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A Framework for Maximality and Interpolation in Abstract Logics with and without Negation

Marta García-Matos[1]

1 Introduction

In the beginning of the study of extensions of first-order logic, the interest in developing such extensions was grounded on the need of studying various mathematical concepts not definable in first-order logic that appear especially in some new fields in mathematics -such as being a countable, a well ordered, or a measurable set. That way, mathematicians studying those fields could benefit from the methods of the model theory of the particular logic adequate to the concepts they handed, just as algebraists benefited from the model theory of first-order logic. But while these extensions have a richer expressive power, it turned out that they lacked interesting properties present in first-order logic. In 1969, Lindström [14] proved a very important maximality theorem: *'First-order logic is a maximal logic satisfying Compactness and Löwenheim-Skolem theorems'.* That is, any logic expressing more things than first-order logic, will loose at least one of those two mentioned valuable model theoretic properties. This phenomenon is interesting by itself; it makes the study of model theoretic languages depart from its original aim, and give rise to abstract model theory, a new field in model theory that will study these languages concentrating in its model theoretic properties. In this field we are not interested anymore in designing a particular language being able to describe a model as having

[1]Helsinki Logic Group - University of Helsinki - Finland

this or that porperty. Instead, we are interested in constructing logics with interesting properties, such as satisfying compactness, Löwenheim-Skolem property, Craig's interpolation theorem [7] [2]. We are happier if we find a logic to be maximal with respect to these properties. We even are satisfied with just proving the existence of such logics, without ever glancing at how do they look like.

This paper is devoted to the study of the relations between interpolation and maximality in first-order logic and its extensions. The framework for this study is hinted, although not fully exploited, in [9]. Caicedo [3], [4], [5] presented several results for extensions of first-order logic with generalized quantifiers that fit within this framework. As he reckons, under very weak assumptions any such extension can be expressed in the form $\mathcal{L}_{\omega\omega}(\bar{Q})$ (where $\mathcal{L}_{\omega\omega}$ denotes first-order logic), for $\bar{Q} = \{Q^i : i \in I\}$ any set of quantifiers. This is also the case for all logics with interpolation. All these logics can be provided with a back-and-forth system [6], so we can find a back-and-forth system for any logic worth to explore[3]. Likewise, all proofs in the above papers concerning interpolation made heavy use of this feature. We therefore use back-and-forth systems as the substrate for our investigation in interpolation.

Section 3 scarcely contains new results. It is more a compilation of several independent results presented under the same framework. That way, we are able to answer a question of Barwise and van Benthem [2] regarding an alternative proof of interpolation for $\mathcal{L}_{\omega_1\omega}$.

Since Section 3 links the proof of interpolation to maximality, in Section 4 we analyze what individual model theoretic properties give rise to orderings of logics with maximal points. We also investigate how this maximality translates in the case we do not have negation. No proof of maximality with or without negation uses back-and-forth methods, and although all these logics have a weaker form of interpolation, proving whether they have interpolation is a very difficult matter: we seem to find ourselves without tools to determine it.

The conclusions of the paper are, in the first place, that for interpolation

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[2]We give their definitions later, by now it suffices to know these properties can be regarded as a measure of a logic as being natural and easy to be treated.

[3]Although it is true that any extension of first-order logic can be written in the form $\mathcal{L}_{\omega\omega}(\bar{Q})$, these extensions are generally divided into two essential kinds: Infinitary logics, $\mathcal{L}_{\kappa\lambda}$ with less than κ conjunction or disjunctions, and strings of existential or universal quantifiers of size less than λ; and extensions by generalized quantifiers, presented for the first time by Mostoski [18], and Lindstrom [13]. Infinitary logics have back-and-forth systems [8] which are described independently of their representation as $\mathcal{L}_{\omega\omega}(\bar{Q})$

to be related to maximality, one should understand interpolation theorem as: "If $\mathbf{K}_1, \mathbf{K}_2$ are disjoint classes belonging to (a certain R-invariant fragment of) $PC(\mathcal{L})$, then they can be separated by an elementary $\mathcal{L}$-class." -one recovers the usual interpolation when he ignores the parenthesis; in the second place, that the proof of interpolation by back-and-forth arguments is only possible when the logic is maximal with respect to some model theoretic properties, and when the invariance under R is among these characterizing properties. These conclusions break down in the case the logic is not closed under negation. Basically what happens is that maximality and interpolation theorems can be stated as corollaries for a so called *separation theorem*, and the proofs of these corollaries have different sensibilities to the lack of negation.

2 Preliminaries

A *vocabulary* τ is a nonempty set that consists of finitary relation symbols $P, R, ...,$ and constant symbols $c, d, ...$ A τ-*structure* $\mathfrak{A}$ is a pair $\langle A, \nu \rangle$, where A, called the domain of $\mathfrak{A}$, is a nonempty set and ν is a map that assigns to every n-ary relation symbol R in τ, an n-ary relation on A^n, and to every constant symbol in τ an element in A. For any symbol $T \in \tau, T^{\mathfrak{A}}$ denotes the interpretation of T on $\mathfrak{A}$. We denote structures by $\mathfrak{A}[\tau] = \langle A, T_i^{\mathfrak{A}} \rangle_{T_i \in \tau}$. Let $Str[\tau]$ denote the class of structures of vocabulary τ. A *logic* is a pair $\langle \mathcal{L}, \models_{\mathcal{L}} \rangle$ where $\mathcal{L}$ is a map $\mathcal{L} : \tau \mapsto \mathcal{L}[\tau]$ such that $\{\mathcal{L}[\tau] : x \in \tau\}$ is a class called the class of $\mathcal{L}$-*sentences* of vocabulary τ, and $\models_{\mathcal{L}}$ (the $\mathcal{L}$-*satisfaction relation*) is a relation between structures and $\mathcal{L}$-sentences such that conditions $1 - 10$ of Definition 1 hold. For φ some $\mathcal{L}$-sentence, and $\mathfrak{A}$ some structure, $\mathfrak{A} \models_{\mathcal{L}} \varphi$ is read "$\mathfrak{A}$ is a model of φ". By $Mod^{\tau}_{\mathcal{L}}(\varphi)$ we denote the class of τ-structures that of $\mathfrak{A} \models_{\mathcal{L}} \phi$.[4] If it's clear from the context which logic we are talking about, we omit the subscript $\mathcal{L}$. If K is the class of models of a sentence φ, then $\bar{K}$ is the class of models of $\neg\varphi$. A *renaming* is a map $\rho : \tau \mapsto \sigma$ that is a bijection from a vocabulary τ to a vocabulary σ, that maps relation symbols to relation symbols of the same arity, and constants to constants. Given a renaming and a structure $\mathfrak{A}$ of vocabulary τ, $\mathfrak{B} = \mathfrak{A}^{\rho}$ is a structure with $B = A$ and $\rho(T)^{\mathfrak{B}} = T^{\mathfrak{A}}$, for all symbols $T \in \tau$. Let $\sigma \subsetneq \tau$, and let $\mathfrak{A} \in Str[\tau]$. We define $\mathfrak{A} \upharpoonright \sigma$,

[4]Then $Mod^{\tau}(\varphi) = \varphi$, and we talk about formulas and model classes indistinctly.

the *reduct* of $\mathfrak{A}$, to be the structure $\mathfrak{B} = \mathfrak{A} \restriction \sigma = (A, T^{\mathfrak{A}})_{T \in \sigma}$, where $T^{\mathfrak{A}} = T^{\mathfrak{B}}$ for $T \in \sigma$. If K is a class of models of vocabulary τ, then $K \restriction \sigma = \{\mathfrak{A} \restriction \sigma : \mathfrak{A} \in K\}$. Let φ be a sentence of vocabulary $\tau \cup \{a\}$ and $\mathfrak{A}$ a structure of vocabulary τ. Then $\varphi^{\mathfrak{A}} = \{a \in A : (\mathfrak{A}, a) \models \varphi\}$. Given a class of models $\mathbf{K}$ in a vocabulary τ, and $\sigma \subseteq \tau$ we denote by $\mathbf{K} \restriction \sigma$ the class $\{\mathfrak{A} \restriction \sigma : \mathfrak{A} \in \mathbf{K}\}$.

Definition 1 (Closure Properties)

1. *Inclusion property. If $\sigma \subseteq \tau$, then $\mathcal{L}[\sigma] \subseteq \mathcal{L}[\tau]$.*

2. *Isomorphism property. If $\mathfrak{A} \models_{\mathcal{L}} \varphi$, and $\mathfrak{A} \cong \mathfrak{B}$, then $\mathfrak{B} \models_{\mathcal{L}} \varphi$.*

3. *Reduct Property. If $\varphi \in \mathcal{L}[\tau]$ and $\tau \subseteq \tau_{\mathfrak{A}}$, then $\mathfrak{A} \models_{\mathcal{L}} \varphi$ iff $\mathfrak{A} \restriction \tau \models_{\mathcal{L}} \varphi$.*

4. *Renaming Property. If $\rho : \sigma \to \tau$ is a renaming , then for each $\varphi \in \mathcal{L}[\sigma]$ there is a sentence ψ^{ρ}, from $\mathcal{L}[\tau]$ such that for all σ-structures $\mathfrak{A}$, $\mathfrak{A} \models_{\mathcal{L}} \varphi$ iff $\mathfrak{A}^{\rho} \models_{\mathcal{L}} \varphi^{\rho}$;*

5. *Substitution Property. Suppose $\sigma \subseteq \tau$, and $\varphi \in L[\tau]$, and for all $R_i \in \tau \backslash \sigma$, we have a sentence $\varphi_i(d_1^i, ..., d_{k_i}^i) \in \mathcal{L}[\sigma \cup \{d_1^i, ..., d_{k_i}^i\}]$, k_i the arity of R_i, and $d_1^i, ..., d_{k_i}^i$ new constants. For any structure $\mathfrak{A} \in Str_{\mathcal{L}}[\sigma]$, let $\mathfrak{A}^* \in Str_{\mathcal{L}}[\tau]$be such that $\mathfrak{A}^* \restriction \sigma = \mathfrak{A}$, and for all $R_i \in \tau \backslash \sigma, R_i^{\mathfrak{A}^*} = \{\langle a_1, ..., a_{k_i}\rangle : (\mathfrak{A}, a_1, ..., a_{k_i}) \models_{\mathcal{L}} \varphi_i(d_1^i, ..., d_{k_i}^i)\}$. Then there exists a sentence $\varphi^* \in \mathcal{L}[\sigma]$ such that for all $\mathfrak{A} \in Str_{\mathcal{L}}[\sigma]$, $\mathfrak{A} \models_{\mathcal{L}} \psi \leftrightarrow \mathfrak{A}^* \models_{\mathcal{L}} \varphi$. We say that ψ is obtained from φ by simultaneously replacing each $R_i \in \tau \backslash \sigma$ by $\varphi_i(d_1^i, d_2^i, ..., d_{k_i}^i)$;*

6. *Atom Property. For all τ and atomic $\varphi \in \mathcal{L}_{\omega\omega}[\tau]$ there is a sentence $\psi \in \mathcal{L}[\tau]$ such that $Mod_{\mathcal{L}}^{\tau}(\psi) = Mod_{\mathcal{L}_{\omega\omega}}^{\tau}(\varphi)$;*

7. *Conjunction Property. For all τ and all $\varphi, \psi, \in \mathcal{L}[\tau]$ there is $\theta \in \mathcal{L}[\tau]$ such that $Mod_{\mathcal{L}}^{\tau}(\varphi) \cap Mod_{\mathcal{L}}^{\tau}(\psi) = Mod_{\mathcal{L}}^{\tau}(\theta)$;*

8. *Disjunction Property. For all τ and all $\varphi, \psi, \in \mathcal{L}[\tau]$ there is $\theta \in \mathcal{L}[\tau]$ such that $Mod_{\mathcal{L}}^{\tau}(\varphi) \cup Mod_{\mathcal{L}}^{\tau}(\psi) = Mod_{\mathcal{L}}^{\tau}(\theta)$;*

9. *Particularization Property. If $c \in \tau$, then for any $\varphi \in \mathcal{L}[\tau]$ there is a sentence $\psi \in \mathcal{L}[\tau \backslash \{c\}]$ such that for all $[\tau \backslash \{c\}]$-structures $\mathfrak{A}, \mathfrak{A} \models_{\mathcal{L}} \psi$ iff $(\mathfrak{A}, a) \models_{\mathcal{L}} \varphi$ for some $a \in A$. In a context of a logic with free variables we write ψ as $\exists x \varphi(x)$;*

10. *Universalization. If $c \in \tau$, then for any $\varphi \in \mathcal{L}[\tau]$ there is a sentence $\psi \in \mathcal{L}[\tau \backslash \{c\}]$ such that for all $[\tau \backslash \{c\}]$-structures $\mathfrak{A}$, $\mathfrak{A} \models_{\mathcal{L}} \psi$ iff $(\mathfrak{A}, a) \models_{\mathcal{L}} \varphi$ for all $a \in A$. In a context of a logic with free variables we write ψ as $\forall x \varphi(x)$;*

11. *Q-Projection. If R_Q is a class of models of vocabulary $\sigma = \{S_1, ..., S_m\}$ disjoint from τ, then for any $\varphi_i \in \mathcal{L}[\tau \cup \{c_0, c_1, ..., c_{k_i - 1}\}]$ there is $\psi_i \in \mathcal{L}[\tau]$ such that $\mathfrak{A} \in Mod^{\tau}_{\mathcal{L}}(\psi - i)$ iff*

    - *There is a structure $\mathfrak{C} \in R_Q$ with $C = A$, and*
    - *For all k_i and all k_i-ary $S_i \in \sigma$, $S_i^{\mathfrak{C}} = \{(a_0, ..., a_{k_i - 1}) \in C^n : (\mathfrak{A}, a_0, a_1, ..., a_{k_i - 1} \models \varphi_i\}$.*

    In a context of a logic with free variables we write ψ as

$$Qx_0^1, ..., x_{k_1-1}^1, ..., x_0^m, ..., x_{k_m-1}^m \varphi_1(x_0^1, ..., x_{k_1-1}^1) ... \\ ... \varphi_m(x_0^m, ..., x_{k_m-1}^m).$$

12. *Negation Property. For all τ and all $\varphi \in \mathcal{L}[\tau]$ there is a sentence $\psi \in \mathcal{L}[\tau]$ such that $Mod^{\tau}_{\mathcal{L}}(\varphi) = Str[\tau] \setminus Mod^{\tau}_{\mathcal{L}}(\psi)$;*

13. *Relativization Property. If $c \notin \tau \cup \sigma, \xi \in \mathcal{L}[\tau \cup c]$ and $\varphi \in \mathcal{L}[\tau]$, then there is a sentence $\psi \in \mathcal{L}[\tau \cup c]$ such that for any $(\tau \cup \sigma)$-structure $\mathfrak{B}$, if the set $\xi^{\mathfrak{B}} = \{b \in B : (\mathfrak{B}, b) \models \xi\}$ is τ-closed, then $\mathfrak{B} \models \psi$ iff $(\mathfrak{B} \upharpoonright \tau)|\xi^{\mathfrak{B}} \models \varphi$.*

14. *PC_{Δ}-operation. If $S = \{\varphi_n : n \in \omega\}$ is a set of $\mathcal{L}[\tau]$-sentences, and if $\sigma \subseteq \tau$, then there is a sentence $\psi \in \mathcal{L}[\sigma]$ such that*

$$Mod^{\sigma}_{\mathcal{L}}(\psi) = (\bigcap_n Mod^{\tau}_{\mathcal{L}}(\varphi_n)) \upharpoonright \sigma.$$

    If S contains just one sentence, the operation is called PC.

If a logic satisfies condition 12, it is called *regular*. Any regular logic contains $\mathcal{L}_{\omega\omega}$. If a logic satisfies condition 13, it is called *relativizing*. **K** is an $\mathcal{L}$-elementary class, in symbols **K** is $EC(\mathcal{L})$, if there is a sentence $\theta \in \mathcal{L}[\tau]$ such that $\mathbf{K} = Mod_{\tau}(\theta)$. **K** is an $\mathcal{L}$-pseudo elementary class, in symbols **K** is $PC(\mathcal{L})$ if there is a vocabulary $\tau' \supseteq \tau$ and a sentence $\theta \in \mathcal{L}[\tau']$ such that $\mathbf{K} = Mod_{\tau'}(\theta) \upharpoonright \tau$.

Let $\mathcal{L}, \mathcal{L}^*$ be logics. We say that $\mathcal{L}^*$ *(properly) extends* (written $\mathcal{L} \leq \mathcal{L}^*$) $\mathcal{L}$, if for every EC-class in $\mathcal{L}[\tau]$, there is an EC-class in $\mathcal{L}^*[\tau]$ with the same models (and also there is an EC-class in $\mathcal{L}^*[\tau]$, such that no EC-class in $\mathcal{L}[\tau]$ has the same models).

A *partial isomorphism* between two models $\mathfrak{A}, \mathfrak{B}$ is a function p from $X \subseteq A$ to $Y \subseteq B$ such that the following holds;

1. For all $n \geq 1$, n-ary $R \in \tau$ and $a_0, ..., a_{n-1} \in X : R^{\mathfrak{A}}(\vec{a})$ iff $R^{\mathfrak{B}}(p(\vec{a}))$;
2. For all $c \in \tau$ and $a \in X : c^{\mathfrak{A}} = a$ iff $c^{\mathfrak{B}} = p(a)$.

$Part(\mathfrak{A}, \mathfrak{B})$ denotes the set of partial isomorphisms between $\mathfrak{A}$ and $\mathfrak{B}$. A *back-and-forth* system for $(\mathfrak{A}, \mathfrak{B})$ is a decreasing sequence $I = (I_\beta)_{\beta \leq \alpha}$ of subsets of $Part(\mathfrak{A}, \mathfrak{B})$ that satisfies the following conditions:

(i) Each I_i is a set of partial isomorphisms.

(ii) $\emptyset \in I_\alpha$

(iii) For $m < \alpha$, if $p \in I_{m+1}$ and $a \in A$, then there is $b \in B$ such that $p \cup \{\langle a, b\rangle\} \in I_m$.

(iv) For $m < \alpha$, if $p \in I_{m+1}$ and $b \in B$, then there is $a \in A$ such that $f \cup \{\langle a, b\rangle\} \in I_m$.

We can generalize this concept for other functions f than partial isomorphisms. The nature of these functions will depend on the modifications on the closure under isomorphism of $\mathcal{L}$. We call the functions f *partial F-isomorphisms*, where F-isomorphism is the new notion of isomorphism. An F-back-and-forth system is a back-and-forth system of partial F-isomorphisms. All notation for partial isomorphisms would be obtained just by ignoring F and f in the following definitions. $Part(\mathfrak{A}, \mathfrak{B})$ has an obvious generalization to $PartF(\mathfrak{A}, \mathfrak{B})$ Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are α-F-isomorphic via I, written $I : \mathfrak{A} \cong^F_\alpha \mathfrak{B}$, iff $I = (I_\beta)_{\beta \leq \alpha}$ is an F-back-and-forth system. $I \subseteq PartF(\mathfrak{A}, \mathfrak{B})$ has the back (forth) property if for each $p \in I$ and $b \in B (a \in A)$ there is $q \in I, p \subseteq q$ with $b \in rg(q) (a \in dom(p))$. Two models are partially F-isomorphic $\mathfrak{A} \cong^F_p \mathfrak{B}$ if in there is $I \subseteq PartF(\mathfrak{A}, \mathfrak{B})$ with the back-and-forth property.

Two models $\mathfrak{A}, \mathfrak{B}$ are $\mathcal{L}$-equivalent, in symbols $\mathfrak{A} \equiv_{\mathcal{L}} \mathfrak{B}$ iff they satisfy the same $\mathcal{L}$-sentences.

Definition 2 (Model Theoretic Properties)

1. *$\mathcal{L}$ is* compact *iff for all countable $\Phi \subseteq \mathcal{L}[\tau]$, if each finite subset of Φ has a model, then Φ has a model. Substitutes of compactness are:*

- Small well ordering number, *where the well ordering number of $\mathcal{L}$ is defined as the supremum of all ordinals α such that for any $\mathcal{L}$-sentence $\phi(<, ...)$ having only models with well ordered $<$, there is a model of ϕ that is a well-ordering of type α.*
- Boundedness *$\mathcal{L}$ is* bounded *if for any $\mathcal{L}$-sentence $\phi(<, ...)$ having only models with well ordered $<$, there is an ordinal α such that the order type of $<$ is always less than α.*

2. *$\mathcal{L}$ has the* Downward Löwenhein-Skolem property *iff each satisfiable sentence has a model of size $\leq \aleph_0$. Substitutes of the Löwenhein-Skolem property are:*

   - *$l(\mathcal{L})$, the* Löwenheim number *of $\mathcal{L}$ is the least cardinal μ such that any satisfiable sentence has a model of power $\leq \mu$. Any logic $\mathcal{L}$ with a set number of classes has $l(\mathcal{L}) = \lambda$ for λ some ordinal. Such logics are called* small *or* set *logics. Otherwise, $l(\mathcal{L}) = \infty$, and $\mathcal{L}$ is called a* big *or* class *logic.*
   - *$l_\Sigma(\mathcal{L})$, the* Löwenheim number for countable sets of sentences *of $\mathcal{L}$ is the least cardinal μ such that any countable satisfiable set of sentences has a model of power $\leq \mu$.*

3. *$\mathcal{L}$ has the* Karp property *if any two partially isomorphic structures are $\mathcal{L}$-equivalent. Karp property can be generalized to partial f-isomorphisms.*

4. *$\mathcal{L}$ satisfies the* Craig interpolation theorem *if given $\varphi, \psi \in \mathcal{L}$, such that $\varphi \models \psi$, there is $\theta \in \mathcal{L}$ such that $\tau(\theta) \subseteq \tau(\varphi) \cap \tau(\psi)$ and $\varphi \models \theta$ and $\theta \models \psi$. Craig's theorem can also be stated as: "Any two disjoint PC-classes in $\mathcal{L}_{\omega\omega}$ can be separated by an EC-class in $\mathcal{L}_{\omega\omega}$.". A substitute of Craig theorem is*

   - *$\mathcal{L}$ satisfies the* Souslin-Kleene *separation property if for each class of models $K \in \mathcal{L}$, if K and $\bar{K}$ are PC in $\mathcal{L}$, then they are EC in $\mathcal{L}$.*

3 Interpolation and maximality

Logics differ from each other in their expressive power, and model-theoretic properties bound this expressive power. For instance, Löwenheim-Skolem property says that any sentence with a model can have a countable model, hence ruling out the possibility of any logic with that property to declare a model uncountable. At the light of this example, one would think that each logic is characterized by its exclusive model theoretic properties, in the sense that if we try to overcome its expressive power, we are in the domains of other logic, with therefore different model theoretic properties. This is the content of Lindström's Theorem. However, it is not true in all logics -the difference between expressive powers not always corresponds to model theoretic properties A close examination of the ingredients of Lindström's proof reveals that it is actually an application of some basic properties of back-and-forth systems. When the proof is broken into its parts, a proof of the interpolation theorem emerges. This has been known in the folklore of the subject [9], but has not been systematically exploited. As I said in the introducion, this paper tries to present this connection over the substrate of back-and-forth systems. Two recent works [2], [20] study the relations between back and forth systems and interpolation. The strategy for proving interpolation theorems begins by finding an appropiate back and forth system for the logic. However, the existence of back and forth systems does not guarantee the success in finding interpolation theorems. As a list of negative results, in the case of extensions with generalized quantifiers, the main result of Caicedo says no extension of first-order logic by means of an arbitrary number of monadic quantifiers satisfies interpolation. Mostowski have a similar result in the case on a finite number of generalized quantifiers of arbitrary type. In this section we address the case of infinitary logics, although the conclusions extend to any logic. $L_{\infty\kappa}$ for $\kappa \geq \omega$ has both a have a back-and-forth system, a Lindström's type theorem, but not interpolation. On the other hand, $\mathcal{L}_{\omega_1\omega}$ has interpolation but no Lindström's type characterization. We try to give a framework that connects interpolation and maximality and yet it is able to explain all these "anomalies".

We start by deriving Lindström's and interpolation theorems for first-order logic from a common source theorem called

Theorem 3 (Separation Theorem) *Let $\mathcal{L}^*$ be a compact logic with Downward Löwenhein-Skolem property. Let K_1 and K_2 be two disjoint $\mathcal{L}^*$-classes. Then there is a first-order sentence θ in the common language of K_1 and K_2 that separates them.*

Sketch of proof Suppose there is no such sentence. We extend the vocabulary with some predicate symbols in order to construct a sentence that says that for all m there are disjoint models $\mathfrak{A}_m$ and $\mathfrak{B}_m$ that satisfy the same sentences of quantifier rank m, and a function from A to B that is a partial isomorphism of length m. By compactness, there are two models that satisfy the same sentences, and between which we can construct a partial isomorphism of length ω. By Downward Löwenhein-Skolem property, we can get these models countable, getting an isomorphism. But this is a contradiction. □

Corollary 4 (Lindström's maximality theorem) *$\mathcal{L}_{\omega\omega}$ is a maximal compact logic satisfying the Löwenheim-Skolem property.*

Proof. Let $\mathcal{L}^*$ be closed under negation. □

Corollary 5 (Craig's interpolation theorem) *[7] Any two disjoint $PC(\mathcal{L}_{\omega\omega}$ classes can be separated by an $EC(\mathcal{L}_{\omega\omega})$-class.*

Proof. Take $\mathcal{L}^*$ to be the logic of $PC(\mathcal{L}_{\omega\omega})$ classes. □

At this point, we notice how important is the fact that first-order logic is closed under negation, for not both above corollaries behave equally if we give negation up. As an illustration, if we take a fragment of first-order logic not closed under negation, a theorem analogous to Theorem 3 [10] gives as corollary an interpolation theorem, but no characterization theorem exists for the logic. Specifically, $\mathcal{L}^P_{\omega\omega}$, the fragment of $\mathcal{L}_{\omega\omega}$ in which a given predicate P appears only positively, has interpolation, but is not maximal with respect to compactness and Löwenheim-Skolem property. Moreover, in next section we will prove no logic without negation is maximal with respect these two properties.

Now we try to generalize these ideas for logics other than $\mathcal{L}_{\omega\omega}$. The following table contains a number of logics and their respective satisfaction of interpolation and generalized forms of Lindström's type maximality theorems. The model theoretic properties in the box for LT are the characterizing ones:

Logic	Lindström's theorem	Interpolation theorem
$\mathcal{L}_{\omega\omega}$	Compactness and Löwenheim-Skolem.	YES
$\mathcal{L}_{\infty\omega}$	Boundedness and Karp property.	NO
$\mathcal{L}_{\kappa,\omega}$ $(\kappa = \beth_\kappa)$	Well ordering number $\leq \kappa$ and Karp property.	NO
$\mathcal{L}_{\omega_1\omega}$	NONE	YES
L^1_κ	Löwenheim-Skolem and strong form of undefinability of well order.	YES
$L^P_{\omega,\omega}$	NONE	YES

Next Definition introduces a relation R involved in the description of the back-and-forth system of a logic.

Definition 6 *Let τ be a vocabulary, R a binary relation between structures, and φ a sentence of a logic $\mathcal{L}$.*

1. *We say that φ is R-invariant if*

$$\mathfrak{A}R\mathfrak{B} \text{ and } \mathfrak{A} \models \varphi \text{ imply } \mathfrak{B} \models \varphi$$

   Denote the class of R-invariant sentences as $\mathcal{L}^R$. In case $\mathcal{L} = \mathcal{L}^R$, we say $\mathcal{L}$ is a logic of R-invariant sentences.

2. *We say that φ entails ψ along R, written $\varphi \models_R \psi$, iff for all τ-structures $\mathfrak{A}$ and $\mathfrak{B}$, if $\mathfrak{A}R\mathfrak{B}$, and $\mathfrak{A} \models \varphi$ then $\mathfrak{B} \models \psi$.*

Next theorem is introduced with aims fo being a generalisation of the separation theorem 3.

Theorem 7 *[9] Suppose there is given for any vocabulary τ a set $\Phi^\tau \subseteq \mathcal{L}[\tau]$ and let $\mathcal{R}^\tau = Mod(\Phi^\tau)$. Assume that R is a binary relation between structures such that $\mathfrak{A}R\mathfrak{B}$ implies $\mathfrak{A}, \mathfrak{B} \in \mathcal{R}$ for some τ. Suppose that*

1. *R restricted to τ-structures is an equivalence relation.*

2. *R is invariant under renamings.*

3. *Given τ, for some $\tau' \supseteq \tau$, there are $\mathcal{L}$-sentences $\varphi_0, \varphi_1, \ldots$ such that for any τ-structures $\mathfrak{A}, \mathfrak{B}$, the following hold:*

   a. *$\mathfrak{A}R\mathfrak{B}$ iff $(\mathfrak{A}, \mathfrak{B}, \ldots) \models \{\varphi_i : i \in \omega\}$ for some choice of ..., and* *b. for $n \in \omega$ the relation R_n on $\mathcal{R}$ given by*

$$\mathfrak{A} R_n \mathfrak{B} \text{ iff } (\mathfrak{A}, \mathfrak{B}, ...) \models \{\varphi_i : i \leq n\} \text{ for some } ...$$

has the following two properties:

a.1. R_n is an equivalence relation on $\mathcal{R}^\tau$;

a.2. For $\mathfrak{A} \in \mathcal{R}^\tau$, there is $\psi^n_{\mathfrak{A}} \in \mathcal{L}[\tau]$ such that for $\mathfrak{B} \in \mathcal{R}^\tau$:

$$\mathfrak{A} R_n \mathfrak{B} \text{ iff } \mathfrak{B} \models \psi^n_{\mathfrak{A}}.$$

The above are the general properties we requiere from R to have. There are some further properties that depend on the particular logic $\mathcal{L}$ we are working with.

Then

Let $\mathcal{L}$ be a logic with any given generalized compactness property. If $\mathcal{L}^* \geq \mathcal{L}$ is a logic of R-invariant sentences with the same generalized compactness property, and not necesarily closed under negation, then any two $\mathcal{L}^*$-classes can be separated by an $\mathcal{L}^R$-class. □

Corollary 8 (Lindström's maximality theorem) *$\mathcal{L}_{\omega\omega}$ is a maximal compact logic with the Löwenheim-Skolem property.*

Proof. Let $\mathcal{L}$ be $\mathcal{L}_{\omega\omega}$, and R be the relation of partial isomorphism $\cong_p$. Let $\mathcal{L}^*$ be compact and closed under negation. It suffices to show that any logic with Löwenheim-Skolem property has the Karp property, as proves the following

Proposition 9 *If $\mathcal{L}$ has the Löwenheim-Skolem property, then $\mathcal{L}$ has the Karp property.*

Proof. By contradiction, suppose $\mathcal{L}[\tau]$ has the Löwenheim-Skolem property, but does not have Karp property. Then, for some $\mathcal{L}$-sentence ϕ we have

$$\mathfrak{A} \cong_p \mathfrak{B}, \ \mathfrak{A} \models \phi \text{ and } \mathfrak{B} \models \neg\phi$$

If A and B are countable we are done, since partially isomorphic countable structures are isomorphic. So, let A and B be uncountable. Let I, V, W be new unary predicates; and G be one new ternary relation. Set $\tau' = \tau \cup \{I, V, W.G\}$. Let ψ be the conjunction of the following $\mathcal{L}[\tau']$-sentences:

"V and W are disjoint"

$\phi^{\{x:V(x)\}}$,

$\neg\phi^{\{x:W(x)\}}$,

"each $p \in I$ is a mapping from V to W" that is,

$$\forall p(I(p) \to \forall x \forall y(Gpxy \to (V(x) \wedge W(y)))$$

"each $p \in I$ is a partial injective mapping" that is,

$$\forall p(I(p) \to \forall x \forall y \forall u \forall v(G(p,x,u) \wedge G(p,y,v) \to (x = y \leftrightarrow u = v))),$$

"each $p \in I$ preserves all the symbols in τ" for example, for binary $T \in \tau$,

$$\forall p(I(p) \to \forall x \forall y \forall u \forall v(G(p,x,u) \wedge G(p,y,v) \to (T(x,y) \leftrightarrow T(u,v))),$$

"the set I is not empty"

"the set I has the forth property", that is,

$$\forall p(I(p) \to \forall x \exists q \exists y(I(q) \wedge G(q,x,y) \wedge \forall z \forall w(G(p,z,w) \to G(q,z,w)))),$$

"the set I has the back property."

Then a model whose relativizations to V and W are isomorphic to $\mathfrak{A}$ and $\mathfrak{B}$, respectively, is a model of ψ, since by hypothesis $\mathfrak{A}$ and $\mathfrak{B}$ are partially isomorphic, $\mathfrak{A} \models \phi$ and $\mathfrak{B} \models \neg\phi$. By Downward Löwenhein-Skolem property ψ has a countable model $\mathfrak{C}$. But then we obtain two countable structures $\mathfrak{A}' = \mathfrak{C}^V$ and $\mathfrak{B}' = \mathfrak{C}^W$, that are partially isomorphic, and therefore isomorphic, such that $\mathfrak{A}' \models \phi$ and $\mathfrak{B}' \models \neg\phi$, a contradiction. □

Note that in this case, we cannot derive as corollary Craig's interpolation theorem, since $PC(\mathcal{L}_{\omega\omega})$ does not preserve Karp property, that is, is not invariant under $\cong_p$[5]. Instead, interpolation theorem holds in $\mathcal{L}_{\omega\omega}$ because $PC(L_{\omega\omega})$ preserves compactness and Löwenheim-Skolem property.

Let's see the case for $\mathcal{L}_{\infty\omega}$. Adding for each ordinal α, a relation R_α with set many equivalence classes, we can prove in a similar way

Corollary 10 (Barwise's characterization for $\mathcal{L}_{\infty\omega}$) *$\mathcal{L}_{\infty\omega}$ is a maximal bounded logic with the Karp property.*

Proof. Let $\mathcal{L} = \mathcal{L}_{\infty\omega}$, and model-theoretic substitute of compactness be boundedness. Let R be the relation of partial isomorphism, and $\mathcal{L}^*$ be closed under negation. □

The following is the associated interpolation theorem. It is not a full interpolation theorem, for $PC(\mathcal{L}_{\infty\omega})$ does not preserve Karp property. Contrary to what happens in the case of first-order logic, $\mathcal{L}_{\infty\omega}$ cannot be characterized by any property preserved by the PC operation, so we don't have any hope of getting the full interpolation in this case.

Corollary 11 (Barwise-van Benthem interpolation theorem for $\mathcal{L}_{\infty\omega}$) *[2] Given $\psi \in \Sigma^1_1(\mathcal{L}_{\infty,\omega})$, and $\phi \in \Pi^1_1(\mathcal{L}_{\infty,\omega})$, the following are equivalent:*

1. *ψ entails ϕ along R.*

2. *There is a sentence θ of $\mathcal{L}_{\infty,\omega}$, such that $\psi \models \theta$ and $\theta \models \phi$.*

The proof of this theorem as given by they authors is very similar to the proof of maximality for $\mathcal{L}_{\infty\omega}$. We give the proof treating this theorem as a corollary of Theorem 7.

Proof. (From the generalized separation theorem) Let R be the relation of partial isomorphism. Let $\mathcal{L}^*$ be the R-invariant fragment of $PC(L_{\infty\omega})$. □

This interpolation theorem says that the only possible $\psi, \phi \in PC(\mathcal{L}_{\infty\omega})$ such that $\psi \models \phi$ has an interpolant are those in which ϕ and ψ are in the R-invariant fragment. On these grounds, Barwise and van Benthem argue interpolation theorem should be understood as:

[5]Indeed, let $\mathfrak{A}$ be the unary structure with uncountable universe and P a predicate which has countable many elements and its complement is uncountable. Let $\mathfrak{B}$ be the unary structure with ω_1 as universe in which the predicate P and its complement are both uncountable. Clearly $\mathfrak{A} \cong_p \mathfrak{B}$. Let φ be the $PC(\mathcal{L}_{\omega\omega})$-sentence "there is a one-one mapping from the complement of P into P". Then $\mathfrak{B} \models \varphi$, but $\mathfrak{A} \not\models \varphi$.

($*$) "A logic $\mathcal{L}$ has interpolation If $\mathbf{K}_1, \mathbf{K}_2$ are $PC(\mathcal{L})$-classes invariant under R, then they can be separated by an $EC(\mathcal{L})$-class."

Similar 'interpolation theorems' could be proved for $\mathcal{L}_{\kappa,\omega}, \kappa = \beth_\kappa$ and $\mathcal{L}_{\omega_1\omega}$. From López-Escobar [17], we know the later has the full interpolation, while the result provided by an adaptation of Theorem 11 for $\mathcal{L}_{\omega_1\omega}$ is only partial. Barwise and van Benthem [2] asked whether it would be possible to make some change in Theorem 11, so that we get the full interpolation theorem for $\mathcal{L}_{\omega_1\omega}$. We see here this is not possible, for separation is essentially Lindström's theorem, and $\mathcal{L}_{\omega_1\omega}$ does not have any Lindström's type characterization[6]. That is, Theorem 11 can be proved for $\mathcal{L}_{\omega_1\omega}$ per se, but it cannot be understood as a corollary from Theorem 7.

Is there any logic besides first-order with a maximallity and a full interpolation theorem? In [21], Shelah and Väänänen construct a new infinitary logic $\mathcal{L}^1_\kappa$ between $\mathcal{L}_{\kappa,\omega}$ and $\mathcal{L}_{\kappa,\kappa}$ characterized by Löwenheim-Skolem property and a substitute of compactness. Both properties are preserved by the PC operation. They achieve this way the new logic satisfying Lindström's and Craig's theorems.

4 Other maximality results

We have seen several cases of logics with model theoretic characterizations. We have seen the importance of Karp properties in the relation between interpolation and maximality. We can now ask what model theoretic properties are able to characterize a logic. That is, we can study the orderings of the family of all logics with respect to every model theoretic property and look for maximal points.

In the literature, there are already some examples. Sgro [16] proved that in the ordering of logics with the Łos' ultraproduct porperty, there is a maximal logic with this property; Lipparini [15] proved that there is a maximal logic that extends a given logic $\mathcal{L}$ and has the same complete extensions as $\mathcal{L}$; Wacławeck [22] proved there is a maximal logic with Löwenheim-Skolem property over any logic with this property. All these maximal logics enjoy

[6]In the case of propositional extensions of $\mathcal{L}_{\omega_1,\omega}$, (see [11], [12]), Harrington [12] proved there are such extensions that continue to have the same model theoretic properties as $\mathcal{L}_{\omega_1,\omega}$, if we restrict to admissible fragments. Gostanian and Hrbacek proved in [11], not restricting to admissible fragments, that among propositional extensions of $\mathcal{L}_{\omega_1,\omega}$, only $\mathcal{L}_{\omega_1,\omega}$ itself warrants interesting model theoretic properties.

Souslin-Kleene separation theorem, which is a weakening of interpolation theorem. It is an open problem whether there is a maximal logic in the ordering of compact logics.

Of particular interest would be the existence of a maximal logic with Karp property[7]. Since Löwenheim-Skolem property implies Karp property, and the converse is true for logics with interpolation (cf. [9] p. 95), it is also of interest the ordering of logics with the Löwenheim-Skolem property. For this reason, as well as for methodological reasons, I reproduce here the proof of Wacławeck on the existence of a maximal logic with respect to the Löwenheim-Skolem property. Then we will prove that it has Souslin-Kleene separation theorem. We will be able to appreciate the essencial differences between these proofs and those of the previous section, as well as to asses the great complexity that a proof of interpolation would mean in this case.

In the next two theorems a logic is considered to be close under negation and conjunction only.

Theorem 12 *Let $(LS, \leq)$ be the ordering of logics that satisfy the Löwenheim-Skolem property. For any logic $\mathcal{L} \in (LS, \leq)$, there is a maximal logic $\mathcal{L}' \in (LS, \leq)$ such that $\mathcal{L} \leq \mathcal{L}'$.*

Proof.

Each sentence in a logic $\mathcal{L}$ with the Löwenheim-Skolem property is determined by its countable models, i.e. two different sentences in $\mathcal{L}$ do not have the same countable models, by definition.

Claim 2: There are at most $2^{\aleph_0}$ countable nonisomorphic models of finite vocabulary.

Let $\tau = \{T_1, ..., T_n\}$ be a vocabulary, and let m_i be the arity of T_i. Let A be a countable set, and let $c_i = |\{f : f \text{ is a function from } A^{m_i} \text{ to } \{0,1\}\}|$. Then the number s of models of vocabulary τ and domain A is $s = \Pi_{i=1,...,n} c_i$ but $c_i = 2^{\aleph_0}$ for all i, so $s = (2^{\aleph_0})^n = 2^{\aleph_0}$.

So there are at most $2^{2^{\aleph_0}}$ possible classes of countable nonisomorphic models, and hence every well-ordered chain in $(LS, \leq)$ has length smaller than $(2^{2^{\aleph_0}})^+$.

Claim 3: The union of an increasing sequence of logics $\mathcal{L}'$ with $\mathcal{L} \leq \mathcal{L}'$ is a supremum of this family of logics.

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[7]However, this is an open problem. $\mathcal{L}_{\infty\omega}$ is a maximal logic with respect to Karp property if we chose the relation of extension between two logics to be:"$\mathcal{L} <_{\equiv} \mathcal{L}'$ iff for all τ and all $\mathfrak{A}, \mathfrak{B} \in Str[\tau]$, if $\mathfrak{A} \equiv_{\mathcal{L}'}$ then $\mathfrak{A} \equiv_{\mathcal{L}}$

Let $\mathcal{L}^* = \bigcup_\alpha \mathcal{L}_\alpha$, where $L_\alpha \in (LS, \leq)$, and $\mathcal{L}_\delta \leq \mathcal{L}_\gamma$ for $\delta \leq \gamma$, and suppose $\mathcal{L}^* \notin (LS, \leq)$. Then there is some $\theta \in \mathcal{L}^*$ with a model but no countable models. But $\theta \in \mathcal{L}_\alpha$ for some α, a contradiction.

By Zorn's Lemma, there is a maximal logic $\mathcal{L}^{**} \in (LS, \leq)$ extending $\mathcal{L}$. □

Theorem 13 *Any maximal logic in with respect to the Löwenheim-Skolem property has the Souslin-Kleene separation property.*

Proof. Let $\mathcal{L}$ be a maximal logic with the Löwenheim-Skolem property, and let K, and $\bar{K}$ be two PC-classes in $\mathcal{L}$. Suppose K is not an $EC(\mathcal{L})$-class. Then we can add K as a new $EC(\mathcal{L})$-class, and get the extension $\mathcal{L}'$ closing under negation and intersection. We prove that $\mathcal{L}'$ satisfies Löwenheim-Skolem theorem, contradicting the hypothesis that $\mathcal{L}$ is a maximal logic in $(LS, \leq)$.

K has countable models, because it is a class of reducts of models of a sentence of $\mathcal{L}$, and similarly for $\bar{K}$, $M \cap K$, and $M \cap \bar{K}$.

In case $\mathcal{L}$ has negation we can also prove the following

Corollary 14 *Any logic has an extension with the same Löwenheim number, and the Souslin-Kleene separation theorem.*

Proof. Let $\mathcal{L}$ be a logic with Löweheim number κ. Consider the ordering of logics with Löwenheim number κ. By an argument similar to the proof of Theorem 12, there is a maximal logic $\mathcal{L}'$ extending $\mathcal{L}$. By an argument similar to that of the proof of Theorem 13, $\mathcal{L}'$ satisfies the Souslin-Kleene separation property. □

4.1 Maximality results for logics not closed under negation

In this section we study the orderings of logics not necessarily closed under negation, with respect to compactness and and Löwenheim-Skolem properties. We see there is no maximal set logic (cf. Definition 2.5) with respect to compactness and Löwenheim-Skolem properties.

Theorem 15 *If we do not assume negation, no small logic is maximal with respect to compactness and Löwenheim-Skolem properties.*

In oder to prove theorem 15, we need some preliminary definitions and theorems.

Definition 16 *Suppose K is a class of models. Suppose $\mathcal{L}$ and $\mathcal{L}$' are logics. We say that $\mathcal{L}$ is partially K-reducible to $\mathcal{L}$' if for any $\phi \in \mathcal{L}$ there is $\phi^* \in \mathcal{L}'$, such that $\phi \to \phi^*$ in all models, and $\phi \leftrightarrow \phi^*$ in models in K.*

In particular, a generalized quantifier Q is partially K-reducible to first-order logic if for any $\phi(\vec{x}) \in \mathcal{L}_{\omega\omega}$ there is $\phi^*(\vec{x}) \in \mathcal{L}_{\omega\omega}$, such that $Q(\vec{x})\phi(\vec{x}) \to \phi^*(\vec{x})$ in all models, and the converse is true in models in K.

Lemma 17 *Let K be a class of models containing all models of cardinality at most λ. Suppose a logic $\mathcal{L}$ is partially K-reducible to a compact logic $\mathcal{L}'$ with $l_\Sigma = \lambda$. Then $\mathcal{L}$ is countably compact.*

Proof. Let Φ be a finitely satisfiable set of $\mathcal{L}$-sentences. Let Φ^* be the set of $\mathcal{L}'$ -sentences that are the reductions of each sentence in Φ. Φ^* is finitely satisfiable, for take any finite set $\Sigma^* \subset \Phi^*$, and look at the corresponding set of sentences $\Sigma \subset \Phi$. By hypothesis, Σ has a model, and it is a model of each of the sentences in Σ^*, so Φ^* is finitely satisfiable. By compactness, it has a model, and as the Löwenheim number for countable sets of sentences of $\mathcal{L}'$ is λ, it has a model $\mathfrak{M}$ of that cardinality. As $\mathfrak{M}$ is a model with cardinality λ of each sentence in Φ^*, it is a model of each sentence in Φ. □

Theorem 18 *Let $\mathcal{L}$ be a compact logic. If we do not assume negation, there is no maximal compact set logic that extends $\mathcal{L}$.*

Proof. Let $\mathcal{L}$ be a compact logic not necessarily closed under negation. Let λ be its Löwenheim number. Add to $\mathcal{L}$ all finite classes plus the class K_κ of models of the generalized quantifier $Q^{\leq\kappa}x(x = x)$, whose interpretation is "there are at most κ elements". We can find $\kappa > \lambda$ such that $K_\kappa \notin \mathcal{L}$, because otherwise $\mathcal{L}$ would be a proper class. Then $\mathcal{L}(Q^{\leq\kappa})$ makes a proper extension of $\mathcal{L}$.

Claim: $\mathcal{L}(Q^{\leq\kappa})$ is compact.

Let K be the class of models of cardinality λ. Then $\mathcal{L}(Q^{\leq\kappa})$ is K-reducible to $\mathcal{L}$ by the sentece $\exists x(x = x)$. □

Lemma 19 *Let K be ta class of models containing all countable models. Suppose $\mathcal{L}$ is partially K-reducible to a logic $\mathcal{L}'$ with the Downward Löwenhein-Skolem property . Then $\mathcal{L}$ satisfies Downward Löwenhein-Skolem property.*

Proof. Let ϕ be an $\mathcal{L}$-sentence with an infinite model. Let ϕ^* be a K-partial reduction of ϕ to $\mathcal{L}'$. By Downward Löwenhein-Skolem theorem, ϕ^* has a countable model, and by definition of partial reduction, ϕ has also a countable model. □

Theorem 20 *Let $\mathcal{L}$ be a logic with with Löweneheim-Skolem theorem. If we do not assume negation, there is no maximal small logic with respect to Löweneheim-Skolem theorem that extends $\mathcal{L}$.*

Proof. Let $\mathcal{L}$ be an the Löwenheim-Skolem property logic not necessarily closed under negation. Add to $\mathcal{L}$ all finite classes plus the class K_κ of models of the generalized quantifier $Q^{\leq\kappa}x(x = x)$, whose interpretation is "there are at most κ elements". We can find $\kappa > \omega$ such that $K_\kappa \notin \mathcal{L}$, because otherwise $\mathcal{L}$ would be a proper class. Then $\mathcal{L}(Q^{\leq\kappa})$ makes a proper extension of $\mathcal{L}$. *Claim:* $\mathcal{L}(Q^{\leq\kappa})$ is the Löwenheim-Skolem property.

Let K be the class of models of cardinality ω. Then $\mathcal{L}(Q^{\leq\kappa})$ is K-reducible to $\mathcal{L}$ by the sentece $\exists x(x = x)$. □

From theorems 17 and 20, we conclude that there is no maximal compact logic with the Löwenheim-Skolem property.

5 Conclusions and further research

Although some of the ideas of this paper were floating around in the literature, we think we have given a framework from which we can understand better the relations between interpolation and maximality. As Prof. Väänänen put it, Lindström's is more than a single theorem, is a phenomenon occuring throughout logics, and it is good to understand it in a framework common to the logics of interest, in this case, logics satisfying also the interpolation theorem.

Understanding the role of negation, we can appreciate that the connections between interpolation and maximality results fade away in case of logics with interpolation. The natural continuation of this investigation, hence, would be to explore how the negative results in interpolation theorems for logics with generalized quantifiers (in particular monadic generalized quantifiers) translates for logics without negation.

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