

since $\{D_{n_k}\}_{k\geq 1}$ is a Følner sequence for H. Thus the sequence $\{C_{n_k}\}_{k\geq 1}$ is a Følner sequence for $G *_A H^\sigma$, and therefore $G *_A H^\sigma$ is in $\mathcal{A}$. □

Remark 4.4.3. 1. The condition (ii) in Definition 4.4.2 is trivially satisfied if the G-action on X is free (which implies that G is amenable).

2. About (iv) in Definition 4.4.2, this condition is used in the proof of the genericity of $\mathcal{O}_1$ and $\mathcal{O}_2$ where, given any two points x and y, we needed to have $|Ax| = |Ay|$ in order to define σ' such that $\sigma'(Ax) = Ay$.

The following lemma shows that given two infinite amenable groups, one can always find Følner sequences having the same cardinality. Precisely, we have:

Lemma 4.19. *Let G, H be infinite amenable groups. Then there exist Følner sequences $\{C_n\}_{n\geq 1}$ of G and $\{D_n\}_{n\geq 1}$ of H such that $|C_n| = |D_n|$, $\forall n \geq 1$.*

Proof. We shall show that for any $\varepsilon > 0$, any finite subset $F \subset G$ and any finite subset $E \subset H$, there exist a finite subset $C' \subset G$ and a finite subset $D' \subset H$ such that C' is (ε, F)-Følner and D' is (ε, E)-Følner verifying $|C'| = |D'|$.

Recall that a finite subset $A \subset G$ is (ε, F)-Følner if

$$|A \vartriangle gA| < \varepsilon |A|, \ \forall g \in F.$$

By amenability of G, there is a finite subset $C_0 \subset G$ which is (ε, F)-Følner. Let $\{D_n\}_{n\geq 1}$ be a Følner sequence of H. Let $n \gg 1$ large enough such that

(1) D_n is $(\varepsilon/4, E)$-Følner;

(2) $|D_n| > \lambda |C_0|$, where $\lambda = \max\{8/\varepsilon, 2\}$.

By Euclidean division, there exist d, $r \in \mathbb{N}$ such that $|D_n| = d|C_0| + r$ with $r < |C_0|$. Let $g_1, \dots, g_d \in G$ such that $\{C_0 g_i\}_i$ are pairwise disjoint. We put $C' := \bigsqcup_{i=1}^{d} C_0 g_i$. Then C' is (ε, F)-Følner and $|C'| = d|C_0|$.

Now let $D' := D_n - \{x_1, \dots, x_r\}$ be a subset of D_n obtained from D_n by deleting any r elements of D_n. Then $|D'| = |D_n| - r = d|C_0| = |C'|$. We claim that D' is (ε, E)-Følner. Indeed, we have

$$\frac{|D_n|}{|D'|} = \frac{|D_n|}{d|C_0|} = \frac{d|C_0| + r}{d|C_0|} = 1 + \frac{r}{d|C_0|} < 1 + \frac{1}{d} \leq 2,$$

since $r < |C_0|$ and $|D_n| = d|C_0| + r > \lambda |C_0|$ so $d > \lambda - 1 \geq 1$ by definition of λ. In addition, we have

$$\frac{|D' \vartriangle D_n|}{|D_n|} = \frac{r}{|D_n|} < \frac{r}{\lambda |C_0|} < \frac{1}{\lambda} \leq \frac{\varepsilon}{8}.$$

Therefore,

$$\begin{aligned} \frac{|D' \vartriangle hD'|}{|D'|} &\leq \frac{|D_n|}{|D'|}\Big(\frac{|D' \vartriangle D_n|}{|D_n|} + \frac{|D_n \vartriangle hD_n|}{|D_n|} + \frac{|hD_n \vartriangle hD'|}{|D_n|}\Big) \\ &< 2\big(\frac{\varepsilon}{8} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8}\big) = \varepsilon, \end{aligned}$$

for every $h \in E$. □

Lemma 4.20. *If there exist amenable actions of G and H, then there exist G-action and H-action such that the actions admit Følner sequences $\{C_n\}_{n\geq 1}$ for the G-action and $\{D_n\}_{n\geq 1}$ for the H-action with $|C_n| = |D_n|$, $\forall n \geq 1$.*

Proof. If $G \curvearrowright X$ amenably, we replace X by a disjoint union of infinitely many copies of X and use the same idea as in the proof of Lemma 4.19; if $C_0 \in X$ is (ε, F)-Følner, we put C' to be d copies of C_0 in d disjoint copies of X. □

Corollary 4.21. *Let G and H be countable groups and A be a common finite subgroup of G and H. If G is an infinite amenable group and there is a H-set Y such that the H-action is amenable and the action of A on Y is free, then $(G, H, A) \in \mathcal{A}'$.*

Proof. Let $\{C_n\}_{n\geq 1}$ be a Følner sequence of G. First of all, we can suppose that the sequence $\{A \cdot C_n\}_{n\geq 1}$ is pairwise disjoint. Indeed, if $\{C_n\}_{n\geq 1}$ is a Følner sequence of G, we define $\{C'_n\}_{n\geq 1}$ inductively on n; let $C'_1 := C_1$ and for every $n \geq 1$, we choose $h_n \in G$ such that $AC_{n+1}h_n \cap (\cup_{i=1}^n AC'_i) = \emptyset$, and we set $C'_{n+1} := C_{n+1}h_n$. Moreover, if $\{D_n\}_{n\geq 1} \subset Y$ is a Følner sequence for H-action, we can also suppose that the sequence $\{A \cdot D_n\}_{n\geq 1}$ is pairwise disjoint. Indeed, let $Y_0 = \sqcup_{i\geq 1} Y_i$ where $Y_i = Y$, $\forall i \geq 1$ be a disjoint union of infinitely many copies of Y. Then the H-action on Y_0 is amenable with a Følner sequence $D_n \subset Y_n$, $\forall n \geq 1$ such that $\{A \cdot D_n\}_{n\geq 1}$ is pairwise disjoint. In addition, by Lemma 4.20 we can suppose that $|C_n| = |D_n|$, $\forall n \geq 1$, so that the condition (iii) in Definition 4.4.2 is verified.

Now we consider the H-action on $Y' := H \sqcup Y_0$. The action satisfies the condition (ii) of Definition 4.4.2 (for the H-action) since the H-action on itself is free, and the action of A on Y' is free (since the A-action on H is clearly free and the A-action on Y_0 is free by assumption). Finally let $G = X$. The G-action on itself is transitive and free. Thus the triple (G, H, A) is in $\mathcal{A}'$. □

Corollary 4.22. *If G is infinite amenable group and H contains a finite index normal subgroup N with $N \cap A = \{1\}$, then $(G, H, A) \in \mathcal{A}'$.*

It follows from Corollary 4.21 by taking $Y = H/N$. So for example if H is residually finite, then $G *_A H$ is in $\mathcal{A}$ for every finite subgroup A of G and H. Besides, with $A = \{1\}$, we find a particular case of the result of Glsner-Monod; if G is amenable, then $G * H \in \mathcal{A}$ for every countable group H.

Corollary 4.23. *Let G, H be amenable groups and let A be a common finite subgroup of G and H. Then the amalgamated free product $G *_A H$ is in $\mathcal{A}$.*

Proof. The cases where G or H is infinite follow from Corollary 4.21 and Proposition 4.18. So let G and H be finite groups. From a result of Baumslag [12], if G and H are finite groups, then the amalgamated free product $G *_A H$ contains a free subgroup of finite index. So $G *_A H$ is in $\mathcal{A}$ since $\mathcal{A}$ is closed under the extension of co-amenable subgroup. □

Remark 4.4.4. When G is a finitely generated group with polynomial growth, Lemma 4.19 can be strengthened. Indeed, in this case G admits Følner sequences of any prescribed size. More precisely, let $\{a_n\}_{n\geq 1}$ be a strictly ascending sequence of positive integers. Let G be an infinite finitely generated group with polynomial growth. Then G has a Følner sequence $\{F_n\}_{n\geq 1}$ such that $|F_n| = a_n$, $\forall n \geq 1$.

Proof. Let S be a finite symmetric generating set of G. Denote $B(k)$ the ball of radius k centered at 1 in the Cayley graph $\mathcal{G}(G, S)$. Let k_n such that $|B(k_n)| \leq a_n < |B(k_n + 1)|$. We choose a finite subset K_n such that $K_n \cap B(k_n) = \emptyset$ and $|K_n| = a_n - |B(k_n)|$, and set $F_n := B(k_n) \cup K_n$, $\forall n \geq 1$. Recall that the boundary ∂A of A is the set of edges (s,t) such that $s \in A$ and $t \notin A$. We have

$$\begin{aligned}\frac{|\partial F_n|}{|F_n|} &\leq \frac{|\partial B(k_n)|}{|F_n|} + \frac{|\partial K_n|}{|F_n|} \leq \frac{|\partial B(k_n)|}{|B(k_n)|} + \frac{|\partial K_n|}{|B(k_n)|} \\ &\leq \frac{|\partial B(k_n)|}{|B(k_n)|} + |S|\frac{|K_n|}{|B(k_n)|} \leq \frac{|\partial B(k_n)|}{|B(k_n)|} + |S|\frac{|B(k_n+1)| - |B(k_n)|}{|B(k_n)|} \\ &\leq \frac{|\partial B(k_n)|}{|B(k_n)|} + |S|\frac{|\partial B(k_n)|}{|B(k_n)|} \leq (1+|S|)\frac{|\partial B(k_n)|}{|B(k_n)|}.\end{aligned}$$

By a result of Pansu [67], a group with polynomial growth with degree d satisfies $\frac{|B(k)|}{k^d} \xrightarrow[k\to\infty]{} C$, for some $C > 0$. Thus in such a group, the sequence of all balls is a Følner sequence. This applies to our case and thus $|\partial F_n|/|F_n| \xrightarrow[k\to\infty]{} 0$. $\square$

4.4.3 Central extensions

From the idea of Lemma 7.3.1 in [66], we have:

Lemma 4.24. *Let $1 \to Z \to G \xrightarrow{p} Q \to 1$ be a central extension. Suppose that there is a co-amenable subgroup $H < Q$ such that $H \in \mathcal{A}$ (so Q is also in $\mathcal{A}$) and the central extension splits over H (i.e. the central extension $1 \to Z \to p^{-1}(H) \xrightarrow{p} H \to 1$ splits). Then $G \in \mathcal{A}$.*

Proof. Since the extension splits over H, the group $p^{-1}(H)$ is a semi-direct product of Z and H, but Z is central so $p^{-1}(H)$ is indeed isomorphic to the direct product $Z \times H$. Since $Z \in \mathcal{A}$ (Z is amenable) and $H \in \mathcal{A}$ by assumption, the group $p^{-1}(H)$ is in $\mathcal{A}$. Moreover, the map $G/p^{-1}(H) \to Q/H$ defined by $gp^{-1}(H) \mapsto p(g)H$ is G-equivariant and bijective, so the co-amenability of H in Q implies the co-amenability of $p^{-1}(H)$ in G, thus $G \in \mathcal{A}$. $\square$

Example 4.4.5. (Example 7.3.4. in [66]) For $p,\ q \geq 2$, a *Torus knot group* $\Gamma_{p,q}$ is the group with the presentation $\Gamma_{p,q} = \langle x, y | x^p = y^q\rangle$. There is a central extension

$$0 \to \mathbb{Z} \to \Gamma_{p,q} \to \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z} \to 1,$$

where $\mathbb{Z} \simeq \langle x^p = y^q\rangle$, $\mathbb{Z}/p\mathbb{Z} \simeq \langle x|x^p = 1\rangle$ and $\mathbb{Z}/q\mathbb{Z} \simeq \langle y|y^q = 1\rangle$.

The free product $\mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$ has a finite index free subgroup $\mathbb{F}$ over which the central extension splits. Thus the group $\Gamma_{p,q}$ is in $\mathcal{A}$ by Lemma 4.24.

Example 4.4.6. Let M be a 3-manifold which is constructed as a fiber bundle over a closed orientable surface with fiber a circle. Such a 3-manifold is an orientable *Seifert fibred space.* There is a central extension

$$0 \to \mathbb{Z} \to \pi_1(M) \xrightarrow{p} \Gamma_g \to 1,$$

where Γ_g is the fundamental group of the closed orientable surface of genus $g \geq 2$. The derived subgroup Γ'_g is free so Γ'_g is co-amenable in Γ_g and $\Gamma'_g \in \mathcal{A}$. Since Γ'_g is free (since $[\Gamma_g : \Gamma'_g]$ is infinite), we have a central extension that splits:

$$1 \to \mathbb{Z} \to p^{-1}(\Gamma'_g) \xrightarrow{p} \Gamma'_g \to 1.$$

Thus $\pi_1(M)$ is in $\mathcal{A}$ by Lemma 4.24.

Remark 4.4.7. Besides Example 4.4.5, other important examples of two generators one-relator groups are the Baumslag-Solitar groups. Recall that these groups are defined by:

$$BS(m,n) := \langle a, b | b^{-1}a^n b = a^m \rangle.$$

Kropholler showed in [49] that the second derived subgroup $BS(m,n)''$ is free for every m and n. It follows that $BS(m,n)$ is in $\mathcal{A}$ since $BS(m,n)''$ is co-amenable in $BS(m,n)$.

Question. If A, $B \in \mathcal{A}$ and $C \simeq \mathbb{Z}$, is $A *_C B$ in $\mathcal{A}$?

Chapter 5

Right-angled Coxeter groups and related topics

In this last chapter, we discuss two classes of groups defined by graphs: these are subclasses of Coxeter groups and Artin groups where the defining relations are commutators, namely right-angled Coxeter groups and right-angled Artin groups. We shall present some partial results on those groups related to amenable actions.

5.1 Right-angled Coxeter groups

5.1.1 Definitions and examples

Throughout this chapter, by a graph we shall mean a finite non-empty undirected graph Γ without loops and without multiple edges.

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph with vertex set $V(\Gamma) = \{s_1, s_2, \ldots, s_m\}$ and the edge set $E(\Gamma) = \{(s_i, s_j) | 1 \leq i \neq j \leq m\}$. The *right-angled Coxeter group* $W(\Gamma)$ associated to the graph Γ is the group with generators $s_1, s_2, \ldots, s_m$ of order 2, and relations $s_i s_j = s_j s_i$ if there is an edge $(s_i, s_j) \in E(\Gamma)$:

$$W(\Gamma) = \langle s_1, s_2, \ldots, s_m | s_i s_j = s_j s_i \text{ if } (s_i, s_j) \in E(\Gamma); s_i^2 = 1, \forall 1 \leq i \leq m \rangle.$$

It is clear that each finite graph defines a unique right-angled Coxeter groups up to isomorphism. The converse is given by Hosaka [42]:

Theorem 5.1. *Every right-angled Coxeter groups determines its defining graph up to isomorphism.*

Thus two right-angled Coxeter groups are isomorphic if and only if the defining graphs are isomorphic. A similar result was obtained also by Radcliffe [70] and Castella [17].

Example 5.1.1. 1. If Γ is a complete graph with m vertices, then $W(\Gamma) \simeq (\mathbb{Z}/2\mathbb{Z})^m$.

2. If Γ is the disjoint union of two graphs Γ_1 and Γ_2, then $W(\Gamma) \simeq W(\Gamma_1) * W(\Gamma_2)$; in particular, if Γ has m vertices without any edges, then

$$W(\Gamma) \simeq \underbrace{\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \cdots \mathbb{Z}/2\mathbb{Z}}_{m \text{ times}} = \big(\mathbb{Z}/2\mathbb{Z}\big)^{*m}.$$

3. If the vertex set of Γ can be divided into two disjoint sets Γ_1 and Γ_2 such that every vertex in Γ_1 is connected to every vertex in Γ_2, then $W(\Gamma) \simeq W(\Gamma_1) \times W(\Gamma_2)$. In particular, if Γ is a complete bipartite graph, then $W(\Gamma) \simeq (\mathbb{Z}/2\mathbb{Z})^{*m} \times (\mathbb{Z}/2\mathbb{Z})^n$ for some m and n.

A subgraph $\Gamma_0 \subset \Gamma$ is *full* if for every edge (v, w) in Γ with $v, w \in V(\Gamma_0)$, it is an edge in Γ_0. Given a set of vertices $s_1, \ldots, s_m$ in Γ, we will denote by $\langle s_1, \ldots, s_m \rangle$ the smallest full subgraph containing the vertices $s_1, \ldots, s_m$.

Remark 5.1.2. If Γ_1, Γ_2 are full subgraphs of Γ such that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_0 = \Gamma_1 \cap \Gamma_2$, we shall denote it by $\Gamma = \Gamma_1 \cup_{\Gamma_0} \Gamma_2$. In this case, we have

$$W(\Gamma) = W(\Gamma_1) *_{W(\Gamma_0)} W(\Gamma_2).$$

Example 5.1.3. Consider the defining graph Γ as in Figure 5.1.

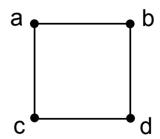


Figure 5.1: The square

It is a complete bipartite graph $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 = \langle a, d \rangle$ and $\Gamma_2 = \langle b, c \rangle$, so that $W(\Gamma) = \big(\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}\big) \times \big(\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}\big)$. Notice that Γ can be decomposed as $\Gamma = \Gamma_1 \cup_{\Gamma_0} \Gamma_2$ with $\Gamma_1 = \langle a, b, c \rangle$, $\Gamma_2 = \langle b, c, d \rangle$ and $\Gamma_0 = \langle b, c \rangle$.

5.1.2 Surjective homomorphism of $W(\Gamma)$ onto $\mathbb{Z}/2\mathbb{Z}$

Let $\phi : W(\Gamma) \to \mathbb{Z}/2\mathbb{Z}$ be a surjective homomorphism defined by $\phi(s_i) = 1$, for every generator s_i in $W(\Gamma)$. The kernel of ϕ is generated by $s_i s_j$, $\forall 1 \leq i \neq j \leq m$. We shall denote $\mathrm{Ker}(\phi) =: KW(\Gamma)$.

Lemma 5.2. *Let Γ be a complete graph with m vertices. Then $KW(\Gamma) \simeq (\mathbb{Z}/2\mathbb{Z})^{m-1}$.*

Proof. Let $\{s_1, s_2, \ldots, s_m\}$ be the generators of $W(\Gamma)$. Since $s_i s_j = s_j s_i$, for all $1 \leq i \leq m$ and $1 \leq j \leq m$, the set $\{s_1 s_2, s_2 s_3, \ldots, s_{m-1} s_m\}$ forms a family of generators of $KW(\Gamma)$, and they commute each other. Furthermore, $(s_i s_{i+1})^2 = 1$ since $s_i^2 = 1$ and $s_i s_{i+1} = s_{i+1} s_i$, $\forall 1 \leq i \leq m$. □

A shorter argument to see the Lemma 5.2 is the following: the group $W(\Gamma)$ is a m-dimensional vector space on $\mathbb{Z}/2\mathbb{Z}$, so $KW(\Gamma)$ is of dimension $m-1$, thus $KW(\Gamma) \simeq (\mathbb{Z}/2\mathbb{Z})^{m-1}$.

Lemma 5.3. *If $W(\Gamma) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, then $KW(\Gamma) \simeq \mathbb{Z}$.*

Proof. If s, t are the generators of $W(\Gamma)$, then the element st is of infinite order, and it generates $KW(\Gamma) \simeq \langle st\rangle$. □

Proposition 5.4. *Let $G = G_1 *_A G_2$ be a free product with amalgamation, and let $\psi : G \to H$ be a group homomorphism such that $\psi(G) = \psi(A)$ (i.e. for all $g \in G$, there is $a \in A$ such that $\psi(g) = \psi(a)$). Then*

$$Ker(\psi) \simeq \big(Ker(\psi) \cap G_1\big) *_{\big(Ker(\psi)\cap A\big)} \big(Ker(\psi) \cap G_2\big).$$

Proof. By the theory of Bass-Serre, G acts on a tree X where the vertices are the left cosets of G_1 and G_2, and the edges are the left cosets of A (Theorem 2.16 in Chapter 2). Let us show that $\mathrm{Ker}(\psi)$ acts on X transitively on the edges. Indeed, if gA is an edge, there is $a \in A$ such that $\psi(a) = \psi(g)$ by assumption. So $h = ag^{-1}$ is in $\mathrm{Ker}(\psi)$ since $\psi(h) = \psi(ag^{-1}) = 1$, and $h \cdot aA = ag^{-1}gA = aA = A$. Since there are two orbits on the vertices of X under the action of $\mathrm{Ker}(\psi)$, we apply again the theory of Bass-Serre (Theorem 2.17 in Chapter 2) to conclude the proof. □

Corollary 5.5. *If $\Gamma = \Gamma_1 \cup_{\Gamma_0} \Gamma_2$, then*

$$KW(\Gamma) = KW(\Gamma_1) *_{KW(\Gamma_0)} KW(\Gamma_2).$$

In particular, if Γ_0 consists of one vertex, then $KW(\Gamma) = KW(\Gamma_1) * KW(\Gamma_2)$ is a free product.

Proof. Since $\phi : W(\Gamma) \to \mathbb{Z}/2\mathbb{Z}$ sends every generator of $W(\Gamma)$ into $1 \in \mathbb{Z}/2\mathbb{Z}$ and Γ_0 is not empty, we have $\phi(W(\Gamma_0)) = \phi(W(\Gamma_0))$; and $KW(\Gamma_i) = KW(\Gamma) \cap W(\Gamma_i)$, for $i = 0$, 1 and 2. So the proof is achieved by Proposition 5.4. □

Corollary 5.6. *If $W(\Gamma)$ is a tree with $n \geq 2$ vertices, then*

$$KW(\Gamma) = \big(\mathbb{Z}/2\mathbb{Z}\big)^{*(n-1)}.$$

Proof. Let us show it by induction. For $n = 2$, there is only one tree Γ with 2 vertices s, t and the associated right-angled Coxeter group $W(\Gamma)$ is $\langle s\rangle \times \langle t\rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. So $KW(\Gamma) = \langle st\rangle \simeq \mathbb{Z}/2\mathbb{Z}$ by Lemma 5.2. If $W(\Gamma)$ has n vertices, choose a vertex s such that there is a unique adjacent vertex t. Let $\Gamma' = \Gamma \setminus \big(\{s\} \cup \{(s,t)\}\big)$ be the $(n-1)$-vertices tree obtained from Γ by deleting the vertex s and the edge (s,t). Then $W(\Gamma) = W(\Gamma') *_{W\langle t\rangle} W\langle s,t\rangle$. So

$$KW(\Gamma) = KW(\Gamma') *_{KW\langle t\rangle} KW\langle s,t\rangle \simeq KW(\Gamma') * \mathbb{Z}/2\mathbb{Z}$$

by Corollary 5.5. By hypothesis of induction, $KW(\Gamma')$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}^{*(n-2)}$ so that $KW(\Gamma) \simeq \mathbb{Z}/2\mathbb{Z}^{*(n-1)}$.

□

5.1.3 Surjective homomorphism of $W(\Gamma)$ onto $(\mathbb{Z}/2\mathbb{Z})^2$

Let Γ be a finite connected graph with $n \geq 2$ vertices. Let us choose a vertex $s_0 \in \Gamma$. We define a surjective homomorphism as follows:

$$\begin{aligned} \phi_{s_0} : W(\Gamma) &\to (\mathbb{Z}/2\mathbb{Z})^2 \\ s &\mapsto (1,0), \text{ if } s = s_0, \\ s &\mapsto (0,1), \text{ if } s \neq s_0. \end{aligned}$$

Let $H = \langle (1,1) \rangle$ be the subgroup of $(\mathbb{Z}/2\mathbb{Z})^2$ generated by $(1,1)$. Then

$$\phi_{s_0}^{-1}(H) = KW(\Gamma),$$

since an element $g \in W(\Gamma)$ is a product of an even number of generators if and only if $g \in \phi_{s_0}^{-1}(H)$.

Now let $\Gamma = \Gamma_1 \cup_{\Gamma_0} \Gamma_2$ such that $|\Gamma_0| \geq 2$. Let $s_0 \in \Gamma_0$. Then

$$\phi_{s_0} : W(\Gamma) = W(\Gamma_1) *_{W(\Gamma_0)} W(\Gamma_2) \to (\mathbb{Z}/2\mathbb{Z})^2$$

is such that $\phi_{s_0}(W(\Gamma)) = \phi_{s_0}(W(\Gamma_0))$. So by Proposition 5.4, we thus have

$$\mathrm{Ker}(\phi_{s_0}) \simeq \big(\mathrm{Ker}(\phi_{s_0}) \cap W(\Gamma_1)\big) *_{\left(\mathrm{Ker}(\phi_{s_0}) \cap W(\Gamma_0)\right)} \big(\mathrm{Ker}(\phi_{s_0}) \cap W(\Gamma_2)\big).$$

By letting $K_1W(\Gamma) := \mathrm{Ker}(\phi_{s_0})$ and $K_1W(\Gamma_i) := \mathrm{Ker}(\phi_{s_0}|_{W(\Gamma_i)})$, we have

$$K_1W(\Gamma) \simeq K_1W(\Gamma_1) *_{K_1W(\Gamma_0)} K_1W(\Gamma_2).$$

Notice that if Γ_0 consists of exactly two adjacent vertices, then $K_1W(\Gamma) \simeq K_1W(\Gamma_1) * K_1W(\Gamma_2)$ is a free product. By letting $K_0W(\Gamma) := KW(\Gamma)$, we have a sequence of normal subgroups of index 2:

$$W(\Gamma) \rhd K_0W(\Gamma) \rhd K_1W(\Gamma).$$

Definition 5.1.4. A graph Γ is a *butterfly* if there are full subgraphs Γ_0, Γ_1 and Γ_2 such that $\Gamma = \Gamma_1 \cup_{\Gamma_0} \Gamma_2$ and Γ_0 consists of either one vertex, or two adjacent vertices (see Figure 5.2).

Corollary 5.7. *If Γ is a butterfly, then $W(\Gamma)$ has a free product subgroup of finite index.*

Proof. Let $\Gamma = \Gamma_1 \cup_{\Gamma_0} \Gamma_2$ be a butterfly. Then there exists $n \in \{0,1\}$ such that $K_nW(\Gamma_0) = 1$. So $K_nW(\Gamma) = K_nW(\Gamma_1) * K_nW(\Gamma_2)$ is a free product, and the index $[W(\Gamma) : K_nW(\Gamma)] = 2^{n+1}$ is finite. □

The free product $K_nW(\Gamma) = K_nW(\Gamma_1) * K_nW(\Gamma_2)$ in the proof of Corollary 5.7 is clearly not trivial (i.e. both factors are not trivial). It is known that the right-angled Coxeter groups are residually finite (see e.g. [7]), so that the factors of $K_nW(\Gamma) = K_nW(\Gamma_1) * K_nW(\Gamma_2)$ satisfies the assumptions of the Theorem of Glasner-Monod (Theorem 4.6 in Chapter 4), thus $K_nW(\Gamma)$ is in $\mathcal{A}$. Since the class $\mathcal{A}$ is closed under extensions by co-amenable subgroup, we have:

Corollary 5.8. *If Γ is a butterfly, then $W(\Gamma)$ is in $\mathcal{A}$.*

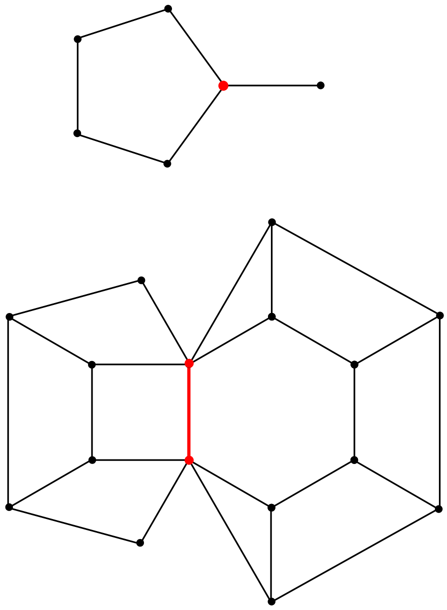


Figure 5.2: Butterflies

5.1.4 n-gon graphs C_n

For $n \geq 2$, let us denote by P_n the graph of n vertices $s_1, \ldots, s_n$ such that (s_i, s_{i+1}) is an edge, $\forall 1 \leq i \leq n-1$ (see Figure 5.3):

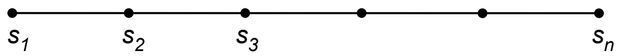


Figure 5.3: Graph P_n

The extreme points s_1 and s_n are called the *endpoints* of P_n. The group $W(P_n)$ has a following presentation:

$$W(P_n) = \langle s_1, \ldots, s_n \,|\, s_i s_{i+1} = s_{i+1} s_i, \forall 1 \leq i \leq n-1;\; s_j^2, \forall 1 \leq j \leq n \rangle$$

and clearly we have

$$W(P_2) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z};$$

$$W(P_3) = \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \times D_\infty;$$

$$W(P_n) = W(P_3) *_{\mathbb{Z}/2\mathbb{Z}} W(P_{n-2}), \forall n \geq 4.$$

For $n \geq 4$, an *n-gon graph* C_n is the graph of n vertices $t_1, \ldots, t_n$ such that (t_i, t_{i+1}) and (t_n, t_1) are the edges, $\forall 1 \leq i \leq n-1$ (see Figure 5.4). It can be

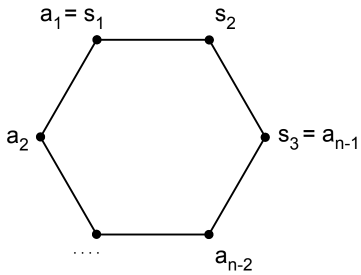


Figure 5.4: n-gon graph C_n

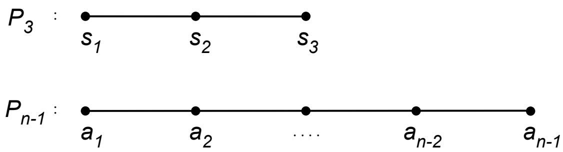


Figure 5.5:

viewed as an amalgamated free product:

$$\begin{aligned} W(C_n) &= W(P_3) *_{W\langle s_1,s_3\rangle} W(P_{n-1}) \\ &= \langle s_1,\, s_2,\, s_3,\, a_1,\, \ldots,\, a_{n-1} \quad | \quad s_1 = a_1,\, s_3 = a_{n-1}; \\ &\qquad s_1s_2 = s_2s_1,\, s_2s_3 = s_3s_2; \\ &\qquad a_ia_{i+1} = a_{i+1}a_i,\, \forall 1 \le i \le n-2; \\ &\qquad s_i^2 = a_j^2 = 1,\, \forall i,j\,\rangle. \end{aligned}$$

where P_3 and P_{n-1} are as in Figure 5.5.

The subgroup $KW(P_3)$ associated to the graph P_3 in Figure 5.5 is generated by $t_1 = s_1s_2$ and $t_2 = s_2s_3$ with $t_i^2 = 1$, $\forall i = 1, 2$; the subgroup $KW(P_{n-1})$ is generated by $b_1 = a_1a_2$, $b_2 = a_2a_3$, $\ldots$, $b_{n-2} = a_{n-2}a_{n-1}$ with $b_i^2 = 1$, $\forall 1 \le i \le n-2$, and $KW\langle s_1, s_3\rangle$ is generated by s_1s_3 which has an infinite order. We have

$$s_1s_3 = s_1s_2s_2s_3 = t_1t_2;$$

and

$$s_1s_3 = a_1a_{n-1} = a_1a_2a_2a_3\cdots a_{n-2}a_{n-2}a_{n-1} = b_1b_2\cdots b_{n-2}.$$

Thus by Corollary 5.5, we have

$$\begin{aligned} KW(C_n) &= KW(P_3) *_{KW\langle s_1,s_3\rangle} KW(P_{n-1}) \\ &= \langle t_1,\, t_2,\, b_1,\, \ldots,\, b_{n-2} \,|\, t_1t_2 = b_1\cdots b_{n-2};\, t_i^2 = b_j^2 = 1, \forall i,j\rangle \\ &\simeq D_\infty *_{\mathbb{Z}} \left(\mathbb{Z}/2\mathbb{Z}^{*(n-2)}\right). \end{aligned}$$

Recall that if a group G surjects onto an amenable group H and $A < G$ is a subgroup such that the restriction on A of the epimorphism is injective and the

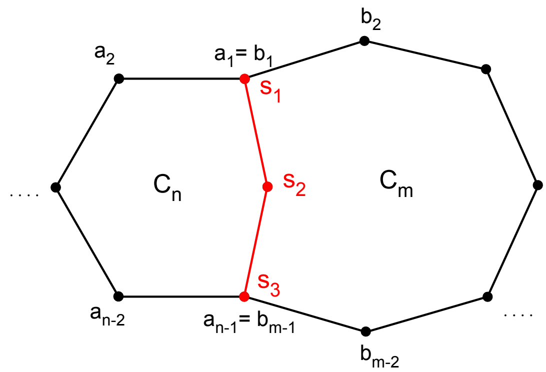


Figure 5.6: Glued two gons

index of the image of A in H is at least 2, then the amalgamated free product $G *_A H$ is in $\mathcal{A}$ (Corollary 4.17 in Chapter 4).

Corollary 5.9. *The right-angled Coxeter group of n-gon $W(C_n)$ is in $\mathcal{A}$, for all $n \geq 4$.*

Proof. For $n \geq 4$, we have seen that $KW(C_n) \simeq D_\infty *_{\mathbb{Z}} \left(\mathbb{Z}/2\mathbb{Z}^{*(n-2)}\right)$, where $D_\infty \simeq \langle t_1, t_2\rangle$, $\mathbb{Z}/2\mathbb{Z}^{*(n-2)} \simeq \langle b_1, \dots, b_{n-2}\rangle$ and $\mathbb{Z} \simeq \langle t_1 t_2\rangle = \langle b_1 \cdots b_{n-2}\rangle$. The homomorphism $f : \mathbb{Z}/2\mathbb{Z}^{*(n-2)} \to D_\infty$ defined by $f(b_1) = t_1$, $f(b_2) = t_2$ and $f(b_i) = 1$, $\forall 3 \leq i \leq n-2$ is surjective and $f|_{\mathbb{Z}}$ is clearly injective. Since D_∞ is amenable, by Corollary 17 in Chapter 3, $KW(C_n)$ is in $\mathcal{A}$. Since $KW(C_n)$ is of finite index in $W(C_n)$, the group $W(C_n)$ is also in $\mathcal{A}$. $\square$

Remark 5.1.5. It is known that the right-angled Coxeter group of C_n contains a surface group of finite index for all $n \geq 4$ (see for example [19]). Indeed, for $n \geq 5$, the right-angled Coxeter group of n-gon $W(C_n)$ is isomorphic to a co-compact subgroup of Isom($\mathbb{H}^2$) generated by reflections in the sides of a right-angled hyperbolic n-gon $\mathcal{P}_n$. So $W(C_n)$ is virtually a hyperbolic surface group. If $n = 4$, $W(C_4)$ contains $\mathbb{Z}^2$ as a finite index subgroup. Since the surface groups are in $\mathcal{A}$ by the result of [56], this gives another argument that the right-angled Coxeter group of n-gon is in $\mathcal{A}$.

Corollary 5.10. *Let n, $m \geq 4$. If Γ is a graph obtained by gluing an n-gon and an m-gon along a path $P_3 = \langle s_1, s_2, s_3\rangle$ as in Figure 5.6, then $W(\Gamma)$ is in $\mathcal{A}$.*

Proof. The graph Γ can be also viewed as $\Gamma = C_{n+m-4} \cup_{\langle s_1, s_3\rangle} P_3$, where $P_3 = \langle s_1, s_2, s_3\rangle$. So $W(\Gamma) = W(C_{n+m-4}) *_{W\langle s_1, s_3\rangle} W(P_3)$, where

$$W(C_{n+m-4}) = W(P_{n-1}) *_{W\langle s_1, s_3\rangle} W(P_{m-1}),$$

with

$$P_{n-1} = \langle a_1 = b_1, a_2, \dots, a_{n-2}, a_{n-1} = b_{m-1}\rangle;$$

$$P_{m-1} = \langle b_1 = a_1, b_2, \ldots, b_{m-2}, b_{m-1} = a_{n-1} \rangle.$$

The subgroup $KW(P_{n-1})$ is generated by

$$t_1 := a_1 a_2,\ t_2 := a_2 a_3,\ \ldots,\ t_{n-2} := a_{n-2} a_{n-1}$$

and $KW(P_{m-1})$ is generated by

$$r_1 := b_1 b_2,\ r_2 := b_2 b_3,\ \ldots,\ r_{m-2} := b_{m-2} b_{m-1},$$

with $t_i^2 = r_i^2 = 1$, $\forall i$, and the subgroup $KW\langle s_1, s_3 \rangle$ is generated by $s_1 s_3 = a_1 a_{n-1} = b_1 b_{m-1}$ with infinite order. Since

$$a_1 a_{n-1} = a_1 a_2 a_2 a_3 \cdots a_{n-2} a_{n-2} a_{n-1} = t_1 t_2 \cdots t_{n-2};$$

and

$$b_1 b_{m-1} = b_1 b_2 \cdots b_{m-2} b_{m-2} b_{m-1} = r_1 r_2 \cdots r_{m-2},$$

the group

$$KW(C_{n+m-4}) = KW(P_{n-1}) *_{KW\langle s_1, s_3 \rangle} KW(P_{m-1})$$

is generated by $t_1, \ldots, t_{n-2}, r_1, \ldots, r_{m-2}$ with the relations

$$\begin{cases} t_1 \cdots t_{n-2} = r_1 \cdots r_{m-2}; \\ t_i^2 = r_j^2 = 1, \forall 1 \leq i \leq n-2, \forall 1 \leq j \leq m-2. \end{cases}$$

Since $KW(P_3)$ is generated by $s_1 s_2 =: p_1$ and $s_2 s_3 =: p_2$ with $p_k^2 = 1$, the group

$$KW(\Gamma) = KW(C_{n+m-4}) *_{KW\langle s_1, s_3 \rangle} KW(P_3)$$

is generated by $t_1, \ldots, t_{n-2}, r_1, \ldots, r_{m-2}, p_1, p_2$ with the relations

$$\begin{cases} t_1 \cdots t_{n-2} = r_1 \cdots r_{m-2} = p_1 p_2\,; \\ t_i^2 = r_j^2 = p_k^2 = 1, \forall 1 \leq i \leq n-2, \forall 1 \leq j \leq m-2, \forall 1 \leq k \leq 2. \end{cases}$$

Thus we see that the surjective homomorphism $\pi : KW(C_{n+m-4}) \to KW(P_3)$ defined by

$$\pi(t_1) = \pi(r_1) = p_1,\ \pi(t_2) = \pi(r_2) = p_2,\ \pi(t_i) = \pi(r_i) = 1,\ \forall 1 \neq i \neq 2,$$

verifies $\pi(s_1 s_3) = \pi(t_1 \cdots t_{n-1}) = p_1 p_2$ and $\pi|_{KW\langle s_1, s_3 \rangle} = \pi|_{\langle t_1 \cdots t_{n-1} \rangle}$ is injective. Therefore the group $KW(\Gamma)$ is in $\mathcal{A}$ by Corollary 4.17 in Chapter 4, and therefore $W(\Gamma)$ is in $\mathcal{A}$. □

Remark 5.1.6. The commensurability of the family of such "glued two gons" is studied by Crisp and Paoluzzi in [19]; the authors showed that if we denote this graph as in Figure 5.6 by $\Gamma_{m-4,n-4}$, then for $5 \leq m \leq n$ and $5 \leq k \leq l$, the groups $W(\Gamma_{m-4,n-4})$ and $W(\Gamma_{k-4,l-4})$ contain isomorphic finite index subgroups if and only if

$$\frac{m-4}{n-4} = \frac{k-4}{l-4}.$$

5.1.5 Handled graph

We shall see that one can use the previous argument for n-gons and glued two gons to a larger class of graphs.

Lemma 5.11. *Let Γ be a graph with at least 3 vertices such that there exist two vertices s, t such that (s,t) is not an edge. Then there exists a surjective homomorphism*

$$\phi_{st} : W(\Gamma) \twoheadrightarrow D_\infty = \langle \alpha, \beta | \alpha^2 = \beta^2 = 1 \rangle$$

such that $\phi_{st}(KW(\Gamma)) = \phi_{st}(W(\Gamma))$ and $\phi_{st}(st) = \alpha\beta$.

Proof. We define $\phi_{st}(s) = \alpha$; $\phi_{st}(t) = \beta$; and $\phi_{st}(u) = 1$, $\forall u \in V(\Gamma) \setminus \{s, t\}$. Then $\phi_{st}(st) = \alpha\beta$. Moreover, let s', t' be vertices such that $s', t' \notin \{s, t\}$. Then clearly $ss' \in KW(\Gamma)$, $tt' \in KW(\Gamma)$; and we have $\phi_{st}(ss') = \alpha$ and $\phi_{st}(tt') = \beta$ which achieve the proof. □

Remark 5.1.7. Any non-complete graph with $n \geq 3$ vertices satisfies the hypothesis of Lemma 5.11.

Definition 5.1.8. A graph Γ is a *handled graph* if it can be obtained from a graph Γ' containing two non-adjacent vertices s and t, by attaching the two endpoints of $P_3 = \langle s_1, s_2, s_3 \rangle$ to the two vertices s, t of Γ'. In other words, Γ is handled if it can be written as $\Gamma = \Gamma' \cup_{\langle s_1, s_3 \rangle} P_3$ (see Figure 5.7).

Corollary 5.12. *For every handled graph Γ, the group $W(\Gamma)$ is in $\mathcal{A}$.*

Proof. Let $\Gamma = \Gamma' \cup_{\langle s_1, s_3 \rangle} P_3$ be a handled graph. By definition, Γ' contains two non-adjacent vertices s, t and they are attached to s_1 and s_3 one another. By Lemma 5.11 there is a surjective homomorphism $\phi : KW(\Gamma') \to D_\infty$ and clearly $\phi|_{\langle st \rangle}$ is injective. So

$$KW(\Gamma) = KW(\Gamma') *_{KW\langle s,t \rangle} KW(P_3) = KW(\Gamma') *_{\langle st \rangle} D_\infty$$

is in $\mathcal{A}$ and therefore $W(\Gamma)$ is also in $\mathcal{A}$. □

Let us summarize this section by giving the following Corollary:

Corollary 5.13. *For the following groups, the associated right-angled Coxeter groups are in $\mathcal{A}$:*

1. *complete graphs;*
2. *disconnected graphs;*
3. *butterflies, including trees;*
4. *handled graphs, including the n-gons, $\forall n \geq 4$;*
5. *graphs whose vertex set can be divided into two disjoint sets Γ_1 and Γ_2 such that every vertex in Γ_1 is connected to every vertex in Γ_2, such that Γ_i is one of the previous four graphs, $\forall i = 1, 2$.*

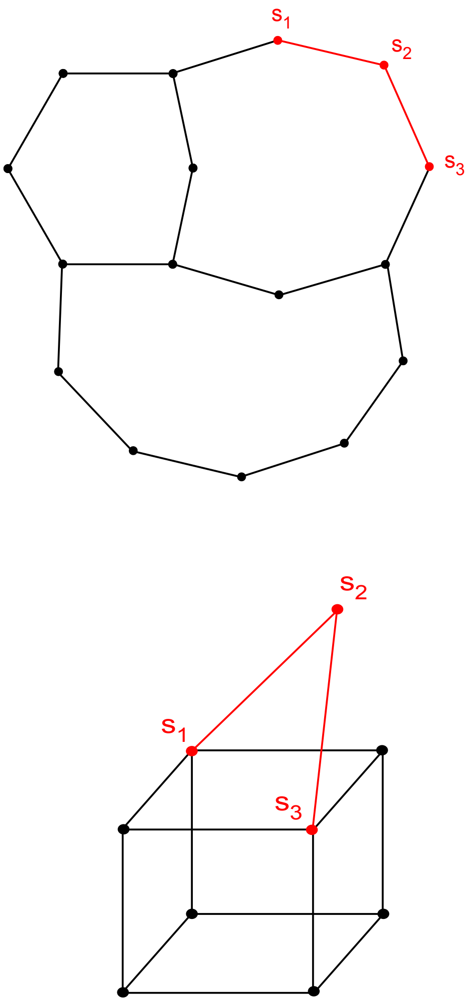


Figure 5.7: Some handled graphs

Question. 1. If $\Gamma = \Gamma_1 *_{\Gamma_0} \Gamma_2$ such that Γ_0 is a complete graph with $n \geq 3$ vertices, is $W(\Gamma)$ in $\mathcal{A}$?

2. If Γ is a cube, is $W(\Gamma)$ in $\mathcal{A}$?

5.2 Right-angled Artin groups

Groups closely related to the right-angled Coxeter groups are the right-angled Artin groups. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph with vertex set $V(\Gamma) = \{s_1, s_2, \ldots, s_m\}$ and the edge set $E(\Gamma) = \{(s_i, s_j) | 1 \leq i \neq j \leq m\}$. The *right-angled Artin group* $A(\Gamma)$ associated to the graph Γ is the group with generators $s_1, s_2, \ldots, s_m$, and relations $s_i s_j = s_j s_i$ if there is an edge $(s_i, s_j) \in E(\Gamma)$:

$$A(\Gamma) = \langle s_1, s_2, \ldots, s_m | s_i s_j = s_j s_i \text{ if } (s_i, s_j) \in E(\Gamma) \rangle.$$

Example 5.2.1. 1. If Γ is a complete graph with m vertices, then $A(\Gamma) \simeq \mathbb{Z}^m$.

2. If Γ is the disjoint union of two graphs Γ_1 and Γ_2, then $A(\Gamma) \simeq A(\Gamma_1) * A(\Gamma_2)$. In particular, if Γ has m vertices without any edges, then

$$A(\Gamma) \simeq \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{m \text{ times}} = \mathbb{Z}^{*m}.$$

Let $\phi : A(\Gamma) \to \mathbb{Z}$ be a surjective homomorphism defined by $\phi(s_i) = 1$, $\forall s_i \in V(\Gamma)$. We shall denote $\mathrm{Ker}(\phi) =: KA(\Gamma)$. Evidently the subgroup $KA(\Gamma)$ is generated by $s_i s_j^{-1}$, $\forall i \neq j$.

Lemma 5.14. *Let Γ be a complete graph with m vertices. Then $KA(\Gamma) \simeq \mathbb{Z}^{m-1}$.*

Proof. If $\{s_1, s_2, \ldots, s_m\}$ are the generators of $W(\Gamma)$, then it is easy to show that $\{s_1 s_2^{-1}, s_2 s_3^{-1}, \ldots, s_{m-1} s_m^{-1}\}$ is a family of generators of $KW(\Gamma)$. □

The following two corollaries can be proved in the analogous way as Corollary 5.5 and Corollary 5.6:

Corollary 5.15. *If $\Gamma = \Gamma_1 \cup_{\Gamma_0} \Gamma_2$, then $A(\Gamma) = A(\Gamma_1) *_{A(\Gamma_0)} A(\Gamma_2)$ and*

$$KA(\Gamma) = KA(\Gamma_1) *_{KA(\Gamma_0)} KA(\Gamma_2).$$

In particular, if Γ_0 consists of one vertex, then $KA(\Gamma) = KA(\Gamma_1) * KA(\Gamma_2)$ is a free product.

Corollary 5.16. *If $A(\Gamma)$ is a tree with $m \geq 2$ vertices, then $KA(\Gamma) = \mathbb{Z}^{*(m-1)}$.*

Let Γ be a finite connected graph with $n \geq 2$ vertices $s_1, \ldots, s_n$. Let us choose a vertex $s_0 \in \Gamma$. As for right-angled Coxeter groups, let us define a surjective homomorphism of $A(\Gamma)$ onto $\mathbb{Z}^2$ as follows:

$$\begin{aligned} \phi_{s_0} : A(\Gamma) &\to \mathbb{Z}^2 \\ s &\mapsto (1,0), \text{ if } s = s_0, \\ s &\mapsto (0,1), \text{ if } s \neq s_0. \end{aligned}$$

Let $H = \langle (1, -1) \rangle$ be the subgroup of $\mathbb{Z}^2$ generated by $(1, -1)$. Then

$$\phi_{s_0}^{-1}(H) = KA(\Gamma).$$

Indeed, let $g \in \phi_{s_0}^{-1}(H)$ and let $r_1^{\varepsilon_1}, \ldots, r_k^{\varepsilon_k}, t_1^{\delta_1}, \ldots, t_m^{\delta_m}$ be the generators or the inverse of the generators occurring in g, with $\varepsilon_i, \delta_i \in \{\pm 1\}$, such that $r_i \in \{s_1, \ldots, s_n\} \setminus \{s_0\}$ and $t_i = s_0$, $\forall i$. Since

$$\phi_{s_0}(g) = \Big(\sum_{i=1}^{m} \delta_i, \sum_{i=1}^{k} \varepsilon_i \Big) \in \langle (1, -1) \rangle,$$

there exists an integer N such that $\sum_{i=1}^{m} \delta_i = N$ and $\sum_{i=1}^{k} \varepsilon_i = -N$. Therefore the sum of the exponents of the generators in g is zero, so $g \in KA(\Gamma)$.

Conversely, if $g \in KA(\Gamma)$, then g can be written as

$$g = (s_{i_1} s_{i_2}^{-1})^{\varepsilon_1} (s_{i_3} s_{i_4}^{-1})^{\varepsilon_2} \cdots (s_{i_{2k-1}} s_{i_{2k}}^{-1})^{\varepsilon_k},$$

with $s_{i_j} \in \{s_1, \ldots, s_n\}$ and $\varepsilon_i \in \{\pm 1\}$. Since $\phi_{s_0}(st^{-1}) = (0, 0)$ if $s \neq s_0 \neq t$, and $\phi_{s_0}(st^{-1}) = \pm(1, -1)$ if exactly only one of s or t is equal to s_0, we have $\phi_{s_0}(g) = (\lambda_1 + \cdots + \lambda_k)(1, -1) \in \langle (1, -1) \rangle$ where $\lambda_i \in \{0, 1, -1\}$. Thus $g \in \phi_{s_0}^{-1}(H)$.

Now let $\Gamma = \Gamma_1 \cup_{\Gamma_0} \Gamma_2$ such that $|\Gamma_0| \geq 2$, and let $s_0 \in \Gamma_0$. Then the homomorphism

$$\phi_{s_0} : A(\Gamma) = A(\Gamma_1) *_{A(\Gamma_0)} A(\Gamma_2) \to \mathbb{Z}^2$$

satisfies $\phi_{s_0}(A(\Gamma)) = \phi_{s_0}(A(\Gamma_0))$. So by letting $K_1 A(\Gamma) := \mathrm{Ker}(\phi_{s_0})$ and $K_1 A(\Gamma_i) := \mathrm{Ker}(\phi_{s_0}) \cap A(\Gamma_i)$, for $i = 0, 1$ and 2, it follows from Proposition 5.4 we have

$$K_1 A(\Gamma) \simeq K_1 A(\Gamma_1) *_{K_1 A(\Gamma_0)} K_1 A(\Gamma_2).$$

Notice that $K_1 A(\Gamma)$ is a normal subgroup with quotient isomorphic to $\mathbb{Z}^2$; and that if Γ_0 consists of exactly two adjacent vertices, then $K_1 A(\Gamma) \simeq K_1 A(\Gamma_1) * K_1 A(\Gamma_2)$ is a free product.

Corollary 5.17. *If Γ is a butterfly, then $A(\Gamma)$ has a co-amenable free product subgroup. In particular, $A(\Gamma)$ is in $\mathcal{A}$.*

Proof. If $\Gamma = \Gamma_1 \cup_{\Gamma_0} \Gamma_2$ is a butterfly, then for $n = 1$ or $n = 2$, we have a non-trivial free product $K_n A(\Gamma) = K_n A(\Gamma_1) * K_n A(\Gamma_2)$ and $K_n A(\Gamma)$ is co-amenable in $A(\Gamma)$. Since all right-angled Artin groups are residually finite by the work of Green [34], the subgroups $K_n A(\Gamma_i)$ are also residually finite, in particular the free product is in $\mathcal{A}$, so $A(\Gamma)$ is in $\mathcal{A}$. □

If Γ is a square as in Example 5.1, the right-angled Artin group of Γ is isomorphic to the direct product of two free groups $\mathbb{F}_2 \times \mathbb{F}_2$, so it is in $\mathcal{A}$. For n-gon graphs in general, the similar argument as right-angled Coxeter groups does not work in right-angled Artin groups. The crucial point in the case of right-angled Coxeter groups was that the subgroup $KW(P_3)$ of the graph $P_3 = \langle s_1, s_2, s_3 \rangle$ is the amenable group D_∞. Unfortunately in the case of right-angled Artin groups, the subgroup $KA(P_3)$ is isomorphic to the free group $\mathbb{F}_2$.

Corollary 5.18. *For the following graphs, the associated right-angled Artin groups are in $\mathcal{A}$:*

1. *complete graphs;*
2. *disconnected graphs;*
3. *butterflies, including trees;*
4. *graphs whose vertex set can be divided into two disjoint sets Γ_1 and Γ_2 such that every vertex in Γ_1 is connected to every vertex in Γ_2, such that Γ_i is one of the previous three graphs, $\forall i = 1, 2$ (including the square).*

Remark 5.2.2. Davis and Januszkiewicz have shown in [21] that for each right-angled Artin group A, there is a right-angled Coxeter group W which contains A as a finite index subgroup. In fact, they show that if Γ is the defining graph for a right-angled Artin group $A(\Gamma)$, then there is a graph Γ' containing Γ as a full subgraph such that $A(\Gamma)$ is a finite index subgroup of $W(\Gamma')$. So if Γ is a n-gon graph C_n, then $W(\Gamma')$ contains a surface group Γ_g (cf. Remark 5.1.5), so that $A(C_n) \cap \Gamma_g$ is a finite index subgroup of Γ_g. Thus $A(C_n)$ has also a surface group $A(C_n) \cap \Gamma_g$ since it is a finite index subgroup of a surface group. Another proof for $A(C_n)$ to have a surface group can be found in [16].

5.3 Braid groups

Another important subclass of Artin groups are braid groups. The *braid group on n strands*, denoted by B_n, is the group admitting a presentation with $n-1$ generators $a_1, \ldots, a_{n-1}$ and relations

- $a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}$, $\forall 1 \leq i \leq n-2$;
- $a_i a_j = a_j a_i$, if $|i-j| \geq 2$.

The group B_1 is trivial and B_2 is isomorphic to $\mathbb{Z}$. For $n = 3$, the group B_3 has the presentation

$$B_3 = \langle a_1, a_2 | a_1 a_2 a_1 = a_2 a_1 a_2 \rangle.$$

By letting $s = a_1 a_2$ and $t = a_1 a_2 a_1$, we obtain the presentation

$$B_3 = \langle s, t | s^3 = t^2 \rangle.$$

So there is a central extension

$$0 \to \mathbb{Z} \to B_3 \to \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} \to 1,$$

and the central extension splits over the finite index free subgroup $\mathbb{F}$ of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. Therefore by Lemma 4.24 in Chapter 4, the group B_3 is in $\mathcal{A}$. Notice that B_3 is a Torus knot group $\Gamma_{2,3}$.

In the presentation of B_n in the beginning of the section, if we add additional relations $a_i = a_i^{-1}$, $\forall i$, then the resulting group is the permutation group $Sym(n)$ of n objects (the generators $a_1, \ldots, a_{n-1}$ become the transpositions $(**)$). So there is a natural surjective homomorphism $\pi : B_n \twoheadrightarrow Sym(n)$. The kernel of π is called the *pure braid group on n strands*, denoted by P_n. It is well-known that there is a splitting extension

$$1 \to \mathbb{F}_{n-1} \to P_n \to P_{n-1} \to 1$$

for every $n \geq 1$. In particular P_3 is in $\mathcal{A}$ since P_2 is infinite cyclic group $\mathbb{Z}$ so that the free group $\mathbb{F}_2$ is co-amenable in P_3. This also shows that B_3 is in $\mathcal{A}$ since $P_3 \in \mathcal{A}$ is co-amenable in B_3.

The above arguments are well-known. It is also known that B_n does not have Property (T), but it is not known whether B_n for $n \geq 4$ has the Haagerup property or not.

Question. Are braid groups in $\mathcal{A}$?

Appendix A

Non-properness of amenable actions on graphs with infinitely many ends

Abstract. We study amenable actions on graphs having infinitely many ends, giving a generalized answer to Ceccherini's question on groups with infinitely many ends.

A.1 Statement of the result

An action of a group G on a set X is *amenable* if there exists a G-invariant mean on X, i.e. a map $\mu : 2^X = \mathcal{P}(X) \to [0,1]$ such that $\mu(X) = 1$, $\mu(A \cup B) = \mu(A) + \mu(B)$ for every pair of disjoint subsets A, B of X, and $\mu(gA) = \mu(A)$, $\forall g \in G$, $\forall A \subseteq X$.

An isometric action of a group G on a metric space (X, d) is *proper* if for some $x_0 \in X$, and every $R > 0$, the set $\{g \in G \mid d(x_0, gx_0) \leq R\}$ is finite.

The aim of this note is to give a short proof of the following result:

Theorem A.1. *Let $X = (V, E)$ be a locally finite graph with infinitely many ends. Let $\overline{X} = V \cup \partial X$ be the end compactification. Let G be a group of automorphisms of X. Assume that the action of G on V is amenable and there exists $x_0 \in V$ such that the closure of Gx_0 contains ∂X. Then there is a unique G-fixed end in ∂X, and the action of G (as a discrete group) on V is not proper.*

A deep result of Stallings [74] says that G has infinitely many ends if and only if G is an amalgamated free product $\Gamma_1 *_A \Gamma_2$ or HNN-extension $HNN(\Gamma, A, \varphi)$ with A finite (with $\min\{[\Gamma_1 : A], [\Gamma_2 : A]\} \geq 2$, not both 2, in the amalgamated product case; and $\min\{[\Gamma : A], [\Gamma : \varphi(A)]\} \geq 2$, not both 2, in the HNN case). In particular, if G has infinitely many ends, it contains non-abelian free subgroups, hence is non amenable. Tullio Ceccherini-Silberstein asked whether non-amenability of G could be proved without appealing to Stallings' theorem. Since a finitely generated group G with infinitely many ends acts properly and transitively on its Cayley graph, our result shows that G is not amenable.

Remarks

1. The density assumption of Theorem A.1 is satisfied when G has finitely many orbits in V. This assumption is necessary; for example the action of $\mathbb{Z}$ on $\mathbb{F}_2 = \langle a, b \rangle$ defined by $n \cdot g = a^n g$, $\forall n \in \mathbb{Z}$, $\forall g \in \mathbb{F}_2$ is amenable and proper.

2. Except for the non-properness statement, our result is contained in a result of Woess (see Theorem 1 in [85]): if $X = (V, E)$ is a locally finite graph and G admits an amenable action on V, then either G fixes a nonempty finite subset of V, or G fixes an end of X, or G fixes a unique pair of ends which are the fixed points of some hyperbolic element in G.

3. There are results on strong isoperimetric inequalities for graphs with infinitely many ends satisfying extra conditions (see Theorem 10.10 in [87]): these give alternative answers to Ceccherini's question.

4. A stronger question is to prove without appealing to Stallings' result that a finitely generated group with infinitely many ends, contains a free group on two generators. Such constructions can be found in the work of Woess (Theorem 3 in [86]), Karlsson and Noskov (Proposition 3 in [46]), and Karlsson (Theorem 1 in [45]).

5. For a finitely generated group with infinitely many ends, Abels shows, using Stallings' theorem, that for G a finitely generated group with infinitely many ends, the compact set of ends is actually a minimal G-space (Theorem 1 in [4]). This is false for compactly generated, non discrete groups. Abels indeed gives the example of the group of affine mappings $(x \mapsto ax + b)$ over $\mathbb{Q}_p$. This group G is $HNN(K, K, \varphi)$, where K is the group of affine mappings over $\mathbb{Z}_p$ and $\varphi : K \to K$ is given by $(x \mapsto ax + b) \mapsto (x \mapsto ax + pb)$. So G has infinitely many ends, but has a unique fixed point on its space of ends[1], which is therefore not G-minimal.

Acknowledgments

We thank T. Ceccherini-Silberstein for suggesting the question, F. Krieger for pointing out a mistake in a previous version, and A. Karlsson for recommending useful references.

[1] This can be seen directly; it also follows from our result, as G is amenable as a discrete group.

A.2 Proof of the theorem

Let X be a countable, discrete set. A *compactification* of X is a compact space $\overline{X} = X \cup \partial X$ in which X is an open dense subset. If G is a group of permutations of X, we say that $\overline{X}$ is a *G-compactification* if the action of G on X extends to an action of G on $\overline{X}$ by homeomorphisms. When X is a locally finite graph (identified with its set of vertices), we will take for ∂X the set of ends of X. In this case, we say that $\overline{X} = X \cup \partial X$ is the *end-compactification* of X (it is an $Aut(X)$-compactification).

Lemma A.2. *Assume that G admits an amenable action without finite orbits, on a countable set X. Let μ be G-invariant mean on X. Let $\overline{X}$ be a G-compactification of X. Then for every subset A of X with $\mu(A) = 1$, the set $\big(\bigcap_{g \in G} \overline{gA}\big) \cap \partial X$ is not empty.*

Proof. By compactness of ∂X, it is enough to show that the family $\big(\overline{gA} \cap \partial X\big)_{g \in G}$ has the finite intersection property. For $g_1, \ldots, g_n \in G$, we have $\mu\big(\bigcap_{i=1}^n g_i A\big) = 1$, while $\mu(F) = 0$ for every finite subset $F \subset X$ since G has no finite orbit. So $\bigcap_{i=1}^n g_i A$ is infinite. Therefore $\left(\overline{\bigcap_{i=1}^n g_i A}\right) \cap \partial X \neq \emptyset$. A fortiori $\bigcap_{i=1}^n \big(\overline{g_i A} \cap \partial X\big) \neq \emptyset$. □

The proof of Theorem A.1 will follow from the four claims below:

Claim 1. Let K be a finite, connected subgraph of X. Let A be an unbounded connected component of $X \setminus K$. Then $gK \subset A$ for infinitely many g in G.

By the assumption, any G-orbit in X has infinite intersection with A (indeed, the assumption implies that Gx is dense in $\overline{X}$ for every vertex x in V since G acts by isometries on X; therefore the intersection of Gx and A is infinite since $\overline{A}$ is a neighborhood of all ends contained in it). So for $x \in K$, one finds a sequence $(g_n)_{n \geq 1}$ in G such that $g_n x$ are pairwise distinct vertices in A. Since $d(g_n x, x) \to \infty$ for $n \to \infty$, we have $g_n K \cap K = \emptyset$ for n sufficiently large. Then $g_n K$ is a connected subset of $X \setminus K$, and $g_n K \cap A \neq \emptyset$. By maximality of A among connected subsets of $X \setminus K$, this implies that $g_n K \subset A$.

If K is a finite connected subgraph of X, we shall say that K is *good* if every connected component of $X \setminus K$ is infinite. Let K be an arbitrary finite connected subgraph of X. Denote by $\widehat{K}$ the union of K and the finite connected components of $X \setminus K$; then $\widehat{K}$ is a good subgraph of X.

Claim 2. Let K be a good subgraph of X such that $X \setminus K$ has at least 3 connected components. Let μ be G-invariant mean on V. Then there exists a unique connected component C_K of $X \setminus K$ such that $\mu(C_K) = 1$.

Indeed, let $A_1, \ldots, A_n$ be the connected components of $X \setminus K$ with $n \geq 3$. Without loss of generality, we may assume that $\mu(A_1) \leq \mu(A_i)$, $\forall i \in \{1, \ldots, n\}$.

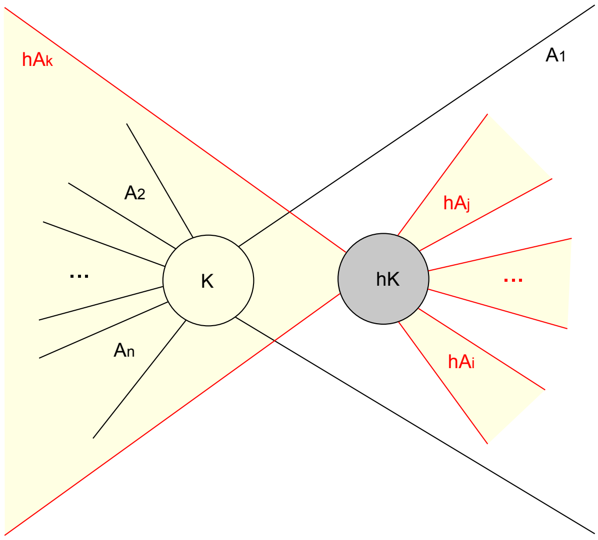


Figure A.1:

By claim 1, we can find $h \in G$ such that $hK \cap K = \emptyset$ and $hK \subset A_1$. Since $hA_1, \ldots, hA_n$ are the connected components of $X \setminus hK$, and K is connected, there exists a unique $k \in \{1, \ldots, n\}$ such that $K \subset hA_k$, so that $hA_i \subset A_1$, $\forall i \neq k$. Hence $\bigsqcup_{i \neq k} hA_i \subset A_1$ (see Figure A.1).

Then by minimality of $\mu(A_1)$,

$$(n-1)\mu(A_1) \leq \sum_{i \neq k} \mu(A_i) = \sum_{i \neq k} \mu(hA_i) = \mu\Big(\bigsqcup_{i \neq k} hA_i\Big) \leq \mu(A_1).$$

Hence $\mu(A_1) = 0$ since $n \geq 3$, and $\mu(A_i) = 0$, $\forall i \neq k$. Since μ is zero on finite subsets of X, we have $1 = \mu(X) = \mu\big(K \cup \bigsqcup_{j=1}^{n} A_j\big) = \mu(A_k)$. We set $A_k = C_K$.

Let x_0 be a base-vertex in V. Denote by B_N the ball of radius N centered at x_0. Let N_0 be such that, for $N \geq N_0$, the complement $X \setminus \widehat{B_N}$ has at least 3 connected components. Set

$$D_N = \Big(\bigcap_{g \in G} \overline{gC_{\widehat{B_N}}}\Big) \cap \partial X.$$

By Lemma A.2, $D_N \neq \emptyset$, and $(D_N)_{N \geq N_0}$ form a decreasing family of closed non-empty subsets of ∂X. So by compactness, $E = \bigcap_{N \geq N_0} D_N$ is non-empty, and obviously G-invariant.

Claim 3. The set E is reduced to one point, and G has no other fixed point in ∂X.

Indeed, if $w \in E$ and $w' \in \partial X$ with $w \neq w'$, then for N large enough w and w' are not in the same closure of a connected component of $X \setminus \widehat{B_N}$. So $w \in \overline{C_{\widehat{B_N}}}$ and $w' \notin \overline{C_{\widehat{B_N}}}$, which means $w' \notin E$.

Let us show that $gw' \neq w'$ for a suitable $g \in G$. Recall (see e.g. Theorem 4 and 9 in [39]) that an automorphism $h \in Aut(X)$ is of exactly one of 3 possible types:

- elliptic, if h stabilizes some finite subset of V.
- parabolic, if h is non-elliptic and fixes exactly one end.
- hyperbolic, if h is non-elliptic and fixes exactly two ends.

Let $A' \neq C_{\widehat{B_N}}$ be a connected component of $X \setminus \widehat{B_N}$ with $w' \in \overline{A'}$. Let A be a connected component of $X \setminus \widehat{B_N}$ distinct from A' and $C_{\widehat{B_N}}$. By claim 1, we can find $g \in G$ such that $gB_N \subset A$. All connected components of $X \setminus \widehat{B_N}$ will be mapped into A by g, except one. This exceptional connected component is necessarily $C_{\widehat{B_N}}$ because $\mu(C_{\widehat{B_N}}) = 1$ and μ is G-invariant. In particular, $gA \subset A$, and this inclusion is strict. So $g^m A \subset A$, $\forall m \geq 1$. The sequence $g^m x_0$ possesses a subsequence $g^{m_k} x_0$ which converges to an end ξ in $\overline{A}$. It is obvious that g fixes ξ; therefore g is hyperbolic fixing exactly ξ and w. In particular, $gw' \neq w'$, as was to be shown.

Claim 4. The action of G (endowed with the discrete topology) on V is not proper.

The proof is inspired by a nice observation due to Karlsson and Noskov (Proposition 4 in [46]; see also Proposition 5 in [45]). As in claim 3, we can find $h \in G$ such that $h^m A' \subset A'$, $\forall m \geq 1$ so that h is hyperbolic and fixes exactly one end η in $\overline{A'}$, apart from w. With the same g as in Claim 3, let $y_n = h^n g h^{-n}$. We claim that $y_n \neq y_m$, $\forall n \neq m$. Suppose by contradiction that there is $n \neq m$ such that $h^n g h^{-n} = h^m g h^{-m}$; so there exists $k \neq 0$ such that $h^k g = g h^k$. Then $h^k g\eta = g h^k \eta = g\eta$ since h fixes η. Since h^k fixes the same ends as h, $g\eta$ has to be η or w. But this is not possible since η, ξ and w are all distinct.

Now, it remains for us to prove that the set $\{y_n x_0 : n \in \mathbb{N}\}$ is bounded. Indeed, for γ a hyperbolic automorphism, let $\ell(\gamma) =: \min\{d(\gamma^k v, v) : k \in \mathbb{Z}\backslash\{0\}, v \in V\}$ be the translation length of γ, and let $L_\gamma =: \{v \in V : d(\gamma v, v) = \ell(\gamma)\}$ be the axis of γ (this is a line in X). We will use one more result of Halin [39]: the end w, being a fixed end of some hyperbolic automorphism, is *thin*, i.e. for $N \gg 1$ the set C_{B_N} contains finitely many disjoint rays. As a consequence, the rays $L_h \cap C_{\widehat{B_N}}$ and $L_g \cap C_{\widehat{B_N}}$ stay within finite distance, i.e. there exists $R > 0$ such that, for every $x \in L_h \cap C_{\widehat{B_N}}$, one can find $x' \in L_g \cap C_{\widehat{B_N}}$ with $d(x, x') \leq R$.

To prove that $\{y_n x_0 : n \in \mathbb{N}\}$ is bounded, we may clearly assume that $x_0 \in L_h$. For n large enough, we have $h^{-n} x_0 \in C_{B_N}$, so we can find $x_n \in L_g$ with $d(h^{-n} x_0, x_n) \leq R$. Then,

$$\begin{aligned} d(y_n x_0, x_0) &= d(g h^{-n} x_0, h^{-n} x_0) \\ &\leq d(g h^{-n} x_0, g x_n) + d(g x_n, x_n) + d(x_n, h^{-n} x_0) \\ &\leq 2R + \ell(g); \end{aligned}$$

this concludes the proof.

Appendix B

Amenable actions of amalgamated free products

Abstract. We prove that the amalgamated free product of two free groups of rank two over a common cyclic subgroup, admits an amenable, faithful, transitive action on an infinite countable set. We also show that any finite index subgroup admits such an action, which applies for example to surface groups and fundamental groups of surface bundles over $\mathbb{S}^1$.

B.1 Introduction

An action of a group G on a set X is *amenable* if there exists a G-invariant mean on X, i.e. a map $\mu : 2^X = \mathcal{P}(X) \to [0,1]$ such that $\mu(X) = 1$, $\mu(A \cup B) = \mu(A) + \mu(B)$ for every pair of disjoint subsets A, B of X, and $\mu(gA) = \mu(A)$, $\forall g \in G$, $\forall A \subseteq X$.

The study of amenability goes back to von Neumann [82] and has spanned over the 20th century in various fields of mathematics, such as geometric group theory, harmonic analysis, graph theory, operator algebra, etc. F. P. Greenleaf asked in [35] whether the presence of a G-invariant mean on a set on which G acts faithfully implies that the group G is amenable (i.e. if the action on itself by left multiplication is amenable), and the first counter example was given in [81], where E. K. van Douwen constructed an interesting amenable action of the non-abelian free group.

The above definition is due to Greenleaf [35]. We should mention that Zimmer [90] has also introduced a notion of amenability for a group action that is different from ours; an action by homeomorphisms of a countable discrete group G on a compact Hausdorff space X is *(topologically) Zimmer amenable* if there exists a sequence of continuous maps $m^n : X \to Proba(G)$ such that $\lim_{n\to\infty} \sup_{x\in X} \|gm^n_x - m^n_{gx}\|_1 = 0$, for all $g \in G$ (cf. [65], [40], [8]). With this definition, a group is amenable if and only if the action on an one-point space is Zimmer amenable, while such an action is always Greenleaf amenable. On the other hand, the action of G on itself by left multiplication is always Zimmer amenable (by taking $m^n : G \to Proba(G)$ defined by $m^n_g = \delta_g$). More generally, the action of G on a homogenous space G/H is Zimmer amenable if and only if the subgroup H is amenable. From now on, we will use the term of an amenable action as mean of Greenleaf amenable action.

For the study of amenable actions of a group G, we should require some restrictions on the G-action in order to avoid trivial cases. One should assume that the action is faithful, otherwise one would take immediately a free group $\mathbb{F}_n$, $n \geq 2$, and any non-trivial normal subgroup $N \lhd \mathbb{F}_n$ such that the quotient group $\mathbb{F}_n/N$ is amenable (e.g. $N = \mathbb{F}'_n$ the commutator subgroup), so that the natural action of $\mathbb{F}_n$ on $\mathbb{F}_n/N$ is amenable but not faithful. In addition, one should require that G acts transitively, otherwise one could take any group G and $X = G \sqcup Y$ where G acts on Y amenably, so that the G-action on X is faithful and amenable (since there is a G-equivariant map from Y into X). In this direction, Y. Glasner and N. Monod [33] proposed to study the class $\mathcal{A}$ of all countable groups which admit a faithful, transitive and amenable action. The class $\mathcal{A}$ is closed under direct products and free products, and a group is in $\mathcal{A}$ if it has a co-amenable subgroup which is in $\mathcal{A}$ (Proposition 1.7 in [33]). On the other hand, in general the class is neither closed under passing to subgroups (the case of finite index subgroups is open), nor closed under semidirect products. As an example for semidirect product, one may take the group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$; while $SL_2(\mathbb{Z})$ is in $\mathcal{A}$ since it contains a free group of finite index, the pair $(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ has the relative property (T) (cf [9]), so that the group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ is not in $\mathcal{A}$ (Lemma 4.3 in [33]). Besides, this group is another example which shows that the class $\mathcal{A}$ is not closed under amalgamated free products; one may see the group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ as the amalgamated free product $G *_A H$ of $G = \mathbb{Z}/4\mathbb{Z} \ltimes \mathbb{Z}^2$ and $H = \mathbb{Z}/6\mathbb{Z} \ltimes \mathbb{Z}^2$ along $A = \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}^2$ and notice that the three groups G, H and A are in $\mathcal{A}$ since they are amenable.

In particular, Y. Glasner and N. Monod showed that the free product of any two countable groups is in $\mathcal{A}$ unless one factor has the fixed point property and the other has the virtual fixed point property[1]; for this, they used an argument of genericity in Baire's sense (Theorem 3.3 in [33]). Let us mention that another construction of amenable action of a non-abelian free group is obtained by R. Grigorchuk and V. Nekrashevych in [36].

The main result of this paper is, motivated by this method of genericity, to give another example of non-amenable group which is in $\mathcal{A}$ (see Theorem B.17 and Theorem B.19):

Theorem. The amalgams $\mathbb{F}_2 *_{\mathbb{Z}} \mathbb{F}_2$ belong to $\mathcal{A}$, where $\mathbb{Z}$ embeds in each factor as subgroup generated by some common word on the generating sets.

[1] A group G has the *fixed point property* if any amenable G-action has a fixed point, and G has the *virtual fixed point property* if it has a finite index subgroup having the fixed point property.

Such amalgams are known as doubles of $\mathbb{F}_2$. The key point of the proof is to fix a transitive permutation β and take a generic element α (i.e. an element in the intersection of countably many generic sets) in order to construct $\mathbb{F}_2 = \langle \alpha, \beta \rangle$ in a way that the amalgamated free product of two copies of $\mathbb{F}_2$ along a cyclic group has the desired properties. Therefore, the difficulty of the proof resides in the choice of the generic sets because they can be very "nasty" (see Proposition B.1).

As we mentioned before, in general it is not known whether the class $\mathcal{A}$ is closed under passing to finite index subgroups or not. But it is true for our case (see Theorem B.20):

Theorem. For any finite index subgroup H of $\mathbb{F}_2 *_{\mathbb{Z}} \mathbb{F}_2$ as above, H belongs to $\mathcal{A}$.

A surface group Γ_g is the fundamental group of a closed oriented surface of genus $g \geq 2$. The group Γ_2 can be viewed as an amalgamated free product of two copies of $\mathbb{F}_2$ along the subgroup generated by the commutator, i.e. $\Gamma_2 = \langle a_1, b_1 \rangle *_{\langle c \rangle} \langle a_2, b_2 \rangle$ where $c = [a_1, b_1] = [a_2, b_2]$. For $g \geq 3$, Γ_g injects into Γ_2 as a finite index subgroup. Therefore, by applying our results, we have the following theorem (see Theorem B.21):

Theorem. The surface groups Γ_g belong to $\mathcal{A}$, $\forall g \geq 2$.

As a corollary, we obtain that the fundamental group of a 3-manifold which virtually fibers over the circle is in $\mathcal{A}$. Indeed, let M be a 3-manifold which fibers over the circle. Then there is a short exact sequence:

$$0 \to \Gamma_g \to \pi_1(M) \to \mathbb{Z} \to 0,$$

so that the subgroup Γ_g is co-amenable in $\pi_1(M)$. Moreover, if M is a 3-manifold which virtually fibers over the circle, then it contains a finite index subgroup which is in $\mathcal{A}$, so that $\pi_1(M)$ is also in $\mathcal{A}$. Some examples of the fundamental group of such manifolds are given in [6], which includes the Bianchi groups $\mathrm{PSL}(2, \mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ with d a positive integer.

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B.2 Baire spaces

For the importance of the idea of generic choice, we briefly discuss Baire spaces in this chapter.

Definition B.2.1. A topological space X is a *Baire space* if every intersection of countably many dense open subsets is dense in X.

Equivalently, X is a Baire space if every union of countably many closed subsets with empty interior has empty interior.

Definition B.2.2. A *Polish space* is a separable completely metrizable topological space, i.e. it is a space homeomorphic to a complete space that has a countable dense subset.

Observe that any closed subspace of a Polish space is Polish.

Let X be an infinite countable set. Equipped with the discrete topology, X is a complete topological space. Let us denote by X^X the set of all self-maps of X and endow it with the topology of pointwise convergence (i.e. α_n converges to α if for all finite subset F of X, there exists n_0 such that $\alpha_n|_F = \alpha|_F$, for all $n \geq n_0$). This is the product of the topologies of X. Hence X^X is complete being a product of complete spaces, and it is separable and metrizable since it is a countable product of separable, metrizable spaces. So X^X is a Polish space and by Baire's theorem, it is a Baire space.

Let us denote by $Sym(X) \subset X^X$ the group of permutations of X. Equipped with the induced topology of X^X, $Sym(X)$ is a topological group. Indeed, let $\{\alpha_n\}_{n\geq 1}$ be a sequence converging to α in $Sym(X)$. Let $F \subset X$ be a finite subset of X. There exists n_0 such that $\alpha_n|_{F\cup\alpha^{-1}F} = \alpha|_{F\cup\alpha^{-1}F}$, $\forall n \geq n_0$. Then for all $x \in F$, we have $\alpha_n(\alpha^{-1}(x)) = \alpha(\alpha^{-1}(x)) = x$, so $\alpha_n^{-1}(x) = \alpha^{-1}(x)$, $\forall n \geq n_0$. Therefore α_n^{-1} converges to α^{-1}, so that the application $\alpha \mapsto \alpha^{-1}$ is continuous. Moreover, let $\{\beta_m\}_{m\geq 1}$ be a sequence converging to β in $Sym(X)$. Let $F \subset X$ be a finite subset of X. There exists n_1 such that $\alpha_n|_{F\cup\beta F} = \alpha|_{F\cup\beta F}$, $\forall n \geq n_1$. In addition, there exists n_2 such that $\beta_m|_F = \beta|_F$, for all $m \geq n_2$. Then for all $x \in F$, $\alpha_n(\beta_m(x)) = \alpha_n(\beta(x)) = \alpha\beta(x)$, for all $m \geq \max\{n_1, n_2\}$. Therefore $\alpha_n\beta_m$ converges to $\alpha\beta$, so that the application $(\alpha, \beta) \mapsto \alpha\beta$ is continuous.

Consequently, the injection $i : Sym(X) \to X^X \times X^X;\ \alpha \mapsto (\alpha, \alpha^{-1})$ is a homeomorphism onto its image which is closed. Thus $Sym(X)$ is a Polish space, in particular it is a Baire space.

Definition B.2.3. A subset $Y \subset Sym(X)$ is called

- *meagre* or *first category* if it is a union of countably many closed subsets with empty interior;

- *generic* or *dense* G_δ if its complement $Sym(X) \setminus Y$ is meagre, i.e. it is an intersection of countably many dense open subsets.

By definition of the topology on $Sym(X)$, a subset $Y \subset Sym(X)$ has empty interior if for all $\alpha' \in Y$ and for all finite subset $F \subset X$, there exists $\alpha \in Sym(X) \setminus Y$ such that $\alpha'|_F = \alpha|_F$.

B.3 Construction of $\mathbb{F}_2$

Let X be an infinite countable set. Let β be a simply transitive permutation of X. Let $c = c(\alpha, \beta)$ be a weakly cyclically reduced word (i.e. if $c = g_m \cdots g_1$, then $g_m \neq g_1^{-1}$) on the alphabet $\{\alpha^{\pm 1}, \beta^{\pm 1}\}$ such that $c \notin \langle \beta \rangle$.

Proposition B.1. *The set*

$$\mathcal{U}_1 = \{\alpha \in Sym(X) \quad | \quad \forall w \in \langle \alpha, \beta \rangle \setminus \langle c \rangle,\ \text{there exist infinitely many } x \in X \text{ such that } cx = x,\ cwx = wx \text{ and } wx \neq x\ \}$$

is generic in $Sym(X)$.

Proposition B.2. *The set*

$$\mathcal{U}_2 = \{\alpha \in Sym(X) \mid \forall k \in \mathbb{Z} \setminus \{0\}, \exists x \in X \text{ such that } c^k x \neq x\}$$

is generic in $Sym(X)$.

Note that $\mathcal{U}_2$ is the set of α's such that c has infinite order.

Definition B.3.1. Let $c = c(\alpha, \beta)$ be a weakly cyclically reduced word. Let $S(\alpha)$ be the sum of exponents of α, and $S(\beta)$ be the sum of exponents of β. We say that c is *special* if c is one of the following types:

(1) $S(\alpha) = S(\beta) = 0$;

(2) $S(\alpha)$ divides $S(\beta)$.

Let $\{A_n\}_{n=1}^{\infty}$ be a pairwise disjoint Følner sequence for β, that is

$$\lim_{n \to \infty} \frac{|A_n \vartriangle \beta \cdot A_n|}{|A_n|} = 0.$$

Proposition B.3. *Let c be a special word. The set*

$$\mathcal{U}_3 = \{\alpha \in Sym(X) \quad | \quad \text{there exists } \{A_{n_k}\}_{k=1}^{\infty} \text{ a subsequence of } \{A_n\}_{n=1}^{\infty} \text{ such that } A_{n_k} \subset \mathrm{Fix}(c),\ \forall k \geq 1 \text{ and } \{A_{n_k}\}_{k=1}^{\infty} \text{ is a Følner sequence for } \alpha\}$$

is generic in $Sym(X)$.

Proposition B.4. *The set*

$$\mathcal{U}_4 = \{\alpha \in Sym(X) \quad | \quad \text{for every finite index subgroup } H \text{ of } \langle \alpha, \beta \rangle, \text{ the action of } H \text{ on } X \text{ is transitive}\ \}$$

is generic in $Sym(X)$.

From the previous four propositions, one deduces immediately:

Corollary B.5. *Let c be a special word on $\{\alpha^{\pm 1}, \beta^{\pm 1}\}$. Let $\alpha \in \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3 \cap \mathcal{U}_4$. Then $\langle \alpha, \beta \rangle \simeq \mathbb{F}_2$ and*

(1) the action of $\mathbb{F}_2$ on X is transitive and faithful;

(2) for all $w \in \langle\alpha,\ \beta\rangle \setminus \langle c\rangle$, there exist infinitely many $x \in X$ such that $cx = x$, $cwx = wx$ and $wx \neq x$. In particular, there are infinitely many fixed points of c in X;

(3) there exists a pairwise disjoint Følner sequence for $\langle\alpha,\ \beta\rangle$ which is fixed by c;

(4) for all finite index subgroup H of $\langle\alpha, \beta\rangle$, the H-action on X is transitive.

B.3.1 Proofs of Propositions B.1 and B.2

Propositions B.1 and B.2 are sufficient conditions for faithfulness of $\mathbb{F}_2$-action with some additional "unnatural looking" properties that will be needed for construction of $\mathbb{F}_2 *_{\mathbb{Z}} \mathbb{F}_2$ in Chapter 4. As we resort to the graph theory for these proofs, we begin by fixing the notations on graphs that will be used in the section. The fundamental notions are based on [72].

Graph extension

A graph G consists of the set of vertices $V(G)$ and the set of edges $E(G)$, and two applications $E(G) \to E(G)$; $e \mapsto \bar{e}$ such that $\bar{\bar{e}} = e$ and $\bar{e} \neq e$, and $E(G) \to V(G) \times V(G)$; $e \mapsto (i(e), t(e))$ such that $i(e) = t(\bar{e})$. An element $e \in E(G)$ is a *directed edge* of G and $\bar{e}$ is the *inverse edge* of e. For all $e \in E(G)$, $i(e)$ is the *initial vertex* of e and $t(e)$ is the *terminal vertex* of e.

Let S be a set. A *labeling* of a graph $G = (V(G), E(G))$ on the set $S^{\pm 1} = S \cup S^{-1}$ is an application

$$l : E(G) \to S^{\pm 1}; e \mapsto l(e)$$

such that $l(\bar{e}) = l(e)^{-1}$. A *labeled graph* $G = (V(G), E(G), S, l)$ is a graph with a labeling l on the set $S^{\pm 1}$. A labeled graph is *well-labeled* if for any edges $e,\ e' \in E(G)$, $\big[i(e) = i(e')$ and $l(e) = l(e')\big]$ implies that $e = e'$. If a group $\Gamma = \langle S\rangle$ acts on X, a labeled graph with set of vertices X and set of edges $S^{\pm 1}$ is well-labeled if and only if it is a Schreier graph.

A word $w = w_m \cdots w_1$ on $\{\alpha^{\pm 1}, \beta^{\pm 1}\}$ is called *reduced* if $w_{k+1} \neq w_k^{-1}$, $\forall 1 \leq k \leq m-1$. A word $w = w_m \cdots w_1$ on $\{\alpha^{\pm 1}, \beta^{\pm 1}\}$ is called *weakly cyclically reduced* if w is reduced and $w_m \neq w_1^{-1}$; this definition allows that w_m and w_1 to be equal. We denote by $|w|$ the word length of w. Given a reduced word, we shall define two finite graphs labeled on $\{\alpha^{\pm 1}, \beta^{\pm 1}\}$ as follows:

Definition B.3.2. Let $w = w_m \cdots w_1$ be a reduced word on $\{\alpha^{\pm 1}, \beta^{\pm 1}\}$. The *path* of w is a finite labeled graph $P(w, v_0)$ consisting of $|w| + 1$ vertices and $|w|$ directed edges $\{e_1, \ldots, e_m\}$ such that

- $i(e_{j+1}) = t(e_j)$, $\forall 1 \leq j \leq m-1$;

- $v_0 = i(e_1) \neq t(e_m)$;
- $l(e_j) = w_j$, $\forall 1 \leq j \leq m$.

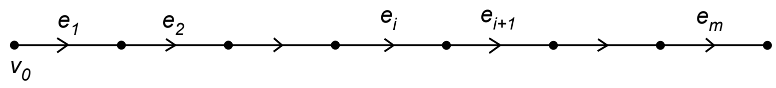

Figure B.1: The path of w

Definition B.3.3. Let $w = w_m \cdots w_1$ be a reduced word on $\{\alpha^{\pm 1}, \beta^{\pm 1}\}$. The *cycle* of w is a finite labeled graph $C(w, v_0)$ consisting of $|w|$ vertices and $|w|$ directed edges $\{e_1, \ldots, e_m\}$ such that

- $i(e_{j+1}) = t(e_j)$, $\forall 1 \leq j \leq m-1$;
- $v_0 = i(e_1) = t(e_m)$;
- $l(e_j) = w_j$, $\forall 1 \leq j \leq m$.

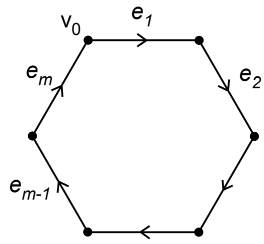

Figure B.2: The cycle of w

Notice that since w is a reduced word, the graph $P(w, v_0)$ is well-labeled. If w is weakly cyclically reduced, then $C(w, v_0)$ is also well-labeled.

Reciprocally, if $P = \{e_1, e_2, \ldots, e_n\}$ is a well-labeled path with $i(e_1) = v_0$, labeled by $l(e_i) = g_i$, $\forall i$, then there exists a unique reduced word $w = g_n \cdots g_1$ such that $P(w, v_0)$ is P. If $C = \{e_1, e_2, \ldots, e_n\}$ is a well-labeled cycle with $t(e_n) = i(e_1) = v_0$, labeled by $l(e_i) = g_i$, $\forall i$, then there exists a unique weakly cyclically reduced word $w = g_n \cdots g_1$ such that $C(w, v_0)$ is C.

Let X be an infinite countable set. Let β be a simply transitive permutation of X. We shall represent the β-action on X as an infinite 2-regular well-labeled graph. The *pre-graph* G_0 is a labeled graph consisting of the set of vertices $V(G_0) = X$ and the set of edges $E(G_0)$ where for all $e \in E(G_0)$, $l(e) \in \{\beta^{\pm 1}\}$ and such that every vertex has exactly one entering edge and one leaving edge. One can imagine G_0 as the Cayley graph of $\mathbb{Z}$ with 1 as a generator.

Definition B.3.4. An *extension* of G_0 is a well-labeled graph G labeled by $\{\alpha^{\pm 1}, \beta^{\pm 1}\}$, containing G_0. We will denote it by $G_0 \subset G$.

In order to have a transitive action with some additional properties of the $\langle\alpha,\beta\rangle$-action on X, we shall extend G_0 by adding finitely many directed edges labeled by α on G_0 where the edges labeled by β are already prescribed. In order that the added edges represent an action on X, we put the edges in such a way that the extended graph is well-labeled, and moreover we put an additional edge labeled by α on every endpoint of the extended edges by α; more precisely, if we have added n edges labeled by α between x_0, x_1, …, x_n successively, we put an α-edge from x_n to x_0 to have a cycle consisting of $n+1$ edges (see Figure B.3). On the points where no α-edges are involved, we put a loop labeled by α; this means that these points are the fixed points of α. In the end, every point has a entering edge and a leaving edge labeled by α (the entering edge is equal to the leaving edge if the edge is a loop), so that the graph represents an $\langle\alpha,\beta\rangle$-action on X, and every α-orbit is finite.

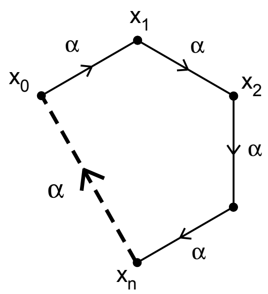

Figure B.3: The α-orbit of x_0 that has the size $n+1$.

Definition B.3.5. Let G, G' be graphs labeled by $\{\alpha^{\pm1}, \beta^{\pm1}\}$. A *homomorphism* $f : G \to G'$ is a map sending vertices to vertices, edges to edges, such that

- $f(i(e)) = i(f(e))$ and $f(t(e)) = t(f(e))$;
- $l(e) = l(f(e))$,

for all $e \in E(G)$.

If there exists an injective homomorphism $f : G \to G'$, we say that f is an *embedding*, and G *embeds* in G'. If there exists a bijective homomorphism $f : G \to G'$, we say that f is an *isomorphism*, and G is *isomorphic* to G'.

Proposition B.6. *Let $w = w_m \cdots w_1$ be a reduced word on $\{\alpha^{\pm1}, \beta^{\pm1}\}$. Let $P(w,v_0) = \{e_1,\ldots,e_m\}$ be the path defined in Definition B.3.2. There exists an extension G of G_0 such that $P(w,v_0)$ embeds in G, and $P(w,v_0)$ is isomorphic to its image by the corresponding embedding. In particular, the image of $P(w,v_0)$ is a path in G.*

Proof. It is enough to consider the case where $w = \alpha^{a_{2n}}\beta^{b_{2n-1}}\cdots\alpha^{a_4}\beta^{b_3}\alpha^{a_2}\beta^{b_1}$, with $m = \sum_{i=1}^{n}\left(|b_{2i-1}| + |a_{2i}|\right)$. Indeed, the other three cases follow from this case by taking n large enough since we are treating all subwords of w. Let $N = \max_j |b_j|$. For $z \in X$, denote by $B_N(z) = \{\beta^l z \mid -N \leq l \leq N\}$ a segment in the β-orbit of z.

Choose $z_0 \in X$. For all $1 \leq k \leq n$, we extend G_0 inductively by applying the following algorithm:

Algorithm (A)

(1) Let $z_{2k-1} = \beta^{b_{2k-1}} z_{2k-2}$;

(2) Choose $z_{2k} \in X$ such that $B_N(z_{2k})$ is outside of the finite set of all used points;

(3) Choose $|a_{2k}| - 1$ points $\{p_1^{(a_{2k})}, \ldots, p_{|a_{2k}|-1}^{(a_{2k})}\}$ outside of the finite set of all points used so far;

(4) Put the directed edges labeled by $\alpha^{sign(a_{2k})}$ from

- z_{2k-1} to $p_1^{(a_{2k})}$;
- $p_j^{(a_{2k})}$ to $p_{j+1}^{(a_{2k})}$, $\forall 1 \leq j \leq |a_{2k}| - 2$;
- $p_{|a_{2k}|-1}^{(a_{2k})}$ to z_{2k},

so that we have $\alpha^{a_{2k}} z_{2k-1} = z_{2k}$.

In the end, we have added $\sum_{i=1}^{n} |a_{2i}|$ new directed edges labeled by α (or α^{-1}) on G_0 (see Figure B.4). Let G be the extended graph of G_0. In this construction, we have considered $|w|+1$ points $\{z_0,\, \beta^{sign(b_1)} z_0,\, \beta^{2sign(b_1)} z_0,\, \ldots,\, \beta^{b_1} z_0 = z_1,$ $\alpha^{sign(a_2)} \beta^{b_1} z_0,\, \ldots,\, \alpha^{a_2}\beta^{b_1} z_0 = z_2,\, \ldots,\, wz_0\}$ in X, that are

$$\{z_0,\, w_1 z_0,\, w_2 w_1 z_0,\, \ldots,\, w z_0\}$$

with $l\big((w_{k-1}\cdots w_1 z_0), (w_k w_{k-1} \cdots w_1 z_0)\big) = w_k$, where (p_1, p_2) symbolizes the edge e with $i(e) = p_1$ and $t(e) = p_2$.

Now, we define an embedding $f : P(w, v_0) \hookrightarrow G$ by

$$\begin{array}{rcll} E(P(w,v_0)) & \to & E(G) & \\ e_1 = (v_0, t(e_1)) & \mapsto & (z_0, w_1 z_0), & \\ e_k = (i(e_k), t(e_k)) & \mapsto & (w_{k-1}\cdots w_1 z_0,\, w_k \cdots w_1 z_0), & \forall 2 \leq k \leq m. \end{array}$$

By construction, $P(w, v_0)$ is isomorphic to its image. $\square$

Proposition B.7. *Let $w = w_m \cdots w_1$ be a weakly cyclically reduced word on $\{\alpha^{\pm 1}, \beta^{\pm 1}\}$ with $w \notin \langle \beta \rangle$. Let $C(w, v_0) = \{e_1, \ldots, e_m\}$ be the cycle defined in Definition B.3.3. There exists an extension G of G_0 such that $C(w, v_0)$ embeds in G, and $C(w, v_0)$ is isomorphic to its image by the corresponding embedding. In particular, the image of $C(w, v_0)$ is a cycle in G.*

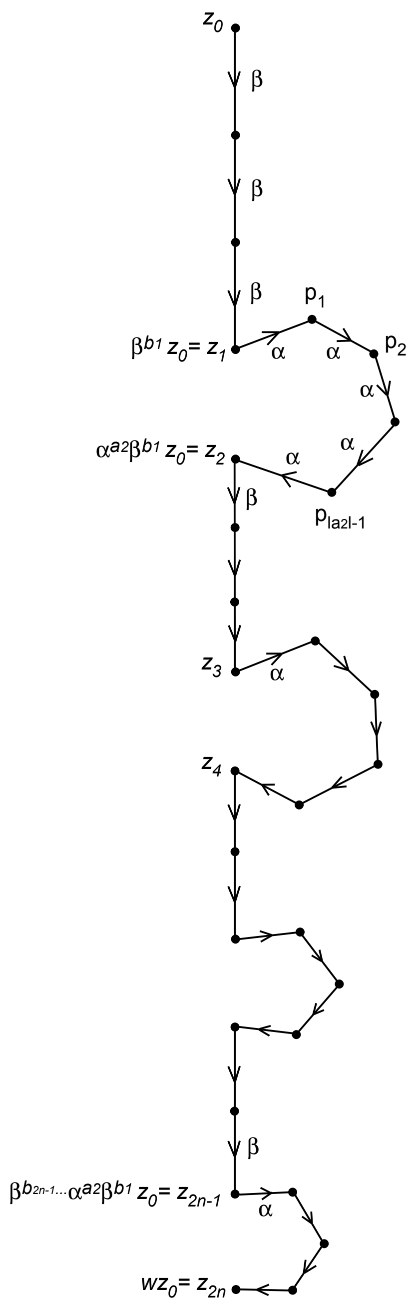

Figure B.4: Construction of a path in G

Proof. It is enough to consider the case where $w = \alpha^{a_{2n}}\beta^{b_{2n-1}}\cdots\alpha^{a_4}\beta^{b_3}\alpha^{a_2}\beta^{b_1}$, with $m = \sum_{i=1}^{n}\left(|b_{2i-1}| + |a_{2i}|\right)$. Let $N = \max_j |b_j|$.

Choose $z_0 \in X$. We extend G_0 inductively by applying Algorithm (A) for $1 \le k \le n-1$. Let $z_{2n-1} = \beta^{b_{2n-1}} z_{2n-2}$. Choose $|a_{2n}| - 1$ points $\{p_1, \ldots, p_{|a_{2n}|-1}\}$ outside of the finite set of all points used so far. Put the directed edges labeled by $\alpha^{sign(a_{2n})}$ from

- z_{2n-1} to p_1;
- p_j to p_{j+1}, $\forall 1 \le j \le |a_{2n}| - 2$;
- $p_{|a_{2n}|-1}$ to z_0.

We define an embedding $f : C(w, v_0) \hookrightarrow G$ by

$$\begin{aligned} E(C(w,v_0)) &\rightarrow E(G) \\ e_1 = (v_0, t(e_1)) &\mapsto (z_0,\, w_1 z_0), \\ e_k = (i(e_k), t(e_k)) &\mapsto (w_{k-1}\cdots w_1 z_0,\, w_k \cdots w_1 z_0), \quad \forall 2 \le k \le m-1, \\ e_m = (i(e_m), v_0) &\mapsto (w_{m-1}\cdots w_1 z_0,\, z_0). \end{aligned}$$

By construction, $C(w, v_0)$ is isomorphic to its image.

□

Corollary B.8. *Let w be a reduced word. Let $F \subset G_0$ be a finite subset of X. There exists an extension G of G_0 such that $P = P(w, v_0)$ embeds in G, the image $\bar{P}$ of P is isomorphic to P, and the intersection of $\bar{P}$ and F is empty. In addition, we can replace $P(w, v_0)$ by $C(w, v_0)$ if w is weakly cyclically reduced and $w \notin \langle\beta\rangle$.*

Proof. The construction of the extension consists of choosing some finite points in X. Therefore, it is enough to choose all considering points far enough outside of F. □

Property (FF)

Let $c = c_m \cdots c_1$ be a weakly cyclically reduced word, such that $c \notin \langle\beta\rangle$. Let $w = w_k \cdots w_1$ be a reduced word, such that $w \notin \langle c\rangle$. Let $C(c, v_0)$ be the cycle defined in Definition B.3.3. Let $P(w, v_0)$ be the path defined in Definition B.3.2 such that every vertex of $P(w, v_0)$ (other than v_0) is distinct from every vertex in $C(c, v_0)$. Let wv_0 be the endpoint of $P(w, v_0)$. Let $C(c, wv_0)$ be the cycle with $i(c_1) = t(c_m) = wv_0$, such that every vertex of $C(c, wv_0)$ (other than wv_0) is distinct from every vertex in $P(w, v_0) \cup C(c, v_0)$ (see Figure B.5). Let us denote by Q_0 the union of $C(c, v_0)$, $P(w, v_0)$ and $C(c, wv_0)$. In general, this finite labeled graph Q_0 is not well-labeled. However, by identifying the successive edges with the same initial vertex and the same label, Q_0 becomes a well-labeled graph Q (See Figure B.6 for an example of the process).

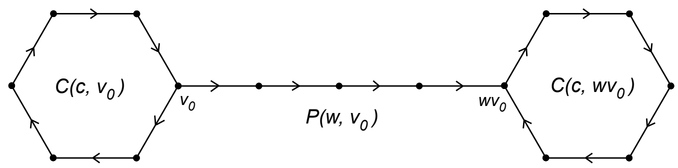

Figure B.5: The graph $Q_0 = C(c, v_0) \cup P(w, v_0) \cup C(c, wv_0)$

In the end of the process of identification of "double edges", Q has fewer edges than Q_0; however, the cycle $C(c, v_0)$ and $C(c, wv_0)$ are not modified, in the sense that the "shapes" of $C(c, v_0)$ and $C(c, wv_0)$ in Q_0 are the same as in Q. In other word, the quotient map $Q_0 \twoheadrightarrow Q$ restricted to $C(c, v_0)$ and to $C(c, wv_0)$ is injective (each one separately).

By construction, in each process, the graph has the following property:

Property (FF)

(1) the starting point of $C(c, v_0)$ is equal to its endpoint which is v_0;

(2) the starting point of $P(w, v_0)$ is different from its endpoint;

(3) the starting point of $C(c, wv_0)$ is equal to its endpoint which is wv_0.

The acronym (FF) stands for "Faithfulness for w and fixed points of c". Notice that (2) comes from the fact that $w \notin \langle c \rangle$. When this process is finished, Q will be one of the following four types (Figure B.7) of well-labeled graph satisfying the property (FF):

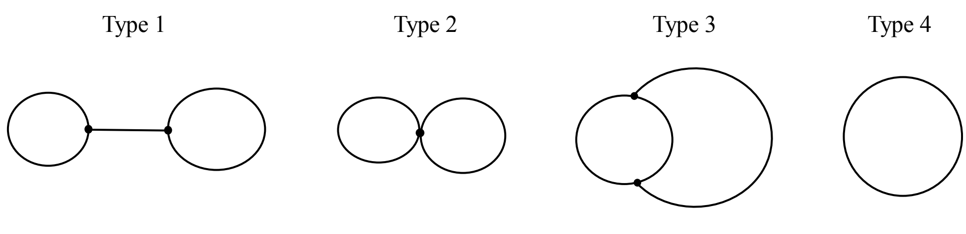

Figure B.7: Four types of Q

Proposition B.9. *For every one of the four types of well-labeled graph $Q = Q(c, w, v_0)$, there exists an extension G of G_0 such that Q embeds in G and the image $Q(c, w, z_0)$ of Q by the embedding has the property (FF), i.e. there exists α such that the word w satisfies*

$$\begin{cases} cz_0 = z_0; \\ wz_0 \neq z_0; \\ cwz_0 = wz_0. \end{cases}$$

where z_0 is the image of v_0 in G.

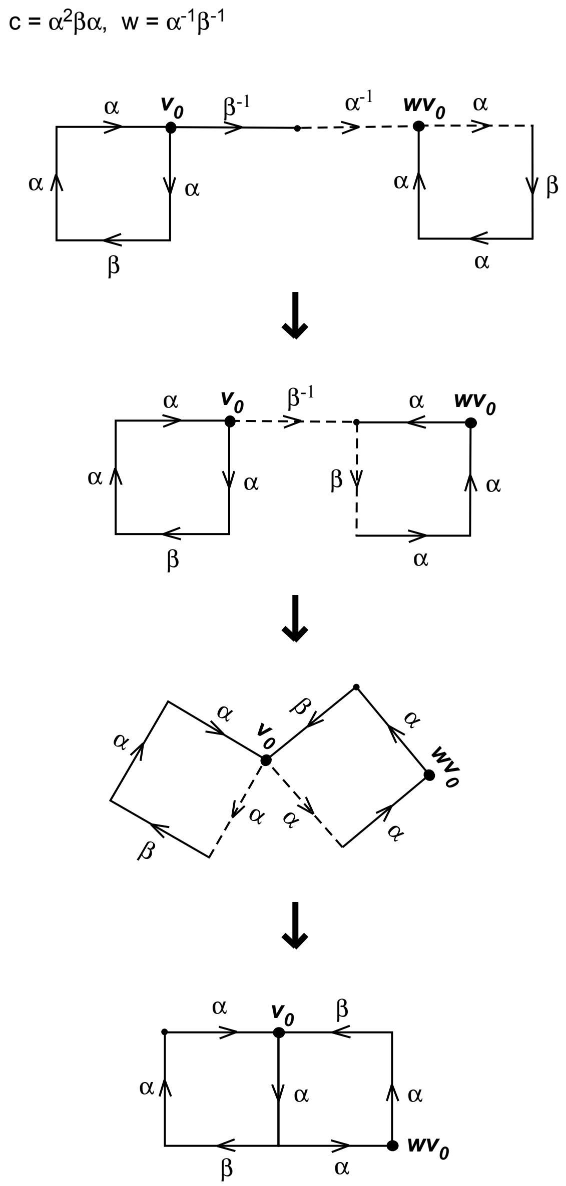

Figure B.6: Example of gluing double edges

We have to prove that every cycle in each types in Figure B.7 contains at least one directed edge labeled by α or α^{-1}. This is clear for the type 1, 2 and 4 since they have at most 2 cycles that represent $C(c, v_0)$ and $C(c, wv_0)$, and $c \notin \langle\beta\rangle$ by assumption. For the type 3, we can read around two subgraphs representing $C(c, v_0)$ and $P(w, v_0)^{-1}C(c, wv_0)P(w, v_0)$ from the vertex v_0. The labeling of the graph $P(w, v_0)^{-1}C(c, wv_0)P(w, v_0)$ is $w^{-1}cw$.

Let us recall the well-known theorem concerning the test for conjugacy of two words (see Theorem 1.3 in [83]):

Theorem B.10. *Two words in the free group $\mathbb{F}_n$ define conjugate elements of $\mathbb{F}_n$ if and only if their cyclic reductions in $\mathbb{F}_n$ are cyclic permutations of one another.*

Lemma B.11. *Let c be a weakly cyclically reduced word, such that $c \notin \langle\beta\rangle$. Let w be a reduced word, such that $w \notin \langle c\rangle$. If c has the form $\gamma\beta^l$ with $\gamma \notin \langle\beta\rangle$, then $w^{-1}cw$ cannot be reduced to neither the form $\gamma\beta^{-k}$, nor the form $\gamma^{-1}\beta^k$ with $sign(k) = sign(l)$, $\forall k \in \mathbb{Z}$.*

Proof. Let $\gamma\beta^l$ with $\gamma = \gamma_n \cdots \gamma_1 \notin \langle\beta\rangle$. By contradiction, let us suppose that $\gamma_n \cdots \gamma_1\beta^l$ is conjugate to $\gamma_n \cdots \gamma_1\beta^{-k}$ with $k, l > 0$. Without loss of generality, we can suppose that $\gamma_1, \gamma_n \notin \{\beta^{\pm1}\}$. There are four types of cyclic permutations of $\gamma_n \cdots \gamma_1\beta^l$, which are $\gamma_n \cdots \gamma_1\beta^l$; $\beta^l\gamma_n \cdots \gamma_1$; $\beta^{l_1}\gamma_n \cdots \gamma_1\beta^{l_2}$ with $l_1 + l_2 = l$; and $\gamma_p \cdots \gamma_1\beta^l\gamma_n \cdots \gamma_{p+1}$ for a certain $1 \leq p \leq n$. Obviously, $\gamma_n \cdots \gamma_1\beta^{-k}$ cannot be of the first three types; so let us suppose that there exists $1 \leq p \leq n$ such that $\gamma_p \cdots \gamma_1\beta^l\gamma_n \cdots \gamma_{p+1} = \gamma_n \cdots \gamma_1\beta^{-l}$ (since the two conjugate elements have the same length). By identification of the l^{th} letter on the right of the two words, we have $\beta^{-1} = \gamma_{p+l} = \gamma_j$, for every j multiple of $p + l$ modulo $n + l$, so in particular $\beta^{-1} = \gamma_{n-p}$. However, by identifying the $(n - p + l)^{\text{th}}$ letter, which is β for the left side, and γ_{n-p} for the right side, we have $\beta = \gamma_{n-p}$ which contradicts with the first identification. The second case can be treated similarly. □

Proof of Proposition B.9. As we mentioned before, it remains us to consider the type 3.

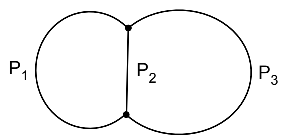

Figure B.8: Type 3 of Q

In this graph, there are three cycles $C = P_1 \cup P_2$, $P_2 \cup P_3$ and $P_1 \cup P_3$.

· **Claim.** If one of the three paths P_1, P_2 and P_3 has only edges labeled by $\beta^{\pm1}$, then the other two paths both contains edges labeled by $\alpha^{\pm1}$.

The claim allows to conclude. In fact, without loss of generality, suppose that P_1 has only edges labeled by $\beta^{\pm1}$ and $P_2 \notin \langle\beta\rangle$ and $P_3 \notin \langle\beta\rangle$. We first take an extension $G_1 \supset G_0$ such that the image of P_1 is a path in G_1. Then we take an extension $G_2 \supset G_1$ such that P_2 is a path in G_2 which connects the starting point and the endpoint of P_1 outside of the finite subset P_1; that is possible since the graph is well-labeled and P_2 contains edges labeled by α. Finally, we take an extension $G_3 \supset G_2$ such that P_3 is a path in G_3 joining these two points outside of $P_1 \cup P_2$.

We now prove the claim. Indeed, if two of these three paths were labeled by $\beta^{\pm 1}$, then c would be the form of $\gamma\beta^l$ up to cyclic permutation and $w^{-1}cw$ would be the form of $\gamma\beta^{-k}$ or $\gamma^{-1}\beta^k$ with $sign(l) = sign(k)$ up to cyclic permutation, which contradicts with Lemma B.11.

□

Corollary B.12. *Let $Q = Q(c, w, v_0)$ be a well-labeled graph. Let $F \subset G_0$ be a finite subset of X. There exists an extension G of G_0 such that the image $Q(c, w, z_0)$ of $Q(c, w, v_0)$ in G preserve the property (FF), and the intersection of $Q(c, w, z_0)$ and F is empty.*

□

Proof of Proposition B.1

Let $c = \alpha^{a_1}\beta^{b_1}\cdots\alpha^{a_n}\beta^{b_n}$ be a weakly cyclically reduced word on $\{\alpha^{\pm 1}, \beta^{\pm 1}\}$ (the other three types are similar). Let $w \in \langle \alpha, \beta\rangle \setminus \langle c\rangle$ be a reduced word on $\{\alpha^{\pm 1}, \beta^{\pm 1}\}$. We shall prove that the set

$$\mathcal{V}_w = \{\alpha \in Sym(X) \quad | \quad \text{there exists a finite number of } x \in X \text{ such that } cx = x,\ cwx = wx, \text{ and } wx \neq x\ \}$$

is meagre. For $K \subset X$ a finite subset of X, let

$$V_{w,K} = \{\alpha \in Sym(X) \mid \big(\mathrm{Fix}(c) \cap w^{-1}\mathrm{Fix}(c) \cap \mathrm{supp}(w)\big) \subseteq K\},$$

where $\mathrm{supp}(w) = \{x \in X \mid wx \neq x\}$.

The set $V_{w,K}$ is closed since if α_n converges to α, then $c(\alpha_n, \beta)$ converges to $c(\alpha, \beta)$ and $w(\alpha_n, \beta)$ converges to $w(\alpha, \beta)$. We shall prove that the interior of $V_{w,K}$ is empty.

Lemma B.13. *Let $\alpha' \in Sym(X)$ and $F \subset X$ be a finite subset of X. There exists $\alpha \in Sym(X)$ such that $\alpha|_F = \alpha'|_F$ and $\mathrm{supp}(\alpha) \subset F \cup \alpha'(F)$.*

Proof. Let us partition F into finitely many pieces $F = \sqcup_{i=1}^m P_i$ according to the orbits of α'. If $\alpha'(P_i) = P_i$, then define $\alpha|_{P_i} = \alpha'|_{P_i}$. If not, write $P_i = \{p_i,\ \alpha'(p_i),\ \ldots,\ \alpha'^{k_i}(p_i)\}$ with $\alpha'^{k_i+1}(p_i) \notin F$. Then define $\alpha|_{P_i} = \alpha'|_{P_i}$ and $\alpha(\alpha'^{k_i+1}(p_i)) = p_i$. □

We see X as the pre-graph G_0, where the $\beta^{\pm 1}$-edges of G_0 are seen as the transitive action of $\beta^{\pm 1}$ on X, which is fixed from the beginning.

Let $\alpha' \in V_{w,K}$ and let $F \subset X$ be a finite subset of X. Let $Y = F \cup \alpha'(F) \cup K$ be a finite subset of X. We construct a well-labeled graph $Q(c, w, v_0)$ as in Section B.3.1. We choose $z_0 \notin Y$ and take α which is defined on F as in Lemma B.13, and which satisfies the property (FF) without touching any point of Y (Corollary B.12). Consequently, $\alpha \notin V_{w,K}$ and $\alpha|_F = \alpha'|_F$.

Proof of Proposition B.2

We want to prove that for all $k \in \mathbb{Z} \setminus \{0\}$, the set

$$\mathcal{V}_k = \{\alpha \in Sym(X) \mid c^k = \mathrm{Id}\}$$

is closed and of empty interior.

Indeed, it is clearly closed. Moreover, let $\alpha' \in \mathcal{V}_k$ and let $F \subset X$ be a finite subset of X. Let $P(c^k, v_0)$ be the path defined in Definition B.3.2. We choose $z_0 \notin F \cup \alpha'(F) =: Y$ and take α which is defined on F as in Lemma B.13, and such that $P(c^k, z_0)$ is a path in X not touching any point of Y. By consequent, $\alpha \notin \mathcal{V}_k$ and $\alpha|_F = \alpha'|_F$.

B.3.2 Proof of Proposition B.3

Let c be a special word. Let $\{A_n\}_{n\geq 1}$ be a pairwise disjoint Følner sequence for β. Let $\{\varepsilon_l\}_{l\geq 1} > 0$ be a sequence tending to 0. Let us write

$$\mathcal{U}_3 = \bigcap_l \bigcap_{N\in\mathbb{N}} \{\alpha \in Sym(X) \mid \text{there exists } k \geq N \text{ such that } A_k \subset \mathrm{Fix}(c) \text{ and } |A_k \vartriangle \alpha A_k| < \varepsilon_l |A_k| \ \}.$$

Set $\varepsilon_l = \varepsilon$. We want to prove that the set

$$\mathcal{V}_N := \{\alpha \in Sym(X) \mid \forall k \geq N,\ A_k \not\subseteq \mathrm{Fix}(c) \text{ or } |A_k \vartriangle \alpha A_k| \geq \varepsilon |A_k| \ \}$$

is closed and of empty interior. We treat the case $c = \alpha^{a_1}\beta^{b_1} \cdots \alpha^{a_n}\beta^{b_n}$ (the other three types are similar). Let $M = \max_j |b_j|$ and set

$$E_k := \cup_{i=-M}^{M} \beta^i(A_k),$$

a finite set of X.

· *$\mathcal{V}_N$ is closed.* Since $\mathcal{V}_N = \cap_{k\geq N} \mathcal{V}_{N,k}$ where

$$\mathcal{V}_{N,k} := \{\alpha \in Sym(X) \mid \ A_k \not\subseteq \mathrm{Fix}(c) \text{ or } |A_k \vartriangle \alpha A_k| \geq \varepsilon |A_k| \ \},$$

it is enough to prove that $\mathcal{V}_{N,k}$ is closed. So let $\{\alpha_n\}_{n\geq 1}$ be a sequence in $\mathcal{V}_{N,k}$ which converges to $\alpha \in Sym(X)$. Since E_k is finite, there exists n_0 such that $\alpha|_{E_k} = \alpha_n|_{E_k}$, $\forall n \geq n_0$. Therefore, $\alpha \in \mathcal{V}_{N,k}$ because $A_k \subset E_k$.

· *$\mathcal{V}_N$ is of empty interior.* Let us distinguish two cases:

First, suppose that $S(\alpha) = S(\beta) = 0$. Let $\alpha' \in \mathcal{V}_N$. Let $F \subset X$ be a finite subset of X. We choose $m \gg N$ such that $(F \cup \alpha'(F)) \cap E_m = \emptyset$. We define $\alpha|_{E_m}$ =Id and $\alpha|_F = \alpha'|_F$. Then $A_m \subset \mathrm{Fix}(c)$ since $S(\beta) = 0$, and $|A_m \vartriangle \alpha A_m| = 0$ since $\alpha(A_m) = A_m$. So $\alpha \notin \mathcal{V}_N$.

Second, suppose that $S(\alpha)$ divides $S(\beta)$. Let $\alpha' \in \mathcal{V}_N$. Let $F \subset X$ be a finite subset of X. We choose $m \gg N$ such that $(F \cup \alpha'(F)) \cap E_m = \emptyset$ and $|A_m \vartriangle \beta^{-\frac{S(\beta)}{S(\alpha)}}(A_m)| < \varepsilon |A_m|$; this is possible as $\{A_m\}$ is a Følner sequence for β. We define

$$\alpha(x) = \beta^{-\frac{S(\beta)}{S(\alpha)}}(x), \quad \forall x \in E_m,$$

and $\alpha|_F = \alpha'|_F$. Then,

$$c(x) = \beta^{-\frac{S(\beta)}{S(\alpha)}a_1}\beta^{b_1}\cdots\beta^{-\frac{S(\beta)}{S(\alpha)}a_n}\beta^{b_n}(x) = \beta^{-\frac{S(\beta)}{S(\alpha)}S(\alpha)}\beta^{S(\beta)}(x) = x,$$

for all $x \in E_m$. In particular, $A_m \subset \mathrm{Fix}(c)$. In addition,

$$|A_m \vartriangle \alpha A_m| = |A_m \vartriangle \beta^{-\frac{S(\beta)}{S(\alpha)}}(A_m)| < \varepsilon |A_m|,$$

so $\alpha \notin \mathcal{V}_N$.

B.3.3 Proof of Proposition B.4

The proof follows from the three claims:

· **Claim 1.** Let G be a group and $H < G$ be a finite index subgroup of G. Then, for all $g \in G$, there exists $n \geq 1$ such that $g^n \in H$.

Indeed, let N be the core of H, that is $N = \bigcap_{x \in G} x^{-1}Hx \subset H$. The subgroup N is a finite index normal subgroup of G. Then for all $g \in G$, $g^m \in N$, where $m = [G : N]$.

· **Claim 2.** The set

$\mathcal{U}_5 = \{\alpha \in Sym(X) \mid \forall n, m \in \mathbb{Z} \setminus \{0\}$, the $\langle \alpha^n, \beta^m \rangle$-action on X is transitive $\}$

is in $\mathcal{U}_4$.

Indeed, let $\alpha \in \mathcal{U}_5$. Let $H < \langle \alpha, \beta \rangle$ be a finite index subgroup. Then by Claim 1, there exist n_0, m_0 such that α^{n_0} and β^{m_0} are in H, so $\langle \alpha^{n_0}, \beta^{m_0} \rangle < H$. Since the $\langle \alpha^{n_0}, \beta^{m_0} \rangle$-action on X is transitive by hypothesis, the H-action on X is also transitive.

· **Claim 3.** The set $\mathcal{U}_4$ is generic in $Sym(X)$.

It is enough to prove that the set $\mathcal{U}_5$ is generic since $\mathcal{U}_5 \subset \mathcal{U}_4$. So let us prove that for all n and m, the set $\mathcal{V}_{n,m} = \{\alpha \in Sym(X) \mid \langle \alpha^n, \beta^m \rangle$-action on X is not transitive $\}$ is closed and it has empty interior.

· $\mathcal{V}_{n,m}$ *is closed.*

$$\begin{aligned}
\mathcal{V}_{n,m} &= \{\alpha \in Sym(X) \mid \exists x, y \in X \text{ such that } \forall w \in \langle \alpha^n, \beta^m \rangle,\ wx \neq y\} \\
&= \{\alpha \in Sym(X) \mid \exists (x_i, x_j) \in S \times S \text{ such that } \forall w \in \langle \alpha^n, \beta^m \rangle,\ wx_i \neq x_j\} \\
&= \bigcup_{(x_i,x_j) \in S \times S} \{\alpha \in Sym(X) \mid \forall w \in \langle \alpha^n, \beta^m \rangle,\ wx_i \neq x_j\}
\end{aligned}$$

where S is a finite family of representatives for β^m-orbits. It is clear that the set $\{\alpha \in Sym(X) \mid \forall w \in \langle \alpha^n, \beta^m \rangle,\, wx_i \neq x_j\}$ is closed. So $\mathcal{V}_{n,m}$ is closed as a finite union of closed sets.

$\cdot$ *$\mathcal{V}_{n,m}$ is of empty interior.* Let $\alpha' \in \mathcal{V}_{n,m}$ and let $F \subset X$ be a finite subset of X. Let $Y := F \cup \alpha'(F)$ be a finite subset of X. We choose representatives for β^m-orbits outside of Y, and form a finite family $S = \{x_1, \ldots, x_m\}$ of X; this is possible since the β^m-orbits are infinite. We define α on F as in Lemma B.13. Inductively on $1 \leq i \leq m-1$, in each β^m-orbit $O(x_i)$ of x_i, we choose $n-1$ points $\{p_{i,1}, p_{i,2}, \ldots, p_{i,n-1}\}$ outside of Y and define

- $\alpha(x_i) = p_{i,1}$;
- $\alpha(p_{i,j}) = p_{i,j+1}$, $\forall 1 \leq j \leq n-2$;
- $\alpha(p_{i,n-1}) = x_{i+1}$.

Then, in $O(x_m)$, we choose $n-1$ points $\{p_{m,1}, \ldots, p_{m,n-1}\}$ outside of Y and define

- $\alpha(x_m) = p_{m,1}$;
- $\alpha(p_{m,j}) = p_{m,j+1}$, $\forall 1 \leq j \leq n-2$;
- $\alpha(p_{m,n-1}) = x_1$.

By construction, $\alpha^n(x_i) = x_{i+1}$, $\forall 1 \leq i \leq m-1$, and $\alpha^n(x_m) = x_1$, so the $\langle \alpha^n, \beta^m \rangle$-action is transitive.

B.4 Construction of $\mathbb{F}_2 *_{\mathbb{Z}} \mathbb{F}_2$

Let X be a countable infinite set. Let $c = c(\alpha, \beta)$ be a special word. Let $G := \mathbb{F}_2 = \langle \alpha, \beta \rangle$ be constructed as in Chapter 3. Let $\{A_n\}_{n=1}^{\infty}$ be a Følner sequence such that $c(A_n) = A_n$, $\forall n \geq 1$. Let $Z_c = \{\sigma \in Sym(X) \mid \sigma c = c\sigma\}$ be the centralizer of c. Let $\alpha' = \sigma^{-1}\alpha\sigma$, $\beta' = \sigma^{-1}\beta\sigma$, and let $H := \langle \alpha', \beta' \rangle$. Let $A = \langle c \rangle$ be the subgroup of G generated by c. We consider $\mathbb{F}_2 *_{\mathbb{Z}} \mathbb{F}_2 = G *_A H$ the amalgamated free product of G and H along A. For all $\sigma \in Z$, the action of $G *_A H$ on X is given by $g \cdot x = g(\alpha, \beta)x = gx$, and $h \cdot x = h(\alpha', \beta')x = \sigma^{-1}h(\alpha, \beta)\sigma x = \sigma^{-1}h\sigma x$, for all $g \in G$ and $h \in H$.

Lemma B.14. *The set Z_c is closed in $Sym(X)$. In particular, Z_c is a Baire space.*

Proof. The application $p : Sym(X) \to Sym(X)$; $\sigma \mapsto [\sigma, c]$ is continuous. So $Z_c = p^{-1}\{\mathrm{Id}\}$ is closed since $\{\mathrm{Id}\}$ is closed in $Sym(X)$. □

Proposition B.15. *The set*

$$\mathcal{O}_1 = \{\sigma \in Z_c \mid \text{ the action of } G *_A H \text{ on } X \text{ is faithful } \}$$

is generic in Z_c.

Proof. For all $w \in G *_A H$, let us denote by w^σ the corresponding element of $Sym(X)$ given by the above action, i.e. if $w = ag_nh_n \cdots g_1h_1$, with $a \in A$, $g_i \neq e \in G \setminus A$ and $h_i \neq e \in H \setminus A$, for all i, then

$$w^\sigma = ag_n\sigma^{-1}h_n\sigma \cdots g_1\sigma^{-1}h_1\sigma.$$

We want to prove that the set

$$\mathcal{O}_1 = \bigcap_{w \neq e \in G *_A H} \{\sigma \in Z_c \mid \exists x \in X \text{ such that } w^\sigma x \neq x \}$$

is generic in Z_c. Therefore, we shall prove that the set

$$\mathcal{V}_w = \{\sigma \in Z_c \mid w^\sigma = \mathrm{Id}_X\}$$

is closed and of empty interior in Z_c.

The set $\mathcal{V}_w$ is closed in Z_c because the application $Z_\sigma \to Sym(X)$; $\sigma \mapsto w^\sigma$ is continuous.

To see that the set $\mathcal{V}_w$ is of empty interior, let $\sigma' \in \mathcal{V}_w$, and let $F \subset X$ be a finite subset of X. Notice that if $F = F_1 \sqcup F_2$ with $F_1 \subset \mathrm{Fix}(c)$ and $F_2 \cap \mathrm{Fix}(c) = \emptyset$, then $\sigma'(F_1) \subset \mathrm{Fix}(c)$ and $\sigma'(F_2) \cap \mathrm{Fix}(c) = \emptyset$ because $\sigma'(\mathrm{Fix}(c)) = \mathrm{Fix}(c)$, for all $\sigma' \in Z_c$. So we define $\sigma|_{F_1} = \sigma'|_{F_1}$ as in Lemma B.13, and $\sigma|_{X \setminus \mathrm{Fix}(c)} = \sigma'|_{X \setminus \mathrm{Fix}(c)}$. Therefore, we have defined σ on $Y := (F \cup \sigma'(F)) \cup (X \setminus \mathrm{Fix}(c))$, and $\sigma|_Y$ commutes with $c|_Y$. Let us now define σ on $X \setminus Y$ in a way that $\sigma \in Z_c \setminus \mathcal{V}_w$. For all $g \in G \setminus A$ and $h \in H \setminus A$, let

$$\widehat{g} = \{x \in X \mid cx = x,\ cgx = gx \text{ and } gx \neq x \},$$

$$\widehat{h} = \{x \in X \mid cx = x,\ chx = hx \text{ and } hx \neq x \}.$$

Recall that we are considering the word $w^\sigma = ag_n\sigma^{-1}h_n\sigma \cdots g_1\sigma^{-1}h_1\sigma$. Choose any $x_0 \in X \setminus Y$. By induction on $1 \leq i \leq n$, we choose $x_{4i-3} \in \widehat{h_i}$ such that x_{4i-3} is different from the finite set of points $x_1, \ldots, x_{4i-4}$ chosen until the $(i-1)^{\text{th}}$ step. This is possible since $\widehat{h_i}$ is infinite by Proposition B.1. Then we define $\sigma x_{4i-4} := x_{4i-3}$ and $\sigma x_{4i-3} := x_{4i-4}$. This is well-defined because x_{4i-4}, $x_{4i-3} \in \mathrm{Fix}(c)$. We set $h_i x_{4i-3} =: x_{4i-2}$ which is different from x_{4i-3} and which is fixed by c, by definition of $\widehat{h_i}$. We choose $x_{4i-1} \in \widehat{g_i}$ such that x_{4i-1} is different from the finite set of points chosen so far. This is again possible since $\widehat{g_i}$ is infinite (Proposition B.1). Then we define $\sigma x_{4i-2} := x_{4i-1}$ and $\sigma x_{4i-1} := x_{4i-2}$. This is also well-defined because x_{4i-2}, $x_{4i-1} \in \mathrm{Fix}(c)$. We finally set $g_i x_{4i-1} =: x_{4i}$. By construction, the $4n$ points defined by the subwords on the right of w^σ are all distinct. In particular, $w^\sigma x_0 = ax_{4n} = x_{4n} \neq x_0$. Besides, this construction works also for the other three types of word w since we are treating all subwords of w. At last, if $w = g \in G \setminus \{\mathrm{Id}\}$, then there exists $x \in X$ such that $gx \neq x$ since G acts faithfully on X. Therefore, σ constructed in this way is beautifully in $Z_c \setminus \mathcal{V}_w$ and $\sigma'|_F = \sigma|_F$.

□

Proposition B.16. *The set*

$$\mathcal{O}_2 = \{\sigma \in Z_c \quad | \quad \text{there exists } \{A_{n_k}\}_{k\geq 1} \text{ a subsequence of } \{A_n\}_{n\geq 1} \text{ such that } \sigma(A_{n_k}) = A_{n_k},\ \forall k \geq 1\ \}$$

is generic in Z_c.

Proof. We want to prove that the set

$$\mathcal{O}_2 = \bigcap_{N\in\mathbb{N}} \{\sigma \in Z_c \mid \exists n \geq N \text{ such that } \sigma(A_n) = A_n\ \}$$

is generic in Z_c. So we shall prove that the set

$$\mathcal{V}_N = \{\sigma \in Z_c \mid \forall n \geq N,\, \sigma(A_n) \neq A_n\}$$

is closed and of empty interior in Z_c.

$\cdot$ $\mathcal{V}_N$ *is closed.* It is enough to prove that the set

$$V_{n,N} = \{\sigma \in Z_c \mid \sigma(A_n) \neq A_n\}$$

is closed since $\mathcal{V}_N = \cap_{n\geq N} V_{n,N}$. Let $\{\sigma_m\}_{m\geq 1} \subset V_{n,N}$ be a sequence converging to σ in Z_c. Since A_n is finite, there exists m_0 such that $\sigma_m(A_n) = \sigma(A_n)$, $\forall m \geq m_0$. Thus we have $\sigma(A_n) \neq A_n$ since $\sigma_m(A_n) \neq A_n$.

$\cdot$ $\mathcal{V}_N$ *is of empty interior.* Let $\sigma' \in \mathcal{V}_N$ and let $F \subset X$ be a finite subset of X. Let $Y := (F \cup \sigma'(F)) \cup (X \setminus \mathrm{Fix}(c))$. Since $A_n \subset \mathrm{Fix}(c)$ (Proposition B.3), there exists $n \geq N$ such that $A_n \cap Y = \emptyset$. We take then $\sigma \in Z_c$ which fixes A_n and $\sigma|_Y = \sigma'|_Y$. Therefore, $\sigma \in Z_c \setminus \mathcal{V}_N$ and $\sigma|_F = \sigma'|_F$. ☐

Let $\sigma \in \mathcal{O}_1 \cap \mathcal{O}_2$. Let $\{A_{n_k}\}_{k\geq 1}$ be a subsequence of $\{A_n\}_{n\geq 1}$ such that $\sigma(A_{n_k}) = A_{n_k}$, $\forall k \geq 1$. We claim that $\{A_{n_k}\}_{k\geq 1}$ is a Følner sequence for $G *_A H$. Indeed, for all $g \in G$ and for all $h \in H$, we have

$$\lim_{k\to\infty} \frac{|A_{n_k} \vartriangle g \cdot A_{n_k}|}{|A_{n_k}|} = \lim_{k\to\infty} \frac{|A_{n_k} \vartriangle g(\alpha,\beta)A_{n_k}|}{|A_{n_k}|} = 0,$$

$$\begin{aligned}
\lim_{k\to\infty} \frac{|A_{n_k} \vartriangle h \cdot A_{n_k}|}{|A_{n_k}|} &= \lim_{k\to\infty} \frac{|A_{n_k} \vartriangle h(\alpha',\beta')A_{n_k}|}{|A_{n_k}|} \\
&= \lim_{k\to\infty} \frac{|A_{n_k} \vartriangle \sigma^{-1}h(\alpha,\beta)\sigma A_{n_k}|}{|A_{n_k}|} \\
&= \lim_{k\to\infty} \frac{|\sigma A_{n_k} \vartriangle h(\alpha,\beta)\sigma A_{n_k}|}{|A_{n_k}|} \\
&= \lim_{k\to\infty} \frac{|A_{n_k} \vartriangle h(\alpha,\beta)A_{n_k}|}{|A_{n_k}|} = 0,
\end{aligned}$$

since $\{A_{n_k}\}$ is Følner for G and $\sigma(A_{n_k}) = A_{n_k}$. Therefore, we have:

Theorem B.17. *There exists a transitive, faithful and amenable action of the group* $\langle\alpha,\beta\rangle *_{\langle c\rangle} \langle\alpha',\beta'\rangle$ *on* X.

Lemma B.18. *Let* $c = c(\alpha,\beta)$ *be any word (not necessarily special) on* $\{\alpha^{\pm 1},\beta^{\pm 1}\}$. *There exists an automorphism* a *of* $\mathbb{F}_2$ *such that* $a(c)$ *is a special word.*

Proof. Let us recall some properties of automorphisms of free groups. The reader can find more details in [53]. Let $\mathbb{F}_n$ be a free group with a finite basis X of n elements. We consider the following endomorphisms of $\mathbb{F}_n$. For any $x \in X$, let φ_x be the endomorphism defined by $\varphi_x : x \mapsto x^{-1};\ y \mapsto y,\ \forall y \in X \setminus \{x\}$. For any $x \neq y \in X$, let $\psi_{xy} : x \mapsto xy;\ z \mapsto z,\ \forall z \in X \setminus \{x\}$. In both cases, the image of X is another basis for $\mathbb{F}_n$, and φ_x and ψ_{xy} are automorphisms of $\mathbb{F}_n$, called the Nielsen generators for $\mathrm{Aut}(\mathbb{F}_n)$, and they generate $\mathrm{Aut}(\mathbb{F}_n)$. Let $\mathbb{F}_n/\mathbb{F}'_n \simeq \mathbb{Z}^n$ be the abelianization of $\mathbb{F}_n$. We have $\mathrm{Aut}(\mathbb{Z}^n) \simeq GL_n(\mathbb{Z})$. The Nielsen generators for $\mathrm{Aut}(\mathbb{F}_n)$ induce the following generators for $\mathrm{Aut}(\mathbb{Z}^n)$:

$$\overline{\varphi}_x : x \mapsto -x; \quad y \mapsto y, \quad \forall y \in X \setminus \{x\};$$

$$\overline{\psi}_{xy} : x \mapsto x+y; \quad z \mapsto z, \quad \forall z \in X \setminus \{x\}.$$

Thus, we conclude that the natural maps from $\mathrm{Aut}(\mathbb{F}_n)$ into $\mathrm{Aut}(\mathbb{Z}^n)$ is an epimorphism. Notice that for a word c to be a special word depends only on its image in $\mathbb{Z}^2$. Therefore, in order to prove the Lemma, it is enough to find a matrix $M \in GL_2(\mathbb{Z})$ such that the exponent sum $S(\alpha)' := S_{a(c)}(\alpha)$ of exponents of α in the word $a(c)$ divides the exponent sum $S(\beta)' := S_{a(c)}(\beta)$ of exponents of β in the word $a(c)$, where $a \in \mathrm{Aut}(\mathbb{F}_2)$ is a reciprocal image of M by the epimorphism $\mathrm{Aut}(\mathbb{F}_2) \to \mathrm{Aut}(\mathbb{Z}^2)$. In fact, once we have $c = c(\alpha,\beta)$ with $S(\alpha)$ dividing $S(\beta)$, we can obtain a weakly cyclically reduced word by conjugating c, and the conjugation is an automorphism of $\mathbb{F}_2$.

If $S(\beta) = 0$, c is already a special word. If $S(\alpha) = 0$ and $S(\beta) \neq 0$, then we apply the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL_2(\mathbb{Z})$ which exchanges $S(\alpha)$ and $S(\beta)$. So suppose that $S(\alpha) \neq 0 \neq S(\beta)$. Let $d = \gcd(S(\alpha), S(\beta))$ be the greatest common divisor of $S(\alpha)$ and $S(\beta)$. By Bézout's identity, there exist relatively prime integers p, q such that $pS(\alpha) + qS(\beta) = d$. Since $\gcd(p,-q) = 1$, there exist r, t such that $rp - tq = 1$ again by Bézout's identity. Then, the matrix $M = \begin{pmatrix} p & q \\ t & r \end{pmatrix}$ is in $GL_2(\mathbb{Z})$ and it sends $\begin{pmatrix} S(\alpha) \\ S(\beta) \end{pmatrix}$ to $\begin{pmatrix} d \\ tS(\alpha) + rS(\beta) \end{pmatrix}$. Therefore, $S(\alpha)' = d$ divides $S(\beta)' = tS(\alpha) + rS(\beta)$. □

From Theorem B.17 and the previous Lemma, we have:

Theorem B.19. *Let* $c = c(\alpha,\beta)$ *be any word on* $\{\alpha^{\pm 1},\beta^{\pm 1}\}$. *Then the group* $\langle\alpha,\beta\rangle *_{\langle c\rangle} \langle\alpha',\beta'\rangle$ *admits a transitive, faithful and amenable action.*

A result of G. Baumslag [12] shows that these groups are residually finite.

Furthermore, let H be a finite index subgroup of $\mathbb{F}_2 *_{\mathbb{Z}} \mathbb{F}_2$. Then $K := H \cap \mathbb{F}_2$ is a finite index subgroup of $\mathbb{F}_2$ so that the H-action on X is transitive since the K-action is transitive by Proposition B.4. Therefore, we have:

Theorem B.20. *For any finite index subgroup H of $\langle \alpha, \beta\rangle *_{\langle c\rangle} \langle \alpha', \beta'\rangle$, H admits a transitive, faithful and amenable action.*

B.5 Applications

Let us recall the class of all countable groups appeared in [33]:

$$\mathcal{A} = \{ \ G \text{ countable} \mid G \text{ admits a faithful transitive amenable action } \}.$$

Let Σ_g be a closed oriented surface of genus $g \geq 2$. It is well-known that the fundamental groups $\Gamma_g = \pi_1(\Sigma_g)$ of Σ_g has a presentation

$$\pi_1(\Sigma_g) = \langle a_1, \, b_1, \, \ldots, \, a_g, \, b_g \ \mid \prod_{i=1}^{g} [a_i, b_i] \, \rangle.$$

In particular, we have $\pi_1(\Sigma_2) = \langle a_1, b_1\rangle *_{\langle c\rangle} \langle a_2, b_2\rangle$ where $c = [a_1, b_1] = [a_2, b_2]$. Therefore, $\pi_1(\Sigma_2) \in \mathcal{A}$ by Theorem B.19 (or already by Theorem B.17 since $c = [a_1, b_1]$ is a special word). Now, let Σ_g be a closed oriented surface of genus $g \geq 3$. Viewing Σ_g as $(g-1)$ tori glued on a central one, the cyclic group $\mathbb{Z}/(g-1)\mathbb{Z}$ acts properly and freely on Σ_g, and the quotient space is Σ_2. Therefore $\pi_1(\Sigma_g)$ injects into $\pi_1(\Sigma_2)$ as a subgroup of index $(g-1)$ (in other words, Σ_g is a $(g-1)$-sheeted regular covering of Σ_2). Consequently, $\pi_1(\Sigma_g)$ is in $\mathcal{A}$ by Theorem B.20. Moreover, the fundamental group of a torus $\pi_1(\mathbb{T}^2) = \pi_1(\Sigma_1)$ is isomorphic to $\mathbb{Z}^2$, an amenable group. Therefore, we have:

Theorem B.21. *Let Σ_g be a closed oriented surface of genus $g \geq 1$. The fundamental group $\Gamma_g = \pi_1(\Sigma_g)$ of Σ_g admits a transitive, faithful and amenable action, for all $g \geq 1$.*

Corollary B.22. *For any compact surface S, the fundamental group $\pi_1(S)$ is in $\mathcal{A}$.*

Proof. First of all, we can suppose that S is oriented. In fact, it is well-known that if S is a non-oriented connected surface, then there exists a oriented 2-sheeted covering space $\tilde{S}$ (cf. [31]). Then $\pi_1(\tilde{S})$ is a subgroup of index 2 of $\pi_1(S)$ so that it is *co-amenable* in $\pi_1(S)$ (a subgroup $H < G$ is co-amenable if the G-action on G/H is amenable). Therefore, in order that $\pi_1(S) \in \mathcal{A}$, it suffices to have $\pi_1(\tilde{S}) \in \mathcal{A}$ by Proposition 1. (vi) in [33].

If S is a closed oriented surface (i.e. without boundary), S is either a sphere or a finite connected sum of tori Σ_g, $g \geq 1$; so $\pi_1(S) \in \mathcal{A}$ in both cases. If S is a surface with boundary components, then $\pi_1(S)$ is a free group (the fundamental group of a sphere with p boundary components is a free group of rank $p-1$, and the fundamental group of Σ_g with p boundary components is a free group of rank $2g + p - 1$, $\forall g \geq 1$), so it is again in $\mathcal{A}$ by van Douwen's theorem. □

Example B.5.1. Surface bundles over $\mathbb{S}^1$

A surface bundle over $\mathbb{S}^1$ is a closed 3-manifold which is constructed as a fiber bundle over the circle with fiber a closed surface. The fundamental group G of such bundle can be viewed as an HNN-extension

$$G = \pi_1(M_\phi) = \langle \Gamma_g, t \mid txt^{-1} = \phi_*(x), \forall x \in \Gamma_g \rangle,$$

where $\phi : \Sigma_g \to \Sigma_g$ is a homeomorphism. Thus, we have a short exact sequence:

$$0 \to \Gamma_g \to G \to \mathbb{Z} \to 0.$$

The subgroup Γ_g is co-amenable in G since it is normal in G and $G/\Gamma_g \simeq \mathbb{Z}$ is amenable. Therefore, we have $G \in \mathcal{A}$.

The Thurston's virtual fibration conjecture states that [77]:

> *Every closed, irreducible, atoroidal 3-manifold M has a finite-sheeted cover which fibres over the circle.*

It follows from the conjecture that the fundamental group $\pi_1(M)$ is in $\mathcal{A}$ since it contains a finite index subgroup which is in $\mathcal{A}$.

Appendix C

Amenable actions of cyclically pinched one-relator groups and generic property

Abstract. We show that the class of cyclically pinched one-relator groups admits amenable, faithful and transitive actions on infinite countable sets. This work generalizes the results on such actions for doubles of free group on 2 generators.

C.1 Introduction

One of the most important classes of torsion-free one-relator groups is the class of cyclically pinched one-relator groups. A group G is *cyclically pinched one-relator group* (defined in [30]) if it admits a presentation

$$G = \langle a_1, \ldots, a_n, b_1, \ldots, b_m | c = d \rangle$$

where $1 \neq c = c(a_1, \ldots, a_n)$ is a cyclically reduced non-primitive word (not part of a basis) in the free group $A = \langle a_1, \ldots, a_n \rangle$, and $1 \neq d = d(b_1, \ldots, b_m)$ is a cyclically reduced non-primitive word in the free group $B = \langle b_1, \ldots, b_m \rangle$.

Such a group is the amalgamated free product of two free groups A and B over the cyclic subgroup generated by $c = d$. Examples of the most familiar cyclically pinched one-relator groups are the *surface groups*, i.e. the fundamental group of a compact surface. The fundamental group of the closed orientable surface of genus g has the presentation

$$\langle a_1, b_1, \ldots, a_g, b_g | [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle.$$

By letting $c = [a_1, b_1] \cdots [a_{g-1}, b_{g-1}]$ and $d = [a_g, b_g]^{-1}$, the group decomposes as the free product of the free group $\mathbb{F}_{2(g-1)}$ on $a_1, b_1, \ldots, a_{g-1}, b_{g-1}$ and the free group $\mathbb{F}_2$ on a_g, b_g amalgamated over the cyclic subgroup generated by c in $\mathbb{F}_{2(g-1)}$ and d in $\mathbb{F}_2$, hence it is a cyclically pinched one-relator group.

The main objective of this paper is to generalize the results of the paper [56] on amenable actions. An action of a countable group G on a set X is *amenable* if there exists a sequence $\{A_n\}_{n\geq 1}$ of finite non-empty subsets of X such that for every $g \in G$, one has

$$\lim_{n\to\infty} \frac{|A_n \vartriangle g \cdot A_n|}{|A_n|} = 0.$$

Such a sequence is called a *Følner sequence* for the action of G on X. Thanks to a result of Følner [52], this definition is equivalent to the existence of a G-invariant mean on subsets of X.

In [56], we have proved that every finite index subgroup of the amalgamated free product of two free groups of rank two over a cyclic subgroup $\mathbb{Z}$, where $\mathbb{Z}$ embeds in each factor as the subgroup generated by some common word on the generating sets, admits an amenable, faithful and transitive action on an infinite countable set (this includes the surface groups). The study of the class $\mathcal{A}$ of all countable groups admitting an amenable, faithful and transitive action is proposed by Glasner and Monod in [33], where many properties of this class were presented.

In this paper, we show the existence of such actions for the class of cyclically pinched one-relator groups (see Theorem C.8):

Theorem C.1. *Let n, $m \geq 1$ be two integers. Let $c = c(a_1, \ldots, a_{n+1})$ be a cyclically reduced non-primitive word in the free group $\mathbb{F}_{n+1} = \langle a_1, \ldots, a_{n+1}\rangle$ and let $d = d(b_1, \ldots, b_{m+1})$ be a cyclically reduced non-primitive word in the free group $\mathbb{F}_{m+1} = \langle b_1, \ldots, b_{m+1}\rangle$. Then any finite index subgroup of the amalgam $\mathbb{F}_{n+1} *_{\mathbb{Z}} \mathbb{F}_{m+1}$, where $\mathbb{Z}$ embeds as the subgroup generated by c (respectively by d) in $\mathbb{F}_{n+1}$ (respectively in $\mathbb{F}_{m+1}$), admits an amenable, faithful and transitive action.*

Remark C.1.1. In the definition of cyclically pinched one-relator group G, if c or d is a primitive word, then G is a free group on $n+m-1$ generators, so G is in $\mathcal{A}$ by the result of Glasner and Monod.

The idea of the proof is to use the Baire's category theorem as in [56]. That is, together with a fixed transitive permutation β, we construct inductively $\mathbb{F}_{n+1}$-actions such that the conjugation of $\mathbb{F}_{n+1}$ by another generic permutation σ defines the desired action of amalgamated free product of two free groups $\mathbb{F}_{n+1}$ and $\mathbb{F}_{m+1}$ along a cyclic subgroup given by the words as in Theorem C.1.

For X an infinite countable set, recall that $Sym(X)$ with the topology of pointwise convergence is a Baire space, i.e. every intersection of countably many dense open subsets is dense in $Sym(X)$. So for every $n \geq 1$, the product space $(Sym(X))^n$ is a Baire space. A subset of a Baire space is called *meagre* if it is a union of countably many closed subsets with empty interior; and *generic* or *dense* G_δ if its complement is meagre.

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C.2 Graph extensions

A graph G consists of the set of vertices $V(G)$ and the set of edges $E(G)$, and two applications $E(G) \to E(G)$; $e \mapsto \bar{e}$ such that $\bar{\bar{e}} = e$ and $\bar{e} \neq e$, and $E(G) \to V(G) \times V(G)$; $e \mapsto (i(e), t(e))$ such that $i(e) = t(\bar{e})$. An element $e \in E(G)$ is a *directed edge* of G and $\bar{e}$ is the *inverse edge* of e. For all $e \in E(G)$, $i(e)$ is the *initial vertex* of e and $t(e)$ is the *terminal vertex* of e.

Let S be a set. A *labeling* of a graph $G = (V(G), E(G))$ on the set $S^{\pm 1} = S \cup S^{-1}$ is an application

$$l : E(G) \to S^{\pm 1}; e \mapsto l(e)$$

such that $l(\bar{e}) = l(e)^{-1}$. A *labeled graph* $G = (V(G), E(G), S, l)$ is a graph with a labeling l on the set $S^{\pm 1}$. A labeled graph is *well-labeled* if for any edges e, $e' \in E(G)$, $\left[i(e) = i(e') \text{ and } l(e) = l(e')\right]$ implies that $e = e'$.

A word $w = w_m \cdots w_1$ on $\{\alpha_n^{\pm 1}, \alpha_{n-1}^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$ is called *reduced* if $w_{k+1} \neq w_k^{-1}$, $\forall 1 \leq k \leq m-1$. A word $w = w_m \cdots w_1$ on $\{\alpha_n^{\pm 1}, \alpha_{n-1}^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$ is called *weakly cyclically reduced* if w is reduced and $w_m \neq w_1^{-1}$; this definition allows w_m and w_1 to be equal. Given a reduced word, we define two finite graphs labeled on $\{\alpha_n^{\pm 1}, \alpha_{n-1}^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$ as follows:

Definition C.2.1. Let $w = w_m \cdots w_1$ be a reduced word on $\{\alpha_k^{\pm 1}, \alpha_{k-1}^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$. The *path* of w is a finite labeled graph $P(w, v_0)$ labeled on $\{\alpha_k^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta\}$ consisting of $m+1$ vertices and m directed edges $\{e_1, \ldots, e_m\}$ such that

- $i(e_{j+1}) = t(e_j)$, $\forall 1 \leq j \leq m-1$;
- $v_0 = i(e_1) \neq t(e_m)$;
- $l(e_j) = w_j$, $\forall 1 \leq j \leq m$.

The point v_0 is the *startpoint* and the point $t(e_m)$ is the *endpoint* of the path $P(w, v_0)$. The two points are the *extreme points* of the path.

Definition C.2.2. Let $w = w_m \cdots w_1$ be a reduced word on $\{\alpha_k^{\pm 1}, \alpha_{k-1}^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$. The *cycle* of w is a finite labeled graph $C(w, v_0)$ labeled on $\{\alpha_k^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta\}$ consisting of m vertices and m directed edges $\{e_1, \ldots, e_m\}$ such that

- $i(e_{j+1}) = t(e_j)$, $\forall 1 \leq j \leq m-1$;

- $v_0 = i(e_1) = t(e_m)$;
- $l(e_j) = w_j$, $\forall 1 \leq j \leq m$.

The point v_0 is the *startpoint* of the cycle $C(w, v_0)$.

Notice that since w is a reduced word, the graph $P(w, v_0)$ is well-labeled. If w is weakly cyclically reduced, then $C(w, v_0)$ is also well-labeled.

Conversely, if $P = \{e_1, e_2, \ldots, e_n\}$ is a well-labeled path with $i(e_1) = v_0$, labeled by $l(e_i) = g_i$, $\forall i$, then there exists a unique reduced word $w = g_n \cdots g_1$ such that $P(w, v_0)$ is P. If $C = \{e_1, e_2, \ldots, e_n\}$ is a well-labeled cycle with $t(e_n) = i(e_1) = v_0$, labeled by $l(e_i) = g_i$, $\forall i$, then there exists a unique weakly cyclically reduced word $w_1 = g_n \cdots g_1$ such that $C(w, v_0)$ is C.

Let X be an infinite countable set. Let β be a simply transitive permutation of X. The *pre-graph* G_0 is a labeled graph consisting of the set of vertices $V(G_0) = X$ and the set of directed edges all labeled by β such that every vertex has exactly one entering edge and one outgoing edge, and $t(e) = \beta(i(e))$. One can imagine G_0 as the Cayley graph of $\mathbb{Z}$ with 1 as a generator.

Definition C.2.3. An *extension* of G_0 is a well-labeled graph G labeled by $\{\alpha_k^{\pm 1}, \alpha_{k-1}^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$, containing G_0, with $V(G) = V(G_0) = X$. We will denote it by $G_0 \subset G$.

In order to have a transitive action with some additional properties of the $\langle \alpha_k, \ldots, \alpha_1, \beta \rangle$-action on X, we shall extend inductively G_0 on $1 \leq i \leq k$ by adding finitely many directed edges labeled by α_i on G_0 where the edges labeled by β are already prescribed. In order that the added edges represent an action on X, we put the edges in such a way that the extended graph is well-labeled, and moreover we put an additional edge labeled by α_i on every endpoint of the extended edges by α_i; more precisely, if we have added n edges labeled by α_i between $x_0, x_1, \ldots, x_n$ successively, we put an α_i-edge from x_n to x_0 to have a cycle consisting of $n + 1$ edges, which corresponds to a α_i-orbit of size $n + 1$. On the points where no α_i-edges are involved, we can put any α_i-edge in a way that the the extended graph is well-labeled and every point has a entering edge and a outgoing edge labeled by α_i (for example we can put a loop labeled by α_i, corresponding to the fixed points). In the end, the graph represents an $\langle \alpha_k, \ldots, \alpha_1, \beta \rangle$-action on X, i.e. G will be a Schreier graph.

Definition C.2.4. Let G, G' be graphs labeled on a set $S^{\pm 1}$. A *homomorphism* $f : G \to G'$ is a map sending vertices to vertices, edges to edges, such that

- $f(i(e)) = i(f(e))$ and $f(t(e)) = t(f(e))$;
- $l(e) = l(f(e))$,

for all $e \in E(G)$.

If there exists an injective homomorphism $f : G \to G'$, we say that f is an *embedding*, and G *embeds* in G'.

Lemma C.2. *Let $k \geq 1$. Let $w_k = w_k(\alpha_k, \alpha_{k-1}, \ldots, \alpha_1, \beta)$ be a reduced word on $\{\alpha_k^{\pm 1},\ \alpha_{k-1}^{\pm 1},\ \ldots,\ \alpha_1^{\pm 1},\ \beta^{\pm 1}\}$. For every finite subset F of G_0, there is an extension G of G_0 on which the path $P(w_k, v_0)$ embeds in G, the image of $P(w_k, v_0)$ in G does not intersect with F, and $G \setminus G_0$ is finite.*

Proof. Let us show this by induction on k. If $k = 1$, it follows from Proposition 6 in [56]. Indeed, in the proof of Proposition 6 in [56], we start by choosing any element $z_0 \in X$ to construct a path. Since the set X is infinite and there is no assumption on the starting point z_0 of the path, there are infinitely many choices for z_0.

For the proof of the induction step, consider the case

$$w_k = \alpha_k^{a_{2m}} w_{k-1}^{2m-1} \alpha_k^{a_{2m-2}} \cdots \alpha_k^{a_4} w_{k-1}^3 \alpha_k^{a_2} w_{k-1}^1.$$

with $w_{k-1}^i = w_{k-1}^i(\alpha_{k-1}, \ldots, \alpha_1, \beta)$ a reduced word on $\{\alpha_{k-1}^{\pm 1},\ \ldots,\ \alpha_1^{\pm 1},\ \beta^{\pm 1}\}$, for all i. To simplify the notation, we assume that a_j is positive, $\forall j$.

Let $F \subset X$ be a finite subset of X. By hypothesis of induction, there is an extension G_1 of G_0 and an embedding f^1 such that $f^1 : P(w_{k-1}^1, v_0) \hookrightarrow G_1$ and the image of $P(w_{k-1}^1, v_0)$ in G_1 does not intersect with F. Let

$$f^1(v_0) = f^1\big(i(P(w_{k-1}^1, v_0))\big) =: z_0$$

and

$$f^1\big(t(P(w_{k-1}^1, v_0))\big) =: z_1.$$

Inductively on each $2 \leq i \leq m$, we apply the following algorithm:

Algorithm

1. Take an extension G_{2i-2} of G_0 such that
   - $P(w_{k-1}^{2i-1}, v_{2i-2})$ embeds in G_{2i-2} such that the image of $P(w_{k-1}^{2i-1}, v_{2i-2})$ does not intersect with F;
   - $G_{2i-2} \cap G_{2i-3} = G_0$ (this is possible since there are infinitely many extensions G'_{2i-2} of G_0 by hypothesis of induction and $G_{2i-3} \setminus G_0$ is finite).
2. Let $f^{2i-1} : P(w_{k-1}^{2i-1}, v_{2i-2}) \hookrightarrow G_{2i-2} \cup G_{2i-3} =: G'_{2i-1}$ with
   - $f^{2i-1}\big(i(P(w_{k-1}^{2i-1}, v_{2i-2}))\big) = f^{2i-1}(v_{2i-2}) =: z_{2i-2}$;
   - $f^{2i-1}\big(t(P(w_{k-1}^{2i-1}, v_{2i-2}))\big) =: z_{2i-1}$.
3. Choose $|a_{2i-2}| - 1$ points $\{p_1^{(a_{2i-2})}, \ldots, p_{|a_{2i-2}|-1}^{(a_{2i-2})}\}$ outside of the finite set of all points appeared until now, and put the directed edges labeled by α_k from
   - z_{2i-3} to $p_1^{(a_{2i-2})}$;
   - $p_j^{(a_{2i-2})}$ to $p_{j+1}^{(a_{2i-2})}$, $\forall 1 \leq j \leq |a_{2i-2}| - 2$;

- $p^{(a_{2i-2})}_{|a_{2i-2}|-1}$ to z_{2i-2},

and let $G_{2i-1} := G'_{2i-1} \cup \{$the additional α_k-edges between z_{2i-3} and $z_{2i-2}\}$.

In the ends, we choose new $|a_{2m}|$ points $\{p^{(a_{2m})}_1, \ldots, p^{(a_{2m})}_{|a_{2m}|}\}$ and put the directed edges labeled by α_k from z_{2m-1} to $p^{(a_{2m})}_1$, and from $p^{(a_{2m})}_j$ to $p^{(a_{2m})}_{j+1}$, $\forall 1 \le j \le |a_{2m}|$, so that we have $\alpha_k^{a_{2m}} z_{2m-1} = z_{2m}$.

By construction, the resulting graph $G_{2m-1} \cup P(\alpha^{a_{2m}}, v_{2m-1}) =: G$ is an extension of G_0 satisfying $P(w_k, v_0) \hookrightarrow G$ such that the image of $P(w_k, v_0)$ does not intersect with F. $\square$

Lemma C.3. *Let $w = w(\alpha_n, \ldots, \alpha_1, \beta)$ be a weakly cyclically reduced word on $\{\alpha_n^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$ such that α_i appears in the word w for some i (i.e. $w \notin \langle \beta \rangle$). For every finite subset F of G_0, there exists an extension G_{n+1} of G_0 such that the cycle $C(w, v_0)$ embeds in G_{n+1} and the image of $C(w, v_0)$ in G_0 does not intersect with F.*

Proof. Let us consider the case

$$w = \alpha_i^{a_{2m}} w_{2m-1} \alpha_i^{a_{2m-2}} \cdots \alpha_i^{a_4} w_3 \alpha_i^{a_2} w_1$$

written as the normal form of $\langle \alpha_n, \ldots, \alpha_{i+1}, \alpha_{i-1}, \ldots, \alpha_1, \beta \rangle * \langle \alpha_i \rangle$.

Since $w' = w_{2m-1} \alpha_i^{a_{2m-2}} \cdots \alpha_i^{a_4} w_3 \alpha_i^{a_2} w_1$ is reduced, by Lemma C.2, there is an extension G'_{n+1} of G_0 and a homomorphism $f : P(w', v_0) \to G'_{n+1}$ such that $f(P(w', v_0))$ is a path in G'_{n+1} outside of F. Let $f(v_0) =: z_0$ be the startpoint of $f(P(w', v_0))$ and $f(w'(z_0)) =: z_{2m-1}$ be the endpoint of $f(P(w', v_0))$.

Choose $|a_{2m}| - 1$ new points $\{p_{a_m}, \ldots, p_{|a_{2m}|-1}\}$ and put the directed edges labeled by $\alpha_i^{sign(a_{2m})}$ from

- z_{2m-1} to p_1;
- p_j to p_{j+1}, $\forall 1 \le j \le |a_{2m}| - 2$;
- $p_{|a_{2m}|-1}$ to z_0.

By construction, the resulting graph $G_{n+1} := G'_{n+1} \cup P(\alpha^{a_{2m}}, v_{2m-1})$ is an extension of G_0 and $C(w, v_0)$ embeds in G_{n+1} outside of F. $\square$

Let $c = c(\alpha_n, \ldots, \alpha_1, \beta)$ be a weakly cyclically reduced word on $\{\alpha_n^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$ such that $c \notin \langle \beta \rangle$ and $w = w(\alpha_n, \alpha_{n-1}, \ldots, \alpha_1, \beta)$ be a reduced word on $\{\alpha_n^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$ such that $w \notin \langle c \rangle$. Let $C(c, v_0)$ be the cycle of c with startpoint at v_0, and let $P(w, v_0)$ be the path of w with the same startpoint v_0 as $C(c, v_0)$ such that every vertex of $P(w, v_0)$ (other than v_0) is distinct from every vertex in $C(c, v_0)$. Let $C(c, wv_0)$ be the cycle of c with startpoint at wv_0 such that every vertex of $C(c, wv_0)$ (other than wv_0) is distinct from every vertex in $P(w, v_0) \cup C(c, v_0)$. Let us denote by $Q_0(c, w)$ the union of $C(c, v_0)$, $P(w, v_0)$ and $C(c, wv_0)$. Let $Q(c, w)$ be the well-labeled graph obtained from $Q_0(c, w)$ by identifying the successive edges with the same initial vertex and the same label.

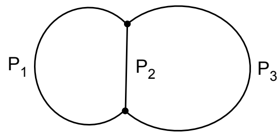

Figure C.1:

Notice that the well-labeled graph $Q(c, w)$ can have one, two or three cycles, and in each type of $Q(c, w)$, the quotient map $Q_0(c, w) \twoheadrightarrow Q(c, w)$ restricted to $C(c, v_0)$ and to $C(c, wv_0)$ is injective (each one separately).

Lemma C.4. *There is an extension G_{n+1} of G_0 such that $Q(c, w)$ embeds in G_{n+1}.*

Proof. By Lemma C.2 and C.3, it is enough to show that every cycle in Q contains edges labeled by $\alpha_i^{\pm 1}$ for some i. For the cases where Q has one or two cycles, it is clear since the cycles in Q represent $C(c, v_0)$ and $C(c, wv_0)$, and $c \notin \langle \beta \rangle$. In the case where $Q(c, w)$ has three cycles, $Q(c, w)$ has three paths P_1, P_2 and P_3 such that $P_1 \cap P_2 \cap P_3$ are exactly two extreme points of P_i's, and $P_1 \cup P_2$, $P_2 \cup P_3$ and $P_1 \cup P_3$ are the three cycles in $Q(c, w)$ (see Figure C.1). So we need to prove that, if one of the three paths has edges labeled only on $\{\beta^{\pm 1}\}$, then the other two paths both contains edges labeled by $\alpha_i^{\pm 1}$ for some i. For this, it is enough to prove:

Claim. If the reduced word $c = \gamma\lambda$ is conjugate to the reduced word $\gamma\lambda'$ via a reduced word w, where $\gamma \in \langle \alpha_n, \alpha_{n-1}, \ldots, \beta \rangle \setminus \langle \beta \rangle$ and $\lambda \in \langle \beta \rangle$, then $wc = cw$. Furthermore, the word c can not be conjugate to the reduced word $\gamma^{-1}\lambda'$ with $\lambda' \in \langle \beta \rangle$.

Let us see how we can conclude Lemma C.4 using the Claim. First of all, notice that c does not commute with w since we are treating the case where Q has three cycles. More precisely, in a free group, two elements commute if and only if they are both powers of the same word. So if $cw = wc$, then $c = \gamma^k$ and $w = \gamma^l$ with $k \neq l$, where γ is a non-trivial word, so that Q has one cycle. Suppose that P_1 consists of edges labeled only on $\{\beta^{\pm 1}\}$. One of the cycles among $P_1 \cup P_2$, $P_2 \cup P_3$ and $P_1 \cup P_3$ consists of edges labeled by the letters of c up to cyclic permutation, let us say $P_1 \cup P_2$ (i.e. if $c = c_1 \cdots c_m$, given any startpoint v_0 in $P_1 \cup P_2$, the directed edges of the cycle $C(c, v_0)$ are labeled on a cyclic permutation of the sequence $\{c_m, \ldots, c_1\}$). Another cycle among $P_2 \cup P_3$ and $P_1 \cup P_3$ consists of edges labeled by the letters of the reduced form of $w^{-1}cw$ up to cyclic permutation. Since $c \notin \langle \beta \rangle$, the path P_2 has edges labeled by $\alpha_i^{\pm 1}$ for some i. Now, if the cycle representing $w^{-1}cw$ is $P_1 \cup P_3$, then the path P_3 has edges labeled by $\alpha_i^{\pm 1}$ since $w^{-1}cw \notin \langle \beta \rangle$ and P_1 has only edges labeled on $\{\beta^{\pm 1}\}$(this is because two words in the free group $\mathbb{F}$ define conjugate elements of $\mathbb{F}$ if and only if their cyclic reduction in $\mathbb{F}$ are cyclic permutations of one another). Suppose now that the cycle representing $w^{-1}cw$ is $P_2 \cup P_3$ and P_3 has edges labeled only on $\{\beta^{\pm 1}\}$. Then, c would be the form $\gamma\lambda$ up to cyclic permutation where $\gamma \in \langle \alpha_n, \alpha_{n-1}, \ldots, \beta\rangle \setminus \langle\beta\rangle$ (representing P_2) and $\lambda \in \langle\beta\rangle$ (representing P_1); and $w^{-1}cw$ would be the form $\gamma^{\pm 1}\lambda'$ up to cyclic permutation where $\lambda' \in \mathbb{F}_n$(representing P_3); but the Claim tells us that this is not possible, therefore P_3 contains edges labeled by $\alpha_i^{\pm 1}$ for some i.

Now we prove the Claim. Let $c = \gamma\lambda$ and $w^{-1}cw = \gamma\lambda'$ such that $\gamma \in \langle \alpha_n, \alpha_{n-1}, \ldots, \beta\rangle \setminus \langle\beta\rangle$ and $\lambda, \lambda' \in \langle\beta\rangle$. Without loss of generality, we can suppose that $\gamma = \gamma_m\lambda_{m-1}\cdots\lambda_1\gamma_1$, with $\gamma_i \in \langle \alpha_n, \alpha_{n-1}, \ldots, \beta\rangle \setminus \langle\beta\rangle$ and $\lambda_i \in \langle\beta\rangle$. Since $\gamma\lambda$ and $\gamma\lambda'$ are conjugate in a free group, there exists $1 \leq k \leq m$ such that

$$\gamma_k\lambda_{k-1}\cdots\lambda_1\gamma_1\lambda\gamma_m\lambda_{m-1}\cdots\gamma_{k+1}\lambda_k = \gamma\lambda' = \gamma_m\lambda_{m-1}\cdots\lambda_1\gamma_1\lambda'.$$

By identification of each letter, one deduces that $\lambda' = \lambda_k = \lambda_j$, for every j multiple of k in $\mathbb{Z}/m\mathbb{Z}$, and $\lambda = \lambda_{m-k}$. In particular, $\lambda = \lambda'$ so that $c = \gamma\lambda = \gamma\lambda' = w^{-1}cw$ and thus $cw = wc$. For the seconde statement, suppose by contradiction that there exists w such that $w^{-1}cw = \gamma^{-1}\lambda'$. Then by the similar identification as above we deduce that $\lambda^{-1} = \lambda'$, so $w^{-1}cw$ would be a cyclic permutation of c^{-1}, which is clearly not possible.

□

C.3 Construction of generic actions of free groups

Let X be an infinite countable set. We identify $X = \mathbb{Z}$. Let β be a simply transitive permutation of X (which is identified to the translation $x \mapsto x + 1$).

Let c be a cyclically reduced word on $\{\alpha_n^{\pm 1}, \alpha_{n-1}^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$ such that the sum $S_c(\beta)$ of the exponents of β in the word c is zero. Thus necessarily c contains α_i for some i.

Let us denote by $S_c^+(\beta)$ the sum of positive exponents of β in the word c; by denoting $S_c^-(\beta)$ the sum of negative exponents of β in the word c, we have $0 = S_c(\beta) = S_c^+(\beta) + S_c^-(\beta)$ (for example, if $c = \alpha_1\beta^{-1}\alpha_2\beta^{-1}\alpha_n^2\beta^2$, then $S_c^+(\beta) = 2$). If c does not contain β, we set $S_c^+(\beta) = 0$.

Let $\{A_m\}_{m\geq 1}$ be a sequence of pairwise disjoint intervals of X such that $|A_m| \geq m + 2S_c^+(\beta)$, $\forall m \geq 1$. Clearly this sequence is a pairwise disjoint Følner sequence for β.

Proposition C.5. *Let c be a cyclically reduced word as above. There exists $\alpha = (\alpha_1, \ldots, \alpha_n) \in (Sym(X))^n$ such that $\langle \alpha_n, \alpha_{n-1}, \ldots, \alpha_1, \beta\rangle$ is free of rank $n + 1$, and*

(1) the action of $\langle \alpha_n, \alpha_{n-1}, \ldots, \alpha_1, \beta\rangle$ on X is transitive and faithful;

(2) for all non trivial word w on $\{\alpha_n^{\pm 1}, \alpha_{n-1}^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$ with $w \notin \langle c\rangle$, there exist infinitely many $x \in X$ such that $cx = x$, $cwx = wx$ and $wx \neq x$;

(3) there exists a pairwise disjoint Følner sequence $\{A_k\}_{k\geq 1}$ *for* $\langle\alpha_n,\, \alpha_{n-1},\, \ldots,\, \alpha_1,\, \beta\rangle$ *which is fixed by c, and* $|A_k| = k$, $\forall k \geq 1$*;*

(4) for all $k \geq 1$*, there are infinitely many* $\langle c\rangle$*-orbits of size k;*

(5) every $\langle c\rangle$*-orbit is finite;*

(6) for every finite index subgroup H of $\langle\alpha_n,\, \alpha_{n-1},\, \ldots,\, \alpha_1,\, \beta\rangle$*, the H-action on X is transitive.*

With the notion of the permutation type, the conditions (4) and (5) mean that the word c has the permutation type $(\infty,\, \infty,\, \ldots,\, ;\, 0)$.

Proof. For the proof, we are going to exhibit six generic subsets of $(Sym(X))^n$ that will do the job.

We start by claiming that the set

$$\mathcal{U}_1 = \{\alpha = (\alpha_1, \ldots, \alpha_n) \in (Sym(X))^n \mid \forall k \in \mathbb{Z}\backslash\{0\}, \exists x \in X \text{ such that } c^k x \neq x\}$$

is generic in $(Sym(X))^n$. Indeed, for every $k \in \mathbb{Z} \setminus \{0\}$, let $\mathcal{V}_k = \{\alpha \in (Sym(X))^n | \forall x \in X,\ c^k x = x\}$. The set $\mathcal{V}_k$ is closed since if $\{\gamma_m\}_{m\geq 1}$ is a sequence in $\mathcal{V}_k$ converging to γ, then $c^k(\gamma_m)$ converges to $c^k(\gamma)$. To see the interior of $\mathcal{V}_k$ is empty, let $\alpha \in \mathcal{V}_k$ and let $F \subset X$ be a finite subset. There is an extension G_{n+1} of G_0 such that $P(c^k(\alpha'), v_0)$ embeds in G_{n+1} outside of F by Lemma C.2. So in particular there is $x \in X \setminus F$ such that $c^k(\alpha')x \neq x$, so $\alpha' \notin \mathcal{V}_k$. By defining $\alpha'|_F = \alpha|_F$, we have shown that $\mathcal{U}_1$ is generic in $(Sym(X))^n$.

Let us show that the set

$$\begin{aligned}\mathcal{U}_2 = \{\alpha = (\alpha_1, \ldots, \alpha_n) \in (Sym(X))^n| &\text{ for every } w \neq 1 \in \langle\alpha_n, \ldots, \alpha_1, \beta\rangle \setminus \langle c\rangle,\\ &\text{there exist infinitely many } x \in X \text{ such that } cx = x,\, cwx = wx \text{ and } wx \neq x\}\end{aligned}$$

is generic in $(Sym(X))^n$.

Indeed, for every non trivial word w in $\langle\alpha_n, \ldots, \alpha_1, \beta\rangle \setminus \langle c\rangle$, let $\mathcal{V}_w = \{\alpha \in (Sym(X))^n|$ there exists a finite subset $K \subset X$ such that $(\mathrm{Fix}(c) \cap w^{-1}\mathrm{Fix}(c) \cap \mathrm{supp}(w)) \subset K\} = \bigcup_{K \text{finite} \subset X} \{\alpha \in (Sym(X))^n | (\mathrm{Fix}(c) \cap w^{-1}\mathrm{Fix}(c) \cap \mathrm{supp}(w)) \subset K\}$. We shall show that the set $\mathcal{V}_w$ is meagre. It is an easy exercise to show that the set

$$\mathcal{V}_{w,K} = \{\alpha \in (Sym(X))^n | (\mathrm{Fix}(c) \cap w^{-1}\mathrm{Fix}(c) \cap \mathrm{supp}(w)) \subset K\}$$

is closed. To show that the interior of $\mathcal{V}_{w,K}$ is empty, let $\alpha \in \mathcal{V}_{w,K}$, and $F \subset X$ be a finite subset. We need to prove that for some α' defined as $\alpha'|_F = \alpha|_F$, we can extend the definition of α' outside of the finite subset such that $\alpha' \notin \mathcal{V}_{w,K}$. By Lemma C.4, we can take an extension G_{n+1} of G_0 such that $Q(c(\alpha'), w)$ embeds in G_{n+1} outside of $F \cup \alpha(F) \cup K$, which proves the genericity of $\mathcal{U}_2$.

Now let us show that the set

$\mathcal{U}_3 = \{\alpha = (\alpha_1, \ldots, \alpha_n) \in (Sym(X))^n |$ there exists $\{A_{m_k}\}_{k\geq 1}$ a subsequence of $\{A_m\}_{m\geq 1}$ such that $A_{m_k} \subset \mathrm{Fix}(\alpha_i)$, $\forall k \geq 1$, $\forall 1 \leq i \leq n\}$

is generic in $(Sym(X))^n$.

Indeed, the set $\mathcal{U}_3$ can be written as $\mathcal{U}_3 = \bigcap_{N\geq 1}\{\alpha = (\alpha_1, \ldots, \alpha_n) \in (Sym(X))^n | \exists k \geq N$ such that $A_k \subset \mathrm{Fix}(\alpha_i)$, $\forall i\}$. We claim that for every $N \geq 1$, the set $\mathcal{V}_N = \{\alpha \in (Sym(X))^n | \forall k \geq N,\ A_k \subsetneq \cap_i \mathrm{Fix}(\alpha_i)\}$ is closed and of empty interior. It is closed since $\mathcal{V}_N = \bigcap_{k\geq N}\{\alpha \in (Sym(X))^n | A_k \subsetneq \cap_i \mathrm{Fix}(\alpha_i)\}$ and the set $\{\alpha \in (Sym(X))^n | A_k \subsetneq \cap_i \mathrm{Fix}(\alpha_i)\}$ is clearly closed. For the emptiness of its interior, let $\alpha \in \mathcal{V}_N$ and let $F \subset X$ be a finite subset. Let $k \geq N$ such that $A_k \cap (F \cup \alpha(F)) = \emptyset$. We can then take $\alpha' \in (Sym(X))^n$ fixing A_k and satisfying $\alpha'|_F = \alpha|_F$.

For (4), we show that the set

$\mathcal{U}_4 = \{\alpha = (\alpha_1, \ldots, \alpha_n) \in (Sym(X))^n | \forall m$, there exist infinitely many $\langle c\rangle$-orbits of size m $\}$

is generic in $(Sym(X))^n$.

For all $m \geq 1$, let $\mathcal{V}_m = \{\alpha \in (Sym(X))^n |$ there exists a finite subset $K \subset X$ such that every $\langle c\rangle$-orbit of size m is contained in $K\} = \bigcup_{K \text{ finite}\subset X} \mathcal{V}_{m,K}$, where

$$\mathcal{V}_{m,K} = \{\alpha \in (Sym(X))^n | \text{ if } |\langle c\rangle \cdot x| = m, \text{ then } \langle c\rangle \cdot x \subset K\}.$$

$\cdot$ *$\mathcal{V}_{m,K}$ is of empty interior.* Let $F \subset X$ be a finite subset. Let $\alpha \in \mathcal{V}_{m,K}$. Take $x \notin (F \cup \alpha(F)) \cup K$. Since c contains α_i for some i, we can construct a cycle $c^m(\alpha')$ outside of $F \cup \alpha(F) \cup K$ such that $\alpha'|_F = \alpha|_F$ (Lemma C.3), so that the orbit of x under α' is of size m and not contained in K.
$\cdot$ *$\mathcal{V}_{m,K}$ is closed.* Let $\{\gamma_l\}_{l\geq 1} \subset \mathcal{V}_{m,K}$ converging to $\gamma \in (Sym(X))^n$. Let $x \in X$ such that $|\langle c(\gamma)\rangle \cdot x| = m$. Since γ_l converges to γ, $c(\gamma_l)$ converges to $c(\gamma)$. Since $\langle c(\gamma)\rangle \cdot x$ is finite, there exists l_0 such that $\langle c(\gamma)\rangle \cdot x = \langle c(\gamma_l)\rangle \cdot x$, $\forall l \geq l_0$. Since $\gamma_l \in \mathcal{V}_{m,K}$ and $m = |\langle c(\gamma)\rangle \cdot x| = |\langle c(\gamma_l)\rangle \cdot x|$, we have $\langle c(\gamma_l)\rangle \cdot x \subset K$, $\forall l \geq l_0$. Therefore $\langle c(\gamma)\rangle \cdot x \subset K$, so that $\gamma \in \mathcal{V}_{m,K}$.

About (5), we prove that the set

$$\mathcal{U}_5 = \{\alpha = (\alpha_1, \ldots, \alpha_n) \in (Sym(X))^n | \forall x \in X,\ \langle c\rangle \cdot x \text{ is finite }\}$$

is generic in $(Sym(X))^n$.

For all $x \in X$, let $\mathcal{V}_x = \{\alpha \in (Sym(X))^n | \langle c\rangle \cdot x$ is infinite $\}$. It is clear that the set $\mathcal{V}_x$ is closed. To see that the interior of $\mathcal{V}_x$ is empty, let $F \subset X$ be a finite subset and let $\alpha \in \mathcal{V}_x$. We shall show that there exists $\alpha' \notin \mathcal{V}_x$ such that $\alpha|_F = \alpha'|_F$. Denote $c = c(\alpha)$ and $c' = c(\alpha')$. We choose $p >> 1$ large enough so that

$$\begin{cases} \left(B(c^{-p-1}x, |c|) \cup B(c^{p+1}x, |c|)\right) \cap (F \cup \alpha(F)) = \emptyset; \\ (F \cup \alpha(F)) \subset B(x, |c^p|), \end{cases}$$

where $|c|$ is the length of c and $B(x, r)$ is the ball centered on x with the radius r.

We construct a path of c' outside of $B(x, |c^p|)$ starting from $c^{p+1}x$ which ends on $c^{-p-1}x$, i.e. $c'(c^{p+1}x) = c^{-p-1}x$. This is possible since c' contains α_i for some i (Lemma C.2). On the points in $B(x, |c^{p+1}|)$, we define

$$\alpha'|_{B(x,|c^{p+1}|)} = \alpha|_{B(x,|c^{p+1}|)}.$$

In particular, $\alpha'|_F = \alpha|_F$, and $|\langle c'\rangle \cdot x|$ is finite.

Finally for (6), let

$$\mathcal{U}_6 = \{\alpha = (\alpha_n, \dots, \alpha_1) \in (Sym(X))^n |\text{ for every finite index subgroup } H \text{ of } \langle \alpha_1, \beta\rangle, \text{ the } H\text{-action on } X \text{ is transitive } \}.$$

By Proposition 4 in [56], the set $\mathcal{W} = \{\alpha_1 \in Sym(X)|$ for every finite index subgroup H of $\langle\alpha_1, \beta\rangle$, the H-action on X is transitive $\}$ is generic in $Sym(X)$. Thus $\mathcal{U}_6$ is generic in $(Sym(X))^n$ since $\mathcal{U}_6 = \mathcal{W} \times (Sym(X))^{n-1}$.

Now let $\alpha = (\alpha_1, \dots, \alpha_n) \in \cap_{i=1}^6 \mathcal{U}_i$. It remains us to prove (3) and (6) in the Proposition. To simplify the notation, let $A_m := A_{m_k}$ be the subsequence of A_m fixed by α_i, $\forall 1 \le i \le n$ (genericity of $\mathcal{U}_3$).

Without loss of generality, let $c = w_1\beta^{b_1}w_2\beta^{b_2}\cdots w_l\beta^{b_l}$, where w_j are reduced words on $\{\alpha_n^{\pm1}, \dots, \alpha_1^{\pm1}\}$, $\forall 1 \le j \le l$. Recall that $\{A_m\}_{m\ge1}$ is a sequence of pairwise disjoint intervals such that $|A_m| \ge m + 2S_c^+(\beta)$. If c does not contain β, then we can take the subinterval A'_m of A_m such that $|A'_m| = m$ for the Følner sequence which is fixed by c. If not, for all $m > S_c^+(\beta)$, let

$$E_m = \beta^{b_1}(A_m) \cap \beta^{b_2+b_1}(A_m) \cap \cdots \cap \beta^{b_{l-1}+b_{l-2}+\cdots+b_1}(A_m) \cap \beta^{b_l+b_{l-1}+\cdots+b_1}(A_m).$$

Notice that $\beta^{b_l+b_{l-1}+\cdots+b_1}(A_m) = A_m$. We claim that the set E_m is not empty. Indeed, for every $1 \le i \le l$, the set

$$\beta^{b_i+b_{i-1}+\cdots+b_1}(A_m) \cap \beta^{b_p+b_{p-1}+\cdots+b_1}(A_m)$$

is not empty, $\forall 1 \le p \le i-1$ since $|b_i + b_{i-1} + \cdots + b_{p+1}| \le S_c^+(\beta) < |A_m|$. Moreover, a family of intervals which meet pairwise, has non-empty intersection so that $E_m \ne \emptyset$.

In addition, let us show that c fixes the elements of E_m. Let $x \in E_m$ and let $1 \le p \le l-1$. There exists $a_{l-p+1} \in A_m$ such that $x = \beta^{b_{l-p}+b_{l-p-1}+\cdots+b_1}(a_{l-p+1})$. Then

$$\begin{aligned}
\beta^{b_{l-p+1}+\cdots+b_{l-1}+b_l}(x) &= \beta^{b_l+b_{l-1}+\cdots+b_{l-p+1}}(x)\\
&= \beta^{b_l+b_{l-1}+\cdots+b_{l-p+1}} \cdot \beta^{b_{l-p}+b_{l-p-1}+\cdots+b_1}(a_{l-p+1})\\
&= a_{l-p+1} \in A_m.
\end{aligned}$$

Since w_j fixes every element in A_m, and the element $\beta^{b_{l-p+1}+\cdots+b_{l-1}+b_l}(x)$ is in A_m for every $1 \le p \le l-1$, the word c fixes x, $\forall x \in E_m$. Clearly the set E_m is a Følner sequence for $\langle\alpha_n, \alpha_{n-1}, \dots, \alpha_1, \beta\rangle$.

Furthermore, we have

$$A_m \cap \beta^{S_c^+(\beta)} A_m \cap \beta^{S_c^-(\beta)} A_m \subseteq E_m,$$

and

$$|A_m \cap \beta^{S_c^+(\beta)} A_m \cap \beta^{S_c^-(\beta)} A_m| = |A_m| - 2S_c^+(\beta) \geq m.$$

So $|E_m| \geq m$, and upon replacing E_m by a subinterval E'_m of E_m such that $|E'_m| = m$, we can suppose that $|E_m| = m$, $\forall m \geq 1$. Thus the sequence $\{E_m\}_{m\geq 1}$ is a Følner sequence satisfying the condition in (3) in the Proposition C.5.

Furthermore, if H is a finite index subgroup of $\langle \alpha_n, \ldots, \alpha_1, \beta \rangle$, then $Q = H \cap \langle \alpha_1, \beta \rangle$ is a finite index subgroup of $\langle \alpha_1, \beta \rangle$, so by the genericity of $\mathcal{U}_6$ the Q-action is transitive and therefore the H-action on X is transitive.

□

C.4 Construction of $\mathbb{F}_{n+1} *_{\mathbb{Z}} \mathbb{F}_{m+1}$-actions, $n, m \geq 1$

Let X be an infinite countable set. Let $G = \langle \alpha_n,\ \alpha_{n-1},\ \ldots,\ \alpha_1,\ \beta \rangle \curvearrowright X$ be the group action constructed as in Proposition C.5 with the pairwise disjoint Følner sequence $\{A_k\}_{k\geq 1}$. For $m \geq 1$, let d be a cyclically reduced word on $\{\alpha_m,\ \alpha_{m-1},\ \ldots,\ \alpha_1,\ \beta\}$ such that $S_d(\beta) = 0$ and d contains α_j for some j. Let $H = \langle \alpha_m,\ \alpha_{m-1},\ \ldots,\ \alpha_1,\ \beta \rangle \curvearrowright X$ be the group action constructed as in Proposition C.5 with the pairwise disjoint Følner sequence $\{B_k\}_{k\geq 1}$. Let $Z = \{\sigma \in Sym(X) | \sigma c = d\sigma\}$. By virtue of the points (4) and (5) of Proposition C.5, the set Z is not empty. Let

$$H^\sigma = \sigma^{-1} H \sigma = \langle \sigma^{-1}\alpha_m\sigma, \sigma^{-1}\alpha_{m-1}\sigma, \ldots, \sigma^{-1}\alpha_1\sigma, \sigma^{-1}\beta\sigma \rangle.$$

For $\sigma \in Z$, consider the amalgamated free product $G *_{\langle c=d \rangle} H^\sigma$ of G and H^σ along $\langle c = d \rangle$. The action of $G *_{\langle c=d \rangle} H^\sigma$ on X is given by $g \cdot x = gx$, and $h \cdot x = \sigma^{-1} h \sigma x$, $\forall g \in G$ and $\forall h \in H$.

Notice that the set Z is closed in $Sym(X)$. In particular, Z is a Baire space.

Proposition C.6. *The set*

$$\mathcal{O}_1 = \{\sigma \in Z \mid \text{ the action of } G *_{\langle c=d \rangle} H^\sigma \text{ on } X \text{ is faithful } \}$$

is generic in Z.

Proof. For every non trivial word $w \in G *_{\langle c=d \rangle} H^\sigma$, let us show that the set

$$\mathcal{V}_w = \{\sigma \in Z | \forall x \in X, w^\sigma x = x\}$$

is closed and of empty interior. It is obvious that the set $\mathcal{V}_w$ is closed. To prove that the set $\mathcal{V}_w$ is of empty interior, let us treat the case where $w = a g_n h_n \cdots g_1 h_1$ with $a \in \langle c \rangle$, $g_i \in G \backslash \langle c \rangle$, and $h_i \in H \backslash \langle d \rangle$, $n \geq 1$. The corresponding element of $Sym(X)$ given by the action is $w^\sigma = a g_n \sigma^{-1} h_n \sigma \cdots g_1 \sigma^{-1} h_1 \sigma$.

Let $\sigma \in \mathcal{V}_w$. Let $F \subset X$ be a finite subset. We shall show that there exists $\sigma' \in Z \setminus \mathcal{V}_w$ such that $\sigma'|_F = \sigma|_F$. For all $g \in G \setminus \langle c\rangle$ and $h \in H \setminus \langle d\rangle$, let

$$\widehat{g} = \{x \in X \mid cx = x,\, cgx = gx \text{ and } gx \neq x\ \},$$

$$\widehat{h} = \{x \in X \mid dx = x,\, dhx = hx \text{ and } hx \neq x\ \}.$$

By (2) of Proposition C.5, these sets are infinite.

Choose any $x_0 \in \mathrm{Fix}(c) \setminus (F \cup \sigma(F))$. By induction on $1 \leq i \leq n$, we choose $x_{4i-3} \in \widehat{h_i}$ such that x_{4i-3}, $h_i x_{4i-3} \notin (F \cup \sigma(F))$ are new points. This is possible since $\widehat{h_i}$ is infinite. Then we define

$$\sigma'(x_{4i-4}) := x_{4i-3} \text{ and } \sigma'(\sigma^{-1}(x_{4i-3})) := \sigma(x_{4i-4}).$$

We set $x_{4i-2} := h_i x_{4i-3}$, which is different from x_{4i-3} and which is fixed by d, by definition of $\widehat{h_i}$. We choose $x_{4i-1} \in \widehat{g_i}$ such that x_{4i-1}, $g_i x_{4i-1} \notin (F \cup \sigma(F))$ are again new points. This is again possible since $\widehat{g_i}$ is infinite. Then we define

$$\sigma'(x_{4i-1}) := x_{4i-2} \text{ and } \sigma'(\sigma^{-1}(x_{4i-2})) := \sigma(x_{4i-1}).$$

We finally set $x_{4i} := g_i x_{4i-1}$. Then every point x on which σ' is defined verifies $\sigma' c(x) = d\sigma'(x)$. Indeed,

- $\sigma' c(x_{4i-4}) = \sigma'(x_{4i-4}) = x_{4i-3} = d(x_{4i-3}) = d\sigma'(x_{4i-4})$ since $x_{4i-4} \in \mathrm{Fix}(c)$ and $x_{4i-3} \in \mathrm{Fix}(d)$;
- $\sigma' c(\sigma^{-1}(x_{4i-3})) = \sigma'(\sigma^{-1}(x_{4i-3})) = \sigma(x_{4i-4}) = d\sigma(x_{4i-4}) = d\sigma'(\sigma^{-1}(x_{4i-3}))$ since $\sigma^{-1}(x_{4i-3}) \in \mathrm{Fix}(c)$ and $\sigma(x_{4i-4}) \in \mathrm{Fix}(d)$ because $\sigma \in Z$;
- $\sigma' c(x_{4i-1}) = \sigma'(x_{4i-1}) = x_{4i-2} = d(x_{4i-2}) = d\sigma'(x_{4i-1})$ since $x_{4i-2} \in \mathrm{Fix}(d)$ and $x_{4i-1} \in \mathrm{Fix}(c)$;
- $\sigma' c(\sigma^{-1}(x_{4i-2})) = \sigma'(\sigma^{-1}(x_{4i-2})) = \sigma(x_{4i-1}) = d\sigma(x_{4i-1}) = d\sigma'(\sigma^{-1}(x_{4i-2}))$ since $\sigma^{-1}(x_{4i-2}) \in \mathrm{Fix}(c)$ and $\sigma(x_{4i-1}) \in \mathrm{Fix}(d)$ because $\sigma \in Z$.

By construction, the $4n$ points defined by the subwords on the right of $w^{\sigma'}$ are all distinct. In particular, $w^{\sigma'} x_0 = x_{4n} \neq x_0$. If $w = h \in H \setminus \{\mathrm{Id}\}$, choose $x_0 \in \mathrm{Fix}(c) \setminus (F \cup \sigma(F))$, $x_1 \in \hat{h} \setminus (F \cup \sigma(F) \cup \{x_0\})$, $x_2 \in \mathrm{Fix}(c) \setminus (F \cup \sigma(F) \cup \{x_0, x_1\})$ and define $\sigma'(x_0) = x_1$, $\sigma'(x_2) = hx_1$, $\sigma'(\sigma^{-1}(x_1)) = \sigma(x_0)$, $\sigma'(\sigma^{-1}(hx_1)) = \sigma(x_2)$ so that $w^{\sigma'} x_0 = x_2 \neq x_0$. At last, if $w = g \in G \setminus \{\mathrm{Id}\}$, then there exists $x \in X$ such that $gx \neq x$ since G acts faithfully on X. For all other points, we define σ' to be equal to σ. Therefore, σ' constructed in this way is in $Z \setminus \mathcal{V}_w$ and $\sigma'|_F = \sigma|_F$. □

Proposition C.7. *The set*

$$\mathcal{O}_2 = \{\sigma \in Z \mid \exists\ \{k_l\}_{l \geq 1} \text{ a subsequence of } k \text{ such that } \sigma(A_{k_l}) = B_{k_l},\ \forall l \geq 1\ \}$$

is generic in Z.

Proof. Let us write $\mathcal{O}_2 = \bigcap_{N\in\mathbb{N}}\{\sigma \in Z |$ there exists $n \geq N$ such that $\sigma(A_n) = B_n\}$. We need to show that for all $N \in \mathbb{N}$, the set $\mathcal{V}_N = \{\sigma \in Z | \forall n \geq N, \sigma(A_n) \neq B_n\}$ is closed and of empty interior.
· *$\mathcal{V}_N$ is of empty interior.* Let $\sigma \in \mathcal{V}_N$. Let $F \subset X$ be a finite subset. Let $n \geq N$ large enough so that $A_n \cap (F \cup \sigma(F)) = \emptyset$ and $B_n \cap (F \cup \sigma(F)) = \emptyset$. This is possible since the sets $\{A_n\}$ (respectively the sets $\{B_n\}$) are pairwise disjoint. Let $A_n = \{a_1, \ldots, a_n\}$ and $B_n = \{b_1, \ldots, b_n\}$. We define $\sigma'(a_i) = b_i$ and $\sigma'(\sigma^{-1}(b_i)) = \sigma(a_i)$, $\forall i$, which is well defined because $a_i \in \mathrm{Fix}(c)$ and $b_i \in \mathrm{Fix}(d)$. For all other points, we define σ' to be equal to σ. Therefore, $\sigma' \in Z \setminus \mathcal{V}_N$ and $\sigma'|_F = \sigma|_F$.
· *$\mathcal{V}_N$ is closed.* We have $\mathcal{V}_N = \bigcap_{n\geq N} \mathcal{W}_n$, where $\mathcal{W}_n = \{\sigma \in Z | \sigma(A_n) \neq B_n\}$. So the set $\mathcal{V}_N$ is closed being the intersection of closed sets. □

Let $\sigma \in \mathcal{O}_1 \cap \mathcal{O}_2$. We claim that $\{A_{k_l}\}_{l\geq 1}$ is a Følner sequence for $G *_{\langle c=d\rangle} H^\sigma$. Indeed, $\{A_{k_l}\}$ is Følner for G, and for all $h \in H$, we have

$$\begin{aligned}\lim_{l\to\infty} \frac{|A_{k_l} \vartriangle h \cdot A_{k_l}|}{|A_{k_l}|} &= \lim_{l\to\infty} \frac{|A_{k_l} \vartriangle \sigma^{-1}h\sigma A_{k_l}|}{|A_{k_l}|} = \lim_{l\to\infty} \frac{|\sigma A_{k_l} \vartriangle h\sigma A_{k_l}|}{|A_{k_l}|} \\ &= \lim_{l\to\infty} \frac{|B_{k_l} \vartriangle hB_{k_l}|}{|B_{k_l}|} = 0,\end{aligned}$$

since $\{B_{k_l}\}$ is Følner for H, $\sigma(A_{k_l}) = B_{k_l}$ and $|A_{k_l}| = |B_{k_l}|$, for all $l \geq 1$.

Furthermore, if H is a finite index subgroup of $\mathbb{F}_{n+1} *_{\langle c=d\rangle} \mathbb{F}_{m+1}$, since every finite index subgroup of $\mathbb{F}_{n+1}$ acts transitively on X, *a fortiori* the H-action on X is transitive.

Therefore, we have:

Theorem C.8. *1. There exists a transitive, faithful and amenable action of the group $\mathbb{F}_{n+1} *_{\langle c=d\rangle} \mathbb{F}_{m+1}$ on X, where $c \in \mathbb{F}_{n+1}$ (respectively $d \in \mathbb{F}_{m+1}$) is a cyclically reduced non-primitive word such that the exponent sum of some generator occurring in c (respectively d) is zero.*

2. Every finite index subgroup of such a group admits transitive, faithful and amenable action on X.

The complete proof of Theorem C.1 is achieved from the following Lemma:

Lemma C.9. *If c is a reduced word in $\mathbb{F}_n$, then there exists an automorphism ϕ of $\mathbb{F}_n$ such that the exponent sum of some generator occurring in $\phi(c)$ is zero.*

Proof. Since there is an epimorphism $\pi : \mathrm{Aut}(\mathbb{F}_n) \twoheadrightarrow \mathrm{Aut}(\mathbb{Z}^n) \simeq GL_n(\mathbb{Z})$, it is enough to find a matrix $M \in GL_n(\mathbb{Z})$ such that the exponent sum $S_{\phi(c)}(t)$ of exponents of some generator t in the word $\phi(c)$ is zero, where $\phi \in \mathrm{Aut}(\mathbb{F}_n)$ is such that $\pi(\phi) = M \in GL_n(\mathbb{Z})$. Denote by $t_1, \ldots, t_n$ the generators of $\mathbb{F}_n$ such that $S_c(t_i) \neq 0$, $\forall 1 \leq i \leq n$. Let $m := lcm(S_c(t_1), S_c(t_2))$ be the least common multiple of $S_c(t_1)$ and $S_c(t_2)$. Then there exist m_1 and m_2 such that $m = m_1 S_c(t_1)$ and $m = m_2 S_c(t_2)$ so that $m_1 S_c(t_1) - m_2 S_c(t_2) = 0$. Moreover, the greatest common divisor $gcd(m_1, m_2)$ of m_1 and m_2 is 1, so by

Bézout's identity, there exist a and b such that $m_1 a + m_2 b = 1$. So by letting $s := bS_c(t_1) + aS_c(t_2)$, the matrix

$$\begin{pmatrix} m_1 & -m_2 & & & \\ b & a & & \text{\Large 0} & \\ & & 1 & & \\ & \text{\Large 0} & & \ddots & \\ & & & & 1 \end{pmatrix}$$

is in $GL_n(\mathbb{Z})$ and it sends $(S_c(t_1), S_c(t_2), \ldots, S_c(t_n))^t$ to $(0, s, \ldots, S_c(t_n))^t$. $\square$

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