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PARADOXICAL SIMPLIFICATION

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In the following paper[1] I will construct a family of alternative propositional modal systems in which the paradoxical "from contradictories" formulas of simplification

p&~p→p and

p&~p→~p

are not derivable. I will start with a short characterisation of what in my opinion are the *main points* in alternative logics in general (Section 1) and sketch a pragmatic point of view concerning inferential situations (Section 2) that is important as a background for the reconstruction of the pretheoretic notion of inferential validity. In Section 3 I will explain, why from this point of view the "from contradictories" formulas of simplification and, as a consequence, the general formulas of simplification should be avoided. In Section 4 I will formulate the requirements an alternative system without general principles of simplification ("WGPS-system") should fulfil. Section 5 to 9 contain the construction and description of a family ("SIM1") of such systems: starting with a weak axiomatic system $SIM1_0$ (Section 5), which satisfies all of the requirements of Section 4, the members of the family are obtained by a stepwise addition of axioms and thus form a sequence ordered by inclusion (of the sets of theses). The system MS1.0 (Section 10) determined by a system of 4-valued matrices can be regarded as an upper bound and thus provides a guideline for the construction of the SIM1 systems.

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1 The paper summarises some results of my Habilschrift "Systems without general principles of simplification", which will appear in 1998.

1. Alternative logics

There are three different striking points in which some alternative logicians like C. I. Lewis, W. Ackermann, A. R. Anderson and N. B. Belnap e.a. are dissatisfied with the conception and the results of the classical propositional calculus of the *Principia Mathematica*[2] and the best way to mark these points is by making a distinction between the pretheoretic understanding of our propositions and inferences on the one hand and the formal reconstruction of these propositions and inferences by a logical theory on the other hand[3]. The first point concerns the reconstruction of *hypothetical propositions* as expressed by indicative "if p, then q" statements for instance "If Peter will marry Jacqueline, then he will have a lot of trouble with his father". For the pretheoretic *truth conditions* for such propositions are *more rigorous* than those of the corresponding reconstruction by means of the material implication connective, i.e. there are hypotheticals that are false in a pretheoretic sense, while the corresponding theoretical reconstruction is true. The second point concerns certain *inferences containing "if p, then q" structures* within the premises or the conclusion. The difference of the truth conditions between statements in which the pretheoretic "if p, then q" structure is used and their reconstruction by means of material implication has consequences with respect to the validity of such inferences. For inferences that are pretheoretically invalid may be valid according to the reconstruction, if the replacement of the "if p, then q" statement by material implication results in a weakening of the conclusion or in a strengthening of the premises; inferences, on the other hand, that are pretheoretically valid, may become theoretically invalid, if the replacement has the

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2 I. e. the propositional calculus developed in the first volume of the *Principia Mathematica* (cf. Whitehead and Russell 1927).

3 For more details and a presentation of alternative logics that differs in some respects from my own presentation, cf. Haack 1996, Routley and Plumwood and Meyer and Brady 1982 (with a critique of Haack's classification of alternative logics 57ff.), Read 1988, Anderson and Belnap 1990, Anderson and Belnap and Dunn 1992.

effect of a strengthening with respect to the conclusion or of a weakening with respect to the premises[4]. The third point concerns the reconstruction of the pretheoretic *notions of inference and/or inferential validity themselves.* Again, the pretheoretic notions are *more rigorous* than the corresponding theoretic ones, i.e. there are inferential structures that are invalid in a pretheoretic sense (or rejected as not representing an inference in a proper sense), while the corresponding theoretic reconstruction of the pretheoretic notions of inference and inferential validity classes them as valid.

Ad 1: Though there may be some ordinary "if p, then q" statements in the pretheoretic sphere whose truth conditions coincide exactly with the truth conditions of material implication – i.e. "if p, then q" statements that are true, if the antecedent p is false or the consequent q is true, and that are false only, if the antecedent p is true and the consequent is false – this does not hold generally. For while the contradictory of such an "if p, then q" statement is the conjunction of the antecedent and the negation of the consequent: "p and not q", there are other "if p, then q" statements where the relation to the "p and not q" statement is only a relation of contrariety and not of contradiction, i.e. it is possible that both statements are false. Even if the truth conditions of this second type of "if p, then q" statements are not sufficiently clear, the difference in the relation to the corresponding conjunction statements shows at least, that statements of this kind cannot be satisfyinglyably reconstructed by material implication.

Ad 2: The most famous examples of such inferences are the so called "paradoxes of material implication":

$$\frac{\sim p}{p \rightarrow q}$$ "ex falso quodlibet" paradox

4 For some examples of inferences that are pretheoretically valid, but invalid according to the classical reconstruction, cf. McCall (1966). In the following I will concentrate on alternative reconstructions that lead to "regular" systems (cf. Definition 3), i.e. systems that do not allow inferences which would be invalid in the classical reconstruction.

$$\frac{q}{p \rightarrow q}$$ "quodlibet ad verum" paradox

but there are also other puzzling examples (due to the fact, that the negation of a material implication statement provides a stronger premise than the negation of the pretheoretic "if p, then q" statement):

$$\frac{\sim(p \supset q)}{p}$$

$$\frac{\sim(p \supset q)}{p \supset \sim q}$$

$$\frac{\sim(p \supset q)}{q \supset p}$$

Ad 3: According to the classical reconstruction of validity, an inference is valid just in case it is impossible for its premises to be true and its conclusion false. But this notion has two strange "limit cases"[5]: first, if it is already impossible for the premises to be true together, then an inference composed of these premises and an arbitrary conclusion is valid. Hence for instance

$$\frac{\begin{array}{l} p \\ \sim p \end{array}}{q}$$

as well as

5 Provided the term "impossible" is used in such a way that the condition "it is impossible for its premises to be true and its conclusion false" is already satisfied if only one of the conditions "it is impossible for its premises to be true" or "it is impossible for its conclusion to be false" holds.

$$\frac{\begin{array}{l}p\\ \sim p\end{array}}{\sim q}$$

is valid according to the classical theory. But according to our pretheoretic understanding such constellations of propositions do not count as valid inferences[6]: it would be a bad idea to start with a contradictory pair of premises *in order to get a new insight* q, (since, beside other things, the 'expected insight' q is annihilated by ~q), and it would be a bad idea as well to refer to a contradictory pair of premises *in order to give a reason for* a proposition q, (since, beside other things, the same reason could be given for the contradictory ~q). Secondly if it is already impossible for the conclusion to be false, then an inference composed of this conclusion and some arbitrary premises is valid. Thus according to the classical theory the following pair

$$\frac{p}{\sim(q\&\sim q)}$$

and

$$\frac{\sim p}{\sim(q\&\sim q)}$$

is valid. But again: according to our pretheoretic understanding such constellations of propositions do not count as valid inferences (at least not by themselves): it would neither be a good idea to start with an arbitrary premise p in order to *derive* the tautological statement ~(q&~q) *as a new insight*, nor would it be a good idea to refer to an arbitrary premise p *in order to give a reason for* a tautological statement.

There are three extreme strategies to settle the conflict between the pretheoretic understanding and the reconstruction given by the classical theory. Either we try a) to keep the classical theory as well as the pretheoretic understanding

6 Though they may, *for other reasons*, be tolerated as *limit cases*.

untouched – then we have to modify the relation between these spheres, i.e. we have to give a reinterpretation of what is reconstructed by the reconstruction. Or we try b) to keep the classical theory as well as the aims of the reconstruction without any restriction – then we have to modify our pretheoretic understanding. Or c) we try to keep our pretheoretic understanding and the aims of the construction – then we have to modify the classical theory.

Classical logicians typically use strategy a) to solve the truth condition problem of "if p, then q" statements (problem 1): the connective of material implication is no longer interpreted as an adequate reconstruction of ordinary "if p, then q" statements *in general*, but as a reconstruction of a special type of such statements, namely of "if p, then q" statements that are (in a pretheoretic sense) equivalent to "not (p and not-q)" statements. Concerning the validity problem (problem 3) they typically use strategy b): a pretheoretic understanding of validity that is in trouble with the just mentioned inferences has to be mistaken, since it is thought that the only *logical* condition that has to be satisfied by a valid inference is that it is impossible for its premises to be true and its conclusion false. Concerning the problem of the "paradoxes of material implication" and other (at least prima facie) strange inferences they typically apply a combination of strategy a) and b): since, on the one hand, the reconstruction is restricted to a special subclass of "if p, then q" statements, namely those that satisfy the truth conditions of material implication, and, on the other hand, the classical reconstruction of the notion of inferential validity is regarded as adequate, these inferences are not paradoxical at all: the validity of these inferences can be proved and the putative paradox or strangeness is thought of as resulting by a misunderstanding due either to the neglect of the intended restriction or to a mistaken idea of what "inferential validity" means (or to a combination of both).

Alternative logicians on the other hand would argue, that by restricting the notion of material implication to the reconstruction of negated conjunction statements (problem 1) the classical theory fails to give an adequate reconstruction of ordinary "if p, then q" statements, though they will willingly admit, that the notion of material implication provides an adequate reconstruction of statements that are (in a pretheoretic sense) equivalent to negated conjunction statements. An adequate reconstruction of ordinary "if p, then q" statements, however, would require a notion of implication that is stronger than that of material implication, so, for instance, C. I. Lewis proposed "strict implication" (Lewis 1959), Ackermann "strong implication" (Ackermann 1956), Anderson, Belnap e. a. different sorts of "entailment" (Anderson 1990), McCall "connexive implication" (McCall 1966) etc.

Concerning the notion of inferential validity (problem 3) C. I. Lewis and W. Ackermann (and other contemporary alternative logicians) did not hold the same opinion, though both rejected material implication as inadequate for the reconstruction of "if p, then q" statements as well as for the definition of inferential validity. While Lewis was convinced and tried to prove that the "from contradictories" inferences must be part of every logical calculus (what he in fact showed was that the formulas corresponding to these inferences must be theorems in every logical calculus in which certain other principles are derivable[7]), Ackermann rejected these inferences as invalid, though, of course, he would have conceded that they satisfy the *classical* criterion of validity. That means, that according to Ackermann the classical criterion formulates only a necessary, but not a sufficient condition, i.e. that we need not only a stronger notion of implication for the reconstruction of "if p, then q" statements, but also a stronger notion of inferential validity for the reconstruction of our pretheoretic understanding of inferential validity.

Since alternative logicians regard material implication as inadequate for the reconstruction of "if p, then q" statements, they reject inferences like the paradoxes of material implication and similar inferences (problem 2) as invalid, if the material implication is replaced by a stronger implicative connective. Moreover, from the point of view of an alternative logician "if p, then q" statements have to be *reconstructed* and the notion of

7 Cf. below footnote 22.

implication has to be *determined* in such a way that these inferences turn out to be invalid. Typically, the determination of valid inferences (and the determination of invalid inferences) in alternative logics is not intended as the result of applying previously defined truth conditions and a previously defined notion of inferential validity, but as the first step in the direction of such definitions: the truth conditions for "if p, then q" statements and the notion of inferential validity has to be defined in such a way that the inferential requirements are fulfilled.

The problem of the paradoxes of material implication (problem 2) requires a solution either by a change with regard to the reconstruction of "if p, then q" statements (problem 1) or with regard to the reconstruction of the notion of inferential validity (problem 3) or by a change of both reconstructions. Indeed, it is sufficient for the solution of problem 2 to solve only one of the other two problems, i.e. it is sufficient either to replace the classical reconstruction of "if p, then q" statements or that of the notion of inferential validity. Thus, for instance, in modal systems like K, D, S4, S5[8] etc. a notion of strict implication can be introduced by the definition: $p \rightarrow q := \sim M(p \supset q)$, which avoids the unwanted parts of the truth conditions (problem 1) as well as the paradoxes (problem 2), while it leaves the notion of inferential validity untouched (problem 3). On the other hand, one can keep the reconstruction of the "if p, then q" statements by material implication (problem 1) and solve problem 2 and 3 by imposing some restrictions on the classical reconstruction of inferential validity[9].

But though it is possible to solve problem 1 and problem 3 separately, it is not only aesthetically more elegant but also philosophically preferable to solve both problems at once, namely by choosing an alternative kind of implication that is a) *stronger* than material implication and that b) can, at the same time, be used for the reconstruction of "If p, then q" statements and as main part within the reconstruction of the notion of

8 Cf. Hughes & Cresswell 1996, 51 et seq.

9 For instance, restrictions similar to those formulated by the von Wright/Geach/Smiley condition (cf. Anderson & Belnap 1990, 215 et seq.).

inferential validity. For though classical propositional logic fails in both cases, its main idea – to reconstruct the notion of inferential validity as a special case of material implication – is philosophically most interesting. Let us briefly re-examine how this idea works. The minimal requirements for the *pretheoretic* truth of "if p, then q" statements (that it is not the case that p is true and q false) and the minimal requirements for the *pretheoretic* notion of inferential validity (that it is for logical reasons not the case that the premises are true and the conclusion false) differ only with respect to the addition "for logical reasons". The classical approach, which reduces the truth conditions and the notion of validity exactly to these minimal requirements, is able to use material implication for the reconstruction of the notion of inferential validity by means of the so called *law of conditionalization*:

> *Law of conditionalization (classical version)*[10]: An inference from premises $x_1, ..., x_n$ to a conclusion y is *valid*, iff the corresponding[11] wff $x_1\& ... \&x_n \supset y$ is a thesis in K_c (where K_c is the classical calculus of propositional logic and the implication symbol "$\supset$" means material implication.)

If we keep the formal connection expressed by the law of conditionalization, but replace the classical propositional calculus by some alternative propositional calculus and the connective of material implication by a stronger kind of implicational connective, we arrive at the following:

> *Law of conditionalization (alternative version)*: An inference from premises $x_1, ..., x_n$ to a conclusion y is

10 Though the term "conditionalization" and a law similar to the principle formulated here are often used in the context of the so-called "deduction theorem" my main point here is, that it functions as a *bridge* principle between the sphere of inferential situations (cf. 12) and the sphere of propositional logic and not between two different formal techniques within the second sphere (cf. also Linneweber-Lammerskitten 1997).

11 Note that there are a lot of problems involved in connection with the formalization of inferences expressed in ordinary language. Some of these problems belong to logic especially the question of what is meant by "form", others belong to the semantics of the ordinary language, in which the inference is formulated. In the following text I silently presuppose that these problems can be solved in an ordinary inferential situation, i.e. that a speaker who claims that he infers something from something (else) is able to give a formal reconstruction of his inference in terms of the formal language.

valid, iff the corresponding wff x_1& ... &x_n→y is a thesis in K_a (where K_a is an alternative calculus of propositional logic and the implication symbol "→" means an alternative type of implication.)

It is essential to distinguish between the different calculi as such and the different versions of the laws of conditionalization. The replacement of the classical calculus by some alternative calculus does not by itself lead to an alternative notion of inference. It is the law of conditionalization that determines whether the notion of inferential validity is alternative or classical.

In its classical version the law of conditionalization *reduces* the notion of inferential validity to a special case of material implication and it is thus understandable that classical logicians tend to use strategy b) with respect to problem 3. The alternative version of the law of conditionalization in contrast cannot be used for such a reduction, since the truth conditions of an alternative kind of implication are not truth-functional. We can in a first step only give a sufficient condition for the falsehood and a necessary (but negative) condition for the truth of such statements. But we can use the law of conditionalization in the other direction and formulate in a second step restrictions to possible truth conditions: if, for instance, the transitivity formula

*1 (p→q)&(q→r)→(p→r)

is a thesis of an alternative calculus K (where "→" stands for the alternative implicative connective of K), then the truth conditions for a simple implication statement p→q must be such that (*1) is a true proposition whatever propositions we choose for the symbols p, q and r. But though it is true that, if (*1) is a thesis of K, this indicates via the law of conditionalization that the corresponding inference is valid according to K, the relation that is more important goes in the other direction: because the inference allowing the transition from implicative premises p→q and q→r to an implicative conclusion p→r is regarded as valid (whatever the exact reconstruction of "if p, then q" statements by the implication "p→q" may be), K has been constructed in such a way, that (*1) becomes a thesis of K and this in turn restricts the possible reconstructions of the "if p, then q" statements.

If we compare the different usage that is made of the law of conditionalization in the classical and in the alternative case, we can say, that classical logic starts with the reconstruction of "if p, then q" statements and uses the law to reconstruct the notion of inferential validity, while alternative logic, typically, starts with the reconstruction of the notion of inferential validity (by singling out certain *inferences* of the classical approach as invalid and others as fundamental) and uses the law to single out certain complex implicative *propositions* as formally true and others as not formally true, and thereby determines the formal truth conditions of the simple implicative statements.

2. Non-classical aspects of inference

The *classical* reconstruction of the notion of inferential validity abstracts from most of the pretheoretic aspects of an inferential situation and concentrates on one single point: that inference is a transition from a set of premises to a conclusion and that the transition is valid iff the logical form of the propositions does not admit inferences from true premises to a false conclusion. However, this point is only one of many other aspects relevant for ordinary inferential situations, or for special subgroups of such situations[12]. Many of these other aspects are logically irrelevant, but there are some non-logical aspects that depend on logical principles and there are also some *logical aspects* beyond the requirement of truth preservation. The most important of the last group, in my opinion, is (premise-)*relevance*: (at least one) of the premises should be relevant for the conclusion, i.e. there should be a logical connection between the (structure of the) premises and the (structure of the) conclusion. How ever this may be interpreted in detail, a necessary condition in the case of *propositional* logic is surely

12 By (the "idealtypus" of) an inferential situation I mean, roughly speaking, a situation, in which some rational beings with different beliefs mutually try to give reasons for their own position and mutually try to understand, what exactly the position of the others consists in and how it is grounded.

that (in the formalization of the inference) the premises and the conclusion must have at least one propositional variable in common, for otherwise this connection could not be expressed by means of propositional logic.

Though many of my considerations have been influenced by works on relevant logic (esp. Anderson 1990 and 1992) there is a non-logical aspect which, in my opinion, deserves some interest, namely that the *theoretical* question of inferential validity has *practical imports* concerning *rationality* and *obligation* in inferential situations. Thus for instance we would say that it is *rational* for someone who accepts some premises $x_1, \ldots, x_n$ to pass to the conclusion y, provided the *inference is valid*; or we might ask "Am I *obliged* to pass from the premises $x_1, \ldots, x_n$, which I granted for the sake of argument, to a certain conclusion y, or not?", "Am I *allowed* to pass from the premises $x_1, \ldots, x_n$, which express my belief at t_0, to the conclusion y, or not?"[13], etc. Though inferential validity is a theoretical matter it *takes its importance* from the fact, that it is relevant for our intellectual *actions,* for what we do in conversation with others as well as for what we do in silent reflection; in turn the practical validity of these actions (in concreto) is dependent on the theoretical validity of the inferences at stake.

In an inferential situation we may be obliged to accept "for the sake of argument" premises to which we normally would not assent – but we are not obliged to accept such premises beyond necessity. We can ask for what kind of inference the premises are needed – and if we do not accept these inferences as valid, we are not obliged to accept the premises, since there is no (accepted) argument for the sake of which the acceptance of the premises could be demanded. If we do not accept certain inferences, we are not allowed to use them and we are not obliged to accept their use by others (we are not obliged to accept the conclusion temporarily, since we are not obliged to accept the premises).

13 Obligation in inferential situation was an important subject in the logic of the fourteenth and fifteenth century (cf. for instance Paul of Venice 1988; there are also treatises on obligation by Walter Burley, Roger Swyneshed, Albert of Saxony (1360-1390) and others) – though it seems that the interest in obligation was less directed to moral duty than to strategies for disputation.

Though the question what kind of obligation and what kind of rights one has in an inferential situation is not a logical question, its answer presupposes some logical investigation, since for the reconstruction of these obligations and rights in detail a logical reconstruction of the notion of inferential validity is needed. Of course, the latter can only be expected to succeed with respect to the non-logical aim, if it takes non-classical aspects of inferential situations serious. But this does not mean that the reconstruction thereby becomes a non-logical (psychological, sociological etc.) enterprise, for it is not the pretheoretic notion of persuasion, of reasoning as psychological act, of arguing as a social event etc. that is at stake here, but it is the pretheoretic notion of inferential validity, that is to be reconstructed.

3. What is wrong with simplification?

In *Principia Mathematica* B. Russell and A. N Whitehead refer to the following three formulas as "principles of simplification"[14] which are theorems of the PM calculus:

*2 $q \rightarrow (p \rightarrow q)$

*3 $(p \& q) \rightarrow p$

*4 $(p \& q) \rightarrow q$

The first expresses one of the well-known paradoxes of material implication that corresponds to the "verum sequitur ad quodlibet"-rule of the scholastic logicians[15], the other two

14 In the notation used by Russell and Whitehead these formulas read (cf. Whitehead and Russell 1927, XI):

* 2·02. ⊢ :q. ⊃ . p ⊃ q Pp. ["Simp"]

* 3·26. ⊢ :p.q . ⊃ . p Pp. ["Simp"]

* 3·27. ⊢ :p.q . ⊃ . q Pp. ["Simp"]

15 It should be noted that this rule formulated for instance in Buridanus' De consequentiis: "Et est notandum quod (...) omnis uera ad omnem aliam sequitur etiam consequentia ut nunc." (Buridanus 1976, 32) is part of a logical theory that differs in some respect from the theories we are accustomed with today. Buridan's rule does *not* mean, that we can pass from a true *premise* q to a true *conclusion* $p \rightarrow q$. It is *not* the rule corresponding to the syllogistic scheme:

formulas are sometimes called formulas of Petrus Hispanus[16] - I will use the name "paradox of material implication" for (*2) and reserve the expression "*simplification*" for expressions of the conjunctive form (*3) and (*4) - the latter will be called "*general formulas of simplification*", if contrasted with its substitution instances for instance with the "*from contradictories formulas of simplification*"

*5 $p\&{\sim}p \rightarrow p$ and

*6 $p\&{\sim}p \rightarrow {\sim}p$

Since the paradox concerning formula (*2) has been one of the main reasons for the development of the systems of strict implication and has got a detailed discussion by C. I. Lewis and his school, I will concentrate on the other two formulas[17] (*3) and (*4).

Prima facie there is nothing paradoxical with the general formulas of simplification (*3) and (*4)[18]. But they have paradoxical consequences in the sense that they can be used together with other theses and rules in order to derive paradoxical formulas like the paradox of material implication (*2). But before I can give a short survey of such hidden traps of paradoxes I must first make precise what is meant by the phrase

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q
———
$p \rightarrow q$
rather it corresponds to the syllogistic scheme:
p
ææ
q
which is not valid with respect to every proposition in every situation, but only with respect to a situation in which the sentence q is actually true. Thus for instance the inference
Socrates runs
———
A white (pale) man runs
is valid in all situations in which a white man runs.

16 This is not quite correct, since they are not mentioned by Petrus Hispanus himself, but appear in a later commentary by a certain Versorius.

17 Despite the fact that formula (*2) is paradoxical in itself, the arguments for the renunciation of (*3) and (*4) will provide another reason for the rejection of (*2). For as long as the importation formula
$(p \rightarrow (q \rightarrow r)) \rightarrow ((p\&q) \rightarrow r)$
is a thesis, (*3) could be derived from (*2).

18 Nelson (1930) held a more radical view.

"formula x has certain (paradoxical) consequences y (as long as certain other rules and theses hold)", for a paradoxical import of a formula depends on the system in which it is a thesis.

Definition 1: An axiomatic system of propositional logic (based on a language L) is a *DCSE-system,* iff the rules of detachment, conjunction, substitution and the equivalence rule[19] hold (i.e. if these rules are part of the axiom-basis or introducible).

Definition 2: Let x, y, $z_1, \dots, z_n$ be formulas of a language L. A formula x has the consequence y, if $z_1, \dots, z_n$ are theses, iff y is a thesis in every DCSE-system (based on the language L) in which $z_1 \dots z_n$ are theses.

Using these definitions we can give some examples of the paradoxical import resulting from the general formulas of simplification:

Theorem 1: The general formula of simplification (*3) has the paradoxical consequence

*7 p→(q→p)

("verum sequitur ad quodlibet" paradox), if the formula of exportation

*8 ((p&q)→r) → (p→(q→r))

is a thesis[20].

Theorem 2: The general formula of simplification (*3) has the paradoxical consequence

*9 (p&~p)→q

("from contradictories" paradox), if the formula of antilogism

*10 ((p&q)→r)→((p&~r)→~q) and the equivalence formula

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19 For an explicit formulation of these rules cf. Section 5.

20 *Proof*: a) ((p&q)→p) → (p→(q→p)) (*8) [p/r]
b) ((p&q)→p) (*2)
c) p→(q→p) (a) (b) X modus ponens

q.e.d.

*11 ~~q ↔ q

are theses[21].

Theorem 3: The general formulas of simplification (*3) and (*4) have the paradoxical consequence

*12 (p&~p)→q

("from contradictories" paradox), if the following formulas

*13 p → (pvq)

*14 ((p→q)&(q→r)) → (p→r)

*15 ((p→q)&(p→r)) → (p→(q&r))

*16 ((pvq)&~p) → q

are theses[22].

These paradoxical consequences are not "absolute consequences", but consequences under the condition that other formulas are theses. This means that the paradoxes can be avoided by renouncing one of the additional theses. Thus Lewis and Langford in their systems of strict implication renounce the *formula of exportation,* but accept the "*from contradictories*" paradox; Ackermann in his system of strong implication

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21 *Proof*: a) ((p&~q)→p)→((p&~p)→~~q) (*10) [~q/q p/r]
b) ((p&~q)→p) (*2) [~q/q]
c) (p&~p) → ~~q (a) (b) X modus ponens
d) (p&~p) → q (c) (*11) X equivalence

q.e.d.

22 The ideas of the following proof are due to Lewis and Langford (1959, 250f.) – The main ideas of the proof were already formulated by Buridanus in the 14th century (cf. Buridanus 1976, 37,169-181).

Proof:
a) (p&~p) → p (*2) [~p/q]
b) p → (pvq) (*13)
c) ((p&~p)→p)&(p→(pvq)) (a) (b) X conjunction
d) ((p&~p)→p)&(p→(pvq)) → ((p&~p)→(pvq) (*14) [(p&~p)/p p/q (pvq)/r]
e) (p&~p) → (p v q) (c) (d) X modus ponens
f) (p&~p) → ~p (*4) [~p/q]
g) ((p&~p)→(pvq)) & ((p&~p)→~p) (e) (f) X conjunction
h) (((p&~p)→(pvq))&((p&~p)→~p)) → ((p&~p) → ((pvq)&~p))
(*15) [(p&~p)/p (pvq)/q ~p/r]
i) (p&~p) → ((pvq)&~p) (g) (h) X modus ponens
j) ((pvq)&~p) → q (*16)
k) ((p&~p)→((pvq)&~p)) & (((pvq)&~p)→q) (i) (j) X conjunction
l) (((p&~p)→((pvq)&~p))&(((pvq)&~p)→q)) → ((p&~p)→q)
(*14) [(p&~p)/p ((pvq)&~p)/q q/r]
m) (p&~p) → q (k) (l) X modus ponens

q.e.d.

renounces the *formula of antilogism*, the *formula of disjunctive syllogism* etc.[23].

But beside these paradoxical consequences conditional on some other theses, the formulas (*3) and (*4) have also *immediate* consequences, which are in a certain sense paradoxical:

*17 (p&~p)→p

*18 (p&~p)→~p

Since these formulas are substitution instances of the general formulas, they can only be avoided by renouncing the general formulas of simplification themselves[24].

In which sense are (*17) and (*18) paradoxical? Obviously they are not only substitution instances of the general formulas of simplification, but also substitution instances of the "*from contradictories*" paradox

*19 (p&~p)→q

This is not by itself a sufficient argument, since substitution instances of a paradoxical formula need not be paradoxical themselves, for instance

*20 (p&~p)→(p&~p)

is a *non-paradoxical* substitution instance of (*19). But it may be helpful to have a look at (*19) in order to get a better understanding of (*17) and (*18). I think there are three main points in which our pretheoretic notion of inference collides[25] with (*19): The *first* point is "missing relevance": we expect that

23 For a detailed survey cf. Linneweber-Lammerskitten 1994.

24 There is still another alternative, namely to give up the calculus rule of substitution: by giving up this rule we could keep (*3) and (*4) as theses and could get rid of (*17) and (*18). But disregarding the technical problems, this solution would undermine the idea that the theses in a logical calculus should not only guarantee the validity of their "material" substitution instances outside the calculus (i.e. the concrete inferences used in everyday life reasoning), but should also guarantee the theoremhood of their "formal" substitution instances within the calculus.

25 Of course, no thesis of the propositional calculus of PM is by itself a paradox: the sort of paradoxes I want to discuss here arise, since by the acceptance of PM as a reasonable systematisation of our intuitions concerning inferences, we are forced to accept inferences that contradict our previous intuitions or some of their consequences.

the conclusions of our inferences have something to do with the premises they are drawn from, but (*19) generates a host of examples of inferences in which the premises have nothing to do with the conclusion. The *second* point is that we expect that whenever there is a valid inference from premises x to a conclusion y then there cannot be also a valid inference from x to the negation of y, but by accepting (*19) we also accept its substitution instance

*21 (p&~p)→~q

hence a formula whose consequent term is the negation of that in (*19). The *third* point is, that we expect – quite contrary to the school tradition of scholastic logic – that from a plain contradiction "as little as possible" follows[26].

Among these three points of paradox concerning the formula (*19), the first vanishes if we pass from (*19) to its substitution instances (*17) or (*18). For p or ~p indeed have something to do with the premise p&~p: the former are constituents of the latter; (*17) and (*18) seem to be expressions of an ordinary analysis with respect to a complex proposition. I will criticise this view later, but I agree that the first objection against (*19) is not an objection against (*17) or (*18). The second and the third argument against (*19), however, can be applied to (*17) and (*18).

According to the second, the two formulas provide an example of inferences that lead from the very same premises to contradicting conclusions. Notice that this does not mean that the conclusions themselves are contradictories – p and ~p may well be contingent propositions – the point is that (*17) and (*18) as principles of inference give rise to two different patterns of inference with the same premise-structure, but with conclusions that cannot be true together. Why do we normally not want to have inference patterns with that property? I think it is because the first thing we expect from inferences is progression – progression concerning our own knowledge or

26 Of course we cannot avoid that the position of a contradiction (cf. *20) follows from itself (at least as long as we accept the identity formula of implication p→p as a thesis). Another formula we cannot avoid (if we keep p→Mp) is p&~p→M(p&~p).

progression concerning the understanding of others or their understanding of us. But the acceptance of inference patterns that lead to contradicting conclusions frustrates the hope for such progress, since a) one such conclusion can at once be abolished by the second pattern and b) a complex proposition x cannot be used as an account for an opinion q, if it could also be used as an account for the opposite opinion. Now it may be objected, that – granted that we normally do not want to have such inference pattern – in this special case we should have them as instrument to detect and abolish contradictions, i.e. we should have them since otherwise we could not have a *reductio ad absurdum* instrument. But this is not true: the systems I want to propose provide the necessary reductio ad absurdum facilities.

The third argument aims at the point, that in discourse we should not be forced beyond necessity to accept (provisionally, for the sake of argument) a plain contradiction ("Imagine p&~p" – this is rather hard to imagine). But if we agree to (*17) and (*18) as principles for inferences in discourse, we are *obliged* to accept premises like "It does and it does not rain", whenever someone wants us to accept it, provided his underlying intention is to pass to one of its contingent parts. I think it would be reasonable to restrict such an obligation to accept (and the corresponding authorisation to demand the acceptance of) a plain contradiction like q&~q beyond necessity. A first step was made by W. Ackermann in abolishing the general formula of "*from contradictories*" paradox (*19) a second should be made by abolishing its substitution instances (*17) and (*18), the general formulas of "*from contradictories*" simplification.

So much to the formulas (*17) and (*18) as substitution instances of the paradoxical formula (*19). The defender of (*17) and (*18) will probably prefer the view that they are (admittedly odd) substitution instances of the general formulas of simplification (*3) and (*4) and thus *principles of inference* since the general formulas are. Now first of all it is true that *if* the general formulas of simplification are principles of inference then (according to our general ideas concerning inference) indeed the formulas (*17) and (*18) are also principles of inference (but on the other hand: if we say that we do not accept the formulas (*17) and (*18) as principles of inference then the general formulas cannot be principles either). Is there an independent reason (concerning our normal understanding of inferences) for saying that the general formulas (and hence all of their formal substitution instances) are principles of inference? I think the most tempting reason is to say that all sorts of formulas of simplification are (in a very special sense of the word) "analytical" principles, since they allow us to pass from a complex whole to each of its parts. But should it be allowed? And is it really the case that "It rains" is part of that what someone means who is apt to hold "It rains and it does not rain". I am not sure whether this is so, but even if it were, the normal reaction to someone performing such an assertion on a rainy day, in my opinion, is to wonder what he is aiming at and not to analyse this assertion in the usual polite way: "I agree with you to the extent that it rains but I cannot see how you could come to the opinion that it does not" for he, being asked, might answer: "For the very same reason: it is a simple consequence of my more general view that it rains and it does not rain." I think that the information the discourse partner gets from someone who asserts "p&~p", (and also the information the latter gets about his beliefs) is in a certain respect greater than the usual information in an assertion of the form x&y. The surplus of information concerns the discovery that there is something seriously wrong with the set of beliefs oneself or the other has, since it contains a contradiction. This discovery is far more important than the "content" of such a complex proposition – its analysis would rather cloud this insight, than focus it. Note that in a normal discourse the discovery of a plain contradiction will cause an interruption and close that part of discussion in order to provide some basis for a reductio ad absurdum argumentation, revisions, reflections etc.

If we look at these two roots, the "*from contradictories*" paradox (*19) on the one hand and the general formulas of simplification (*3) and (*4) on the other hand, the first root exhibits that something is wrong with the "*from contradictories*" formulas of simplification (*17) and (*18), since it is (like every principle of inference with a contradiction as premise) useless in the sense that it cannot be used to pass from true premises to a true conclusion, and, for that and the other reasons indicated, is in conflict with our pretheoretic notion of inference. At first, the second root drives us in the opposite direction: it seems that these formulas, since they can be understood as special cases of the general principles of simplification, content special principles of "analysis" that are tolerable. My point is that if we arrive at p&~p it is not analysis that is needed, but a new beginning, and that therefore the second root does not really give a reason in favour of these principles – the conflict with our pretheoretic notion of inference remains.

This is the reason why I think it worthwhile to study systems that try to give up (*17) and (*18). But in doing so, we should be modest and should not postulate that every substitution instance of (*17) and (*18) is not a thesis, for this, probably, would lead us to very weak systems. I think it is already an advance to construct and analyse systems in which the general formulas of "*from contradictories*" simplification are not derivable[27].

The family of systems I want to propose are neither the only nor the first systems in which the general formulas of simplification and some of their substitution instances are rejected[28], but they are motivated by different reasons and thus lead to different results.

4. Minimal requirements for a system without general principles of simplification

In order to determine which properties an alternative propositional system without general principles of simplification (short: "WGPS-system"), in my opinion, should at least have, it

27 In the family of SIM1 systems I found the following substitution instances of the general "from contradictories" formulas of simplification:

Mp&~Mp→Mp	but not:	Mp&~Mp→~Mp
p&~Lp→~Lp	but not:	Lp&~Lp→Lp
(p&~p)&~(p&~p)→(p&~p)	but not:	(p&~p)&~(p&~p)→~(p&~p)

28 Systems in which the general principles of simplification are not derivable have been proposed for instance by Nelson 1930, McCall 1966, Angell 1962, Cheng 199), et al.

is essential to distinguish by definition two different relations of inclusion that may or may not hold between the classical and the alternative WGPS-system:

Definition 3: Let S be an alternative WGPS-system with conjunction, negation, implication and necessity as primitives and T be a reformulation of the PM-calculus with conjunction, negation and material implication as primitives. Let

ϕ: WFF(S) ————> WFF(T) be a mapping such that

$\phi(p) = p$ for every atomic formula p

$\phi(\sim x) = \sim\phi(x)$

$\phi(x\&y) = \phi(x)\&\phi(y)$

$\phi(x\rightarrow y) = \phi(x)\supset\phi(y)$

$\phi(Lx) = \phi(x)$

then S is a *regular system* iff for every thesis x of S $\phi(x)$ is a thesis of T[29].

In a certain sense a *regular* WGPS-system *is included* in the *PM-calculus*, namely if the necessity operator is ignored and the implication operator is identified with material implication. In quite another sense the PM-calculus *is included* in a WGPS-system, if the PM-calculus is a *subsystem* of a WGPS-system (cf. Definition 4), namely in the sense that every formula of the PM-calculus can also be derived in the alternative system (but with an important difference concerning the law of conditionalization: a material implication thesis in a WGPS-system does not count as a principle of inference.

Definition 4: Let S be a WGPS-system with conjunction, negation, implication and necessity as primitives and let T be a reformulation of the PM-calculus with conjunction and negation as primitives. Let

ψ: WFF(T) ————> WFF(S) be a mapping such that

$\psi(p) = p$ for every atomic formula p

29 Cf. Hughes and Cresswell 1990, 267.

$\psi(\sim x) \quad = \sim\psi(x)$

$\psi(x \& y) = \psi(x) \& \psi(y)$

then *PM is a subsystem of S* (or *'S is a supersystem of PM'*) iff for every thesis x of T, ψ(x) is a thesis of S.

The following definition concerning weakest and strongest formulas generalises the idea expressed by the paradoxical formulas (*30) and (*31).

Definition 5: A wff x of a propositional system S is called a *weakest formula*, iff it is implied by every other term; x is called a *strongest formula*, iff it implies every other term.[30]

By using these definitions we can formulate the following requirements:

(i) None of the above mentioned paradoxes should be derivable, i.e. a WGPS-system should not have the "from contradictories" formulas of simplification

*22 p&~p→p and *23 p&~p→~p

as theses and therefore cannot have the general formulas of simplification

*24 (p&q)→p *25 (p&q)→q

Also, it should not have any of the paradoxes of material or strict implication,

*26 ~p→(p→q) *27 q→(p→q)

*28 ~Mp→(p→q) *29 Lq→(p→q)

nor the "from contradictories" paradox or its conversion

*30 (p&~p)→q *31 p→(qv~q)

(ii) A WGPS-system should not contain any *weakest* or *strongest* formula, for in the pretheoretic sense there is no premise from which every arbitrarily chosen proposition follows as its conclusion, and no conclusion that can be drawn from every arbitrarily chosen premise.

30 These definitions have been introduced by T. Sugihara 1955.

(iii) It should be a *regular* system, for it should not lead to inferences that must be regarded as invalid from a classical point of view, i.e. it should, concerning the notion of inferential validity, be more restrictive than, but still compatible with the classical system.

(iv) It should be a *supersystem* of the PM-calculus, for it should respect the truth functional relations between the truth-functional connectives.

(v) It should guarantee the following equivalences as theses:

*32	$p\leftrightarrow p$	*33	$p\leftrightarrow\sim(\sim p)$
*34	$(p\&q)\leftrightarrow(q\&p)$	*35	$(pvq)\leftrightarrow(qvp)$
*36	$(p\&(q\&r))\leftrightarrow((p\&q)\&r)$		
*37	$(pv(qvr))\leftrightarrow((pvq)vr)$		
*38	$(p\&p)\leftrightarrow p$	*39	$(pvp)\leftrightarrow p$
*40	$(p\rightarrow\sim q)\leftrightarrow(q\rightarrow\sim p)$	*41	$(p\rightarrow q)\leftrightarrow(\sim q\rightarrow\sim p)$
*42	$Mp\leftrightarrow\sim L\sim p$	*43	$Lp\leftrightarrow\sim M\sim p$
*44	$(p\rightarrow q)\leftrightarrow L(p\rightarrow q)$		

(vi) It should guarantee the following implications as theses:

*45	$((p\rightarrow q)\&(q\rightarrow r))\rightarrow(p\rightarrow r)$		
*46	$((q\rightarrow r)\&(p\rightarrow q))\rightarrow(p\rightarrow r)$		
*47	$((p\rightarrow q)\&(p\rightarrow r))\rightarrow(p\rightarrow(q\&r))$		
*48	$((p\rightarrow r)\&(q\rightarrow r))\rightarrow((pvq)\rightarrow r)$		
*49	$((p\rightarrow q)\&p)\rightarrow q$		
*50	$((p\rightarrow q)\&\sim q)\rightarrow\sim p$		
*51	$(p\rightarrow(q\rightarrow r))\rightarrow((p\&q)\rightarrow r)$		
*52	$(p\rightarrow q)\rightarrow\sim(p\&\sim q)$		
*53	$(p\rightarrow\sim p)\rightarrow\sim p$		
*54	$((p\rightarrow q)\&(p\rightarrow\sim q))\rightarrow\sim p$		
*55	$p\rightarrow Mp$	*56	$Lp\rightarrow p$

*57 ~Lp→M~p

*58 Mp→~L~p

*59 ((p→q)&~Mq)→~Mp

*60 ((p→q)&Lp)→Lq

*61 (p→q)→~M(p&~q)

*62 Mp→~(p→~p)

5. The minimal system $SIM1_0$

The system $SIM1_0$ is based on the following axioms[31]:

Ax1) (p&q)→(q&p)
Ax2) (p&(q&r))→((p&q)&r)
Ax3) p→p
Ax4) (p→q)&(q→r)→(p→r)
Ax5) (p→q)&(p→r)→(p→(q&r))
Ax6) (p→(q→r))→((p&q)→r)
Ax7) (p→~q)→(q→~p)
Ax8) (~p→q)→(~q→p)
Ax9) p→Mp
Ax10) (p→q)→~M(p&~q)
Ax11) (p&p)→p
Ax12) M(p&q)→Mp&Mq
Ax13) Mp&Mq→Mp[32]

31 It can be shown, that the axioms Ax1, Ax2, Ax4, Ax5, Ax6, Ax7, Ax8, Ax9, Ax10, Ax13, Ax14 and Ax16 (taken separately) are independent of all the other axioms and rules of $SIM1_0$.

32 Though (Ax13) is not only a substitution instance of, but also has much resemblance with the general formula of simplification (and thus could be called "general formula of possibility simplification") it should be noted that there are no "*from contradictories*" formulas of simplification that could be obtained from it by mere substitution. For the rule of substitution allows only formulas like
Mp&M~p→Mp and Mp&M~p→M~p
On the other hand (Ax13) together with (Ax16) (and some other axioms) indeed leads to a paradoxical formula of simplification (cf. footnote 27).

Ax14)	(p→q)→~M~(~q→~p)
Ax15)	((p→q)&~Mq)→~Mp
Ax16)	MMp→Mp
Ax17)	~M(~(p&~q)&((p&r)&~(q&r)))

together with the following definitions for "↔", "v", "L" and "⊃" (material implication) and calculus rules:

DEF."↔"	: p↔q	:= (p→q)&(q→p)
DEF."v"	: pvq	:= ~(~p&~q)
DEF."L"	: Lp	:= ~M~p
DEF."⊃"	: p⊃q	:= ~(p&~q)

Rule of substitution 1: If x is a thesis and y differs from x only in so far as all occurrences of a propositional variable of x are replaced in y by a certain term z, then y is a thesis.

Rule of substitution 2 ("equivalence rule"): If x is a thesis and y differs from x only in so far as some or all occurrences of a certain term z of x are replaced in y by a certain term z', then y is a thesis, provided that z↔z' is a thesis.

Rule of detachment for implication[33] ("modus ponens rule"): If x→y is a thesis and x is a thesis, then y is a thesis.

Rule of detachment for material implication: If x⊃y is a thesis and x is a thesis, then y is a thesis[34].

33 The rule of detachment *for implication* can be introduced by means of the rule of detachment *for material implication* as soon as it is shown that (p→q) ⊃ (p⊃q) is a thesis, for instead of using the rule of detachment for implication with respect to some theses x→y and x we could proceed in the following way:
a) (p→q) ⊃ (p⊃q) thesis of SIM1
b) (x→y) ⊃ (x⊃y) (a) X [x/p y/q]
c) (x→y) condition for the application of the rule
d) (x⊃y) (b) (c) X detachment for material implication
e) x condition for the application of the rule
f) y (d) (e) X detachment for material implication
but in order to show that (a) indeed is a thesis of SIM1 the stronger rule of detachment for implication is needed.

34 This rule is (with respect to the SIM1 systems) interchangeable with the *Rule of disjunctive syllogism*: "If x and ~xvy are theses, then y is a thesis", which is known as "Ackermann's γ– rule", since it is used in (Ackermann 1956, 119).

Rule of conjunction: If x is a thesis and y is a thesis, then x&y is a thesis.

That $SIM1_0$ fulfils the "positive" requirements (iv), (v) and (vi) can be shown by straight forward derivations – the fulfilment of (ii) by a finite adaptation of the Sugihara matrices[35], for (i) and (iii) cf. the last section.

6. The system $SIM1_1$

The system $SIM1_0$ is designed for the connectives "&", "~", "→" and "M", and does not contain distribution formulas which could establish a connection between the basic operators of $SIM1_0$ and the operators "v" and "L" defined in the usual way. To get a system with distributive properties concerning these connectives, i.e. a system, in which

*63 L(p&q)↔Lp&Lq

*64 M(pvq)↔MpvMq

*65 ((pvr)&(qvr))↔((p&q)vr)

*66 ((p&r)v(q&r))↔((pvq)&r)

can be derived, some additional axioms must be chosen, for instance,

Ax18) (~M~p&~M~q)→~M~(p&q)

Ax19) ~M~(p&q)→~M~p

Ax20) (~(~p&~r)&~(~q&~r))→~(~(p&q)&~r)

Ax21) ~(~(p&q)&~r)→(~(~r&~p)&~(~r&~q))

which together with the axioms, definitions and rules of $SIM1_0$ constitute the system $SIM1_1$.

It should be noticed, that though (*65) and (*66) are theses in SIM1 the following formula, which has some similarities with distribution and is derivable in the system E) is *not* a thesis of

35 Cf. for instance Linneweber-Lammerskitten 1988, 278.

$SIM1_1$ (or any other member of the SIM1 family) since it does not satisfy the set of matrices SM1.0:

*67 (p&(qvr))→((p&q)vr)

Since the idea of distribution concerning conjunction and disjunction is fully expressed by the formulas (*65) and (*66), there must be something else beyond distribution hidden in (*67), but up to now I have no good idea what it is.

7. The system $SIM1_2$

It is widely agreed among modal logicians that our pretheoretic notions of necessity, possibility and conjunction are such, that a relation of distribution holds between necessity and conjunction, but not between possibility and conjunction, i.e. that (*68) should be a thesis, while (*69) should not,

*68 L(p&q)↔(Lp&Lq)

*69 M(p&q)↔(Mp&Mq)

The reason for rejecting (*69) is that for two contingent propositions p and q (for instance for a contingent proposition p and its negation ~p) the corresponding possibility statements Mp and Mq are both true, while the statement M(p&q) might be false (as in the example M(p&~p)), thus the formula

*70 (Mp&Mq)→M(p&q)

in most of the modal systems[36] is not a thesis. The same argument can be used for the rejection of (*71) – in both cases the second part of the conjunction in the antecedent is too weak.

*71 (Mp&q)→M(p&q)

If, however, this second part is replaced by Lq, the antecedent should be strong enough to imply M(p&q), thus by adding

36 A different position was held by Lukasiewicz in his paper *A system of modal logic*, 1952.

Ax22) $(Mp\&\sim M\sim q) \rightarrow M(p\&q)$

to $SIM1_1$ we get a system ("$SIM1_2$") in which the formula

*72 $(Mp\&Lq) \rightarrow M(p\&q)$

is a thesis, which cannot be derived in $SIM1_1$. In $SIM1_2$ the rule of necessitation ("If x is a thesis, Lx is a thesis") is introducible and it can be shown, that it is a normal modal system, i.e. that

(i) every theorem of the PM–calculus (based on "~" and "&") is a thesis
(ii) $L(p \supset q) \supset (Lp \supset Lq)$ is a thesis.
(iii) it is closed under the rule of material detachment.
(iv) it is closed under the rule of necessitation.

8. The system $SIM1_3$

One of the disadvantages the system $SIM1_2$ still has, is that only restricted versions of the following rules[37]

(i) *Becker's rule 1*: If $x \rightarrow y$ is a thesis then also $Lx \rightarrow Ly$ is a thesis.
(ii) *Becker's rule 2*: If $x \rightarrow y$ is a thesis then also $Mx \rightarrow My$ is a thesis.
(iii) *Becker's rule 3*: If $x \rightarrow y$ is a thesis then also $LMx \rightarrow LMy$ is a thesis.
(iv) *Becker's rule 4*: If $x \rightarrow y$ is a thesis then also $MLx \rightarrow MLy$ is a thesis.

are introducible[38] and that it contains a lot of hidden modalities[39], e.g. conjunctions such as (Lp&p), which cannot be reduced. By adding the following axioms

37 This rule was originally formulated by O. Becker (cf. Hughes 1996, 200).
38 I.e. the rules can only be applied, if in addition to $x \rightarrow y$ also the corresponding simplification formula $x\&y \rightarrow x$ is a thesis.
39 We can interpret any *first degree modal function of a single variable* i.e. a wff that contains (several occurrences of) a single variable and one or more modal operators, such that none of these modal operators is in the scope of any other, as a hidden modality. It has been shown that even a system like S4 has infinitely many distinct modal functions of a single

Ax23) (p→(p&q))→((p&q)→p)

Ax24) (p→q)→((p&q)→q)

to $SIM1_2$ we arrive at the system $SIM1_3$, in which Becker's rules above are introducible. $SIM1_3$ has 14 different modalities, namely p, Mp, Lp, MLp, LMp, MLMp, LMLp and their negations. Most of the hidden modalities built up by conjunction (e.g. Lp&p etc.) can be reduced using Becker's rules.

9. The matrix based system MS1.0

By means of the following set of matrices SM1.0, one can show that the requirements (i) and (iii) are fulfilled[40] by $SIM1_0$:

K	*	1 2 3 4	C	2 4 4 4	N	4	M 1
	*	2 2 3 4		1 2 4 4		3	1
		3 3 3 4		2 4 2 4		2	3
		4 4 4 4		1 2 1 2		1	4

(designated values 1 and 2):

But this set of matrices SM1.0 can also be useful in another respect: it can be regarded as defining a limit case for the construction of a family of systems, where each member is an extension of the $SIM1_0$ system satisfying all of the WGPS-requirements and allowing some additional plausible inferences. For the set of matrices SM1.0 together with appropriate definitions for the matrices of equivalence, disjunction, necessity etc. determines a matrix based system MS1.0 in the

variable (cf. Hughes and Cresswell 1990, 57). With respect to SIM1 conjunctive formulas are, of course, of main interest and I confine myself to these.

40 One has to show a) that the axioms and rules of $SIM1_0$ satisfy this set of matrices, i.e. that the axioms take only designated values and that the application of the calculus rules preserves this property, b) that the paradoxical formulas under (i) take at least one undesignated value, c) that $SIM1_0$ satisfies the 2-valued characteristic set of matrices of the PM-calculus extended by identity matrices for the modal notions and matrices for $SIM1_0$ implication (equivalence) that are congruent with that for material implication (equivalence). $SIM1_0$ satisfies the 2-valued characteristic set of matrices of the PM-calculus extended by identity matrices for the modal notions and matrices for $SIM1_0$ implication (equivalence) that are congruent with that for material implication (equivalence). This can be seen at once by replacing the designated values 1 and 2 in the set of matrices above by the truth-value t and the undesignated values 3 and 4 by the truth-value f.

canonical way: a wff x is a thesis of MS1.0, iff x takes a designated value under SM1.0. The matrix based system MS1.0 is a limit case for a sequence (or in general for a net) of systems satisfying the SM1.0 matrices and containing $SIM1_0$.
As MS1.0 is stronger than each member of the SIM1 family and has the advantage of being decidable, it may appear to be preferable. But though it satisfies all of the positive (iv)-(vi) and the negative requirements (i) and (iii), it contains some weakest (e.g. M(p→p)) and some strongest formulas (e.g. ~M(p→p)). Like all the systems based on a *finite valued* set of matrices MS1.0 has the additional disadvantage that it allows unwanted equivalences. Thus for instance

*73 (p→p)↔((q&(r&s))→((q&r)&s))

is a thesis, though its constituents (p→p) and ((q&(r&s))→((q&r)&s)) have a different number of (different) propositional variables.

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