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Variations of Frege's Grundgesetze

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Abstract

We retain the Grundgesetz V, but replace impredicative comprehension with certain ramified comprehension principles.

1 Introduction

If f is a unary function we write $x.f(x)$ for its Werthverlauf, which is a Gegenstand (object). In this notation, Frege's Grundgesetz V is

$$(GG\,V):\ \ x.f(x) = y.g(y) \ \leftrightarrow\ \forall z\,(\,f(z) = g(z)\,)$$

The operation $f \mapsto x.f(x)$ is according to the left-to-right direction ($GG\,V\ \ \rightarrow$) of the Grundgesetz ($GG\,V$) an injection from the class of functions into the class of objects. Therefore, if we have at least two objects and enough functions in the domain of function quantifiers, we get a contradiction, as was shown by Cantor around 1874. Frege (perhaps) learned that Cantorian cause of the inconsistency of his *Grundgesetze der Arithmetik*

(GGA) [5] from Russell in 1902, although the Nachwort to [5] 1903 indicates that Frege did not completely get the point.

Thus the question arises: what should be done with Frege's *GGA*? The best advice is to throw *GGA* out of the window since all what Frege intended to do with *GGA*, can surely be done by other systems which are more elegant and probably consistent. More precisely, the kind of mathematics Frege wanted to capture in *GGA*, or *begründen* (as the Germans like to say), does not transcend a certain version of PA^3, i.e. third-order Peano Arithmetic.

However, there remain several things to say about the relation $x.f(x) = a$ itself which was operative in forming Frege's ideas about the mathematical universe, ideas which are, at least according to standard conceptions, quite incorrect; the rest of *GGA* is just a two-sorted quantification theory.

The relation $x.f(x) = a$ may be read as: the object a *represents* the function f, or as: the function f *reduces or is contracted to* the object a. That is to say, a thing of a higher type, like the function f is contracted to a thing of a lower type, viz. the object a. We shall use the term *contraction*, because it is vague and unspecific enough. For Frege, functions and especially his *Begriffe*, seemed to belong to his famous Realm of the Objective Unreal, whereas the Werthverlauf $x.f(x)$ of the function f was perhaps imagined by Frege, at least in a case like $\lambda x.x^2 : [0,1] \to [0,1]$, as the graph of f which can be written down on the blackboard as an object consisting of chalk particles and should thus belongs to the Realm of the Objective Real.

With hindsight we can say that both Frege and Russell had a *type theory*; both wanted to found mathematics (or at least large parts of it) on a so-called *logic of types*. While Frege had just two types, objects and functions, his logic of types aimed to create mathematical strength by contracting functions from objects to objects — to objects, in some reversible or reconstructible, perhaps injective way. That's precisely the task of $(GG\ V)$.

Russell, on the other hand, tried to obtain mathematical strength by introducing more and more types, e.g. after the type 0, the type of objects, for each type n the type $n+1$ of predicates of things of type n. Because of some type-theoretic *horror infiniti*, he did not want to have infinite ordinals ω, $\omega+1$, $\omega^2, \ldots$ as types.

This is all that I want to say about the *difference* between Fregean and Russellian logicism. Observe, that the types 0,1,2,3 are more than enough for ordinary mathematics *provided* one has an axiom of infinity and some not too weak principles of comprehension. There are at least two possibilities, viz. in my papers [3] and [4], respectively, to give something like a *purely logical proof* of the axiom of infinity.

Therefore, I think that Russell's logicism can be and has been vindicated.

We shall now go back to Frege's logicism and explore possible vindications of it. Perhaps Frege's logicism is already vindicated by replacing $(GG\ V)$ with Hume's Principle for cardinalities. Here too we have a kind of contraction relation: a concept F is contracted to its cardinality $\#_x F(x)$, i.e. $\#_x F(x) = c$, where c is an object.

$$(\textit{hume card}): \quad \#_x F(x) = \#_y F(y) \leftrightarrow F \textit{ equipollent to } G$$

The consistency, mathematical strength, and elegance of Hume's Principle for cardinalities are admirable facts which should be elaborated further. For instance, one should work out subsystems, e.g. predicative ones, of the system with (*hume* *card*). By the way, $\forall y \exists F : \#_x F(x) = y$ can be consistently added. What about third-order extensions[1], etc? Moreover, one should consider versions of Hume's Principle for cardinals in the context of set theory, e.g.

$$(set\ hume[A]) :$$
$$\exists H : P(A)\backslash\{\emptyset\} \to A\, \forall X, Y\ [\ H(X) = H(Y) \leftrightarrow X\, equip.\, Y\ ]$$

For instance, $ZFC \vdash \forall Z\, (set\ hume[Z])$, but on the other hand $ZF \nvdash \forall Z\, (set\ hume[Z])$ (consider amorphous sets).

I have played around with a Hume's Principle for ordinal numbers, without any success.

Observe, that Frege defined the humean contraction relation $\#_x F(x) = c$ in terms of his primitive and fundamental contraction relation $x.f(x) = a$. Thus we go back to $x.f(x) = a$.

First we generalize (or weaken) the idea inherent in $x.f(x) = a$ as follows. Our formal language will be syntactically simpler than Frege's but of equal expressiveness.

We let capital letters $A, B, \ldots$ (*free*) and $X, Y, \ldots$ (*bound*) range over *predicates* (or notions or classes or properties or whatever you may call these things); and small letters $a, b, \ldots$, $x, y, \ldots$ range over (so-called) individuals, or *objects*. Our primitive relations are (1) $B(a)$ [*predication*] with the usual intended meaning, and $A \searrow b$ [*contraction*] whose meaning is yet to be found. We only know that $A \searrow b$ should mean that the predicate A reduces or is contracted to the object b. Although equality between individuals could be defined (e.g. by $a = b\ :=\ \forall X(X(a) \to X(b))$, we adopt also the primitive relation $a = b$.

It is a sad fact that we have no clear ideas at all about

[1] Here there is a "plausible" version which implies $\neg CH$

possible axioms for the relation $\searrow$. We have to make experiments and look for the results. Let me illustrate our situation by considering the following axiom

$$(sub) :: \quad \exists x : A \searrow x \wedge \forall z(B(z) \rightarrow A(z)) \rightarrow \exists y : B \searrow y$$

Is (sub) a plausible axiom? Suppose that A is a recursive set of natural numbers, and A can be contracted to a natural number x (which codes a definition of A). But if the subset B of A is not even hyperarithmetic, why should B be contractible to any number y at all? Moreover, if B contracts to the number y, then not necessarily by the same mechanism that contracted A to the number x.

Two further question. (1) If not every predicate can be contracted, then which predicates can? (2) How much of the logical (internal) structure of a predicate is permitted to vanish during contraction?

2 On the real nature of Russell's paradox in *GGA*

I will now turn to an account of Russell's paradox in *GGA* which may be not wholly faithful to what really happened when Russell derived his paradox from *GGA*. But I will show that it should have been the way I shall describe. In particular, I sacrifice historic faithfulness in order to better discuss both the *contraction* relation $A \searrow b$ and the issue of *predicativity*.

Frege's $(GG\ V)$ reads in our notation:

$$(gg\ 5) \quad \exists x\,(\,A \searrow x \wedge B \searrow x\,) \leftrightarrow \forall x\,(\,A(x) \leftrightarrow B(x)\,)$$

Together with (*gg* 5), the following *impredicative* case of comprehension engenders a contradiction, using in addition only pure logic.

$$(russell) \quad \exists Y \forall x \, (Y(x) \leftrightarrow \neg \exists Z \, (Z \searrow x \wedge Z(x)))$$

Now, we can see that (*gg* 5) and (*russell*) are consistent separately. To repeat: as an isolated formula of our Fregelike language, Frege's (*GG V*) is consistent; and, what is perhaps also unknown, Russell's paradoxical formula, again as the isolated formula (*russell*) of our Fregelike language, is equally consistent: interpret it simply as

$$\exists y \forall x \, (x \in y \leftrightarrow \neg \exists z \, (x \in z \wedge x \in z))$$

Of course, the set y is the empty set $\emptyset$.

Thus, the formula (*russell*) *is definitely not* the (trivially inconsistent) formula $\exists y \forall x \, (x \in y \leftrightarrow \neg x \in x)$.

And the formula (*gg* 5) transforms into the satisfiable formula $\exists x(a = x = b) \leftrightarrow \forall x(a = x \leftrightarrow b = x)$.

Digression. A not wholly trivial but purely logical derivation of a contradiction from (*gg* 5) and (*russell*) runs as follows; it uses only intuitionistic laws [2]:

Using (*russell*), let A be such that

(1) $\forall x \, (A(x) \leftrightarrow \forall Z \neg (Z \searrow x \wedge Z(x)))$.

But $\forall x \, (A(x) \leftrightarrow A(x))$ is logically true; by (*gg* 5) there is an a such that $A \searrow a$. Using (1) we have:

[2] This is of course no heroic deed, since if $\neg(\varphi \wedge \psi)$ is classically provable, then also intuitionistically.

$A(a) \to \forall Z \neg (Z \searrow a \wedge Z(a)) \to \neg (A \searrow a \wedge A(a))$.
But we have also $A(a) \to (A \searrow a \wedge A(a))$. Hence $\neg A(a)$. Using (gg 5) we get $A \searrow a \wedge B \searrow a \wedge B(a). \to .A(a)$. Hence by pure logic we get also $\neg A(a) \wedge A \searrow a. \to .\forall Z \neg (Z \searrow a \wedge Z(a))$.

Applying (1) once more we have $\neg A(a) \wedge A \searrow a. \to .A(a)$. Then $A(a)$ follows by cuts. Thus we have derived both $\neg A(a)$ and $A(a)$.

We add the remark that we have not used the fact that $\searrow$ is total in the sense that $\forall X \, \exists z : X \searrow z$, whereas $\forall f \, \exists z : x.f(x) = z$ follows logically from $x.f(x) = x.f(x)$.

The main steps in our argument are buried in the Frege's book *GGA* vol. I, up to p. 75; these steps are used there to derive the general comprehension principle $a \in \{x : F(x)\} \leftrightarrow F(a)$, which Russell could use immediately for his paradox. It was thus Frege himself who did the lion's share in the derivation of Russell's paradox in *GGA* via the just given proof.
End of Digression.

In 1983 I defined in [1] a system called *praeKid* which has (besides some equality axioms) the axioms $\forall X \, \exists z : X \searrow z$, and ($gg$ 5), and the *strictly predicative* comprehension schema:

$$(praecomp) \quad : \quad \exists Y \forall x \, (Y(x) \leftrightarrow \mathcal{F}[x])$$
$\mathcal{F}$ has no bound predicate variables

In *praeKid*, one can deduce besides some simple set-combinatorial laws eight of the nine axioms of Hailperin for Quine's NF, e.g. by using the definition

$$[Def \ \in] \quad a \in b \ := \ \exists X \, (X \searrow b \wedge X(a))$$

for the (type-homogeneous) membership relation *which makes* *the individuals to sets*; this is incidentally Frege's own definition of the membership relation. Recall that the formula-part $\neg\exists Z\,(\,Z \searrow x \wedge Z(x)\,)$ in (*russell*) is really nothing else but $\neg x \in x$.

Another definition of the membership relation would be

$$[DEF\ \in]\quad A \in B\ :=\ \exists x\,(\,A \searrow x \wedge B(x)\,)$$

which tries to *make the predicates into sets.* One gets the same theorems in *praeKid* under both definitions of the membership relation.

Unfortunately, I could not prove the consistency of *praeKid* yet; but I believe that I am missing just a trivial trick. Since *praeKid* proves

$$\forall u\,(\,\neg\exists Z(Z \searrow u \wedge Z(u)) \leftrightarrow \exists Z(Z \searrow u \wedge \neg Z(u))),$$

not even Δ^1_1-comprehension is consistent. This phenomenon is important with respect to the (to my mind) not yet fully elucidated status of predicativity. In the usual second-order arithmetic, Δ^1_1 is the limit of predicativity, although such comprehensions are already slightly impredicative. In *praeKid* the limit of predicativity is much lower, viz. Δ^0_∞-comprehension. Regarding first-order arithmetic under the standard definitions of $0, S, +, \times$, one gets in *praeKid* certain rich extensions of Robinson Arithmetic; the amount of induction we can get I have not yet determined.

Suppose that the principles of *praeKid* are *sound and even true*, then we have sound and true principles *that are contradictory with rather weak forms of impredicativity.* Within second-order arithmetic there are no sound principles which would, e.g., contradict Π^1_2-comprehension. Therefore, to make consistency proofs easier (more constructive) seems to be at present the only advantage of predicativity.

Notes. (1) The precise form of *praeKid* can be recaptured from the system $GGA[\Delta]$ below. — (2) By the way, if one *defines* a type-homogeneous (!) membership relation by making reference to more than one type instead of taking it as primitive, what has been done in the definitions [$Def\ in$] and [$DEF\ \in$], then it is an interesting philosophical problem *to elucidate those notions through which the membership relation is defined.*

We want now to give a more refined version of predicativity, viz. through *ramification by levels.* Levels are nothing else but ordinals.

Let Δ be a limit ordinal, like $\omega, \omega^2, \omega^\omega, \omega^{(\omega^\omega)} \ldots$, but not much larger than ϵ_0. Then we define a system $GGA_0[\Delta]$ with levels $< \Delta$ in the next section.

3 The system $GGA_0[\Delta]$ and its extensions

The letters $\alpha, \beta, \xi, \eta, \ldots$ range over ordinals $< \Delta$; these ordinals are called *levels.* We define the language $\mathcal{L}_\Delta$ as follows.

Object variables $a, b, \ldots x, y \ldots$ have no levels. Predicate variables of level ξ are denoted by $A^\xi, \ldots, Y^\xi, \ldots$. Next we assign levels to formulae and to more complicated predicate terms. We use λ as a symbol for predicate-abstraction, not for function-abstraction.

Every expression $a = b$ is a formula with level 0.

The formula $T^\alpha(a)$ and the formula $T^\alpha \searrow b$ have level α, where T^α is a predicate-term of level α.

Let $\mathcal{F}[a]$ be a formula of level α such that the bound variable

x does not occur in $\mathcal{F}[a]$. Then the predicate-term $\lambda x \mathcal{F}[x]$ has also level α; And the formulae $\forall x \mathcal{F}[x]$ and $\exists x \mathcal{F}[x]$ have both also the level α.

Applying $\neg$ does not change the level. In the case of $\wedge, \vee, \rightarrow$ we take the maximum as the new level.

If in the formula $\mathcal{F}[A^\xi]$ of level α the bound predicate-variable X^ξ does not occur, then the formulae $\forall X^\xi \mathcal{F}[X^\xi]$ and $\exists X^\xi \mathcal{F}[X^\xi]$ have both the level $\rho = max\{\xi + 1, \alpha\}$.

Besides the free object-variable $a, b, \ldots$ we may also introduce some constants, $k, l, k_1, \ldots$ for objects, and some n-ary function constants (for functions from objects to objects). We denote the arising object-terms by $s, t, \ldots$

We define now $GGA_0[\Delta]$ as a classical sequent calculus over the language $\mathcal{L}_\Delta$. We may also consider, of course, its intuitionistic subsystem.

(1) Initial sequents $\varphi \Longrightarrow \varphi$.

(2) The usual structural inference rules, including the cut rule.

(3) The usual rules for $\neg, \wedge, \vee, \rightarrow$ and for $\forall$ and $\exists$ *over objects.*

(4) The rules for λ introduction, viz.

$$(\lambda \Longrightarrow) \quad :: \quad \frac{\mathcal{F}[t], \Phi \Longrightarrow \Psi}{\lambda x \mathcal{F}[x](t), \Phi \Longrightarrow \Psi}$$

The right rule $(\Longrightarrow \lambda)$ is defined analogously.

(5) The rules for predicate-quantifiers.

$$(\forall pred \Longrightarrow) \quad \frac{\mathcal{F}[T^{\alpha}], \Phi \Longrightarrow \Psi}{\forall X^{\beta}\mathcal{F}[X^{\beta}], \Phi \Longrightarrow \Psi}; \quad \alpha \leq \beta$$

$$(\Longrightarrow \forall pred) \quad \frac{\Phi \Longrightarrow \Psi, \mathcal{F}[A^{\beta}]}{\Phi \Longrightarrow \Psi, \forall X^{\alpha}\mathcal{F}[X^{\alpha}]}; \quad \alpha \leq \beta$$

The rules for $\exists pred$ are dual. In ($\Longrightarrow \forall pred$) the eigenvariable A^{β} is not allowed in the conclusion.

The *levels* are used *cumulatively* in the quantifier inferences because there are upward and downward shifts involved; but we have only two types: predicates and objects. [If we have more than two types; and if the *types* are taken *cumulatively* in the quantifiers inferences, then there is no (good) cut-elimination theorem, in contrast to Theorem 1 below. See e.g. [4] for this phaenomenon.]

The inference ($\Longrightarrow \exists pred$), equivalently ($\forall pred \Longrightarrow$) in classical logic, may be called a *ramified comprehension principle*; with its help and the help of some other rules we can derive

$$(ram\ comp) \quad : \quad \Longrightarrow \exists X^{\beta}\forall y\,(\,X^{\beta}(y) \leftrightarrow \mathcal{F}[y]\,)$$

when $\lambda y\mathcal{F}[y]$ has level α with $\alpha \leq \beta$.

Observe that we have in $GGA_0[\Delta]$ no rules for $=$, although we admit the symbol $=$ in the formulae of the language $\mathcal{L}_{\Delta}$.

Theorem 1. *If* $GGA_0[\Delta] \vdash S$, *then* $GGA_0[\Delta] \setminus \{cut\} \vdash S$

The proof of Theorem 1 is constructive and more or less as usual. If follows that $GGA_0[\Delta]$ is consistent.

How is $GGA_0[\Delta]$ related to Frege's *GGA*? If one takes away (*GG V*), and replaces the impredicative comprehension law with something like the inference ($\Longrightarrow \exists pred$) or the principle (*ram comp*), then one gets almost precisely the system $GGA_0[\Delta]$.

The system $GGA_0[\Delta]$ is just a logic within which one may formalize possibly nonlogical theories. Of course, some of these theories may also be called logics; we do not discuss in the present paper the distinction between *a logic* and *a theory formalized in a logic.*

To apply a sequent calculus to formulate a theory T one may state the axioms of T by sequents of the form $\Longrightarrow \varphi$, where φ contains not free variables. We now list a set of such axioms, called the *contraction laws.* We write simply φ for the sequent $\Longrightarrow \varphi$.

$(=)$: $\forall x : x = x \ \wedge\ \forall x, y, z(x = z \wedge y = z \rightarrow x = y)$

$(cong()\ \alpha)$: $\forall Z^\alpha, x, y\,(\,Z^\alpha(x) \wedge x = y \rightarrow Z^\alpha(y)\,) \quad \alpha < \Delta$

$(cong() \searrow\ \alpha)$: $\forall X^\alpha, Y^\alpha, z, u\,(\,X^\alpha \approx Y^\alpha \ \wedge\ X^\alpha \searrow z \ \wedge$
$Y^\alpha \searrow u \rightarrow z = u\,) \quad \alpha < \Delta$
where $A^\alpha \approx B^\alpha \ := \ \forall x\,(\,A^\alpha(x) \ \leftrightarrow\ B^\alpha(x)\,)$

$(cong \searrow\ \alpha)$: $\Longrightarrow \forall Z^\alpha, x, y\,(\,Z^\alpha \searrow x \wedge x = y \rightarrow Z^\alpha \searrow y\,)$
$\alpha < \Delta$

$(updown)$: $\Longrightarrow \forall z\, \exists X^\alpha : X^\alpha \searrow z \ \wedge$
$\forall X^\alpha\, \exists z : X^\alpha \searrow z \quad \alpha < \Delta$

$(\searrow\ fun)$: $\forall X^\alpha, y, z\,(\,X^\alpha \searrow y \ \wedge\ X^\alpha \searrow z \ \rightarrow\ y = z\,)$
$\alpha < \Delta$

$$(rev \searrow fun) : \forall x, Y^\alpha, Z^\alpha \,(Y^\alpha \searrow x \;\wedge\; Z^\alpha \searrow x \rightarrow Y^\alpha \approx Z^\alpha \,)\; \alpha < \Delta$$

And finally a version of $(GG\ V)$:

$$(gg5\ \alpha) : \forall X^\alpha, Y^\alpha \,[\, \exists z \,(\, X^\alpha \searrow z \wedge Y^\alpha \searrow z \,) \leftrightarrow X^\alpha(u) \approx Y^\alpha \,]\; \alpha < \Delta$$

What these (infinitely many) axioms say seems to be very dangerous. They say that given a level α, the predicates of level α stand in an "isomorphic" relation with the objects. But the danger is only apparent. Consider these axiom as naked and isolated formulae of our language $\mathcal{L}_\Delta$ and interpret the contraction relation $\searrow$ and the predication relation ..(..) both as equality $=$, and all variables as ranging over a one-element domain. In particular, the most dangerous axiom $(gg5\ \alpha)$ simply becomes

$$\forall x, y \,[\, \exists z \,(\, x = z = y \,) \leftrightarrow \forall u \,(\, x = u \leftrightarrow y = u \,)\,]$$

We call the just described system $GGA[\Delta]$; it contains the previously mentioned system *praeKid*, when we use only predicates of level 0: every formula $\mathcal{F}[a]$ without predicate quantifiers has also level 0; therefore, $\lambda x \mathcal{F}[x]$ has also level 0, and we can prove (*praecomp*) in the form $\exists Y^0 \forall x \,(\, Y^0(x) \leftrightarrow \mathcal{F}[x]\,)$. However, if we arrange our assignment of levels more strictly, e.g. by letting an object-quantifier $\forall x$ raise the level by 1, then *praeKid* seems to be at least not directly contained in the resulting system.

For getting intermediate systems between $GGA_0[\Delta]$ and $GGA[\Delta]$ we have to choose judiciously certain subsets of the contraction laws; here a wide field of experimentation opens before our eyes — and we may play with our intuition for the contraction relation $\searrow$.

Let me add some scattered remark about possible developments within $GGA[\Delta]$.

We define a *leveled membership relation* on the objects

$$[Def\ \in_\alpha] \quad a \in_{(\alpha)} b := \exists X^\alpha (X^\alpha(a) \wedge X^\alpha \searrow b)$$

Proposition 2. *If $\lambda x\mathcal{F}[x]$ has level α, then we can prove in $GGA[\Delta]$: $\lambda x\mathcal{F}[x] \searrow b \implies (a \in_{(\alpha)} b \leftrightarrow \mathcal{F}[a])$.*

Proof. Let $\lambda x\mathcal{F}[x] \searrow b$, and let B^α be such that $B^\alpha(a)$ and $B^\alpha \searrow b$. Then by ($gg5\ \alpha$) and ($\forall pred \implies$) we have $\forall z (\lambda x\mathcal{F}[x](z) \leftrightarrow B^\alpha(z))$. Applying the λ-rule we get, in view of $B^\alpha(a)$, the formula $\mathcal{F}[a]$.

The other way around. From $\mathcal{F}[a]$ we get $\lambda x\mathcal{F}[x](a)$. And then by ($\implies \exists pred$) the formula $a \in_{(\alpha)} b$. QED

What can we do with our leveled membership relations? I do not yet know; I have yet to carry out some more experiments with them.

Using Proposition 2 we have also a leveled powerset schema.

$[powerset\ \alpha]$:
$$\forall u \exists y \forall x (x \in_{(\alpha+1)} y \leftrightarrow \forall z (z \in_{(\alpha)} x \to z \in_{(\alpha)} u))$$

Again by Proposition 2 we get things like

$$\bigwedge_{1 \le i < j \le n} \neg a_i = a_j \implies \exists y \forall x (x \in_{(0)} y \leftrightarrow \bigvee_{1 \le i \le n} x = a_i)$$

We finish this section with the derivation of a nontrival law which contains the relation $a \in b$, or equivalently $a \in_{(0)} b$, on both sides of an $\leftrightarrow$.

$$[single] \quad : \quad \exists y \forall x\, (\, x \in y \,\leftrightarrow\, \exists z \forall u\, (\, u \in x \,\leftrightarrow\, u = z\,)\,)$$

The sentence [*single*] says that the set of all singletons exists; this is wildly false in Zermelo-like set theories.

Proof. Consider the predicate-term
$T \;:=\; \lambda x \exists z\, [\, (\lambda v : v = z) \searrow x\,]$.
The term T has level 0. In view of Proposition 2, to prove [*single*], it is sufficient to show the equivalence

$$\exists z\, [\, (\lambda v : v = z) \searrow a\,] \;\leftrightarrow\; \exists z \forall u\, (\, u \in a \,\leftrightarrow\, u = z\,)$$

The direction $\rightarrow$. Let c be such that $(\lambda v : v = c) \searrow a$. Our aim is to show, that the right-hand z can be taken as c. So let $u \in a$, i.e. there is an A such that $A \searrow a$ and $A(u)$. By (gg5 0) [equivalently by (gg 5)] the terms $\lambda v : v = c$ and A have the same elements. Hence $u = c$. If $u = c$, then $A(u)$. Hence $u \in a$.

Now for the direction $\leftarrow$. Let c such that (1) $\forall u\, (\, u \in a \leftrightarrow u = c\,)$. We have to show that $(\lambda v : v = c) \searrow a$. From (1) we have $c \in a$. Hence there is an A with $A \searrow a$ and $A(c)$. But $(\lambda v : v = c) \approx A$. By the axiom (*updown*) there is a d such that $(\lambda v : v = c) \searrow d$. Finally, by ($cong() \searrow$ 0) we get $(\lambda v : v = c) \searrow a$. QED

4 Conclusion

We have replaced the impredicative comprehension principle of *GGA* by the ramified principle (*ram comp*), and have strengthened Frege's (*GG V*) to the infinitely many contraction laws.

Appendix on Logicism versus Ontologicism

Logicism is not sharply determined by the statement that (Standard) Mathematics is "part of" Logic, or even"the same as" Logic. There are too many precise versions of the possible relationship between mathematics and logic.

However, let us (for the sake of discussion) assume that the formal system PM is a good vindication of logicism. Here PM is the unramified version of *Principia Mathematica* together with an axiom of infinity and some choice principles. The formal system PM is quite logical, it can be interpreted in systems which are even more logical, see [3] and [4].

The adjective *logical* is intended (by me) to denote *non-ontological*, or *ontologically non-committal*, or *ontologically thin and neutral.* Seen from the point of view of this distinction, Frege's *GGA* does definitely not belong to logicism, but rather to ontologicism, despite Frege's own terminology (and imagination). For Frege's objects (Gegenstände) are ontologically very heavy, even ontologically overloaded. Let us make this entirely clear. If f is the Fregean concept of being a real number, then its Werthverlauf $x.f(x)$ is an object, and this object is identical with the set $\mathbb{R}$ of real numbers, i.e. $\mathbb{R} = x.f(x)$, or $f \searrow \mathbb{R}$ in our notation. Also, the set of all functions from the reals to the reals is an Fregean object.

In this way, all entities are either objects or a contracted to objects. Therefore, ontologicism is a better term for what

GGA tried to implement. The effective implementation of the ideas behind *GGA* amounts to an ontological or rather *physical* theory of the contraction relation $A \searrow b$.

Suppose now we had a theory T of the contraction relation such that T is more or less mathematically equivalent to PM. Does then PM cease to belong to logicism? One may say NO since the infinitely many types of PM could delute somehow the ontological substance and heaviness of the objects in T.

Some of the ideas in this Appendix were inspired by [6].

References

[1] J. W. Degen : *Systeme der kumulativen Logik*, Philosophia Verlag, 1983

[2] J. W. Degen : *Two formal vindications of logicism*, in: Philosophy of Mathematics, Proceed. 15th Intern. Wittgenstein Symp. ed. J. Czermak, Wien 1993, 243–250

[3] J. W. Degen : *Complete infinitary type logics*, Studia Logica 63 (1999) 85–119

[4] J. W. Degen and Jan Johannsen : *Cumulative higher-order logic as a foundation for set theory*, Math. Log. Quart. 46 (2000) 2, 147–170

[5] G. Frege : *Grundgesetze der Arithmetik* Jena (1893, 1903).

[6] H. T. Hodes : *Where do sets come from?* J.S.L. 56 (1991) 1, 150–175