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GÖDEL'S INCOMPLETENESS THEOREM AND THE PHILOSOPHY OF OPEN SYSTEMS

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1. Introduction

In recent years a number of criticisms have been raised against the formal systems of mathematical logic. The latter, qualified as *closed systems*, have been contrasted with systems of a new kind, called *open systems*, whose main feature is that they are always subject to unanticipated outcomes in their operation and can receive new information from outside at any time (cf. Hewitt 1991). While Gödel's incompleteness theorem has been widely used to refute the main contentions of Hilbert's program, it does not seem to have been generally used to point out the inadequacy of a basic ingredient of that program - the concept of formal system as a closed system - and to stress the need to replace it by the concept of formal system as an open system.

A partial exception seems to be provided by van Heijenoort who states:

> «The notion of formal system, introduced by Frege in 1879, had become by then the accepted standard of precision in the foundations of mathematics. It seemed to embody the Aristotelian ideal of a perfect deduction from first principles. Gödel's results, by showing that mathematics cannot be completely and consistently formalized in one system, shattered this ideal. The bounds of mathematics cannot be those of one formal system. Since mathematics has often been regarded as the standard of rational knowledge that other sciences should strive to attain, Gödel's theorems seem to acquire significance for the whole body of human knowledge; they certainly establish that the old ideal of a deductive system cannot be maintained» (1967: 356).

Here van Heijenoort rightly points out the impact of Gödel's result on the concept of formal system as a closed system but does not suggest any alternative to such a concept.

In this paper, on the one hand, I want to stress the role of Gödel's incompleteness theorem in showing the inadequacy of the concept of formal system as a closed system, and, on the other hand, I want to point out the interest of the concept of formal system as an open system, which is essential both for current developments in artificial intelligence and for the emergence of a new paradigm of logic, alternative to mathematical logic: *computational logic* (see also Cellucci 1990, 1992).

2. The logicist concept of formal system

At least in its most elementary form, the notion of open system is not a new one. One of its ancestors is provided by Frege's concept of formal system, but it has also much older ancestors (see section 9 below).

It is usually claimed that «Frege was the first to present, with all the necessary accuracy, the notion of formal system» (van Heijenoort 1985: 11). The value of such claim depends on how it is interpreted. It is true if it is meant in the sense that Frege was the first to present a notion of formal system. It is false if it is meant in the sense that he introduced the notion of formal system which has been used in mathematical logic since the Thirties.

The latter interpretation is given, for instance, by Dummett in his book on Frege. Dummett acknowledges that Frege's aim in constructing his formal systems was not to make proofs themselves the objects of mathematical investigation, but only «to attain the ideal of that rigour to which the whole of nineteenth-century mathematics had been striving» (1981: *xxxiv-xxxv*).

On the other hand Dummett also claims that «Frege's calculus was a formal system in the modern sense» (1981: *xxxiv*). This view, that Frege's notion of formal system was the same as the notion of formal system of current mathematical logic, seems to me quite misleading. A basic difference stems from the fact that it is an important part of Frege's conception of logic that nothing can be, or has to be, said outside the system: since logic is the system, anything that can be said must be said within the system. This was called by Sheffer the 'logocentric predicament' of logicism (cf. Sheffer 1926). A consequence of this position is that, since logic is the universal system within which every rational discourse must be made, there is no law court outside logic by which logic can be judged: no metasystematic question on logic can be raised (cf. van Heijenoort 1985: 13; Goldfarb 1979; Cellucci 1987).

Indeed, no metasystematic question on logic was ever raised by Frege. Of course Frege was fully aware that, in order to introduce a formal system, one needs formal rules that are not expressed in the language of the system, but considered such rules as void of any intuitive logic. Indeed, in the *Begriffsschrift* he introduces a number of fundamental principles of thought

> «in order to transform them into rules for the use of our signs. These rules and the laws whose transforms they are cannot be expressed in the ideography because they form its basis» (1967: 28).

Since such rules cannot be expressed in the ideography, they are just rules for the use of signs, void of any intuitive logic (cf. van Heijenoort 1985: 13).

A similar position was held by Russell. But, as regards the concept of formal system, Russell took a backward step with respect to Frege. This was stressed by Gödel maintaining that the system of *Principia Mathematica* is greatly lacking in formal precision, not only because it provides no precise statement of the syntax of the formalism, but also because it omits syntactical considerations, even in cases where they are necessary for the cogency of the proofs (Gödel 1964).

In one respect, however, Russell went even further than Frege: not only he never raised any metasystematic question on logic, he even considered impossible to raise such a question.

For example, in the first edition of *The Principles of Mathematics*, Russell claims that the method used in geometry to establish the independence of the axiom of parallels cannot be used to establish the independence of his axioms for propositional calculus, because all such axioms

> «are principles of deduction; and if they are true, the consequences which appear to follow from the employment of an opposite principle will not really follow» (1937: 15).

Furthermore, in the introduction to the second edition of *The Principles of Mathematics*, Russell says:

> «It is difficult to see any way of proving that the system resulting from a given set of premisses is complete, in the sense of embracing everything that we would wish to include among logical propositions» (1937: *xii*).

To appreciate the interest of this statement, note that it was made a number of years after the discovery of Gödel's result. The reason for Russell's statement was that, from the viewpoint of Frege and Russell by which logic is the universal system within which every rational discourse must be made, the question of completeness, in the metasystematic sense of current mathematical logic, simply did not arise.

The idea that logic is the system and that anything that can be said must be said within the system, together with its corollary that no metasystematic question can be raised, was also shared by Wittgenstein. Indeed it is one of the main themes of the *Tractatus* where it is stated that, since the limits of the world are also the limits of logic, one cannot say in logic that this there is in the world while that there is not since «otherwise logic must get outside the limits of the world: only in this way one could consider these limits from the other side also» (Wittgenstein 1961: 5.61). Here Wittgenstein makes fully explicit the implications of Frege's conception of logic as a total system for which no metasystematic question can be raised, and carries them to extremes: raising metasystematic questions would entail getting outside the limits of the world.

3. The logicist view of logic as an open system

The view of Frege, Russell and Wittgenstein on logic as the universal system implies that logic is an open system. This can be seen as follows.

Frege's first motivation in the *Begriffsschrift* was to establish the fundamental principles of mathematics: «I became aware of the need for a *Begriffsschrift* when I was looking for the fundamental principles or axioms upon which the whole of mathematics rests» (1969: 1). This raises the problem: once a certain number of principles have been laid out as the fundamental principles or axioms upon which the whole of mathematics rests, how can one make sure that such principles are complete?

Russell's answer to this problem was that the only question of completeness that could be raised was completeness in an empirical sense: do the rules exhaust the intuitive modes of reasoning actually used in science?

All the works of Frege and Russell, from the *Begriffsschrift* to *Principia Mathematica*, can be regarded as steps towards establishing completeness experimentally. This concept of completeness in an empirical sense is stated in the preface to *Principia Mathematica* by claiming that the evidence in favour of any theory on the principles of mathematics «must always be inductive, i.e. it must lie in the fact that the theory in question enables us to deduce ordinary mathematics» (Whitehead and Russell 1927: *v*).

The problem of completeness of *Principia Mathematica* had for Russell this empirical meaning, not the metasystematic meaning familiar from current mathematical logic. For him, existing mathematics was a datum to be explained in logical terms: one need only show that, starting from the axioms of *Principia Mathematica*, «such datum can be reconstructed» (Russell 1937: *xii*). What worried Russell was not that the system of *Principia Mathematica* could turn out to be incomplete in the metasystematic sense, but only that it could turn out to be incomplete in an empirical sense, i.e. that some parts of ordinary mathematics could not be reconstructed from the axioms of *Principia Mathematica.*

The discovery of parts of mathematics not derivable from these axioms would raise the problem: are such parts incorrect, or are the axioms inadequate? In the latter case the axioms would have to be modified in order to accomodate also such parts: here lies the character of open system of *Principia Mathematica.* From this viewpoint Gödel's incompleteness theorem raised no problem: a formally undecidable Gödel sentence simply did not belong to ordinary mathematics.

On the other hand, while the existence of formally undecidable metasystematic sentences, like Gödel sentences, was irrelevant to the question of completeness of *Principia Mathematica*, the existence of formally undecidable *mathematical* sentences would be quite relevant to that question. Now, in the last decade several examples of mathematical sentences formally undecidable in elementary number theory have been discovered, such as Goodstein's theorem, the Paris-Harrington variant of the finite Ramsey's theorem and the Friedman-Takeuti miniature version of Kruskal's theorem (see e.g. Takeuti 1987: 120-147).

These examples show elementary number theory to be incomplete in an empirical sense. It has been objected that such sentences, while being mathematical, are not strictly number-theoretical, because there would be a sense to 'number-theoretical' different from expressibility in the language of elementary number theory, but the alleged alternative sense that has been proposed does not seem to be quite clear or convincing (cf. Isaacson 1987). In any case, these sentences would be no problem for Russell because they are provable in *Principia Mathematica.*

4. The formalist concept of formal system

I have already mentioned that, as regards the concept of formal system, Russell took a backward step with regard to Frege. The notion of formal system was again brought into the forefront by Hilbert in the Twenties, but with an essential twist: it was no longer assumed that there is a universal system in the sense of Frege, Russell and Wittgenstein, and formal systems were no longer conceived as open systems. This was stated by Hilbert as follows:

> «No more than any other science can mathematics be founded by logic alone; rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought (...) All the propositions that constitute mathematics are converted into formulas, so that mathematics proper becomes an inventory of formulas (...) Wherever the axiomatic method is used it is incumbent upon us to prove the consistency of the axioms» (1967: 464-465, 472).

By stating that, no more than any other science, can mathematics be founded by logic alone - and hence denying the system of *Principia Mathematica* the status of the all-embracing logic - Hilbert claims that no universal system is possible. By stating that, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation, he denies that the inference rules are void of any intuitive logic. By stating that all propositions that constitute mathematics are converted into formulas, Hilbert claims that the content of every mathematical theory can be given by its representation in a formal system. Finally, by considering the task of giving a consistency proof, he appears to have a metasystematic standpoint.

While for Frege and Russell the rules of inference were the rules of conduct for the study of mathematics, for Hilbert they became the object of study. Thus he was led to a proof theory which deals with the proofs themselves. As in the concrete intuitive number theory the numbers are something which is objective and can be exhibited, in proof theory «it is just proof that is something which is concrete and can be exhibited; contentual arguments are only about proofs» (Hilbert 1922: 169-170).

Making proofs the object of study was Hilbert's way of implementing his metasystematic standpoint.

5. The formalist view of logic as a closed system

Hilbert's approach entails that formal systems are no longer conceived as open systems. They are formula games carried out according to certain definite rules which «form a closed system that can be discovered and definitively stated» (1967: 475).

Hilbert's influence was such that the concept of formal system became synonymous with closed system. Thus Hertz uses 'closed system of propositions' instead of 'formal system' (cf. Hertz 1929). Zermelo uses 'logically closed system' in the same sense (cf. Zermelo 1929). Tarski follows on: «Every set of sentences which contains all its consequences is called a deductive system, or possibly a closed system, or simply a system» (1956: 69-70).

It is true that Hilbert admits that formal systems are constantly extended: «The axiom system is constantly extended and the formal construction, corresponding to our constructive tendency, becomes more and more complete» (1922: 169). In particular, formal systems are constantly extended by adding new concepts. However Bernays, in his 1930 paper on Hilbert's program, stresses that an extension of a formal system by new concepts may be conservative, i.e. it may not lead to new results in the original language of the system.

This condition is satisfied when the formal system is deductively closed, i.e. complete in the syntactical sense. Now, «as regards number theory, as determined by Peano's axioms with the addition of recursive definitions, we believe that it is, in such sense, deductively closed» (Bernays 1976: 59). This shows that Bernays considered the possibility of constantly extending a formal system to be perfectly compatible with its deductive closure and with its character of closed system.

Indeed he had to admit such a compatibility if he wanted to preserve Hilbert's view of formal systems as closed systems. Such a view was implicit in the fact that Hilbert had a metasystematic standpoint: asking metasystematic questions about a formal system S involves referring to the collection of theorems of S as to a given totality, and hence considering it as a closed system.

Gödel's incompleteness theorem shows that the concept of formal system as a closed system is inadequate for mathematics. Indeed it establishes that each formal system for number-theoretical truth must admit proper extensions, and hence the choice of any particular formal system for number-theoretical truth would be intrinsically provisional, subject to an eventual need to go beyond it. Therefore one may conclude that the concept of formal system as a closed system is inadequate for number-theoretical truth and hence is inadequate for mathematics.

As anticipated by the remarks of Bernays already mentioned, this conclusion seems to be in conflict with an authoritative interpretation of Gödel's incompleteness theorem: the one provided by Bernays in his 1930 paper on Hilbert's program and taken over by Hilbert and Bernays in their joint work *Grundlagen der Mathematik*.

6. The Hilbert-Bernays interpretation of Gödel's incompleteness theorem

In the second volume of the *Grundlagen der Mathematik* Hilbert and Bernays state that, by Gödel's result,

> «the idea of characterizing the whole of mathematics as a deductive formalism, as sometimes suggested by the logistic systems, appears inadequate (...) We have avoided introducing the idea of a total system for mathematics (...) Instead we have contented ourselves with characterizing the actually existing system of analysis and set theory as providing an adequate framework for accomodating the geometrical and physical disciplines. A formalism may correspond to this aim even without having the property of full deductive closure. Our conception (...) very well agrees with the deductive openness of this system: the inference modes in the system are oriented in accordance with the representation of a closed, totally determined reality and give formal expression to this representation; but from this it does not

> follow that the deductive structure (...) resulting from these inference modes should have that property of total closure» (1970: 289-290).

Such an interpretation seems inadequate in at least two respects. First, Hilbert and Bernays claim that Gödel's incompleteness theorem refutes the logicist view that logic is the system and that anything that can be said must be said within the system. Now, as we have already seen, the logicist view has as corollary that no metasystematic question can be raised and that the only question of completeness that can be raised is completeness in an empirical sense: hence the logicist view is unaffected by Gödel's result.

Secondly, while reasserting that formal systems are closed systems, Hilbert and Bernays claim that their incompleteness is compatible with their adequacy for mathematical practice, at least relative to the part of mathematics used in geometrical and physical applications. This amounts to saying that such systems, while incomplete in the metasystematic sense, may be complete in an empirical sense, which clashes with Hilbert's aim «to eliminate once and for all the questions regarding the foundations of mathematics» (1967: 489).

In order to achieve such an aim no empirical approach seems to be possible: how could one make sure that the next theorem discovered in mathematical practice would be provable in the formal system? In order to obtain a definitive solution, nothing less than a reduction of all mathematical reasoning to the mathematically secure finitary reasoning would be required.

Hilbert felt that such a reduction could be obtained, thus definitively eliminating the foundational problem: indeed, in 1930 he even felt that he had already achieved such an aim:

> «I believe that, through my proof theory, I have completely achieved what I wanted and had promised: the problem of foundations of mathematics, as such, is thereby definitively eliminated» (Hilbert 1931: 494).

But, in order to obtain such a definitive solution, finitary reasoning must involve only concrete-intuitive objects, i.e. objects such that they can be surveyed

> «completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects» (Hilbert 1967: 464-465).

Thus, finitary reasoning must not involve abstract objects. Depending only on concrete-intuitive objects, Hilbert's definitive solution to the problem of foundations of mathematics would achieve that separation of mathematics from philosophy which would allow him to claim that «mathematics is a presuppositionless science» (Hilbert 1967: 479).

The currently prevailing interpretation of the Gödel-Hilbert-Bernays theorem on unprovability of consistency is that, while all finitary proofs can be expressed in the formalism of elementary number theory, the proof of its consistency cannot be carried out within that formalism and hence cannot be finitary. However, in his 1931 paper, speaking about the system *P* for which he established his theorem, Gödel suggested that «it is conceivable that there exist finitary proofs that cannot be expressed in the formalism of *P*» (1986: 195).

Such conclusion was considered to be somewhat paradoxical by Hilbert and Bernays who stated that Gödel's incompleteness theorem entails that,

> «in case of success of a finitary consistency proof for the formalism of analysis and set theory, at the same time also a finitary proof of a theorem of recursive number theory would be made possible which is not derivable in that formalism. It appears paradoxical that the methods of finitary proof theory should, under a certain respect, be superior to those of analysis and set theory in proving number-theoretical propositions. We are thus led to the question of the range of finitary methods» (1970: 290).

No convincing solution to this question was provided by Hilbert and Bernays. On this subject Bernays' position seems to be wavering. In 1935 he acknowledged that by Gödel's result one cannot prove by elementary combinatorial methods the consistency of a formal system in which every elementary combinatorial proof of an arithmetical proposition can be represented. An example of such a system is provided by elementary number theory. Indeed,

> «no attempt made up to now has given us any example of an elementary combinatorial proof which cannot be expressed in this formalism, and the methods by which one can, in the cases considered, translate a proof into the aforementioned formalism, seem to suffice in general» (Bernays 1964: 285).

Here Bernays seems to be rejecting the possibility, suggested by Gödel 1931, that it is conceivable that there exist finitary proofs which cannot be expressed in the formalism of elementary number theory. Indeed he recognizes that «means more powerful than elementary combinatorial methods are necessary to prove the consistency of the axiomatic theory of numbers» (Bernays 1964: 285). In his view such a more powerful method is provided by the Gödel-Gentzen result that the consistency of intuitionistic number theory entails the consistency of classical number theory. In order to conclude from this result that classical number theory is consistent, it suffices to assume the consistency of intuitionistic number theory. This proof of the consistency of classical number theory shows us that «intuitionism, by its abstract arguments, goes essentially beyond elementary combinatorial methods» (1964: 285-286).

Here Bernays acknowledges that, in order to prove the consistency of elementary number theory, abstract methods are required and seems to be willing to accept the intuitionistic methods to this effect. However, in the second volume of the *Grundlagen der Mathematik*, Hilbert and Bernays reject the proof of the consistency of elementary number theory via translations into intuitionistic number theory as totally removed from Hilbert's methodological views on proof theory (cf. Hilbert and Bernays 1970: 372). On the other hand, in the Introduction to that volume Bernays stresses the necessity to extend the previous delimitation of the 'finitary standpoint' and claims that «the discussion of the extension of the finitary standpoint leads to consider Gentzen's new consistency proof for the number-theoretical formalism» (1970: *vii*). Here, then, Bernays seems to be willing to extend the finitary standpoint so as to include Gentzen's proof.

In no sense, however, Gentzen's proof can be considered to be finitary, if finitary reasoning must involve no abstract objects. As stressed by Gödel in his 1958 *Dialectica* paper, the validity of recursion up to the first epsilon number cannot be made immediately intuitive because it is impossible to visualize the many different ways in which descending sequences may be structured, and hence it is impossible to intuitively know that each such sequence must break off. In particular, «if one comes to know it by moving step by step from lower to higher ordinals, this will not constitute knowing it by *inspection*. It will merely be an abstract knowledge» (Gödel 1980: 134).

The fact that Gentzen's proof involves essentially abstract knowledge entails that it cannot be considered as finitary. Indeed, this opinion seems to be shared by Gödel in his *Dialectica* paper, where he implicitly rejects his previous 1931 suggestion that it is conceivable that there exist finitary proofs which cannot be expressed in the formalism of elementary number theory. Indeed, apparently approvingly, he mentions Bernays' 1935 view that, in order to prove the consistency of the number-theoretical formalism, it is necessary to extend Hilbert's finitary standpoint by admitting certain abstract concepts in addition to the concrete-intuitive ones. With the only difference that, while the abstract concepts considered by Bernays 1935 in order to go beyond finitary reasoning were those of intuitionism, Gödel instead intended to use the concept of primitive recursive functional of finite type.

Both Gödel's 1931 view and *Dialectica* view have a devastating impact on Hilbert's program. The 1931 view raises the problem that, for any formal system **S** which is supposed to be adequate for elementary number theory, there must be a finitary proof in the extended sense which cannot be carried out in **S**. Thus the finitary methods in the extended sense cannot be fixed once and for all, but must be suitably chosen for any specific formal system **S** considered, i.e. they are dependent on **S**. This contradicts Hilbert's assumption that, in order to obtain a definitive solution to the question of foundations of mathematics, the finitary methods must be absolute, not relative to S.

On the other hand, the *Dialectica* view raises the problem that the question of the consistency of any specific S which is supposed to be adequate for elementary number theory - and hence adequate for finitary reasoning - cannot be settled conclusively, because the consistency of S can be expressed in the language of S but cannot be decided by S. Hence the choice of S becomes problematic because, by Gödel's incompleteness theorem, the completeness criterion is not available in all the essential cases. This contradicts Hilbert's aim to obtain a definitive solution to the question of foundations of mathematics.

Quite rightly, Hilbert and Bernays considered the possibility suggested by Gödel 1931 - that there exist finitary proofs which cannot be expressed in the formalism of elementary number theory - to be somewhat paradoxical. An obvious alternative is to interpret Gödel's incompleteness theorem as providing a conclusive refutation of Hilbert's program and of the underlying assumption that formal systems must be closed systems. By denying this natural conclusion, Hilbert and Bernays were lead to consider the former paradoxical possibility and were unable to produce a sense of 'finitary' which, while substantiating that possibility, would be compatible with Hilbert's aim of providing a definitive solution to the problem of foundations of mathematics.

7. Logic programming and open systems

The proper interpretation of Gödel's incompleteness theorem seems to be that the choice of any particular formal system for number-theoretical truth is intrinsically provisional, subject to an eventual need to go beyond it: hence the concept of formal system as a closed system is inadequate for number-theoretical truth and must be replaced by that of formal system as an open system.

This conclusion, however natural, did not find its way easily. For a long time people preferred to preserve the concept of formal system as a closed system and tried to cope with the situation raised by Gödel's result by pushing the openness requirement outside the formal system - using Turing's ordinal logics or Feferman's recursive progressions of axiom systems. This, however, was no escape from incompleteness because, for any recursive progression satisfying certain natural conditions, there exists a true number-theoretical sentence which is not provable in the progression (see Feferman and Spector 1962).

The notion of formal system as an open system has been brought again into the forefront only recently, thanks to the emergence of a new paradigm of logic, alternative to mathematical logic: *computational logic*. The origin of such a paradigm can be dated back to the early Seventies, when a programming language of an entirely new kind was developed: *Prolog*. Although it was not realized at the time, an accessory and somewhat marginal feature of Prolog introduced an essentially new concept of proof.

According to the concept of proof underlying Hilbert's proof theory, proofs start from given axioms and lead to theorems by given rules of inference: axioms and rules of inference are fixed beforehand and do not change in the course of proof. The concept of proof introduced in Prolog deviates from Hilbert's in one essential respect: thanks to predicates for manipulating the data base, such as **assert** and **retract**, axioms and rules of inference may change in the course of proof. Such predicates modify the program, so relations holding at a certain time may no longer hold at another time, hence the same question can have different answers at different times.

This capability of Prolog of managing the data base dynamically, is essential for simulating the real world and is one of the main features which distinguish Prolog from the formal systems of mathematical logic. Prolog makes fully explicit what was only implicit in the logicist concept of formal system: the possibility of modifying the system whenever required. While in the logicist concept of formal system this possibility was allowed but was, so to say, accidental and exceptional, in Prolog it becomes part of the definition of formal system itself.

As already mentioned, the fact that computational logic represents a change of paradigm with respect to Hilbert's concept of proof was not realized, however, from the very beginning. The mathematical logic paradigm was so deeply rooted that the innovative changes of Prolog have long been mistaken for its defects. E.g., according to what it is claimed in the first and by now classical book on Prolog:

> «These operations violate the simple self-contained nature of Predicate Calculus propositions (...) the use of **assert** means that the rule is talking about adding something to the set of axioms. In logic, each fact or rule states an independent truth, independent of what other facts and rules there may be. Here we have a rule that violates that principle. Also, if we use this rule, we will be in a position of having a different set of axioms at differents times of the proof! (...) The ultimate goal of a logic programming language has not, then, been achieved with Prolog» (Clocksin and Mellish 1987: 241-242).

Here, what is an important theoretical innovation of Prolog is rightly seen as a deviation from the mathematical logic paradigm, but such a deviation is considered dangerous and to be amended as soon as possible to comply with that paradigm. On the contrary, the proper interpretation of the situation seems to be: almost accidentally and unintentionally, Prolog ran into a concept of proof which is essentially innovative with respect to the concepts of proof of Euclid and Hilbert.

8. Concurrent object-logic programming and open systems

Introducing a new concept of proof, however, was not a main aim of Prolog, but only a byproduct of an accidental and somewhat marginal feature of the language: the inclusion of predicates for manipulating the data base such as **assert** and **retract**. On the other hand, this aim became central in various attempts to combine logic, object-oriented and concurrent programming which have been made in the last decade. In such attempts two distinct approaches can be identified.

The first one is exemplified by languages, like *Concurrent Prolog*, in which concurrency occurs at the micro-level of the single Horn clause (cf. Shapiro and Takeuchi 1987). Such languages, while designed for fast execution on parallel architectures, do not however provide a good syntax for expressing the abstractions of object-oriented programming. At best they may be considered as machine-oriented object-logic languages (cf. Kahn, Tribble, Miller and Bobrow 1987).

The second approach is exemplified by languages, like *logObjects*, in which concurrency occurs at the macro-level of logical systems (cf. Welsch and Barth 1989; see also Mello and Natali 1987). Such languages seem to provide a better integration between logic, object-oriented and concurrent programming.

In languages like logObjects a program is intended as a collection of logical objects. Each logical object consists 1) of a database which represents a chunk of knowledge about a particular domain and 2) of a theorem prover which is able to answer proof requests. Logical objects can communicate by sending messages: a message sent to a particular logical object may be conceived either as a request to modify the database or as a request to prove a goal using the given database. Both kinds of requests trigger a proof search procedure. However, while the answer to a request to modify the database consists either in an acceptance or in a refusal, the answer to a request to prove a goal consists either in the proof of the goal or in the notification of a failure. Thanks to the possibility of making requests to other logical objects, each logical object can be modeled as an open world.

At least potentially, this results into a concept of proof which is even more innovative than that provided by Prolog. In order to show such an innovative character let us consider two examples.

Our first example is given by the flight reservation system, which consists of several concurrent objects, e.g. flights and agents. Flights will register the reservation of seats and ensure that no seats are reserved by two agents at the same time. Both flights and agents may be constructed as logical objects. Now, suppose that two agents are concurrently attempting to reserve seats in the same flight but the number of seats is insufficient to satisfy both requests. The operation will be serialized so that one of the requests will be accepted and the other will be rejected. However, it is indeterminate which of them will be accepted and which will be rejected. The outcome cannot be deductively decided even from complete knowledge of all the circumstances at the time when the two agents send their respective requests. In order to manage conflicts between requests, a negotiation is needed to establish an order of acceptance.

This example illustrates a situation which is typical of concurrent systems in general: they have the property that often the behavior of the system is critically affected by the arrival order of communications. Generally, the decisions on the arrival order in concurrent systems are not deductively derivable, hence logical deduction does not provide an adequate basis for decision making in computations in concurrent systems (see Hewitt 1991).

Our second example is provided by setting the price of an industrial product. The following two principles suggest themselves: 1) greater profitability is better than lower profitability; 2) greater market share is better than lower market share. Such two principles, however, are incompatible: increasing the price generally increases profitability but decreases market share. Two formal systems including 1) and 2), respectively, among their axioms would be inconsistent. Logical deduction provides no mechanism for resolving this conflict: what is needed is a negotiation yielding restrictions on 1) and 2), but such a negotiation cannot be based on logical principles only.

This suggests the need for a concept of formal system far removed from Hilbert's. The correctness of proofs in Hilbert's formal systems can be algorithmically checked without having to make any observations on the external world or to consult any external sources of information. The algorithm is fixed in advance and no other computation is needed other than to read the proof and apply the algorithm. Thus proof checking proceeds in a closed world in which the axioms and the rules of inference have been laid out explicitly beforehand.

Hilbert's concept of formal system has the advantage that proofs in the system have a timeless and acontextual character and can be checked only from the text of the proof. This allows them to be checked by many people at different times and places. However, just because of their timeless and acontextual character, such proofs do not provide an adequate basis for decision making in concurrent systems because decision making is usually crucially time dependent.

For example, in the flight reservation system mentioned earlier, the capability to process conflicts in which multiple agents concurrently attempt to reserve seats in the same flight goes beyond the capabilities of logical deduction. Of course a record can be made of what happened, but this record is not the same as a proof: proofs are produced in advance of any conflict and their correctness can be algorithmically checked.

Whenever conflicts arise, logical deduction does not allow to infer the outcome, because the processes which actually produce the outcome are left out. Decisions in concurrent systems may be justified by a negotiation: such is the case both in the flight reservation system and in setting the price of an industrial product. But justifying a decision by negotiation is essentially different from justifying it by logical proof.

Logical deduction is concerned only with the internal structure of Hilbert's formal systems. As we have already mentioned, its great strength stems from the fact that proofs in such systems have a timeless and acontextual character and can be checked only from the text of the proof. But, by concentrating only on the internal structure of formal systems, logical deduction leaves out something which plays a fundamental role in concurrent systems: communication. Thus the very features of formal systems that give them their great strength turn out to also be their greatest weakness.

The development of a theory of communication has already started (see e.g. Hoare 1985; Milner 1989). In the theory of communication one deals with conflicts, cooperation, negotiation, i.e. concepts which are essential for modeling the real world, as shown by the above examples of the flight reservation system or setting the price of an industrial product (cf. Hewitt 1991). There is, moreover, the problem of integrating logical deduction with communication, which requires further elaboration.

An adequate concept of formal system must include, in addition to logical deduction, also communication: logical deduction is an important component of the concept of formal system but is only a component and has inherent limitations in concurrent systems (cf. Hewitt 1987).

9. Open systems: past and present

In the *Republic*, Plato makes a fierce attack against contemporary geometers. What is the reason for that attack, given that in his previous works, up to and including *Phaedo*, geometry was a research model for him? Most likely, the reason is to be found in the circumstance that, between *Phaedo* and the *Republic*, geometry underwent a radical transformation: the axiomatic revolution.

Geometry, as described in Plato's works before the *Republic*, especially in *Meno* and *Phaedo*, is an open system, based on the analytic method in the form already used by Hippocrates of Chios in his work on the quadrature of lunes. For Plato the axioms of geometry are not the starting point of the proof: the starting point is the proposition to be proved. The aim of the proof is to reduce such a proposition to another one already proved.

While the axiomatic method assumes that axioms are fixed beforehand and hence the system is a closed system, the analytic method does not make such an assumption. Proofs in the axiomatic method establish a global ordering between propositions. On the contrary, the aim of proofs in the analytic method is to establish a local ordering between propositions through a backwards search. The ordering is local because it is only relative to the proposition to be proved: different propositions to be proved may require different orderings. The axioms are not fixed beforehand. They are unknown at the beginning of the backwards search and are found only at the end of the process. It may very well turn out that, in order to prove a given proposition, new axioms must be introduced: thus geometry is an open system (cf. Cambiano 1967, 1991).

Notwithstanding Plato's fierce opposition, the view of geometry as a closed axiomatic system prevailed and was elaborated by Aristotle in his *Posterior Analytics*. Apparently this depended on a misjudgement: Aristotle believed that most sciences, including geometry, were complete or nearly complete, hence the question was not how to acquire new knowledge but how best to arrange and impart this vast and perfect body of knowledge (cf. Barnes 1975). In any case, as a result of the axiomatic viewpoint prevailing, the analytic method of Hippocrates of Chios and Plato was abandoned and replaced by a different form of the method, subordinate to the axiomatic method, as described by Pappus. On his own part, after the *Republic*, starting from *Phaedrus*, Plato replaced geometry by medicine as his research model (see Cambiano 1967).

An attempt to revive the analytic method in its original form, and hence the concept of open system, was made by Galileo, Descartes, Newton at the beginning of the modern age (see e.g. Engfer 1982). But in the Nineteenth century, as a result of the non-euclidean revolution, the view of geometry as a closed axiomatic system again prevailed.

Ultimately, by his celebrated result, Gödel showed that the concept of formal system as a closed system is inadequate for mathematics. But the logical community failed to learn Gödel's lesson - that Hilbert's concept of formal system is inadequate - and continued to use it as if there were no incompleteness theorem. Contemporary work on computational logic tries to reverse this trend, by developing a new concept of formal system as an open system, capable of handling information in continuous change and evolution. One may hope that this work will succeed where Gödel failed, convincing people to replace Hilbert's concept of formal system, still used in mathematical logic, by a richer concept more adequate for applications to the real world.

Contrary to what was suggested by Turing's ordinal logics or Feferman's recursive progressions of axiom systems, there is no global axiomatic theory of the real world that gradually becomes more and more complete. More modestly, each problematical situation is dealt with by establishing ways of negotiation between the available, usually conflicting, theories concerning that situation. This seems to be the only possible approach to the real world.

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