

On Two Conjectures about Practical Numbers

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A positive integer m is said to be a practical number if every integer n , with $1 \leq n \leq \sigma(m)$, is a sum of distinct positive divisors of m . In this note we prove two conjectures of Margenstern:

- (i) every even positive integer is a sum of two practical numbers;
- (ii) there exist infinitely many practical numbers m such that $m - 2$ and $m + 2$ are also practical. © 1996 Academic Press, Inc.

INTRODUCTION

A positive integer m is said to be *practical* if every n with $1 \leq n \leq \sigma(m) = \sum_{d|m} d$ is a sum of distinct positive divisors of m .

In [4] B. M. Stewart showed that $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, with primes $p_1 < p_2 < \cdots < p_k$ and integers $\alpha_i \geq 1$, is practical if and only if either $m = 1$ or $p_1 = 2$ and for every $i = 2, 3, \dots, k$, $p_i \leq \sigma(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{i-1}^{\alpha_{i-1}}) + 1$. In 1950 Erdős [1] announced that practical numbers have zero asymptotic density. Moreover Hausman and Shapiro [2] showed that if P is the set of practical numbers and if

$$P(x) = \sum_{m \leq x, m \in P} 1,$$

then

$$P(x) = o\left(\frac{x}{(\log x)^\beta}\right)$$

with $\beta < \frac{1}{2}(1 - 1/\log 2)^2 \simeq 0.0979$. In [3] Margenstern conjectured that

$$P(x) \sim \lambda \frac{x}{\log x}$$

with $\lambda \simeq 1.341$. Many other properties of practical numbers similar to those of primes are proved in [3]: for example, a statement similar to Dirichlet's theorem holds for practical numbers.

Margenstern [3] also states two "Goldbach-type" conjectures for practical numbers:

- (i) every even positive integer is a sum of two practical numbers;
- (ii) there exist infinitely many practical numbers m such that $m-2$ and $m+2$ are also practical.

The aim of this note is to prove these conjectures.

1

In this section we prove that every even positive integer is a sum of two practical numbers.

LEMMA 1. *If m is a practical number and n is an integer such that $1 \leq n \leq \sigma(m) + 1$, then mn is a practical number. In particular, for $1 \leq n \leq 2m$, mn is practical.*

Proof. The first assertion easily follows from Stewart's structure theorem; see also [3] p. 6. Since $m-1$ is a sum of distinct divisors of m , we have $m + (m-1) \leq \sigma(m)$, thus proving the second assertion. ■

LEMMA 2. *If m and $m+2$ are two practical numbers, then every even integer $2n$ with $m^2 \leq 2n \leq 3m^2$ is a sum of two practical numbers.*

Proof. We split up the interval $[m^2, 3m^2]$ into the union of three subintervals.

- (i) We have

$$m^2 = \frac{m}{2}m + \frac{m}{2}m,$$

$$m^2 + 2m = m + (m+1)m,$$

and for $1 \leq k \leq m-1$

$$m^2 + 2m - 2k = km + (m-k)(m+2),$$

whence, by Lemma 1, every $2n$ satisfying $m^2 \leq 2n \leq m^2 + 2m$ is a sum of two practical numbers.

(ii) Every $2n$ satisfying $m^2 + 2m < 2n \leq 2m^2$ can be obviously represented as $2m^2 - 2mh - 2j$ for $0 \leq h \leq m/2 - 2$ and $0 \leq j \leq m - 1$. Therefore

$$2n = 2m^2 - 2mh - 2j = (m - 2(h + 1) + j)m + (m - j)(m + 2),$$

whence, again by Lemma 1, $2n$ is a sum of two practical numbers.

(iii) Let now $2m^2 < 2n \leq 3m^2$. We can represent $2n$ as $2m^2 + 2mh + 2j$ for $0 \leq h \leq m/2 - 1$ and $1 \leq j \leq m$. We have

$$2n = 2m^2 + 2mh + 2j = (m + 2(h - 1) - j)m + (m + j)(m + 2),$$

whence, by Lemma 1, $2n$ is a sum of two practical numbers, except for four exceptional cases which we deal with as follows:

$$2m^2 + 2m - 4 = (m + 2)m + (m - 2)(m + 2)$$

$$2m^2 + 2m - 2 = (m + 1)m + (m - 1)(m + 2)$$

$$2m^2 + 2m = m \cdot m + m(m + 2)$$

$$2m^2 + 4m = m(m + 2) + m(m + 2). \quad \blacksquare$$

THEOREM 1. *Every even positive integer is a sum of two practical numbers.*

Proof. Suppose we have a sequence $\{m_n\}_{n \in \mathbb{N}}$ such that for every $n \in \mathbb{N}$

- (i) m_n is practical
- (ii) $m_n + 2$ is practical
- (iii) $1 < m_{n+1}/m_n < \sqrt{3}$.

Since, by (iii), the intervals $[m_n^2, 3m_n^2]$ and $[m_{n+1}^2, 3m_{n+1}^2]$ overlap, every even positive integer $2n \geq m_1^2$ is a sum of two practical numbers by Lemma 2. In [3], Margenstern verified that every even positive integer $2n < 100000$ is a sum of two practical numbers. Hence to prove the theorem it suffices to construct a sequence verifying (i), (ii), (iii) with $m_1 < 100000^{1/2}$. We shall actually construct a sequence $\{m_n\}$ satisfying (i), (ii) and a condition somewhat stronger than (iii), i.e. $1 < m_{n+1}/m_n < \frac{3}{2}$.

Let $S_0 = \{28, 40, 54, 78, 88, 126, 160, 208, 306, 448, 558, 810, 868\}$. It is easy to check that for every $r \in S_0$, r and $r + 2$ are practical numbers. Denote $S_0 = \{r_{0,1}, r_{0,2}, \dots, r_{0,h_0}\}$ with $r_{0,1} < r_{0,2} < \dots < r_{0,h_0}$. One easily sees that $r_{0,i} < \frac{3}{2}r_{0,i-1}$ ($i = 2, 3, \dots, h_0$) and $r_{0,h_0} = r_{0,1}^2 + 3r_{0,1}$. Define, for $k = 1, 2, \dots$,

$$S_k = \{r_{k-1,i}^2 + 3r_{k-1,i}, \frac{3}{2}r_{k-1,i}^2 + 4r_{k-1,i}\}_{i=1, 2, \dots, h_{k-1}} = \{r_{k,1}, r_{k,2}, \dots, r_{k,h_k}\}$$

with $r_{k,1} < r_{k,2} < \dots < r_{k,h_k}$. Moreover, let $S = \bigcup_{k=0}^{\infty} S_k$. If we write $S = \{m_n\}_{n \in \mathbb{N}}$, with $m_n < m_{n+1}$ for every $n \in \mathbb{N}$, we have to show that $\{m_n\}$ satisfies (i), (ii), and $m_{n+1} < \frac{3}{2}m_n$. We have already checked this for the set S_0 . Since $r^2 + 3r = r(r+3)$, $r^2 + 3r + 2 = (r+2)(r+1)$, $\frac{3}{2}r^2 + 4r = r(\frac{3}{2}r+4)$, $\frac{3}{2}r^2 + 4r + 2 = (r+2)(\frac{3}{2}r+1)$, (i) and (ii) hold for every set S_k by induction.

We now show, by induction on k , that $r_{k,i} < \frac{3}{2}r_{k,i-1}$ for all $k \geq 0$ and $i = 2, 3, \dots, h_k$. This is true for $k=0$. Assuming that $r_{k,l} < \frac{3}{2}r_{k,l-1}$ for some k and $l = 2, 3, \dots, h_k$, we have, for any fixed $i \geq 2$, either $r_{k+1,i} = \frac{3}{2}r_{k,j}^2 + 4r_{k,j}$ or $r_{k+1,i} = r_{k,l}^2 + 3r_{k,l}$ for some $j \geq 1$ or $l \geq 2$ respectively. If $r_{k+1,i} = \frac{3}{2}r_{k,j}^2 + 4r_{k,j}$, then

$$\frac{r_{k+1,i}}{r_{k+1,i-1}} \leq \frac{\frac{3}{2}r_{k,j}^2 + 4r_{k,j}}{r_{k,j}^2 + 3r_{k,j}} = \frac{3}{2} \cdot \frac{1 + (8/3)r_{k,j}}{1 + (3/r_{k,j})} < \frac{3}{2}.$$

If $r_{k+1,i} = r_{k,l}^2 + 3r_{k,l}$, then either

$$\frac{3}{2}(r_{k,l-1}^2 + 3r_{k,l-1}) > r_{k,l}^2 + 3r_{k,l},$$

whence

$$\frac{r_{k+1,i}}{r_{k+1,i-1}} \leq \frac{r_{k,l}^2 + 3r_{k,l}}{r_{k,l-1}^2 + 3r_{k,l-1}} < \frac{3}{2},$$

or

$$\frac{3}{2}(r_{k,l-1}^2 + 3r_{k,l-1}) \leq r_{k,l}^2 + 3r_{k,l},$$

whence

$$\frac{3}{2}r_{k,l-1}^2 + 4r_{k,l-1} < \frac{3}{2}(r_{k,l-1}^2 + 3r_{k,l-1}) \leq r_{k,l}^2 + 3r_{k,l},$$

which implies that

$$\frac{r_{k+1,i}}{r_{k+1,i-1}} \leq \frac{r_{k,l}^2 + 3r_{k,l}}{\frac{3}{2}r_{k,l-1}^2 + 4r_{k,l-1}} < \frac{\frac{9}{4}r_{k,l-1}^2 + \frac{9}{2}r_{k,l-1}}{\frac{3}{2}r_{k,l-1}^2 + 4r_{k,l-1}} = \frac{3}{2} \cdot \frac{1 + (2/r_{k,l-1})}{1 + (8/3r_{k,l-1})} < \frac{3}{2}$$

by the inductive assumption. This proves that $m_{n+1} < \frac{3}{2}m_n$, provided that $m_{n+1} = r_{k,i}$ for some $k \geq 0$ and $i \geq 2$. To complete the proof of the theorem, we must prove that $m_{n+1} < \frac{3}{2}m_n$ when $m_{n+1} = r_{k,1} = r_{k-1,1}^2 + 3r_{k-1,1}$ for some $k \geq 1$. In this case we show, by induction on k , that

$$r_{k,1} = r_{k-1,1}^2 + 3r_{k-1,1} \in S_{k-1}.$$

This is true for $k=1$ since $r_{1,1} = r_{0,1}^2 + 3r_{0,1} = r_{0,h_0} \in S_0$. Assuming that $r_{k,1} \in S_{k-1}$ for some k , we have, by definition of S_k ,

$$r_{k+1,1} = r_{k,1}^2 + 3r_{k,1} \in S_k.$$

This shows that $r_{k,1} = r_{k-1,j}$ for some $j \geq 2$. Hence, by the previous argument, $m_{n+1} = r_{k,1} = r_{k-1,j} < \frac{3}{2}m_n$. ■

2

THEOREM 2. *There exist infinitely many practical numbers m such that both $m - 2$ and $m + 2$ are also practical.*

Proof. We shall prove that, for every $k \in \mathbb{N}$, $2(3^{3^k \cdot 70} - 1)$, $2 \cdot 3^{3^k \cdot 70}$, $2(3^{3^k \cdot 70} + 1)$ are practical numbers.

By Stewart's structure theorem, $2 \cdot 3^{3^k \cdot 70}$ is obviously practical. We separately show, by induction on k , that $2(3^{3^k \cdot 70} - 1)$ and $2(3^{3^k \cdot 70} + 1)$ are practical. We have

$$2(3^{70} - 1) = 2^4 \cdot 11^2 \cdot 61 \cdot 71 \cdot 547 \cdot 1093 \cdot 2664097031 \cdot 374857981681$$

and, by the structure theorem, this is a practical number. Suppose that $2(3^{3^k \cdot 70} - 1)$ is practical for some k . Then

$$2(3^{3^{k+1} \cdot 70} - 1) = 2(3^{3^k \cdot 70} - 1)(3^{3^k \cdot 70} - 3^{3^k \cdot 35} + 1)(3^{3^k \cdot 70} + 3^{3^k \cdot 35} + 1)$$

whence, by Lemma 1 applied twice, $2(3^{3^{k+1} \cdot 70} - 1)$ is practical.

We now have

$$2(3^{70} + 1) = 2^2 \cdot 5^2 \cdot 29 \cdot 1181 \cdot 16493 \cdot 28596961 \cdot 32839661 \cdot 94373861$$

and, by the structure theorem, this is a practical number. Suppose that $2(3^{3^k \cdot 70} + 1)$ is practical. Then

$$2(3^{3^{k+1} \cdot 70} + 1) = 2(3^{3^k \cdot 70} + 1) \phi_{12}(3^{3^k}) \phi_{60}(3^{3^k}) \phi_{84}(3^{3^k}) \phi_{420}(3^{3^k})$$

where $\phi_d(x)$ is the cyclotomic polynomial for $\exp(2\pi i/d)$. Here

$$\phi_{12}(x) = x^4 - x^2 + 1$$

$$\phi_{60}(x) = x^{16} + x^{14} - x^{10} - x^8 - x^6 + x^2 + 1$$

$$\phi_{84}(x) = x^{24} + x^{22} - x^{18} - x^{16} + x^{12} - x^8 - x^6 + x^2 + 1$$

$$\begin{aligned} \phi_{420}(x) &= x^{96} - x^{94} + x^{92} + x^{86} - x^{84} + 2x^{82} - x^{80} + x^{78} + x^{72} - x^{70} + x^{68} \\ &\quad - x^{66} + x^{64} - x^{62} - x^{56} - x^{52} - x^{48} - x^{44} - x^{40} - x^{34} + x^{32} - x^{30} \\ &\quad + x^{28} - x^{26} + x^{24} + x^{18} - x^{16} + 2x^{14} - x^{12} + x^{10} + x^4 - x^2 + 1. \end{aligned}$$

Applying four times Lemma 1, we see that $2(3^{3^{k+1} \cdot 70} + 1)$ is practical, and the theorem is proved. ■

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