IMPRIMATUR POUR LA THESE

K-theories, C*algebras and assembly maps

de M. Michel Matthey

UNIVERSITE DE NEUCHATEL

FACULTE DES SCIENCES

La Faculté des sciences de l'Université de Neuchâtel sur le rapport des membres du jury,

MM. U. Suter (directeur de thèse),
A. Valette (co-directeur de thèse), P. Jolissaint,
P. Julg (Strasbourg F) et
N. Higson (Pennsylvania USA)

autorise l'impression de la présente thèse.

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J.-P. Derendinger

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Michel Matthey

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Professeur Ulrich Suter
et la co-direction du
Professeur Alain Valette

Institut de Mathématiques
Université de Neuchâtel
Rue Emile-Argand 13
CH-2007 Neuchâtel

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Introduction

During my quest through the treasureland of mathematics, I have had the opportunity to discover some of the many beauties of the subject. Unfortunately (or fortunately?) it is impossible for me to have an overall picture. Nevertheless, the present work is an opportunity for me to share my passion for mathematics. Many different topics are studied, however, in my opinion, only in a too superficial manner.

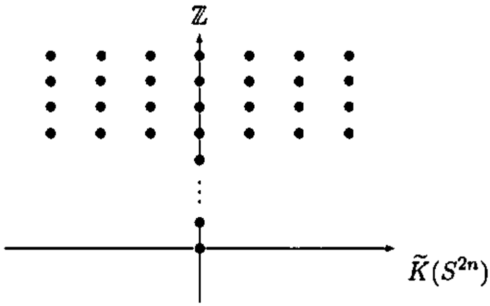
This thesis is concerned with K-theories and is essentially subdivided into two parts. The first one deals with algebraic topology, or more precisely with the positive cone in topological K-theory of spaces. The second is concerned with analytic K-theory of operator algebras, and more precisely with the Baum-Connes conjecture and related topics, such as group algebras, group C^*-algebras, assembly maps, Hochschild and cyclic homology, and algebraic K-theory.

Historical account of the thèsis

At the origin of this work, is the following question posed by G. A. Elliott in a seminar at Oberwolfach in April 1996: "What is the positive cone in the K-theory of the spheres S^4 and S^6 (and more generally of the even dimensional spheres)?" He was motivated by his classification of unital C^*-algebras of type AF by means of K-theoretical invariants, namely the K-theory, the positive cone and the K-theory class [1] of the unit. My co-advisor Alain Valette was in the audience and has transmitted this question to my advisor Ueli Suter and me. Within a couple of days, U. Suter was able to compute the positive cone in the K-theory of all even dimensional spheres. Let us now explain what is the positive cone in the K-theory of a compact space X, say a connected finite CW-complex. First, the K-theory of X is by definition the abelian group $K(X) := \mathcal{G}(\mathrm{Vect}(X))$, where $\mathrm{Vect}(X)$ is the abelian semi-group of isomorphism classes of finite dimensional complex vector bundles over X, and $\mathcal{G}(-)$ is the Grothendieck construction (also called the group completion). The positive cone $K_+(X)$ is the image of the canonical homomorphism of semi-groups $\theta : \mathrm{Vect}(X) \longrightarrow K(X)$. It is therefore a sub-semi-group of $K(X)$, verifying the equality $K_+(X) - K_+(X) = K(X)$. If $K_+(X)$ satisfies the property $K_+(X) \cap (-K_+(X)) = \{0\}$ (this condition cannot be fulfilled if $K(X)$ has torsion, but holds for $X = S^{2n}$, as we will see), then $K_+(X)$ defines a translation-invariant partial ordering on $K(X)$, by requiring

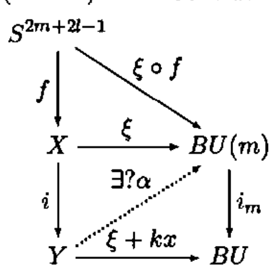
$$\xi \leq \eta \iff \eta - \xi \in K_+(X).$$

The problem posed by Elliott is to compute $K_+(X)$ explicitly as a sub-semi-group of $K(X)$ for particular compact spaces, such as spheres. After the celebrated computations of Bott, it is well-known that $K(S^{2n}) = \mathbb{Z} \oplus \widetilde{K}(S^{2n}) \cong \mathbb{Z}^2$ for any $n \geq 1$. In this sense, the K-theory itself does *not* suffice to distinguish between the different spheres. However, it turns out that the positive cones are non-isomorphic (as monoids) and easily allow to make the distinction. The picture is as follows:



(with the first "complete row" at height n).

In October 1996, after I had attended his 1995-1996 course on topological K-theory, my advisor Ueli Suter asked me to pass from the subject of my 1994-1995 diploma work, namely the global structure of compact Lie groups, to topological K-theory and homotopy theory. More precisely, he proposed to establish general properties of the positive cone, to compute it for some products of spheres, and to use characteristic classes of vector bundles to detect non-zero elements in the homotopy groups of the Grassmannians $BU(n)$, i.e. the classifying space of the unitary group $U(n)$. Let us make the latter idea explicit. Let $f : S^{2m+2l-1} \longrightarrow X$ be a continuous map, with values in a finite CW-complex X of dimension $\leq 2m+1$, where $l \geq 2$. Consider the cone $Y = C_f = X \cup_f e^{2m+2l}$ of the map f, and i the inclusion of X in Y. Let $\xi \in \widetilde{K}^0(X) = [X, BU]$ denote a virtual vector bundle over X. For well-known dimensional reasons, ξ can be considered as (the homotopy class of) a continuous map $X \longrightarrow BU(m)$. One has $\widetilde{K}(Y) \cong \widetilde{K}(X) \oplus \mathbb{Z} \cdot x$, where x is the Bott generator of $\widetilde{K}(S^{2m+2l}) \cong \mathbb{Z}$. Consider the diagram



that commutes up to homotopy, where $k \in \mathbb{Z}$ is a parameter. By basic homotopy theory, an extension α of ξ exists if and only if the compose map $\xi \circ f$ represents the trivial element in the group $\pi_{2m+2l-1}(BU(m))$. If such an α exists, then there exists a unique $k \in \mathbb{Z}$ such that $i_m \circ \alpha = \xi + kx \in \widetilde{K}(Y)$. Then, the existence of α amounts to saying that the geometric dimension $\mathrm{g-dim}(\xi + kx)$ of $\xi + kx$ is $\leq m$. Characteristic classes (such as Chern classes and γ-operations) allow to give lower bounds for the geometric dimension of a given virtual vector bundle. This explains how this program proposes to detect the non-triviality of $\xi \circ f$ in $\pi_{2m+2l-1}(BU(m))$. A more subtle question is: Can one detect by the same method the precise order $o(\xi \circ f)$ of $\xi \circ f$ in this group? The answer is yes:

$$\mathrm{o}(\xi \circ f) = \min\{n \in \mathbb{N} \setminus \{0\} \mid \exists k \in \mathbb{Z} \text{ such that } \mathrm{g-dim}(n\xi + kx) \leq m\}.$$

(Here, we have implicitly used the well-known fact that the groups $\pi_{2i+1}(BU(j))$ are all finite.) Consequently, this program consists in choosing appropriate X, $f \in \pi_{2m+2l-1}(X)$ and $\xi \in \widetilde{K}(X)$, and then hope to get precise information on the geometric dimension of the virtual vector bundles $n\xi + kx \in \widetilde{K}(Y)$ by means of Chern classes and/or γ-operations. Of course, the aim is to have $o(\xi \circ f)$ as large as possible.

A few positive results have been obtained in this direction. We have decided to include only one of them in this text (by far the most interesting one); it is stated as theorem 2.6.4 in chapter 2. It predicts for $2 \leq 2k \leq m - 1$ and any s such that $m \leq s \leq m + 2k - 1$, an explicit non-zero element in $\pi_{2m+4k-1}(BU(s))$, whose order is given by

$$\begin{cases} \operatorname{denom}\left(\frac{B_k}{4k}\right), & \text{if } k \text{ is even} \\ \operatorname{denom}\left(\frac{B_k}{4k}\right) \text{ or } \frac{1}{2}\operatorname{denom}\left(\frac{B_k}{4k}\right), & \text{if } k \text{ is odd}, \end{cases}$$

where B_k denotes the k-th Bernoulli number (we could not completely settle the case where k is odd). For example, when $k = 6$ and $m = s = 13$, this defines an element of order 65520 in $\pi_{49}(BU(13))$. Chronologically speaking, this result is my first one in this thesis (hence the particular attention I pay to it in this introduction). The first part of this thesis, i.e. chapters 1, 2 and 3, are concerned with this kind of problems, in other words, the subject is algebraic topology.

During summer 1997, Hela Bettaieb, at that time PhD student of Alain Valette, asked me many questions about compact surfaces and integral homology, and more precisely the second integral homology group of a group. She was motivated by the Baum-Connes conjecture. She woke up my curiosity and I asked her to explain me what this "famous conjecture" was about. After I had quickly read a book about the K-theory of C^*-algebras (I did not know before what a C^*-algebra was), she succeeded in this difficult task. The statement of the conjecture is that given a countable discrete group Γ, the analytic K-theory of its reduced C^*-algebra, denoted by $K_*(C_r^*\Gamma)$, is isomorphic to a group of topological (and geometrical) nature associated to Γ, and denoted by $K_*^{top}(\Gamma)$. More precisely, there is a map

$$\mu_*^\Gamma : K_*^{top}(\Gamma) \longrightarrow K_*(C_r^*\Gamma),$$

called the analytic assembly map (or the Baum-Connes assembly map, or the Γ-index map), and *it is conjectured that precisely this map realizes the isomorphism.* My interest for this conjecture grew in such proportions that half of this thesis is concerned with it and with surrounding subjects.

Let me now describe the contents of the different chapters.

Organization of the chapters and appendices

Chapter 1 is a joint work with Ueli Suter. Besides his computation of the positive cone of spheres, it contains our calculation of the positive cone of the products of an odd-dimensional sphere by a sphere, products of the 2-sphere by a sphere and of the products $S^4 \times S^4$, $S^4 \times S^6$, $S^6 \times S^6$ and $S^6 \times S^8$. This chapter also includes general properties of the positive cone and of approximations of it, namely the c-cone (based on Chern classes) and the γ-cone (based on γ-operations). For torsion-free spaces, i.e. spaces without torsion in their integral homology, the three notions of cones coincide. In chapter 2, we compute the c-cone of CW-complexes with two cells, and also the positive cone in particular cases.

In chapter 3, we compute, by means of the Atiyah-Hirzebruch spectral sequence, K-theory (resp. K-homology) of low-dimensional spaces in terms of integral homology (resp. cohomology). As an application, for any connected CW-complex X, we define natural maps

$$\beta_j^X : H_j(X;\mathbb{Z}) \longrightarrow K_j(X) \qquad (0 \le j \le 2),$$

where K_j is the 2-periodic K-homology with compact supports. The three maps are rationally injective, since we prove that, after tensoring with $\mathbb{Q}$, they are right inverses of the usual Chern character in K-homology. Finally, we prove that β_0^X is a split-injection, that β_1^X is injective, and we show by an example that β_2^X is generally *not* injective. We also establish injectivity of β_2^X if Γ has its reduced integral homology concentrated in even degree, except possibly for H_1 and H_3, as for example if there is a model for the classifying space $B\Gamma$ that is a CW-complex of dimension ≤ 4. The results about β_1^X and β_2^X are also of interest in the framework of the K-theory of group C^*-algebras and more precisely of the Baum-Connes conjecture, as we will soon explain.

Chapter 4 is devoted to giving a short introduction and a superficial overall picture of the conjecture. In particular, we define $K_*^{top}(\Gamma)$, with some details, as the Γ-equivariant K-homology with compact supports of the universal example for proper Γ-spaces:

$$K_*^{top}(\Gamma) := RK_*^\Gamma(\underline{E}\Gamma).$$

Since we will only work with K-homology with compact supports, from now on, we write $K_*^\Gamma(\underline{E\Gamma})$ in place of $RK_*^\Gamma(\underline{E\Gamma})$. In the same chapter, we also quote families of groups for which the conjecture is known to hold.

To describe chapter 5, we let $F\Gamma$ be the $\mathbb{C}$-module freely generated by the set of torsion elements in the group Γ, with the action of Γ by conjugation. Among other things, Hela Bettaieb has made me acquainted with "Alain Valette's program", namely to use the isomorphism

$$ch_*^\Gamma \otimes Id_{\mathbb{C}} : K_*^\Gamma(\underline{E\Gamma}) \otimes \mathbb{C} \xrightarrow{\cong} H_*(\Gamma; F\Gamma)$$

given by the delocalized equivariant Chern character ch_*^Γ of Baum and Connes, and to describe explicitly the composition $\mu_*^\Gamma \circ (ch_*^\Gamma \otimes Id_{\mathbb{C}})^{-1}$, in order to study the rational injectivity of μ_*^Γ. As a first step in this direction, in her thesis, Hela Bettaieb has defined maps

$$\beta_t^{loc} = \beta_t^{(j),loc} : H_j(\Gamma; \mathbb{Z}) \longrightarrow K_j(B\Gamma) \qquad (0 \leq j \leq 2).$$

(She denotes them by β_t; the subscript "t" stands for "topological", and the exponent "loc" for "localized".) In chapter 5, after identifying the integral homologies of Γ and of $B\Gamma$, we prove that

$$\beta_t^{(j),loc} = \beta_j^{B\Gamma} \qquad (0 \leq j \leq 2).$$

She has also defined explicit maps

$$\beta_a^{loc} = \beta_a^{(j),loc} : H_j(\Gamma; \mathbb{Z}) \longrightarrow K_j(C_r^*\Gamma) \qquad (0 \leq j \leq 2)$$

(that she denotes by β_a, with the subscript "a" standing for "analytical"), such that the diagram

$$\begin{array}{ccccc} K_j(B\Gamma) & \longrightarrow & K_j^\Gamma(\underline{E\Gamma}) & \xrightarrow{\mu_j^\Gamma} & K_j(C_r^*\Gamma) \\ & \nwarrow {\scriptstyle \beta_t^{loc}} & & \nearrow {\scriptstyle \beta_a^{loc}} & \\ & & H_j(\Gamma; \mathbb{Z}) & & \end{array}$$

commutes, for $j = 0, 1$ and 2. Here, the map $K_j(B\Gamma) \longrightarrow K_j^\Gamma(\underline{E\Gamma})$ is the canonical map; as is well-known, it is rationally injective, and for torsion-free groups, it is even an isomorphism (see chapter 4). Motivated by these results, Alain Valette has posed the following problem: Can one "delocalize" the above constructions, in other words, can one define "topological" maps

$$\beta_t = \beta_t^{(j)} : H_j(\Gamma; F\Gamma) \longrightarrow K_j^\Gamma(\underline{E\Gamma}) \otimes \mathbb{C}$$

and "analytical" maps

$$\beta_a = \beta_a^{(j)} : H_j(\Gamma; F\Gamma) \longrightarrow K_j(C_r^*\Gamma) \otimes \mathbb{C},$$

such that the diagram

$$\begin{array}{ccc} & K_j^\Gamma(\underline{E\Gamma}) \otimes \mathbb{C} \xrightarrow{\mu_j^\Gamma \otimes Id_{\mathbb{C}}} & K_j(C_r^*\Gamma) \otimes \mathbb{C} \\ {\scriptstyle \beta_t^{loc} \otimes Id_{\mathbb{C}}} \nearrow & \uparrow {\scriptstyle \beta_t} & \nearrow {\scriptstyle \beta_a} \\ H_j(\Gamma; \mathbb{C}) \xhookrightarrow{i_*} & H_j(\Gamma; F\Gamma) & \end{array}$$

commutes, for $j = 0$, 1 and 2. (The injection i_* is induced by the inclusion $i : \mathbb{C} \hookrightarrow F\Gamma$, $\lambda \longmapsto \lambda \cdot e$ of Γ-modules). This amounts to trying to express the rational Baum-Connes assembly map in "low homological degrees". We have succeeded in constructing these maps. This is a joint work with H. Bettaieb (for $j = 0$), and it is the content of chapter 5. An important feature of our construction of the maps β_t and β_a is that we need a "classifying family" of groups for $H_j(\Gamma; F\Gamma)$, that is a family of groups $\{G_i\}_{i \in I}$ (indexed by a certain set I) such that for any group Γ, any homology class in $H_j(\Gamma; F\Gamma)$ arises from a "fundamental class" in $H_j(G_i, FG_i)$, for a certain G_i, through a homomorphism $G_i \longrightarrow \Gamma$. In the three considered cases, the "classifying family" is given by

$$\begin{aligned} &\text{for } j = 0 : \ \{\mathbb{Z}/n\}_{n \geq 1} \\ &\text{for } j = 1 : \ \{\mathbb{Z} \times \mathbb{Z}/n\}_{n \geq 1} \\ &\text{for } j = 2 : \ \{\Gamma_g \times \mathbb{Z}/n\}_{g,\, n \geq 1}\,, \end{aligned}$$

where Γ_g is the fundamental group of a closed oriented surface of genus g.

In chapter 6, we generalize the construction of the map β_a out of β_a^{loc}, by establishing a "delocalization property" for the Baum-Connes assembly map μ_*^Γ. More precisely, we show how one can glue together simpler assembly maps, namely Novikov assembly maps associated to centralizers of torsion elements in Γ, to build μ_*^Γ. The name "delocalization" comes from the fact, explained in chapter 4, that the Novikov assembly map can be considered as the Baum-Connes assembly map localized at the identity e of Γ. The main idea to perform this gluing is to use the spectral projectors, in the complex group algebra $\mathbb{C}\Gamma$, associated to the finite order elements in Γ.

It turns out that the statement of the delocalization property makes sense for assembly maps in other areas than the K-theory of group C^*-algebras. In chapter 7, we discuss Hochschild homology and cyclic homology of group algebras, and recall the corresponding assembly maps. We then establish the delocalization property for these maps. As an application of the delocalization in Hochschild homology, by delocalizing a known assembly map in algebraic K-theory, we prove the following theorem:

1 Theorem. *Let Γ be a discrete group, and let $\mathbb{Z}_\Gamma$ be the ring extension of the integers by the set*

$$\left\{ \frac{e^{2\pi i/n}}{n} \,\middle|\, \exists \gamma \in \Gamma \text{ of order } n \right\}.$$

Let R be a ring such that $\mathbb{Z}_\Gamma \subseteq R \subseteq \mathbb{C}$. Then there is an injective assembly map

$$H_*(\Gamma;\, F\Gamma) \hookrightarrow K_*^{alg}(R\Gamma) \otimes \mathbb{C},$$

where $K_*^{alg}(R\Gamma)$ is the algebraic K-theory "à la Quillen" of the group ring $R\Gamma$.

In chapter 8, a joint work with Hervé Oyono-Oyono, we prove that for $0 \leq j \leq 2$, the map $\beta_a^{(j),loc} : H_j(\Gamma;\, \mathbb{Z}) \longrightarrow K_j(C_r^*\Gamma)$ factorizes through a map

$$H_j(\Gamma;\, \mathbb{Z}) \longrightarrow K_j^{alg}(\mathbb{Z}\Gamma)/\Delta_j\,,$$

where Δ_0 and Δ_1 are the trivial subgroup of $K_0^{alg}(\mathbb{Z}\Gamma)$ and $K_1^{alg}(\mathbb{Z}\Gamma)$ respectively, and

$$\Delta_2 := \langle \{\gamma,\, \gamma\} \,|\, \gamma \in \Gamma \rangle$$

is the subgroup of $K_2^{alg}(\mathbb{Z}\Gamma)$ generated by the Steinberg symbols $\{\gamma,\, \gamma\}$. This answers a question posed by Nigel Higson and Pierre Julg. By delocalizing this result, we deduce that, for $0 \leq j \leq 2$, the map $\beta_a^{(j)} : H_j(\Gamma;\, F\Gamma) \longrightarrow K_j(C_r^*\Gamma) \otimes \mathbb{C}$ factorizes through a map

$$H_j(\Gamma;\, F\Gamma) \longrightarrow K_j^{alg}(\mathbb{Z}_\Gamma\Gamma) \otimes \mathbb{C}.$$

Denoting by $\mathbb{T}^2$ the 2-torus, the proof is a lengthy computation showing that the Bott generator in $K_2(C_r^*\mathbb{Z}^2) \cong K^0(\mathbb{T}^2)$ coincides with the image in $K_2(C_r^*\mathbb{Z}^2)$ of the Steinberg symbol $\{a,\, b\} \in K_2^{alg}(\mathbb{Z}[\mathbb{Z}^2])$, where a and b denote the prescribed generators of $\mathbb{Z}^2$, viewed as classes in $K_1^{alg}(\mathbb{Z}[\mathbb{Z}^2])$.

Together with the results of chapter 3, this allows to prove the following two theorems.

2 Theorem. *Let Γ is a countable discrete group, and let A be a ring such that $\mathbb{Z}\Gamma \subseteq A \subseteq C_r^*\Gamma$. If Γ is torsion-free and the Baum-Connes assembly map μ_*^Γ is injective, then the maps*

$$H_j(\Gamma;\, \mathbb{Z}) \longrightarrow K_j^{alg}(A)/\Delta_i \quad \text{and} \quad H_j(\Gamma;\, \mathbb{Z}) \longrightarrow K_j(C_r^*\Gamma)$$

are injective in the following two cases:

i) For $j = 0$ and 1, without further assumption on Γ.

ii) *For $j = 2$, if Γ has its reduced integral homology concentrated in even degree, except possibly for H_1 and H_3, as for example if there is a model for the classifying space $B\Gamma$ that is a CW-complex of dimension ≤ 4.*

The second of these theorems is a delocalized version of the first one.

3 Theorem. *Let Γ is a countable discrete group, and let A be a ring such that $\mathbb{Z}_\Gamma\Gamma \subseteq A \subseteq C^*_r\Gamma$. If the Baum-Connes assembly map μ^Γ_* is rationally injective, then the map*

$$H_j(\Gamma; F\Gamma) \longrightarrow K_j^{alg}(A) \otimes \mathbb{C}$$

is injective, for $j = 0$, 1 and 2.

In the final chapter, namely chapter 9, we prove that the product structures in algebraic and in analytic K-theory of a unital Banach algebra are compatible in total degree ≤ 2. This is an application of the computations performed in chapter 8, and it answers by the positive an open question of John Milnor in his famous book [75].

In appendix A, we define six notions of cone related to K-theory tensored by $\mathbb{Q}$. We prove that four of them coincide and that the other two give an upper and a lower bound. More precisely, the final result reads:

$$K_+^{\mathbb{Q}}(X) \subseteq K_\gamma^{\mathbb{Q}}(X) = K_c^{\mathbb{Q}}(X) = K_{\bar{c}}^{\mathbb{Q}}(X) = K_+(j_X; \mathbb{Q}) \subseteq K_+(X; \mathbb{Q})$$

(see this appendix for the definition of these objects). In appendix B, we review the Grothendieck construction of semi-groups and semi-rings, and define (in an obvious way) a notion of tensor product of semi-modules over semi-rings. One of the main results says that if R, S and T are three semi-rings, M is an (R, S)-semi-bimodule, and N is an (S, T)-semi-bimodule, then there is a canonical isomorphism of $(\mathcal{G}(R), \mathcal{G}(T))$-bimodules

$$\mathcal{G}(M \otimes_S N) \xrightarrow{\cong} \mathcal{G}(M) \otimes_{\mathcal{G}(S)} \mathcal{G}(N).$$

This is useful in appendix A, where we set

$$K(X, \mathbb{Q}) := \mathcal{G}(\mathrm{Vect}(X) \otimes_{\mathbb{N}} \mathbb{Q}),$$

and then deduce that there are canonical isomorphisms

$$K(X, \mathbb{Q}) \cong K(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q} \oplus \widetilde{K}(X; \mathbb{Q}).$$

In appendix C, we study the Moore-Postnikov tower of the canonical fibration $BSU(3) \longrightarrow BSU(5)$ to re-prove that the positive cone of $S^6 \times S^6$ coincides with its c-cone. This computation is performed by means of the Leray-Serre spectral sequence, and, being extremely lengthy, we have decided to include it as an appendix. The first proof of this result for $S^6 \times S^6$, contained in chapter 1, is based on deep result on higher homotopy groups of spheres. The second proof is independent of this kind of results. We also compute the c-cone of the quaternionic projective space $\mathbb{H}P^3$, and as an application of this long calculation, we give some precise piece of information on its positive cone.

In appendix D, we discuss briefly the connection between the lifting problem related to the computation of the geometric dimension of a stable class of vector bundles, and the first obstruction from homotopy theory.

Finally, in *appendix* E, *we give a second proof of the injectivity of* the map β_1^X considered in chapter 3. It is based on homology approximations of simply-connected CW-complexes and on the universal coefficient theorem for K-homology. The map β_2^X is also shortly considered.

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First of all, I want to express my infinite gratitude to my master and advisor, Professor Ueli Suter. I remember clearly that from the first time I met him (this was during my very first minutes at university), I was admiring and instantly eager to work with him. His way of teaching is, in my opinion, perfect. In particular, he taught me everything I know in linear algebra, differential geometry, Lie group theory, representation theory, algebraic topology, homotopy theory and topological K-theory. During my PhD thesis, each time I needed his help, he was available and welcoming. I posed him many stupid or boring questions: he was always listening carefully and patiently, and then answering with passion and in details. He never refused to re-explain the most difficult or tricky arguments. Without his tremendous help, this thesis would be skeletical or non-existent. Among other things, he has improved and corrected many results (without accepting that I mention it in the course of the text!). I am thankful for his detailed rereading of all this work. Finally, I thank him for having accepted me as a PhD student (especially when I started to work on the Baum-Connes conjecture), and for the great confidence he has permanently had.

The second person I want to thank is Professor Alain Valette, in particular for having accepted the thankless (and not clearly defined) task of being my "co-advisor". Since the first courses he taught me, namely general topology and general algebra in the second year at university, I could appreciate his huge pedagogical ability. Already since then, he has expressed permanently a deep confidence in me. I could also appreciate to be introduced by him to many mathematicians from all over the world. I am also thankful for his thorough rereading of my thesis.

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Chapter 1

The positive cone of spheres and some products of spheres

Motivated by Elliott's K-theoretic classification of C^*-algebras of type AF, we compute the positive cone of the K-theory of some spaces. These include the spheres, products of an odd-dimensional sphere by a sphere, products of the 2-sphere by a sphere and of the products $S^4 \times S^4$, $S^4 \times S^6$, $S^6 \times S^6$ and $S^6 \times S^8$. This amounts to compute the geometric dimension of stable classes of complex vector bundles over these spaces. We establish a few general properties of the positive cone and of approximations of it, the γ-cone and the c-cone. We also get information on the Whitehead product structure in the homotopy groups of $BU(n)$. Moreover, we prove a "doubling formula" for Stirling numbers of the second kind. This chapter is a joint work with Ueli Suter.

1.1 Introduction

Let $\mathcal{G}(S)$ be the Grothendieck group completion of an abelian semigroup S, and let $\theta : S \longrightarrow \mathcal{G}(S)$ be the corresponding universal homomorphism. The image of θ, denoted by $\mathcal{G}_+(S)$, is a sub-semigroup of $\mathcal{G}(S)$. It induces a translation invariant order on $\mathcal{G}(S)$; the elements of $\mathcal{G}_+(S)$ are called *positive* and $\mathcal{G}_+(S)$ is called the *positive cone* (see [41], [15]). The pair $(\mathcal{G}(S), \mathcal{G}_+(S))$ is an isomorphism invariant for S, and a basic question is: To what extend does this invariant characterize the semigroup S?

The above notions are of interest in connection with the classification problem of C^*-algebras. For a unital C^*-algebra A, let $S = V(A)$ be the semigroup of equivalence classes of projectors in the matrix algebra $\mathbb{M}_\infty(A)$. The K-theory of A, denoted by $K_0(A)$ or $K(A)$, is by definition the group $\mathcal{G}(V(A))$. The positive cone in K-theory is $\mathcal{G}_+(V(A))$ and it is denoted by $K_+(A)$. In [41], Elliott has put forward a program to classify a large class of unital C^*-algebras by invariants of K-theoretic nature, such as $K(A)$, $K_+(A)$, [1] (the K-theory class of the unit), etc. (see also [15]). For a compact space X, the algebra of continuous complex valued functions $C(X)$ is a unital C^*-algebra and its K-theory coincides with the topological K-theory $K^0(X)$ of the space X (according to the Swan-Serre theorem). In view of Elliott's program and to shed light on various conjectures, it is of great interest to determine for such spaces the positive cone $K_+(X) = \mathcal{G}_+(V(C(X)))$.

The problem of computing the positive cone of some spaces and in particular of spheres has been communicated to us by Alain Valette, after a question asked by G. A. Elliott in Oberwolfach.

This chapter is organized as follows. In section 1.2, we recall the basic facts from topological K-theory needed in the sequel. Among other things, we review γ-operations. The computation of these operations for even-dimensional spheres puts Stirling numbers of the second kind on stage. In section 1.3, we define what we call the γ-cone and the c-cone (the latter is defined in terms of Chern classes), and we explain in what sense they are approximations of the positive cone. We illustrate by examples that the three notions of cones are different in general, although the γ-cone and the c-cone coincide for torsion-free spaces.

In section 1.4, we compute the positive cones of the spheres, by using some standard homotopy theory. Section 1.5 is devoted to the naturality properties of the three cones. The positive cone of the products $S^n \times S^{2m-1}$ is computed in section 1.6. The γ-cone of the products $S^{2n} \times S^{2m}$ is easily calculated in section 1.7 by means of Chern classes. In that section, we also compute the positive cone of $S^2 \times S^{2n}$.

The Whitehead product structure on the homotopy of the classifying space $BU(n)$ is closely related to the problem of determining the positive cone of the product of two even-dimensional spheres, as is explained in section 1.8. This allows us to improve slightly a result of Bott on this structure, and gives some precise information on the positive cone of such a product of spheres.

In section 1.9, we perform the computation of the positive cones of $S^4 \times S^4$, $S^4 \times S^6$, $S^6 \times S^6$ and of $S^6 \times S^8$. This is achieved by using some well-known results on the homotopy groups of the unitary groups. In section 1.10, we show that for spaces "with only one high-dimensional cell" the γ-cone is "blind" in some sense to be made precise there.

Section 1.11 is devoted to explicitly compute the γ-operations for the products $S^{2n} \times S^{2m}$. As a consequence of these calculations, we establish a "doubling-formula" for Stirling numbers of the second kind. Moreover, we are led to conjecture that the same formula holds for Stirling numbers of the first kind. (It has now been proved by Al Lundell.)

1.2 Preliminaries

We start by reviewing some topological K-theory. Our basic references are the books by Atiyah [4] and by Husemoller [54].

Let X be a *connected finite* CW-complex. (We assume all spaces and maps to be pointed.) For $n \geq 0$, let $\mathrm{Vect}_n(X)$ be the set of isomorphism classes of complex n-plane vector bundles over X, and $\mathrm{Vect}(X)$ their disjoint union. There are well-known bijections

$$\mathrm{Vect}_n(X) \approx [X, BU(n)] \qquad (n \geq 0)$$

where $BU(n)$ is the classifying space of the unitary group $U(n)$ and $[.,.]$ stands for the set of homotopy classes of maps. For an n-plane vector bundle ξ over X, i.e. $\xi \in \mathrm{Vect}_n(X)$, we write $\mathrm{rk}(\xi) = n$ (the rank of ξ). The direct sum (also called Whitney sum) and the tensor product of vector bundles endow $\mathrm{Vect}(X)$ with a semiring structure. The K-theory of X is the ring $K(X)$, also denoted by $K^0(X)$, obtained by applying the Grothendieck construction to $\mathrm{Vect}(X)$, i.e. $K(X) = \mathcal{G}(\mathrm{Vect}(X))$. An element of $K(X)$ is sometimes called a virtual vector bundle. There is a ring isomorphism

$$K(X) \cong [X, \mathbb{Z} \times BU],$$

where BU is the infinite Grassmannian, i.e. the direct limit of the classifying spaces $BU(n)$. We identify both rings from now on. There is a canonical splitting $K(X) = \mathbb{Z} \oplus [X, BU] = \mathbb{Z} \oplus \widetilde{K}(X)$, where $\widetilde{K}(X) = \widetilde{K}^0(X)$ is the subring of stable classes of vector bundles, and $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ is represented by the n-dimensional trivial vector bundle. Clearly, the Grothendieck construction gives rise to maps $\theta : \mathrm{Vect}(X) \longrightarrow K(X)$ and $\theta_n : \mathrm{Vect}_n(X) \longrightarrow n \times \widetilde{K}(X)$ (by restriction of θ).

1.2.1 Definition. *i) The positive cone of X, denoted by $K_+(X)$, is the image of θ. An element $\xi \in K(X)$ is called positive if it lies in the positive cone.*

ii) The geometric dimension of a class $x \in \widetilde{K}(X)$, denoted by $\mathrm{g{-}dim}(x)$, is the smallest integer n such that (n, x) lies in the image of θ_n, i.e. the least integer n such that the stable class x is represented by an n-dimensional vector bundle.

Since θ is a semiring homomorphism, it is clear that $K_+(X)$ is a sub-semiring of $K(X)$. Notice that it is equivalent to determine the positive cone or the map $\mathrm{g{-}dim} : \widetilde{K}(X) \longrightarrow \mathbb{Z}$; in fact, we have

$$K_+(X) = \{(n, x) \in \mathbb{Z} \oplus \widetilde{K}(X) \mid n \geq \mathrm{g{-}dim}(x)\}.$$

Let us also notice that an element $x \in \widetilde{K}(X)$, considered as a homotopy class of maps $X \longrightarrow BU$, has geometric dimension $\leq n$ if and only if x has a lifting

$x_n : X \longrightarrow BU(n)$, i.e.

$$\begin{array}{ccc} & & BU(n) \\ & \overset{x_n}{\nearrow} & \downarrow i_n \\ X & \xrightarrow{\quad x \quad} & BU \end{array} \qquad i_n \circ x_n = x$$

(Here, we identify maps with the homotopy class they represent.) Recall that i_n is a fibration with fiber $U/U(n)$, where $U = \varinjlim U(n)$ is the infinite unitary group (and BU is really its classifying space).

The image of θ_n is equal to the image of the composition

$$[X, BU(n)] \overset{(i_n)_*}{\longrightarrow} [X, BU] \hookrightarrow n \times [X, BU], \; y \longmapsto (n, (i_n)_*(y)).$$

We write $K^*(X) = K^0(X) \oplus K^1(X)$, where the K^1-group is defined by

$$K^1(X) := [X, U].$$

For a pair of connected finite CW-complexes (X, Y), there is the famous six-term exact sequence:

$$\begin{array}{ccccc} \widetilde{K}^0(X/Y) & \xrightarrow{q^*} & \widetilde{K}^0(X) & \xrightarrow{i^*} & \widetilde{K}^0(Y) \\ \uparrow & & & & \downarrow \\ K^1(Y) & \xleftarrow{i^*} & K^1(X) & \xleftarrow{q^*} & K^1(X/Y) \end{array}$$

where $i : Y \hookrightarrow X$ is the inclusion and $q : X \twoheadrightarrow X/Y$ is the quotient map.

The n-th exterior power operation for complex vector spaces induces an operation on vector bundles denoted by $\xi \longmapsto \lambda^n \xi$, and endows $K(X)$ with a natural λ-ring structure. For $\xi \in K(X)$, one defines

$$\lambda_t(\xi) := \sum_{n \geq 0} (\lambda^n \xi) \cdot t^n \in K(X)[[t]]$$

(the latter being the ring of formal power series with coefficients in $K(X)$).

The function λ_t is exponential, i.e. $\lambda_t(\xi + \eta) = \lambda_t(\xi) \cdot \lambda_t(\eta)$. Associated to the λ-operations are the γ-operations or Grothendieck operations $\gamma^n(\xi)$, which are defined by their generating series as follows:

$$\sum_{n \geq 0} \gamma^n(\xi) \cdot t^n = \gamma_t(\xi) := \lambda_{t/1-t}(\xi).$$

In particular, $\gamma^0(\xi) = 1$ and $\gamma^1(\xi) = \xi$. Again, the function γ_t is exponential, which implies that

$$\gamma^n(\xi + \eta) = \sum_{k=0}^{n} \gamma^k(\xi) \cdot \gamma^{n-k}(\eta).$$

The importance of the γ-operations in our context is illustrated by the following fact (see [4], prop. 3.1.1):

$$\text{Let } x \in \widetilde{K}(X)\,;\text{ if } \mathrm{g-dim}(x) \leq n\,,\text{ then } \gamma^k(x) = 0 \text{ for } k > n\,.$$

(Assume that $(n,\, x) \in \mathbb{Z}\oplus\widetilde{K}\,(X)$ is represented by an n-dimensional vector bundle ξ. Then $\lambda_s(\xi)$ is a polynomial of degree n in s. By the exponential property, $\lambda_s(\xi) = \lambda_s(1)^n \cdot \lambda_s(x) = (1+s)^n \cdot \lambda_s(x)$. Letting $s = t/(1-t)$, we see that $\gamma_t(x) = \lambda_{t/1-t}(x) = (1-t)^n \lambda_{t/1-t}(\xi)$ is a polynomial of degree $\leq n$ in t.)

The representable K-theory of $BU(n)$, i.e. $[BU(n),\, \mathbb{Z} \times BU]$, is given by

$$K(BU(n)) = \mathbb{Z}[[\widetilde{\gamma}^1,\, \dots\,,\, \widetilde{\gamma}^n]]$$

where $\widetilde{\gamma}^k = \gamma^k(\widetilde{\rho}_n)$ for $1 \leq k \leq n$, $\widetilde{\rho}_n$ being the stable class represented by the universal n-plane bundle ρ_n over $BU(n)$. Note that $\gamma^k(\widetilde{\rho}_n) = 0$ for all $k > n$, and that the homomorphism j^*, induced by $j : BU(n) \longrightarrow BU(n+l)$ in K-theory, maps $\widetilde{\rho}_{n+l}$ to $\widetilde{\rho}_n$ for any $l \geq 0$.

For a complex vector bundle ξ over X the n-th Chern class $c_n(\xi)$ is a $2n$-dimensional integral cohomology class of X, i.e. $c_n(\xi) \in H^{2n}(X;\, \mathbb{Z})$. One has $c_0(\xi) = 1$ for any ξ. The element $c(\xi) = \sum_{n\geq 0} c_n(\xi) \in H^*(X;\, \mathbb{Z})$, called the total Chern class, is exponential, i.e. it satisfies

$$c(\xi + \eta) = c(\xi) \cdot c(\eta)\,.$$

The basic properties of Chern classes (see [54]) imply the following facts:

i) Two stably equivalent bundles over X have the same Chern classes. Hence, for an element $x \in \widetilde{K}(X)$, the Chern class $c_n(x) \in H^{2n}(X;\, \mathbb{Z})$ is well-defined.

ii) If $n > \mathrm{rk}(\xi)$, then $c_n(\xi) = 0$.

iii) Let $x \in \widetilde{K}(X)$; if $\mathrm{g-dim}(X) \leq n$ then $c_k(x) = 0$ for $k > n$.

Let us also formally define the polynomial

$$c_\xi(t) := \sum_{n\geq 0} c_n(\xi) \cdot t^n \in H^*(X;\, \mathbb{Z})[t]\,,$$

which, by ii) above, indeed is a polynomial. It is also exponential.

A central feature of Chern classes is that the cohomology ring of $BU(n)$ is given by

$$H^*(BU(n);\, \mathbb{Z}) = \mathbb{Z}[\widetilde{c}_1,\, \dots\,,\, \widetilde{c}_n]\,,$$

where $\widetilde{c}_k = c_k(\widetilde{\rho}_n)$ for $1 \leq k \leq n$. Moreover, $c_k(\widetilde{\rho}_n) = 0$ for any $k > n$. On the combinatorial point of view, for any $n \geq 0$, the Grassmannians $BU(n)$ and BU admit CW-decompositions with the same $(2n+1)$-skeleton, in other words, such that $BU(n)^{[2n+1]} = BU^{[2n+1]}$. (This can be proved by adapting section 6 of [76] to the complex case.)

The Chern character ch is a multiplicative natural transformation from K-theory to rational cohomology

$$ch : K(X) \longrightarrow H^{ev}(X;\mathbb{Q}) = \bigoplus_{q\geq 0} H^{2q}(X;\mathbb{Q}),\ \xi \longmapsto ch(\xi) = \sum_{q\geq 0} ch_{2q}(\xi),$$

where $ch_{2q}(\xi) \in H^{2q}(X;\mathbb{Q})$ (X being a connected *finite* CW-complex). It relates γ-operations and Chern classes as given in the following well-known proposition. Before stating it, we introduce some notation. For $x \in \widetilde{K}(X)$, we let $\bar{c}_j(x)$ be the image of $c_j(x)$ under the coefficient homomorphism $H^{2j}(X;\mathbb{Z}) \longrightarrow H^{2j}(X;\mathbb{Q})$, and let $I_x(\bar{c}_k, \ldots, \bar{c}_n)$ be the ideal in $H^{ev}(X;\mathbb{Q})$ generated by $\bar{c}_k(x), \ldots, \bar{c}_n(x)$, where $x \in \widetilde{K}(X)$ and $n \geq k$.

1.2.2 Proposition. *Let X be a connected finite CW-complex of dimension $\leq 2n$, $x \in \widetilde{K}(X)$, and $k \leq n$. One then has*

$$ch(\gamma^k(x)) = \bar{c}_k(x) + P_{k+1}(\bar{c}_1(x), \ldots, \bar{c}_n(x)),$$

where P_{k+1} is a polynomial in $\bar{c}_1(x), \ldots, \bar{c}_n(x)$ contained in the ideal

$$I_x(\bar{c}_k, \ldots, \bar{c}_n) \bigcap \left(\bigoplus_{q\geq k+1} H^{2q}(X;\mathbb{Q}) \right).$$

In particular,

$$ch(\gamma^n(x)) = \bar{c}_n(x) \in H^{2n}(X;\mathbb{Q}).$$

Proof. Since for a line bundle $\eta = 1 + y$ one has $ch(\gamma_t(y)) = 1 + (e^{c_1(y)} - 1)\cdot t$ in $H^{ev}(X;\mathbb{Q})[[t]]$, the result follows readily for the Whitney sum of n universal line bundles over $\mathbb{C}P^\infty \times \ldots \times \mathbb{C}P^\infty$ (n factors) and hence for any element x represented by a Whitney sum of line bundles. The general case is obtained by invoking the splitting principle. □

(Proposition 1.2.2 shows that $\gamma^k(x)$ is of filtration $\geq k$, in the terminology of [5].)

Recall that a finite CW-complex is called torsion-free if its integral homology (or equivalently integral cohomology) contains no torsion. The Chern classes give some "sharp information" on the geometric dimension, as the next fundamental theorem shows.

1.2.3 Theorem. *Let X be a connected finite CW-complex of dimension $\leq 2n$, and $x \in \widetilde{K}(X)$. Then*

$$\text{g-dim}(x) < n \iff c_n(x) = 0.$$

If moreover X is torsion-free, then this is also equivalent to $\gamma^n(x) = 0$.

Proof. If $\mathrm{g-dim}(x) < n$, as already mentioned, $c_n(x) = 0$. The converse is a consequence of theorem 41.5 on page 210 of [100] (the Chern classes of a vector bundle are the same as the Chern classes of the associated spherical bundle as defined in [100]). The last statement follows from proposition 1.2.2 and injectivity of the Chern character for a torsion-free X. □

Let us recall the K-theory of the spheres:

$$\begin{array}{ll} K^0(S^{2n}) \cong \mathbb{Z} \oplus \mathbb{Z} & K^0(S^{2n+1}) \cong \mathbb{Z} \\ K^1(S^{2n}) = 0 & K^1(S^{2n+1}) \cong \mathbb{Z}. \end{array}$$

(For technical reasons, we will always implicitly exclude the 0-sphere.) The multiplicative structure on $\widetilde{K}(S^{2n}) = \mathbb{Z} \cdot x_{2n}$ is given by $x_{2n}^2 = 0$. The γ-operations and the Chern classes are as given in the next proposition.

1.2.4 Proposition. *Let x_{2n} be a generator of $\widetilde{K}(S^{2n}) \cong \mathbb{Z}$. Then*

i) $\gamma^k(x_{2n}) = (-1)^{k-1}(k-1)!\, S(n,k) \cdot x_{2n}$, *where $S(n,k)$ is a Stirling number of the second kind.*

ii) $c_n(x_{2n}) = (-1)^{n-1}(n-1)! \cdot a_{2n}$, *where a_{2n} is a suitable generator of the cohomology group $H^{2n}(S^{2n}; \mathbb{Z}) \cong \mathbb{Z}$.*

Proof. It is well-known that $\lambda^k(x_{2n}) = (-1)^{k+1}k^{n-1} \cdot x_{2n}$, for $k \geq 1$ (see proposition 2.5 and theorem 11.2 in chapter 13 of [54]). We thus get

$$\begin{aligned}
\gamma_t(x_{2n}) = \lambda_{t/1-t}(x_{2n}) &= 1 + \left(\sum_{k\geq 1}(-1)^{k+1}k^{n-1}t^k(1-t)^{-k}\right) \cdot x_{2n} \\
&= 1 - \left(\sum_{k\geq 1}\sum_{j\geq 0}(-1)^{k+j}\binom{-k}{j}k^{n-1} \cdot t^{k+j}\right) \cdot x_{2n} \\
&= 1 - \left(\sum_{k\geq 1}\sum_{j\geq 0}(-1)^{k}\binom{k+j}{k}\frac{k^n}{k+j} \cdot t^{k+j}\right) \cdot x_{2n} \\
&\overset{(*)}{=} 1 - \left(\sum_{m\geq 1}\left(\sum_{k=1}^{m}(-1)^{k}\binom{m}{k}\frac{k^n}{m}\right) \cdot t^{m}\right) \cdot x_{2n} \\
&= 1 + \left(\sum_{m\geq 1}(-1)^{m-1}(m-1)!\, S(n,m) \cdot t^{m}\right) \cdot x_{2n},
\end{aligned}$$

since $S(n,m) = \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\frac{k^n}{m!}$ (see (6.19) on page 251 of [47]), hence the first formula. (The equality $(*)$ is obtained by substituting $m := k + j$.)

The second formula follows from tbeorem 9.6 and corollary 9.8 (and its proof) in chapter 20 of [54]. □

Let us finally state a lemma relating γ-operations and Chern classes. We will need it later on.

1.2.5 Lemma. *Let Y be a connected CW-complex (possibly infinite). Then, for an element $x \in \widetilde{K}(Y) = [Y, BU]$, one has*

$$c_n(\gamma^n(x)) = (-1)^{n-1}(n-1)!\, c_n(x) \in H^{2n}(Y; \mathbb{Z}).$$

Proof. Let $i : BU(n-1) \longrightarrow BU$ be the canonical map and

$$i^* : H^*(BU; \mathbb{Z}) = \mathbb{Z}[\widetilde{c}_1, \widetilde{c}_2, \dots] \longrightarrow \mathbb{Z}[\widetilde{c}_1, \dots, \widetilde{c}_{n-1}] = H^*(BU(n-1); \mathbb{Z})$$

the induced map. Since $c_n(\widetilde{\gamma}^n) \in \operatorname{Ker}(i^*) \cap H^{2n}(BU; \mathbb{Z}) = \mathbb{Z} \cdot \widetilde{c}_n$, there exists an integer q_n such that $c_n(\widetilde{\gamma}^n) = q_n \cdot \widetilde{c}_n$. Recalling that $S(n, n) = 1$, an easy computation (based on proposition 1.2.4) for the sphere S^{2n} shows that one has $q_n = (-1)^{n-1}(n-1)!$, as claimed. □

1.3 The γ-cone and the c-cone

In general, the problem of computing the geometric dimension of vector bundles is very complicated, as is any general lifting problem in homotopy theory. So, the same is true for the positive cone. That is why we now introduce what we call the γ-cone and the c-cone. They are supposed to be easier to compute and might be good approximations of the positive cone. As we will see, these two cones coincide for torsion-free spaces.

1.3.1 Definition. *i) The γ-cone of X is defined by*

$$K_\gamma(X) := \{(n, x) \in \mathbb{Z} \oplus \widetilde{K}(X) \,|\, \gamma^k(x) = 0 \ \text{for all} \ k > n\}.$$

The γ-dimension of a class $x \in \widetilde{K}(X)$, denoted by $\gamma{-}\dim(x)$, is the least integer n such that $\gamma^k(x) = 0$ for all $k > n$, in other words, it is the degree (in the variable t) of the polynomial $\gamma_t(x)$.

ii) The c-cone of X is defined by

$$K_c(X) := \{(n, x) \in \mathbb{Z} \oplus \widetilde{K}(X) \,|\, c_k(x) = 0 \ \text{for all} \ k > n\}.$$

The c-dimension of a class $x \in \widetilde{K}(X)$, denoted by $c{-}\dim(x)$, is the least integer n such that $c_k(x) = 0$ for all $k > n$, in other words, it is the degree (in the variable t) of the polynomial $c_x(t)$.

Let us point out that the "lower boundary" of the positive cone $K_+(X)$, as a subset of $\mathbb{Z}\oplus\widetilde{K}(X)$, coincides with the graph of the geometric dimension function $\mathrm{g-dim}:\widetilde{K}(X)\longrightarrow\mathbb{Z}$ (the positive elements consisting exactly of the boundary and the points located above it). The analogous statements hold for the γ-cone and the c-cone with respect to the corresponding dimension function.

The following results on these objects follow readily from our preliminaries on K-theory.

1.3.2 Proposition. *Let X be a connected finite CW-complex. Then*

i) $\mathrm{g-dim}(x)\leq\dim(X)/2$ *for any* $x\in\widetilde{K}(X)$;

ii) $\gamma\mathrm{-dim}(x)\leq\mathrm{g-dim}(x)$ *for any* $x\in\widetilde{K}(X)$;

iii) $K_+(X)\subseteq K_\gamma(X)$;

iv) $\mathrm{c-dim}(x)\leq\mathrm{g-dim}(x)$ *for any* $x\in\widetilde{K}(X)$;

v) $K_+(X)\subseteq K_c(X)$.

This proposition shows that the γ-cone and the c-cone are approximations of the positive cone, more precisely, that they constitute upper bounds of the latter.

It turns out that the γ-cone and the c-cone coincide for torsion-free spaces, i.e. those spaces having no torsion in their integral cohomology.

1.3.3 Proposition. *Let X be a connected finite CW-complex. If X is torsion-free, then*

$$K_\gamma(X)=K_c(X).$$

Proof. The result follows immediately from proposition 1.2.2 and injectivity of the Chern character for a torsion-free X. □

1.3.4 Remark. *It is also possible to define the notions of $\bar{c}$-cone and of $\bar{c}$-dimension based on the rational Chern classes $\bar{c}_n$. However, a direct application of proposition 1.2.2 and of the definitions shows that for any connected finite CW-complex, the $\bar{c}$-cone is a coarser approximation of the positive cone than the γ-cone and the c-cone, in other words the γ-cone and the c-cone are always contained in the $\bar{c}$-cone (and they all coincide for torsion-free spaces). However, the $\bar{c}$-cone might generally be the simplest to compute.*

It is worth mentioning that there is no general comparison statement for the γ-cone and the c-cone, i.e. there are spaces with torsion for which the γ-cone is not contained in the c-cone, and spaces with torsion for which the c-cone is not contained in the γ-cone. Moreover, there exist spaces for which the γ-cone and the

c-cone strictly contain the positive cone (the product $S^4 \times S^4$ is such an example as we will later see). We now illustrate the situation by three examples.

Examples.

i) Let $j : BSU(3) \longrightarrow BU(3)$ be the map induced by the inclusion of the special unitary group $SU(3)$ in $U(3)$. Then the composition map

$$BSU(3) \xrightarrow{j} BU(3) \xrightarrow{\widetilde{\gamma}_3} BU$$

lifts to a map $f : BSU(3) \longrightarrow BSU$. Consider W the homotopy fiber of f. It enters in a pull-back diagram

$$\begin{array}{ccc} SU & = & SU \\ \downarrow & & \downarrow \\ W & \longrightarrow & PBSU \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ BSU(3) & \xrightarrow{f} & BSU \end{array}$$

where $SU \simeq \Omega BSU \hookrightarrow PBSU \to BSU$ is the path-loop fibration of BSU. The Leray-Serre spectral sequence in cohomology for this fibration is well-known and maps via f^* to the corresponding spectral sequence for the fibration π. By lemma 1.2.5, one has

$$f^*(\widetilde{c}_3) = j^* \circ \widetilde{\gamma}_3^*(c_3(\widetilde{\rho}_3)) = c_3(\widetilde{\gamma}_3) = 2\widetilde{c}_3 .$$

Similarly, $f^*(\widetilde{c}_2) = c_2(\widetilde{\gamma}_3)$, which is easily seen to vanish. For the cohomology of W in degree ≤ 6, letting $a_4 := \pi^*(\widetilde{c}_2)$ and $b_6 := \pi^*(\widetilde{c}_3)$, we have computed that $x_3^2 = 0$ and

$$H^{\leq 6}(W;\mathbb{Z}) = \mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot x_3 \oplus \mathbb{Z}\cdot a_4 \oplus \mathbb{Z}\cdot x_5 \oplus \underbrace{\mathbb{Z}\cdot b_6}_{\cong \mathbb{Z}/2} \cong \mathbb{Z}^4 \oplus \mathbb{Z}/2 ,$$

where $\deg(x_{2j+1}) = 2j+1$. The inclusion $i : Y := W^{[7]} \hookrightarrow W$ of the 7-skeleton of W induces isomorphisms in cohomology up to degree 6. If we let $x := i^* \circ \pi^* \circ j^*(\widetilde{\rho}_3) \in \widetilde{K}(Y)$, we find $c_3(x) = b_6 \neq 0$, whereas $\gamma^k(x) = 0$ for all $k \geq 3$. Indeed, this is clear for $k \geq 4$ since then $\gamma^k(\widetilde{\rho}_3) = 0$, and $\gamma^3(x) = 0$ because its classifying map is the composition $f \circ \pi \circ i$, which is homotopically trivial. Thus c$-$dim$(x) = 3$ and $\gamma-$dim$(x) \leq 2$. Consequently, Y is a connected finite CW-complex with a strict inclusion

$$K_c(Y) \subsetneq K_\gamma(Y) .$$

ii) Consider the Moore space $M = M(\mathbb{Z}/2, 5)$, in other words, the mapping cone of a continuous map $f : S^5 \xrightarrow{2} S^5$ of degree two, or more explicitly, $M = C_f = S^5 \cup_2 e^6$. The exact sequences in cohomology and in K-theory of the cofibration $S^5 \hookrightarrow M \overset{q}{\twoheadrightarrow} M/S^5 \simeq S^6$ give epimorphisms

$$q^* : \mathbb{Z} \cong H^6(S^6; \mathbb{Z}) \twoheadrightarrow H^6(M; \mathbb{Z}) \cong \mathbb{Z}/2$$
$$q^* : \mathbb{Z} \cong \widetilde{K}(S^6) \twoheadrightarrow \widetilde{K}(M) \cong \mathbb{Z}/2\,.$$

Let x and a be suitable generators of $\widetilde{K}(S^6)$ and $H^6(S^6; \mathbb{Z})$ respectively. Let $\bar{x} := q^*(x)$ and $\bar{a} := q^*(a)$. For obvious dimensional reasons, $c_1(\bar{x}) = 0$ and $c_2(\bar{x}) = 0$. Moreover $c_3(\bar{x}) = q^*(c_3(x)) = q^*(2a) = 0$ (see proposition 1.2.4). Hence $\mathrm{c}-\dim(\bar{x}) = 0$. On the other hand, we have $\gamma^1(\bar{x}) = \bar{x} \neq 0$, so $\gamma-\dim(\bar{x}) \geq 1$; more precisely, $\gamma^2(\bar{x})$ is $q^*(-S(3, 2) \cdot x) = q^*(-3x) = \bar{x} \neq 0$ and $\gamma^3(\bar{x}) = q^*(2S(3, 3) \cdot x) = 0$, so $\gamma-\dim(\bar{x}) = 2$. Consequently M is a connected finite CW-complex with a strict inclusion

$$K_\gamma(M) \subsetneq K_c(M)\,.$$

iii) Let $Z = Y \vee M$ be the wedge of the preceding two examples. It is an 7-dimensional finite connected CW-complex for which none of $K_\gamma(Z)$ and $K_c(Z)$ contains the other one. (The product $Y \times M$ would also do.)

To end the present section, we prove that the cones are semigroups and homotopy invariants.

1.3.5 Proposition. *The positive cone, the γ-cone and the c-cone of a connected finite CW-complex X are sub-semigroups of $K(X)$ and homotopy invariants of X. Moreover, the positive cone is a sub-λ-semiring of $K(X)$.*

Proof. The homotopy invariance is obvious for the three cones.

We have already mentioned in the preliminaries that the positive cone is a sub-semiring of $K(X)$. It is also clear that it is a sub-λ-semiring. The "exponentiality" of γ_t and of c (the total Chern class) immediately show that the γ-cone and the c-cone are sub-semigroups of $K(X)$. □

We do not know if in general the γ-cone and the c-cone are sub-λ-semirings of $K(X)$. Notice that even if the positive cone is a sub-semiring of $K(X)$, it is generally *not* true that $\mathrm{g}-\dim(xy) \leq \mathrm{g}-\dim(x) \cdot \mathrm{g}-\dim(y)$, for x and y in $\widetilde{K}(X)$; see section 2.5 for a counter-example when X is the complex projective plane $\mathbb{C}P^2$. (The same holds for the c-dimension and the γ-dimension.)

1.4 The positive cone of the spheres

We now intend to compute the positive cone of the spheres. For odd dimensional spheres, there is nothing to do since $\widetilde{K}(S^{2n+1}) = 0$. Whereas for even dimensional spheres, one has $\widetilde{K}(S^{2n}) = \mathbb{Z}\cdot x \cong \mathbb{Z}$, so we only have to compute $\mathrm{g-dim}(lx)$ for all integers l.

By proposition 1.2.4, we have

$$c(lx) = c(x)^l = (1 + (-1)^{n-1}(n-1)!\cdot a)^l = 1 + (-1)^{n-1}l(n-1)!\cdot a\,,$$

where a is the orientation class of S^{2n}. Therefore, by proposition 1.3.2, we deduce that, for $l \neq 0$,

$$n = \mathrm{c-dim}(lx) \le \mathrm{g-dim}(lx) \le \dim(S^{2n})/2 = n\,,$$

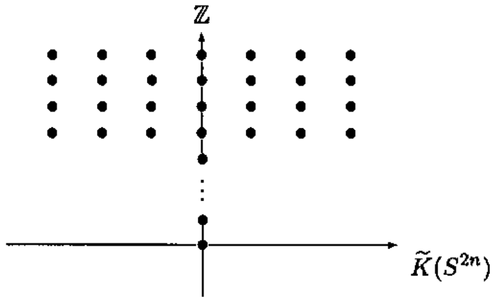
and this shows that $\mathrm{c-dim}(lx) = \mathrm{g-dim}(lx) = n$. The sphere S^{2n} being a torsion-free space, the following theorem follows from proposition 1.3.3.

1.4.1 Theorem. *Let x be a generator of $\widetilde{K}(S^{2n}) \cong \mathbb{Z}$. Then, for $l \in \mathbb{Z}$,*

$$\mathrm{g-dim}(lx) = \begin{cases} 0\,, & \text{if } l = 0 \\ n\,, & \text{otherwise.} \end{cases}$$

Moreover the positive cone, the c-cone and the γ-cone of S^{2n} coincide:

$$K_+(S^{2n}) = K_c(S^{2n}) = K_\gamma(S^{2n}) = \mathbb{N}\times 0 \cup \{(l,\, x)\,|\, l \ge n\} \subset \mathbb{Z}\times\widetilde{K}(S^{2n})\,.$$



There is another proof, purely homotopical, of the theorem. It is based on Bott's celebrated results on the homotopy groups of $BU(n)$ and Serre's computation of the rational homotopy groups of spheres. Let us present this proof. We have

$$[S^{2n},\, BU(k)] = \pi_{2n}(BU(k)) \;\text{ and }\; \widetilde{K}(S^{2n}) = [S^{2n},\, BU] = \pi_{2n}(BU)\,.$$

Consider the long exact sequence in homotopy of the fibration $BU(k) \xrightarrow{i_k} BU$:

$$\ldots \to \pi_{2n}(U/U(k)) \to \pi_{2n}(BU(k)) \xrightarrow{(i_k)_*} \pi_{2n}(BU) \to \pi_{2n-1}(U/U(k)) \to \ldots$$

The fiber $U/U(k)$ of i_k is $2k$-connected and it follows that $(i_k)_*$ is an isomorphism for $n \le k$. According to Bott [21], we have $\pi_{2n}(BU) \cong \mathbb{Z}$. It is well-known that for $k < n$, the group $\pi_{2n}(BU(k))$ is finite. Let us give a short proof of this result.

1.4.2 Lemma. *For $m \geq 2k+1$, the group $\pi_m(BU(k))$ is finite.*

Proof. We fix $m \geq 3$. The fibration $BU(k-1) \longrightarrow BU(k)$, with fiber S^{2k-1}, gives the following long exact sequence in homotopy:

$$\ldots \longrightarrow \pi_m(S^{2k-1}) \longrightarrow \pi_m(BU(k-1)) \longrightarrow \pi_m(BU(k)) \longrightarrow \pi_{m-1}(S^{2k-1}) \longrightarrow \ldots$$

By Serre [94], $\pi_j(S^{2k-1})$ is finite for $j \neq 2k-1$, and we can conclude by induction over k (with $k \geq 1$ and $2k+1 \leq m$), since $\pi_m(BU(1)) = \pi_{m-1}(U(1)) = 0$ for $m \geq 3$. □

From this, we now infer that the image of $(i_k)_*$ is zero for $k < n$. This implies that $\mathrm{g-dim}(lx) = n$ when $l \neq 0$, and concludes the second proof.

1.4.3 Remarks. *i) Since we were motivated by Elliott's classification of unital C^*-algebras of type AF by means of their K-theory, their positive cone and the K-theory class* [1] *of the unit (see [15]), it is important to single out the fact that the positive cone of S^{2n} and of S^{2m} are non-isomorphic as monoids if n is different from m. (There is no need here to distinguish the K-theory class 1 of the trivial one-dimensional bundle.) Let us provide with a short proof of this claim. For $n \geq 1$, let M_n denote the positive cone of S^{2n} (identified as above with a sub-monoid of $\mathbb{Z}^2$). The abelian monoid M_n has a minimal set A_n of generators, in other words a generating set (as monoid) that is contained in any other generating set, namely*

$$A_n = \{(0,\, 1)\} \cup \{(k,\, n) \mid k \in \mathbb{Z} \setminus \{0\}\}.$$

Now, consider the function $\sigma : A_n \longrightarrow \{2,\, 3,\, \ldots\}$ defined, for $x \in A_n$, by

$$\sigma(x) := \min\{l \geq 2 \mid lx \text{ decomposes as a sum of elements of } A_n \setminus \{x\}\}\,.$$

It is clear that such an l exists for any $x \in A_n$ and that $\sigma(A_n) = \{2,\, 2n\}$. Since A_n and σ are isomorphism invariants of M_n, this proves our claim.

ii) For odd-dimensional spheres the positive cone is "trivial", in other words, $K(S^{2n-1}) = \mathbb{Z}$ and $K_+(S^{2n-1}) = \mathbb{N}$.

1.5 Further properties of the cones

We now investigate naturality properties and behaviour under products of the positive cone, the γ-cone and the c-cone.

The following result is obvious.

1.5.1 Proposition. *Let $f : X \longrightarrow Y$ be a map between connected finite CW-complexes. Let $f^* : K(Y) \longrightarrow K(X)$ be the λ-homomorphism induced by f. Then, for any $y \in \widetilde{K}(Y)$,*

$$\begin{aligned} \mathrm{g}-\dim(f^*(y)) &\leq \mathrm{g}-\dim(y) \\ \gamma-\dim(f^*(y)) &\leq \gamma-\dim(y) \\ \mathrm{c}-\dim(f^*(y)) &\leq \mathrm{c}-\dim(y), \end{aligned}$$

and in particular,

$$\begin{aligned} f^*(K_+(Y)) &\subseteq K_+(X) \\ f^*(K_\gamma(Y)) &\subseteq K_\gamma(X) \\ f^*(K_c(Y)) &\subseteq K_c(X). \end{aligned}$$

If f^* is an isomorphism, then

$$f^*(K_\gamma(Y)) = K_\gamma(X).$$

For the next corollary we need a new definition.

1.5.2 Definition. *Let X and Y be two connected finite CW-complexes. A map $f : X \longrightarrow Y$ is called a K^0-equivalence (or K-equivalence for short) if there exists a map $g : Y \longrightarrow X$ such that on the level of the K^0-groups,*

$$f^* \circ g^* = Id_{K^0(X)} \text{ and } g^* \circ f^* = Id_{K^0(Y)}.$$

Note that a K-equivalence is *not* necessarily a homotopy equivalence: There are homotopically non-trivial finite CW-complexes X for which $\widetilde{K}(X) = 0 = \widetilde{K}(pt)$; see example i) below.

1.5.3 Proposition. *If $f : X \longrightarrow Y$ is a K-equivalence, then f induces the following isomorphisms of semigroups:*

$$K_+(Y) \overset{f^*}{\cong} K_+(X) \text{ and } K_\gamma(Y) \overset{f^*}{\cong} K_\gamma(X).$$

Proof. Applying twice proposition 1.5.1, we get (in the notations of definition 1.5.2)

$$K_+(X) = f^* \circ g^*(K_+(X)) \subseteq f^*(K_+(Y)) \subseteq K_+(X).$$

This establishes the first isomorphism, whereas the second is obvious. ☐

The following result is more technical to state.

1.5.4 Corollary. *Let X and Y be two connected finite CW-complexes. Assume that $K^1(X) = 0$ and $\widetilde{K}^0(Y) = 0$. Then the projection $p : X \times Y \longrightarrow X$ induces isomorphisms*

$$K_+(X) \overset{p^*}{\cong} K_+(X \times Y) \text{ and } K_\gamma(X) \overset{p^*}{\cong} K_\gamma(X \times Y).$$

Proof. Invoking the Künneth theorem for K-theory, our hypotheses imply that $p^* : K^0(X) \longrightarrow K^0(X \times Y)$ is an isomorphism with inverse i^*, where i is the inclusion of X in $X \times Y$. Consequently, p^* is a K-equivalence. □

The following is a useful result.

1.5.5 Proposition. *Let X and Y be a connected finite CW-complexes. Assume that the positive cone and the γ-cone of Y coincide, and let $f : X \longrightarrow Y$ be a map inducing an isomorphism $f^* : K(Y) \longrightarrow K(X)$. Then f induces an isomorphism of positive cones, and the γ-cone of X coincides with the positive cone:*

$$K_+(Y) \overset{f^*}{\cong} K_+(X) = K_\gamma(X) .$$

Proof. By proposition 1.5.1 we have $f^*(K_+(Y)) = f^*(K_\gamma(Y)) = K_\gamma(X)$ and $f^*(K_+(Y)) \subseteq K_+(X)$, hence $K_\gamma(X) \subseteq K_+(X)$. We conclude with iii) of proposition 1.3.2. □

Examples.

i) Let X be a connected finite CW-complex of dimension ≤ 3. Since for suitable CW-decompositions $BU(1)^{[3]} = BU^{[3]}$ and since $BU(1) = \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$, any $x \in \widetilde{K}(X) = [X, BU]$ lifts to $[X, BU(1)]$, giving an isomorphism $\widetilde{K}(X) \cong H^2(X; \mathbb{Z})$ mapping x to $c_1(x)$. It follows that the positive cone coincides with the c-cone and is given by

$$K_+(X) = \mathbb{N} \times \{0\} \cup \mathbb{N}^* \times \widetilde{K}(X) \subset \mathbb{Z} \times \widetilde{K}(X).$$

ii) Example i) applies to a closed oriented surface Σ_g of genus g. Since it is torsion-free, its positive cone coincides with its c-cone and with its γ-cone. Moreover, let $f : \Sigma_g \longrightarrow S^2$ be a map of degree 1 (it exists, since both the 2-sphere and Σ_g are quotients of the square $[0, 1]^2$). Then f not only induces an isomorphism in K-theory, but also an isomorphism of positive cones, as follows from proposition 1.5.1.

iii) Let X and Y denote the Moore spaces $M(\mathbb{Z}/3, 2q+11) = S^{2q+11} \cup_3 e^{2q+12}$ and $M(\mathbb{Z}/3, 2q-1) = S^{2q-1} \cup_3 e^{2q}$ respectively. In [2], Adams shows that for q large enough, there exists a map $A : X = \Sigma^{12} Y \longrightarrow Y$ such that the induced map $A^* : \widetilde{K}(Y) \longrightarrow \widetilde{K}(X)$ is an isomorphism (take $p = m = 3$, $f = 1$ and $r = 6$ in theorem 1.7 and in lemmas 12.4 and 12.5 of [2]). Therefore, A is a K-equivalence between simply connected finite CW-complexes, but it is *not* a weak homotopy equivalence. The mapping cone C_A is a non-contractible finite CW-complex with $\widetilde{K}(C_A) = 0$. (It is non-contractible because its homology is non-trivial.)

iv) In [48], pp. 203-206, a CW-complex $X = (S^1 \vee S^2) \cup e^3$ is defined, with the property that the inclusion $i : S^1 = X^{[1]} \hookrightarrow X$ of the 1-skeleton induces an isomorphism in integral homology (and on the level on fundamental groups); however, i is *not* a homotopy equivalence (not even a weak homotopy equivalence) since $\pi_2(X) \neq 0$. Consequently, by the universal coefficient theorem (see cor. V.7.2 in [24]), i induces an isomorphism in integral cohomology, and, by a direct application of the Atiyah-Hirzebruch spectral sequence, also in K-theory. In particular, i is a K-equivalence, but *not* an equivalence. (The quotient space $X/X^{[1]}$ has vanishing $\widetilde{K}$, however it is the closed 3-ball and is therefore contractible.)

In appendix A, we study the positive cone, the c-cone and the γ-cone from the rational point of view, and we consider rational K-theory.

1.6 The cones of the products $S^n \times S^{2m-1}$

In this section, we will compute the cones for the products $S^{2n} \times S^{2m-1}$ and $S^{2n-1} \times S^{2m-1}$.

Let us begin with $S^{2n} \times S^{2m-1}$. Since $\widetilde{K}(S^{2m-1}) = 0$ and $K^1(S^{2n}) = 0$, the answer immediately follows from proposition 1.5.5.

1.6.1 Theorem. *The projection $p : S^{2n} \times S^{2m-1} \longrightarrow S^{2n}$ induces an isomorphism of positive cones, and, for $S^{2n} \times S^{2m-1}$, the γ-cone and the c-cone coincide with the positive cone:*

$$K_+(S^{2n}) \overset{p^*}{\cong} K_+(S^{2n} \times S^{2m-1}) = K_\gamma(S^{2n} \times S^{2m-1}).$$

We now turn to the product $S^{2n-1} \times S^{2m-1}$. From the six-term exact sequence of the pair $(S^{2n-1} \times S^{2m-1}, S^{2n-1} \vee S^{2m-1})$, with quotient $S^{2n-1} \wedge S^{2m-1}$ homeomorphic to $S^{2m+2n-2}$, we get an isomorphism

$$q^* : \widetilde{K}(S^{2m+2n-2}) \longrightarrow \widetilde{K}(S^{2n-1} \times S^{2m-1})$$

induced by the quotient map $q : S^{2n-1} \times S^{2m-1} \longrightarrow S^{2m+2n-2}$. By theorem 1.4.1, the space $Y = S^{2n+2m-2}$ satisfies the hypothesis of proposition 1.5.5 and we deduce the

1.6.2 Theorem. *The map $q : S^{2n-1} \times S^{2m-1} \longrightarrow S^{2m+2n-2}$ induces an isomorphism of positive cones, and, for $S^{2n-1} \times S^{2m-1}$, the γ-cone and the c-cone coincide with the positive cone:*

$$K_+(S^{2m+2n-2}) \overset{q^*}{\cong} K_+(S^{2n-1} \times S^{2m-1}) = K_\gamma(S^{2n-1} \times S^{2m-1}).$$

1.6.3 Remark. *According to Blackadar ([16], 6.10.2), the positive cone of the n-torus $(S^1)^n$ has been partially computed by Villadsen.*

1.7 The γ-cone of $S^{2n} \times S^{2m}$ and the positive cone of $S^2 \times S^{2n}$

The positive cone was rather easy to compute for a product of an odd dimensional sphere by any sphere, whereas the case of a product of two even dimensional spheres is much more involved. On the other hand, the γ-cone of such a product is in the scope of the present chapter. We perform this calculation by computing the c-cone and appealing to proposition 1.3.3.

By the Künneth theorem, we have an isomorphism

$$K(S^{2n}) \otimes K(S^{2m}) \longrightarrow K(S^{2n} \times S^{2m}),\ \xi \otimes \eta \longmapsto p^*(\xi) \cdot q^*(\eta),$$

where p and q are the projections onto the factors. Writing $\widetilde{K}(S^{2n}) = \mathbb{Z} \cdot x_1$ and $\widetilde{K}(S^{2m}) = \mathbb{Z} \cdot x_2$, and letting $y_1 := p^*(x_1)$ and $y_2 := q^*(x_2)$, we deduce that

$$\widetilde{K}(S^{2n} \times S^{2m}) = \mathbb{Z} \cdot y_1 \oplus \mathbb{Z} \cdot y_2 \oplus \mathbb{Z} \cdot y_1 y_2 .$$

The product structure on $\widetilde{K}(S^{2n} \times S^{2m})$ is given by $y_1^2 = 0$ and $y_2^2 = 0$. Moreover, one has $y_1 y_2 = \pi^*(y)$, where $\pi : S^{2n} \times S^{2m} \longrightarrow S^{2n} \wedge S^{2m} \cong S^{2n+2m}$ and y is a suitable generator of $\widetilde{K}(S^{2n+2m})$.

Let $i : S^{2n} \hookrightarrow S^{2n} \times S^{2m}$ and $j : S^{2m} \hookrightarrow S^{2n} \times S^{2m}$ be the inclusions. One has $i^*(y_1) = x_1$ and $j^*(y_2) = x_2$, therefore (by theorem 1.4.1 and a double application of proposition 1.5.1), for any $k \in \mathbb{Z} \backslash \{0\}$, one has $\mathrm{g-dim}(ky_1) = \mathrm{g-dim}(kx_1) = n$; similarly $\mathrm{g-dim}(ky_2) = \mathrm{g-dim}(kx_2) = m$. This justifies that, from now on, we write x_1 and x_2 for y_1 and y_2 respectively.

Let $a_1 \in H^{2n}(S^{2n}; \mathbb{Z})$ and $a_2 \in H^{2m}(S^{2m}; \mathbb{Z})$ be suitable generators (referring to proposition 1.2.4). As before, it is justified to write

$$\widetilde{H}^*(S^{2n} \times S^{2m}; \mathbb{Z}) = \mathbb{Z} \cdot a_1 \oplus \mathbb{Z} \cdot a_2 \oplus \mathbb{Z} \cdot a_1 a_2 .$$

Let us assume $n \leq m$. Consider an element $x = ax_1 + bx_2 + lx_1x_2$ in the group $\widetilde{K}(S^{2n} \times S^{2m})$. For the Chern classes, invoking proposition 1.2.4 and "exponentiality" of the total Chern class, we compute

$$\begin{aligned} c(x) &= c(ax_1)c(bx_2)c(lx_1x_2) \\ &= 1 + (-1)^{n-1}a(n-1)! \cdot a_1 + (-1)^{m-1}b(m-1)! \cdot a_2 + \\ &\quad +(-1)^{n+m}(ab(n-1)!\,(m-1)! - l(n+m-1)!) \cdot a_1 a_2 . \end{aligned}$$

This immediately gives the γ-cone (which coincides with the c-cone) in terms of the γ-dimension function.

1.7.1 Theorem. *For $n \leq m$, the γ-dimension function on $\widetilde{K}(S^{2n} \times S^{2m})$ is given as follows: For $x = ax_1 + bx_2 + lx_1x_2 \in \widetilde{K}(S^{2n} \times S^{2m})$, one has*

$$\gamma\text{-dim}(x) = \begin{cases} 0 & , \text{ if } a = b = l = 0 \\ n & , \text{ if } a \neq 0,\ b = l = 0 \\ m & , \text{ if } b \neq 0,\ l = ab\,(n-1)!\,(m-1)!/(n+m-1)! \\ n+m & , \text{ if } l \neq ab\,(n-1)!\,(m-1)!/(n+m-1)! \end{cases}$$

Moreover, for $k \neq 0$, one has

$$\text{g-dim}(kx_1) = n \quad \text{and} \quad \text{g-dim}(kx_2) = m\,.$$

This theorem allows to give interesting information on the positive cone of the product $S^{2n} \times S^{2m}$. We will state the result as theorem 1.8.2 in the following section, because the tools developed there allow to make a crucial improvement.

Combined with theorem 1.2.3, theorem 1.7.1 enables to compute completely the positive cone of $S^2 \times S^{2n}$.

1.7.2 Theorem. *For the product $S^2 \times S^{2n}$, we have*

$$K_+(S^2 \times S^{2n}) = K_c(S^2 \times S^{2n}) = K_\gamma(S^2 \times S^{2n})\,.$$

The latter is given by theorem 1.7.1.

1.8 The Whitehead product and the positive cone

We will establish an interesting connection between the positive cone of a product $S^{2n} \times S^{2m}$ and the Whitehead product structure on the homotopy groups of the spaces $BU(k)$. As an application we will get some precise information on the positive cone of $S^{2n} \times S^{2m}$.

Let us first recall the basic properties of the Whitehead product (the reader may refer to [112]). The product $S^p \times S^q$ has a cell structure obtained by attaching a $(p+q)$-cell to $S^p \vee S^q$. More precisely, there exists a suitable pointed map $f : S^{p+q-1} \longrightarrow S^p \vee S^q$ such that $S^p \times S^q$ is homeomorphic to the mapping cone of f:

$$S^p \times S^q = C_f = (S^p \vee S^q) \cup_f e^{p+q}$$

Given a pointed map $g = \alpha \vee \beta : S^p \vee S^q \longrightarrow X$, where X is a CW-complex, there exists (up to homotopy) an extension $\bar{g} : S^p \times S^q \longrightarrow X$ of g if and only if the composition $g \circ f$ is homotopically trivial. Now, considering α and β as elements of the homotopy groups $\pi_p(X)$ and $\pi_q(X)$ respectively, the composition $(\alpha \vee \beta) \circ f$ determines an element in the homotopy group $\pi_{p+q-1}(X)$. This defines a map

$$\pi_p(X) \times \pi_q(X) \longrightarrow \pi_{p+q-1}(X), \ (\alpha, \beta) \longmapsto [\alpha, \beta] := (\alpha \vee \beta) \circ f,$$

which by definition is the Whitehead product. One can show that it is $\mathbb{Z}$-bilinear (provided that $p, q \geq 2$), i.e.

$$\begin{aligned} [\alpha_1 + \alpha_2, \beta] &= [\alpha_1, \beta] + [\alpha_2, \beta] \\ [\alpha, \beta_1 + \beta_2] &= [\alpha, \beta_1] + [\alpha, \beta_2]. \end{aligned}$$

Moreover the Whitehead product is natural with respect to pointed maps, i.e. if $f : X \longrightarrow Y$ is a pointed map between CW-complexes, then

$$[f_*(\alpha), f_*(\beta)] = f_*([\alpha, \beta]).$$

We now want to study the case where $X = BU(l)$. Let x_1 and x_2 be two generators of $\widetilde{K}(S^{2n})$ and $\widetilde{K}(S^{2m})$ respectively, and assume $1 \leq n \leq m$. By theorem 1.4.1, we know that $\mathrm{g-dim}(x_1) = n$ and $\mathrm{g-dim}(x_2) = m$. Letting $q \geq m$, we consider x_1 and x_2 as maps from S^{2n} (respectively S^{2m}) to BU that lift to $BU(q)$. The element $x_1 + x_2$ of $\widetilde{K}(S^{2n} \vee S^{2m}) = \widetilde{K}(S^{2n}) \oplus \widetilde{K}(S^{2m})$ can be represented by the map $x_1 \vee x_2 : S^{2n} \vee S^{2m} \longrightarrow BU$, and it also lifts to a map $z : S^{2n} \vee S^{2m} \longrightarrow BU(q)$.

Claim. For $k \in \{m, m+1, \ldots, m+n-1\}$, there is no extension of the map $z = x_1 \vee x_2 : S^{2n} \vee S^{2m} \longrightarrow BU(k)$ to a map $S^{2n} \times S^{2m} \longrightarrow BU(k)$.

Let $y : S^{2n} \times S^{2m} \longrightarrow BU(s)$ be an extension of z for some $s \geq m$. Let x be the composition of y with the map $i_s : BU(s) \longrightarrow BU$. This means that $\mathrm{g-dim}(x) \leq s$ and that $\iota^*(x) = x_1 + x_2 \in \widetilde{K}(S^{2n} \vee S^{2m})$, where ι is the inclusion of $S^{2n} \vee S^{2m}$ in $S^{2n} \times S^{2m}$. Recall that $(\iota^*)^{-1}(x_1 + x_2) = x_1 + x_2 + \mathbb{Z} \cdot x_1 x_2 \subset \widetilde{K}(S^{2n} \times S^{2m})$. So, there exists an integer l such that $x = x_1 + x_2 + l x_1 x_2$, and consequently

$$\gamma^{n+m}(x) = (-1)^{n+m-1}(l\,(n+m-1)! - (n-1)!\,(m-1)!) \cdot x_1 x_2 \neq 0.$$

We conclude that $s \geq \mathrm{g-dim}(x) \geq \gamma\mathrm{-dim}(x) \geq n + m$. This proves the claim.

As a direct consequence, by considering x_1 and x_2 as elements (in fact generators) of $\pi_{2n}(BU(k))$ and $\pi_{2m}(BU(k))$ respectively, we get the following result on the Whitehead product:

$$[x_1, x_2] \neq 0 \ \text{ in } \ \pi_{2n+2m-1}(BU(k)) \ \text{ for } \ m \leq k < n + m.$$

We would now like to get some information on the order of $[x_1, x_2]$ in the homotopy group $\pi_{2n+2m-1}(BU(k))$. By $\mathbb{Z}$-bilinearity of the Whitehead product, we have $ab[x_1, x_2] = [ax_1, bx_2]$ for any integers a and b. Replacing x_1 by ax_1 and x_2 by

bx_2 in the preceding computation (in particular $x = ax_1 + bx_2 + lx_1x_2$ for some integer l), one easily verifies that

$$\left.\begin{array}{c} ab[x_1, x_2] = 0 \\ \text{in } \pi_{2n+2m-1}(BU(k)) \\ \text{for } m \leq k < n+m \end{array}\right\} \Longrightarrow l\,(n+m-1)! - ab(n-1)!\,(m-1)! = 0 \qquad (*)$$

and this implies that ab is a multiple of $(n+m-1)!/((n-1)!\,(m-1)!)$. Notice that $[x_1, x_2] \in \pi_{2n+2m-1}(BU(k))$ has to be a torsion element. Indeed, by lemma 1.4.2, the group $\pi_{2n+2m-1}(BU(m))$ is finite, and the result follows from naturality of the Whitehead product. (In fact, one can show that any group $\pi_{2i+1}(BU(j))$ is finite; this is proved like lemma 1.4.2, by appealing to a result of Borel and Hirzebruch: see remark i) in section 1.9). We have thus obtained the following theorem.

1.8.1 Theorem. *Let $1 \leq n \leq m$ and $m \leq k < n+m$. Let x_1 and x_2 be generators of the homotopy groups $\pi_{2n}(BU(k)) \cong \mathbb{Z}$ and $\pi_{2m}(BU(k)) \cong \mathbb{Z}$ respectively. Then the Whitehead product*

$$[x_1, x_2] \in \pi_{2n+2m-1}(BU(k))$$

is non-zero. Moreover its order is a multiple of $\dfrac{(n+m-1)!}{(n-1)!\,(m-1)!}$.

The implication $(*)$ also means that if $ab[x_1, x_2] = 0$ in $\pi_{2n+2m-1}(BU(k))$ for some k such that $m \leq k < n+m$, then for $l = ab\,(n-1)!\,(m-1)!/(n+m-1)!$, the geometric dimension of $x := ax_1 + bx_2 + lx_1x_2$ is $\leq k$ (and for any other value of l, the geometric dimension of x is $m+n$, provided that $ab \neq 0$). Surprisingly, this condition only depends on l and on the product ab. Consequently, from theorem 1.2.3 together with theorem 1.7.1, we get the following information on the positive cone of $S^{2n} \times S^{2m}$ in terms of the geometric dimension function:

1.8.2 Theorem. *For $1 \leq n \leq m$, the geometric dimension on $\widetilde{K}(S^{2n} \times S^{2m})$ is given as follows: For $x = ax_1 + bx_2 + lx_1x_2 \in \widetilde{K}(S^{2n} \times S^{2m})$, one has*

$$\text{g-dim}(x) = \begin{cases} 0 & , \text{ if } a = b = l = 0 \\ n & , \text{ if } a \neq 0,\, b = l = 0 \\ m & , \text{ if } a = 0,\, b \neq 0,\, l = 0 \\ s(ab) & , \text{ if } b \neq 0,\, l = ab\,(n-1)!\,(m-1)!/(n+m-1)! \\ n+m & , \text{ if } l \neq ab\,(n-1)!\,(m-1)!/(n+m-1)! \end{cases}$$

where $s(ab) \in \{m, m+1, \ldots, n+m-1\}$ only depends on the product ab (for fixed n and m).

As a direct consequence of theorems 1.8.1 and 1.8.2, we have

1.8.3 Corollary. **(Bott, [22])** *The order of $[x_1, x_2]$ in $\pi_{2n+2m-1}(BU(n+m-1))$ is exactly $(n+m-1)!/((n-1)!\,(m-1)!)$.*

A few remarks have to be done here.

1.8.4 Remarks. *i) This result has been established only using information on the γ-cone of $S^{2n} \times S^{2m}$ (and Serre's theorem on the rational homotopy of spheres). If one is able to compute its positive cone, one then can easily compute the exact order of $[x_1, x_2]$ in the various homotopy groups $\pi_{2n+2m-1}(BU(k))$, for $m \leq k < n+m$: it is given by*

$$\min\left\{l \geq 1 \,\middle|\, \mathrm{g\!-\!dim}\left(l\frac{(n+m-1)!}{(n-1)!\,(m-1)!}\,x_1 + x_2 + lx_1x_2\right) \leq k\right\}.$$

ii) In 1960, Bott has proved corollary 1.8.3 by different methods (see [22]).

iii) We elaborate on these results in appendix D, by comparing them with obstruction theory (see in particular theorem D.0.4).

1.9 The positive cone of some products of even dimensional spheres

In this section, using known results from the theory of homotopy groups of spheres, we compute the positive cone of $S^4 \times S^4$, $S^4 \times S^6$, $S^6 \times S^6$ and $S^6 \times S^8$. This computation will in particular show that the positive cone and the γ-cone do not coincide for $S^4 \times S^4$! Keeping notations as in section 1.7, we describe the positive cone in terms of the geometric dimension function.

A) We start with the case of $S^4 \times S^4$.

1.9.1 Theorem. *The geometric dimension function on $\widetilde{K}(S^4 \times S^4)$ is given as follows: For $x = ax_1 + bx_2 + lx_1x_2 \in \widetilde{K}(S^4 \times S^4)$, one has*

$$\mathrm{g\!-\!dim}(x) = \begin{cases} 0 & , \text{ if } a = b = l = 0 \\ 2 & , \text{ if } a \neq 0,\, b = l = 0 \\ 2 & , \text{ if } b \neq 0,\, l = ab/6,\, l \text{ even} \\ 3 & , \text{ if } b \neq 0,\, l = ab/6,\, l \text{ odd} \\ 4 & , \text{ if } l \neq ab/6 \end{cases}$$

Proof. Theorem 1.8.2 reduces the problem to the computation of the function $s = s(ab)$, i.e. to calculating $\mathrm{g\!-\!dim}(x)$ for $x = ax_1 + bx_2 + (ab/6)x_1x_2$ (where ab is a multiple of 6), or equivalently the order of $[x_1, x_2]$ in both groups $\pi_7(BU(3))$ and $\pi_7(BU(2))$ (with a little abuse of notation, we write both Whitehead products in the same way). By Samelson [93], one has

$$\pi_7(BU(2)) \cong \pi_6(U(2)) \cong \pi_6(SU(2)) \cong \pi_6(S^3) \cong \mathbb{Z}/12,$$

precisely generated by $[x_1, x_2]$. This shows that for these particular values of x, $\mathrm{g\!-\!dim}(x) = 2$ if and only if ab is a multiple of 12. This completes the proof. □

1.9.2 Remark. *i) Borel and Hirzebruch in [17] (p. 355), applying Bott's results of [20], have proved that*

$$\pi_{2n+1}(BU(n)) \cong \pi_{2n}(SU(n)) \cong \mathbb{Z}/n! \ \ (n \geq 2),$$

hence $\pi_7(BU(3)) \cong \mathbb{Z}/6$. *Moreover, corollary 1.8.3 shows that the order of* $[x_1, x_2]$ *in* $\pi_7(BU(3))$ *is* 6 *; it is consequently a generator.*

ii) As already alluded to, we have just proved that $S^4 \times S^4$ *has its positive cone strictly contained in its* γ*-cone, although it is a torsion-free space.*

B) As for $S^4 \times S^4$, classical results from the theory of homotopy groups of the unitary groups allow to compute the positive cone of $S^4 \times S^6$. In this case, it coincides with the γ-cone.

1.9.3 Theorem. *For the product* $S^4 \times S^6$, *one has*

$$K_+(S^4 \times S^6) = K_c(S^4 \times S^6) = K_\gamma(S^4 \times S^6).$$

The latter is described in theorem 1.7.1.

Proof. By Lundell's tables [73] (see also [77]) and by remark i) above, one has

$$\pi_9(BU(3)) \cong \mathbb{Z}/12 \ \text{ and } \ \pi_9(BU(4)) \cong \mathbb{Z}/24.$$

Corollary 1.8.3 shows that $[x_1, x_2]$ is of order 12 in $\pi_9(BU(4))$. By naturality of the Whitehead product, the homomorphism $j_* = \pi_9(j)$, induced by the map $j : BU(3) \longrightarrow BU(4)$, takes $[x_1, x_2] \in \pi_9(BU(3))$ to $[x_1, x_2] \in \pi_9(BU(4))$. This implies that $[x_1, x_2]$ is of order 12 in $\pi_9(BU(3))$ too, and that $[ax_1, bx_2]$ vanishes in $\pi_9(BU(3))$ precisely when it is zero in $\pi_9(BU(4))$.

Together with theorem 1.8.2, this completes the proof. □

1.9.4 Remark. *This proof shows in particular that* $[x_1, x_2]$ *is a generator of* $\pi_9(BU(3)) \cong \mathbb{Z}/12$ *and that the map* $j_* : \pi_9(BU(3)) \longrightarrow \pi_9(BU(4))$ *is injective.*

C) By similar methods, we now show that the positive cone and the γ-cone coincide for $S^6 \times S^6$ and for $S^6 \times S^8$.

1.9.5 Theorem. *For the product* $S^6 \times S^6$, *one has*

$$K_+(S^6 \times S^6) = K_c(S^6 \times S^6) = K_\gamma(S^6 \times S^6).$$

The latter is given by theorem 1.7.1.

Proof. By Lundell's tables [73] (see also [77]), one has

$$\pi_{11}(BU(3)) \cong \mathbb{Z}/30 \text{ and } \pi_{11}(BU(5)) \cong \mathbb{Z}/120\,.$$

Corollary 1.8.3 shows that $[x_1, x_2]$ is of order 30 in $\pi_{11}(BU(5))$. By naturality, the map $j_* = \pi_{11}(j)$, induced by $j : BU(3) \longrightarrow BU(5)$, takes the Whitehead product $[x_1, x_2] \in \pi_{11}(BU(3))$ to $[x_1, x_2] \in \pi_{11}(BU(5))$. This implies that $[x_1, x_2]$ is of order 30 in $\pi_{11}(BU(3))$ too, and that $[ax_1, bx_2]$ vanishes in $\pi_{11}(BU(3))$ precisely when it is zero in $\pi_{11}(BU(5))$.

Together with theorem 1.8.2, this completes the proof. □

1.9.6 Remark. *i) This shows that* $[x_1, x_2]$ *generates* $\pi_{11}(BU(3)) \cong \mathbb{Z}/30$ and *that the map* $j_* : \pi_{11}(BU(3)) \longrightarrow \pi_{11}(BU(5))$ *is injective.*

ii) We were also able to prove this theorem without appealing to results on homotopy groups of $BU(n)$. *Using spectral sequences arguments, we have computed the first few stages of the Moore-Postnikov tower of the map* $BSU(3) \longrightarrow BSU(5)$. *This computation, being extremely lengthy, is given in appendix C.*

We pass to $S^6 \times S^8$.

1.9.7 Theorem. *For the product* $S^6 \times S^8$, *one has*

$$K_+(S^6 \times S^8) = K_c(S^6 \times S^8) = K_\gamma(S^6 \times S^8)\,.$$

The latter is described in theorem 1.7.1.

Proof. By Lundell's tables [73] (see also [77]), one has

$$\pi_{13}(BU(4)) \cong \mathbb{Z}/60 \text{ and } \pi_{13}(BU(6)) \cong \mathbb{Z}/720\,.$$

Corollary 1.8.3 shows that $[x_1, x_2]$ is of order 60 in $\pi_{13}(BU(6))$. By naturality, the map $j_* = \pi_{13}(j)$, induced by $j : BU(4) \longrightarrow BU(6)$, takes the Whitehead product $[x_1, x_2] \in \pi_{13}(BU(4))$ to $[x_1, x_2] \in \pi_{13}(BU(6))$. This implies that $[x_1, x_2]$ is of order 60 in $\pi_{13}(BU(4))$ too, and that $[ax_1, bx_2]$ vanishes in $\pi_{13}(BU(4))$ precisely when it is zero in $\pi_{13}(BU(6))$.

Together with theorem 1.8.2, this completes the proof. □

1.9.8 Remark. *This proof shows in particular that* $[x_1, x_2]$ *is a generator of* $\pi_{13}(BU(4)) \cong \mathbb{Z}/60$ *and that the map* $j_* : \pi_{13}(BU(4)) \longrightarrow \pi_{13}(BU(6))$ *is injective.*

1.10 "Gaps in cohomology" and the γ-cone

In the present section, we are interested in spaces having a "gap in cohomology", more precisely we look at spaces obtained by attaching a single large-dimensional cell to a finite CW-complex Y. For such spaces, the integral cohomology is zero between the dimension of Y and the top dimensional class. The products $S^n \times S^m$ are typical examples (see section 1.8). For this kind of spaces, the c-cone obviously cannot give information in the dimensions corresponding to the gap. At first sight, one could think that the γ-cone is more powerful in this range. Unfortunately, this is not the case: we show that the γ-cone (or equivalently the γ-dimension function) is also "blind" in some sense. Here is the precise statement.

1.10.1 Proposition. *Consider a connected finite CW-complex Y of dimension $\leq 2n$; let $X = C_f = Y \cup_f e^{2n+2m}$ be the mapping cone of a continuous map $f : S^{2n+2m-1} \longrightarrow Y$, with $m \geq 1$. Then, for any $x \in \widetilde{K}(X)$, one has*

$$\gamma^{n+m}(x) = 0 \implies \gamma^{n+l}(x) = 0 \ \text{ for all } \ l = 1, \ldots, m.$$

In other words, if $\gamma-\dim(x) < n+m$, then $\gamma-\dim(x) \leq n$.

Proof. By assumption, one obviously has $H^k(X;\mathbb{Z}) = 0$ for $2n < k < 2n+2m$ and $H^{2n+2m}(X;\mathbb{Z}) \cong \mathbb{Z}$.

Let $x \in \widetilde{K}(X)$ such that $\gamma^{n+m}(x) = 0$. By proposition 1.2.2, keeping the same notation, we have

$$ch(\gamma^k(x)) = \bar{c}_k(x) + P_{k+1}(\bar{c}_1(x), \ldots, \bar{c}_{n+m}(x)),$$

and $0 = ch(\gamma^{n+m}(x)) = \bar{c}_{n+m}(x)$. Due to the "gap" in the cohomology of X, we find that, for $k > n$, we have

$$ch(\gamma^k(x)) = 0.$$

By the particular cohomological properties of X, the Chern character is injective for elements of filtration $> n$ in $\widetilde{K}(X)$ (see [5]). Being zero or of filtration $\geq k$ (as proposition 1.2.2 shows), $\gamma^k(x)$ has to vanish for $k > n$.

This concludes the proof. □

1.11 A "doubling formula" for Stirling numbers of the second kind

In this section, we calculate the γ-operations for the product $S^{2n} \times S^{2m}$. From this computation and proposition 1.10.1, we deduce again the γ-cone, as appearing in theorem 1.7.1. This example illustrates that computing the c-cone is in general easier than computing the γ-cone. On the other hand, the latter calculation leads to an interesting "doubling formula" for Stirling numbers of the second kind. We will also conjecture the analogous formula for Stirling numbers of the first kind.

Keeping notations of section 1.7, we have

$$\widetilde{K}(S^{2n} \times S^{2m}) = \mathbb{Z} \cdot x_1 \oplus \mathbb{Z} \cdot x_2 \oplus \mathbb{Z} \cdot x_1 x_2 .$$

We still assume $n \leq m$. Using the known γ-operations for even dimensional spheres, one can easily calculate γ^k for $S^{2n} \times S^{2m}$: For $x = ax_1 + bx_2 + lx_1x_2$, one has clearly $\gamma^k(x) = \gamma^k(ax_1 + bx_2) + \gamma^k(lx_1x_2)$ and this allows to compute

$$\gamma^{m+q}(x) = (-1)^{m+q-1}(m+q-1)! \cdot \\ \cdot \left(l\, S(m+n,\, m+q) - ab \sum_{k=q}^{n} \frac{S(n,\,k)S(m,\,m+q-k)}{k\binom{m+q-1}{k}} \right) \cdot x_1 x_2$$

for $q \geq 1$; in particular

$$\gamma^{n+m}(x) = (-1)^{n+m-1}(l\,(n+m-1)! - ab\,(n-1)!\,(m-1)!) \cdot x_1 x_2 .$$

For γ^m, we have to distinguish the case $n = m$ from the case $n < m$. One gets

$$\gamma^m(x) = (-1)^{m-1} a\,(m-1)! \cdot x_1 \;+\; (-1)^{m-1} b\,(m-1)! \cdot x_2 \;+\; (-1)^{m-1} \cdot \\ \cdot\, (m-1)! \left(l\, S(2m,\, m) - ab \sum_{k=1}^{m-1} \frac{S(m,\,k)S(m,\,m-k)}{k\binom{m-1}{k}} \right) \cdot x_1 x_2$$

when $n = m$, whereas

$$\gamma^m(x) = (-1)^{m-1} b\,(m-1)! \cdot x_2 \;+\; (-1)^{m-1}(m-1)! \cdot \\ \cdot \left(l\, S(n+m,\, m) - ab \sum_{k=1}^{m-1} \frac{S(n,\,k)S(m,\,m-k)}{k\binom{m-1}{k}} \right) \cdot x_1 x_2$$

when $n < m$.

We want to compute the γ-dimension of $x = ax_1 + bx_2 + lx_1x_2$. If $l = 0$, the result is clear. We now assume $l \neq 0$. If l is different from $ab\,(n-1)!\,(m-1)!/(n+m-1)!$, we see that $\gamma{-}\dim(x) = n + m$. On the other side, if l has precisely this value, then $\gamma^m(x) \neq 0$, because in this case $b \neq 0$, and by proposition 1.10.1 we get $\gamma{-}\dim(x) = m$ precisely.

This gives another proof of theorem 1.7.1.

Let us now pass to the "doubling formula".

1.11.1 Theorem. *Let $q \leq n \leq m$ be positive integers; then*

$$S(m+n,\, m+q) = n\binom{n+m-1}{n} \sum_{k=q}^{n} \frac{S(n,\,k)S(m,\,m+q-k)}{k\binom{m+q-1}{k}} .$$

We called this a "doubling formula" because, particularizing to $n = m$, we get an expression allowing to compute $S(2n, n+q)$ in terms of the numbers $S(n, k)$ with $q \le k \le n-1$.

Proof. This is an immediate consequence of proposition 1.10.1 and the above computations. □

An alternative proof would be to invoke theorem 1.7.1 rather than proposition 1.10.1.

After trying to verify on a computer the analogous formula for Stirling numbers of the first kind, namely

$$s(n, k) = \sum_{j=0}^{n-k} \binom{-k}{n-k+j} \binom{2n-k}{n+j} S(n-k+j, j),$$

we were led to conjecture it:

1.11.2 Conjecture. *Let $q \le n \le m$ be positive integers; then*

$$s(m+n, m+q) = n\binom{n+m-1}{n} \sum_{k=q}^{n} \frac{s(n, k)s(m, m+q-k)}{k\binom{m+q-1}{k}}.$$

1.11.3 Remark. *After we had informed him about the above theorem and conjecture, Al Lundell has sent us a proof of the latter. The elegant proof is "elementary" in the following sense: it only uses some basic formulas for Stirling numbers (such as generating functions) and a contour argument in the computation of an integral, but no K-theory. Moreover, his proof encompasses both the Stirling numbers of the first and of the second kind in a unified way.*

Chapter 2

On the positive cone of CW-complexes with two cells

We compute the c-cone of the CW-complexes with two even dimensional cells. This involves the Hopf invariant and the Adams e-invariant. We determine the positive cone in several situations. We discuss in detail the case of the projective planes $\mathbb{R}P^2$, $\mathbb{C}P^2$, $\mathbb{H}P^2$ and $\mathbb{C}aP^2$ (the octonionic projective plane). As an application of these computations and of the work on the J-homomorphism due to Dyer [38], Adams [2] and Quillen [84], we construct some elements of "large order" in the homotopy groups of $BU(n)$.

2.1 Introduction

We are interested in the positive cone of connected CW-complexes with two cells of positive dimension, that is, spaces having the homotopy type of the mapping cone of a map $f : S^q \longrightarrow S^n$ (with $q \geq n \geq 1$), i.e.

$$C_f = S^n \cup_f e^{q+1} = \mathrm{e}^0 \cup \mathrm{e}^n \cup \mathrm{e}^{q+1}\,,\ q \geq n \geq 1\,.$$

If f is homotopic to the constant map, then C_f has the same homotopy type as $S^n \vee S^{q+1}$.

The positive cone of a wedge of two spheres was one of the subjects of chapter 1: the c-cone and the γ-cone were computed, and the positive cone was completely determined. So, in what follows, only the case where f is *not* homotopic to the constant map will be of interest.

This chapter is organized as follows: In section 2.2, we compute the cohomology and the K-theory of the mapping cone of a map $f : S^q \longrightarrow S^n$. We determine the positive cone in all cases, except when n is even and q is odd, or when $n = q$ is odd and X is a Moore space $M(\mathbb{Z}/d,\, n)$. For the latter, we compute the c-cone and the γ-cone in section 2.3. We can determine the positive cone in some particular cases, as for example when $n \in \{1,\, 3\}$, or when d is prime and $n \leq 2d-1$. We also explain that for $M(\mathbb{Z}/2,\, 5)$, the positive cone and the γ-cone coincide, whereas the c-cone is "informationless". In section 2.4, for n even and q odd, we compute the Chern character and the Chern classes in terms of the Hopf invariant and the Adams e-invariant of the map f. We conclude this section with an integrality relation between these two invariants. We calculate the c-cone and determine partially the positive cone of the four projective planes in section 2.5. The results are complete only for $\mathbb{R}P^2$ and $\mathbb{C}P^2$. To achieve this, we use known results on the J-homomorphism to find the e-invariant of the Hopf fibrations. Finally, in section 2.6, we show that the Chern classes allow to detect elements of "large order" in some homotopy groups of the Grassmannians $BU(s)$. To give explicit results, we need some classical results on the J-homomorphism.

2.2 Preliminary computations

In this section, we compute the cohomology and the K-theory of the mapping cone of a pointed map $f : S^q \longrightarrow S^n$, with $q \geq n \geq 1$. We also determine the positive cone, except when n is even and q is odd, or when $q = n$.

We keep notations as in chapter 1.

Let X denote the mapping cone C_f of f. Consider the Barratt-Puppe sequence

$$S^q \xrightarrow{f} S^n \overset{i}{\hookrightarrow} X \overset{p}{\twoheadrightarrow} X/S^n \simeq S^{q+1} \simeq \Sigma S^q \xrightarrow{\Sigma f} \Sigma S^n \simeq S^{n+1},$$

where p is the projection onto the quotient space. It induces the long exact sequence

$$\ldots \longrightarrow \widetilde{H}^m(S^{q+1};\mathbb{Z}) \xrightarrow{p^*} \widetilde{H}^m(X;\mathbb{Z}) \xrightarrow{i^*} \widetilde{H}^m(S^n;\mathbb{Z}) \xrightarrow{\partial} \widetilde{H}^{m+1}(S^{q+1};\mathbb{Z}) \longrightarrow \ldots$$

in reduced integral cohomology, and similarly in reduced rational cohomology. The boundary homomorphism ∂ is simply the composition of the suspension isomorphism with $(\Sigma f)^*$. The homotopy type of X depends only on the homotopy class of f, i.e. of $[f] \in \pi_q(S^n)$. If $q = n$, then $\pi_n(S^n) \cong \mathbb{Z}$, and f corresponds under this identification to its degree d. It follows that, in this case, the map

$$\mathbb{Z} \cong \widetilde{H}^n(S^n;\mathbb{Z}) \xrightarrow{\partial} \widetilde{H}^{n+1}(S^{q+1};\mathbb{Z}) \cong \mathbb{Z}$$

is multiplication by the degree of Σf, which coincides with d, and then X is either a wedge $S^n \vee S^{n+1}$ (when $d = 0$) or a Moore space $M(\mathbb{Z}/d,\, n)$ (when $d \neq 0$). On the other hand, we have the 6-term exact sequence in reduced K-theory

$$\begin{array}{ccccc} \widetilde{K}^0(S^{q+1}) & \xrightarrow{p^*} & \widetilde{K}^0(X) & \xrightarrow{i^*} & \widetilde{K}^0(S^n) \\ \partial \uparrow & & & & \downarrow \partial \\ \widetilde{K}^1(S^n) & \xleftarrow{i^*} & \widetilde{K}^1(X) & \xleftarrow{p^*} & \widetilde{K}^1(S^{q+1}) \end{array}$$

Under the Chern character

$$ch : \widetilde{K}^0(Y) \longrightarrow \widetilde{H}^{ev}(Y;\mathbb{Q}) \quad \text{and} \quad ch : \widetilde{K}^1(Y) \longrightarrow \widetilde{H}^{odd}(Y;\mathbb{Q}),$$

defined for any connected finite CW-complex Y, the 6-term exact sequence in reduced K-theory maps to the corresponding sequence in reduced rational cohomology. Since the Chern character is injective for spheres, it follows readily that the boundary homomorphisms ∂ in the 6-term exact sequence for K-theory are zero, except when $q = n$, and then they are multiplication by d, the degree of f.

The following proposition is a direct consequence of theorem 1.4.1 and of propositions 1.5.5 and 1.3.3.

2.2.1 Proposition. *Let $q \geq n \geq 1$, and let X be the mapping cone of the pointed map $f : S^q \longrightarrow S^n$. Denote by $S^n \overset{i}{\hookrightarrow} X \overset{p}{\twoheadrightarrow} X/S^n \simeq S^{q+1}$ the obvious maps. Then*

- *i) If n and q are even and if $q > n$, then $\widetilde{K}(X) \overset{i^*}{\cong} \widetilde{K}(S^n) \cong \mathbb{Z}$, and i^* induces an isomorphism of positive cones ($K_+(S^n)$ being described in theorem 1.4.1).*
- *ii) If n is even and $q = n$, then $\widetilde{K}(X) = 0$, consequently, the positive cone is $\mathbb{N} \subset \mathbb{Z} = K(X)$.*
- *iii) If n is odd and q is even, then $\widetilde{K}(X) = 0$, consequently, the positive cone is $\mathbb{N} \subset \mathbb{Z} = K(X)$.*
- *iv) If n and q are odd and if $q > n$, then $\widetilde{K}(X) \overset{p^*}{\cong} \widetilde{K}(S^{q+1}) \cong \mathbb{Z}$, and p^* induces an isomorphism of positive cones ($K_+(S^{q+1})$ being described in theorem 1.4.1).*

Moreover, in the four cases, the γ-cone and the c-cone of X coincide with the positive cone.

This proposition deals successfully with all cases, except when n is even and q is odd, or when $n = q$ is odd and X is a Moore space $M(\mathbb{Z}/d,\, n) = S^n \cup_d e^{n+1}$, i.e. the mapping cone of a map $f : S^n \longrightarrow S^n$ of degree $d \neq 0$. The case $q = n$ is the subject of section 2.3, and the remaining case is treated in section 2.4.

2.3 The positive cone of $M(\mathbb{Z}/d,\, n) = S^n \cup_d e^{n+1}$

We compute the positive cone of the Moore space $M(\mathbb{Z}/d,\, n) = S^n \cup_d e^{n+1}$, i.e. the mapping cone of a map $f : S^n \longrightarrow S^n$ of degree $d \neq 0$ (with $n \geq 1$). The case where n is even was treated as case ii) in proposition 2.2.1. Let us therefore assume that $n = 2m-1$ is odd (where $m \geq 1$).

Working with the exact sequences of section 2.2, we deduce that the homomorphisms

$$p^* : \mathbb{Z} \cong \widetilde{K}(S^{2m}) \longrightarrow \widetilde{K}(X) \quad \text{and} \quad p^* : \mathbb{Z} \cong \widetilde{H}^{2m}(S^{2m}) \longrightarrow \widetilde{H}^{2m}(X)$$

are both surjective with kernel $d\mathbb{Z}$; in particular, $\widetilde{K}(X) = \mathbb{Z} \cdot p^*(x) \cong \mathbb{Z}/d$, and $\widetilde{H}^{2m}(X) = \mathbb{Z} \cdot p^*(y) \cong \mathbb{Z}/d$, where x and y are generators of $\widetilde{K}(S^{2m})$ and $\widetilde{H}^{2m}(S^{2m})$ respectively. The other reduced cohomology groups of X vanish. It follows from proposition 1.2.4 that for any $l \in \mathbb{Z}$ (and suitable choices of x and y),

$$\gamma^k(l \cdot p^*(x)) = (-1)^{k-1} l (k-1)! S(m,\, k) \cdot p^*(x)\,, \text{ for } 1 \leq k \leq m$$
$$c_m(l \cdot p^*(x)) = (-1)^{m-1} l (m-1)! \cdot p^*(y)\,,$$

all the other γ-operations and Chern classes being zero. From theorem 1.2.3, we can now deduce the

2.3.1 Proposition. *Consider the Moore space $M(\mathbb{Z}/d,\, 2m-1) = S^{2m-1} \cup_d e^{2m}$, with $m \geq 1$ and $d \neq 0$, and the obvious projection map $p : M(\mathbb{Z}/d,\, 2m-1) \twoheadrightarrow S^{2m}$. Then $\widetilde{K}(M(\mathbb{Z}/d,\, 2m-1)) = \mathbb{Z} \cdot p^*(x) \cong \mathbb{Z}/d$, where x is a generator of $\widetilde{K}(S^{2m})$. The c-cone contains the γ-cone; they are given by*

$$\mathrm{c-dim}(l \cdot p^*(x)) = \begin{cases} 0\,, & \text{if } l(m-1)! \equiv 0 \pmod d \\ m\,, & \text{otherwise}, \end{cases}$$

$$\gamma\mathrm{-dim}(l \cdot p^*(x)) = \max\{k \leq m \,|\, l(k-1)! S(m,\, k) \not\equiv 0 \pmod d\}\,,$$

where $l \in \mathbb{Z}$. If $l(m-1)! \equiv 0 \pmod d$, then $\mathrm{g-dim}(l \cdot p^*(x)) \leq m-1$. In particular, if d and $(m-1)!$ are prime to each other (e.g. if $m \leq 2$, or if d is a prime and $m \leq d$), then the positive cone coincides with the c-cone and with the γ-cone. Moreover, for $(m-1)! \equiv 0 \pmod d$, the c-cone is as large as possible, in other words, it is given by $\mathbb{N} \times \widetilde{K}(M(\mathbb{Z}/d,\, 2m-1))$.

Recall that the γ-dimension is always ≥ 1, except for the zero virtual vector bundle. Let us illustrate the situation by two examples.

Examples.

i) For $M(\mathbb{Z}/2,\, 5)$, one has $\widetilde{K}(M(\mathbb{Z}/2,\, 5)) = \mathbb{Z} \cdot p^*(x) \cong \mathbb{Z}/2$, and

$$\mathrm{c-dim}(p^*(x)) = 0 \text{ and } \gamma\mathrm{-dim}(p^*(x)) = 2\,,$$

since $S(3, 2) = 3$. Consequently, the positive cone and the γ-cone coincide, whereas the c-cone strictly contains the positive cone and is as large as possible. (This example was already considered in section 1.3.)

ii) For $M(\mathbb{Z}/5, 9)$, one has $\widetilde{K}(M(\mathbb{Z}/5, 9)) = \mathbb{Z} \cdot p^*(x) \cong \mathbb{Z}/5$, and

$$\text{c}-\text{dim}(z) = 0 \text{ and } \gamma-\text{dim}(z) = 1,$$

for any nonzero $z \in \widetilde{K}(M(\mathbb{Z}/5, 9))$, since $S(5, 2) = 15$, $S(5, 3) = 25$ and $S(5, 4) = 10$. Consequently, the c-cone contains the γ-cone strictly, and we can say that both are as large as possible.

2.4 On the positive cone of $S^{2m} \cup_f e^{2m+2l}$

We compute the c-cone of the mapping cone $X = S^{2m} \cup_f e^{2m+2l}$ of a pointed map $f : S^{2m+2l-1} \longrightarrow S^{2m}$, where $m, l \geq 1$. The Hopf invariant $H(f)$ and the Adams e-invariant $e(f)$ play a crucial role. We conclude this section with an integrality relation between these two invariants.

We keep notations as in section 2.2. The integral cohomology of X is given by

$$H^*(X, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \cdot y \oplus \mathbb{Z} \cdot z \cong \mathbb{Z}^3,$$

with y in degree $2m$ corresponding under i^* to a generator of $H^{2m}(S^{2m}; \mathbb{Z})$, and z in degree $2m + 2l$ corresponding under p^* to a generator of $H^{2m+2l}(S^{2m+2l}; \mathbb{Z})$. The ring structure is as follows: If $m \neq l$, the products y^2, yz and z^2 all vanish; if $m = l$, then $yz = 0$, $z^2 = 0$ and $y^2 = \mu z$, for some integer μ, called the Hopf invariant $H(f)$ of f. This integer only depends on the homotopy class of f, i.e. on $[f] \in \pi_{4m-1}(S^{2m})$. In order to simplify the forthcoming formulas, we set $H(f) := 0$ when $m \neq l$; we can then write $y^2 = H(f)z$ in any case. In reduced K-theory, we have a short exact sequence

$$0 \longrightarrow \mathbb{Z} \cdot x_{2m+2l} = \widetilde{K}(S^{2m+2l}) \xrightarrow{p^*} \widetilde{K}(X) \xrightarrow{i^*} \widetilde{K}(S^{2m}) = \mathbb{Z} \cdot x_{2m} \longrightarrow 0.$$

Taking $\xi \in (i^*)^{-1}(x_{2m})$ and $\eta := p^*(x_{2m+2l})$, we get

$$K(X) \cong \mathbb{Z} \oplus \mathbb{Z} \cdot \xi \oplus \mathbb{Z} \cdot \eta \cong \mathbb{Z}^3.$$

Notice that η is uniquely determined up to addition of an integral multiple of ξ. The Chern character is as follows:

$$ch(\xi) = y + \lambda z \quad \text{and} \quad ch(\eta) = z,$$

for some rational number λ. Because of the different possible choices for η, the rational number λ is only determined modulo 1, i.e. it represents a unique element

$e(f)$ in the group $\mathbb{Q}/\mathbb{Z}$, called the Adams e-invariant of f (also denoted by $e_{\mathbb{C}}(f)$). It only depends on the homotopy class of f. Without loss of generality, we can consider $e(f)$ as an element of $\mathbb{Q}\cap]-\frac{1}{2}, \frac{1}{2}]$ (uniquely determined). (See [2], pp. 321-323 for some more details on the e-invariant.) Since ch is an injective ring homomorphism (X being torsion-free), the product in $\widetilde{K}(X)$ is given by

$$\xi^2 = H(f)\eta\,,\ \xi\eta = 0 \ \text{ and } \ \eta^2 = 0\,.$$

We would like to compute the Chern classes. They are closely related to the Chern character, as we now recall. For a connected finite CW-complex Y, we denote by ch_{2k} the component of ch in $H^{2k}(Y;\mathbb{Q})$. One has $ch_{2k} = \frac{1}{k!}s_k(c_1, \dots, c_k)$ (for $k \geq 1$), where the s_k's are the Newton polynomials. They are defined by the relation

$$s_k(\sigma_1(t_1, \dots, t_k), \dots, \sigma_k(t_1, \dots, t_k)) = t_1^k + \dots + t_k^k\,,$$

the σ_j's being the elementary symmetric polynomials (see for example [58], p. 255). One immediately sees that

$$s_1(c_1) = c_1 \ \text{ and } \ s_2(c_1, c_2) = c_1^2 - 2c_2\,.$$

We will however need some more values of these polynomials. Fortunately, we have the Newton formula ([52], p. 92)

$$s_k - c_1 s_{k-1} + c_2 s_{k-2} - + \dots + (-1)^{k-1} c_{k-1} s_1 + (-1)^k k \cdot c_k = 0\,.$$

With the first few values given above, this gives for example

$$s_3 = c_1^3 - 3c_1c_2 + 3c_3 \qquad s_4 = c_1^4 - 4c_1^2c_2 + 2c_2^2 + 4c_1c_3 - 4c_4\,.$$

Coming back to X, it is straightforward to check that

$$c_m(\xi) = (-1)^{m-1}(m-1)!\,y \ \text{ and } \ c(\eta) = 1 + (-1)^{m+l-1}(m+l-1)!\,z\,.$$

It is also clear that $s_m(\xi) = m!\,y$ and $c_j(\xi) = s_k(\xi) = 0$, for $j \notin \{m, m+l\}$ and $1 \leq k \leq m-1$. In the Newton formula for s_{m+l}, the only possible nonzero contributions are $(-1)^{m+l}(m+l)c_{m+l}$ and, if $m = l$, the product $(-1)^m c_m s_m$. After a short computation, we get

$$c(\xi) = 1 + (-1)^{m-1}(m-1)!\,y + (-1)^{m+l}\left(\frac{(m-1)!\,m!}{m+l}H(f) - (m+l-1)!\,e(f)\right) z\,.$$

By means of the Newton binomial formula, we find, for $\zeta = a\xi + b\eta \in \widetilde{K}(X)$ with $a, b \in \mathbb{Z}$,

$$\begin{aligned} c(\zeta) = 1 + (-1)^{m-1}(m-1)!\,a \cdot y + (-1)^{m+l}\Big(&(-1)^{m+l}H(f)\tfrac{((m-1)!)^2}{2}\,a(a-1) + \\ &+ H(f)\tfrac{(m-1)!\,m!}{m+l}\,a - e(f)(m+l-1)!\,a - (m+l-1)!\,b\Big) \cdot z\,. \end{aligned} \qquad (*)$$

It is clear that if $c_m(\zeta) = 0$, then a must be zero, and if $c_m(\zeta) = c_{m+l}(\zeta) = 0$, then $\zeta = 0$. We have now proved the following proposition:

2.4.1 Proposition. *Let X be the mapping cone of a map $f : S^{2m+2l-1} \longrightarrow S^{2m}$.*

i) Then $\widetilde{K}(X) = \mathbb{Z}\cdot\xi \oplus \mathbb{Z}\cdot\eta \cong \mathbb{Z}^2$, and $\widetilde{H}^*(X;\mathbb{Z}) = \mathbb{Z}\cdot y \oplus \mathbb{Z}\cdot z \cong \mathbb{Z}^2$, with ξ, η, y and z as above, and subject to the relations $\xi^2 = H(f)\eta$, $y^2 = H(f)z$ (the other products being zero). Furthermore, $ch(\xi) = y + e(f)z$ and $ch(\eta) = z$.

ii) The c-cone and the γ-cone coincide and are given by

$$\mathrm{c-dim}(\zeta) = \begin{cases} 0, & \text{if } \zeta = 0 \\ m, & \text{if } c_{m+l}(\zeta) = 0 \text{ and } \zeta \neq 0 \\ m+l, & \text{otherwise} \end{cases} \qquad (\zeta = a\xi + b\eta,\ a, b \in \mathbb{Z});$$

$c(\zeta)$ is given by formula $(*)$ above.

As a by-product of our computation, noticing that the coefficients of $c(\xi)$ above must be integers, we get a proof of the following result.

2.4.2 Proposition. *Let $[f] \in \pi_{2m+2l-1}(S^{2m})$, where $m, l \geq 1$. Then*

$$(m-1)!\, m!\, H(f) - (m+l)!\, e(f) \equiv 0 \pmod{m+l}.$$

(This result is not the best possible: compare with thm. 1 in [38] and with prop. 7.9 in [2].)

2.5 On the positive cone of the projective planes

We compute the c-cone and the γ-cone, and determine partially the positive cone, of the projective planes $\mathbb{R}P^2$, $\mathbb{C}P^2$, $\mathbb{H}P^2$ and $\mathbb{C}\mathrm{a}P^2$.

Let us first remark that in chapter 1, we have already computed the positive cone of the four projective lines. Indeed, there are well-known homeomorphisms

$$\mathbb{R}P^1 \cong S^1,\ \mathbb{C}P^1 \cong S^2,\ \mathbb{H}P^1 \cong S^4 \text{ and } \mathbb{C}\mathrm{a}P^1 \cong S^8.$$

For these spaces, the positive cone coincides with the γ-cone and with the c-cone.

Let us now compute the positive cone for $X := \mathbb{R}P^2$. Since $\widetilde{K}(X) \cong \mathbb{Z}/2$, generated by $x = \zeta \otimes \mathbb{C} - 1$, where ζ is the canonical real line bundle (see [74], prop. 4.3.11), one has $\mathrm{g-dim}(x) = 1$ and $K_+(X) = \mathbb{N} \oplus \widetilde{K}(X) \cong \mathbb{N} \oplus \mathbb{Z}/2$. On the other hand, one has $\mathrm{c-dim}(x) = 1$ (by virtue of thm. 1.2.3) and $\gamma\mathrm{-dim}(x) = 1$ (since $\gamma^1 = Id_K$). This proves the

2.5.1 Theorem. *For the real projective plane $\mathbb{R}P^2$, the positive cone, the c-cone and the γ-cone coincide. Moreover,*

$$K_+(\mathbb{R}P^2) = \mathbb{N} \oplus \widetilde{K}(\mathbb{R}P^2),$$

with $\widetilde{K}(\mathbb{R}P^2) \cong \mathbb{Z}/2$ being generated by $x = \zeta \otimes \mathbb{C} - 1$.

For another proof of this result, we invoke proposition 2.3.1, because $\mathbb{R}P^2$ is a Moore space $M(\mathbb{Z}/2, 1)$.

The real projective plane is an example of a space with torsion for which the three cones coincide.

The other three projective planes $\mathbb{K}P^2$, where $\mathbb{K}$ is one of the division algebras $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{C}a$, are the mapping cones of the Hopf fibrations $S^3 \longrightarrow S^2$, $S^7 \longrightarrow S^4$ and $S^{15} \longrightarrow S^8$ respectively (see [101], p. 12). They are defined as follows: Let n denote the dimension (over $\mathbb{R}$) of $\mathbb{K}$. Consider S^{2n-1} as the set of pairs $(\alpha, \beta) \in \mathbb{K}^2$ with $||\alpha||^2 + ||\beta||^2 = 1$, and S^n as the one-point compactification of $\mathbb{K}$; then the corresponding Hopf fibration (with fiber S^{n-1}) is the map

$$f : S^{2n-1} \longrightarrow S^n, \, (\alpha, \beta) \longmapsto \begin{cases} \bar{\alpha}^{-1}\beta, & \text{if } \alpha \neq 0 \\ \infty, & \text{otherwise} \end{cases}$$

(see [112], p. 677). These maps have Hopf invariant one, and by Adams [1], these values of n are the only ones for which there are maps with Hopf invariant one.

The projective plane $\mathbb{K}P^2$ has the cell structure $\mathbb{K}P^2 = S^n \cup_f e^{2n} = e^0 \cup e^n \cup e^{2n}$, and the cohomology ring is $H^*(\mathbb{K}P^2, \mathbb{Z}) = \mathbb{Z}[y]/(y^3)$, with y in degree n, and the K-theory ring is $K^*(\mathbb{K}P^2) = \mathbb{Z}[\xi]/(\xi^3)$, as we have shown in section 2.4. We can choose y and ξ so that

$$ch(\xi) = y + e(f)y^2,$$

where $e(f) \in \mathbb{Q} \cap]-\frac{1}{2}, \frac{1}{2}]$ is the Adams e-invariant of f. The values of $e(f)$ for the Hopf fibrations are as follows:

$$\mathbb{C}P^2 : e(f) = \frac{1}{2} \qquad \mathbb{H}P^2 : e(f) = \frac{1}{12} \qquad \mathbb{C}aP^2 : e(f) = -\frac{1}{240}.$$

Let us explain how to find these values. In the complex case, this is given by example 7.4 in [2]. It will be reproved below, when we discus $\mathbb{C}P^2$ in more details. In the other two cases, we use some more machinery, that we now introduce. The real Hopf-Whitehead J-homomorphism $J : \pi_i(\mathrm{SO}(n)) \longrightarrow \pi_{n+i}(S^n)$ (see [86], p. 5 for the definition) gives rise, after stabilization, to a homomorphism (denoted by the same letter)

$$J : \pi_i(\mathrm{SO}) \longrightarrow \pi_i^S,$$

with values in the stable stem π_i^S, i.e. in the i-th stable homotopy group of the sphere S^0. By real Bott periodicity [20], $\pi_{4k-1}(\mathrm{SO}) \cong \mathbb{Z}$. By theorem 7.16 and proposition 7.14 of [2] (see also theorem 1 and pp. 370-371 of [38]), if a map $f : S^{2n+4k-1} \longrightarrow S^{2n}$, with $k \geq 1$, is the standard generator of the image of J in π_{4k-1}^S, then $e(f) = (-1)^{k-1} b_k \frac{B_k}{4k} \in \mathbb{Q}/\mathbb{Z}$, with b_k equal to 1 (resp. 2) for k even (resp. odd), and B_k denoting the k-th Bernoulli number. It is defined by $B_k := (-1)^{k-1}\beta_{2k}$, for $k \geq 1$, where the β_j's are the rational numbers given by $\frac{x}{e^x - 1} = \sum_{j \geq 0} \frac{\beta_j}{j!} x^j$. For later reference, let us denote by M_k the denominator of

$\frac{B_k}{4k}$ expressed in lowest terms. The first few values are

k	1	2	3	4	5	6	7	8	9	10
B_k	$\frac{1}{6}$	$\frac{1}{30}$	$\frac{1}{42}$	$\frac{1}{30}$	$\frac{5}{66}$	$\frac{691}{2730}$	$\frac{7}{6}$	$\frac{3617}{510}$	$\frac{43867}{798}$	$\frac{174611}{330}$
$(-1)^{k-1}b_k\frac{B_k}{4k}$	$\frac{1}{12}$	$-\frac{1}{240}$	$\frac{1}{252}$	$-\frac{1}{480}$	$\frac{1}{132}$	$-\frac{691}{65520}$	$\frac{1}{12}$	$-\frac{3617}{16320}$	$\frac{43867}{14364}$	$-\frac{174611}{13200}$
M_k	24	240	504	480	264	65520	24	16320	28728	13200
$\frac{M_k}{b_k}$	12	240	252	480	132	65520	12	16320	14364	13200

The three Hopf fibrations $S^3 \longrightarrow S^2$, $S^7 \longrightarrow S^4$ and $S^{15} \longrightarrow S^8$ generate a copy of $\mathbb{Z}$ in the groups $\pi_3(S^2) \cong \mathbb{Z}$, $\pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}/12$ and $\pi_{15}(S^8) \cong \mathbb{Z} \oplus \mathbb{Z}/120$ respectively, and they represent the standard generators of the corresponding stable homotopy groups $\pi_1^S \cong \mathbb{Z}/2$, $\pi_3^S \cong \mathbb{Z}/24$ and $\pi_7^S \cong \mathbb{Z}/240$; moreover, these groups coincide with the image of the J-homomorphism (see [86], pp. 4 and 6). This proves that the Adams e-invariant takes the announced values for these maps. (In the quaternionic case, we will also give another explanation below.)

Now that we know the Hopf invariant and the Adams e-invariant of the Hopf fibrations, we can deduce the Chern character and the Chern classes for $\mathbb{K}P^2$, using the general computations made in section 2.4. For a, $b \in \mathbb{Z}$, we find

$$ch(a\xi + b\xi^2) = \begin{cases} ay + \frac{a+2b}{2}\, y^2\,, & \text{for } \mathbb{C}P^2 \\ ay + \frac{a+12b}{12}\, y^2\,, & \text{for } \mathbb{H}P^2 \\ ay + \frac{-a+240b}{240}\, y^2\,, & \text{for } \mathbb{C}\mathrm{a}P^2\,, \end{cases}$$

and

$$c(a\xi + b\xi^2) = \begin{cases} 1 + ay + \frac{a^2-a-2b}{2}\, y^2\,, & \text{for } \mathbb{C}P^2 \\ 1 - ay + \frac{a^2-a-12b}{2}\, y^2\,, & \text{for } \mathbb{H}P^2 \\ 1 - 6ay + (18a^2 + 21a - 5040b)y^2\,, & \text{for } \mathbb{C}\mathrm{a}P^2\,. \end{cases}$$

To end this section, let us consider the three projective planes $\mathbb{K}P^2$ independently, and state the results we find for their c-cone and positive cone.

We first consider the complex projective plane $X := \mathbb{C}P^2$. Recall that the integral cohomology and the K-theory are given by $H^*(X, \mathbb{Z}) = \mathbb{Z}[y]/(y^3)$, with y a generator in degree 2, and $K(X) \cong \mathbb{Z}[\xi]/(\xi^3)$, where $\xi = \zeta - 1$, with ζ being the canonical complex line bundle. The first Chern class is given by $c_1(\zeta) = y$ (see [74], prop. 4.3.1 and its proof). Since ζ is a line bundle, $ch(\xi) = ch(\zeta) - 1 = e^{c_1(\zeta)} - 1 = y + \frac{y^2}{2}$, and we recover the fact that the Adams e-invariant is $\frac{1}{2}$.

The above formula for the total Chern class determines the c-cone. The γ-cone coincides with it, since $\mathbb{C}P^2$ is torsion-free (see prop. 1.3.3). It follows readily from theorem 1.2.3 that the positive cone is equal to the c-cone. We have thus proved the following theorem.

2.5.2 Theorem. *For the complex projective plane $\mathbb{C}P^2$, the positive cone, the c-cone and the γ-cone coincide. In $K(\mathbb{C}P^2) = \mathbb{Z}[\xi]/(\xi^3)$, the positive cone is given, in terms of the geometric dimension function, by*

$$\mathrm{g-dim}(a\xi + b\xi^2) = \begin{cases} 0\,, & \text{if } a = 0 \text{ and } b = 0 \\ 1\,, & \text{if } b = \frac{a(a-1)}{2} \neq 0 \\ 2\,, & \text{otherwise} \end{cases} \qquad (a,\, b \in \mathbb{Z})\,.$$

Notice that the geometric dimension is *not* sub-multiplicative, in other words, the formula $\mathrm{g-dim}(\xi_1\xi_2) \leq \mathrm{g-dim}(\xi_1) \cdot \mathrm{g-dim}(\xi_2)$, for $\xi_1,\, \xi_2 \in \widetilde{K}(X)$, is generally *false*, as was mentioned in section 1.3. Indeed, taking $\xi_1 = \xi_2 = 2\xi + \xi^2 \in \widetilde{K}(\mathbb{C}P^2)$, we get $\mathrm{g-dim}(\xi_1) = 1$, $\xi_1^2 = 4\xi^2$ and $\mathrm{g-dim}(\xi_1^2) = 2$. The same holds for the c-dimension and the γ-dimension (for the same example).

The complex projective plane is an example of a compact manifold X for which the "boundary" of the positive cone is *not* given by a finite number of "affine equations" in $K(X)$ (indeed, the above formulas for the geometric dimension function are quadratic in a and b). This answers by the negative a question posed independently by Nigel Higson and by Etienne Ghys.

For the quaternionic projective plane $\mathbb{H}P^2$, we compute the total Chern class in section C.4 of appendix C; we have $c(a\xi + b\xi^2) = 1 - ay + \frac{a^2-a-12b}{2} y^2$. Comparing this result with the one obtained in section 2.4, one sees that the Adams e-invariant is $\frac{1}{12}$, confirming the result above. In the present case, $\xi = \zeta - 2$, where ζ is the canonical quaternionic line bundle, considered as a complex bundle of rank 2.

We see that $c_4(a\xi + b\xi^2) = 0$ if and only if $b = a(a-1)/12$, and that c_2 and c_4 both vanish if and only if $a = b = 0$. Together with theorem 1.2.3 and proposition 1.3.3, this proves:

2.5.3 Theorem. *For the quaternionic projective plane $\mathbb{H}P^2$, the K-theory ring is $K(\mathbb{H}P^2) = \mathbb{Z}[\xi]/(\xi^3)$, and the c-cone coincides with the γ-cone and is given, in terms of the c-dimension function, by*

$$\mathrm{c-dim}(a\xi + b\xi^2) = \begin{cases} 0\,, & \text{if } a = 0 \text{ and } b = 0 \\ 2\,, & \text{if } b = \frac{a(a-1)}{12} \neq 0 \\ 4\,, & \text{otherwise} \end{cases} \qquad (a,\, b \in \mathbb{Z})\,.$$

Furthermore, the positive cone is partially determined by the following information on the geometric dimension function:

$$\mathrm{g-dim}(a\xi + b\xi^2) = \begin{cases} 0\,, & \text{if } a = 0 \text{ and } b = 0 \\ 2 \text{ or } 3\,, & \text{if } b = \frac{a(a-1)}{12} \neq 0 \\ 4\,, & \text{otherwise} \end{cases} \qquad (a,\, b \in \mathbb{Z})\,.$$

We finally consider the Cayley plane $\mathbb{C}aP^2$, which is the mapping cone of the Hopf fibration $f : S^{15} \longrightarrow S^7$ discussed earlier. The cell structure is $\mathbb{C}aP^2 = e^0 \cup e^8 \cup e^{16}$. There are (at least) two other definitions of $\mathbb{C}aP^2$ (the interested reader may refer to [112], pp. 695-715, or to [83], pp. 281-295). The first one defines $\mathbb{C}aP^2$ as the set of primitive idempotents (i.e. of trace 1) in the 27-dimensional real exceptional Jordan algebra $\mathcal{J}$ of hermitian (3×3)-matrices over $\mathbb{C}a$, equipped with the Jordan multiplication $X \circ Y = \frac{1}{2}(XY + YX)$, which is commutative but *not* associative. It is sometimes called the Albert algebra, and denoted by $\mathbb{A}l$. The second description involves the exceptional compact 1-connected simple Lie group of type F_4, which is the group of automorphisms of the real algebra $\mathcal{J}$. All the automorphisms preserve the trace, therefore F_4 acts on the set $\mathbb{C}aP^2$ of primitive idempotents. The action is transitive, with stabilizer a closed connected subgroup of maximal rank of F_4. This subgroup turns out to be isomorphic to $\mathrm{Spin}(9)$. This shows that there is a homeomorphism

$$\mathbb{C}aP^2 \cong F_4/\,\mathrm{Spin}(9).$$

The integral cohomology ring of $\mathbb{C}aP^2$ is given by $H^*(\mathbb{C}aP^2; \mathbb{Z}) = \mathbb{Z}[y]/(y^3)$, where y is in degree 8. The K-theory ring is $K(\mathbb{C}aP^2) = \mathbb{Z}[\xi]/(\xi^3)$. The virtual bundle ξ may be described as follows. Let $\Delta_9 : \mathrm{Spin}(9) \longrightarrow U(16)$ be the Spin-representation. This defines a bundle $B\Delta_9 : B\mathrm{Spin}(9) \longrightarrow BU(16)$ of (complex) rank 16. The inclusion of the closed connected subgroup $\mathrm{Spin}(9)$ in F_4 induces a fibration

$$\mathbb{C}aP^2 = F_4/\,\mathrm{Spin}(9) \overset{\iota}{\hookrightarrow} B\mathrm{Spin}(9) \longrightarrow BF_4.$$

The bundle ξ is $\zeta - 16$, where ζ is the composition

$$\zeta = B\Delta_9 \circ \iota : \mathbb{C}aP^2 \hookrightarrow B\,\mathrm{Spin}(9) \longrightarrow BU(16).$$

This is due to Minami [78] (thm. 7.1), and is based on a theorem of Pittie [82]. (The crucial hypotheses are that F_4 is simply connected, and $\mathrm{Spin}(9)$ is of maximal rank in F_4.)

For $\eta = a\xi + b\xi^2$, we have seen that $c(\eta) = -6ay + (18a^2 + 21a - 5040b)y^2$. We deduce that $c_8(\eta) = 0$ if and only if $b = (6a^2+7a)/1680$. If moreover $c_4(\eta) = 0$, this implies that $\eta = 0$. It is not difficult to show that the equation $b = (6a^2+7a)/1680$ has infinitely many (integral) solutions, as for example, for any $k \in \mathbb{Z}$, the integers

$$a = -672 - 3360k \ \text{ and } \ b = 1610 + 16114k + 40320k^2.$$

(There are many other infinite families of solutions of the same kind.) This means that for these values of a and b, one has $\mathrm{c}{-}\dim(a\xi + b\xi^2) = 4$. Consequently, theorem 1.2.3 with proposition 1.3.3 give the

2.5.4 Theorem. *For the octonionic projective plane $\mathbb{C}aP^2$, the K-theory ring is $K(\mathbb{C}aP^2) = \mathbb{Z}[\xi]/(\xi^3)$, the c-cone coincides with the γ-cone and is given, in terms* *of the c-dimension function, by*

$$\mathrm{c-dim}(a\xi + b\xi^2) = \begin{cases} 0\,, & \text{if } a = 0 \text{ and } b = 0 \\ 4\,, & \text{if } b = \frac{6a^2+7a}{1680} \neq 0 \\ 8\,, & \text{otherwise} \end{cases} \qquad (a,\, b \in \mathbb{Z})\,,$$

and all cases occur. Moreover, the positive cone is partially determined by the following information on the geometric dimension function:

$$\mathrm{g-dim}(a\xi + b\xi^2) = \begin{cases} 0\,, & \text{if } a = 0 \text{ and } b = 0 \\ 4,\, 5,\, 6 \text{ or } 7\,, & \text{if } b = \frac{6a^2+7a}{1680} \neq 0 \\ 8\,, & \text{otherwise} \end{cases} \qquad (a,\, b \in \mathbb{Z})\,.$$

2.6 Application to some homotopy groups of the Grassmannian $BU(n)$

We use the computation of the c-cone for the mapping cone $X = C_f$ of a map $f : S^{2m+2l-1} \longrightarrow S^{2m}$, to construct elements "of large order" in some homotopy groups $\pi_{2m+2l-1}(BU(s))$. Our computation is based on the results of section 2.4 and some known facts relating the Adams e-invariant and the J-homomorphism.

We first collect some classical results from homotopy theory that we will need.

2.6.1 Proposition. *Let X be pointed space; let r, $m \geq 0$ and n, $q \geq 1$. Denote by f_k (resp. g_k) a pointed map of degree $k \in \mathbb{Z}$ on S^n (resp. S^{n+1}).*

i) The suspension homomorphism $\sigma : \pi_r(S^n) \longrightarrow \pi_{r+1}(S^{n+1})$ is an isomorphism if $r \leq 2n-2$ and an epimorphism if $r = 2n-1$. In particular, $\pi_m^S = \pi_{j+m}(S^j)$ for any $j \geq m+2$.

ii) For $\alpha \in \pi_n(X)$, the composition $\alpha \circ f_k$ represents $k\alpha$ (resp. α^k, if $n = 1$) in $\pi_n(X)$. More generally, for any fixed pointed map $g : S^n \longrightarrow S^q$, the application $\pi_q(X) \longrightarrow \pi_n(X)$, $\beta \longmapsto g \circ \beta$ is a group homomorphism.

iii) For $\alpha \in \pi_m(S^n)$, the composition $g_k \circ \sigma(\alpha)$ represents $k\sigma(\alpha)$ in $\pi_{m+1}(S^{n+1})$.

Proof. i) is the Freudenthal suspension theorem (see [51], thm. VI.2.10), and ii) is a direct consequence of the definition of the product in the homotopy groups. For a proof of iii), we refer to [51], thm. VI.2.3, or to [45], pp. 416-417. □

Let now $f : S^{2m+2l-1} \longrightarrow S^{2m}$ be a pointed map, with $m > l \geq 1$. We also choose an integer s such that $m \leq s \leq m+l-1$.

We keep notations as in section 2.4. In particular, we have

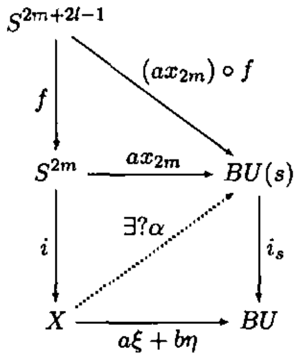
$$\pi_{2m}(BU(s)) \overset{(i_s)_*}{\cong} \pi_{2m}(BU) = \widetilde{K}(S^{2m}) = \mathbb{Z}\cdot x_{2m} \cong \mathbb{Z},$$

where $i_s : BU(s) \longrightarrow BU$ is the canonical fibration, with $2s$-connected fibre $U/U(s)$. From now on, we identify these groups. The mapping cone $S^{2m} \cup_f e^{2m+2l}$ of f is denoted by X, and i designates the inclusion of S^{2m} into X. The K-theory of X is given by

$$\widetilde{K}(X) = \mathbb{Z}\cdot\xi \oplus \mathbb{Z}\cdot\eta \cong \mathbb{Z}^2 ,$$

and $i^*(a\xi + b\eta) = ax_{2m}$ $(a,\, b \in \mathbb{Z})$. Since $l \le m-1$, the Hopf invariant $H(f)$ of f vanishes.

We now fix some integer a, and consider the following diagram representing a lifting and extension problem:



At this point, b is an unknown integral parameter. It is clear that, up to homotopy, there exists an extension of ax_{2m} to X if and only if $(ax_{2m})\circ f$ is zero in $\pi_{2m+2l-1}(BU(s))$. In this case, the composition $i_s \circ \alpha \in \widetilde{K}(X)$ is a virtual vector bundle ζ over X such that $i^*(\zeta) = ax_{2m}$ and with $\mathrm{g-dim}(\zeta) \le s$. It follows that there exists an integer b (our parameter!) such that $\zeta = a\xi + b\eta$ and $\mathrm{c-dim}(\zeta) \le s \le m+l-1$, and therefore $c_{m+l}(\zeta) = 0$. We have thus proved that

$$(ax_{2m})\circ f = 0 \in \pi_{2m+2l-1}(BU(s)) \implies \exists b \in \mathbb{Z} \text{ s.t. } c_{m+l}(a\xi + b\eta) = 0. \qquad (\spadesuit)$$

In section 2.4, we have computed the Chern classes for X. Since $H(f) = 0$, we see that the condition $c_{m+l}(a\xi + b\eta) = 0$ amounts to

$$\exists b \in \mathbb{Z} \text{ such that } a\cdot \mathrm{e}(f) + b = 0 .$$

This means that the denominator of the Adams invariant $e(f) \in \mathbb{Q}\cap\,]-\frac{1}{2},\, \frac{1}{2}]$, expressed in lowest terms, must divide a. (Let us remark that the denominator of $\mathrm{e}(f)$ considered as an element of $\mathbb{Q}/\mathbb{Z}$ is well-defined: it is simply its order in this group.) When $s = m + l - 1$, condition (♠) is an equivalence, as a consequence of theorem 1.2.3.

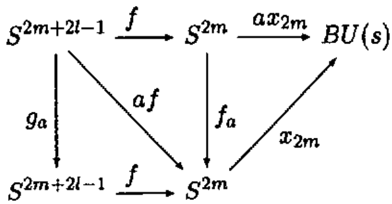
Apparently, we have no control on the behaviour of $(ax_{2m}) \circ f \in \pi_{2m+2l-1}(BU(s))$ with respect to a. Fortunately, the following lemma comes at our rescue.

2.6.2 Lemma. *Under the above hypotheses, in particular, $l \leq m-1$, one has*

$$(ax_{2m}) \circ f = a \cdot (x_{2m} \circ f) \in \pi_{2m+2l-1}(BU(s)),$$

for any $a \in \mathbb{Z}$.

Proof. Let f_a (resp. g_a) be a pointed map of degree a on S^{2m} (resp. $S^{2m+2l-1}$). Consider the following diagram:



By proposition 2.6.1 ii), the left-hand and the right-hand triangles commute. By 2.6.1 iii), the middle triangle commutes, provided that $f \in \sigma(\pi_{2m+2l-2}(S^{2m-1}))$. From i) in the same proposition, and from the condition $l \leq m-1$, it follows that f is always a suspension. Finally, from 2.6.1 ii), we find $x_{2m} \circ f \circ g_a = a \cdot (x_{2m} \circ f)$, and this concludes the proof. □

We now understand that the condition $l \leq m-1$ is crucial.

Noticing that the group $\pi_{2m+2l-1}(BU(s))$ is finite (as follows from lemma 1.4.2, since $1 \leq s \leq m+l-1$), we can collect these results in a theorem.

2.6.3 Theorem. *For $1 \leq l \leq m-1$, let $f : S^{2m+2l-1} \longrightarrow S^{2m}$ be a map, and let x_{2m} be a generator of $\widetilde{K}(S^{2m})$. Then, for any s such that $m \leq s \leq m+l-1$, the composition $x_{2m} \circ f$ represents a non-zero element in $\pi_{2m+2l-1}(BU(s))$, whose order is a multiple of* $\mathrm{denom}(\mathrm{e}(f))$, *the denominator of the Adams invariant $e(f)$ of f expressed in lowest terms. For $s = m+l-1$, the order of $x_{2m} \circ f$ is precisely* $\mathrm{denom}(e(f))$.

Of course, this theorem becomes interesting once we know how large the denominator of $\mathrm{e}(f)$ can be. Another question arises: What is the precise order of $x_{2m} \circ f$ in $\pi_{2m+2l-1}(BU(s))$?

It turns out that for l even, say $l = 2k$, we can partially answer this question. First by proposition 2.6.1 i), the condition $l \leq m-1$ implies that the maps $\pi_{2m+2l-1}(S^{2m}) \longrightarrow \pi^S_{2l-1}$ and $\pi_{2l-1}(\mathrm{SO}(2m)) \longrightarrow \pi_{2l-1}(\mathrm{SO})$ are isomorphisms (the fiber $\mathrm{SO}/\mathrm{SO}(2m)$ of the canonical fibration $B\mathrm{SO}(2m) \longrightarrow B\mathrm{SO}$ is $2m$-connected). For $i > 0$, $\pi_i(\mathrm{SO}) = \pi_i(\mathrm{O})$, and by real Bott periodicity [20],

$$\pi_j(\mathrm{O}) \cong \begin{cases} \mathbb{Z}, & \text{if } j \equiv 3 \pmod 4 \\ \mathbb{Z}/2, & \text{if } j \equiv 0 \text{ or } 1 \pmod 8 \\ 0, & \text{otherwise.} \end{cases}$$

We see tbat the case of most interest in our framework is when l is even, say $l = 2k$. Then, the J-homomorphism is a map

$$J : \mathbb{Z} \cong \pi_{4k-1}(\mathrm{SO}) \cong \pi_{4k-1}(\mathrm{SO}(2m)) \longrightarrow \pi_{2m+4k-1}(S^{2m}) \cong \pi^S_{4k-1}.$$

Following Adams [2], we denote the image of a (chosen) generator of the cyclic group $\pi_{4k-1}(\mathrm{SO}(2m))$ by $j_{4k-1} \in \pi_{2m+4k-1}(S^{2m})$. By Adams [2] and Quillen [84], the image of J is a direct summand in $\pi_{2m+4k-1}(S^{2m})$ and is of order exactly M_k, the latter being defined in section 2.5 (see also [103], p. 488). This means that j_{4k-1} is of order M_k and generates a direct summand. On the other side, by theorem 1 of [38], the Adams e-invariant $\mathrm{e}(j_{4k-1})$ (expressed in lowest terms) has denominator $\frac{M_k}{b_k}$. (This result is also a consequence of [2], prop. 7.14 and thm. 7.16.)

It follows from proposition 2.6.1 ii) that the application

$$\pi_{2m+4k-1}(S^{2m}) \longrightarrow \pi_{2m+4k-1}(BU(s)),\ g \longmapsto g \circ x_{2m}$$

is a group homomorphism. From theorem 2.6.3, we deduce the following result.

2.6.4 Theorem. *Let $j_{4k-1} \in \pi_{2m+4k-1}(S^{2m}) \cong \pi^S_{4k-1}$ denote the image of a generator of the group $\pi_{4k-1}(\mathrm{SO}(2m)) \cong \pi_{4k-1}(\mathrm{SO}) \cong \mathbb{Z}$ under the J-homomorphism, where $2 \leq 2k \leq m-1$. Let x_{2m} be a generator of $\widetilde{K}(S^{2m})$. Then, for any s such that $m \leq s \leq m+2k-1$, the composition $x_{2m} \circ j_{4k-1}$ represents a non-zero element in $\pi_{2m+4k-1}(BU(s))$, whose order is given by*

$$\begin{cases} \operatorname{denom}\left(\frac{B_k}{4k}\right), & \text{if } k \text{ is even} \\ \operatorname{denom}\left(\frac{B_k}{4k}\right) \text{ or } \frac{1}{2}\operatorname{denom}\left(\frac{B_k}{4k}\right), & \text{if } k \text{ is odd}, \end{cases}$$

where B_k denotes the k-th Bernoulli number. When k is odd *and $s = m+2k-1$, the order of $x_{2m} \circ f$ is $\frac{1}{2}$ denom $\left(\frac{B_k}{4k}\right)$.*

Unfortunately, we were unable to determine the exact order when k is odd. However, notice that for a given k, the order may depend on s. (We could not settle this question.)

Before giving some numerical examples, we would like to insist on the fact that the element we have constructed in $\pi_{2m+4k-1}(BU(s))$, namely $x_{2m} \circ j_{4k-1}$, can be written down explicitly by means of the real and the complex Bott periodicity isomorphisms.

Examples.

i) For $k = 1$ and $m = 3$, we have denom $\left(\frac{B_1}{4}\right) = 24$ and we can take $s = 3$ or 4; the corresponding groups are

$$\pi_9(BU(3)) \cong \mathbb{Z}/12 \text{ and } \pi_9(BU(4)) \cong \mathbb{Z}/24.$$

We see that $x_6 \circ j_3$ is a generator of the former, but only generates a subgroup of index 2 in the latter. If we change m, we obtain less interesting results: For $m = 4$, the groups under consideration are $\pi_{11}(BU(4)) \cong \mathbb{Z}/120 \oplus \mathbb{Z}/2$ and $\pi_{11}(BU(5)) \cong \mathbb{Z}/120$, and for $m = 5$, they are $\pi_{13}(BU(5)) \cong \mathbb{Z}/360$ and $\pi_{13}(BU(6)) \cong \mathbb{Z}/720$.

ii) For $k = 2$ and $m = 5$, our results give an element of order denom $\left(\frac{B_2}{8}\right) = 24$ in the groups

$$\begin{array}{ll} \pi_{17}(BU(5)) \cong \mathbb{Z}/5040 \oplus \dots & \pi_{17}(BU(6)) \cong \mathbb{Z}/5040 \oplus \dots \\ \pi_{17}(BU(7)) \cong \mathbb{Z}/20160 \oplus \dots & \pi_{17}(BU(8)) \cong \mathbb{Z}/40320 \oplus \dots \end{array}$$

iii) In the theorem, the cases of most interest are those for which m and s are as small as possible for a fixed $k \geq 1$, namely $m = s = 2k + 1$: it predicts that $x_{4k+2} \circ j_{4k-1}$ is of order denom $\left(\frac{B_k}{4k}\right)$ (or possibly half of it, if k is odd) in the group $\pi_{8k+1}(BU(2k+1))$. As an illustration, for $k = 6$, we get the element $x_{26} \circ j_{23}$ of order 65520 in $\pi_{49}(BU(13))$.

Chapter 3

K-homology and K-theory of low-dimensional spaces

We partially compute K-homology and K-theory of CW-complexes of dimension ≤ 3 in terms of integral homology and cohomology. This is performed by means of the Atiyah-Hirzebruch spectral sequence. As an application, for $0 \leq j \leq 2$ and any CW-complex X, we define natural homomorphisms $\beta_j^X : H_j(X;\mathbb{Z}) \longrightarrow K_j(X)$ that are rationally right inverses of the usual Chern character "à la Atiyah" in K-homology. We give four equivalent constructions of these maps; one is based on Spin^c-bordism, and a fifth is given in appendix E. We prove injectivity of β_1^X for any CW-complex X. This is of interest in the framework of the Baum-Connes conjecture and of algebraic K-theory of group C^*-algebras (see chapters 4 and 8).

3.1 Introduction

Having spent some time on topological K-theory in the previous chapters, in other words on topological K-cohomology, we would like to study its dual theory, namely K-homology.

As for any cohomology theory on the category of finite CW-complexes, there is a spectrum associated to topological K-theory: the BU-spectrum. Topological K-homology can be defined by means of this Ω-spectrum, and it is a 2-periodic homology theory. However, there is an alternative definition, due to Baum and Douglas, that we now recall. Let X be a finite CW-complex. We denote its K-homology by $K_*(X) = K_0(X) \oplus K_1(X)$. An element of $K_j(X)$ is a suitable equivalence class of triples (M, ξ, f), where M is a closed Spin^c-manifold of dimension $m \equiv j \pmod 2$ (hence oriented, but not necessarily connected), ξ is a virtual vector-bundle over M, i.e. an element of $K^0(M)$, and $f : M \longrightarrow X$ is a continuous map. The sum is given by disjoint union, and the homomorphism induced by a continuous map $g : X \longrightarrow Y$ is defined by composition, i.e. one has

$$g_*([M, \xi, f]) := [M, \xi, f \circ g] \in K_*(Y).$$

Notice that for X connected, any element in $K_*(X)$ is represented by a sum of triples (M_i, ξ_i, f_i) with M_i connected and $f_i : M_i \longrightarrow X$ pointed. For details on this "bordism-type" definition (in particular on the equivalence relation), we refer the reader to Baum-Douglas [8] and to Jakob [56].

It is well-known that for a finite CW-complex X, the Chern character in K-theory,

$$ch : K^*(X) \longrightarrow H^*(X; \mathbb{Q}),$$

is natural and induces an isomorphism after tensoring with $\mathbb{Q}$. It is also a classical result that in the "dual" theory, i.e. K-homology, there is a dual Chern character denoted by the same symbol

$$ch : K_*(X) \longrightarrow H_*(X; \mathbb{Q}),$$

that is natural and an isomorphism after tensoring with $\mathbb{Q}$. In the bordism-type description of K-homology, ch was constructed in [8], and is explicitly given by

$$ch([M, \xi, f]) = f_*(ch(\xi) \cup Td(TM) \cap [M]) \in H_*(X; \mathbb{Q}),$$

where $ch(\xi) \in H^{ev}(M; \mathbb{Q})$ is the usual K-theory Chern character of the virtual vector bundle ξ, $Td(TM) \in H^{ev}(M; \mathbb{Q})$ is the Spin^c Todd class of the tangent bundle TM of M, $[M] \in H_m(M; \mathbb{Q})$ is the fundamental class of M, and finally $f_* : H_*(M; \mathbb{Q}) \longrightarrow H_*(X; \mathbb{Q})$ is induced by f. Notice that $ch([M, \xi, f])$ belongs to H_{ev} or H_{odd} according to the parity of j (i.e. of m, the dimension of M).

For an arbitrary CW-complex X (i.e. not necessarily finite), we will only consider K-homology with compact supports. In other words, we define

$$K_*(X) := \varinjlim_{Y} K_*(Y),$$

where the direct limit runs over all finite sub-CW-complexes Y of X. Since for CW-complexes, cellular homology coincides with singular homology, it has automatically compact supports. Consequently, the Chern character in K-homology is rationally an isomorphism for arbitrary CW-complexes. (In fact, the bordism-type description of K-homology has automatically compact supports, since $f(M)$ has to be compact.)

In this framework, two natural questions arise: 1) "For which CW-complexes is there an isomorphism between K-homology and integral homology?" 2) "Are there integral Chern characters, i.e. with values in integral homology?"

We will prove that for a 2-dimensional connected CW-complex X, there is a natural isomorphism

$$ch^{\mathbb{Z}} : K_*(X) \xrightarrow{\cong} H_*(X; \mathbb{Z}).$$

Moreover, we show that for a 3-dimensional CW-complex, there is a similar isomorphism, but whose naturality is not at all clear. We also discuss the case of simply-connected 4-dimensional CW-complexes. Similarly, for K-theory, we prove that for a connected *finite* CW-complex of dimension ≤ 3, there is a natural isomorphism

$$ch^{\mathbb{Z}} : K^*(X) \xrightarrow{\cong} H^*(X; \mathbb{Z}).$$

We also consider finite CW-complexes of dimension ≤ 5.

For any CW-complex X, we define maps

$$\beta_j^X : H_j(X, \mathbb{Z}) \longrightarrow K_j(X)$$

for $0 \leq j \leq 2$. For classifying spaces of (discrete) groups, this was already done by H. Bettaieb and A. Valette. They showed that, in this case, β_j^X is rationally a right inverse of the Chern character; β_j^X is consequently rationally injective. We extend this result to the case of arbitrary CW-complexes, and prove that β_0^X is split-injective for any CW-complex (this is trivial) and that β_1^X is injective. This result is much more involved as will be seen. We show by an example that β_2^X is generally *not* injective.

Throughout this chapter, we assume all CW-complexes and all maps between them to be pointed. Moreover, for the spectral sequence arguments, we suppose that the 0-skeleton of any connected CW-complex is reduced to the base-point (up to homotopy equivalence, this is no restriction).

We are indebted to Hervé Oyono-Oyono for many helpful discussions and notably for having detected an irreparable error in a (completely different) previous version of this chapter.

The organization of the chapter is as follows. In section 3.2, we define the map β_0^X and we express the K-homology of CW-complexes of dimension ≤ 3 in terms of integral homology. We also consider the case of simply-connected CW-complexes of dimension 4. Section 3.3 contains two definitions of β_1^X and β_2^X, and a proof of the injectivity of these maps for CW-complexes of dimension ≤ 3. It is also shown that both maps are rationally right inverses of the Chern character. Moreover, this section contains a proof that oriented bordism coincides with integral homology up to degree 3 (degrees 4 and 5 are also briefly discussed). In section 3.4, using the particular structure of the **BU**-spectrum, we define groups $SK^0(X)$ and $SK^1(X)$ and establish natural splittings of K-theory. This allows for expressing the K-theory of a finite CW-complex of dimension ≤ 5 in terms of its integral cohomology. A bordism-type description of the maps β_1^X and β_2^X is given in section 3.5; it is based on Spin^c-bordism. In section 3.6, we state the Künneth theorem in K-homology, and we establish useful "Künneth-type" results for those two maps. We prove, in section 3.7, injectivity of β_1^X for any CW-complex X. The proof is based on spectral sequence arguments. As a by-product, a fourth description of β_1^X and of β_2^X is given, and a criterion (in terms of the Atiyah-Hirzebruch spectral sequence) is established to determine whether or not β_2^X is injective for a given X. The final section 3.7 contains an example of a finite CW-complex for which β_2^X is *not* injective.

3.2 β_0^X and K-homology of CW-complexes of dimension ≤ 3

We start by fixing our notations and conventions. We then define the map β_0^X and show that it is a split-injection. Next, we compute the K-homology of CW-complexes of dimension ≤ 3 in terms of integral homology.

For a CW-complex X, let us denote by $ch_n : K_*(X) \longrightarrow H_n(X; \mathbb{Q})$ the component of ch of degree n. In the sequel, $H_*(X; \mathbb{Z}) \longrightarrow H_*(X; \mathbb{Q})$ will always denote the canonical coefficient homomorphism, induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. In order to simplify some formulas, we will sometimes denote the integral homology of X by $H_*(X)$, and write $H_{ev} := \oplus_{n\geq 0} H_{2n}$, $H_{odd} := \oplus_{n\geq 0} H_{2n+1}$, $H_{\leq n} := \oplus_{k=0}^{n} H_k$ and $ch_{\leq n} := \oplus_{k=0}^{n} ch_k$. It is the right place to recall that ch maps K_0 to H_{ev} and K_1 to H_{odd}; this justifies the notation $ch = ch_{ev} \oplus ch_{odd}$.

We will say that a map $\varphi : K_j(X) \longrightarrow H_n(X; \mathbb{Z})$ (with $j \equiv n \bmod 2$) is *compatible with the Chern character* if the diagram

$$\begin{array}{ccc} K_j(X) & & \\ \varphi \downarrow & \searrow^{ch_n} & \\ H_n(X; \mathbb{Z}) & \longrightarrow & H_n(X; \mathbb{Q}) \end{array}$$

is commutative (we do not require φ to be defined for any CW-complex, nor to be natural). We use the same terminology for a map in the reverse direction.

For the sphere S^n, one has $\widetilde{K}_j(S^n) \cong \mathbb{Z}$, where $j \in \{0, 1\}$ is the reduction modulo 2 of n, with a canonical generator given by the Bott element $[S^n]_K$ (the "fundamental K-homology class", or "K-orientation class" of S^n), that is characterized by the equality $ch([S^n]_K) = [S^n] \in H_n(S^n; \mathbb{Q})$. (In the bordism-type description of K-homology, it is given by the class $[S^n, 1_{S^n}, Id_{S^n}]$, where, for $n \neq 2$, S^n is equipped with the unique Spin^c-structure inducing the canonical orientation, and S^2 is equipped with the Spin^c-structure associated to its canonical complex structure; 1_{S^n} stands for the trivial 1-dimensional bundle over S^n.) Therefore, one has a canonical isomorphism

$$ch_n^{\mathbb{Z}} : \widetilde{K}_j(S^n) \xrightarrow{\cong} \widetilde{H}_n(S^n; \mathbb{Z}), \ [S^n]_K \longmapsto [S^n].$$

Clearly, this maps extends to wedges of spheres. We would like to define analogous "integral Chern characters" for arbitrary CW-complexes.

The easiest case is to construct a map $ch_0^{\mathbb{Z}}$, and as a by-product this allows for a definition of β_0^X.

3.2.1 Lemma. *For a connected CW-complex X, there is a homotopy-invariant split-injection*

$$\beta_0^X : H_0(X;\, \mathbb{Z}) \longrightarrow K_0(X)$$

that is compatible with the Chern character, and whose section will be denoted by $ch_0^{\mathbb{Z}}$. Both maps are canonical and natural. This yields in particular the canonical and natural splitting

$$K_0(X) = \mathbb{Z}\cdot[1] \oplus \widetilde{K}_0(X) = \mathbb{Z} \oplus \widetilde{K}_0(X)\,,$$

with the element $[1]$ representing $[\{x_0\},\, 1_{x_0},\, i_{x_0}]$, where i_{x_0} is the inclusion of the base-point x_0 of X.

Proof. The map β_0^X is defined as to be the composition

$$H_0(X;\, \mathbb{Z}) \xrightarrow{\cong} H_0(\{x_0\};\, \mathbb{Z}) \xrightarrow{\cong} K_0(\{x_0\}) \longrightarrow K_0(X)\,.$$

The projection $X \twoheadrightarrow \{x_0\}$ yields a splitting of the last map in the above composition. The rest is trivial. □

In the sequel, we will need an explicit homological description of the K-homology of CW-complexes of dimension ≤ 2.

3.2.2 Lemma. *For any connected CW-complex X of dimension ≤ 2, there are canonical isomorphisms*

$$ch_{ev}^{\mathbb{Z}} := ch_0^{\mathbb{Z}} \oplus ch_2^{\mathbb{Z}} : K_0(X) \xrightarrow{\cong} H_0(X;\, \mathbb{Z}) \oplus H_2(X;\, \mathbb{Z})$$
$$ch_1^{\mathbb{Z}} : K_1(X) \xrightarrow{\cong} H_1(X;\, \mathbb{Z})\,,$$

that are natural for such complexes, and compatible with the Chern character. For $X = S^1$, one has $ch_1^{\mathbb{Z}}([S^1]_K) = [S^1]$, and for $X = \Sigma_g$, a closed oriented surface of genus g, we get an isomorphism

$$ch_2^{\mathbb{Z}} : \widetilde{K}_0(\Sigma_g) \xrightarrow{\cong} H_2(\Sigma_g;\, \mathbb{Z}) = \mathbb{Z}\cdot[\Sigma_g] \cong \mathbb{Z}\,.$$

We write $[\Sigma_g]_K := (ch_2^{\mathbb{Z}})^{-1}([\Sigma_g]) \in \widetilde{K}_0(\Sigma_g)$: this is the "fundamental K-homology class of Σ_g" (or "K-orientation class"). In the bordism-type description of K-homology, $[\Sigma_g]_K$ is the class $[\Sigma_g,\, 1_{\Sigma_g},\, Id_{\Sigma_g}]$, where Σ_g is equipped with the Spin^c-structure canonically associated to *any* complex structure on Σ_g, and 1_{Σ_g} is the trivial 1-bundle over Σ_g. (By connectedness of the Teichmüller space and by means of the Riemann-Roch-Hirzebruch formula, one can show that for any complex structure on Σ_g, the K-homology class $[\bar{\partial}_g] \in K_0(\Sigma_g)$ associated to the corresponding Dolbeault operator $\bar{\partial}_g$ is same and that $[\Sigma_g]_K = [\bar{\partial}_g] + (1-g)[1]$ (see [96]).)

Before the proof of the lemma, let us define a 2-periodic homology theory h_* on the category of finite CW-complexes by setting, for $n \in \mathbb{Z}$,

$$h_n(X) := \begin{cases} H_{ev}(X;\mathbb{Q}), & \text{if } n \text{ is even} \\ H_{odd}(X;\mathbb{Q}), & \text{if } n \text{ is odd}. \end{cases}$$

Proof. Since we are working with compact supports, we can assume that X is a finite CW-complex. The Atiyah-Hirzebruch spectral sequence for reduced K-homology, namely $E^2_{p,q} = \widetilde{H}_p(X; K_q(pt)) \Longrightarrow \widetilde{K}_{p+q}(X)$, is trivial, i.e. all the differentials vanish. This yields immediately the desired natural isomorphisms. The compatibility with the Chern character ch follows by comparing this spectral sequence with the corresponding Atiyah-Hirzebruch spectral sequence for the homology theory h_*. Indeed, by naturality, ch induces a morphism of spectral sequences, that coincides, at the level of the E^2-pages, with the coefficient homomorphism induced by $\mathbb{Z} \hookrightarrow \mathbb{Q}$.

The statement about the circle and the surfaces follows readily. This completes the proof. □

Let us now consider the case of dimension ≤ 3.

3.2.3 Proposition. *For any connected CW-complex X of dimension ≤ 3, there are isomorphisms*

$$ch^{\mathbb{Z}}_{ev} := ch^{\mathbb{Z}}_0 \oplus ch^{\mathbb{Z}}_2 : K_0(X) \xrightarrow{\cong} H_0(X;\mathbb{Z}) \oplus H_2(X;\mathbb{Z})$$
$$ch^{\mathbb{Z}}_{odd} := ch^{\mathbb{Z}}_1 \oplus ch^{\mathbb{Z}}_3 : K_1(X) \xrightarrow{\cong} H_1(X;\mathbb{Z}) \oplus H_3(X;\mathbb{Z}),$$

that are compatible with the Chern character. For $n = 0$, 2 and 3 the maps $ch^{\mathbb{Z}}_n$ are natural for CW-complexes of dimension ≤ 3. For X of dimension ≤ 2, these maps coincide with those considered in lemma 3.2.2.

Proof. As in the proof of lemma 3.2.2, the Atiyah-Hirzebruch spectral sequence for K-homology is trivial and yields the natural isomorphism $ch^{\mathbb{Z}}_{ev}$, and a natural short exact sequence

$$0 \longrightarrow H_1(X) \xrightarrow{\iota} \widetilde{K}_1(X) \longrightarrow H_3(X) \longrightarrow 0,$$

that must split (naturally or not) since $H_3(X)$ is a free-abelian group. We define $ch^{\mathbb{Z}}_3$ to be the above surjection, and $ch^{\mathbb{Z}}_1$ to be any choice of a retraction of the injection ι. The compatibility with the Chern character follows from the comparison between the spectral sequences for K-homology and for the homology theory h_*. □

We do not know if there is a choice of the retraction $ch^{\mathbb{Z}}_1$ that is natural and compatible with the Chern character.

For CW-complexes of dimension 4, the following result is proved similarly. (By 1-connected, we mean connected and simply-connected.)

3.2.4 Proposition. *For any 1-connected CW-complex of dimension ≤ 4, there is a natural short exact sequence*

$$0 \longrightarrow H_2(X;\mathbb{Z}) \longrightarrow \widetilde{K}_0(X) \xrightarrow{ch_4^{\mathbb{Z}}} H_4(X;\mathbb{Z}) \longrightarrow 0.$$

There are also isomorphisms

$$ch_0^{\mathbb{Z}} \oplus ch_2^{\mathbb{Z}} \oplus ch_4^{\mathbb{Z}} : K_0(X) \xrightarrow{\cong} H_0(X;\mathbb{Z}) \oplus H_2(X;\mathbb{Z}) \oplus H_4(X;\mathbb{Z})$$
$$ch_3^{\mathbb{Z}} : K_1(X) \xrightarrow{\cong} H_3(X;\mathbb{Z}),$$

that are compatible with the Chern character. For $n = 0$, 3 and 4, the maps $ch_n^{\mathbb{Z}}$ are natural for such CW-complexes.

Notice that it is also possible to prove lemma 3.2.2 and proposition 3.2.3 by carefully comparing the long exact sequences of the pair (X, A), where A is the skeleton of codimension 1 of X, for K-homology, integral homology and rational homology, with the help of the integral Chern characters for spheres (because X/A has the homotopy type of a wedge of spheres). This argument is slightly longer that the one presented here, but it has the advantage of avoiding the intensive use of spectral sequences (see also [13]).

3.2.5 Remark. *In general, for a non simply-connected CW-complex X of dimension 4, one cannot expect an isomorphism $K_1(X) \cong H_1(X;\mathbb{Z}) \oplus H_3(X;\mathbb{Z})$. For example, the real projective space $\mathbb{R}P^4$ satisfies $H_1(\mathbb{R}P^4;\mathbb{Z}) \cong \pi_1(\mathbb{R}P^4)^{ab} \cong \mathbb{Z}/2$ and $H_3(\mathbb{R}P^4;\mathbb{Z}) \cong \mathbb{Z}/2$, whereas $K_1(\mathbb{R}P^4) \cong \mathbb{Z}/4$. See also theorem 3.7.4 for an improvement of the latter proposition.*

3.3 Definition and first properties of β_1^X and β_2^X

We first define the maps β_1^X and β_2^X, and next, we prove that they are natural homomorphisms, and rationally right inverses of the Chern character. We then give a second construction, and show that it is equivalent. This will imply that our maps coincide with those defined by Hela Bettaieb and Alain Valette for classifying spaces of discrete groups. Their construction will be given in section 5.2.

Let us first recall that the inclusion of the n-skeleton $X^{[n]}$ of a CW-complex X induces an isomorphism in integral homology and in integral cohomology up to degree $n-1$, and that any continuous map between two CW-complexes X and Y is homotopic to a cellular map, i.e. a map that takes, for each n, the n-skeleton of X into the n-skeleton of Y (this is the Whitehead cellular approximation theorem).

Now, we make the following definition:

3.3.1 Definition. *Let X be a connected CW-complex, and let i_n denote the inclusion of its n-skeleton $X^{[n]}$. Then the map β_1^X is defined as the composition*

$$\beta_1^X : H_1(X;\mathbb{Z}) \xrightarrow[\cong]{(i_2)_*^{-1}} H_1(X^{[2]};\mathbb{Z}) \xrightarrow[\cong]{(ch_1^{\mathbf{Z}})^{-1}} K_1(X^{[2]}) \xrightarrow{(i_2)_*} K_1(X),$$

where the isomorphism $ch_1^{\mathbf{Z}}$ is given by lemma 3.2.2. Similarly, the map β_2^X is defined as the composition

$$\beta_2^X : H_2(X;\mathbb{Z}) \xrightarrow[\cong]{(i_3)_*^{-1}} H_2(X^{[3]};\mathbb{Z}) \xrightarrow[\cong]{(ch_2^{\mathbf{Z}})^{-1}} \widetilde{K}_0(X^{[3]}) \xrightarrow{(i_3)_*} \widetilde{K}_0(X) \hookrightarrow K_0(X),$$

where the isomorphism $ch_2^{\mathbf{Z}}$ is given by proposition 3.2.3.

Since $ch_1^{\mathbf{Z}}$ and $ch_2^{\mathbf{Z}}$ are natural for CW-complexes of dimension ≤ 2 and ≤ 3 respectively, these maps are well-defined (i.e. independent of the CW-decomposition of X), natural, and compatible with the Chern character. Together with lemma 3.2.1, this proves the following proposition.

3.3.2 Proposition. *For a connected CW-complex X and $0 \leq j \leq 2$, the maps β_j^X are well-defined natural homomorphisms. They are rationally right inverses of the Chern character, i.e.*

$$(ch \otimes Id_{\mathbf{Q}}) \circ (\beta_j^X \otimes Id_{\mathbf{Q}}) = Id_{H_j(X;\mathbf{Q})}.$$

In particular, the maps β_j^X are rationally injective for $0 \leq j \leq 2$. The map β_0^X is split-injective, with splitting $ch_0^{\mathbf{Z}}$.

Here is tbe first non-trivial injectivity result for these maps.

3.3.3 Proposition. *For a connected CW-complex X of dimension ≤ 3, one has*

$$ch_j^{\mathbf{Z}} \circ \beta_j^X = Id_{H_j(X;\mathbf{Z})}, \quad 0 \leq j \leq 2,$$

implying that β_j^X is split-injective. (For $j = 1$, this is independent of the choice of the retraction $ch_1^{\mathbf{Z}}$.) If X is 1-connected of dimension 4, then the same holds for $j = 0$ and 2. (For $j = 2$, this is independent of the choice of the retraction $ch_2^{\mathbf{Z}}$.)

Proof. Let us first assume that X is 3-dimensional. For $j = 0$ and 2 the result is obvious. For $j = 1$, we have to be more careful, because of the choice of the retraction $ch_1^{\mathbf{Z}}$ in the short exact sequence

$$0 \longrightarrow H_1(X) \underset{ch_1^{\mathbf{Z}}}{\overset{\iota}{\rightleftarrows}} K_1(X) \longrightarrow H_3(X) \longrightarrow 0$$

(see the proof of proposition 3.2.3). By naturality of the latter sequence for CW-complexes of dimension ≤ 3, the diagram

$$\begin{array}{ccc} H_1(X^{[2]}) & \xrightarrow[\cong]{\iota^{X^{[2]}} = (ch_1^{\mathbf{Z}})^{-1}} & K_1(X^{[2]}) \\ {\scriptstyle (i_2)_*}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle (i_2)_*} \\ H_1(X) & \xrightarrow{\iota^X} & K_1(X) \end{array}$$

commutes. Since by definition $\beta_1^X = (i_2)_* \circ (ch_1^{\mathbf{Z}})^{-1} \circ (i_2)_*^{-1} = \iota^X$, we see that for any choice of the retraction $ch_1^{\mathbf{Z}}$ of ι^X, we have $ch_1^{\mathbf{Z}} \circ \beta_1^X = Id_{H_1(X;\mathbf{Z})}$.

The case where X is of dimension 4 is completely similar. □

We denote the trivial n-dimensional vector bundle over a space X simply by n_X. Recall that the fundamental K-homology class of the sphere S^n is denoted by $[S^n]_K$; it is characterized by the equality $ch([S^n]_K) = [S^n] \in H_n(S^n; \mathbb{Q})$. Since a given closed connected Spin^c-manifold M admits several such structures, parameterized by $H^1(M; \mathbb{Z}/2) \oplus 2 \cdot H^2(M; \mathbb{Z})$ (see [65], p. 392), the circle admits precisely one inducing the canonical orientation. We always endow S^1 with this structure. We now give another construction of β_1^X.

3.3.4 Lemma. *For any connected CW-complex X, the map*

$$\widetilde{\alpha}_1^X : \pi_1(X) \longrightarrow K_1(X), \ [f] \longmapsto [S^1, 1_{S^1}, f] = f_*([S^1]_K)$$

is a natural homomorphism. Factorizing through the Hurewicz homomorphism h_1^X in degree 1, it defines a natural map $\alpha_1^X : H_1(X; \mathbb{Z}) \longrightarrow K_1(X)$ fitting in the commutative diagram

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{\widetilde{\alpha}_1^X} & K_1(X) \\ {\scriptstyle h_1^X}\downarrow & \nearrow{\scriptstyle \alpha_1^X} & \\ H_1(X;\mathbb{Z}) & & \end{array}$$

Moreover, the maps α_1^X and β_1^X coincide.

Proof. Since the bordism-type description of K-homology is homotopy-invariant (by the "bordism relation": see [56]), $\widetilde{\alpha}_1^X$ is well-defined. Let us show that it is a homomorphism. Let $[f], [g] \in \pi_1(X)$. The product $[f] \cdot [g]$ in $\pi_1(X)$ is given by the class $[f \cdot g]$ of the composition

$$f \cdot g : S^1 \twoheadrightarrow S^1/S^0 \xrightarrow{\cong} S^1 \vee S^1 \xrightarrow{f \vee g} X .$$

On the other hand, one has

$$[S^1, 1_{S^1}, f] + [S^1, 1_{S^1}, g] = [S^1 \amalg S^1, 1_{S^1 \amalg S^1}, f \amalg g] .$$

It is easy to show that there is a continuous map $h : M \longrightarrow X$, where M is a pair of pants, i.e. a compact, connected and orientable surface with boundary $\partial M = S^1 \amalg S^1 \amalg S^1$, such that the restrictions of h to the three components of ∂M are f, g and $f \cdot g$ respectively. This shows that

$$\widetilde{\alpha}_1^X([f \cdot g]) = [S^1, 1_{S^1}, f \cdot g] = [S^1 \amalg S^1, 1_{S^1 \amalg S^1}, f \amalg g] = \widetilde{\alpha}_1^X([f]) + \widetilde{\alpha}_1^X([g]).$$

The naturality of $\widetilde{\alpha}_1^X$ and of α_1^X is clear. Let us now prove that α_1^X and β_1^X coincide. First, observe that any homology class $x \in H_1(X; \mathbb{Z})$ is "Steenrod-representable". This simply means that there exists a pointed continuous map $f : S^1 \longrightarrow X$ such that $f_* : H_1(S^1; \mathbb{Z}) \longrightarrow H_1(X; \mathbb{Z})$ maps $[S^1]$ to x: this is a direct consequence of the surjectivity of the Hurewicz map $h_1^X : \pi_1(X) \longrightarrow H_1(X; \mathbb{Z})$. By naturality of α_1^X and of β_1^X, it is therefore enough to check that

$$\alpha_1^{S^1}([S^1]) = \beta_1^{S^1}([S^1]).$$

The class $[S^1]$ is "Steenrod-represented" by the identity map on S^1, therefore $\alpha_1^{S^1}([S^1]) = [S^1]_K$. The equality $\beta_1^{S^1}([S^1]) = [S^1]_K$ is a consequence of lemma 3.2.2. This completes the proof. □

Notice that $\widetilde{\alpha}_1 : \pi_1(X) \longrightarrow K_1(X)$ is nothing but the Hurewicz homomorphism in K-homology.

Before passing to the second definition of β_2^X, we need two classical theorems. We first recall the basics on oriented bordism.

For a connected CW-complex X, oriented hordism $\Omega_*^{\mathrm{SO}}(X) = M\mathrm{SO}_*(X)$ is the set of equivalence classes of pairs (M, f), where M is a closed oriented smooth manifold (not necessarily connected), and $f : M \longrightarrow X$ is a continuous map. Two such pairs (M, f) and (M', f') are said equivalent if they are oriented-cobordant. This means that M and M' have the same dimension, say m, and that there exists a pair (W, F), where W is an oriented compact smooth manifold of dimension $m+1$, whose *oriented* boundary is $\partial W = M \amalg -M'$, and $F : W \longrightarrow X$ is a continuous map whose restriction to M and M' is f and f' respectively. This set is an abelian group for the disjoint union, with zero element given by the empty manifold. The group $\Omega_*^{\mathrm{SO}}(X)$ is $\mathbb{Z}$-graded by the dimension of the manifolds. The equivalence class of a pair (M, f) is denoted by $[M, f]$, and we write Ω_*^{SO} for $\Omega_*^{\mathrm{SO}}(pt)$. It turns out that $\Omega_*^{\mathrm{SO}}(-)$ is a homology theory on the category of CW-complexes.

3.3.5 Theorem. *For a connected CW-complex X, the map*

$$\mu_j^X : \Omega_j^{\mathrm{SO}}(X) \longrightarrow H_j(X; \mathbb{Z}), \ [M, f] \longmapsto f_*([M]),$$

where $f_* : H_*(M; \mathbb{Z}) \longrightarrow H_*(X; \mathbb{Z})$, is a natural isomorphism for $j \leq 3$.

This theorem is well-known. The existence of such an isomorphism follows readily from the classical equalities $\Omega_n^{\mathrm{SO}} = 0$ for $0 \neq n \leq 3$, and $\Omega_0^{\mathrm{SO}} \cong \mathbb{Z}$ (see [76],

p. 203), together with an application of the Atiyah-Hirzebruch spectral sequence. The explicit description of the isomorphism is established in [32] (thm. 7.2, p. 17).

The same ideas and the known isomorphisms $\Omega_4^{\mathrm{SO}} \cong \mathbb{Z}$ and $\Omega_5^{\mathrm{SO}} \cong \mathbb{Z}/2$ (see [110] and [76], p. 203) imply the existence of a natural isomorphism

$$p_* \oplus \mu_4^X : \Omega_4^{\mathrm{SO}}(X) \xrightarrow{\cong} \mathbb{Z} \oplus H_4(X;\mathbb{Z}),$$

where $p : X \longrightarrow pt$, and of a natural short exact sequence

$$0 \longrightarrow H_1(X;\mathbb{Z}/2) \longrightarrow \Omega_5^{\mathrm{SO}}(X) \xrightarrow{\mu_5^X} H_5(X;\mathbb{Z}) \longrightarrow 0,$$

for any connected CW-complex X. This shows that any integral homology class of degree ≤ 5 is "Steenrod-representable".

Here comes the second classical theorem.

3.3.6 Theorem. *Let X be a connected CW-complex. Then any homology class $x \in H_2(X;\mathbb{Z})$ is "Steenrod-representable", in other words, there exists a surface Σ_g of genus $g \geq 1$ and a pointed continuous map $f : \Sigma_g \longrightarrow X$ (with both g and f depending on x) such that $f_* : H_2(\Sigma_g;\mathbb{Z}) \longrightarrow H_2(X;\mathbb{Z})$ maps the fundamental class $[\Sigma_g]$ to x.*

For $x \in H_2(X;\mathbb{Z})$, we write $x = [\Sigma_g, f]$ to express the fact that x is "Steenrod-represented" by $f : \Sigma_g \longrightarrow X$.

Proof. It is an immediate consequence of theorem 3.3.5, noticing that we can (for our comfort) assume $g \geq 1$. Indeed, the 2-sphere S^2 is a suitable quotient of the 2-torus $\mathbb{T}^2$, and the quotient map takes $[\mathbb{T}^2]$ to $[S^2]$. □

Notice that this theorem is independent of the highly non-trivial fact, used in the sketch of proof of theorem 3.3.5, that $\Omega_3^{\mathrm{SO}} = 0$. Indeed, we only use the isomorphism $\Omega_2^{\mathrm{SO}}(X) \cong H_2(X;\mathbb{Z})$. We should mention that in [71] (lemma 2.2.4), there is an "elementary" proof of theorem 3.3.6 for $X = B\Gamma$, the classifying space of a discrete group Γ. We also mention Zimmermann's theorem [114] and Thom's theorem [104]: they are stated as theorems 5.2.2 and 5.2.1 in chapter 5.

Recall that we always consider Σ_g as equipped with the Spin^c-structure canonically associated to any complex structure. This defines uniquely a K-orientation class $[\Sigma_g]_K \in K_0(\Sigma_g)$, satisfying $ch_2^{\mathbb{Z}}([\Sigma_g]_K) = [\Sigma_g]$. We can now pass to the second construction of β_2^X.

3.3.7 Proposition. *For any connected CW-complex X, the map β_2^X is given as follows:*

$$\beta_2^X : H_2(X;\mathbb{Z}) \longrightarrow K_0(X),\ f_*([\Sigma_g]) = [\Sigma_g, f] \longmapsto [\Sigma_g, 1_{\Sigma_g}, f] = f_*([\Sigma_g]_K).$$

Proof. The result follows from the following computation:

$$\begin{aligned}\beta_2^X([\Sigma_g, f]) &= \beta_2^X \circ f_*([\Sigma_g, Id_{\Sigma_g}]) = \beta_2^X \circ f_*([\Sigma_g]) = f_* \circ \beta_2^{\Sigma_g}([\Sigma_g]) \\ &= f_*([\Sigma_g]_K) = f_*([\Sigma_g, 1_{\Sigma_g}, Id_{\Sigma_g}]) = [\Sigma_g, 1_{\Sigma_g}, f].\end{aligned}$$

□

3.4 The groups SK^0 and SK^1, and K-theory of CW-complexes of dimension ≤ 4

In this section, we compute K-theory of low-dimensional CW-complexes in terms of integral cohomology. These results are analogous to those obtained in section 3.2 for K-homology in terms of integral homology. To achieve this, we need known natural decompositions of K-theory as products involving groups that we denote by SK^0 and SK^1. This is of independent interest, but also applies to one of the main objectives in the present chapter, namely to show that β_1^X is injective. Indeed, in the second proof, contained in appendix E, we need these computations.

For cohomology and K-theory, we use similar conventional notations as we did in section 3.2 for homology and K-homology (in particular, we denote the component of the Chern character in $H^n(X; \mathbb{Q})$ by ch_n).

Let X be a connected CW-complex. Its representable K-theory is

$$K^0(X) = [X, \mathbb{Z} \times BU] \quad \text{and} \quad K^1(X) = [X, U].$$

The first natural decomposition is trivial, namely

$$K^0(X) = \mathbb{Z} \oplus \widetilde{K}^0(X), \text{ where } \widetilde{K}^0(X) := [X, BU].$$

We get a second decomposition in the following way. The fibration of groups

$$SU \overset{i}{\hookrightarrow} U \underset{s}{\overset{\det}{\rightleftarrows}} S^1$$

has a section s which is a group homomorphism. On the level of classifying spaces, this gives rise to a fibration

$$BSU \overset{Bi}{\hookrightarrow} BU \underset{Bs}{\overset{B\det}{\rightleftarrows}} BS^1 = \mathbb{C}P^\infty$$

with section Bs. All spaces under consideration are homotopy commutative H-groups, and all maps, *except* Bs, are H-maps. Let μ denote the multiplication on U. If follows that the composition

$$S^1 \times SU \overset{s \times i}{\hookrightarrow} U \times U \xrightarrow{\mu} U$$

is a homotopy equivalence of homotopy commutative H-groups, and the composition

$$\mathbb{C}P^\infty \times BSU \xrightarrow{Bs \times Bi} BU \times BU \xrightarrow{B\mu} BU$$

is a homotopy equivalence of spaces. (For details on all these points, we refer to Adams [3].)

Let us now make the following definition.

3.4.1 Definition. *For a connected CW-complex X, we let*

$$SK^0(X) := [X, BSU] \text{ and } SK^1(X) := [X, SU].$$

The sets $SK^0(X)$ and $SK^1(X)$ are abelian groups, U and BSU being homotopy-commutative H-groups.

The above consideration immediately proves

3.4.2 Proposition. *For a connected CW-complex X, there are natural decompositions*

$$\widetilde{K}^0(X) \approx H^2(X; \mathbb{Z}) \oplus SK^0(X) \text{ and } K^1(X) \cong H^1(X; \mathbb{Z}) \oplus SK^1(X).$$

The latter decomposition is a group isomorphism, but the first is only a bijection between abelian groups. More precisely,

$$0 \longrightarrow SK^0(X) \longrightarrow \widetilde{K}^0(X) \longrightarrow H^2(X; \mathbb{Z}) \longrightarrow 0$$

is an exact sequence of abelian groups, with the projection onto $H^2(X; \mathbb{Z})$ corresponding to the first Chern class c_1.

In cohomology and K-theory, we have to be more careful concerning supports than in homology and K-homology. Therefore, we will from time to time assume that the CW-complexes we consider are finite.

In the K-theoretical context, we will use the terminology "compatible with the Chern character" in a completely similar way as we did it for K-homology.

Now, we consider the Atiyah-Hirzebruch spectral sequences for reduced K-theory, namely

$$E_2^{p,q} = \widetilde{H}^p(X; K^q(pt)) \Longrightarrow \widetilde{K}^{p+q}(X).$$

3.4.3 Proposition. *Let X be a connected CW-complex of finite type. In the Atiyah-Hirzebruch spectral sequence for $\widetilde{K}$-theory, one has*

$$\widetilde{H}^j(X; \mathbb{Z}) = E_2^{j,0} \cong E_\infty^{j,0}, \quad j = 1, 2,$$

in other words, all differentials emanating from $E_{\geq r}^{j,0}$, with $r \geq 2$ and $j = 1, 2$, are zero.

Proof. Let $a \in H^1(X; \mathbb{Z}) \cong [X, S^1]$, and let $\alpha : X \longrightarrow S^1$ be a map representing a. One has $\alpha^*(y) = a$, where y is a generator of $H^1(S^1; \mathbb{Z}) \cong \mathbb{Z}$. The map α induces a morphism $E(\alpha)$ of spectral sequences for which

$$E(\alpha) : E_2^{1,0}(S^1) = H^1(S^1; \mathbb{Z}) \xrightarrow{\alpha^*} H^1(X; \mathbb{Z}) = E_2^{1,0}(X);$$

in particular, $E(\alpha)(y) = a$. Since the spectral sequence for S^1 is trivial, it follows that $d_r(a) = 0$ for any $r \geq 2$.

We proceed in the same way with $b \in H^2(X; \mathbb{Z}) \cong [X; \mathbb{C}P^\infty]$, knowing that the spectral sequence for $\mathbb{C}P^\infty$ is also trivial. □

We thank Ueli Suter for drastically simplifying our original proof.

3.4.4 Proposition. *For a connected finite CW-complex X of dimension ≤ 5, there are natural short exact sequences*

$$0 \longrightarrow H^4(X; \mathbb{Z}) \longrightarrow \widetilde{K}^0(X) \xrightarrow{c_1} H^2(X; \mathbb{Z}) \longrightarrow 0.$$

and

$$0 \longrightarrow H^5(X; \mathbb{Z}) \longrightarrow SK^1(X) \longrightarrow H^3(X; \mathbb{Z}) \longrightarrow 0.$$

Proof. The result follows from proposition 3.4.3, since, in this case, obviously all differentials emanating and reaching $E_r^{j,q}$ are zero, for $3 \leq j \leq 5$. □

This proposition has two important immediate corollaries.

3.4.5 Corollary. *Let X be a connected finite CW-complex of dimension ≤ 3. Then there are canonical isomorphisms*

$$ch_{ev}^{\mathbb{Z}} := ch_0^{\mathbb{Z}} \oplus ch_2^{\mathbb{Z}} : K^0(X) \xrightarrow{\cong} H^0(X; \mathbb{Z}) \oplus H^2(X; \mathbb{Z})$$
$$ch_{odd}^{\mathbb{Z}} := ch_1^{\mathbb{Z}} \oplus ch_3^{\mathbb{Z}} : K^1(X) \xrightarrow{\cong} H^1(X; \mathbb{Z}) \oplus H^3(X; \mathbb{Z}),$$

that are natural for such CW-complexes, and compatible with the Chern character. The map $ch_2^{\mathbb{Z}}$ coincides with the first Chern class c_1.

Notice that this is simpler than the corresponding situation in K-homology, except for the finiteness assumption (the crucial difference being the presence of proposition 3.4.2; compare with proposition 3.2.3). Another proof of the isomorphism $\widetilde{K}^0(X) \cong H^2(X; \mathbb{Z})$ of the proposition is to invoke the fact that for suitable CW-decompositions, one has $BU(1)^{[3]} = BU^{[3]}$ (see chapter 1).

For CW-complexes of dimension ≤ 4, we have

3.4.6 Corollary. *Let X be a connected finite CW-complex of dimension ≤ 4. Then there are canonical bijections*

$$ch_{ev}^{\mathbb{Z}} := ch_0^{\mathbb{Z}} \oplus ch_2^{\mathbb{Z}} \oplus ch_4^{\mathbb{Z}} : K^0(X) \xrightarrow{\approx} H^0(X; \mathbb{Z}) \oplus H^2(X; \mathbb{Z}) \oplus H^4(X; \mathbb{Z})$$
$$ch_{odd}^{\mathbb{Z}} := ch_1^{\mathbb{Z}} \oplus ch_3^{\mathbb{Z}} : K^1(X) \xrightarrow{\cong} H^1(X; \mathbb{Z}) \oplus H^3(X; \mathbb{Z}),$$

that are natural for such CW-complexes, and compatible with the Chern character. The map $ch_{odd}^{\mathbb{Z}}$ is a group isomorphism, but $ch_{ev}^{\mathbb{Z}}$ is only a bijection between abelian groups. The maps $ch_0^{\mathbb{Z}}$ and $ch_2^{\mathbb{Z}}$ are group homomorphisms, with $ch_2^{\mathbb{Z}} = c_1$, the first Chern class. Moreover, if $H^4(X;\mathbb{Z}) = 0$, then $ch_{ev}^{\mathbb{Z}}$ is a group isomorphism. If $H^2(X;\mathbb{Z})$ is torsion-free, then there is an abstract isomorphism $K^0(X) \cong H^{ev}(X;\mathbb{Z})$.

All statements concerning K^0 extend to the category of connected finite CW-complexes of dimension ≤ 5.

3.4.7 Remark. *For the 4-dimensional real projective space $\mathbb{R}P^4$, there are well-known isomorphisms $K^0(\mathbb{R}P^4) \cong \mathbb{Z}/4$ and $H^2(\mathbb{R}P^4) \cong H^4(\mathbb{R}P^4) \cong \mathbb{Z}/2$. This shows that $ch_{ev}^{\mathbb{Z}}$ is generally not a group isomorphism, but merely a bijection. It also follows that $SK^0(\mathbb{R}P^4) \cong \mathbb{Z}/2$. In general, we do not know if $ch_{ev}^{\mathbb{Z}}$ realizes an isomorphism for connected finite CW-complexes of dimension ≤ 5 having a torsion-free H^2.*

3.5 A bordism-type description of the maps β_j^X

We give a description of the maps β_j^X based on Spin^c-bordism, for $0 \leq j \leq 2$.

Recall the short introduction to oriented bordism in section 3.3. The Spin^c-bordism $\Omega_*^{\mathrm{Spin}^c}(X) = M\mathrm{Spin}_*^c(X)$ is defined as $\Omega_*^{\mathrm{SO}}(X)$, but with Spin^c-manifolds, that are automatically orientable, and also K-orientable. Such a manifold M, of dimension m say, consequently has two fundamental classes, depending on the chosen Spin^c-structure, namely $[M] \in H_m(M;\mathbb{Z})$ and $[M]_K = [M, 1_M, Id_M] \in K_m(M)$. The "first four" coefficient groups of Spin^c-bordism are

$$\Omega_0^{\mathrm{Spin}^c} \cong \mathbb{Z} \qquad \Omega_1^{\mathrm{Spin}^c} = 0 \qquad \Omega_2^{\mathrm{Spin}^c} \cong \mathbb{Z} \qquad \Omega_3^{\mathrm{Spin}^c} = 0$$

(see [46]). Therefore, for any connected CW-complex X, there is a graded natural map

$$\varphi_*^X : \Omega_*^{\mathrm{Spin}^c}(X) \longrightarrow H_*(X;\mathbb{Z}),\ [M, f] \longmapsto f_*([M]),$$

and the Atiyah-Hirzebruch spectral sequence for $\Omega_*^{\mathrm{Spin}^c}(-)$ yields natural isomorphisms

$$\varphi_j^X : \Omega_j^{\mathrm{Spin}^c}(X) \xrightarrow{\cong} H_j(X;\mathbb{Z}), \quad j = 0, 1$$
$$p_* \oplus \varphi_2^X : \Omega_2^{\mathrm{Spin}^c}(X) \xrightarrow{\cong} \mathbb{Z} \oplus H_2(X;\mathbb{Z}),$$

where $p : X \twoheadrightarrow pt$ (compare with 7.2 on page 17 of [32]). We will write $(\varphi_2^X)^{-1}$ for the restriction of $(p_* \oplus \varphi_2^X)^{-1}$ to $H_2(X;\mathbb{Z})$. On the other hand, there is a natural map

$$\kappa_*^X : \Omega_*^{\mathrm{Spin}^c}(X) \longrightarrow K_*(X),\ [M, f] \longmapsto f_*([M]_K).$$

The following result is easily checked.

3.5.1 Proposition. *For a connected CW-complex X and for $0 \leq j \leq 2$, one has the equality*

$$\beta_j^X = \kappa_*^X \circ (\varphi_j^X)^{-1} : H_j(X; \mathbb{Z}) \longrightarrow K_j(X)\,,\; f_*([M]) \longmapsto f_*([M]_K)\,.$$

One could equally well replace $\Omega_*^{\mathrm{Spin}^c}(X)$ by complex bordism $\Omega_*^U(X) = MU_*(X)$ in the above consideration (it has the same "first six" coefficient groups (see [46])).

The fact that $\Omega_2^{\mathrm{Spin}^c} \neq 0$ illustrates the difficulties one faces in trying to define a natural map β_3^X : there is a natural short exact sequence

$$0 \longrightarrow H_1(X; \mathbb{Z}) \longrightarrow \Omega_3^{\mathrm{Spin}^c}(X) \xrightarrow{\varphi_3^X} H_3(X; \mathbb{Z}) \longrightarrow 0\,,$$

but we do not know if it splits naturally.

3.6 The Künneth theorem for K-homology

In section 3.7, we will prove that β_1^X is always injective. One of the crucial tools will be a "Künneth-type" result for β_1 that we prove in the present section. Before establishing it, we state the Künneth theorem in K-homology. We also include a useful "Künneth-type" formula for β_2.

For M and N two (not necessarily connected) finite CW-complexes, for which representable K-theory coincides with vector bundle K-theory, the external tensor product of vector bundles

$$\mathrm{Vect}(M) \times \mathrm{Vect}(N) \longrightarrow \mathrm{Vect}(M \times N)\,,\; (\xi, \eta) \longmapsto \xi \boxtimes \eta$$

induces the external cup product

$$K^0(M) \times K^0(N) \longrightarrow K^0(M \times N),\, (\xi, \eta) \longmapsto \xi \cup \eta\,.$$

In the bordism-type description of K-homology, for two connected CW-complexes X and Y, the external cross product is given by

$$\begin{aligned} K_i(X) \times K_j(Y) &\longrightarrow K_{i+j}(X \times Y) \\ (x, y) = ([M, \xi, f], [N, \eta, g]) &\longmapsto x \times y = [M \times N, \xi \cup \eta, f \times g]\,, \end{aligned}$$

where $M \times N$ is endowed with the product Spin^c-structure (see [55], p. 18).

Let us explain our notation concerning the gradings: $[K_*(X) \otimes K_*(Y)]_1$ denotes $(K_0(X) \otimes K_1(Y)) \oplus (K_1(X) \otimes K_0(Y))$, and similarly, $[\mathrm{Tor}(K_*(X);\, K_*(Y))]_1$ stands for $\mathrm{Tor}(K_0(X);\, K_1(Y)) \oplus \mathrm{Tor}(K_1(X);\, K_0(Y))$. The other cases are analogous.

We are now in position to state the following theorem (see [23]).

3.6.1 Theorem. (Künneth theorem in K-homology) *For X and Y two connected CW-complexes and for any $n \in \mathbb{Z}$, there is a natural short exact sequence*

$$0 \longrightarrow [K_*(X) \otimes K_*(Y)]_n \xrightarrow{\times} K_n(X \times Y) \longrightarrow [\mathrm{Tor}(K_*(X);\, K_*(Y))]_{n-1} \longrightarrow 0\,.$$

We do not know if the sequence splits. Remembering the splitting of K_0 given by lemma 3.2.1, we obtain injections

$$\iota_j : \underbrace{(K_j(X) \otimes 1) \oplus (1 \otimes K_j(Y))}_{=K_j(X) \oplus K_j(Y)} \hookrightarrow K_j(X \times Y)\,, \quad j = 1,\, 2$$

$$([M,\, \xi,\, f],\, [N,\, \eta,\, g]) \longmapsto [M,\, \xi,\, f \times c_{y_0}] + [N,\, \eta,\, c_{x_0} \times g] =$$

$$= [M \amalg N,\, \xi \amalg \eta,\, (f \times c_{y_0}) \amalg (c_{x_0} \times g)]\,,$$

where c_{x_0} and c_{y_0} are the constant maps taking the base-point of X and Y respectively as value. If we denote by j_X and j_Y the inclusions (with respect to the corresponding base-points) of X and Y in $X \times Y$ respectively, then ι_j coincides with the composition

$$K_j(X) \oplus K_j(Y) \xrightarrow{(j_X)_* \times (j_Y)_*} K_j(X \times Y) \oplus K_j(X \times Y) \xrightarrow{+} K_j(X \times Y)\,.$$

Clearly, there is a canonical isomorphism $\varphi : \pi_1(X \times Y) \xrightarrow{\cong} \pi_1(X) \times \pi_1(Y)$, and by abelianizing, we get $\varphi^{ab} : H_1(X \times Y; \mathbb{Z}) \xrightarrow{\cong} H_1(X; \mathbb{Z}) \oplus H_1(Y; \mathbb{Z})$. Let us finally recall that we have defined the Hurewicz map for K-homology $\widetilde{\alpha}_1$ in lemma 3.3.4. We can now state a "Künneth-type" lemma for the maps $\widetilde{\alpha}_1$ and β_1.

3.6.2 Lemma. *Let X and Y be two connected CW-complexes. Then the map $\widetilde{\alpha}_1^{X \times Y}$ fits into the commutative diagram*

$$\begin{array}{ccc}
\pi_1(X \times Y) & \xrightarrow{\widetilde{\alpha}_1^{X \times Y}} & K_1(X \times Y) \\
\varphi \downarrow \cong & & \uparrow \iota_1 \\
\pi_1(X) \times \pi_1(Y) & \xrightarrow{\widetilde{\alpha}_1^{X} \times \widetilde{\alpha}_1^{Y}} & K_1(X) \oplus K_1(Y)
\end{array}$$

In particular, $\beta_1^{X \times Y} = \iota_1 \circ (\beta_1^X \times \beta_1^Y) \circ \varphi^{ab}$, and $\beta_1^{X \times Y}$ is injective if and only if so are β_1^X and β_1^Y.

Proof. Let μ denote the product map on $\pi_1(X \times Y)$. The diagram

$$\begin{array}{ccc}
\pi_1(X \times Y) & \xrightarrow{\widetilde{\alpha}_1^{X \times Y}} & K_1(X \times Y) \\
\mu \uparrow & & \uparrow + \\
\pi_1(X \times Y) \times \pi_1(X \times Y) & \xrightarrow{\widetilde{\alpha}_1^{X \times Y} \times \widetilde{\alpha}_1^{X \times Y}} & K_1(X \times Y) \oplus K_1(X \times Y) \\
(j_X)_* \times (j_Y)_* \uparrow & & \uparrow (j_X)_* \times (j_Y)_* \\
\pi_1(X) \times \pi_1(Y) & \xrightarrow{\widetilde{\alpha}_1^{X} \times \widetilde{\alpha}_1^{Y}} & K_1(X) \oplus K_1(Y)
\end{array}$$

commutes. For the upper square, this is the fact that $\widetilde{\alpha}_1^{X\times Y}$ is a group homomorphism, and for the bottom square, it follows from the naturality of $\widetilde{\alpha}_1$ (see lemma 3.3.4). The left-hand composition is φ^{-1} and the right-hand composition is ι_1. The rest is clear from lemma 3.3.4. □

For the next statement, by means of the decomposition $K_0 = \mathbb{Z} \oplus \widetilde{K}_0$ of lemma 3.2.1, we make the following identification:

$$[K_*(X)\otimes K_*(Y)]_0 = \mathbb{Z} \oplus \widetilde{K}_0(X) \oplus \widetilde{K}_0(Y) \oplus (\widetilde{K}_0(X) \otimes \widetilde{K}_0(Y)) \oplus (K_1(X) \otimes K_1(Y)).$$

We also recall that β_2 maps H_2 in $\widetilde{K}_0$. This will make the following statement clear.

3.6.3 Lemma. *Let X and Y be two connected CW-complexes. Then the map $\beta_2^{X\times Y}$ fits into the commutative diagram*

$$\begin{array}{ccc}
(H_1(X)\otimes H_1(Y)) \oplus H_2(X) \oplus H_2(Y) & \xrightarrow{(\beta_1^X \otimes \beta_1^Y)\oplus \beta_2^X \oplus \beta_2^Y} & [K_*(X)\otimes K_*(Y)]_0 \\
\times \downarrow \cong & & \times \downarrow \\
H_2(X\times Y) & \xrightarrow{\quad\beta_2^{X\times Y}\quad} & K_0(X\times Y)
\end{array}$$

In particular, $\beta_2^{X\times Y}$ is injective if and only if the three maps $\beta_1^X \otimes \beta_1^Y$, β_2^X and β_2^Y are injective.

Proof. The vertical maps are given by the corresponding Künneth theorem. We split the proof of the commutativity into two parts. First, by naturality and additivity of β_2, the diagram

$$\begin{array}{ccc}
H_2(X) \oplus H_2(Y) & \xrightarrow{\quad\beta_2^X \oplus \beta_2^Y\quad} & \widetilde{K}_0(X) \oplus \widetilde{K}_0(Y) \\
(j_X)_* \oplus (j_Y)_* \downarrow & & \downarrow (j_X)_* \oplus (j_Y)_* \\
H_2(X\times Y) \oplus H_2(X\times Y) & \xrightarrow{\beta_2^{X\times Y} \oplus \beta_2^{X\times Y}} & \widetilde{K}_0(X\times Y) \oplus \widetilde{K}_0(X\times Y) \\
+ \downarrow & & \downarrow + \\
H_2(X\times Y) & \xrightarrow{\quad\beta_2^{X\times Y}\quad} & K_0(X\times Y)
\end{array}$$

commutes. Both vertical compositions coincide with the corresponding restrictions of the vertical maps in the statement.

For the second part, let $x = [S^1, f] \in H_1(X)$ and $y = [S^1, g] \in H_1(Y)$ be "Steenrod-represented" homology classes. By naturality of the external cross product in homology, the left-hand square of the following diagram commutes:

$$\begin{array}{ccccc}
H_1(S^1)\otimes H_1(S^1) & \xrightarrow{f_*\otimes g_*} & H_1(X)\otimes H_1(Y) & \xrightarrow{\beta_1^X \otimes \beta_1^Y} & K_1(X)\otimes K_1(Y) \\
\times \downarrow & & \times \downarrow & & \times \downarrow \\
H_2(\mathbb{T}^2) & \xrightarrow{(f\times g)_*} & H_2(X\times Y) & \xrightarrow{\beta_2^{X\times Y}} & K_0(X\times Y)
\end{array}$$

We have to show that the right-hand square commutes. For one composition, one has

$$\beta_1^X([S^1, f]) \times \beta_1^Y([S^1, g]) = [S^1, 1_{S^1}, f] \times [S^1, 1_{S^1}, g] = [\mathbb{T}^2, 1_{\mathbb{T}^2}, f \times g].$$

Since $[S^1] \times [S^1] = [\mathbb{T}^2]$, for the second composition, we first compute

$$[S^1, f] \times [S^1, g] = f_*([S^1]) \times g_*([S^1]) = (f \times g)_*([\mathbb{T}^2]) = [\mathbb{T}^2, f \times g],$$

from which we deduce that

$$\beta_2^{X\times Y}([S^1, f] \times [S^1, g]) = \beta_2^{X\times Y}([\mathbb{T}^2, f \times g]) = [\mathbb{T}^2, 1_{\mathbb{T}^2}, f \times g].$$

In total, this proves that the diagram of the lemma commutes. The statement about injectivity is clear. This ends the proof. □

3.7 Injectivity of β_1^X

We prove injectivity of β_1^X for any connected CW-complex X. Roughly speaking, the idea is to first prove injectivity for $X = B\mathbb{Q}$ and for $X = B(\mathbb{Q}/\mathbb{Z})$. This poses no difficulty, since their reduced integral homology is concentrated in odd degree. Then, by a "Künneth-type" argument, one deduces injectivity for $B((\mathbb{Q}\oplus\mathbb{Q}/\mathbb{Z})^n)$. If $H_1(X; \mathbb{Z})$ is of finite type, then it injects in $(\mathbb{Q}\oplus\mathbb{Q}/\mathbb{Z})^n$, for some n. This allows for proving the result for such spaces, by a suitable simple argument. The general case then follows by taking direct limits over finite sub-complexes of X. As an application, we describe K-homology of CW-complexes of dimension ≤ 4 in terms of integral homology.

3.7.1 Theorem. *The map $\beta_1^X : H_1(X; \mathbb{Z}) \longrightarrow K_1(X)$ is injective for any connected CW-complex X.*

We prove it at the end of this section.

The following classical lemma is a key ingredient.

3.7.2 Lemma. *The groups $\mathbb{Z}$, $\mathbb{Z}/m$ with $m \geq 2$, $\mathbb{Z}/p^\infty$ with p a prime, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ have their homology concentrated in odd degree.*

Proof. For $\mathbb{Z}$, this is clear. For $\mathbb{Z}/m$, one has

$$\widetilde{H}_{2q}(\mathbb{Z}/m; \mathbb{Z}) = 0 \text{ and } H_{2q+1}(\mathbb{Z}/m; \mathbb{Z}) \cong \mathbb{Z}/m,$$

for any q. Since $\mathbb{Z}/p^\infty = \varinjlim_k \mathbb{Z}/p^k$ and since homology is continuous (i.e. commutes with direct limits), one gets

$$\widetilde{H}_{2q}(\mathbb{Z}/p^\infty; \mathbb{Z}) = 0 \text{ and } H_{2q+1}(\mathbb{Z}/p^\infty; \mathbb{Z}) \cong \mathbb{Z}/p^\infty,$$

for any q. The isomorphism $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p\in P} \mathbb{Z}/p^\infty$, where P is the set of primes, together with the Künneth theorem in homology and the continuity of homology, of the tensor product and of Tor yield

$$\widetilde{H}_*(\mathbb{Q}/\mathbb{Z};\, \mathbb{Z}) = H_1(\mathbb{Q}/\mathbb{Z};\, \mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}.$$

Finally, one has $\widetilde{H}_*(\mathbb{Q};\, \mathbb{Z}) = H_1(\mathbb{Q};\, \mathbb{Z}) \cong \mathbb{Q}$ (see [48], pp. 84-85). This completes the proof. □

In the proof of theorem 3.7.1, we need the forthcoming proposition that gives a fourth description of the maps β_j^X for $j = 1$ and $j = 2$. We have to recall some points before stating it. We have defined a 2-periodic integral homology theory h_* for the proof of lemma 3.2.2. Since we are working with compact supports, it is defined for all connected CW-complexes. Consider the Atiyah-Hirzebruch spectral sequence $E^*_{**} = \{E^r_{p,q}\}$ satisfying

$$E^2_{p,q} = \widetilde{H}_p(X;\, h_q(pt)) \implies \widetilde{K}_{p+q}(X).$$

The convergence means that $E^\infty_{p,q} = J_{p,q}/J_{p-1,q+1}$, where $\{J_{p,q}\}$ is the filtration

$$0 = J_{0,n} \subseteq J_{1,n-1} \subseteq \ldots \subseteq J_{p,n-p} \subseteq \ldots \subseteq J_{n,0} \subseteq \ldots \subseteq \bigcup_{p+q=n} J_{p,q} = \widetilde{K}_n(X)$$

defined by $J_{p,q} := \operatorname{Im}(\widetilde{K}_{p+q}(X^{[p]}) \longrightarrow \widetilde{K}_{p+q}(X))$. Notice that for $1 \le j \le 3$, there is a natural epimorphism $H_j(X;\, \mathbb{Z}) = E^2_{j,0} \twoheadrightarrow E^\infty_{j,0} = J_{j,0}/J_{j-1,1}$, with $J_{0,1} = 0$ (since $X^{[0]} = pt$, as a standing assumption) and $J_{1,1} = 0$ because, by lemma 3.2.2, $\widetilde{K}_0(X^{[1]}) = 0$. We therefore get a natural map

$$\delta_j^X : H_j(X;\, \mathbb{Z}) = E^2_{j,0} \twoheadrightarrow E^\infty_{j,0} = J_{j,0} = \operatorname{Im}(\widetilde{K}_j(X^{[j]}) \longrightarrow \widetilde{K}_j(X)) \hookrightarrow \widetilde{K}_j(X),$$

for $j = 1$ and 2.

3.7.3 Proposition. *Let X be a connected CW-complex. For $j = 1$ and 2, the map δ_j^X coincides with β_j^X. In particular, β_j^X is injective if and only if no non-zero differential reaches $E^{\geq 2}_{j,0}$ in the spectral sequence. Consequently, if the reduced integral homology of X is concentrated in odd (resp. even degree), except possibly for H_2 (resp. H_1 and H_3), then β_1^X (resp. β_2^X) is injective.*

Proof. By naturality, it is enough to check the equality between both maps in the case of S^1 for $j = 1$, and in the case of the surfaces Σ_g for $j = 2$. This is obvious. The result about injectivity is clear. □

This proposition also illustrates the difficulty in defining a map β_3^X : the group $J_{2,1}$ is generally non-zero, since it is isomorphic to $H_1(X;\, \mathbb{Z})$, by virtue of lemma 3.2.2.

We can now pass to the main proof.

Proof of theorem 3.7.1. By lemma 3.7.2 and proposition 3.7.3, β_1^X is injective for $X = B\mathbb{Q}$ and for $X = B(\mathbb{Q}/\mathbb{Z})$. By the "Künneth-type" lemma 3.6.2, it is also injective for $X = B((\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z})^n)$. If $H := H_1(X;\mathbb{Z})$ is of finite type (as for example if X is finite), then H is a finite direct sum of cyclic groups, and therefore injects in $G := (\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z})^n$, for some n. Let p denote the projection of $\pi := \pi_1(X)$ onto $H = \pi^{ab}$. The universal covering space $\widetilde{X}$ of X is a principal π-fibre bundle over X. It is consequently classified by a map $f : X \longrightarrow B\pi$, unique up to homotopy. The map $f_* : \pi_1(X) \longrightarrow \pi_1(B\pi) = \pi$ is the identity, and this means that the composition $X \xrightarrow{f} B\pi \xrightarrow{Bp} BH$ induces the map p on the level of fundamental groups, and the identity on first integral homology groups. Let $i : H \hookrightarrow G$ be the inclusion in the abelian group G. By naturality of β_1, we obtain a commutative diagram

$$\begin{array}{ccc}
H_1(X;\mathbb{Z}) & \xrightarrow{\beta_1^X} & K_1(X) \\
{\scriptstyle Id_H}\downarrow & & \downarrow{\scriptstyle f_*} \\
H_1(B\pi;\mathbb{Z}) & \xrightarrow{\beta_1^{B\pi}} & K_1(B\pi) \\
{\scriptstyle Id_H}\downarrow & & \downarrow{\scriptstyle Bp_*} \\
H_1(BH;\mathbb{Z}) & \xrightarrow{\beta_1^{BH}} & K_1(BH) \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle Bi_*} \\
H_1(BG;\mathbb{Z}) & \xhookrightarrow{\beta_1^{BG}} & K_1(BG)
\end{array}$$

Consequently, β_1^X is injective is this case too. Since we are working with compact supports, $H_1(X;\mathbb{Z})$ and $K_1(X)$ coincide with the direct limits over the connected finite sub-complexes of X, of the corresponding groups. Since the direct limit over a directed set is an exact functor, it preserves injectivity (see thm. 2.18 in [90]). This shows that β_1^X is injective for any X, and completes the proof. □

The following result follows immediately from theorem 3.7.1 and from proposition 3.7.3. (It generalizes proposition 3.2.4.)

3.7.4 Theorem. *For a connected CW-complex X of dimension ≤ 4, there are natural short exact sequences*

$$0 \longrightarrow H_1(X;\mathbb{Z}) \longrightarrow K_1(X) \xrightarrow{ch_3^{\mathbb{Z}}} H_3(X;\mathbb{Z}) \longrightarrow 0.$$

and

$$0 \longrightarrow H_2(X;\mathbb{Z}) \longrightarrow \widetilde{K}_0(X) \xrightarrow{ch_4^{\mathbb{Z}}} H_4(X;\mathbb{Z}) \longrightarrow 0.$$

The latter sequence splits and yields an (abstract) isomorphism

$$ch_0^{\mathbb{Z}} \oplus ch_2^{\mathbb{Z}} \oplus ch_4^{\mathbb{Z}} : K_0(X) \xrightarrow{\cong} H_0(X;\mathbb{Z}) \oplus H_2(X;\mathbb{Z}) \oplus H_4(X;\mathbb{Z}).$$

The second short exact sequence is also valid and natural for X of dimension 5.

3.8 Non-injectivity of β_2^X

We summarize the injectivity results we have so far for β_2^X, and we show by a simple example that it is generally *not* injective.

We have seen in proposition 3.3.3 that β_2^X is split-injective for CW-complexes of dimension ≤ 3 and for simply-connected CW-complexes of dimension 4. We can improve this result as follows. We have proved in proposition 3.7.3 that it is injective if and only if no non-zero differential reaches $E_{2,0}^{\geq 2}$ in the Atiyah-Hirzebruch spectral sequence for $\widetilde{K}_*(X)$. This implies in particular that it is injective if X has its reduced integral homology concentrated in even degree, except possibly for H_1 and H_3. This applies to CW-complexes of dimension 4.

The "Künneth-type" lemma 3.6.3 allows for constructing an example of a finite CW-complex for which β_2^X is not injective.

We claim that the map $\beta_2^{\mathbb{R}P^2 \times \mathbb{R}P^4}$ is *not* injective. For $\mathbb{R}P^2$, one has

$$\beta_1^{\mathbb{R}P^2} : \mathbb{Z}/2 \cong H_1(\mathbb{R}P^2; \mathbb{Z}) \xrightarrow{\cong} K_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\,, \ 1 \longmapsto 1\,,$$

and for $\mathbb{R}P^4$,

$$\beta_1^{\mathbb{R}P^4} : \mathbb{Z}/2 \cong H_1(\mathbb{R}P^4; \mathbb{Z}) \hookrightarrow K_1(\mathbb{R}P^4) \cong \mathbb{Z}/4\,, \ 1 \longmapsto 2$$

(where we write the groups $\mathbb{Z}/n$ additively). It follows that

$$\beta_1^{\mathbb{R}P^2} \otimes \beta_1^{\mathbb{R}P^4} : \mathbb{Z}/2 \cong \mathbb{Z}/2 \otimes \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \otimes \mathbb{Z}/4 \cong \mathbb{Z}/2\,, \ 1 = 1 \otimes 1 \longmapsto 1 \otimes 2 = 0\,.$$

Now, the non-injectivity of $\beta_2^{\mathbb{R}P^2 \times \mathbb{R}P^4}$ follows from lemma 3.6.3, as claimed.

Since in the sequel we will mainly be interested in classifying spaces of countable discrete groups, it is worth mentioning that the same argument shows that the map $\beta_2^{\mathbb{R}P^\infty \times \mathbb{R}P^\infty}$ is *not* injective. The space $\mathbb{R}P^\infty \times \mathbb{R}P^\infty$ is nothing but $B(\mathbb{Z}/2 \oplus \mathbb{Z}/2)$.

Chapter 4

An introduction to the Baum-Connes conjecture

We quickly review the basic notions related to the Baum-Connes conjecture, and fix our notations for the sequel. These include the classifying space for proper actions of a group Γ, equivariant K-homology, the maximal and reduced C^*-algebras of Γ, the algebra $\ell^1\Gamma$, the Baum-Connes assembly map, the Baum-Connes conjecture and a conjecture sometimes called the "Bost conjecture", and the delocalized equivariant Chern character. We also quote some consequences of the Baum-Connes conjecture in different areas of mathematics, and some classes of groups for which the conjecture has been proved. This chapter contains no new result and has no pretension in completeness.

4.1 Introduction

Let Γ be a countable discrete group. The Baum-Connes conjecture is the statement that the analytical assembly map (or Baum-Connes assembly map, or Γ-index map)

$$\mu_*^\Gamma : K_*^\Gamma(\underline{E\Gamma}) \longrightarrow K_*(C_r^*\Gamma)$$

is an isomorphism. The left-hand side is the K-homology with compact supports of the classifying space for proper actions of Γ. It is a group of geometric nature associated to Γ, whose precise definition will be given later in this chapter. The right-hand side is the K-theory of the reduced C^*-algebra of Γ. The goal of the conjecture is to express the latter K-theory group, which is a rather complicated object of analytical nature, in simpler terms, namely by means of geometry and topology. The rational Baum-Connes conjecture is the weaker statement that the assembly map $\mu_*^\Gamma \otimes Id_{\mathbb{C}}$ is an isomorphism.

The Baum-Connes conjecture has many implications in topology, geometry, analysis and algebra. For example, it implies the Novikov conjecture, the stable Gromov-Lawson-Rosenberg conjecture, the conjecture of idempotents (absence of non-trivial idempotents in $\mathbb{C}\Gamma$ for Γ torsion-free), and the more general Kaplansky-Kadison conjecture (absence of non-trivial idempotents in $C_r^*\Gamma$ for Γ torsion-free); see section 4.7 for some details.

The origin of the Baum-Connes conjecture is the work of Kasparov on the index of Γ-invariant elliptic differential operators on Γ-vector bundles over a Γ-compact manifold M: He proved in [60] that there is a well-defined notion of index of such an operator D, with value in $K_*(C_r^*\Gamma)$. The pair (M, D) also defines an element of $K_*^\Gamma(\underline{E\Gamma})$. The assembly map μ_*^Γ is defined so that

$$\mu_*^\Gamma(M, D) = \text{Index}(D).$$

The surjectivity of μ_*^Γ means that any K-theory class of the reduced C^*-algebra of Γ arises in this way, and the injectivity that it arises essentially in a unique way.

Our references for this chapter are [6], [7], [107] and [108].

Throughout this chapter, we assume all maps to be pointed and continuous.

This chapter is organized as follows: In section 4.2, we review proper actions and classifying spaces for proper actions. In section 4.3, we recall briefly the definition of equivariant K-homology; this gives a description of the domain of the Baum-Connes assembly map, namely the group $K_*^\Gamma(\underline{E\Gamma})$. The group algebras $C^*\Gamma$, $C_r^*\Gamma$ and $\ell^1\Gamma$, and their K-theory are the subject of section 4.4. In section 4.5, we compare the Baum-Connes assembly map with the Novikov assembly map, we state the Baum-Connes and the strong Novikov conjectures, as well as the so-called "Bost conjecture". We also discuss the functoriality of the Baum-Connes assembly map. The delocalized equivariant Chern character is introduced in section 4.6, and we briefly describe the consequences in various areas of the Baum-Connes conjecture in section 4.7. Finally, in section 4.8 we quote some classes of groups for which the conjecture is proved.

4.2 Classifying spaces for proper actions

We recall the definition of a proper action of a discrete group, and give some classical results for this kind of action. We then define the classifying space for proper actions of a given group. For details on the subject of this section, the reader can refer to [7] and to [108].

Let Γ denote a countable discrete group. Let X denote a Hausdorff space on which Γ acts continuously, in other words, by homeomorphisms (on the left, say). (In this section, we assume all spaces to be Hausdorff.)

4.2.1 Definition. *The action of the group Γ on X is proper if for every pair of points x, $y \in X$, there exist neighbourhoods U_x and U_y of x and y in X respectively, such that the set*

$$\{\gamma \in \Gamma \mid \gamma \cdot U_x \cap U_y \neq \varnothing\}$$

is finite. Then X is called a proper Γ-space.

Since Γ is discrete, this amounts to requiring that the action has a Hausdorff quotient, and that for any $x \in X$, there exists a triple (U, H, ρ), where U is a Γ-invariant neighbourhood of x in X, H is a *finite* subgroup of Γ, and $\rho : U \longrightarrow \Gamma/H$ is a Γ-equivariant map (see p. 25 in [62]).

The following proposition is classical.

4.2.2 Proposition. *i) If $p : X \longrightarrow Y$ is a covering space with X and Y Hausdorff and structure group Γ, then the action of Γ on X is proper and free.*

ii) If Γ is finite, any action on a Hausdorff space is proper.

iii) If Γ acts simplicially on a simplicial complex X, then the action is proper if and only if the stabilizers of the vertices are finite.

In the framework of locally compact spaces, there is a simpler criterion to determine whether or not an action is proper, as the following theorem shows (see [81]).

4.2.3 Theorem. *If X is locally compact, the action of Γ on X is proper if and only if for any pair of compact subsets K and L of X, the set*

$$\{\gamma \in \Gamma \mid \gamma \cdot K \cap L \neq \varnothing\}$$

is finite. It is also equivalent to requiring that the action map

$$\Gamma \times X \longrightarrow X \times X\,,\ (\gamma, x) \longmapsto (\gamma \cdot x, x)$$

is proper, i.e. the inverse image of any compact subspace is compact.

4.2.4 Definition. *A subspace Y of a proper Γ-space X is Γ-compact (resp. Γ-paracompact) if it is a Γ-subspace of X and if the orbit space $\Gamma\backslash Y$ is compact (resp. paracompact).*

The following result is well-known.

4.2.5 Proposition. *Let X be a proper Γ-space. Then*

i) The orbit space $\Gamma\backslash X$ is Hausdorff.

ii) If X is metrizable, then so is $\Gamma\backslash X$.

iii) If X is locally compact, then so is $\Gamma\backslash X$.

iv) If X is Γ-compact, then X is locally compact.

We now pass to the definition of the classifying space for proper actions of a given group.

4.2.6 Definition. *A proper Γ-space X is universal if it satisfies the following universal property: For every proper Γ-space Y, there exists a continuous Γ-equivariant map $Y \longrightarrow X$, that is unique up to Γ-equivariant homotopy. Such a space, if it exists, is unique up to Γ-equivariant homotopy equivalence, and is denoted by $\underline{E\Gamma}$ (or by $\underline{E}\Gamma$). It is also called the classifying space for proper actions of Γ. The orbit space $\Gamma\backslash\underline{E\Gamma}$ is denoted by $\underline{B\Gamma}$ (or by $\underline{B}\Gamma$), and is uniquely determined up to homotopy equivalence; it is called the classifying space for proper Γ-bundles.*

We will see that $\underline{E\Gamma}$ always exists. In fact, there are metrizable Γ-paracompact models, and also models that are Γ-CW-complexes. There are very general existence theorems for this kind of classifying spaces (see thm. I.6.6 in [105]). As a standard abuse of notation, when we write $\underline{E\Gamma} = X$, we mean that a given proper Γ-space X is a model for $\underline{E\Gamma}$. Let us give a few examples.

Examples.

i) If H is a subgroup of Γ, then $\underline{E\Gamma}$ is a model for $\underline{EH}$.

ii) If Γ is a torsion-free group, every proper action of Γ is free, and we have $\underline{E\Gamma} = E\Gamma = \widetilde{B\Gamma}$, the universal covering space of the classifying space $B\Gamma$ of Γ, and $\underline{B\Gamma} = B\Gamma$.

iii) If $\Gamma = \mathbb{Z}^n$, then $\underline{E\Gamma} = \mathbb{R}^n$ and $\underline{B\Gamma} = \mathbb{T}^n$, the n-torus. For Γ finite, we have $\underline{E\Gamma} = \underline{B\Gamma} = pt$.

iv) If $\Gamma = \Gamma_g$, the fundamental group of an orientable surface Σ_g of genus g, then $\underline{E\Gamma} = \mathbb{R}^2$ (or the hyperbolic upper half-plane) and $\underline{B\Gamma} = \Sigma_g$.

v) If Γ is a discrete subgroup of a Lie group G with finitely many connected components, then $\underline{E\Gamma} = G/K$ and $\underline{B\Gamma} = \Gamma\backslash G/K$, where K is a maximal compact subgroup of G. If moreover Γ is torsion-free, then $\underline{B\Gamma} = B\Gamma$ is a manifold.

vi) If Γ acts simplicially on a tree T with finite stabilizers of vertices, then $\underline{E\Gamma} = T$. This applies to any amalgamated product $\Gamma_1 *_H \Gamma_2$ with Γ_1, Γ_2 and H finite, as for example $SL_2(\mathbb{Z}) = (\mathbb{Z}/4) *_{\mathbb{Z}/2} (\mathbb{Z}/6)$.

vii) The space $E(\mathcal{F}in) = E_{\mathcal{F}in}(\Gamma)$ studied in [105] and [63] is a (generally non-metrizable) model for $\underline{E\Gamma}$ (see the end of the present section).

viii) Any model for $\underline{E}\Gamma$ is a contractible space, but *not* Γ-equivariantly (except for Γ finite).

The next proposition is an important result allowing to recognize when a given proper Γ-space is a model for $\underline{E}\Gamma$. It is proved in [7].

4.2.7 Proposition. *A metrizable proper Γ-space X with paracompact orbit space $\Gamma\backslash X$ is universal if and only if the following two conditions are satisfied:*

i) For every finite subgroup H of Γ, there exists $x \in X$ stabilized by H, i.e. such that $H \cdot x = \{x\}$.

ii) The two projection maps $p_1, p_2 : X \times X \longrightarrow X$ are Γ-equivariantly homotopic.

This proposition allows to prove that a metrizable Γ-paracompact universal proper Γ-space exists for any countable discrete group Γ. Indeed, the set

$$\left\{ \mu : \Gamma \longrightarrow [0,\,1] \;\middle|\; \mathrm{supp}(\mu) \text{ is finite and } \sum_{\gamma\in\Gamma} \mu(\gamma) = 1 \right\}$$

of finitely supported probability measures on Γ, with the metric of uniform convergence $\|\mu - \nu\|_\infty = \sup_{\gamma\in\Gamma} |\mu(\gamma) - \nu(\gamma)|$, is a model for $\underline{E}\Gamma$. We call it the standard model for $\underline{E}\Gamma$, and denote it by $\underline{E}\Gamma^{std}$. More geometrically, $\underline{E}\Gamma^{std}$ is the geometric realization of the simplicial set of nonempty finite subsets of Γ, or equivalently the standard simplex in the real group algebra $\mathbb{R}\Gamma$ of Γ (in both cases, with the topology of uniform convergence). The action of Γ is defined by $\gamma \cdot \mu(\gamma') := \mu(\gamma^{-1}\gamma')$.

Since any free and proper action is proper and since $E\Gamma$ classifies free and proper actions of Γ, there exists a Γ-equivariant map $E\Gamma \longrightarrow \underline{E}\Gamma$ that is unique up to Γ-equivariant homotopy, and fitting in a commutative diagram

$$\begin{array}{ccc} E\Gamma & \longrightarrow & \underline{E}\Gamma \\ \downarrow & & \downarrow \\ B\Gamma & \longrightarrow & \underline{B}\Gamma \end{array}$$

In chapter 6, we will need explicit descriptions of these maps. For this purpose we now describe Segal's model for the classifying space for a family $\mathcal{F}$ of subgroups of Γ. For details, the reader is referred to [105] pp. 46-50, and to [68]. Let $\mathcal{F}$ be a non-empty family of subgroups of Γ that is closed under taking subgroups and conjugates, as for example $\mathcal{T}$, the trivial family consisting in the trivial subgroup $\{e\}$, and $\mathcal{F}in$, the family of finite subgroups. Consider the Γ-set

$$\Delta_{\mathcal{F}} := \coprod_{H\in\mathcal{F}} \Gamma/H$$

(the disjoint union of the corresponding cosets, with action of Γ by left translation). Let $X_\bullet^{\mathcal{F}} = \{X_n^{\mathcal{F}}\}_{n\geq 0}$ be the simplicial Γ-set of tuples of elements of $\Delta_{\mathcal{F}}$, more precisely

$$X_n^{\mathcal{F}} = (\Delta_{\mathcal{F}})^{n+1},$$

with the obvious face and degeneracy maps. Segal's model for the classifying space for the family $\mathcal{F}$ is the geometric realization of $X_\bullet^{\mathcal{F}}$:

$$E_{\mathcal{F}}(\Gamma) := |X_\bullet^{\mathcal{F}}|.$$

It is clearly a Γ-CW-complex with one n-cell for each $(n+1)$-tuple of elements of $\Delta_{\mathcal{F}}$. Its main properties are that the cell stabilizers are members of $\mathcal{F}$, and that for any $H \in \mathcal{F}$, the fixed point set $E_{\mathcal{F}}(\Gamma)^H$ is contractible. It follows that $E_{\mathcal{F}}(\Gamma)$ is always contractible, that $E_{\mathcal{T}}(\Gamma)$ is a model for $E\Gamma$, and that $E_{\mathcal{F}in}(\Gamma)$ is a model for $\underline{E\Gamma}$. The space $E_{\mathcal{F}}(\Gamma)$ is metrizable if and only if it is locally finite (see prop. 1.5.17 in [44]), but this is never the case. For example, for the trivial group, one has

$$E_{\mathcal{F}}(\{e\}) = \mathrm{e}^0 \cup \mathrm{e}^1 \cup e^2 \cup \ldots = S^\infty .$$

The map $E\Gamma = E_{\mathcal{T}}(\Gamma) \longrightarrow \underline{E\Gamma} = E_{\mathcal{F}in}(\Gamma)$ we are looking for is "simplicially induced" by the canonical inclusion of Γ-sets

$$\Delta_{\mathcal{T}} = \Gamma/\{e\} = \Gamma \hookrightarrow \Delta_{\mathcal{F}in} = \coprod_{H\in\mathcal{F}in} \Gamma/H .$$

In chapter 6, we will also need information on the cellular chain complex of $E_{\mathcal{F}}(\Gamma)$. Let R be a commutative ring with unit. The cellular chain complex of $E_{\mathcal{F}}(\Gamma)$ with coefficients in R is given by

$$C_\bullet(E_{\mathcal{F}}(\Gamma);\, R) : \quad \ldots \longrightarrow C_1(E_{\mathcal{F}}(\Gamma);\, R) \xrightarrow{d_1} C_0(E_{\mathcal{F}}(\Gamma);\, R) \longrightarrow 0 \longrightarrow \ldots$$

where $C_n(E_{\mathcal{F}}(\Gamma);\, R) = \left(\bigoplus_{H\in\mathcal{F}} R[\Gamma/H]\right)^{\otimes_R n+1}$, and the map d_1 is

$$d_1 : \lambda\cdot\gamma H \otimes \lambda'\cdot\gamma' H' = \lambda\lambda'(\gamma H\otimes\gamma' H') \longmapsto \lambda\lambda'(\gamma H - \gamma' H') = \lambda\lambda'\cdot\gamma H - \lambda\lambda'\cdot\gamma' H'$$

($\lambda,\, \lambda' \in R$, $\gamma,\, \gamma' \in \Gamma$, and $H,\, H' \in \mathcal{F}$). Since $E_{\mathcal{F}}(\Gamma)$ is contractible, its homology is concentrated in degree 0, isomorphic to R, and the sequence of $R\Gamma$-modules

$$\ldots \longrightarrow C_1(E_{\mathcal{F}}(\Gamma);\, R) \xrightarrow{d_1} C_0(E_{\mathcal{F}}(\Gamma);\, R) \xrightarrow{\varepsilon} R \longrightarrow 0$$

is exact, where ε is the augmentation

$$\varepsilon : \sum_{H\in\mathcal{F}} \sum_{\gamma H\in\Gamma/H} \lambda_{\gamma H}\cdot\gamma H \longmapsto \sum_{H\in\mathcal{F}} \sum_{\gamma H\in\Gamma/H} \lambda_{\gamma H} \in R .$$

4.3 Equivariant K-homology

We briefly review the definition of the equivariant K-homology of a locally compact proper Γ-space.

Let X denote a locally compact proper Γ-space. Let $C_0(X)$ be the C^*-algebra of continuous complex valued functions on X, vanishing at infinity (i.e. for every positive ε, there exists a compact subset K of X, such that the function is in module $\leq \varepsilon$ outside K).

A generalized elliptic Γ-operator over X (or Kasparov triple over X, or cycle over X) is a triple $(\mathcal{H}, \pi, F)$, where $\mathcal{H}$ is a Hilbert space endowed with a unitary representation U of Γ, π is a $*$-representation of $C_0(X)$ by bounded operators on $\mathcal{H}$ (in particular, $\pi(\bar{f}) = \pi(f)^*$), and F is a bounded self-adjoint operator on $\mathcal{H}$. It is moreover assumed that the representation π is covariant (with respect to U), in other words

$$\pi(f \circ \gamma^{-1}) = U_\gamma \pi(f) U_\gamma^*, \ \forall f \in C_0(X), \ \forall \gamma \in \Gamma,$$

and that, for any $f \in C_0(X)$ and any $\gamma \in \Gamma$, the operators

$$\pi(f)[U_\gamma, F], \ \pi(f)(F^2 - 1) \ \text{and} \ [\pi(f), F]$$

are compact. As shown in [107], we can always assume that F is Γ-equivariant, i.e. $U_\gamma F = F U_\gamma$, for any $\gamma \in \Gamma$. Such a Kasparov triple $(\mathcal{H}, \pi, F)$ is even if the Hilbert space $\mathcal{H}$ is $\mathbb{Z}/2$-graded and if U and π preserve the grading, whereas F reverses it. It is odd otherwise. If X is compact (and then Γ is finite), in the above definition of a Kasparov triple, the last two conditions can be weakened as to become: $(F^2 - 1)$ and $[\pi(f), F]$ are compact, for any $f \in C_0(X)$. (In particular, F is a Fredholm operator.) There is an obvious notion of direct sum of two Kasparov triples α and β of the same parity; it is denoted by $\alpha \oplus \beta$.

A cycle $(\mathcal{H}, \pi, F)$ is said to be *degenerate* if $\pi(f)(F^2 - 1) = 0$ and $[\pi(f), F] = 0$, for any $f \in C_0(X)$. Two cycles $\alpha_0 = (\mathcal{H}, \pi, F)$ and $\alpha_1 = (\mathcal{H}', \pi', F')$ over X are called *homotopic* if $\mathcal{H} = \mathcal{H}'$, if the corresponding unitary representations U and U' are equal, if $\pi = \pi'$, and if there exists a norm-continuous path $(F_t)_{t \in [0,1]}$ of operators connecting F to F' such that the triple $\alpha_t = (\mathcal{H}, \pi, F_t)$ is a cycle of the same parity, for any $t \in [0, 1]$. Finally, two cycles α and β of the same parity are defined to be *equivalent*, and we write $\alpha \sim \beta$, if there exists two degenerate cycles α' and β' (still of the same parity), such that $\alpha \oplus \alpha'$ is homotopic to $\beta \oplus \beta'$. The equivalence class of a cycle $(\mathcal{H}, \pi, F)$ is denoted by $[\mathcal{H}, \pi, F]$.

4.3.1 Definition. *The set of equivalence classes of even cycles over X is denoted by $K_0^\Gamma(X)$, and the set of equivalence classes of odd cycles by $K_1^\Gamma(X)$.*

It turns out that the direct sum of cycles (of the same parity) endows $K_0^\Gamma(X)$ and $K_1^\Gamma(X)$ with a structure of abelian group, and we set

$$K_*^\Gamma(X) := K_0^\Gamma(X) \oplus K_1^\Gamma(X).$$

It is called the Γ-equivariant K-homology of X.

The groups $K_i^\Gamma(X)$ depend functorially on X in the following sense: If $f : X \longrightarrow Y$ is a continuous, proper and Γ-equivariant map between two locally compact proper Γ-spaces, then it induces a group homomorphism

$$f_* : K_i^\Gamma(X) \longrightarrow K_i^\Gamma(Y),\ [\mathcal{H}, \pi, F] \longmapsto [\mathcal{H}, \pi \circ f^*, F],$$

where $f^* : C_0(Y) \longrightarrow C_0(X)$ is the $*$-homomorphism given by $\phi \longmapsto \phi \circ f$. If f and g are two continuous, proper Γ-equivariant maps that are Γ-homotopic (i.e. Γ-equivariantly homotopic), then $f_* = g_*$. (This is a deep result of Kasparov.)

4.3.2 Remark. *The group $K_*^\Gamma(X)$ we have just defined coincides with the Kasparov KK-group $KK_*^\Gamma(C_0(X), \mathbb{C})$.*

We now define equivariant K-homology with compact supports.

4.3.3 Definition. *Let X be a Hausdorff Γ-space. The Γ-equivariant K-homology with compact supports (or representable Γ-equivariant K-homology) of X is the group*

$$RK_*^\Gamma(X) = RK_0^\Gamma(X) \oplus RK_1^\Gamma(X) := \varinjlim_{Y} K_*^\Gamma(Y),$$

where the direct limit runs over all Γ-compact subspaces Y of X. Since throughout this work we will only consider K-homology with compact supports, we will simply denote it by $K_*^\Gamma(X)$. If Γ is the trivial group, we denote $K_*^\Gamma(X)$ simply by $K_*(X)$ (this makes sense for any Hausdorff space X).

Notice that this is well-defined, since given two Γ-compact subspaces $Y_1 \subseteq Y_2$ of X (which is Hausdorff), the inclusion $i : Y_1 \hookrightarrow Y_2$ induces a homomorphism $i_* : K_*^\Gamma(Y_1) \longrightarrow K_*^\Gamma(Y_2)$. Indeed, Y_1 and Y_2 are closed in X, so i is a proper continuous Γ-equivariant map.

(A more accurate denomination for representable Γ-equivariant K-homology would be "K-homology with co-compact (or Γ-compact) supports". We will however follow the most commonly used terminology.)

For Hausdorff spaces, the K-homology with compact supports considered here coincides with the one studied in chapter 3. More precisely, there is a canonical natural equivalence between both functors. We will identify them.

As expected, Γ-equivariant K-homology with compact supports is functorial for proper Γ-equivariant maps between Hausdorff Γ-spaces. In the sequel, when we refer to the functoriality of K_i^Γ, it is always with respect to this kind of maps.

The following proposition is important.

4.3.4 Proposition. *If Γ acts freely on a proper Γ-space X (as for example when Γ is torsion-free), then, for $i = 0, 1$, there is a canonical and natural isomorphism*

$$K_i^\Gamma(X) \xrightarrow{\cong} K_i(\Gamma\backslash X).$$

In particular, there is a canonical isomorphism $K_*^\Gamma(E\Gamma) \xrightarrow{\cong} K_*(B\Gamma)$. It is natural in the group Γ.

A proof (together with an explicit construction of the isomorphism) can be found in [107]. Notice that we do *not* need to assume X to be locally compact, because we are working with compact supports.

At this point, we see how the left-hand side in the Baum-Connes conjecture, namely $K_*^\Gamma(\underline{E\Gamma})$, is defined. Since equivariant K-homology is invariant under Γ-homotopies, after the remark ending section 4.2, we see that there are canonical maps

$$K_*(B\Gamma) \xrightarrow{\cong} K_*^\Gamma(E\Gamma) \longrightarrow K_*^\Gamma(\underline{E\Gamma}).$$

In particular, if Γ is *torsion-free*, then there is a canonical isomorphism

$$K_*(B\Gamma) \xrightarrow{\cong} K_*^\Gamma(\underline{E\Gamma}).$$

An important feature of the K-homology group $K_*^\Gamma(\underline{E\Gamma})$ is that it is functorial in Γ. Indeed, as shown by Alain Valette in [107], a group homomorphism $\varphi : \Gamma_1 \longrightarrow \Gamma_2$ induces a homomorphism

$$\varphi_* : K_*^{\Gamma_1}(\underline{E\Gamma}_1) \longrightarrow K_*^{\Gamma_2}(\underline{E\Gamma}_2).$$

Let us mention that this functoriality is not at all obvious, and that the proof given in [107] splits the problem into two parts: first, the case of a surjection, and then the case of an injection (the general case readily following). As a particular application, one can show that the map $K_*(B\Gamma) \longrightarrow K_*^\Gamma(\underline{E\Gamma})$ is natural in the group Γ.

Examples.

i) Let H be a finite subgroup of a countable discrete group Γ. Let ρ denote a unitary representation of H on a finite dimensional complex vector space V_ρ. We will associate to these data a Kasparov triple $(\mathcal{H}, \pi, F)$ over $X_H := \Gamma/H$, and representing an element in $K_0^\Gamma(\underline{E\Gamma})$. First, X_H identifies with the subset of $\underline{E\Gamma}^{std}$ consisting of uniform probability measures on the left cosets of H in Γ. (For $H = \{e\}$, this is the set of Dirac measures.) Let

$$\mathcal{H} := \{\xi \in \ell^2(\Gamma, V_\rho) \,|\, \xi(\gamma{\cdot}h) = \rho(h^{-1})\xi(\gamma),\ \forall h \in H,\ \forall \gamma \in \Gamma\} \cong \ell^2\Gamma \otimes_{\mathbb{C}H} V_\rho,$$

the Hilbert space of the induced representation $U := \mathrm{Ind}_H^\Gamma(\rho)$, endowed with the trivial $\mathbb{Z}/2$-graduation. This is also the Hilbert space of ℓ^2-sections of the induced vector bundle $\Gamma \times_H V_\rho$ over X_H. Let π denote the action of $C_0(X_H)$ by pointwise multiplication on sections; this is clearly a Γ-covariant representation. For each $f \in C_0(X_H)$, one checks that $\pi(f)$ is a compact operator. Finally, take $F = 0$, the zero operator on $\mathcal{H}$. This defines

$$x_{H,\rho} := [\mathcal{H}, \pi, F] \in K_0^\Gamma(\underline{E\Gamma}).$$

ii) For a finite group Γ, one has $\underline{E\Gamma} = pt$, and one shows that there are isomorphisms

$$K_0^\Gamma(\underline{E\Gamma}) \cong R(\Gamma) \text{ and } K_1^\Gamma(\underline{E\Gamma}) = 0,$$

where $R(\Gamma)$ is (the underlying additive group of) the complex representation ring of the group Γ. The first isomorphism is realized by mapping a representation ρ of Γ to $x_{\Gamma,\rho} \in K_0^\Gamma(\underline{E\Gamma})$, as in example i).

iii) For $\Gamma = \mathbb{Z}$, one has $\underline{E\mathbb{Z}} = \mathbb{R}$ (it is $\mathbb{Z}$-compact), and one verifies that

$$K_0^{\mathbb{Z}}(\underline{E\mathbb{Z}}) = \mathbb{Z} \cdot x_{\{e\},1} \cong \mathbb{Z} \text{ and } K_1^{\mathbb{Z}}(\underline{E\mathbb{Z}}) = \mathbb{Z} \cdot y \cong \mathbb{Z},$$

where $x_{\{e\},1}$ is "induced from" the trivial representation of the trivial subgroup of $\mathbb{Z}$ (as in example i)), and y is the "Toeplitz" generator, i.e. the class of the Kasparov triple $(L^2(S^1), M, F)$, where $U_m(\xi)(z) := \xi(z^m)$, for $m \in \mathbb{Z}$, $\xi \in L^2(S^1)$ and $z \in S^1$, and where M is the representation of $C(S^1)$ by pointwise multiplication on $L^2(S^1)$, and F is the operator $\mathrm{Diag}(\mathrm{sign}(n))_{n\in\mathbb{Z}}$ in the trigonometric basis $(e^{2\pi i n\theta})_{n\in\mathbb{Z}}$ of $L^2(S^1)$.

4.4 The Banach algebras $C^*\Gamma$, $C_r^*\Gamma$ and $\ell^1\Gamma$

The three unital Banach algebras $C^*\Gamma$, $C_r^*\Gamma$ and $\ell^1\Gamma$ are suitable completions of the complex group algebra $\mathbb{C}\Gamma$. The first two are C^*-algebras, called the maximal and reduced C^*-algebras of Γ respectively, and are related by a canonical surjective $*$-homomorphism

$$\lambda_\Gamma : C^*\Gamma \twoheadrightarrow C_r^*\Gamma.$$

One of the equivalent definitions of amenability of the discrete group Γ is that this map is an isomorphism (see also section 4.8). There are also injective unital Banach algebra morphisms

$$j^\Gamma : \ell^1\Gamma \hookrightarrow C^*\Gamma \text{ and } j_r^\Gamma : \ell^1\Gamma \hookrightarrow C_r^*\Gamma.$$

The maximal C^*-algebra and the Banach algebra $\ell^1\Gamma$ depend functorially on the group (and j^Γ is natural), but the reduced C^*-algebra does *not*, as explicit examples show. (The inclusion of the trivial group in Γ produces a copy of $\mathbb{C}$ as a direct summand in $C_r^*\Gamma$ if and only if Γ is amenable.)

The analytical K-theory of a unital Banach algebra A is defined by

$$K_0(A) := K_0^{alg}(A)$$

(the algebraic K_0-group of the ring A), and, for $j \geq 1$,

$$K_j(A) := \pi_{j-1}(\mathrm{GL}(A)) \cong \pi_j(B\mathrm{GL}(A)),$$

where $\mathrm{GL}(A)$ (also denoted by $\mathrm{GL}_\infty(A)$) is defined as the direct limit of the topological groups $\mathrm{GL}_n(A)$ (with topology inherited from the Banach algebra $\mathbb{M}_n(A)$). It is worth mentioning that these groups do *not* coincide with higher algebraic K-groups (as for example $K_1^{alg}(\mathbb{C}) \cong \mathbb{C}^\times$, whereas $K_1(\mathbb{C}) = 0$). By Bott periodicity, there are canonical isomorphisms

$$K_{j+2}(A) \cong K_j(A), \text{ for all } j \geq 0.$$

This means that analytical K-theory of (complex) Banach algebras is 2-periodic, and we write $K_*(A) = K_0(A) \oplus K_1(A)$. The K-groups are functorial for unital Banach algebra morphisms. We have in particular group homomorphisms

$$(j_r^\Gamma)_* : K_*(\ell^1\Gamma) \xrightarrow{j_*^\Gamma} K_*(C^*\Gamma) \xrightarrow{(\lambda_\Gamma)_*} K_*(C_r^*\Gamma).$$

We now have an idea of what is the right-hand side in the Baum-Connes conjecture, namely $K_*(C_r^*\Gamma)$.

Let us illustrate this by a few simple examples. If Γ is finite, then the three algebras $\ell^1\Gamma$, $C^*\Gamma$ and $C_r^*\Gamma$ coincide (as abstract algebras) with the complex group algebra $\mathbb{C}\Gamma$, and $C^*\Gamma = C_r^*\Gamma$ as C^*-algebras. However, if Γ is not the trivial group, then $\ell^1\Gamma$ does *not* coincide with them as a Banach algebra. If $\mathcal{A}$ denotes one of these Banach algebras, then one has $K_0(\mathcal{A}) \cong R(\Gamma)$, where $R(\Gamma)$ is (the underlying additive group of) the complex representation ring of Γ. On the other hand, one checks that $K_1(\mathcal{A}) = 0$. As a second example, when Γ is abelian, then $C^*\Gamma = C_r^*\Gamma \cong C(\hat{\Gamma})$, where $\hat{\Gamma} = \mathrm{Hom}(\Gamma, S^1)$ is the Pontryagin dual of the group Γ (with the compact-open topology, i.e. the topology of uniform convergence on finite subsets of Γ, for which $\hat{\Gamma}$ is compact), and where the isomorphism is given by Fourier transform. By the Swan-Serre theorem, we see that $K_*(C_r^*\Gamma) \cong K_*(C(\hat{\Gamma})) \cong K^*(\hat{\Gamma})$. In particular, $C^*\mathbb{Z}^n = C_r^*\mathbb{Z}^n \cong C(\mathbb{T}^n)$, with $K^0(\mathbb{T}^n) \cong \mathbb{Z}^{2^{n-1}} \cong K^1(\mathbb{T}^n)$. More explicitly, for $n = 1$, the generators of $K_0(C(S^1))$ and of $K_1(C(S^1))$ are $[1]$ and $[z]$, the K-theory classes of the unit and of the identity function z on the circle respectively.

4.5 The Baum-Connes and the Novikov assembly maps

We state the Baum-Connes and the strong Novikov conjectures, as well as a conjecture that some people call the Bost conjecture. We also see how they are related.

With the goal of computing $K_*(C_r^*\Gamma)$ in mind, Paul Baum and Alain Connes have defined a (graded) map

$$\mu_*^\Gamma : K_*^\Gamma(\underline{E\Gamma}) \longrightarrow K_*(C_r^*\Gamma),$$

called the analytic assembly map (or Baum-Connes assembly map, or Γ-index map), and were led to the following conjecture (see [6] and [7]):

4.5.1 Conjecture. (Baum-Connes) *For any countable discrete group* Γ, *the assembly map* $\mu_*^\Gamma : K_*^\Gamma(\underline{E\Gamma}) \longrightarrow K_*(C_r^*\Gamma)$ *is an isomorphism.*

The left hand-side is, as we have seen, of topological and geometrical nature, and is supposed to be simpler to compute (at least rationally, as we will see in section 4.6).

For the definition of μ_*^Γ, we refer to [107], where it is also proved that there exists an assembly map

$$\tilde{\mu}_*^\Gamma : K_*^\Gamma(\underline{E\Gamma}) \longrightarrow K_*(C^*\Gamma)$$

such that $\mu_*^\Gamma = (\lambda_\Gamma)_* \circ \tilde{\mu}_*^\Gamma$, where $(\lambda_\Gamma)_*$ is induced, at the level of K-theory, by the canonical surjection λ_Γ. It is also shown that the map $\tilde{\mu}_*^\Gamma$ is natural with respect to group homomorphisms.

Notice that the Baum-Connes conjecture (or even just the split-surjectivity of μ_*^Γ) would imply that $(\lambda_\Gamma)_*$ is split-surjective and that $K_*(C_r^*\Gamma)$ is functorial for group homomorphisms. These two potential consequences are still unproved. (About the former problem, let us mention that for K-amenable groups, $(\lambda_\Gamma)_*$ is always an isomorphism, and at the opposite, that for infinite groups with Kazhdan's property (T), it is never injective; see section 4.8 for the definition of property (T).)

It turns out that the composition map

$$\tilde{\beta}_*^\Gamma : K_*(B\Gamma) \xrightarrow{\cong} K_*^\Gamma(E\Gamma) \longrightarrow K_*^\Gamma(\underline{E\Gamma}) \xrightarrow{\tilde{\mu}_*^\Gamma} K_*(C^*\Gamma)$$

coincides with the Novikov assembly map considered by Mishchenko and Kasparov. They have formulated the following conjecture:

4.5.2 Conjecture. (Strong Novikov conjecture) *For any countable discrete group* Γ, *the Novikov assembly map*

$$\tilde{\beta}_*^\Gamma : K_*(B\Gamma) \longrightarrow K_*(C^*\Gamma)$$

is rationally injective, in other words it is an injection after tensoring with $\mathbb{Q}$ *(or equivalently* $\mathbb{C}$*).*

Notice that one can show that this conjecture implies the usual Novikov conjecture on the homotopy invariance of higher signatures in topology, hence its name (see also section 4.7).

We denote the composition $(\lambda_\Gamma)_* \circ \tilde{\beta}_*^\Gamma : K_*(B\Gamma) \longrightarrow K_*(C_r^*\Gamma)$ by β_*^Γ.

We now state a useful result allowing to relate the strong Novikov conjecture to the Baum-Connes conjecture (see [43], pp. 44-45).

4.5.3 Proposition. *For any countable discrete group* Γ, *the map*

$$K_*(B\Gamma) \longrightarrow K_*^\Gamma(E\Gamma) \longrightarrow K_*^\Gamma(\underline{E\Gamma})$$

is rationally injective.

(See also section 4.6 about this proposition; section 6.7 contains a proof.)

4.5.4 Corollary. *If the Baum-Connes assembly map* μ_*^Γ *is rationally injective, then the Novikov assembly map* $\tilde{\beta}_*^\Gamma$ *is also rationally injective, i.e.* Γ *satisfies the strong Novikov conjecture. (The rational injectivity of* $\tilde{\mu}_*^\Gamma$ *suffices.)*

As already mentioned, if Γ is *torsion-free*, the map $K_*(B\Gamma) \longrightarrow K_*^\Gamma(\underline{E\Gamma})$ is an isomorphism, and therefore, $\tilde{\beta}_*^\Gamma$ is rationally injective if and only if so is $\tilde{\mu}_*^\Gamma$.

In [64], Vincent Lafforgue has shown that the Baum-Connes assembly map further factorizes, more precisely, he defines an assembly map

$$\hat{\mu}_*^\Gamma : K_*^\Gamma(\underline{E\Gamma}) \longrightarrow K_*(\ell^1\Gamma)$$

such that $\mu_*^\Gamma = (j_r^\Gamma)_* \circ \hat{\mu}_*^\Gamma$, and $\tilde{\mu}_*^\Gamma = j_*^\Gamma \circ \hat{\mu}_*^\Gamma$. This map is also natural with respect to group homomorphisms. Apparently, Jean-Benoît Bost was led to the following conjecture (at least for discrete subgroups of connected Lie groups; see [98], p. 11):

4.5.5 Conjecture. *For any countable discrete group* Γ, *the assembly map*

$$\hat{\mu}_*^\Gamma : K_*^\Gamma(\underline{E\Gamma}) \longrightarrow K_*(\ell^1\Gamma)$$

is an isomorphism.

For simplicity, in the sequel, we will refer to this conjecture as the "Bost conjecture".

We see that if a group Γ satisfies at the same time the Baum-Connes conjecture and the Bost conjecture, then the inclusion $\ell^1\Gamma \hookrightarrow C_r^*\Gamma$ induces an isomorphism in K-theory.

Examples.

i) Let H be a finite subgroup of Γ. In example i) of section 4.3, we have associated to any finite dimensional complex representation (V_ρ, ρ) of H a K-homology class $x_{H,\rho} = [\mathcal{H}, \pi, 0] \in K_0^\Gamma(\underline{E\Gamma})$. If ρ is irreducible, then there exists a unique projector p_ρ in $\mathbb{C}H$ such that $V_\rho \cong \mathbb{C}H \cdot p_\rho$, as a left

$\mathbb{C}H$-module, and one has $\pi(C_c(\Gamma/H))\cdot\mathcal{H}\cong\mathbb{C}\Gamma\otimes_{\mathbb{C}H}V_\rho\cong\mathbb{C}\Gamma\cdot p_\rho$. Let ρ^* denote the dual (or contragredient) representation of ρ. The map

$$\mathbb{C}\Gamma\longrightarrow\mathbb{C}\Gamma\,,\ \xi=\sum_{\gamma\in\Gamma}\lambda_\gamma\gamma\longmapsto\check{\xi}:=\sum_{\gamma\in\Gamma}\lambda_\gamma^{-1}\gamma$$

takes p_ρ to $\check{p}_\rho=p_{\rho^*}$ and induces a bijection $\mathbb{C}\Gamma\cdot p_\rho\approx p_{\rho^*}\cdot\mathbb{C}\Gamma$ transforming the left $\mathbb{C}\Gamma$-module structure into the right $\mathbb{C}\Gamma$-module structure. This implies that

$$\tilde{\mu}_0^\Gamma(x_{H,\rho})=[p_{\rho^*}]\in K_0(C^*\Gamma)\ \text{ and }\ \mu_0^\Gamma(x_{H,\rho})=[p_{\rho^*}]\in K_0(C_r^*\Gamma)\,,$$

the K-theory class of p_{ρ^*}, viewed as a projector in the maximal (resp. reduced) C^*-algebra of Γ. In particular, if $H=\{e\}$ and ρ is the trivial 1-dimensional representation of H, then $\tilde{\mu}_0^\Gamma(x_{\{e\},1})=[1]\in K_0(C^*\Gamma)$, the K-theory class of the unit.

ii) If Γ is finite, the above consideration allows to prove explicitly that μ_0^Γ is an isomorphism. On the other hand, μ_1^Γ and $\hat{\mu}_1^\Gamma$ are obviously isomorphisms, since their domain and range are 0. Consequently, any finite group satisfies both the Baum-Connes and the Bost conjectures.

iii) For $\Gamma=\mathbb{Z}$, we have seen in example iii) of section 4.3 that $K_0^{\mathbf{Z}}(\underline{E\mathbb{Z}})$ and $K_1^{\mathbf{Z}}(\underline{E\mathbb{Z}})$ are infinite cyclic, generated by $x_{\{e\},1}$ and the Toeplitz generator y respectively. By example i), $\mu_0^\Gamma(x_{\{e\},1})=[1]\in K_0(C_r^*\mathbb{Z})$, showing that $\mu_0^{\mathbf{Z}}$ is an isomorphism. A more involved computation shows that if a denotes the generator of $\mathbb{Z}$, then $\mu_1^{\mathbf{Z}}(y)=-[a]=[a^{-1}]\in K_1(C_r^*\mathbb{Z})$, and this corresponds to $[z\mapsto z^{-1}]\in K_1(C(S^1))$ (see [107]). We see that under the canonical identifications $K_1^{\mathbf{Z}}(\underline{E\mathbb{Z}})\cong K_1(S^1)$ and $K_1(C_r^*\mathbb{Z})\cong K^1(S^1)$, the assembly map $\mu_1^{\mathbf{Z}}$ takes the Bott generator to the inverse of the Bott generator. In particular, $\mu_1^{\mathbf{Z}}$ is also an isomorphism. Therefore, $\mathbb{Z}$ satisfies the Baum-Connes conjecture. Similarly, one shows that the group $\mathbb{Z}^2$ satisfies the Baum-Connes conjecture, and that $\mu_0^{\mathbf{Z}^2}$ takes the Bott generator of $K_0^{\mathbf{Z}^2}(\underline{E\mathbb{Z}^2})\cong K_0(\mathbb{T}^2)\cong\mathbb{Z}^2$ to the Bott generator of $K_0(C_r^*\mathbb{Z}^2)\cong K_0(C(\mathbb{T}^2))\cong K^0(\mathbb{T}^2)\cong\mathbb{Z}^2$.

iv) By the Wiener lemma (see [91], 11.6), for any $n\geq 1$, $\ell^1\mathbb{Z}^n$ is stable under holomorphic functional calculus in $C_r^*\mathbb{Z}^n$. Therefore, by the density theorem (see [34], prop. 3, pp. 285-286), the inclusion $\ell^1\mathbb{Z}^n\hookrightarrow C_r^*\mathbb{Z}^n$ induces an isomorphism $K_*(\ell^1\mathbb{Z}^n)\cong K_*(C_r^*\mathbb{Z}^n)$. It follows from example iii) that the group $\mathbb{Z}$ satisfies the Bost conjecture, and that $\hat{\mu}_1^{\mathbf{Z}}(y)=[a^{-1}]\in K_1(\ell^1\mathbb{Z})$.

Let us also point out that there is a generalized version of the Baum-Connes conjecture, called the Baum-Connes conjecture with coefficients. The coefficients in question are C^*-algebras on which the group Γ acts by $*$-automorphisms.

4.6 The delocalized equivariant Chern character

We consider the usual Chern character in K-homology, and the delocalized equivariant Chern character defined by Paul Baum and Alain Connes, and explain their main properties and how they are related. We also introduce some notations that we will use in forthcoming chapters.

In section 3.1, we have explained that the usual Chern character in K-homology with compact supports induces an isomorphism $ch \otimes Id : K_*(X) \otimes \mathbb{C} \xrightarrow{\cong} H_*(X; \mathbb{C})$, for an arbitrary CW-complex X. In particular, for a discrete group Γ, we get an isomorphism

$$ch \otimes Id : K_*(B\Gamma) \otimes \mathbb{C} \xrightarrow{\cong} H_*(B\Gamma; \mathbb{C}) \cong H_*(\Gamma; \mathbb{C}) .$$

In [6], Baum and Connes have defined a new Chern character, called the delocalized equivariant Chern character. It is a group homomorphism

$$ch_*^\Gamma : K_*^\Gamma(\underline{E\Gamma}) \longrightarrow H_*(\Gamma; F\Gamma) ,$$

where Γ is a countable discrete group, and $F\Gamma$ is the $\mathbb{C}$-module freely generated by the set of torsion elements in Γ, on which Γ acts by conjugation. It maps K_0^Γ to H_{ev} and K_1^Γ to H_{odd}. The following two results are announced in [6] and in [7] (and are generally considered as well-known), but we do not know of any complete proof.

4.6.1 Theorem. *The delocalized equivariant Chern character ch_*^Γ is a rational isomorphism (i.e. an isomorphism after tensoring its domain with $\mathbb{C}$), and is natural with respect to group homomorphisms.*

4.6.2 Proposition. *For any countable discrete group Γ, there is a commutative diagram*

$$\begin{array}{ccc} K_*(B\Gamma) & \longrightarrow & K_*^\Gamma(\underline{E\Gamma}) \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch_*^\Gamma} \\ H_*(\Gamma; \mathbb{C}) & \xhookrightarrow{i_*} & H_*(\Gamma; F\Gamma) \end{array}$$

where we have identified $H_*(B\Gamma; \mathbb{C})$ with $H_*(\Gamma; \mathbb{C})$, and the bottom map is induced by the inclusion $i : \mathbb{C} \hookrightarrow F\Gamma$, $\lambda \longmapsto \lambda \cdot e$ of Γ-modules.

The above two results show that the map $K_*(B\Gamma) \longrightarrow K_*^\Gamma(\underline{E\Gamma})$ is rationally injective (this was the content of proposition 4.5.3). We also see that for a torsion-free group, $F\Gamma = \mathbb{C}$ and the Chern character ch_*^Γ "coincides" with the usual one.

4.6.3 Remark. *We have already mentioned that we do not know of any complete proof of the above theorem and proposition. There is however another major difficulty: The statement of the Baum-Connes conjecture presented here (and considered nowadays) is the one of [7], whereas in [6], the left-hand side was a group $K^*(pt,\Gamma)$ defined in a completely different fashion. The relation between $K_*^\Gamma(\underline{E}\Gamma)$ and $K^*(pt,\Gamma)$ (and between the two corresponding assembly maps) is far from clear. (See also sections 5.1.1 and 5.1.6 in J.-L. Tu's thesis [106] on this point.) Moreover, the Chern character ch_*^Γ was defined in [6], with domain $K^*(pt,\Gamma)$. We will therefore adopt the following point of view (shared with most workers in this field): We accept the existence of a delocalized equivariant Chern character with domain $K_*^\Gamma(\underline{E}\Gamma)$, and consider theorem 4.6.1 and proposition 4.6.2 as true. Section 6.7 contains partial proofs of these results. (Complete proofs can probably be deduced from the work of J. Davis, W. Lück and B. Oliver, see [37], [67] and [66].)*

Let $\langle\Gamma\rangle$ be the set of conjugacy classes of Γ, partitioned as

$$\langle\Gamma\rangle = \langle\Gamma\rangle^{ell} \amalg \langle\Gamma\rangle^{\infty},$$

where $\langle\Gamma\rangle^{ell}$ is the (non-empty) set of conjugacy classes of elliptic elements of Γ, i.e. of finite order, and $\langle\Gamma\rangle^{\infty}$ is the (possibly empty) set of conjugacy classes of hyperbolic elements, i.e. of infinite order. Let $\{\gamma_C\}_{C\in\langle\Gamma\rangle}$ be a once and for all chosen set of representatives of the conjugacy classes, i.e. $\gamma_C \in C$. Let us denote $Z_\Gamma(\gamma_C)$, the centralizer of γ_C in Γ, by Z_C. (There is a slight abuse of notation since Z_C does not only depend on the class C, but on its chosen representative γ_C. Changing the representative amounts to taking a conjugate of Z_C.) Notice tbat $Z_{\{e\}} = \Gamma$.

The Γ-module $F\Gamma$ decomposes canonically as a direct sum of sub-Γ-modules parameterized by the set of elliptic conjugacy classes. Therefore, by a direct application of the Shapiro lemma, we have another interpretation of the target of the delocalized equivariant Chern character ch_*^Γ:

4.6.4 Proposition. *For any discrete group Γ, there is a canonical graded isomorphism*

$$\nu : H_*(\Gamma; F\Gamma) \xrightarrow{\cong} \bigoplus_{C\in\langle\Gamma\rangle^{ell}} H_*(Z_C;\mathbb{C}),$$

where the latter are homology groups with trivial coefficients $\mathbb{C}$. Moreover, the restriction of ν^{-1} to the direct summand corresponding to $C=\{e\}$ is the map i_* of proposition 4.6.2.

In chapter 7, we will see that there is also a canonical graded isomorphism (due in particular to Burghelea [27])

$$\bigoplus_{C\in\langle\Gamma\rangle^{ell}} H_*(Z_C;\mathbb{C}) \xrightarrow{\cong} HH_*^{ell}(\mathbb{C}\Gamma),$$

where the latter group is the "elliptic part" (a suitable direct summand) of the Hochschild homology of the complex group algebra $\mathbb{C}\Gamma$ (see theorem 7.2.1).

4.7 Consequences of the Baum-Connes conjecture

We quote a few classical conjectures that would be implied by the Baum-Connes conjecture. Roughly speaking, injectivity of the Baum-Connes assembly map has consequences in topology, and surjectivity in analysis.

First, let us state the Novikov conjecture from topology.

4.7.1 Conjecture. (Novikov) *For a closed oriented manifold M with fundamental group Γ, the higher signatures (coming from $H^*(\Gamma;\mathbb{Q})$) are oriented homotopy invariants of M.*

In section 4.4, we have stated the strong Novikov conjecture. It is clear that rational injectivity of the Baum-Connes assembly map for a countable discrete group Γ implies the strong Novikov conjecture for this group. Mishchenko has proved that the strong Novikov conjecture for a group Γ implies the Novikov conjecture for any closed oriented manifold having Γ as fundamental group.

Due to Stephan Stolz, there is another consequence, in geometry, of injectivity (rational injectivity is probably not enough, according to him) of the Baum-Connes assembly map for a countable discrete group Γ, namely the following conjecture. Before the statement, we let B be a fixed "Bott manifold", i.e. a simply-connected closed Spin-manifold of dimension 8 whose index is a generator of $KO_8(pt)$, in other words, such that $\hat{A}(B) = 1$.

4.7.2 Conjecture. (Stable Gromov-Lawson-Rosenberg conjecture) *Let Γ be the fundamental group of a closed* Spin-*manifold M of dimension m. Then there exists $k \geq 0$ such that $M \times B^k$ is endowed with a metric of positive scalar curvature if and only if a suitable topological invariant $\alpha(M) \in KO_m(C^*_{\mathbb{R},r}\Gamma)$ vanishes (where $C^*_{\mathbb{R},r}\Gamma$ is the real reduced C^*-algebra of Γ).*

In fact, the Baum-Connes conjecture implies the "if part" of the above conjecture, and the "only if part" is a theorem of Rosenberg, generalizing a result of Hitchin (see [89] and [102]).

In algebra, there is a famous conjecture, that we now state, on the group algebra of a torsion-free group.

4.7.3 Conjecture. (Conjecture of idempotents) *Let Γ be a discrete group. If Γ is torsion-free, then the complex group algebra $\mathbb{C}\Gamma$ contains no non-trivial* *idempotents, in other words if* $x \in \mathbb{C}\Gamma$ *satisfies* $x^2 = x$, *then* $x = 0$ *or* 1.

This conjecture is an immediate consequence of the following much deeper conjecture in analysis:

4.7.4 Conjecture. (Kaplansky-Kadison) *If* Γ *is a discrete torsion-free group, then the reduced* C^**-algebra* $C_r^*\Gamma$ *contains no idempotents other than* 0 *and* 1.

It is known that surjectivity of the Baum-Connes assembly map for a discrete torsion-free group Γ implies the Kaplansky-Kadison conjecture for Γ (and therefore also the conjecture of idempotents for this group).

4.8 Status of the Baum-Connes conjecture

We quote a few classes of groups for which the Baum-Connes conjecture has been proved.

First, we have seen in section 4.5 that all finite groups satisfy the Baum-Connes and the Bost conjectures.

Recall that a locally compact group G is amenable if the left regular representation of G on L^2G almost contains invariant vectors, or equivalently if the trivial representation is weakly contained in the left regular representation. (This is in fact one of the numerous equivalent definitions. Another would be to ask that the canonical surjection $\lambda_G : C^*G \twoheadrightarrow C_r^*G$ is an isomorphism.) Among countable discrete groups, one has the following chain of inclusions of classes of groups:

$$\text{abelian} \subsetneq \text{nilpotent} \subsetneq \text{solvable} \subsetneq \text{amenable}$$

Clearly, any finite group is amenable (and in fact, so is any compact group). Among discrete groups, affine Coxeter groups and virtually solvable groups are amenable. On the other hand, free groups with two or more generators, the groups $SL_n(\mathbb{Z})$ for $n \geq 2$, and the surface-groups Γ_g with $g \geq 2$ are non-amenable (they all contain a free group on two generators as a subgroup of finite index, the standard obstruction to being amenable). It is known that amenable groups satisfy the Baum-Connes and the Bost conjectures. In fact, the same holds for a much larger class of groups, namely groups with the Haagerup property (also called a-T-menable groups).

4.8.1 Definition. *A locally compact group* G *has the Haagerup property if it admits an isometrically proper action by affine isometries on an affine Hilbert space* $\mathcal{H}$. *(Here, "isometrically proper" means that for any pair* A, B *of bounded subsets of* $\mathcal{H}$, *the set* $\{g \in G \mid g \cdot A \cap B \neq \varnothing\}$ *is relatively compact, i.e. it has compact closure in* G; *it is enough to require this for* $A = B$.*)*

One can show that a countable discrete amenable group has the Haagerup property (see [9]). The surface-groups, the discrete subgroups of $\mathrm{SO}(n, 1)$ and $SU(n, 1)$, the free groups with finitely many generators, and the group $SL_2(\mathbb{Z})$ have the Haagerup property, the groups $SL_n(\mathbb{Z})$ do not when $n \geq 3$. When Γ is a countable discrete group with the Haagerup property, one can take the affine Hilbert space $\mathcal{H}$ of the definition as a model for $\underline{E\Gamma}$. We now state the result alluded to above.

4.8.2 Theorem. (Higson-Kasparov; Lafforgue) *If Γ is a countable discrete group having the Haagerup property, then it satisfies the Baum-Connes and the Bost conjectures. (It even satisfies the Baum-Connes conjecture with coefficients.)*

This was proved in 1996 and is contained in the paper [50] (see also the survey [57]). The part concerning the Bost conjecture is due to Lafforgue. For example, the free abelian groups $\mathbb{Z}^n$, the surface-groups Γ_g and the free groups of finite type all satisfy the Baum-Connes and the Bost conjectures.

In 1981, G. Kasparov has proved injectivity of the Baum-Connes assembly map for discrete subgroups of real Lie groups with finitely many connected components. These includes for example the groups $SL_n(\mathbb{Z})$. Eight years later, he and G. Skandalis have obtained the same result to discrete subgroups of p-adic groups.

Concerning Gromov hyperbolic groups, it was first proved that the Baum-Connes assembly map is rationally injective for fundamental groups of compact Riemannian manifolds of non-positive scalar curvature (by A. Mishchenko and G. Kasparov, 1974 and 1988), and then, in 1988, A. Connes and H. Moscovici have proved the strong Novikov conjecture for Gromov hyperbolic groups (see [35] and [36]). In fact, for these groups, the Baum-Connes assembly map is injective. This was proved independently in 1993 by N. Higson and J. Roe, S. Hurder, and G. Kasparov and G. Skandalis. The same holds for groups acting properly on euclidean buildings (by G. Kasparov and G. Skandalis, 1991).

In 1991, P. Julg and G. Kasparov have proved that a group Γ satisfies the Baum-Connes conjecture if Γ is a discrete subgroup of a connected Lie group G whose Levi-Malcev decomposition is $G = RS$, where R the radical and S is the semi-simple part assumed to be locally isomorphic to a product

$$K \times \mathrm{SO}(n_1, 1) \times \ldots \times \mathrm{SO}(n_k, 1) \times SU(m_1, 1) \times \ldots \times SU(m_l, 1),$$

with K a compact Lie group.

The Baum-Connes conjecture also holds for one-relator groups, as was shown by C. Béguin, H. Bettaieb and A. Valette in their 1999 paper [13]. (This gives another proof for surface-groups.) This applies to the braid group B_3.

H. Oyono-Oyono and J.-L. Tu have independently proved in 1997 that fundamental groups of Haken 3-manifolds (including knot groups) satisfy the Baum-Connes and the Bost conjectures. In the same year, H. Oyono-Oyono has shown that if a countable discrete group Γ acts (simplicially) on a tree with edge *and* vertex stabilizers satisfying the Baum-Connes conjecture *with coefficients*, then so does Γ. Two years later, he has established that if a group Γ fits in a short exact sequence $0 \longrightarrow \Gamma_0 \longrightarrow \Gamma \longrightarrow \Gamma_1 \longrightarrow 0$, with Γ_1 and all subgroups H of Γ that contain Γ_0 as a subgroup of finite index satisfying the Baum-Connes conjecture *with coefficients*, then so does Γ. In particular, this conjecture is stable under taking semi-direct products with torsion-free "acting group". For example, this implies that the pure braid groups P_n satisfy the Baum-Connes conjecture.

Recall that a locally compact group Γ with a countable base has Kazhdan's property (T) if every action of Γ by affine isometries on an affine Hilbert space has a fixed point. (This is in fact one of the numerous equivalent definitions.) It is easy to see that a locally compact group with a countable base having at the same time the Haagerup property and Kazhdan's property (T) is necessarily compact, and hence finite if it is discrete.

It was only in 1998 that the Baum-Connes conjecture was proved for infinite groups having Kazhdan's property (T). Indeed, V. Lafforgue has proved that the Baum-Connes conjecture is true for Gromov hyperbolic groups acting properly co-compactly on a CAT(0) space. These include all co-compact lattices in the simple Lie groups $\mathrm{SO}(n, 1)$, $SU(n, 1)$, $Sp(n, 1)$ and $F_4(-20)$ (all having real rank one). The lattices in $Sp(n, 1)$ and in $F_4(-20)$ have property (T). In the same year, he has established that the co-compact lattices in $SL_3(F)$ also satisfy the Baum-Connes conjecture, where F is $\mathbb{R}$, $\mathbb{C}$ or a p-adic field $\mathbb{Q}_p$.

Up to now (July 2000), it is still not known if $SL_n(\mathbb{Z})$ satisfies the Baum-Connes conjecture, when $n \geq 3$.

Chapter 5

Low-dimensional group homology and the Baum-Connes assembly map

For products of a group by a finite cyclic group, we carefully describe the Baum-Connes assembly map, its domain, its range and the delocalized equivariant Chern character. As an application, we compute very explicitly the rational Baum-Connes assembly map in "low homological degrees".

5.1 Introduction

Throughout this chapter, we keep notations as in chapter 4. In particular, we write $K_j = K_{j+2}$ for K-homology of spaces and K-theory of Banach algebras.

In chapter 3, we have defined natural maps

$$\beta_j^X : H_j(X; \mathbb{Z}) \longrightarrow K_j(X),$$

for $j = 0$, 1 and 2, where X is any connected CW-complex, and $K_j(X)$ is the 2-periodic K-homology with compact supports. For $X = B\Gamma$, the classifying space of a countable discrete group, this gives maps

$$\beta_t^{loc} = \beta_t^{(j), loc} := \beta_j^{B\Gamma} : H_j(\Gamma; \mathbb{Z}) \longrightarrow K_j(B\Gamma),$$

where we bave identified the integral homologies of Γ and of $B\Gamma$. (The subscript "t" stands for "topological", and the exponent "loc" for "localized".) In her thesis [10], Hela Bettaieb has defined maps with same domain and same range, and denoted by β_t (see also [11]). We will show that her maps coincide with ours, even if her construction was completely different. Following a program initiated by Alain Valette, she has also defined explicit maps

$$\beta_a^{loc} = \beta_a^{(j),loc} : H_j(\Gamma;\, \mathbb{Z}) \longrightarrow K_j(C_r^*\Gamma)$$

(that she denotes by β_a, with the subscript "a" standing for "analytical"), such that the diagram

$$\begin{array}{ccccc} K_j(B\Gamma) & \longrightarrow & K_j^{\Gamma}(\underline{E}\Gamma) & \xrightarrow{\mu_j^{\Gamma}} & K_j(C_r^*\Gamma) \\ & \nwarrow {\scriptstyle \beta_t^{loc}} & & {\scriptstyle \beta_a^{loc}} \nearrow & \\ & & H_j(\Gamma;\, \mathbb{Z}) & & \end{array}$$

commutes, for $j = 0$, 1 and 2. We will recall the definition of β_a^{loc} in section 5.2. Of course, the interest is to construct β_a^{loc} directly, and not just as the composition of β_t^{loc} with the horizontal map in the above diagram (which is the Novikov assembly map composed with $(\lambda_\Gamma)_*$). G. A. Elliott and T. Natsume [40] (and independently H. Bettaieb and A. Valette) have proved the following result.

5.1.1 Theorem. *For any countable discrete group* Γ, *the maps* $\beta_a^{(0),loc}$ *and* $\beta_a^{(1),loc}$ *are rationally injective.*

Motivated by these results, Alain Valette has posed the following problem: Can one "delocalize" the above constructions, in other words, can one define "topological" maps

$$\beta_t = \beta_t^{(j)} : H_j(\Gamma;\, F\Gamma) \longrightarrow K_j^{\Gamma}(\underline{E}\Gamma) \otimes \mathbb{C}$$

and "analytic" maps

$$\beta_a = \beta_a^{(j)} : H_j(\Gamma;\, F\Gamma) \longrightarrow K_j(C_r^*\Gamma) \otimes \mathbb{C}\,,$$

such that the diagram

$$\begin{array}{ccccc} & & K_j^{\Gamma}(\underline{E}\Gamma) \otimes \mathbb{C} & \xrightarrow{\mu_j^{\Gamma} \otimes Id_{\mathbb{C}}} & K_j(C_r^*\Gamma) \otimes \mathbb{C} \\ & {\scriptstyle \beta_t^{loc} \otimes Id_{\mathbb{C}}} \nearrow & \uparrow {\scriptstyle \beta_t} & {\scriptstyle \beta_a} \nearrow & \\ H_j(\Gamma;\, \mathbb{C}) & \xhookrightarrow{i_*} & H_j(\Gamma;\, F\Gamma) & & \end{array}$$

commutes, for $j = 0$, 1 and 2 (see propositions 4.6.2 and 4.6.4 for i_*). This amounts to trying to express the rational Baum-Connes assembly map in "low homological degrees". In this chapter, we present a positive answer to this question, and in chapters 6 and 7, we will establish two kinds of generalizations of the construction of β_a in various situations.

The construction of $\beta_a^{(j)}$ when $j = 0$ and partially for $j = 1$ is a joint work with Hela Bettaieb. For these values of j, the presentation given here varies from the one of [10] in that it is a bit more general and allows for lifting some ambiguities on particular values of the delocalized equivariant Chern character.

An important feature of our construction of the maps β_t and β_a is that we need a "classifying family" of groups for $H_j(\Gamma; F\Gamma)$, that is a family of groups $\{G_i\}_{i\in I}$ (indexed by a certain set I) such that for any group Γ, any homology class in $H_j(\Gamma; F\Gamma)$ arises from a "fundamental class" in $H_j(G_i, FG_i)$, for a certain G_i, through a homomorphism $G_i \longrightarrow \Gamma$. In the three considered cases, the "classifying family" is given by

$$\begin{aligned} \text{for } j = 0 &: \ \{\mathbb{Z}/n\}_{n\geq 1} \\ \text{for } j = 1 &: \ \{\mathbb{Z}\times\mathbb{Z}/n\}_{n\geq 1} \\ \text{for } j = 2 &: \ \{\Gamma_g\times\mathbb{Z}/n\}_{g,\, n\geq 1}\,, \end{aligned}$$

where Γ_g is the fundamental group of a closed oriented surface of genus g.

We should explain in a few words our choice for the terminology "localized" and "delocalized". In view of proposition 4.6.2 (together with the fact that both occurring Chern characters induce isomorphisms after tensoring their respective domain with $\mathbb{C}$), the Novikov assembly map $\tilde{\beta}_*^\Gamma$ can be considered as the part of $\tilde{\mu}_*^\Gamma$ localized at the identity e of Γ. The other way round, $\tilde{\mu}_*^\Gamma$ is the delocalized counterpart of $\tilde{\beta}_*^\Gamma$. (These statements are accurate at least rationally.) We will therefore call the maps β_t and β_a the delocalized counterparts of β_t^{loc} and β_a^{loc} respectively.

This chapter is organized as follows: In section 5.2, after recalling their definition, we prove that the maps $\beta_t^{(j),\,loc}$ constructed by H. Bettaieb coincide with ours. We also recall the definition of her maps $\beta_a^{(j),\,loc}$, and their main properties. In section 5.3, we introduce the spectral projectors associated to finite order elements in a group. They play a crucial role in the sequel. In section 5.5, we compute explicitly the delocalized equivariant Chern character ch_*^Γ for $\Gamma = G\times\mathbb{Z}/n$, with G a torsion-free group having a smooth manifold as a model for $\underline{E\Gamma}$. This computation is based on a technical result, established in section 5.4, on the periodic topological cyclic cohomology of the algebra $C_c^\infty(M, \Gamma)$ associated to a proper action of Γ on a manifold M. The concrete determination of the assembly map μ_*^Γ for any product $\Gamma = G\times\mathbb{Z}/n$ is the subject of section 5.6. Finally, in the remaining three sections, we construct the maps $\beta_t^{(j)}$ and $\beta_a^{(j)}$, for $j = 0$, 1 and 2, and determine their properties.

5.2 The maps β_t^{loc} and β_a^{loc}

We first recall the definition of the maps $\beta_t^{(j),loc}$ and $\beta_a^{(j),loc}$ considered by Hela Bettaieb in [10]. We then show that for any discrete group Γ, the maps $\beta_t^{(j),loc}$ coincide with our maps $\beta_j^{B\Gamma}$. We also give the main properties of these maps.

For $j = 0$, one has $H_0(\Gamma;\mathbb{Z}) = \mathbb{Z}$, and

$$\beta_t^{(0),loc} : H_0(\Gamma;\mathbb{Z}) \longrightarrow K_0(B\Gamma)\,,\ 1 \longmapsto [1]\,,$$

where $[1] := i_*(1)$, with i denoting the inclusion of the base-point in $B\Gamma$, and 1 being the canonical generator of $K_0(pt) = \mathbb{Z}$. On the other hand, we set

$$\hat{\beta}_a^{(0),loc} : H_0(\Gamma;\mathbb{Z}) \longrightarrow K_0(\ell^1\Gamma)\,,\ 1 \longmapsto [1]\,,$$

where [1] is the K-theory class of the unit (viewed as an idempotent (1×1)-matrix over $\ell^1\Gamma$). The map considered by H. Bettaieb is given by

$$\beta_a^{(0),loc} = (j_r^\Gamma)_* \circ \hat{\beta}_a^{(0),loc}\,,$$

where j_r^Γ is the inclusion of $\ell^1\Gamma$ in $C_r^*\Gamma$.

For $j = 1$, one has a canonical and natural isomorphism $H_1(\Gamma;\mathbb{Z}) \cong \Gamma^{ab}$, the abelianization of Γ. We write γ^{ab} for the class in Γ^{ab} of an element $\gamma \in \Gamma$. By definition, one has

$$\beta_t^{(1),loc} : H_1(\Gamma;\mathbb{Z}) \longrightarrow K_1(B\Gamma)\,,\ \gamma^{ab} \longmapsto \gamma_*([S^1]_K)\,,$$

where γ is considered as a the homotopy class of a pointed map $S^1 \longrightarrow B\Gamma$, where γ_* is the map induced at the level of K_1-homology, and $[S^1]_K$ is the canonical generator of $K_1(S^1) \cong \mathbb{Z}$. (The fact that $\beta_t^{(1),loc}$ is a well-defined homomorphism is not obvious. The reader can refer to [10] for a proof of analytical flavour, or to [11] for a more geometric proof. Another proof consists in identifying this map with $\beta_1^{B\Gamma}$ (see proposition 5.2.3 below) and then to invoke the results of chapter 3.) On the other hand, we set

$$\hat{\beta}_a^{(1),loc} : H_1(\Gamma;\mathbb{Z}) \longrightarrow K_1(\ell^1\Gamma)\,,\ \gamma^{ab} \longmapsto -[\gamma] = [\gamma^{-1}]\,,$$

where $[\gamma^{\pm 1}]$ is the K-theory class of $\gamma^{\pm 1}$ (viewed as an invertible (1×1)-matrix over $\ell^1\Gamma$). The map considered by H. Bettaieb is given by

$$\beta_a^{(1),loc} = (j_r^\Gamma)_* \circ \hat{\beta}_a^{(1),loc}\,.$$

(More precisely, H. Bettaieb considers $-\beta_a^{(1),loc}$ in her thesis [10].)

For $j = 2$, the situation is more involved. In [114], B. Zimmermann considers the set $S(\Gamma)$ of pointed maps $\Sigma_g \longrightarrow B\Gamma$ (with $g \geq 1$ varying arbitrary), inducing a surjection at the level of fundamental groups. Clearly, this set is empty if Γ is not finitely generated; let us therefore assume, for a moment, that Γ has finite type. Recall that the surface-group Γ_g admits the presentation

$$\Gamma_g = \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g \;\middle|\; \prod_{j=1}^{g} [a_j, b_j] \right\rangle .$$

It follows that the free group F_g on g generators is the quotient of Γ_g by the normal subgroup generated by the $a_i b_i^{-1}$'s $(1 \leq i \leq g)$. It follows that any finitely generated group is a quotient of some surface-group. Zimmermann considers a suitable equivalence relation on the set $S(\Gamma)$, and shows that the quotient set $\Omega(\Gamma)$ is in canonical bijection with $H_2(\Gamma; \mathbb{Z})$. Explicitly, the bijection is

$$\Omega(\Gamma) \longrightarrow H_2(\Gamma; \mathbb{Z}), \ [f : \Sigma_g \to B\Gamma] \longmapsto f_*([\Sigma_g]),$$

where $[\Sigma_g] \in H_2(\Sigma_g; \mathbb{Z})$ is the orientation class. Under this identification, the addition in homology corresponds to connected sum (for which $\Omega(\Gamma)$ becomes a group). This construction is only valid for finitely generated groups, and has the disadvantage of rendering the functoriality of this bordism type description of $H_2(-; \mathbb{Z})$ obscure. We now explain how to avoid these problems. First, we recall a 1954 theorem of René Thom (see thm. II.3 and cor. III.7 in [104]; compare with theorem 3.3.6.).

5.2.1 Theorem. (Thom) *For a CW-complex X, any class x in $H_n(X; \mathbb{Z})$, for $n \leq 6$, is Steenrod-representable, in other words there exists a smooth, closed and oriented manifold M of dimension n, and a continuous map $f : M \longrightarrow X$ (both depending on x) such that $x = f_*([M])$. Moreover, there is a class in $H_7(B(\mathbb{Z}/3)^{[7]} \times B(\mathbb{Z}/3)^{[7]})$ that is not Steenrod-representable.*

Notice that the manifold M need not be connected. Let us also point out that the above theorem also holds for infinite CW-complexes because we are working with compact supports.

We will denote a homology class $f_*([M])$ as in the theorem also by $[M, f]$. By applying this theorem for $n = 2$, and by dropping the surjectivity hypothesis everywhere in Zimmermann's paper [114], we get a proof of the

5.2.2 Theorem. (Zimmermann) *For any group Γ, consider two "bordism" pairs (Σ_{g_1}, f_1) and (Σ_{g_2}, f_2), where $f_i : \Sigma_{g_i} \longrightarrow B\Gamma$ is a continuous map for $i = 1$ and 2, and $g_1, g_2 \geq 1$. Then these two pairs represent the same class in $H_2(\Gamma; \mathbb{Z})$, i.e. $(f_1)_*([\Sigma_{g_1}]) = (f_2)_*([\Sigma_{g_2}])$, if and only if they are stably equivalent in the sense that there exists $g \geq g_1, g_2$ and an orientation-preserving homeomorphism h of*

Σ_g, *such that the triangle*

$$\begin{array}{ccc} \Sigma_g = \Sigma_{g_1}\#\Sigma_{g-g_1} & & \\ & \searrow^{\bar{f}_1} & \\ h \big\downarrow \cong & & B\Gamma \\ & \nearrow_{\bar{f}_2} & \\ \Sigma_g = \Sigma_{g_2}\#\Sigma_{g-g_2} & & \end{array}$$

commutes up to homotopy, where, for $i = 1$ and 2, $\bar{f}_i = f_i \# x_0$ and x_0 *is the constant map from* Σ_{g-g_i} *to* $B\Gamma$ *taking the base-point of* $B\Gamma$ as *value (the symbol "#" stands for the oriented* connected *sum.)*

Recall that to prove this result, we only need the easy part of theorem 5.2.1, namely for $n = 2$. See also theorems 3.3.5 and 3.3.6 for a sketch of proof in this case. As already mentioned in chapter 3, for an "elementary" proof of Thom's theorem for $n = 2$ and for $X = B\Gamma$, the classifying space of a discrete group Γ (the only case to which we apply the theorem in the sequel), the reader can consult lemma 2.2.4 in [71].

The functoriality of $\Omega(\Gamma)$ is then simply given by composition with the homomorphism; more explicitly, for a group homomorphism $\varphi : \Gamma_1 \longrightarrow \Gamma_2$ and a stable equivalence class $[\Sigma_g, f]$ (in the sense of theorem 5.2.2), one has

$$\varphi_*([\Sigma_g, f]) = [\Sigma_g, B\varphi \circ f],$$

where $B\varphi : B\Gamma_1 \longrightarrow B\Gamma_2$ is induced by φ. Let us finally recall that the sum is given by the oriented connected sum:

$$[\Sigma_{g_1}, f_1] + [\Sigma_{g_2}, f_2] = [\Sigma_{g_1}\#\Sigma_{g_2}, f_1\#f_2].$$

From now on, we identify the group $H_2(\Gamma; \mathbb{Z})$ with the set $\Omega(\Gamma)$ of equivalence classes of bordism classes $[M, f]$ as in the above version of Zimmermann's theorem.

Coming back to the construction of $\beta_t^{(2),loc}$, the K-homology of Σ_g is given by

$$K_0(\Sigma_g) = \mathbb{Z}\cdot[1] \oplus \mathbb{Z}\cdot[\Sigma_g]_K \cong \mathbb{Z}^2 \quad\text{and}\quad K_1(\Sigma_g) \cong \mathbb{Z}^{2g},$$

where $[1]$ and $[\Sigma_g]_K$ are determined by the relations $ch([1]) = 1 \in H_0(\Sigma_g; \mathbb{Q})$ and $ch([\Sigma_g]_K) = [\Sigma_g] \in H_2(\Sigma_g; \mathbb{Q})$ (see lemma 3.2.2). By definition, one has

$$\beta_t^{(2),loc} : H_2(\Gamma; \mathbb{Z}) = \Omega(\Gamma) \longrightarrow K_0(B\Gamma),\ [\Sigma_g, f] \longmapsto f_*([\Sigma_g]_K).$$

To have a better insight into this definition, let us point out that in [10] and in [11], it is shown that $[\Sigma_g]_K = [\bar{\partial}_g] + (g-1)\cdot[1]$, where $[\bar{\partial}_g]$ is the K-homology class represented by the Dolbeault operator on Σ_g. In the analytical framework, the proof that $\beta_t^{(2),loc}$ is a well-defined group homomorphism is highly technical and rather involved (see [11]). Another proof consists in identifying this map with $\beta_2^{B\Gamma}$ (see proposition 5.2.3 below) and then to invoke the results of chapter 3. On the other hand, for Γ not necessarily countable, we set

$$\hat{\beta}_a^{(2),loc} : H_2(\Gamma;\mathbb{Z}) = \Omega(\Gamma) \longrightarrow K_0(\ell^1\Gamma)\,,\ [\Sigma_g, f] \longmapsto f_*(\hat{\mu}_0^{\Gamma_g}([\Sigma_g]_K))\,.$$

(Here, the elements $\hat{\mu}_0^{\Gamma_g}([\Sigma_g]_K) \in K_0(\ell^1\Gamma_g)$ can be thought about as "universal classes" parameterized by $g \geq 1$.) The map considered by H. Bettaieb is given by

$$\beta_a^{(1),loc} = (j_r^\Gamma)_* \circ \hat{\beta}_a^{(1),loc}\,.$$

For $j = 0$, 1 and 2, we set $\tilde{\beta}_a^{(j),loc} := j_*^\Gamma \circ \hat{\beta}_a^{(j),loc}$, where $j^\Gamma : \ell^1\Gamma \hookrightarrow C^*\Gamma$.

To be precise, the maps $\beta_t^{(2),loc}$ and $\beta_a^{(2),loc}$ of H. Bettaieb were only defined for finitely generated groups. However, as we have seen above, the constructions can be extended without major modification to the class of all countable discrete groups. On the other hand, the map β_2^X of chapter 4 is defined for any connected CW-complex X.

5.2.3 Proposition. *For any discrete group Γ, the map $\beta_t^{(j),loc}$ defined by H. Bettaieb in [10] coincides with $\beta_j^{B\Gamma}$, for $j = 0$, 1 and 2. It is in particular natural in Γ, rationally injective, and rationally a right inverse of the Chern character, in other words,*

$$ch \circ (\beta_t^{(j),loc} \otimes Id_{\mathbb{C}}) = Id_{H_j(\Gamma;\mathbb{C})}\,.$$

Proof. The statement for $j = 0$ is obvious. For $j = 1$ and 2, this is a direct consequence of lemma 3.3.4 and of proposition 3.3.7 respectively. □

The following result is an obvious modification of a theorem proved in [10] and in [11]. (In the framework of the Baum-Connes conjecture, we have to assume that Γ is countable.)

5.2.4 Theorem. *For any countable discrete group Γ, the diagram*

$$\begin{array}{ccccc} K_j(B\Gamma) & \longrightarrow & K_j^\Gamma(\underline{E\Gamma}) & \xrightarrow{\hat{\mu}_j^\Gamma} & K_j(\ell^1\Gamma) \\ & {\scriptstyle \beta_t^{(j),loc}} \nwarrow & & \nearrow {\scriptstyle \hat{\beta}_a^{(j),loc}} & \\ & & H_j(\Gamma;\mathbb{Z}) & & \end{array}$$

commutes, for $j = 0$, 1 and 2.

As already mentioned in the introduction, H. Bettaieb and A. Valette have shown that $\beta_a^{(j),loc}$ is rationally injective for $j = 0$ and 1. The case $j = 0$ is easy, since the canonical trace τ on $C_r^*\Gamma$ induces a homomorphism $\tau_* : K_0(C_r^*\Gamma) \longrightarrow \mathbb{R}$ mapping $[1]$ to 1. The case $j = 1$ is dealt with in [10] and in [12] and is based on the theory of C^*-algebras and their K-theory, more precisely, on the Pimsner-Voiculescu 6-term exact sequence. (In [10], a much simpler proof is presented for the injectivity of $\tilde{\beta}_a^{(1),loc}$.) As noticed by Alain Valette in [11], for a 2-dimensional group (i.e. a group having a classifying space of dimension 2), the Baum-Connes conjecture is equivalent to the statement that the maps $\beta_a^{(0),loc} \oplus \beta_a^{(2),loc}$ and $\beta_a^{(1),loc}$ are isomorphisms. (This follows readily from lemma 3.2.2 and proposition 4.3.4.)

5.3 The spectral projectors of an elliptic element

In the construction of the map $\beta_a^{(j)} : H_j(\Gamma; F\Gamma) \longrightarrow K_j(C_r^*\Gamma) \otimes \mathbb{C}$, we will need some particular idempotents in the group algebra $\mathbb{C}\Gamma$. In the present section, we will define these idempotents (the spectral projectors) and establish some of their properties.

Let γ_C be one of the fixed representative elements of the elliptic conjugacy class C. Let n_C be the (finite) order of γ_C in Γ, and let $\omega_C := e^{2\pi i/n_C}$ be one of the primitive n_C-th root of unity in $\mathbb{C}$.

5.3.1 Definition. *The spectral projectors associated to the elliptic element γ_C are*

$$P_l^{(C)} := \frac{1}{n_C} \sum_{s=0}^{n_C-1} (\omega_C^l \gamma_C)^s \in \mathbb{C}\Gamma\,,\ l \in \mathbb{Z}.$$

For $\Gamma = \mathbb{Z}/n$, generated by b, we denote the projector $P_l^{((b))}$ simply by P_l.

Taking $\Gamma = H = \mathbb{Z}/n$, in the notation of example i) of section 4.5, P_l is the projector $p_{\bar{\omega}^l} = \tilde{p}_{\omega^l}$, associated to the 1-dimensional representation $\bar{\omega}^l$, dual of ω^l. To establish the fundamental properties of these spectral projectors, we need the following well-known lemma.

5.3.2 Lemma. *Let ω be a primitive n-th root of unity in $\mathbb{C}$. Then*

$$\frac{1}{n}\sum_{s=0}^{n-1} \omega^{sl} = \begin{cases} 1\,, & \text{if } l \equiv 0 \pmod n \\ 0\,, & \text{otherwise.} \end{cases}$$

Proof. Each complex number ω^l is a root of the polynomial

$$x^n - 1 = (x-1)(1 + x + x^2 + \ldots + x^{n-1})\,,$$

but it is a root of $x-1$ if and only if $l \equiv 0 \pmod{n}$. □

Recall that the complex group algebra $\mathbb{C}\Gamma$ is equipped with the canonical involution given by

$$\Big(\sum_{\gamma\in\Gamma} \lambda_\gamma \cdot \gamma\Big)^* = \sum_{\gamma\in\Gamma} \overline{\lambda}_\gamma \cdot \gamma^{-1}.$$

By a straightforward application of the above lemma, one proves the

5.3.3 Proposition. *The spectral projectors associated to the elliptic element γ_C of Γ, of order n_C, satisfy the following properties:*

i) The spectral projector $P_l^{(C)}$ is indeed a projector (= self-adjoint idempotent), in other words

$$(P_l^{(C)})^2 = P_l^{(C)} = (P_l^{(C)})^*.$$

ii) The set of spectral projectors of γ_C decompose the identity as a sum of projectors, more precisely

$$\sum_{l=0}^{n_C-1} P_l^{(C)} = 1 \in \mathbb{C}\Gamma.$$

iii) For $k \not\equiv l \pmod{n_C}$, the spectral projectors $P_k^{(C)}$ and $P_l^{(C)}$ are "orthogonal", in other words

$$P_k^{(C)} \cdot P_l^{(C)} = 0.$$

iv) In $\mathbb{C}Z_C$ and in $\mathbb{C}\Gamma$, the following formula holds:

$$\sum_{l=0}^{n_C-1} \omega_C^{sl} P_l^{(C)} = \gamma_C^{-s}, \ \forall s \in \mathbb{Z},$$

where $\omega_C = e^{2\pi i/n_C}$.

As an immediate consequence, since γ_C is central in $\mathbb{C}Z_C$, $\ell^1 Z_C$, $C^* Z_C$ and in $C_r^* Z_C$, we get the

5.3.4 Corollary. *For any elliptic element $\gamma_C \in C$, one has a decomposition of the group algebra $\mathbb{C}Z_C$ as a direct sum of algebras with unit:*

$$\mathbb{C}Z_C = \bigoplus_{l=0}^{n_C-1} P_l^{(C)} \cdot \mathbb{C}Z_C,$$

the unit in the algebra $P_l^{(C)} \cdot \mathbb{C}Z_C$ being $P_l^{(C)}$. The same holds for the group C^*-algebras $C^* Z_C$ and $C_r^* Z_C$ (all summands being unital C^*-algebras), and for $\ell^1 Z_C$ (all summands being unital Banach algebras).

Before stating a useful technical lemma, for any positive integer n, we let G_n be the finite cyclic group $\mathbb{Z}/n$, generated by b, and we set

$$Q_{l,k}^{(m)} := \frac{1}{m}\sum_{s=0}^{m-1}(\omega_m^l b^k)^s \in \mathbb{C}G_n \quad (k,\, l,\, m \in \mathbb{Z}),$$

where $\omega_m = \mathrm{e}^{2\pi i/m}$. For example, when m is the order of b^k in $G_n = \mathbb{Z}/n$, $Q_{l,k}^{(m)}$ is the spectral projector $P_l^{(\langle b^k\rangle)}$ associated to b^k.

5.3.5 Lemma. *Let $k \in \mathbb{Z}$, and m be the order of b^k in $G_n = \mathbb{Z}/n$ (in other words, $m = \frac{n}{\mathrm{GCD}(n,k)} = \frac{\mathrm{LCM}(n,k)}{k}$). Then, in the tensor product $\mathbb{C}G_n \otimes_{\mathbb{Z}} \mathbb{C}$, one has*

$$\sum_{l=0}^{n-1} Q_{l,k}^{(n)} \otimes \bar{\omega}_n^l = \sum_{l=0}^{m-1} Q_{l,k}^{(m)} \otimes \bar{\omega}_m^l,$$

for any $l \in \mathbb{Z}$.

Proof. First, letting $a := \mathrm{GCD}(n, k) = \frac{n}{m}$, one has $n = am$, $\omega_a = \mathrm{e}^{2\pi i/a} = \omega_n^m$ and $\omega_n^a = \mathrm{e}^{2\pi i a/n} = \omega_m$. Writing $l = pa + i$ and $s = qm + j$, we have $b^{ks} = b^{kj}$, and, by lemma 5.3.2, we find

$$\frac{1}{a}\sum_{q=0}^{a-1}\omega_a^{(pa+i)q} = \begin{cases} 1, & \text{if } pa + i \equiv 0 \pmod a), \text{ i.e. if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Applying this formula, we compute

$$\begin{aligned}
\sum_{l=0}^{n-1} Q_{l,k}^{(n)} \otimes \bar{\omega}_n^l &= \frac{1}{n}\sum_{l=0}^{n-1}\sum_{s=0}^{n-1}\omega_n^{ls} b^{ks} \otimes \bar{\omega}_n^l \\
&= \frac{1}{m}\sum_{i=0}^{a-1}\sum_{p=0}^{m-1}\sum_{j=0}^{m-1}\omega_n^{(pa+i)j}\left(\frac{1}{a}\sum_{q=0}^{a-1}\omega_a^{(pa+i)q}\right) b^{kj} \otimes \bar{\omega}_n^{pa+i} \\
&= \frac{1}{m}\sum_{p=0}^{m-1}\sum_{j=0}^{m-1}\omega_n^{paj} b^{kj} \otimes \bar{\omega}_n^{pa} \\
&= \frac{1}{m}\sum_{p=0}^{m-1}\sum_{j=0}^{m-1}(\omega_m^p b^k)^j \otimes \bar{\omega}_m^p \\
&= \frac{1}{m}\sum_{p=0}^{m-1} Q_{p,k}^{(m)} \otimes \bar{\omega}_m^p .
\end{aligned}$$

This completes the proof. □

Notice that the formula of lemma 5.3.5 is completely obvious in $\mathbb{C}G_n \otimes_{\mathbb{C}} \mathbb{C}$, since a direct application of lemma 5.3.2 shows that under the canonical isomorphism $\mathbb{C}G_n \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}G_n$, both terms correspond to b^k.

5.4 On the topological algebra $C_c^\infty(M,\Gamma)$

The first goal of this section is to define carefully the topological algebra $C_c^\infty(M,\Gamma)$, where Γ is a discrete group acting properly on the manifold M. As a second topic, we will establish a technical lemma concerning the computation (due to Baum and Connes [6]) in topological terms of the periodic cyclic cohomology of this algebra, in the particular case where Γ is a product $G\times\mathbb{Z}/n$. This result will be fundamental in the following section, when we compute explicitly the delocalized equivariant Chern character for such a product. This will also single out the "raison d'être" of the "delocalization" formulas that will occur in the sequel. More precisely, many forthcoming formulas contain lots of roots of unity, whose origin is exactly in the lemma of this section.

Throughout this part, we use similar notations to those used by Baum and Connes in [6], except that we still consider all actions of groups on spaces as left actions.

Let Γ be a countable discrete group acting, on the left, properly by diffeomorphisms on a smooth manifold M without boundary. The group Γ acts, on the left, on the commutative algebra $C_c^\infty(M)$ by

$$\Gamma\times C_c^\infty(M)\longrightarrow C_c^\infty(M)\,,\ (\gamma,f)\longmapsto({}^{\gamma}\!f:x\mapsto f(\gamma^{-1}\cdot x))\,.$$

By definition, the algebra $C_c^\infty(M,\Gamma)$ is the set of finite formal sums $\sum_{\gamma\in\Gamma}f_\gamma\cdot[\gamma]$, where $f_\gamma\in C_c^\infty(M)$ for any $\gamma\in\Gamma$ (and all but finitely many of the f_γ's are zero). It is endowed with the obvious structure of complex vector space, and the product is the "twisted product" given by

$$(f\cdot[\gamma])(\tilde f\cdot[\tilde\gamma])=f\,{}^{\gamma}\!\tilde f\cdot[\gamma\tilde\gamma]\,.$$

The map $C_c^\infty(M,\Gamma)\longrightarrow C_c^\infty(M\times\Gamma)$ defined by $\left(\sum_{\gamma\in\Gamma}f_\gamma\cdot[\gamma]\right)(x,\tilde\gamma)=f_{\tilde\gamma}(x)$ is an isomorphism of vector spaces (but generally *not* of algebras). The C^∞-topology on $C_c^\infty(M\times\Gamma)$ determines, via this isomorphism, the topology on $C_c^\infty(M,\Gamma)$, and it becomes a topological algebra. One of its main interests is that it is a dense sub-algebra of the reduced crossed product $C_0(M)\rtimes_r\Gamma$ that is closed under holomorphic functional calculus (see [6], lem. 7.5). In particular, by the density theorem, the inclusion $C_c^\infty(M,\Gamma)\hookrightarrow C_0(M)\rtimes_r\Gamma$ induces an isomorphism

$$K_0(C_c^\infty(M,\Gamma))\xrightarrow{\cong}K_0(C_0(M)\rtimes_r\Gamma)\,.$$

As in [6], we denote by $H^*(\mathcal{A})=H^{ev}(\mathcal{A})\oplus H^{odd}(\mathcal{A})$ the ($\mathbb{Z}/2$-graded) periodic topological cyclic cohomology of the topological algebra $\mathcal{A}$. In [6], Baum and Connes have computed $H^*(C_c^\infty(M,\Gamma))$ as some equivariant topological homology of the manifold M. Now, we explain this result in some details, since we will need to make explicit computations involving their isomorphism.

First, we let $\widehat{M} := \{(x, \gamma) \in M \times \Gamma \,|\, \gamma \cdot x = x\}$. It is a smooth sub-manifold of $M \times \Gamma$ equipped with the following proper action of Γ by diffeomorphisms:

$$\Gamma \times \widehat{M} \longrightarrow \widehat{M}, \ (\tilde{\gamma}, (x, \gamma)) \longmapsto (\tilde{\gamma}x, \tilde{\gamma}\gamma\tilde{\gamma}^{-1}).$$

The orbit space $\Gamma\backslash\widehat{M}$ is an orbifold. Of course, if the action of Γ on M is also free, then $\widehat{M} = M \times \{e\} = M$ as a Γ-manifold.

Let $\Omega_j^\Gamma(\widehat{M})$ denote the vector space of all complex j-dimensional de Rham currents on $\widehat{M}$ which are fixed by Γ. The j-th homology group of the associated de Rham complex

$$0 \longrightarrow \Omega_n^\Gamma(\widehat{M}) \xrightarrow{\partial} \Omega_{n-1}^\Gamma(\widehat{M}) \xrightarrow{\partial} \dots \xrightarrow{\partial} \Omega_0^\Gamma(\widehat{M}) \longrightarrow 0$$

is the j-th homology group of the space $\Gamma\backslash\widehat{M}$ using countable locally finite chains with complex coefficients, and denoted by $H_j^\infty(\Gamma\backslash\widehat{M};\, \mathbb{C})$. This is also isomorphic to the j-th Borel-Moore homology of $\Gamma\backslash\widehat{M}$ with unrestricted supports and coefficients $\mathbb{C}$ (see [6], p. 187, and [18]).

To define Baum and Connes' isomorphism, we need to introduce an analogue of $C_c^\infty(M, \Gamma)$ for differential forms. Let $\Omega_c^*(M)$ be the graded differential algebra of compactly supported complex smooth differential forms on M. It is endowed with the action

$$\Gamma \times \Omega_c^m(M) \longrightarrow \Omega_c^m(M), \ (\gamma, \eta) \longmapsto {}^\gamma\eta, \ {}^\gamma\eta(v_1, \dots, v_m) = \eta(\gamma^{-1}v_1, \dots, \gamma^{-1}v_m),$$

where $(v_1, \dots, v_m) \in T_xM$ and $x \in M$. By definition, the complex graded differential algebra $\Omega_c^*(M, \Gamma)$ is the set of finite formal sums $\sum_{\gamma\in\Gamma} \eta_\gamma \cdot [\gamma]$, where $\eta_\gamma \in \Omega_c^*(M)$ for $\gamma \in \Gamma$, equipped with the obvious structure of graded complex vector space and the obvious differential, namely

$$d : \Omega_c^m(M, \Gamma) \longrightarrow \Omega_c^{m+1}(M, \Gamma), \ \sum_{\gamma\in\Gamma} \eta_\gamma \cdot [\gamma] \longmapsto \sum_{\gamma\in\Gamma} d\eta_\gamma \cdot [\gamma].$$

The graded multiplication is the "twisted product" defined by

$$(\eta \cdot [\gamma])(\tilde{\eta} \cdot [\tilde{\gamma}]) = \eta \wedge {}^\gamma\tilde{\eta} \cdot [\gamma\tilde{\gamma}].$$

As an algebra, $\Omega_c^0(M, \Gamma) = C_c^\infty(M, \Gamma)$, and the map $\Omega^m(M, \Gamma) \xrightarrow{\xi} \Omega_c^m(M \times \Gamma)$ given by

$$\xi\Big(\sum_{\gamma\in\Gamma} \eta_\gamma \cdot [\gamma]\Big)(v_1, \dots, v_m) = \eta_{\tilde{\gamma}}(v_1, \dots, v_m),$$

where $v_1, \dots, v_m \in T_{(x,\tilde{\gamma})}(M \times \Gamma)$, is an isomorphism of complex vector spaces for any m. This identification and the inclusion $\widehat{M} \subseteq M \times \Gamma$ allow for defining a pairing

$$\Omega_m^\Gamma(\widehat{M}) \times \Omega_c^m(M, \Gamma) \longrightarrow \mathbb{C}, \ \Big(Z, \sum_{\gamma\in\Gamma} \eta_\gamma \cdot [\gamma]\Big) \longmapsto \int_Z \xi\Big(\sum_{\gamma\in\Gamma} \eta_\gamma \cdot [\gamma]\Big).$$

We are now in position to state the Baum-Connes theorem on the $\mathbb{Z}/2$-graded periodic topological cyclic cohomology of the topological algebra $C_c^\infty(M, \Gamma)$ (see thm. 7.14 in [6]).

5.4.1 Theorem. (Baum-Connes) *Let Γ be a countable discrete group acting properly by diffeomorphisms on a smooth manifold M without boundary. Then the following map is an isomorphism:*

$$\Phi^{C_c^\infty(M,\Gamma)} : \bigoplus_{m\in\mathbb{N}} H_m^\infty(\Gamma\backslash\widehat{M}; \mathbb{C}) \longrightarrow H^*(C_c^\infty(M, \Gamma)), \ [Z] \longmapsto [\varphi_Z],$$

where, for $Z \in \Omega_m^\Gamma(\widehat{M})$, $[\varphi_Z]$ is the class of the continuous cyclic cocycle

$$\varphi_Z : C_c^\infty(M, \Gamma)^{m+1} \longrightarrow \mathbb{C}, \ (b_0, b_1, \ldots, b_m) \longmapsto \int_Z \xi(b_0 db_1 \cdots db_m).$$

The map $\Phi^{C_c^\infty(M,\Gamma)}$ is $\mathbb{Z}/2$-graded in the obvious sense.

For our crucial lemma, we suppose that G is a countable discrete group acting properly by diffeomorphisms on a smooth manifold M without boundary, and that $\Gamma = G \times \mathbb{Z}/n$. The trivial action of $\mathbb{Z}/n$ on M determines a proper action of Γ on M by diffeomorphims. We let

$$\widehat{M_G} := \{(x, g) \in M \times G \,|\, g \cdot x = x\} \ \text{ and } \ \widehat{M_\Gamma} := \{(x, \gamma) \in M \times \Gamma \,|\, \gamma \cdot x = x\},$$

endowed with their respective action of G and Γ. It is clear that

$$\widehat{M_\Gamma} = \coprod_{j=0}^{n-1} \widehat{M_G} \times b^j = \coprod_{j=0}^{n-1} \widehat{M_G} \ \text{ and } \ \Gamma\backslash\widehat{M_\Gamma} = \coprod_{j=0}^{n-1} G\backslash\widehat{M_G}.$$

There is consequently an obvious canonical isomorphism

$$I : H_*^\infty(G\backslash\widehat{M_G}; \mathbb{C})^{\oplus n} \xrightarrow{\cong} H_*^\infty(\Gamma\backslash\widehat{M_\Gamma}; \mathbb{C}), \ ([Z_0], \ldots, [Z_{n-1}]) \longmapsto \sum_{j=0}^{n-1}[Z_j].$$

As in corollary 5.3.4, there is a decomposition

$$C_c^\infty(M, \Gamma) = \bigoplus_{j=0}^{n-1} \check{P}_j \cdot C_c^\infty(M, \Gamma) \cong \bigoplus_{j=0}^{n-1} \check{P}_j \cdot C_c^\infty(M, G) \cong C_c^\infty(M, G)^{\oplus n},$$

with

$$\check{P}_j := \frac{1}{n}\sum_{s=0}^{n-1} \bar{\omega}^{sj} \cdot [b^s] \in C_c^\infty(M, \Gamma) \ \text{ and } \ \check{P}_j(f_g \cdot [g]) := \frac{1}{n}\sum_{s=0}^{n-1} \bar{\omega}^{sj} f_g \cdot [gb^s],$$

where $\omega := e^{2\pi i/n}$, $f_g \in C_c^\infty(M)$ and $g \in G$; see also example i) in section 4.5 for the notation $\check{P}_j$ (and the equality $\check{P}_j = P_{-j}$). It induces an isomorphism

$$\Theta : H^*(C_c^\infty(M, \Gamma)) \xrightarrow{\cong} H^*(C_c^\infty(M, G))^{\oplus n}, \; \varphi \longmapsto (\Theta_0(\varphi), \dots, \Theta_{n-1}(\varphi)),$$

where for $\varphi \in H^m(C_c^\infty(M, \Gamma))$, $\Theta_j(\varphi)$ is the class of the continuous cyclic cocycle

$$C_c^\infty(M, G)^{m+1} \longrightarrow \mathbb{C}, \; (a_0, \dots, a_m) \longmapsto \varphi(\check{P}_j a_0, \dots, \check{P}_j a_m).$$

5.4.2 Remark. *At first sight, the choice of the above decomposition, namely with the $\check{P}_j$'s in place of the P_j's, might seem incoherent with the choices we make in all the other cases. However, this will be justified a posteriori, when we compute the delocalized equivariant Chern character ch_*^Γ for a product $\Gamma = G \times \mathbb{Z}/n$: see remark 5.5.5 and proposition 5.5.4.*

Now, we can state the lemma.

5.4.3 Lemma. *Let G be a countable discrete group acting properly by diffeomorphisms on a smooth manifold M without boundary, and let $\Gamma = G \times \mathbb{Z}/n$. Consider the map*

$$\Omega : H^*(C_c^\infty(M, G))^{\oplus n} \longrightarrow H^*(C_c^\infty(M, G))^{\oplus n}, \; \vec{\varphi} = {}^t(\varphi_0, \dots, \varphi_{n-1}) \longmapsto A \cdot \vec{\varphi},$$

where A is the Vandermonde block $(n \times n)$-matrix

$$A = \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \bar{\omega} & \bar{\omega}^2 & \dots & \bar{\omega}^{n-1} \\ 1 & \bar{\omega}^2 & \bar{\omega}^4 & \dots & \bar{\omega}^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \bar{\omega}^{n-1} & \bar{\omega}^{(n-1)\cdot 2} & \dots & \bar{\omega}^{(n-1)^2} \end{pmatrix}.$$

Then Ω is a $\mathbb{Z}/2$-graded isomorphism, that, with the above convention, fits in the following commutative diagram

$$\begin{array}{ccccc} H_*^\infty(G\backslash\widehat{M}_G; \mathbb{C})^{\oplus n} & \xrightarrow[\cong]{(\Phi^{C_c^\infty(M,G)})^{\oplus n}} & H^*(C_c^\infty(M, G))^{\oplus n} & \xrightarrow[\cong]{\Omega} & H^*(C_c^\infty(M, G))^{\oplus n} \\ \cong\downarrow I & & & & \cong\uparrow\Theta \\ H_*^\infty(\Gamma\backslash\widehat{M}_\Gamma; \mathbb{C}) & & \xrightarrow[\cong]{\Phi^{C_c^\infty(M,\Gamma)}} & & H^*(C_c^\infty(M, \Gamma)) \end{array}$$

Proof. We show that $(\Phi^{C_c^\infty(M,G)})^{\oplus n} = \Omega^{-1} \circ \Theta \circ \Phi^{C_c^\infty(M,\Gamma)} \circ I$, where Ω^{-1} has $A^{-1} = n \cdot \bar{A} = (\omega^{st})_{0 \le s,t \le n-1}$ as matrix. Let $Z = (Z_0, \dots, Z_{n-1}) \in \Omega_m^G(\widehat{M}_G)^{\oplus n}$ be an n-tuple of G-invariant de Rham currents of degree m on $\widehat{M}_G$ representing a class $[Z]$ in $H_m^\infty(G\backslash\widehat{M}_G)^{\oplus n}$. Then $\left(\Phi^{C_c^\infty(M,G)}\right)^{\oplus n}([Z]) \in H^m(C_c^\infty(M,G))^{\oplus n}$ is represented by the n-tuple of continuous cyclic cocycles whose l-th component is

$$C_c^\infty(M,G)^{m+1} \longrightarrow \mathbb{C},\ (a_0, a_1 \dots, a_m) \longmapsto \int_{Z_l} \xi(a_0 da_1 \cdots da_m).$$

On the other hand, $\Phi^{C_c^\infty(M,\Gamma)} \circ I([Z]) \in H^m(C_c^\infty(M,\Gamma))$ is represented by the continuous cyclic cocycle

$$C_c^\infty(M,\Gamma)^{m+1} \longrightarrow \mathbb{C},\ (b_0, b_1 \dots, b_m) \longmapsto \int_{Z} \xi(b_0 db_1 \cdots db_m).$$

Therefore, for any j, $\Theta_j \circ \Phi^{C_c^\infty(M,\Gamma)} \circ I([Z]) \in H^m(C_c^\infty(M,G))$ is represented by the continuous cyclic cocycle

$$C_c^\infty(M,G)^{m+1} \longrightarrow \mathbb{C},\ (a_0, a_1 \dots, a_m) \longmapsto \int_{Z} \xi((\check{P}_j a_0) d(\check{P}_j a_1) \cdots d(\check{P}_j a_m)).$$

Writing $a_k = \sum_{g\in G} f_{g,k}\cdot[g] \in C_c^\infty(M,G)$, with $f_{g,k} \in C_c^\infty(M)$, one has

$$da_k = \sum_{g\in G} df_{g,k}\cdot[g] \ \text{ and }\ \check{P}_j a_k = \sum_{g\in G}\sum_{s=0}^{n-1} \frac{\bar{\omega}^{sj}}{n} f_{g,k}\cdot[gb^s].$$

It follows that

$$d(\check{P}_j a_k) = \sum_{g\in G}\sum_{s=0}^{n-1} \frac{\bar{\omega}^{sj}}{n} df_{g,k}\cdot[gb^s].$$

For $f \in C_c^\infty(M)$ and $\gamma\in\Gamma$, to make the formulas shorter, we denote ${}^{\gamma}(df)$ simply by df^{γ}. It is straightforward to compute that

$$a_0 da_1 \cdots da_m = \sum_{g_0\in G} \cdots \sum_{g_m\in G} f_{g_0,0} \wedge df^{g_0}_{g_1,1} \wedge \cdots \wedge df^{g_{m-1}}_{g_m,m} \cdot [g_0 \cdots g_m]$$

and that

$$\begin{aligned}&(\check{P}_j a_0) d(\check{P}_j a_1) \cdots d(\check{P}_j a_m) = \\ &= \sum_{g_0\in G} \cdots \sum_{g_m\in G}\sum_{s_0=0}^{n-1} \cdots \sum_{s_m=0}^{n-1} \frac{\bar{\omega}^{j(s_0+\ldots+s_m)}}{n^{m+1}} f_{g_0,0} \wedge df^{g_0}_{g_1,1} \wedge \cdots \wedge df^{g_{m-1}}_{g_m,m} \cdot \\ &\qquad\qquad \cdot [g_0 \cdots g_m b^{s_0+\ldots+s_m}]\end{aligned}$$

From the equality $\int_Z \xi(\eta\cdot[gb^s]) = \int_{Z_s} \xi(\eta\cdot[g])$, for $g\in G$, it follows readily that

$$\int_Z \xi((\check{P}_j a_0) d(\check{P}_j a_1) \cdots d(\check{P}_j a_m)) = \sum_{s_0=0}^{n-1} \cdots \sum_{s_m=0}^{n-1} \frac{\bar{\omega}^{j(s_0+\ldots+s_m)}}{n^{m+1}} \cdot \int_{Z_{s_0+\ldots+s_m}} \xi(a_0 da_1 \cdots da_m),$$

where the subscript in $Z_{s_0+\ldots+s_m}$ is computed modulo n. Finally, we find that $\Omega^{-1}\circ\Theta\circ\Phi^{C_c^\infty(M,\Gamma)}\circ I([Z])\in H^m(C_c^\infty(M,G))^{\oplus n}$ is represented by the n-tuple of continuous cyclic cocycles whose l-th component is given by

$$\sum_{j=0}^{n-1}\omega^{jl}\int_Z\xi((\check{P}_ja_0)d(\check{P}_ja_1)\cdots d(\check{P}_ja_m))=$$

$$=\sum_{j=0}^{n-1}\sum_{s_0=0}^{n-1}\cdots\sum_{s_m=0}^{n-1}\frac{\bar{\omega}^{j(s_0+\ldots+s_m-l)}}{n^{m+1}}\cdot\int_{Z_{s_0+\ldots+s_m}}\xi(a_0da_1\cdots da_m)$$

$$=\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}n^m\frac{\bar{\omega}^{j(k-l)}}{n^{m+1}}\cdot\int_{Z_k}\xi(a_0da_1\cdots da_m)$$

$$=\sum_{k=0}^{n-1}\left(\sum_{j=0}^{n-1}\frac{\omega^{j(l-k)}}{n}\right)\cdot\int_{Z_k}\xi(a_0da_1\cdots da_m)$$

$$=\int_{Z_l}\xi(a_0da_1\cdots da_m)\,,$$

where the second equality is obtained by grouping the indices $(s_0,\ldots,s_m)$ according to their sum $k=s_0+\ldots+s_m$ modulo n, and the last equality follows from lemma 5.3.2.

This proves that $\left(\Phi^{C_c^\infty(M,G)}\right)^{\oplus n}([Z])=\Omega^{-1}\circ\Theta\circ\Phi^{C_c^\infty(M,\Gamma)}\circ I([Z])$ and completes the proof. □

5.5 The delocalized equivariant Chern character for $\Gamma=G\times\mathbb{Z}/n$

To define the map $\beta_t^{(j)}:H_j(\Gamma;F\Gamma)\longrightarrow K_j^\Gamma(\underline{E\Gamma})\otimes\mathbb{C}$, we will use the identification of the source of the map, given by the Shapiro lemma, with

$$\bigoplus_{C\in\langle\Gamma\rangle^{ell}}H_j(Z_C;\mathbb{C})\,,$$

where the homology groups are taken with trivial coefficients $\mathbb{C}$. It is therefore possible to focus on one summand at a time. The idea to delocalize from the summand corresponding to the trivial element $e\in\Gamma$ to the one corresponding to γ_C is in some sense to consider a cohomology class $x\in H_j(Z_C;\mathbb{C})$ as a pair (x,γ_C) (this will be made precise later, when we construct $\beta_t^{(j)}$). For this reason, we are led to consider groups of the form $\Gamma=G\times\mathbb{Z}/n$ (with G torsion-free). For such groups, we need to compute $K_*^\Gamma(\underline{E\Gamma})$ in terms of $K_*^G(\underline{EG})\cong K_*(BG)$ and $H_*(\Gamma;F\Gamma)$ in terms of $H_*(G; FG) = H_*(G; \mathbb{C})$. Moreover we need to express the delocalized equivariant Chern character ch_*^Γ in terms of ch_*^G, which coincides with the usual Chern character "à la Atiyah" $ch_* : K_*(BG) \longrightarrow H_*(G; \mathbb{C})$ (for G torsion-free). This section will therefore be a very first step in the direction of a Künneth formula in the framework of the delocalized equivariant Chern character.

Let G be a countable group (with or without torsion). Let $\Gamma = G \times \mathbb{Z}/n$, with $\mathbb{Z}/n$ generated by b ($n \geq 1$). As a model for $\underline{E\Gamma}$, we can obviously choose $\underline{EG}$.

Our first aim is to determine $K_*^\Gamma(\underline{E\Gamma})$ in terms of $K_*^G(\underline{EG})$. Recall that $K_*^G(\underline{EG})$ is defined as the direct limit, over the G-compact subspaces X of $\underline{EG}$, of the Kasparov groups $KK_*^G(C_0(X), \mathbb{C})$, and that an element of the latter group is given by the homotopy class of a triple $x = (\mathcal{H}, \pi, F)$, where $\mathcal{H}$ is a Hilbert space equipped with an action of G (see section 4.3). Clearly, for each $l = 0, \dots, n-1$, one gets a Kasparov triple $x[l] = (\mathcal{H}_l, \pi, F)$, where $\mathcal{H}_l = \mathcal{H}$ endowed with the action of $\Gamma = G \times \mathbb{Z}/n$ defined by the same action of G as before, and by the action of $\mathbb{Z}/n$ given by

$$b \cdot \xi := \omega^l \xi\,, \text{ for } \xi \in \mathcal{H}_l = \mathcal{H}\,,$$

where $\omega := e^{2\pi i/n}$. This passes to the direct limit and allows to associate unambiguously to each class $x \in K_*^G(\underline{EG})$ a set of n classes $x[l] \in K_*^\Gamma(\underline{EG}) = K_*^\Gamma(\underline{E\Gamma})$, for $l = 0, \dots, n-1$.

5.5.1 Proposition. *If for* $\Gamma = G \times \mathbb{Z}/n$, *we choose* $\underline{E\Gamma} := \underline{EG}$, *then the map*

$$K_*^G(\underline{EG})^{\oplus n} \longrightarrow K_*^\Gamma(\underline{E\Gamma})\,,\ (x_0, \dots, x_{n-1}) \longmapsto \sum_{l=0}^{n-1} x_l[l]$$

is an isomorphism, natural in G. *This generalizes to an arbitrary proper* G*-space* X *to yield an isomorphism* $K_*^G(X)^{\oplus n} \cong K_*^\Gamma(X)$.

Proof. Let x be a class in $K_*^\Gamma(\underline{E\Gamma})$. We can assume that x is given by a Kasparov triple $(\mathcal{H}, \pi, F)$ (over some Γ-compact subspace of $\underline{E\Gamma}$), with F Γ-equivariant (see section 4.3). (In fact, the assumption that F is $\mathbb{Z}/n$-equivariant would be enough in our argument.) One then decomposes the Hilbert space $\mathcal{H}$ as $\mathcal{H} = \oplus_{l=0}^{n-1} \mathcal{H}_l$, where $\mathcal{H}_l := \{\xi \in \mathcal{H} \mid b \cdot \xi = \omega^l \xi\}$. It is easy to check that F and π map $\mathcal{H}_l$ to itself (for F, this is a consequence of the Γ-equivariance, and for π, of the covariance). One then sets $x_l := [\mathcal{H}_l, \pi, F] \in K_*^G(\underline{EG})$, and the correspondence $x \longmapsto (x_0, \dots, x_{n-1})$ yields the inverse of the map in the statement. The naturality in G and the generalization to the case of an arbitrary proper G-space X are clear. □

This proposition obviously extends to the case where $\Gamma = G \times H$ with H finite, the left-hand side being replaced by $K_*^G(X) \otimes_{\mathbb{Z}} R(H)$, where $R(H)$ is the underlying additive group of the complex representation ring of H.

The following lemma is trivial.

5.5.2 Lemma. *Let G be a torsion free group, and let $\Gamma = G \times \mathbb{Z}/n$. Then the G-module FG is $\mathbb{C}$ with the trivial action of G, and the Shapiro lemma yields a canonical decomposition*

$$H_*(\Gamma;\, F\Gamma) = \bigoplus_{l=0}^{n-1} H_*(Z_\Gamma(b^l);\, \mathbb{C}) = H_*(\Gamma;\, \mathbb{C})^{\oplus n}.$$

We would now like to understand the delocalized equivariant Chern character ch_*^Γ. Since in the literature that we know, this map is only explicitly defined with a domain different from $K_*^\Gamma(\underline{E\Gamma})$, and denoted by $K^*(pt, \Gamma)$ in [6], we would like to determine a class of groups for which $K_*^\Gamma(\underline{E\Gamma})$ and $K^*(pt, \Gamma)$ coincide (see also the remark in section 4.6). In the following lemma, we show that this is the case when $\Gamma = G \times \mathbb{Z}/n$ and BG is a closed manifold.

5.5.3 Lemma. *Let G be a countable discrete group acting freely and properly by diffeomorphisms on a smooth manifold M, such that the orbit space $G\backslash M$ is a smooth closed manifold, and let $\Gamma := G \times \mathbb{Z}/n$. Then there are natural (Poincaré duality) isomorphisms*

$$\varphi_G : K_*^G(M) \xrightarrow{\cong} K_*(C_0(T^*M) \rtimes_r G)$$
$$\varphi_\Gamma : K_*^\Gamma(M) \xrightarrow{\cong} K_*(C_0(T^*M) \rtimes_r \Gamma)$$

such that the diagram

$$\begin{array}{ccc} K_*^\Gamma(M) & \xrightarrow[\cong]{\varphi_\Gamma} & K_*(C_0(T^*M) \rtimes_r \Gamma) \\ \psi \downarrow \cong & & \downarrow \cong \\ K_*^G(M)^{\oplus n} & \xrightarrow[\cong]{\varphi_G^{\oplus n}} & K_*(C_0(T^*M) \rtimes_r G)^{\oplus n} \end{array}$$

commutes, where ψ is the inverse of the isomorphism given by proposition 5.5.1. In particular, if BG is a smooth closed manifold (and therefore G is torsion-free), this holds for $M := \widetilde{BG} = EG = \underline{EG} = \underline{E\Gamma}$, and then there are isomorphisms

$$K_*(C_0(T^*M) \rtimes_r G) \cong K^*(pt, G) \text{ and } K_*(C_0(T^*M) \rtimes_r \Gamma) \cong K^*(pt, \Gamma).$$

Proof. It is clear that G acts freely and properly by diffeomorphisms on the cotangent bundle T^*M of M. There is a homeomorphism $G\backslash T^*M \cong T^*(G\backslash M)$, and consequently an isomorphism $C_0(G\backslash T^*M) \cong C_0(T^*(G\backslash M))$. Since G acts freely and properly on M, there is a canonical isomorphism $K_*^G(M) \cong K_*(G\backslash M)$ (see [107]). By compactness and absence of boundary of the manifold $G\backslash M$, Poincaré duality implies that the C^*-algebra $C(G\backslash M)$ is K-dual to $C_0(T^*(G\backslash M))$ (see 6.9

in [97]). On the other hand, since G acts freely and properly by diffeomorphisms on the smooth manifold T^*M, the C^*-algebra $C_0(G\backslash T^*M)$ is strongly Morita equivalent to the unreduced crossed product $C_0(T^*M) \rtimes G$, which, in this case, coincides with the reduced crossed product $C_0(T^*M) \rtimes_r G$ (see [34], p. 112). Putting these results together, we get

$$\begin{aligned} K_*^G(M) &\cong K_*(G\backslash M) = KK_*(C(G\backslash M), \mathbb{C}) \cong KK_*(\mathbb{C}, C_0(T^*(G\backslash M))) \\ &= K_*(C_0(G\backslash T^*M)) \cong K_*(C_0(G\backslash T^*M)) \cong K_*(C_0(T^*M) \rtimes_r G). \end{aligned}$$

This composition is the isomorphism φ_G we are looking for. For the group Γ, following the convention used in section 5.4, we write $\check{P}_j := P_{-j}$. As in corollary 5.3.4, there is a decomposition

$$\begin{aligned} C_0(T^*M) \rtimes_r \Gamma &= \bigoplus_{l=0}^{n-1} \check{P}_l \cdot (C_0(T^*M) \rtimes_r \Gamma) \cong \bigoplus_{l=0}^{n-1} \check{P}_l \cdot (C_0(T^*M) \rtimes_r G) \\ &\cong (C_0(T^*M) \rtimes_r G)^{\oplus n}. \end{aligned}$$

The isomorphism φ_Γ is simply defined by requiring commutativity in the diagram of the statement.

Finally, if BG is a closed manifold, then $M := \widetilde{BG} = EG$ satisfies the hypotheses of the first part of the lemma, and the equality $M = \underline{EG} = \underline{E\Gamma}$ is clear. Moreover, under our assumption, $(M, M \to pt)$ is a terminal object in the categories $\mathscr{C}(pt, G)$ and $\mathscr{C}(pt, \Gamma)$ of proper G-manifolds (resp. proper Γ-manifolds) considered in [6] for the definition of $K^*(pt, G)$ and of $K^*(pt, \Gamma)$. This yields isomorphisms

$$K_*(C_0(T^*M) \rtimes_r G) \cong K^*(pt, G) \text{ and } K_*(C_0(T^*M) \rtimes_r \Gamma) \cong K^*(pt, \Gamma),$$

and completes the proof. □

As can be expected from this lemma, we make the assumption that G is torsion-free and that BG is a closed manifold. We can then prove the

5.5.4 Proposition. *Let $\Gamma := G \times \mathbb{Z}/n$, where G is a countable discrete torsion-free group such that BG is a closed manifold. Let $x \in K_*^G(\underline{EG}) \cong K_*(BG)$ be a K-homology class, and let $\alpha := ch(x) \in H_*(G; \mathbb{C})$ be the associated homology class. Then, under the identifications given by lemmas 5.5.2 and 5.5.3, one has*

$$ch_*^\Gamma(x[l]) = \left(\frac{1}{n}\alpha, \frac{\omega^l}{n}\alpha, \ldots, \frac{\omega^{(n-1)l}}{n}\alpha\right) \in \bigoplus_{j=0}^{n-1} H_*(Z_\Gamma(b^j); \mathbb{C}),$$

for any $l \in \mathbb{Z}$, where $\omega = e^{2\pi i/n}$.

5.5.5 Remark. *It is implicit in this statement that we use the same convention as in section 5.4 and in lemma 5.5.3, namely we decompose the algebras $C_0(M) \rtimes_r \Gamma$ and $C_c^\infty(M, \Gamma)$ as a direct sum of sub-algebras parameterized by the $\check{P}_j$'s (and not* *by the P_j's). This is justified by the fact that for $\Gamma = \mathbb{Z}/n = \langle b \rangle$, any element y in $K_0^\Gamma(\underline{E\Gamma}) = K_0^\Gamma(pt)$ is represented by a Γ-vector bundle of rank one, in other words, it is a 1-dimensional representation ρ of $\mathbb{Z}/n$. With the above choice, the component of $ch_*^\Gamma(y)$ in $H_{ev}(Z_\Gamma(b^j); \mathbb{C}) \cong \mathbb{C}$ is simply $\rho(b^j)$ (instead of its complex conjugate): this is in accordance with Baum and Connes' formula 1.13 in [6], except for the factor $\frac{1}{n}$ that they have forgotten. Our choice is also motivated by analogy with some formulas we will establish in the framework of Hochschild and cyclic homology (see chapters 7 and 8).*

Proof of proposition 5.5.4. During this proof, we keep notations as in [6] and in section 5.4. We first deal with the even Chern character

$$ch_*^\Gamma : K_0^\Gamma(\underline{E\Gamma}) \longrightarrow H_{ev}(\Gamma; F\Gamma).$$

Again, M denotes $\widetilde{BG} = EG = \underline{EG} = \underline{E\Gamma}$, and we let $\tau^* = T^*M$ with the proper action of G and of Γ induced by their action on M. As in corollary 5.3.4, there are decompositions

$$C_0(\tau^*) \rtimes_r \Gamma = \bigoplus_{j=0}^{n-1} \check{P}_j \cdot (C_0(\tau^*) \rtimes_r \Gamma) \cong \bigoplus_{j=0}^{n-1} \check{P}_j \cdot (C_0(\tau^*) \rtimes_r C) \cong (C_0(\tau^*) \rtimes_r G)^{\oplus n}$$
$$C_c^\infty(\tau^*, \Gamma) = \bigoplus_{j=0}^{n-1} \check{P}_j \cdot C_c^\infty(\tau^*, \Gamma) \cong \bigoplus_{j=0}^{n-1} \check{P}_j \cdot C_c^\infty(\tau^*, G) \cong C_c^\infty(\tau^*, G)^{\oplus n}$$

(see section 5.4 for the definition of $C_c^\infty(\tau^*, G)$ and of $C_c^\infty(\tau^*, \Gamma)$). Proposition 5.5.3 yields isomorphisms

$$\varphi_G : K_0^G(\underline{EG}) \xrightarrow{\cong} K_0(C_0(T^*M) \rtimes_r C) = K^0(pt, C)$$
$$\varphi_\Gamma : K_0^\Gamma(\underline{E\Gamma}) \xrightarrow{\cong} K_0(C_0(T^*M) \rtimes_r \Gamma) = K^0(pt, \Gamma),$$

that fit in the commutative diagram

$$\begin{array}{ccccc} K_0^\Gamma(\underline{E\Gamma}) & \underset{\cong}{\xrightarrow{\varphi_\Gamma}} & K_0(C_0(\tau^*) \rtimes_r \Gamma) & \xleftarrow{\cong} & K_0(C_c^\infty(\tau^*, \Gamma)) \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ K_0^G(\underline{EG})^{\oplus n} & \underset{\cong}{\xrightarrow{\varphi_G^{\oplus n}}} & K_0(C_0(\tau^*) \rtimes_r G)^{\oplus n} & \xleftarrow{\cong} & K_0(C_c^\infty(\tau^*, G))^{\oplus n} \end{array}$$

where the horizontal isomorphisms on the right come from the density theorem (see also 7.5 in [6], and section 5.4). Given a topological algebra $\mathcal{A}$, we denote, as in [6], by $H^{ev}(\mathcal{A})^*$ the dual vector space of the periodic topological cyclic cohomology of even degree of $\mathcal{A}$, and by $ch^{\mathcal{A}} : K_0(\mathcal{A}) \longrightarrow H^{ev}(\mathcal{A})^*$ the Chern character due to

Connes ([33], [34]) and used in [6]. Starting from the definition of $ch^{\mathcal{A}}$, it is easy to verify that the diagram

$$\begin{array}{ccccc}
K_0(C_c^\infty(\tau^*,\Gamma)) & \xrightarrow{\cong} & \bigoplus_{j=0}^{n-1} K_0(\check{P}_j \cdot C_c^\infty(\tau^*,\Gamma)) & \xrightarrow{\cong} & K_0(C_c^\infty(\tau^*,G))^{\oplus n} \\
\Big\downarrow {\scriptstyle ch^{C_c^\infty(\tau^*,\Gamma)}} & & \Big\downarrow {\scriptstyle ch^{\check{P}_j \cdot C_c^\infty(\tau^*,G)}} & & \Big\downarrow {\scriptstyle \left(ch^{C_c^\infty(\tau^*,G)}\right)^{\oplus n}} \\
H^{ev}(C_c^\infty(\tau^*,\Gamma))^* & \xrightarrow{\cong} & \bigoplus_{j=0}^{n-1} H^{ev}(\check{P}_j \cdot C_c^\infty(\tau^*,\Gamma))^* & \xrightarrow{\cong} & (H^{ev}(C_c^\infty(\tau^*,G))^*)^{\oplus n}
\end{array}$$

commutes. Now, to relate these periodic cyclic cohomology groups to topology, consider the isomorphism $\Omega : H^{ev}(G_c^\infty(\tau^*,G))^{\oplus n} \xrightarrow{\cong} H^{ev}(C_c^\infty(\tau^*,G))^{\oplus n}$ of lemma 5.4.3. By passing to the dual vector space, under the obvious identification, we get an isomorphism $\Omega^* : (H^{ev}(C_c^\infty(\tau^*,G))^*)^{\oplus n} \xrightarrow{\cong} (H^{ev}(C_c^\infty(\tau^*,G))^*)^{\oplus n}$ described by the Vandermonde matrix $A^* = \bar{A} = \frac{1}{n}(\omega^{st})_{0 \le s,t \le n-1}$. By taking duals of the vector spaces involved in the diagram of lemma 5.4.3, we get the following commutative diagram

$$\begin{array}{ccccc}
H^{ev}(C_c^\infty(\tau^*,\Gamma))^* & & \xrightarrow[\cong]{\left(\Phi^{G_c^\infty(M,\Gamma)}\right)^*} & & H^\infty_{ev}(\Gamma\backslash\hat{\tau}^*_\Gamma;\mathbb{C}) \\
\Theta^* \Big\downarrow \cong & & & & \cong \Big\downarrow I^* \\
(H^{ev}(C_c^\infty(\tau^*,G))^*)^{\oplus n} & \xrightarrow[\cong]{\Omega^*} & (H^{ev}(C_c^\infty(M,G))^*)^{\oplus n} & \xrightarrow{\cong} & (H^\infty_{ev}(G\backslash\widehat{\tau^*_G};\mathbb{C})^*)^{\oplus n}
\end{array}$$

where the bottom right isomorphism is $\left(\left(\Phi^{C_c^\infty(M,G)}\right)^*\right)^{\oplus n}$, and as in section 5.4, one has decompositions

$$\widehat{\tau^*_G} = \tau^*\,,\quad \hat{\tau}^*_\Gamma = \coprod_{l=0}^{n-1} \tau^* \quad\text{and}\quad \Gamma\backslash\hat{\tau}^*_\Gamma = \coprod_{l=0}^{n-1} G\backslash\widehat{\tau^*_G}\,,$$

that yield an obvious isomorphism $J_c : H_c^{ev}(\Gamma\backslash\hat{\tau}^*_\Gamma;\mathbb{C}) \xrightarrow{\cong} H_c^{ev}(G\backslash\widehat{\tau^*_G};\mathbb{C})^{\oplus n}$ and similarly in homology with countable locally finite chains with complex coefficients (see section 5.4), $I : H^\infty_{ev}(G\backslash\widehat{\tau^*_G};\mathbb{C})^{\oplus n} \xrightarrow{\cong} H^\infty_{ev}(\Gamma\backslash\hat{\tau}^*_\Gamma;\mathbb{C})$, whose dual is I^* (see also 5.4). The isomorphisms I^* and J_c enter in the diagram

$$\begin{array}{ccccc}
H^\infty_{ev}(\Gamma\backslash\hat{\tau}^*_\Gamma;\mathbb{C})^* & \longrightarrow & H_c^{ev}(\Gamma\backslash\hat{\tau}^*_\Gamma;\mathbb{C}) & \xrightarrow{\cong} & H_{ev}(\Gamma;F\Gamma) \\
I^* \Big\downarrow \cong & & J_c \Big\downarrow \cong & & \Big\downarrow \cong \\
(H^\infty_{ev}(G\backslash\widehat{\tau^*_G};\mathbb{C})^*)^{\oplus n} & \longrightarrow & H_c^{ev}(G\backslash\widehat{\tau^*_G};\mathbb{C})^{\oplus n} & \xrightarrow{\cong} & H_{ev}(G;FG)^{\oplus n}
\end{array}$$

that is also commutative, where the right-hand vertical isomorphism is given by the Shapiro lemma (see lemma 5.5.2 and notice that $FG = \mathbb{C}$), and the horizontal maps, from left to right, come from [6], sections 7.17-7.20 and 15 respectively.

Remark that commutativity of the right-hand square follows from 15.8-15.11 in [6], and that for the direct limit argument in 7.18-7.20 of [6], one chooses, at the same time for G and for Γ, open subsets $X_i \subseteq \tau^*$ which are G-invariant (hence Γ-invariant), and such that

$$X_1 \subseteq X_2 \subseteq \ldots \subseteq \bigcup_{i\geq 1} X_i = \tau^* \quad \text{and} \quad \dim_{\mathbb{C}} H^{\infty}_{ev}(G\backslash X_i;\, \mathbb{C}) < \infty\,.$$

Following the definition [6], one easily computes the "delocalized equivariant Todd genus" $Td(\tau^*,\, \Gamma)$ in terms of $Td(\tau^*,\, G)$: one obtains

$$J(Td(\tau^*,\, \Gamma)) = (Td(\tau^*,\, G),\, \ldots\, ,\, Td(\tau^*,\, G)) \in H^*(G\backslash\widehat{\tau^*_G};\, \mathbb{C})^{\oplus n}\,,$$

where $J : H^*(\Gamma\backslash\widehat{\tau^*_\Gamma};\, \mathbb{C}) \xrightarrow{\cong} H^*(G\backslash\widehat{\tau^*_G};\, \mathbb{C})^{\oplus n}$ is the obvious map, for cohomology with unrestricted supports, corresponding to J_c.

Since $ch^\Gamma_* : K^\Gamma_0(\underline{E\Gamma}) \longrightarrow H_{ev}(\Gamma;\, F\Gamma)$ is defined by requiring commutativity in the diagram

$$\begin{array}{ccccc}
K^\Gamma_0(\underline{E\Gamma}) & \xrightarrow{\cong} & K_0(G^\infty_c(\tau^*,\, \Gamma)) & \xrightarrow{ch^{G^\infty_c(\tau^*,\Gamma)}} & H^{ev}(G^\infty_c(\tau^*,\, \Gamma))^* \\
{\scriptstyle ch^\Gamma_*}\downarrow & & & & \downarrow \\
H_{ev}(\Gamma;\, F\Gamma) & \xleftarrow{\cong} & H^{ev}_c(\Gamma\backslash\widehat{\tau^*_\Gamma};\, \mathbb{C}) & \xleftarrow{\cup Td(\tau^*,\, \Gamma)} & H^{ev}_c(\Gamma\backslash\widehat{\tau^*_\Gamma};\, \mathbb{C})
\end{array}$$

and similarly for G, one gets the announced result by combining these two diagrams with the four preceding ones, and using the above computation of $Td(\tau^*,\, \Gamma)$.

The case of the odd Chern character $ch^\Gamma_* : K^\Gamma_1(\underline{E\Gamma}) \longrightarrow H_{odd}(\Gamma;\, F\Gamma)$ is completely similar (one also works with the even Chern character, but with $M \times \mathbb{R}$ in place of M, where G and Γ act trivially on the factor $\mathbb{R}$).

This completes the proof. □

5.6 The Baum-Connes assembly map for a product $\Gamma = G \times \mathbb{Z}/n$

In the present section, we continue the study started in section 5.5, and devoted to the case of a direct product $\Gamma = G \times \mathbb{Z}/n$. Given $x \in K^G_*(\underline{EG})$, we compute, in terms of $\mu^G_*(x)$, the image by the assembly map μ^Γ_* of the element of $K^\Gamma_*(\underline{E\Gamma})$ denoted by $x[l]$ in proposition 5.5.1. As a by-product, this gives an explicit proof of the well-known fact that if the group G satisfies the Baum-Connes conjecture, then so does the group $\Gamma = G \times \mathbb{Z}/n$.

Let $G_n = \mathbb{Z}/n$, generated by b. By corollary 5.3.4, one has clearly a decomposition of $\mathbb{C}G_n$, as a direct sum of 1-dimensional algebras with unit, given by

$$\mathbb{C}G_n = \bigoplus_{l=0}^{n-1} P_l \cdot \mathbb{C}G_n = \bigoplus_{l=0}^{n-1} \mathbb{C} \cdot P_l .$$

The following lemma is now obvious (it is a particular case of corollary 5.3.4).

5.6.1 Lemma. *One has a decomposition*

$$C^*\Gamma \cong C^*G \otimes_{\mathbb{C}} \mathbb{C}G_n = \bigoplus_{l=0}^{n-1} P_l \cdot C^*G \cong C^*G^{\oplus n},$$

where the latter isomorphism (of unital C^*-algebras) is given by

$$P_l \cdot C^*G \longrightarrow C^*G, \; P_l \cdot z \longmapsto (0, \ldots, 0, z, 0, \ldots, 0)$$

with z in l-th position. The corresponding statement holds for $\mathbb{C}\Gamma$, $\ell^1\Gamma$ and $C_r^*\Gamma$.

(See the proof of proposition 5.6.4 for a very explicit description of the isomorphism $\mathbb{C}\Gamma \cong \mathbb{C}G^{\oplus n}$ of the lemma.)

5.6.2 Corollary. *The K-theory of the maximal C^*-algebra of Γ splits as*

$$K_*(C^*\Gamma) \cong \bigoplus_{l=0}^{n-1} K_*(P_l \cdot C^*G) \cong K_*(C^*G)^{\oplus n} .$$

The corresponding statement holds for $\ell^1\Gamma$ and $C_r^*\Gamma$.

Let us recall the following well-known lemma:

5.6.3 Lemma. *Let A and B be two algebras (resp. Banach algebras) with unit 1_A and 1_B respectively. Let $\alpha : A \longrightarrow B$ be a non-unital homomorphism (resp. continuous homomorphism). Then the induced map $\alpha_* : K_1^{alg}(A) \longrightarrow K_1^{alg}(B)$ (resp. $\alpha_* : K_1(A) \longrightarrow K_1(B)$) is given, for $u \in \mathrm{GL}_1(A)$, by*

$$[u] \longmapsto [\alpha(u) + 1_B - \alpha(1_A)] .$$

We arrive now at the main result in this section.

5.6.4 Proposition. *Let $x \in K_*^G(\underline{EG})$ and $l \in \{0, \ldots, n-1\}$. Let $x[l] \in K_*^\Gamma(\underline{E\Gamma})$ be the corresponding element. Then, under the isomorphism of corollary 5.6.2, $\tilde{\mu}_*^\Gamma(x[l]) \in K_*(C^*\Gamma)$ maps to*

$$(0, \ldots, 0, \tilde{\mu}_*^G(x), 0, \ldots, 0) \in K_*(C^*G)^{\oplus n},$$

with $\tilde{\mu}_*^G(x)$ in l-th position. In other words, one has

$$\tilde{\mu}_*^\Gamma(x[l]) = \begin{cases} [P_l \cdot \tilde{\mu}_*^G(x)], & \text{if } * = 0 \\ [P_l \cdot \tilde{\mu}_*^G(x) + 1 - P_l], & \text{if } * = 1 . \end{cases}$$

Proof. Assume that x is the K-homology class of a Kasparov triple $(\mathcal{H}, \pi, F)$ over a G-compact subspace X of $\underline{EG}$. By the (first) definition of $\tilde{\mu}_*^G$ and $\tilde{\mu}_*^\Gamma$ given in [107], and the construction of $x[l]$ (see section 5.5), we have to compare the Hausdorff completions of $\pi(C_c(X))\mathcal{H}$ for the (not necessarily positive definite) scalar products

$$\langle \xi_1 \,|\, \xi_2 \rangle_G = \sum_{g\in G} \langle \xi_1 \,|\, g\xi_2 \rangle_{\mathcal{H}} \cdot g \in \mathbb{C}G$$

$$\langle \xi_1 \,|\, \xi_2 \rangle_\Gamma = \sum_{\gamma\in\Gamma} \langle \xi_1 \,|\, \gamma\xi_2 \rangle_{\mathcal{H}} \cdot \gamma \in \mathbb{C}\Gamma = \bigoplus_{j=0}^{n-1} P_j \cdot \mathbb{C}G$$

(where $\xi_1, \xi_2 \in \pi(C_c(X))\mathcal{H}$). For this purpose, we need to give an explicit formula for the isomorphism $\psi : \mathbb{C}\Gamma \xrightarrow{\cong} \mathbb{C}G^{\oplus n}$ of lemma 5.6.1. Let $\omega = e^{2\pi i/n}$. By proposition 5.3.3 iv), one has, for $\gamma = g \cdot b^s \in G \times G_n = \Gamma$,

$$\gamma = \sum_{j=0}^{n-1} P_j \cdot \bar{\omega}^{sj} g \, .$$

Therefore, the composition $\psi_j : \mathbb{C}\Gamma \xrightarrow{\cong} \mathbb{C}G^{\oplus n} \xrightarrow{\pi_j} \mathbb{C}G$, where π_j is the projection onto the j-th factor, is given by

$$\sum_{\gamma = g\cdot b^s \in\Gamma} \lambda_\gamma \cdot \gamma \longmapsto \sum_{g\in G} \sum_{s=0}^{n-1} \bar{\omega}^{sj} \lambda_{g\cdot b^s} \cdot g \, .$$

Since by construction of $x[l]$, one has $\langle \xi_1 \,|\, \gamma\xi_2 \rangle_{\mathcal{H}} = \omega^{ls} \langle \xi_1 \,|\, g\xi_2 \rangle_{\mathcal{H}}$ for $\gamma = gb^s$, under ψ_j, the element $\langle \xi_1 \,|\, \xi_2 \rangle_\Gamma$ maps to

$$\sum_{g\in G} \sum_{s=0}^{n-1} \omega^{s(l-j)} \cdot \langle \xi_1 \,|\, g\xi_2 \rangle_{\mathcal{H}} \cdot g = \delta_{lj} \cdot n \sum_{g\in G} \langle \xi_1 \,|\, g\xi_2 \rangle_{\mathcal{H}} \cdot g = \delta_{lj} \cdot n \, \langle \xi_1 \,|\, \xi_2 \rangle_G$$

(where the middle equality comes from lemma 5.3.2).

This means that we complete $\pi(C_c(X))\mathcal{H}$ with respect to the zero scalar product in all except one component, in which we complete with respect to the same scalar product (up to a trivial renormalization) as for G. This establishes the first formula of the statement.

For the other two formulas, we have to compose with the map induced in K-theory by the non-unital inclusion $P_l \cdot C^*G \hookrightarrow C^*\Gamma$. The equality $\tilde{\mu}_0^\Gamma(x[l]) = [P_l \cdot \tilde{\mu}_0^G(x)]$ is obvious, and $\tilde{\mu}_1^\Gamma(x[l]) = [P_l \cdot \tilde{\mu}_1^G(x) + 1 - P_l]$ follows directly from lemma 5.6.3.

This completes the proof. □

As already mentioned, this proposition has an easy consequence.

5.6.5 Corollary. *If for a countable discrete group G, the Baum-Connes assembly map μ_*^G is (rationally or not) an isomorphism (resp. an injection, or a split-injection, or a surjection), then so is the assembly map μ_*^Γ for $\Gamma = G \times \mathbb{Z}/n$.*

5.7 The maps β_t and β_a for $H_0(\Gamma; F\Gamma)$

Let Γ be a countable discrete group. In the present section, we construct the homomorphisms

$$\beta_t^{(0)} : H_0(\Gamma; F\Gamma) \longrightarrow K_0^{\Gamma}(\underline{E\Gamma}) \otimes \mathbb{C} \text{ and } \beta_a^{(0)} : H_0(\Gamma; F\Gamma) \longrightarrow K_0(C_r^*\Gamma) \otimes \mathbb{C},$$

we show that $\beta_t^{(0)}$ is a right inverse of the delocalized equivariant Chern character $ch_*^{\Gamma} \otimes Id$, and that $(\mu_0^{\Gamma} \otimes Id) \circ \beta_t^{(0)} = \beta_a^{(0)}$. We also prove that $\beta_a^{(0)}$ factorizes through $K_0(l^1\Gamma) \otimes \mathbb{C}$, and that the corresponding map is split-injective.

In this section, to lighten the notations, we write β_t and β_a for $\beta_t^{(0)}$ and $\beta_a^{(0)}$ respectively (and similarly for the other maps). This will cause no confusion.

We begin by defining β_t. Let n be a positive integer. We denote by ω the n-th primitive root of 1 in $\mathbb{C}$ given by $e^{2\pi i/n}$. Let G_n be the finite cyclic group $\mathbb{Z}/n$, generated by b. By lemma 5.5.2, one has (and this is in fact trivial)

$$H_0(G_n; FG_n) = \bigoplus_{l=0}^{n-1} H_0(Z_{G_n}(b^l); \mathbb{C}) = \bigoplus_{l=0}^{n-1} \mathbb{C} \cdot [a]_l,$$

where $[a]_l$ is the canonical generator for which $H_0(Z_{G_n}(b^l); \mathbb{Z}) = \mathbb{Z} \cdot [a]_l$. On the other hand, by proposition 5.5.1 or example 4.3 ii), one has

$$K_0^{G_n}(\underline{EG_n}) = R(G_n) = \bigoplus_{l=0}^{n-1} \mathbb{Z} \cdot x[l] \cong \mathbb{Z}^n,$$

where $R(G_n)$ is the (underlying group of the) representation ring of G_n, and $x[l]$ is the 1-dimensional representation of G_n given by ω^l. It follows from proposition 5.5.4 (and lemma 5.3.2) that

$$ch_*^{G_n} \otimes Id : K_0^{G_n}(\underline{EG_n}) \otimes \mathbb{C} \longrightarrow H_0(G_n; FG_n)$$

is an isomorphism (independently of the general result of [6], partially proved by a spectral sequence argument). Let

$$z_n := (ch_*^{G_n} \otimes Id)^{-1}([a]_1) \in K_0^{G_n}(\underline{EG_n}) \otimes \mathbb{C} \cong \bigoplus_{l=0}^{n-1} \mathbb{C} \cdot x[l].$$

One can easily check (by a direct application of proposition 5.5.4 and of lemma 5.3.2) that

$$z_n = (A^*)^{-1} \cdot {}^t(0, 1, 0, \dots, 0) = {}^t(x[0], \bar{\omega} \cdot x[1], \dots, \bar{\omega}^{n-1} \cdot x[n-1]),$$

where A is the Vandermonde $(n \times n)$-matrix

$$A = \frac{1}{n} \cdot \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \bar{\omega} & \bar{\omega}^2 & \dots & \bar{\omega}^{n-1} \\ 1 & \bar{\omega}^2 & \bar{\omega}^4 & \dots & \bar{\omega}^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \bar{\omega}^{n-1} & \bar{\omega}^{(n-1)\cdot 2} & \dots & \bar{\omega}^{(n-1)^2} \end{pmatrix}$$

of lemma 5.4.3, that satisfies $(A^*)^{-1} = n \cdot A$.

Now let $C \in \langle\Gamma\rangle^{ell}$ be an elliptic conjugacy class in Γ. Recall that we have chosen a representative element $\gamma_C \in C$, whose order is denoted by n_C. Consider now the group homomorphism

$$\alpha_{\gamma_C} : G_{n_C} \longrightarrow \Gamma\,,\ b \longmapsto \gamma_C\,.$$

5.7.1 Definition. *The map*

$$\beta_t : H_0(\Gamma; F\Gamma) \cong \bigoplus_{C \in \langle\Gamma\rangle^{ell}} H_0(Z_C; \mathbb{C}) \longrightarrow K_0^\Gamma(\underline{E\Gamma}) \otimes \mathbb{C}$$

is defined by $\beta_t([\gamma_C]) := (\alpha_{\gamma_C})_*(z_{n_C})$, *where* $[\gamma_C]$ *is the canonical generator of* $H_0(Z_C; \mathbb{C})$ *for which* $H_0(Z_C; \mathbb{Z}) = \mathbb{Z} \cdot [\gamma_C]$, *and*

$$(\alpha_{\gamma_C})_* : K_0^{G_{n_C}}(\underline{EG_{n_C}}) \otimes \mathbb{C} \longrightarrow K_0^\Gamma(\underline{E\Gamma}) \otimes \mathbb{C}$$

is induced by the homomorphism α_{γ_C}.

Since the delocalized equivariant Chern character is natural, since γ_C and Z_C are uniquely defined up to conjugation by an element of Γ, and since any conjugation in Γ induces the identity on $K_0^\Gamma(\underline{E\Gamma})$, the above definition of β_t immediately yields

5.7.2 Proposition. *The map* β_t *is a well-defined natural homomorphism, and it is a right inverse of the delocalized equivariant Chern character:*

$$(ch_*^\Gamma \otimes Id) \circ \beta_t = Id_{H_0(\Gamma; F\Gamma)}\,.$$

Notice that the map β_t is an extension of the map

$$\beta_t^{loc} \otimes Id : H_0(\Gamma; \mathbb{C}) = H_0(\Gamma; \mathbb{Z}) \otimes \mathbb{C} \longrightarrow K_0^\Gamma(\underline{E\Gamma}) \otimes \mathbb{C}$$

defined in section 5.2.

The description of the functoriality of $K_0^\Gamma(\underline{E\Gamma})$ in the group Γ to be found in [107] allows for giving an explicit formula for β_t, namely

$$\beta_t([\gamma_C]) = \sum_{l=0}^{n_C-1} \mathrm{Ind}_{\langle\gamma_C\rangle}^\Gamma(\omega_C^l) \otimes \bar{\omega}_C^l \in K_0^\Gamma(\underline{E\Gamma}) \otimes \mathbb{C}\,,$$

where $\mathrm{Ind}_{\langle\gamma_C\rangle}^{\Gamma}(\omega_C^l)$ is the representation of Γ induced from the one-dimensional representation of its cyclic subgroup $\langle\gamma_C\rangle$, generated by γ_C, given by multiplication by ω_C^l. As we have seen in example i) of section 4.3, this really determines an element of $K_0^{\Gamma}(\underline{E\Gamma})$.

In section 5.2, we have defined a map

$$\beta_a^{loc} = \beta_a^{\Gamma, loc} : \mathbb{Z} = H_0(\Gamma;\, \mathbb{Z}) \longrightarrow K_0(C_r^*\Gamma)\,,\ 1 \longmapsto [1]\,,$$

where $[1]$ is the class of the unity of $C_r^*\Gamma$. The idea to define the map

$$\beta_a = \beta_a^{\Gamma} : H_0(\Gamma;\, F\Gamma) \longrightarrow K_0(C_r^*\Gamma) \otimes \mathbb{C}$$

is to "delocalize" the above map by means of the spectral projectors $P_l^{(C)}$ described in section 5.3. More precisely, we will present a way (kindly suggested to Hela Bettaieb by Paul Baum) of gluing together the maps $\beta_a^{Z_C, loc}$, for C running through $\langle\Gamma\rangle^{ell}$. Before proceeding, we need to recall that K-theory is also functorial for non-unital homomorphisms of algebras, and that the K-theory of a finite direct sum of algebras is canonically (and naturally) isomorphic to the direct sum of the corresponding K-theory groups.

In order to simplify some formulas, we will sometimes denote by $G_{\mathbb{C}}$ the tensor product $G \otimes_{\mathbb{Z}} \mathbb{C}$, where G is an abelian group.

5.7.3 Definition. *The map $\beta_a : H_0(\Gamma;\, F\Gamma) \longrightarrow K_0(C_r^*\Gamma) \otimes \mathbb{C}$ is defined on the direct summand $H_0(Z_C;\, \mathbb{C})$ by the following composition of homomorphisms:*

$$H_0(Z_C;\, \mathbb{C}) \xrightarrow{\ \beta_a^{Z_C, loc} \otimes Id\ } K_0(C_r^* Z_C)_{\mathbb{C}} \xrightarrow{\ \cong\ } \bigoplus_{l=0}^{n_C - 1} K_0(P_l^{(C)} \cdot C_r^* Z_C)_{\mathbb{C}} \xrightarrow{\ \sum_l (\iota_l^{(C)})_* \otimes \bar{\omega}_C^l\ } K_0(C_r^*\Gamma)_{\mathbb{C}}\,,$$

where $(\iota_l^{(C)})_* : K_0(P_l^{(C)} \cdot C_r^* Z_C) \longrightarrow K_0(C_r^*\Gamma)$ is the homomorphism induced by the non-unital inclusion of C^*-algebras $\iota_l^{(C)} : P_l^{(C)} \cdot C_r^* Z_C \hookrightarrow C_r^* Z_C \hookrightarrow C_r^*\Gamma$. In other words, we have

$$\beta_a([\gamma_C]) = \sum_{l=0}^{n_C - 1} [P_l^{(C)}] \otimes \bar{\omega}_C^l \in K_0(C_r^*\Gamma) \otimes \mathbb{C}\,,$$

where $[P_l^{(C)}]$ is the K-theory class of the projector $P_l^{(C)} \in C_r^*\Gamma$.

Notice that the formula describing $\beta_a([\gamma_C])$ is purely algebraic (more precisely, in the definition of β_a, we can replace everywhere $C_r^* Z_C$ and $C_r^*\Gamma$ by $\mathbb{C}Z_C$ and $\mathbb{C}\Gamma$ respectively). Therefore, since a conjugation by an element of Γ induces the identity on $K_0(\mathbb{C}\Gamma)$, one has immediately the

5.7.4 Proposition. *The map β_a is a well-defined homomorphism, and if $\mathcal{A}$ is any unital algebra equipped with two unital algebra morphisms $\mathbb{C}\Gamma \xrightarrow{f} \mathcal{A} \xrightarrow{g} C_r^*\Gamma$ such that $g \circ f$ is the inclusion (as for example $\mathbb{C}\Gamma$, $\ell^1\Gamma$ or $C^*\Gamma$), then β_a factorizes as*

$$\begin{array}{ccc} H_0(\Gamma; F\Gamma) & \xrightarrow{\beta_a} & K_0(C_r^*\Gamma) \otimes \mathbb{C} \\ & {\scriptstyle \beta_{a,\mathcal{A}}} \searrow \quad \nearrow {\scriptstyle g_* \otimes Id} & \\ & K_0(\mathcal{A}) \otimes \mathbb{C} & \end{array}$$

For $\mathcal{A} = \ell^1\Gamma$ (resp. $C^*\Gamma$), we denote $\beta_{a,\mathcal{A}}$ by $\hat{\beta}_a$ (resp. $\bar{\beta}_a$).

We will soon prove that $\hat{\beta}_a$ is injective.

Notice that the map β_a is an extension of the map

$$\beta_a^{loc} \otimes Id : H_0(\Gamma; \mathbb{C}) \longrightarrow K_0(C_r^*\Gamma) \otimes \mathbb{C}$$

defined in section 5.2.

Now, we would like to investigate the naturality of β_a. Since the functoriality of $K_0(C_r^*\Gamma)$ in the group Γ is still an open problem (see section 4.5), as in the case of the Baum-Connes assembly map, we focus on the case of the "'functorial group algebras" $\ell^1\Gamma$ and $C^*\Gamma$, and also $\mathbb{C}\Gamma$. Therefore, we now prove the

5.7.5 Proposition. *The map $\beta_{a,\mathcal{A}} : H_0(\Gamma; F\Gamma) \longrightarrow K_0(\mathcal{A}) \otimes \mathbb{C}$ is a natural homomorphism for $\mathcal{A} = \mathbb{C}\Gamma$, $\ell^1\Gamma$ and $C^*\Gamma$.*

Proof. It is enough to check naturality in the case of $\mathcal{A} = \mathbb{C}\Gamma$. Let $\varphi : \Gamma_1 \longrightarrow \Gamma_2$ be a group homomorphism, and let $\mathcal{A}_1 = \mathbb{C}\Gamma_1$ and $\mathcal{A}_2 = \mathbb{C}\Gamma_2$. We have to prove that the diagram

$$\begin{array}{ccc} H_0(\Gamma_1; F\Gamma_1) & \xrightarrow{\beta^{\Gamma_1}_{a,\mathcal{A}_1}} & K_0(\mathbb{C}\Gamma_1) \otimes \mathbb{C} \\ \varphi_* \downarrow & & \downarrow \varphi_* \otimes Id \\ H_0(\Gamma_2; F\Gamma_2) & \xrightarrow{\beta^{\Gamma_2}_{a,\mathcal{A}_2}} & K_0(\mathbb{C}\Gamma_2) \otimes \mathbb{C} \end{array}$$

commutes. Let $C_1 \in \langle\Gamma_1\rangle^{ell}$ and $\gamma_{C_1} \in C_1$ be a representative element. Since β_a is well-defined (i.e. independent of these choices), the explicit formula given for β_a shows that $(\varphi_* \otimes Id) \circ \beta^{\Gamma_1}_{a,\mathcal{A}_1}([\gamma_{C_1}]) = \beta^{\Gamma_2}_{a,\mathcal{A}_2} \circ \varphi_*([\gamma_{C_1}])$ whenever the order of γ_{C_1} is equal to the order of $\varphi(\gamma_{C_1})$ in Γ_2.

For the general case, by construction of β_a, it is enough to check naturality for the "classifying family" of groups $\{G_n = \mathbb{Z}/n\}_{n\geq 1}$ for the homology group $H_0(\Gamma; F\Gamma)$. It is even enough to check the case of a surjection

$$\varphi : \Gamma_1 = G_{n_1} = \langle b_1 \rangle \twoheadrightarrow \Gamma_2 = G_{n_2} = \langle b_2 \rangle \ , \ b_1 \longmapsto b_2 \quad (n_2 \text{ dividing } n_1)$$

and simply for $\gamma_{C_1} = b_1$ (the generator), or equivalently for

$$\varphi : \Gamma = G_n = \langle b \rangle \longrightarrow \Gamma \ , \ b \longmapsto b^k$$

and $\gamma_{C_1} = b$. In this case, we have (with the notations of lemma 5.3.5)

$$(\varphi_* \otimes Id) \circ \beta_a([a]_1) = \sum_{l=0}^{n-1} \frac{1}{n} \sum_{s=0}^{n-1} (\omega_n^l b^k)^s \otimes \bar{\omega}_n^l = \sum_{l=0}^{n-1} Q_{l,k}^{(n)} \otimes \bar{\omega}_n^l$$

$$\beta_a \circ \varphi_*([a]_1) = \sum_{l=0}^{m-1} \frac{1}{m} \sum_{s=0}^{m-1} (\omega_m^l b^k)^s \otimes \bar{\omega}_m^l = \sum_{l=0}^{m-1} Q_{l,k}^{(m)} \otimes \bar{\omega}_m^l \, ,$$

where m is the order of b^k in $G_n = \mathbb{Z}/n$, $\omega_n = e^{2\pi i/n}$, and $\omega_m = e^{2\pi i/m}$. By lemma 5.3.5, both expressions coincide, and we are done. □

We prove now that β_t and β_a fulfill the desired compatibility property with respect to the Baum-Connes assembly map.

5.7.6 Proposition. *The following natural triangle commutes:*

$$\begin{array}{ccc} K_0^\Gamma(\underline{E\Gamma}) \otimes \mathbb{C} & \xrightarrow{\hat{\mu}_0^\Gamma \otimes Id} & K_0(\ell^1\Gamma) \otimes \mathbb{C} \\ \uparrow \beta_t & \nearrow \hat{\beta}_a & \\ H_0(\Gamma; F\Gamma) & & \end{array}$$

We use in this proposition the fact, mentioned in section 4.5, that the Baum-Connes assembly map $\mu_*^\Gamma : K_*^\Gamma(\underline{E\Gamma}) \longrightarrow K_*(C_r^*\Gamma)$ factorizes through $K_*(\ell^1\Gamma)$, and we denote the corresponding (natural) map by $\hat{\mu}_*^\Gamma$.

Proof. By construction of β_t and of β_a, and by naturality of β_t, $\hat{\beta}_a$ and of $\hat{\mu}_0^\Gamma$, it is enough to check it in the case of the "classifying family" $\{G_n\}_{n\geq 1}$ and for the particular generator $[a]_1$. But this follows readily from example i) in section 4.6, where it is explained that, for the 1-dimensional representation ω^l of G_n, one has $\mu_0^{G_n}(\omega^l) = [p_{\bar{\omega}^l}] = [P_l] \in K_0(C_r^*G_n) = K_0(\ell^1 G_n)$ (this is also a consequence of the proof of proposition 5.6.4). Indeed, since $\mu_*^{G_n} = \hat{\mu}_*^{G_n}$, one deduces that

$$(\hat{\mu}_*^{G_n} \otimes Id) \circ \beta_t([a]_1) = \sum_{l=0}^{n-1} \hat{\mu}_*^{G_n}(\omega^l) \otimes \bar{\omega}^l = \sum_{l=0}^{n-1} [P_l] \otimes \bar{\omega}^l = \hat{\beta}_a([a]_1) \, .$$

□

As promised, to end the present section, we prove that $\hat{\beta}_a$ is injective.

5.7.7 Theorem. *The map* $\hat{\beta}_a : H_0(\Gamma;\, F\Gamma) \longrightarrow K_0(\ell^1\Gamma)\otimes\mathbb{C}$ *is injective.*

(This result was independently proved by Hela Bettaieb in [10].)

Proof. To each elliptic conjugacy class $C \in \langle\Gamma\rangle^{ell}$, we associate the trace

$$\tau_C : \ell^1\Gamma \longrightarrow \mathbb{C},\ \sum_{\gamma\in\Gamma}\lambda_\gamma\cdot\gamma \longmapsto \sum_{\gamma\in C}\lambda_\gamma\,.$$

This map induces a homomorphism $(\tau_C)_* : K_0(\ell^1\Gamma) \longrightarrow \mathbb{C}$, which maps the class of an idempotent (1×1)-matrix x over $\ell^1\Gamma$ to $\tau_C(x)$.

Let $D \in \langle\Gamma\rangle^{ell}$ and $\gamma_D \in D$ be a representative element, of order n_D. We denote the (possibly empty) set $\{s \,|\, 0\le s\le n_D-1\,,\ s\neq 1 \text{ and } \gamma_D^s\in C\}$ by A_{CD}. One then computes (by means of lemma 5.3.2):

$$((\tau_C)_*\otimes Id)\circ\hat{\beta}_a([\gamma_D]) = \frac{1}{n_D}\sum_{l=0}^{n_D-1}\bar{\omega}_D^l\left(\delta_{CD}\cdot\omega_D^l+\sum_{s\in A_{CD}}\omega_D^{ls}\right) = \delta_{CD}\,,$$

where $\omega_D = e^{2\pi i/n_D}$, and δ_{CD} is a Kronecker symbol. This concludes the proof. □

It would of course be of great interest if one could prove injectivity of $\tilde{\beta}_a$, or even of β_a itself (β_a is injective if the Baum-Connes assembly map is rationally injective).

5.8 The maps β_t and β_a for $H_1(\Gamma;\, F\Gamma)$

This section is devoted to the construction of the homomorphisms

$$\beta_t^{(1)} : H_1(\Gamma;\, F\Gamma) \longrightarrow K_1^\Gamma(\underline{E\Gamma})\otimes\mathbb{C} \ \text{ and } \ \beta_a^{(1)} : H_1(\Gamma;\, F\Gamma) \longrightarrow K_1(C_r^*\Gamma)\otimes\mathbb{C}.$$

We show that $\beta_t^{(1)}$ is a right inverse of the equivariant delocalized Chern character $ch_*^\Gamma\otimes Id$, and that $(\mu_1^\Gamma\otimes Id)\circ\beta_t^{(1)} = \beta_a^{(1)}$.

In this section, to simplify the notations, we write β_t and β_a for $\beta_t^{(1)}$ and $\beta_a^{(1)}$ respectively (and similarly for the other maps). This will cause no confusion.

The basic ideas in the present case are the same as those used in the case of $H_0(\Gamma;\, F\Gamma)$. Namely, we look for a "classifying family" of groups for the homology $H_1(\Gamma;\, F\Gamma)$ and we define β_t by "inverting" the delocalized equivariant Chern character for the members of this family, and finally we extend β_t for all groups by naturality. On the other hand, to define β_a, we delocalize the map

$$\beta_a^{loc} : H_1(\Gamma;\, \mathbb{Z}) \longrightarrow K_1(C_r^*\Gamma)$$

defined in section 5.2.

Let n be a positive integer. Let $G_n = \mathbb{Z} \times \mathbb{Z}/n$, with generators a for $\mathbb{Z}$ and b for $\mathbb{Z}/n$. By lemma 5.5.2, one has a decomposition

$$H_1(G_n;\, FG_n) = \bigoplus_{l=0}^{n-1} H_1(Z_{G_n}(b^l);\, \mathbb{C}) = \bigoplus_{l=0}^{n-1} G_n \otimes \mathbb{C} = \bigoplus_{l=0}^{n-1} \mathbb{C} \cdot [a]_l\,,$$

where $[a]_l$ denotes the element $a \otimes 1$ in the tensor product $Z_{G_n}(b^l) \otimes \mathbb{C}$. On the other hand, by proposition 5.5.1, one has

$$K_1^{G_n}(\underline{E}G_n) = \bigoplus_{l=0}^{n-1} \mathbb{Z} \cdot x[l] \cong \mathbb{Z}^n\,,$$

where x denotes the "Toeplitz" generator of $K_1^{\mathbb{Z}}(\underline{E}\mathbb{Z}) \cong K_1(S^1) \cong \mathbb{Z}$ (see example iii) in section 4.3). Since $ch_*^{\mathbb{Z}} = ch_* : K_1^{\mathbb{Z}}(\underline{E}\mathbb{Z}) \longrightarrow H_1(\mathbb{Z};\, \mathbb{C})$ is an isomorphism after tensoring with $\mathbb{C}$, it follows from proposition 5.5.4 (and lemma 5.3.2) that

$$ch_*^{G_n} \otimes Id : K_1^{G_n}(\underline{E}G_n) \otimes \mathbb{C} \longrightarrow H_1(G_n;\, FG_n)$$

is an isomorphism (independently of the general result of [6]). We proceed by letting

$$z_n := (ch_*^{G_n} \otimes Id)^{-1}([a]_1) \in K_1^{G_n}(\underline{E}G_n) \otimes \mathbb{C} \cong \bigoplus_{l=0}^{n-1} \mathbb{C} \cdot x[l]\,.$$

By a similar computation as in the case of $H_0(\Gamma;\, F\Gamma)$ (see section 5.7), we find that

$$z_n = (x[0], \bar{\omega} \cdot x[1], \ldots, \bar{\omega}^{n-1} \cdot x[n-1])\,.$$

In order to define β_t, we let $C \in \langle\Gamma\rangle^{ell}$ be an elliptic conjugacy class of Γ, with representative element $\gamma_C \in C$, of order n_C. Let $\gamma \in Z_C$ be an element whose class in $H_1(Z_C;\, \mathbb{C}) = Z_C^{ab} \otimes \mathbb{C}$ is $\gamma^{ab} \otimes 1$. Consider the group homomorphism

$$\alpha_{\gamma,\gamma_C} : G_{n_C} = \mathbb{Z} \times \mathbb{Z}/n_C \longrightarrow \Gamma\,,\ a \longmapsto \gamma\,,\ b \longmapsto \gamma_C\,.$$

Clearly, the map $(\alpha_{\gamma,\gamma_C})_* : H_1(G_{n_C};\, FG_{n_C}) \longrightarrow H_1(\Gamma;\, F\Gamma)$ takes $[a]_1$ to $\gamma^{ab} \otimes 1$ (after the identification given by the Shapiro lemma). This leads naturally to the following definition:

5.8.1 Definition. *The map*

$$\beta_t : H_1(\Gamma;\, F\Gamma) \cong \bigoplus_{C \in \langle\Gamma\rangle^{ell}} H_1(Z_C;\ \mathbb{C}) \longrightarrow K_1^{\Gamma}(\underline{E}\Gamma) \otimes \mathbb{C}$$

is defined by $\beta_t(\gamma^{ab}) := (\alpha_{\gamma_C,\gamma})_*(z_{n_C})$, *for* $\gamma \in Z_C$, *where*

$$(\alpha_{\gamma,\gamma_C})_* : K_1^{G_{n_C}}(\underline{E}G_{n_C}) \otimes \mathbb{C} \longrightarrow K_1^{\Gamma}(\underline{E}\Gamma) \otimes \mathbb{C}$$

is induced by the homomorphism α_{γ,γ_C}.

Since the delocalized equivariant Chern character is natural, and since any conjugation of Γ induces the identity on $K_1^{\Gamma}(\underline{E}\Gamma)$, one has

5.8.2 Proposition. *The map β_t is a well-defined natural homomorphism, and it is a right inverse of the delocalized equivariant Chern character:*

$$(ch_*^{\Gamma} \otimes Id) \circ \beta_t = Id_{H_1(\Gamma; F\Gamma)} .$$

Notice that the map β_t is an extension of the map

$$\beta_t^{loc} \otimes Id : H_1(\Gamma; \mathbb{C}) = H_1(\Gamma; \mathbb{Z}) \otimes \mathbb{C} \longrightarrow K_1^{\Gamma}(\underline{E}\Gamma) \otimes \mathbb{C}$$

defined in section 5.2.

Since the homomorphism $\alpha_{\gamma, \gamma_C}$ is injective (for $\gamma^{ab} \otimes 1 \neq 0$), the description of the functionality of $K_*^{\Gamma}(\underline{E}\Gamma)$ in Γ given in [107] allows to make, in concrete cases, explicit computations for $\beta_t(\gamma^{ab} \otimes 1) \in K_1^{\Gamma}(\underline{E}\Gamma) \otimes \mathbb{C}$.

Now, to define $\beta_a : H_1(\Gamma; F\Gamma) \longrightarrow K_1(C_r^*\Gamma) \otimes \mathbb{C}$, we recall the definition of its localized version constructed in section 5.2:

$$\beta_a^{loc} = \beta_a^{\Gamma, loc} : H_1(\Gamma; \mathbb{Z}) = \Gamma^{ab} \longrightarrow K_1(C_r^*\Gamma) , \; \gamma^{ab} \longmapsto -[\gamma] = [\gamma^{-1}] ,$$

where $[\gamma]$ is the K-theory class of $\gamma \in \Gamma$. As for $H_0(\Gamma; F\Gamma)$ (see section 5.7), we are going to delocalize $\beta_a^{\Gamma, loc}$, more precisely, we are going to glue together the different maps $\beta_a^{Z_C, loc}$, for $C \in \langle \Gamma \rangle^{ell}$.

5.8.3 Definition. *The map $\beta_a : H_1(\Gamma; F\Gamma) \longrightarrow K_1(C_r^*\Gamma) \otimes \mathbb{C}$ is defined on the direct summand $H_1(Z_C; \mathbb{C})$ by the following composition of homomorphisms:*

$$\begin{array}{ccccc} H_1(Z_C; \mathbb{C}) & \xrightarrow{\beta_a^{Z_C, loc} \otimes Id} & K_1(C_r^* Z_C)_{\mathbb{C}} & \xrightarrow{\cong} & \bigoplus_{l=0}^{n_C - 1} K_1(P_l^{(C)} \cdot C_r^* Z_C)_{\mathbb{C}} \\ & & & & \Big\downarrow {\scriptstyle \sum_l (\iota_l^{(C)})_* \otimes \bar{\omega}_C^l} \\ & & & & K_1(C_r^*\Gamma)_{\mathbb{C}} , \end{array}$$

where $(\iota_l^{(C)})_* : K_1(P_l^{(C)} \cdot C_r^* Z_C) \longrightarrow K_1(C_r^*\Gamma)$ is the homomorphism induced by the non-unital inclusion of C^*-algebras $\iota_l^{(C)} : P_l^{(C)} \cdot C_r^* Z_C \hookrightarrow C_r^* Z_C \hookrightarrow C_r^*\Gamma$.

The following proposition follows readily from the above definition and from lemma 5.6.3.

5.8.4 Proposition. *For $\gamma^{ab} \in H_1(Z_C; \mathbb{Z})$ representing $\gamma \in Z_C$, one has*

$$\beta_a(\gamma^{ab}) = \sum_{l=0}^{n_C - 1} [P_l^{(C)} \cdot \gamma^{-1} + 1 - P_l^{(C)}] \otimes \bar{\omega}_C^l \in K_1(C_r^*\Gamma) .$$

Let $\mathcal{A}$ denote a unital Banach algebra. Recall that its algebraic K_1-theory is by definition $K_1^{alg}(\mathcal{A}) = \mathrm{GL}(\mathcal{A})/[\mathrm{GL}(\mathcal{A}), \mathrm{GL}(\mathcal{A})]$, where $\mathrm{GL}(\mathcal{A}) = \mathrm{GL}_\infty(\mathcal{A})$. The topological K_1-theory of $\mathcal{A}$ is the group $K_1(\mathcal{A}) = \pi_0(\mathrm{GL}(\mathcal{A})) = \mathrm{GL}(\mathcal{A})/\mathrm{GL}(\mathcal{A})_0$, where $\mathrm{GL}(\mathcal{A})_0$ denotes the arc connected component of the identity of the topological group $\mathrm{GL}(\mathcal{A})$. It is classical that any commutator in $\mathrm{GL}(\mathcal{A})$ is in $\mathrm{GL}(\mathcal{A})_0$ (see for example prop. 3.4.1 in [16]). (This shows in particular that $K_1(\mathcal{A})$ is abelian.) There is consequently a canonical (and natural) map

$$j_{\mathcal{A}} : K_1^{alg}(\mathcal{A}) \longrightarrow K_1(\mathcal{A}).$$

Notice that the formula describing $\beta_a([\gamma^{ab} \otimes 1])$ is purely algebraic (more precisely, in the definition of β_a, we can replace everywhere $C_r^* Z_C$ and $C_r^*\Gamma$ by $\ell^1 Z_C$ and $\ell^1\Gamma$ respectively). It is moreover possible to work with K_1^{alg} in place of K_1, and then also with the group algebras $\mathbb{C}Z_C$ and $\mathbb{C}\Gamma$; indeed, the map $\beta_a^{Z_C, loc}$ clearly factorizes through $K_1^{alg}(\mathbb{C}Z_C)$. Therefore, since a conjugation by an element of Γ induces the identity on $K_1^{alg}(\mathbb{C}\Gamma)$, one has immediately the

5.8.5 Proposition. *The map* β_a *is a well-defined homomorphism, and if* $\mathcal{A}$ *is any unital algebra equipped with two unital algebra morphisms* $\mathbb{C}\Gamma \xrightarrow{f} \mathcal{A} \xrightarrow{g} C_r^*\Gamma$ *such that* $g \circ f$ *is the inclusion (as for example* $\mathbb{C}\Gamma$, $\ell^1\Gamma$, $C^*\Gamma$ *or* $C_r^*\Gamma$*), then* β_a *factorizes as*

$$\begin{array}{ccc} H_1(\Gamma; F\Gamma) & \xrightarrow{\beta_a} & K_1(C_r^*\Gamma) \otimes \mathbb{C} \\ & {\scriptstyle \alpha_{a,\mathcal{A}}} \searrow \quad \nearrow {\scriptstyle (j_{C_r^*\Gamma} \circ g_*) \otimes Id} & \\ & K_1^{alg}(\mathcal{A}) \otimes \mathbb{C} & \end{array}$$

Moreover, if $\mathcal{A}$ *is any unital Banach algebra equipped with two unital Banach algebra morphisms* $\ell^1\Gamma \xrightarrow{f} \mathcal{A} \xrightarrow{g} C_r^*\Gamma$ *such that* $g \circ f$ *is the inclusion* j_r^Γ *(as for example* $\ell^1\Gamma$ *or* $C^*\Gamma$*), then* β_a *factorizes as*

$$\begin{array}{ccc} H_1(\Gamma; F\Gamma) & \xrightarrow{\beta_a} & K_1(C_r^*\Gamma) \otimes \mathbb{C} \\ {\scriptstyle \alpha_{a,\mathcal{A}}} \downarrow & & \uparrow {\scriptstyle g_* \otimes Id} \\ K_1^{alg}(\mathcal{A}) \otimes \mathbb{C} & \xrightarrow{j_{\mathcal{A}} \otimes Id} & K_1(\mathcal{A}) \otimes \mathbb{C} \end{array}$$

where $j_{\mathcal{A}}$ *is the canonical homomorphism. For* $\mathcal{A} = \ell^1\Gamma$ *(resp.* $C^*\Gamma$*), we denote* $\beta_{a,\mathcal{A}} := (j_{\mathcal{A}} \otimes Id) \circ \alpha_{a,\mathcal{A}}$ *by* $\hat{\beta}_a$ *(resp.* $\tilde{\beta}_a$*).*

We now investigate the naturality of β_a.

5.8.6 Proposition. *The map $\beta_{a,\mathcal{A}} : H_1(\Gamma; F\Gamma) \longrightarrow K_1(\mathcal{A}) \otimes \mathbb{C}$ is a natural homomorphism for $\mathcal{A} = \ell^1\Gamma$ and $C^*\Gamma$. The same holds for the map $\alpha_{a,\mathcal{A}}$, when $\mathcal{A} = \mathbb{C}\Gamma$, $\ell^1\Gamma$ or $C^*\Gamma$.*

Proof. It is enough to check it for $\alpha_{a,\mathbb{C}\Gamma} : H_1(\Gamma; F\Gamma) \longrightarrow K_1^{alg}(\mathbb{C}\Gamma) \otimes \mathbb{C}$. As in the case of $H_0(\Gamma; F\Gamma)$ (see the proof of proposition 5.7.5), it is sufficient to verify the naturality for the homomorphism

$$\varphi : G_n = \mathbb{Z} \times \mathbb{Z}/n \longrightarrow G_n \,,\ a \longmapsto a \,,\ b \longmapsto b^k \,,$$

and for the generator $[a]_1$ of $H_1(Z_{G_n}(b); \mathbb{C}) = G_n \otimes \mathbb{C}$. In that case, we compute (with the notations of lemma 5.3.5):

$$(\varphi_* \otimes Id) \circ \beta_a([a]_1) = \sum_{l=0}^{n-1} [Q_{l,k}^{(n)} \cdot a^{-1} + 1 - Q_{l,k}^{(n)}] \otimes \bar{\omega}_n^l$$

$$\beta_a \circ \varphi_*([a]_1) = \sum_{l=0}^{m-1} [Q_{l,k}^{(m)} \cdot a^{-1} + 1 - Q_{l,k}^{(m)}] \otimes \bar{\omega}_m^l \,,$$

where m is the order of b^k in $G_n = \mathbb{Z}/n$, $\omega_n = e^{2\pi i/n}$, and $\omega_m = e^{2\pi i/m}$. By observing that $\mathrm{GL}_1(\mathbb{C}G_n) \otimes_{\mathbb{Z}} \mathbb{C} \subset \mathbb{C}G_n \otimes_{\mathbb{Z}} \mathbb{C}$, it follows from lemmas 5.3.2 and 5.3.5 that both expressions coincide. □

We now relate β_t and β_a to the Baum-Connes assembly map.

5.8.7 Proposition. *The following natural triangle commutes:*

$$\begin{array}{ccc}
K_1^\Gamma(\underline{E\Gamma}) \otimes \mathbb{C} & \xrightarrow{\hat{\mu}_1^\Gamma \otimes Id} & K_1(\ell^1\Gamma) \otimes \mathbb{C} \\
\uparrow{\scriptstyle \beta_t} & \nearrow{\scriptstyle \hat{\beta}_a} & \\
H_1(\Gamma; F\Gamma) & &
\end{array}$$

Proof. By construction of β_t and of β_a, and by naturality of β_t, $\hat{\beta}_a$ and $\hat{\mu}_1^\Gamma$, it is enough to check it for the "classifying family" $\{G_n\}_{n \geq 1}$ and for the generator $[a]_1 \in H_1(Z_{G_n}(b); \mathbb{C}) = G_n \otimes \mathbb{C}$. We have

$$\beta_t([a]_1) = z_n = (x[0], \bar{\omega} \cdot x[1], \ldots, \bar{\omega}^{n-1} \cdot x[n-1]) \in \bigoplus_{l=0}^{n-1} \mathbb{C} \cdot x[l] \cong K_1^{G_n}(\underline{EG_n})_{\mathbb{C}} \,,$$

where $x \in K_1^{\mathbb{Z}}(\underline{E\mathbb{Z}})$ is the Toeplitz generator, whose image in $K_1(C^*\mathbb{Z})$, by $\tilde{\mu}_1^{\mathbb{Z}}$, is $[a^{-1}]$, with a the prescribed generator of $\mathbb{Z}$ (see example iii) in section 4.5).

Therefore, by proposition 5.6.4, we get

$$(\tilde{\mu}_1^{G_n} \otimes Id) \circ \beta_t([a]_1) = \sum_{l=0}^{n-1} [P_l \cdot \tilde{\mu}_1^{\mathbb{Z}}(x) - 1 + P_l] \otimes \bar{\omega}^l = \sum_{l=0}^{n-1} [P_l \cdot a^{-1} - 1 + P_l] \otimes \bar{\omega}^l$$
$$= \beta_{a, C^*\Gamma}([a]_1) \in K_1(C^*G_n) \otimes \mathbb{C}.$$

As mentioned in example 4.5 iv), by the Wiener lemma, $\ell^1\mathbb{Z}$ is a dense subalgebra of $C^*\mathbb{Z}$ and is stable under holomorphic functional calculus. Therefore, by the density theorem, the inclusion $\ell^1\mathbb{Z} \hookrightarrow C^*\mathbb{Z}$ induces an isomorphism in K-theory. By corollary 5.6.2, the same holds for the groups G_n in place of $\mathbb{Z}$. This shows that $(\hat{\mu}^{G_n} \otimes Id) \circ \beta_t([a]_1) = \beta_{a,\ell^1\Gamma}([a]_1) \in K_1(\ell^1\Gamma) \otimes \mathbb{C}$. □

(Notice that for the above proof, we did not need to establish an analogue of proposition 5.6.4 for the algebra $\ell^1\Gamma$.)

5.9 The maps β_t and β_a for $H_2(\Gamma;\, F\Gamma)$

In the present section, we construct the homomorphisms

$$\beta_t^{(2)} : H_2(\Gamma;\, F\Gamma) \longrightarrow K_0^{\Gamma}(\underline{E\Gamma}) \otimes \mathbb{C} \quad \text{and} \quad \beta_a^{(2)} : H_2(\Gamma;\, F\Gamma) \longrightarrow K_0(C_r^*\Gamma) \otimes \mathbb{C},$$

we show that $\beta_t^{(2)}$ is a right inverse of the delocalized equivariant Chern character $ch_*^{\Gamma} \otimes Id$, and that $(\mu_0^{\Gamma} \otimes Id) \circ \beta_t^{(2)} = \beta_a^{(2)}$. It will also follow from the definition that $\beta_a^{(2)}$ factorizes through $K_0(\ell^1\Gamma) \otimes \mathbb{C}$. We do *not* need to assume that Γ is of finite type.

In this section, to lighten the notations, we write β_t and β_a for $\beta_t^{(2)}$ and $\beta_a^{(2)}$ respectively (and similarly for the other maps). This will cause no confusion.

The ideas are the same as for $H_0(\Gamma;\, F\Gamma)$ and $H_1(\Gamma;\, F\Gamma)$. Namely, let n be a positive integer, $\omega = e^{2\pi i/n}$, and $G_{g,n} := \Gamma_g \times \mathbb{Z}/n$, where Γ_g is the fundamental group of the closed oriented surface Σ_g of genus g, and denote by b the (chosen) generator of $\mathbb{Z}/n$. By lemma 5.5.2, one has

$$H_2(G_{g,n};\, FG_{g,n}) = \bigoplus_{l=0}^{n-1} H_2(Z_{G_{g,n}}(b^l);\, \mathbb{C}) = \bigoplus_{l=0}^{n-1} H_2(\Gamma_g;\, \mathbb{C}) = \bigoplus_{l=0}^{n-1} \mathbb{C} \cdot [a_g]_l\,,$$

where, for any l, $[a_g]_l$ denotes the generator $[\Sigma_g]$ of $H_2(\Gamma_g;\, \mathbb{Z}) \cong \mathbb{Z}$ (the second equality above comes from the Künneth theorem and the classical fact that $\widetilde{H}_*(\mathbb{Z}/n;\, \mathbb{C}) = 0$). On the other hand, by proposition 5.5.1, one has

$$K_0^{G_{g,n}}(\underline{EG}_{g,n}) = \bigoplus_{l=0}^{n-1} (\mathbb{Z} \cdot y_g[l] \oplus \mathbb{Z} \cdot x_g[l])\,,$$

where y_g is the class of the trivial 1-dimensional bundle in $K_0(\Sigma_g) \cong K_0^{\Gamma_g}(\underline{E\Gamma}_g)$, and x_g denotes $[\Sigma_g]_K = ch_*^{-1}([\Sigma_g]) \in K_0(\Sigma_g) \cong K_0^{\Gamma_g}(\underline{E\Gamma}_g)$. It follows from proposition 5.5.4 (and lemma 5.3.2) that

$$ch_*^{G_{g,n}} \otimes Id : K_0^{G_{g,n}}(\underline{EG}_{g,n}) \otimes \mathbb{C} \longrightarrow H_0(G_{g,n};\, FG_{g,n}) \oplus H_2(G_{g,n};\, FG_{g,n})$$

is an isomorphism. We now let

$$z_{g,n} := (ch_*^{G_{g,n}} \otimes Id)^{-1}([a_g]_1) \in K_0^{G_{g,n}}(\underline{EG}_{g,n}) \otimes \mathbb{C} \cong \bigoplus_{l=0}^{n-1}(\mathbb{C} \cdot y_g[l] \oplus \mathbb{C} \cdot x_g[l]).$$

As in the case of $H_0(\Gamma;\, F\Gamma)$ and $H_1(\Gamma;\, F\Gamma)$, one computes that

$$z_{g,n} = (x_g[0],\, \bar{\omega} \cdot x_g[1],\, \ldots\, ,\, \bar{\omega}^{n-1} \cdot x_g[n-1]).$$

To define β_t, let $C \in \langle\Gamma\rangle^{ell}$ with $\gamma_C \in C$ of order n_C. If $x \in H_2(Z_C;\, \mathbb{Z})$, let $f : \Sigma_g \longrightarrow BZ_C$ be a map in $S(Z_C)$ such that $f_*([\Sigma_g]) = x$ (see section 5.2, and in particular theorem 5.2.2). Consider the group homomorphism

$$\alpha_{x,\gamma_C} : G_{g,n_C} = \Gamma_g \times \mathbb{Z}/n_C \longrightarrow \Gamma\,,\ (\alpha_{x,\gamma_C})_{|\Gamma_g} = \pi_1(f)\,,\ b \longmapsto \gamma_C\,.$$

Clearly, the map $(\alpha_{x,\gamma_C})_* : H_2(G_{g,n_C};\, FG_{g,n_C}) \longrightarrow H_2(\Gamma;\, F\Gamma)$ takes $[a_g]_1$ to $x \otimes 1$ in $H_2(Z_C;\, \mathbb{Z}) \otimes \mathbb{C} = H_2(Z_C;\, \mathbb{C})$. Therefore, we say that the groups $G_{g,n}$, for $g,\, n \geq 1$, constitute a "classifying family" for $H_2(\Gamma;\, F\Gamma)$, and we are led to the following

5.9.1 Definition. *The map*

$$\beta_t : H_2(\Gamma;\, F\Gamma) \cong \bigoplus_{C \in \langle\Gamma\rangle^{ell}} H_2(Z_C;\, \mathbb{C}) \longrightarrow K_0^\Gamma(\underline{E\Gamma}) \otimes \mathbb{C}$$

is defined by $\beta_t(x \otimes 1) := (\alpha_{x,\gamma_C})_*(z_{g,n_C})$, *for* $x \in H_2(Z_C;\, \mathbb{Z})$, *where*

$$(\alpha_{x,\gamma_C})_* : K_0^{G_{g,n_C}}(\underline{EG}_{g,n_C}) \otimes \mathbb{C} \longrightarrow K_0^\Gamma(\underline{E\Gamma}) \otimes \mathbb{C}$$

is induced by the homomorphism α_{x,γ_C}.

Since the delocalized Chern character is natural, $(ch_*^\Gamma \otimes Id) \circ \beta_t = Id_{H_2(\Gamma;\, F\Gamma)}$, which, together with the fact that $ch_*^\Gamma \otimes Id$ is an isomorphism(see theorem 4.6.1), implies that β_t is well-defined (i.e. independent of the choice of $f \in S(Z_C)$: see section 5.2). Since any conjugation of Γ induces the identity on $K_0^\Gamma(\underline{E\Gamma})$, we have just proved the

5.9.2 Proposition. *The map β_t is a well-defined natural homomorphism, and it is a right inverse of the delocalized equivariant Chern character:*

$$(ch_*^\Gamma \otimes Id) \circ \beta_t = Id_{H_2(\Gamma;\, F\Gamma)}\,.$$

Notice that the map β_t is an extension of the map

$$\beta_t^{loc} \otimes Id : H_2(\Gamma;\, \mathbb{C}) = H_2(\Gamma;\, \mathbb{Z}) \otimes \mathbb{C} \longrightarrow K_0^{\Gamma}(\underline{E\Gamma}) \otimes \mathbb{C}$$

defined in section 5.2.

Now, to define $\beta_a : H_2(\Gamma;\, F\Gamma) \longrightarrow K_0(C_r^*\Gamma) \otimes \mathbb{C}$, recall that we have defined its localized version in section 5.2 as a homomorphism

$$\beta_a^{\Gamma, loc} : H_2(\Gamma;\, \mathbb{Z}) \longrightarrow K_0(C_r^*\Gamma)\,.$$

We are again going to delocalize $\beta_a^{\Gamma, loc}$ to build β_a.

5.9.3 Definition. *The map $\beta_a : H_2(\Gamma;\, F\Gamma) \longrightarrow K_2(C_r^*\Gamma) \otimes \mathbb{C}$ is defined on the direct summand $H_2(Z_C;\, \mathbb{C})$ by the following composition of homomorphisms:*

$$\begin{array}{ccccc} H_2(Z_C;\, \mathbb{C}) & \xrightarrow{\ \beta_a^{Z_C, loc} \otimes Id\ } & K_0(C_r^* Z_C)_{\mathbb{C}} & \xrightarrow{\ \cong\ } & \displaystyle\bigoplus_{l=0}^{n_C - 1} K_0(P_l^{(C)} \cdot C_r^* Z_C)_{\mathbb{C}} \\ & & & & \Big\downarrow {\scriptstyle \sum_l (\iota_l^{(C)})_* \otimes \bar{\omega}_C^l} \\ & & & & K_0(C_r^*\Gamma)_{\mathbb{C}}\,, \end{array}$$

where $(\iota_l^{(C)})_* : K_1(P_l^{(C)} \cdot C_r^* Z_C) \longrightarrow K_1(C_r^*\Gamma)$ is the homomorphism induced by the non-unital inclusion of C^*-algebras $\iota_l^{(C)} : P_l^{(C)} \cdot C_r^* Z_C \hookrightarrow C_r^* Z_C \hookrightarrow C_r^*\Gamma$.

By construction, the map $\beta_a^{Z_C, loc}$ factorizes through $K_0(\ell^1 Z_C)$. We can therefore replace everywhere in the above definition $C_r^* Z_C$ and $C_r^*\Gamma$ by $\ell^1 Z_C$ and $\ell^1\Gamma$ respectively. Since the conjugation by an element of Γ induces the identity on $K_0(\ell^1\Gamma)$, one has the

5.9.4 Proposition. *The map β_a is a well-defined homomorphism, and if $\mathcal{A}$ is any unital Banach algebra equipped with two continuous unital algebra morphisms $\ell^1\Gamma \xrightarrow{f} \mathcal{A} \xrightarrow{g} C_r^*\Gamma$ such that $g \circ f$ is the inclusion j_r^{Γ} (as for example $\ell^1\Gamma$ or $C^*\Gamma$), then β_a factorizes as*

$$\begin{array}{ccc} H_2(\Gamma;\, F\Gamma) & \xrightarrow{\qquad \beta_a \qquad} & K_0(C_r^*\Gamma) \otimes \mathbb{C} \\ & {\scriptstyle \beta_{a,\mathcal{A}}} \searrow \qquad \nearrow {\scriptstyle g_* \otimes Id} & \\ & K_0(\mathcal{A}) \otimes \mathbb{C} & \end{array}$$

For $\mathcal{A} = \ell^1\Gamma$ (resp. $C^*\Gamma$), we denote $\beta_{a,\mathcal{A}}$ by $\hat{\beta}_a$ (resp. $\tilde{\beta}_a$).

It was conjectured by Pierre Julg and by Nigel Higson that the homomorphism $\beta_a^{loc}\otimes Id : H_2(\Gamma;\mathbb{C}) \longrightarrow K_0(C_r^*\Gamma)\otimes\mathbb{C}$ factorizes through $K_2^{alg}(\mathbb{C}\Gamma)\otimes\mathbb{C}$. In chapter 8 (a joint work with Hervé Oyono-Oyono), we will prove this, and show that the same holds for β_a. We will moreover establish the appropriate statement, for β_a^{loc}, without tensoring with $\mathbb{C}$.

Concerning naturality, we have the

5.9.5 Proposition. *For $\mathcal{A} = \ell^1\Gamma$ and $C^*\Gamma$, the map $\beta_{a,\mathcal{A}}$ is natural.*

Proof. We can restrict to the case where $\mathcal{A} = \ell^1\Gamma$.

Since any class $[f]$ in $\Omega(\Gamma)$ can be "stabilized" to get a function $\tilde{f}$ equivalent to f and defined on a surface Σ_g of arbitrarily large genus g, we can restrict, as in the case of $H_0(\Gamma;\, F\Gamma)$ (see the proof of proposition 5.7.5), to the case of a homomorphism

$$\varphi : G_{g,n} = \Gamma_g \times \mathbb{Z}/n \longrightarrow G_{g,n}\,,\ \varphi_{|\Gamma_g} = Id\,,\ b \longmapsto b^k\,,$$

and for the generator $[a_g]_1$ of $H_2(Z_{G_{g,n}}(b);\, \mathbb{C}) = \mathbb{C}\cdot[a_g]_1$. We let

$$c_g := \hat{\mu}_0^{\Gamma_g}(x_g) \in K_0(\ell^1\Gamma_g) \ \text{ and } \ c_g[l] := \hat{\mu}_0^{G_{g,n}}(x_g[l]) \in K_0(\ell^1 G_{g,n})\,,\ l\in\mathbb{Z}.$$

Notice that $c_g[0] = i_*(c_g)$, where i_* is induced by the inclusion $i : \Gamma_g \hookrightarrow G_{g,n}$. It is clear that $\varphi_*(x_g[l]) = x_g[kl]$, and by naturality of $\hat{\mu}_0^{G_{g,n}}$, we get $\varphi_*(c_g[l]) = c_g[kl]$, in particular $\varphi_*(c_g[0]) = c_g[0]$. By using this fact, we compute

$$(\varphi_*\otimes Id)\circ\beta_a([a_g]_1) = \sum_{l=0}^{n-1}[Q_{l,k}^{(n)}\cdot\varphi_*(c_g[0])]\otimes\bar{\omega}_n^l = \sum_{l=0}^{n-1}[Q_{l,k}^{(n)}\cdot c_g[0]]\otimes\bar{\omega}_n^l\,.$$

On the other hand, we have

$$\beta_a\circ\varphi_*([a_g]_1) = \beta_a([a_g]_k) = \sum_{l=0}^{m-1}[Q_{l,k}^{(m)}\cdot c_g[0]]\otimes\bar{\omega}_m^l\,,$$

where m is the order of b^k in $\mathbb{Z}/n$. By lemma 5.3.5, we get the desired equality. □

Let us now prove that β_t composed with the Baum-Connes assembly map coincides with β_a.

5.9.6 Proposition. *The following natural triangle commutes:*

$$\begin{array}{ccc} K_0^\Gamma(\underline{E\Gamma})\otimes\mathbb{C} & \xrightarrow{\hat{\mu}_0^\Gamma\otimes Id} & K_0(\ell^1\Gamma)\otimes\mathbb{C} \\ \uparrow{\scriptstyle \beta_t} & \nearrow{\scriptstyle \hat{\beta}_a} & \\ H_2(\Gamma;\, F\Gamma) & & \end{array}$$

Proof. By construction of β_a and of β_t, and by naturality of the triangle, it is enough to prove it for the "classifying family" $\{G_{g,n}\}_{g,n\geq 1}$ and for the generator $[a_g]_1$. One computes (as in the proof of proposition 5.9.5)

$$\beta_a([a_g]_1) = \sum_{l=0}^{n-1}[P_l\cdot i_*(c_g)]\otimes\bar{\omega}_n^l = \sum_{l=0}^{n-1}c_g[l]\otimes\bar{\omega}_n^l = (\hat{\mu}_0^{\Gamma_{g,n}}\otimes Id)\circ\beta_t([a_g]_1)$$

where $c_g = \hat{\mu}_0^{\Gamma_g}(y_g)\in K_0(\ell^1\Gamma_g)$ and i_* is induced by the inclusion $i:\Gamma_g\hookrightarrow G_{g,n}$, and where the second equality follows from the fact that

$$[P_l\cdot i_*(c_g)] = c_g[l]\in K_0(\ell^1 G_{g,n}),$$

which is easily deduced from proposition 5.6.4 and lemma 5.6.1. □

The preceding two propositions yield the following formulas for $x\in H_2(Z_C;\,\mathbb{Z})$:

$$\hat{\beta}_a(x\otimes 1) = ((\alpha_{x,\gamma_C})_*\otimes Id)(c_{g,n_C}) = \sum_{l=0}^{n_C-1}[P_l^{(C)}\cdot(\alpha_{x,\gamma_C})_*(c_g)]\otimes\bar{\omega}_C^l\in K_0(\ell^1\Gamma),$$

where $\alpha_{x,\gamma_C}: G_{g,n_C}\longrightarrow\Gamma$ "represents" x, and where $c_g := \hat{\mu}_0^{\Gamma_g}(x_g)\in K_0(\ell^1\Gamma_g)$ and, for $n\geq 1$, $c_{g,n} := \hat{\mu}_0^{G_{g,n}}(z_{g,n})\in K_0(\ell^1 G_{g,n})$. These are "universal classes" in the sense that they are independent of Γ, C and x.

Chapter 6

A delocalization property for the Baum-Connes assembly map

We prove that, for any countable discrete group Γ, the Baum-Connes assembly map (tensored by $\mathbb{C}$) admits a decomposition in terms of the Novikov assembly maps corresponding to the centralizers of finite order elements in Γ. This expression is based on the spectral projectors associated to torsion elements in Γ. This generalizes the results of chapter 5, and gives very useful information on the delocalized equivariant Chern character. We call this property the delocalization for the Baum-Connes assembly map. We formulate a conjecture stating that $K_*^\Gamma(\underline{E\Gamma}) \otimes \mathbb{C}$ decomposes as a direct sum of the K-homology groups $K_*(BZ_c) \otimes \mathbb{C}$, parameterized by the elliptic conjugacy classes in Γ. We prove that this conjecture would imply the existence of a well-defined, natural delocalized equivariant Chern character, that would be an isomorphism after tensoring with $\mathbb{C}$. We also prove the rational injectivity of the canonical map $K_*(B\Gamma) \longrightarrow K_*^\Gamma(\underline{E\Gamma})$, by constructing a retraction.

6.1 Introduction

In chapter 5, for any countable discrete group Γ and for $0 \leq j \leq 2$, we have constructed maps $\beta_a^\Gamma : H_j(\Gamma; F\Gamma) \longrightarrow K_j(C_r^*\Gamma) \otimes \mathbb{C}$ such that the diagram

$$\begin{array}{ccc} K_j^\Gamma(\underline{E\Gamma}) \otimes \mathbb{C} & \xrightarrow{\mu_j^\Gamma \otimes Id_{\mathbb{C}}} & K_j(C_r^*\Gamma) \otimes \mathbb{C} \\ {\scriptstyle ch_j^\Gamma}\downarrow & \nearrow{\scriptstyle \beta_a^\Gamma} & \\ H_j(\Gamma; F\Gamma) & & \end{array}$$

commutes. This was performed by "delocalizing" simpler assembly maps, namely $\beta_a^{Z_C, loc} : H_j(Z_C; \mathbb{Z}) \longrightarrow K_j(C_r^* Z_C)$, where C runs through the set of elliptic conjugacy classes of Γ (see section 5.3 for the notations). The assembly map $\beta_a^{\Gamma, loc}$ is, in some sense, the localized counterpart of β_a^Γ, more precisely, localized at the identity of Γ.

We would like to generalize these results in the following sense. The delocalized equivariant Chern character $ch_*^\Gamma : K_*^\Gamma(\underline{E}\Gamma) \longrightarrow H_*(\Gamma; F\Gamma)$ is an isomorphism after tensoring with $\mathbb{C}$. By the Shapiro lemma, the recipient $H_*(\Gamma; F\Gamma)$ decomposes as a direct sum parameterized by the set of elliptic conjugacy classes $\langle\Gamma\rangle^{ell}$ (see proposition 4.6.4). We would therefore like to find an explicit decomposition of the vector space $K_*^\Gamma(\underline{E}\Gamma) \otimes \mathbb{C}$ as a direct sum running over $\langle\Gamma\rangle^{ell}$ and then express the rational Baum-Connes assembly map $\mu_*^\Gamma \otimes Id_\mathbb{C}$ as the delocalization of assembly maps that are simpler, i.e. localized at the identity. By virtue of proposition 4.6.2, the natural candidate for the localized assembly map is the Novikov assembly map $\beta_*^\Gamma : K_*(B\Gamma) \longrightarrow K_*(C_r^*\Gamma)$. In this chapter, we partially solve this problem. First, we propose such a decomposition for $K_*^\Gamma(\underline{E}\Gamma) \otimes \mathbb{C}$ by constructing an explicit map. However, we are not able to prove that it is an isomorphism, we only conjecture this, and give some positive results in this direction. In a second part, we can prove a delocalization result for the rational Baum-Connes assembly map. This result would have more insight if the mentioned conjecture was true.

We keep notations as in chapter 4 and section 5.3.

The goal of section 6.2 is to propose a decomposition of the left-hand side in the Baum-Connes conjecture, namely $K_*^\Gamma(\underline{E}\Gamma) \otimes \mathbb{C}$, as a direct sum, parameterized by $\langle\Gamma\rangle^{ell}$, of $K_*(BZ_C) \otimes \mathbb{C}$, the K-homology groups of the classifying space of the centralizers Z_C of the representative elliptic elements $\gamma_C \in \Gamma$. An explicit map is constructed and conjectured to be an isomorphism. Partial results in this direction are established in section 6.5. They are based on some preparatory material on models for $\underline{E}Z_C$ with a trivial action of a given γ_C. The delocalization property for the Baum-Connes assembly map is stated and proved in section 6.3. The proof is an easy computation based on naturality properties and results of chapter 5. In section 6.6, we prove rational injectivity of $K_*(B\Gamma) \longrightarrow K_*^\Gamma(\underline{E}\Gamma)$. In the final section 6.7, we prove that our conjecture would imply the existence of a well-defined natural delocalized equivariant Chern character, that would be an isomorphism after tensoring with $\mathbb{C}$.

6.2 A decomposition of $K_*^\Gamma(\underline{E}\Gamma) \otimes \mathbb{C}$ in terms of elliptic elements in Γ

In order to state the delocalization property for the (rational) Baum-Connes assembly map, we need to express its domain $K_*^\Gamma(\underline{E}\Gamma) \otimes \mathbb{C}$ as a direct sum running over the set of elliptic conjugacy classes in the group Γ. The purpose of the present section is to propose such a decomposition and to formulate a conjecture about it. Recall that by definition, $K_*^\Gamma(\underline{E}\Gamma) = \varinjlim_X KK_*^\Gamma(C_0(X), \mathbb{C})$, where the direct limit runs over all Γ-compact subspaces X of $\underline{E}\Gamma$. For $i = 0$ or 1, an element in $KK_i^\Gamma(C_0(X), \mathbb{C})$ is given by the homotopy class of a triple $(\mathcal{H}, \pi, F)$, where $\mathcal{H}$ is a Hilbert space endowed with a unitary representation of Γ, π is a covariant representation of $C_0(X)$ on $\mathcal{H}$, and where F is a bounded self-adjoint operator on $\mathcal{H}$, such that $[F, \pi(f)]$, $\pi(f)(F^2-1)$ and $\pi(f)[\gamma, F]$ are compact operators on $\mathcal{H}$, for any $f\in C_0(X)$ and any $\gamma\in\Gamma$. For $i=0$, the Hilbert space $\mathcal{H}$ is moreover required to be $\mathbb{Z}/2$-graded, the representation of Γ to be by degree 0 operators, and F to be a degree 1 operator. We can moreover assume that the operator F is Γ-equivariant, in other words that $[F,\gamma]=0$ for any $\gamma\in\Gamma$, and tbat F is properly supported. (See section 4.3, [107] and [61] for details.)

Let $n\geq 1$. Given such a Kasparov triple $x = (\mathcal{H}, \pi, F)$ (defined over $X\subseteq \underline{E}\Gamma$), for $l\in\mathbb{Z}$, we define a Kasparov triple $x[l] = (\mathcal{H}_l, \pi, F)$ (still defined over X) for the group $\Gamma\times\mathbb{Z}/n$ by the following requirements: first, as a model for $\underline{E}(\Gamma\times\mathbb{Z}/n)$, we take $\underline{E}\Gamma$ (it is easily seen to be a possible choice), second, $\mathcal{H}_l$ is defined to be equal to $\mathcal{H}$ and is endowed with the action of $\Gamma\times\mathbb{Z}/n$ defined by the same action of Γ and by the action of $\mathbb{Z}/n$ given by $b\cdot\xi = \omega_n^l\xi$, for $\xi\in\mathcal{H}_l$, where b is the (chosen) generator of $\mathbb{Z}/n$ and $\omega_n := e^{2\pi i/n}$. See sections 5.5 and 5.6 (in particular propositions 5.5.1 and 5.6.4) for the main properties of these elements.

Let us now define the different maps appearing in the decomposition we are aiming at. Let C be an elliptic conjugacy class in the group Γ. Recall that as a model for $\underline{E}Z_C$ and for $\underline{E}(Z_C\times\mathbb{Z}/n_C)$ we can choose $\underline{E}\Gamma$. Consider the map

$$\begin{array}{rccc} \psi_C: & K_*^{Z_C}(\underline{E}Z_C)\otimes\mathbb{C} & \longrightarrow & K_*^{Z_C\times\mathbb{Z}/n_C}(\underline{E}(Z_C\times\mathbb{Z}/n_C))\otimes\mathbb{C} \\ & x\otimes\lambda & \longmapsto & \displaystyle\sum_{l=0}^{n_C-1} x[l]\otimes\bar{\omega}_C^l\cdot\lambda \end{array}$$

and the surjective group homomorphism

$$p_C: Z_C\times\mathbb{Z}/n_C \longrightarrow Z_C\,,\ p_{C|Z_C} = Id_{Z_C}\,,\ b\longmapsto\gamma_C\,,$$

where b is the (chosen) generator of $\mathbb{Z}/n_C$. We denote by j_C the inclusion of Z_C in Γ. These two group homomorphisms induce a map

$$(j_C\circ p_C)_* = (j_C)_*\circ(p_C)_*: K_*^{Z_C\times\mathbb{Z}/n_C}(\underline{E}(Z_C\times\mathbb{Z}/n_C)) \longrightarrow K_*^\Gamma(\underline{E}\Gamma)$$

for any C. Now, we define a map $\alpha: \bigoplus_{C\in\langle\Gamma\rangle^{ell}} K_*^{Z_C}(\underline{E}Z_C)\otimes\mathbb{C}\longrightarrow K_*^\Gamma(\underline{E}\Gamma)\otimes\mathbb{C}$ on the direct summand $K_*^{Z_C}(\underline{E}Z_C)\otimes\mathbb{C}$ by the composition

$$K_*^{Z_C}(\underline{E}Z_C)\otimes\mathbb{C}\xrightarrow{\psi_C}K_*^{Z_C\times\mathbb{Z}/n_C}(\underline{E}(Z_C\times\mathbb{Z}/n_C))\otimes\mathbb{C}\xrightarrow{(p_C)_*\otimes Id}K_*^{Z_C}(\underline{E}Z_C)\otimes\mathbb{C}\xrightarrow{(j_C)_*\otimes Id}K_*^{\Gamma}(\underline{E}\Gamma)\otimes\mathbb{C}$$

In other words, $\alpha := \sum_{C\in\langle\Gamma\rangle^{ell}} ((j_C\circ p_C)_*\otimes Id)\circ\psi_C$.

We are now in position to state the following conjecture.

6.2.1 Conjecture. *For any countable discrete group* Γ *, the composed map*

$$\phi:\bigoplus_{C\in\langle\Gamma\rangle^{ell}}K_*(BZ_C)\otimes\mathbb{C}\hookrightarrow\bigoplus_{C\in\langle\Gamma\rangle^{ell}}K_*^{Z_C}(\underline{E}Z_C)\otimes\mathbb{C}\xrightarrow{\alpha}K_*^{\Gamma}(\underline{E}\Gamma)\otimes\mathbb{C}$$

is an isomorphism, where the injection is given by proposition 4.5.3.

In section 6.4 we present the preparatory material (on models for $\underline{E}Z_C$) in order to prove, in section 6.5, that the composition $((p_C)_*\otimes Id)\circ\psi_C$ is an isomorphism by giving a very explicitly description and by exhibiting its inverse (that turns out to be very similar to it). In section 6.6, we construct a retraction of the injection given by proposition 4.5.3. In the remark ending section 6.7, we explain why the conjecture is plausible.

6.3 The delocalization for the Baum-Connes assembly map

We prove here the delocalization property for the Baum-Connes assembly map. The statement uses the map ϕ of conjecture 6.2.1, but is completely independent of its truth. The proof is an easy chase around the definitions, based on the naturality of the assembly map $\tilde{\mu}_*$ and on proposition 5.6.4.

Given a conjugacy class $C\in\langle\Gamma\rangle^{ell}$, we denote by

$$\tilde{\iota}_l^{(C)}:P_l^{(C)}\cdot C^*Z_C\hookrightarrow C^*\Gamma \text{ and } \iota_l^{(C)}:P_l^{(C)}\cdot C_r^*Z_C\hookrightarrow C_r^*\Gamma$$

the non-unital inclusions.

With all these notations, we can now state the main result in this chapter. It is in some sense a generalization of the construction of the maps β_a^Γ of chapter 5.

6.3.1 Theorem. (Delocalization for the Baum-Connes assembly map) *For any countable discrete group* Γ, *the diagram*

$$\begin{array}{ccc}
\bigoplus_{C\in\langle\Gamma\rangle^{ell}} K_*(BZ_C)\otimes\mathbb{C} & \xrightarrow{\quad\phi\quad} & K_*^{\Gamma}(\underline{E}\Gamma)\otimes\mathbb{C} \\
\Big\downarrow {\scriptstyle \bigoplus_C \tilde{\beta}_*^{Z_C}\otimes Id} & & \Big\downarrow {\scriptstyle \tilde{\mu}_*^{\Gamma}\otimes Id} \\
\bigoplus_{C\in\langle\Gamma\rangle^{ell}} K_*(C^*Z_C)\otimes\mathbb{C} & & \\
\Big\downarrow {\scriptstyle \cong} & & \\
\bigoplus_{C\in\langle\Gamma\rangle^{ell}}\ \bigoplus_{l=0}^{n_C-1} K_*(P_l^{(C)}\cdot C^*Z_C)\otimes\mathbb{C} & \xrightarrow{\sum_C\sum_l(\tilde{\iota}_l^{(C)})_*\otimes\tilde{\omega}_C^l} & K_*(C^*\Gamma)\otimes\mathbb{C}
\end{array}$$

commutes. There is a similar commutative diagram for the reduced C^**-algebras* $C_r^*Z_C$ *and* $C_r^*\Gamma$, *with the maps* $\iota_l^{(C)}$ *and* $\beta_*^{Z_C}$ *in place of* $\tilde{\iota}_l^{(C)}$ *and of* $\tilde{\beta}_*^{Z_C}$ *respectively.*

As we will see in chapter 7, this result in not just a particular property of the Baum-Connes assembly map. Many other assembly maps share a similar property. We will prove it for the assembly maps in Hochschild homology and in cyclic homology. We will also construct a new assembly map in algebraic K-theory by imposing this property.

Proof of theorem 6.3.1. During tbe proof, we identify $K_*(BZ_C)\otimes\mathbb{C}$ with its image in $K_*^{Z_C}(\underline{E}Z_C)\otimes\mathbb{C}$. We start with an element $x\in K_*(BZ_C)$, for some $C\in\langle\Gamma\rangle^{ell}$. If we let $y:=\tilde{\beta}_*^{Z_C}(x)$ in $K_*(C^*Z_C)$, then, by means of lemma 5.6.3, we compute

$$\begin{array}{llclcl}
 & K_j(C^*Z_C) & \xrightarrow{\cong} & \bigoplus_l K_j((P_l^{(C)}\cdot C^*Z_C) & \xrightarrow{\vartheta} & K_j(C^*Z_C)\otimes\mathbb{C} \\
j=0: & y & \longmapsto & \sum_l [P_l^{(C)}\cdot y] & \longmapsto & \sum_l [P_l^{(C)}\cdot y]\otimes\tilde{\omega}_C^l \\
j=1: & y & \longmapsto & \sum_l [P_l^{(C)}\cdot y] & \longmapsto & \sum_l [P_l^{(C)}\cdot y+1-P_l^{(C)}]\otimes\tilde{\omega}_C^l
\end{array}$$

where $\vartheta:=\sum_l(\tilde{\iota}_l^{(C)})_*\otimes\tilde{\omega}_C^l$. By pushing forward these elements in $K_*(C^*\Gamma)\otimes\mathbb{C}$ by $(j_C)_*$, we get the same formulas, according to the value of j. On the other hand, by naturality of $\tilde{\mu}_*^{\Gamma}$ (see section 4.5) and since $y=\tilde{\beta}_*^{Z_C}(x)=\tilde{\mu}_*^{Z_C}(x)$, for $j=0$, we get

$$\begin{aligned}(\tilde{\mu}_*^\Gamma \otimes Id)\circ\phi(x) &= (\tilde{\mu}_*^\Gamma \otimes Id)\circ((j_C\circ p_C)_*\otimes Id)\circ\psi_C(x)\\ &= \sum_l(\tilde{\mu}_*^\Gamma \otimes Id)\circ((j_C\circ p_C)_*\otimes Id)(x[l]\otimes\bar{\omega}_C^l)\\ &= \sum_l((j_C\circ p_C)_*\otimes Id)\circ(\tilde{\mu}_*^{Z_C\times\mathbb{Z}/n_C}\otimes Id)(x[l]\otimes\bar{\omega}_C^l)\\ &= \sum_l((j_C\circ p_C)_*\otimes Id)(P_l\cdot y\otimes\bar{\omega}_C^l)\\ &= \sum_l P_l^{(C)}\cdot y\otimes\bar{\omega}_C^l\,,\end{aligned}$$

where the previous to last equality is due to proposition 5.6.4, and the last one is a consequence of the trivial fact that $(p_C)_*(P_l) = P_l^{(C)} \in K_*(G^*Z_C)$. Similarly, for $j=1$, we get

$$(\tilde{\mu}_*^\Gamma \otimes Id)\circ\phi(x) = \sum_l (P_l^{(C)}\cdot y + 1 - P_l^{(C)})\otimes\bar{\omega}_C^l\,,$$

and this completes the proof. □

6.4 Models for $\underline{E}Z_C$ with trivial action of γ_C

The goal of this section is to prove the following result:

6.4.1 Proposition. *Let G be a countable discrete group, and H be a finite central subgroup of G. Then for any model X for $\underline{E}G$, the quotient space $H\backslash X$ is also a model for $\underline{E}G$, on which H acts trivially. Moreover, the canonical projection $p: X \twoheadrightarrow H\backslash X$ is a G-equivariant homotopy equivalence.*

Given a countable discrete group Γ, this result applies to $G = Z_C$ and to $H = \langle\gamma_C\rangle$, for any $C \in \langle\Gamma\rangle^{ell}$. In particular, the quotient $\langle\gamma_C\rangle\backslash\underline{E}\Gamma^{std}$ is a model for $\underline{E}Z_C$ with a trivial action of γ_C, where

$$\underline{E}\Gamma^{std} := \left\{\mu:\Gamma\longrightarrow[0,\,1]\;\middle|\;\text{supp}(\mu)\text{ is finite, and }\sum_{\gamma\in\Gamma}\mu(\gamma)=1\right\}$$

is the "standard" model for $\underline{E}\Gamma$.

Proof of proposition 6.4.1. Let us first show that $H\backslash X$ is a metrizable proper G-space. It is a G-space because H is central in G. Since G acts properly on the metrizable space X, so does its subgroup H, therefore, the orbit-space $H\backslash X$ is metrizable. Similarly, $G\backslash(H\backslash X) = G\backslash X$ is metrizable, hence Hausdorff. The space

X being G-proper, for any $x \in X$, there exist an open neighbourhood U_x of x in X, a finite subgroup H_x of G, and a G-equivariant continuous map $\rho_x : U_x \longrightarrow G/H_x$. The canonical projection $p : X \twoheadrightarrow H\backslash X$ is a G-equivariant, continuous, closed and open map (H being finite). Consequently, for any $\bar{x} = p(x) \in H\backslash X$, $U_{\bar{x}} := p(U_x)$ is an open neighbourhood of $\bar{x}$ in $H\backslash X$, $H_{\bar{x}} := \langle H_x, H\rangle$ is a *finite* subgroup of G (H is finite and central in G), and $\rho_{\bar{x}} : U_{\bar{x}} \longrightarrow G/H_{\bar{x}}$, $\bar{y} = p(y) \longmapsto q \circ \rho_x(y)$ is a G-equivariant continuous map, where $q : G/H_x \twoheadrightarrow G/H_{\bar{x}}$. This shows that $H\backslash X$ is a proper G-space.

Since X is a model for $\underline{EG}$, there is, up to G-homotopy, a unique G-equivariant map $f : H\backslash X \longrightarrow X$, and the composition $f \circ p : X \longrightarrow X$ has to be G-equivariantly homotopic to the identity. To see that $H\backslash X$ is a universal proper G-space, it is enough to prove that $p \circ f$ is G-homotopic to the identity. By universality of $\underline{EG}$, there are G-homotopy equivalences $X \leftrightarrows \underline{EG}^{std}$ inducing G-homotopy equivalences $H\backslash X \leftrightarrows H\backslash \underline{EG}^{std}$. We can therefore assume that $X = \underline{EG}^{std}$. The G-homotopy we are looking for is simply given by

$$\begin{aligned} L : (H\backslash \underline{EG}^{std}) \times [0, 1] &\longrightarrow H\backslash \underline{EG}^{std} \\ (p(\mu), t) &\longmapsto p(f \circ p(t \cdot \mu) + ((1-t) \cdot \mu)), \end{aligned}$$

where $\mu \in \underline{EG}^{std}$ (the multiplications by t and $1-t$, and the addition above are simply performed in the vector space of functions from G to $\mathbb{R}$).

We come back to the case where X is any model for $\underline{EG}$. Since $H\backslash X$ is a model for $\underline{EG}$, the composition $p \circ f : H\backslash X \longrightarrow H\backslash X$ has to be G-homotopic to the identity. Consequently, f and p are G-homotopy equivalences, inverse of each other.

This completes the proof. □

6.5 Description of the map $((p_C)_* \otimes Id) \circ \psi_C$

We first give a new decomposition of the group $K_*^{Z_C}(\underline{E}Z_C)$ in terms of subgroups consisting in homotopy classes of Kasparov triples endowed with a given action of the element γ_C. For this purpose, we need the existence (established in section 6.4) of models for $\underline{E}Z_C$ on which γ_C acts trivially. We then give an explicit description of $((p_C)_* \otimes Id) \circ \psi_C$ in terms of this decomposition, and show that it is an isomorphism, by exhibiting its inverse. It appears that this inverse has a description very similar to that of $((p_C)_* \otimes Id) \circ \psi_C$.

Let us make the following standing assumption: For any countable discrete group G and any G-compact space X, in a Kasparov triple $(\mathcal{H}, \pi, F)$, the operator F is self-adjoint, G-equivariant, properly supported, and $\mathcal{H}$ essential, in the sense that the subspace $\pi(C_c(X)) \cdot \mathcal{H}$ is dense in $\mathcal{H}$; moreover all homotopies between Kasparov triples are performed within this class. This is no restriction.

Let X be a Z_C-compact space. For $l = 0, \dots, n_C - 1$, let us define

$$K_*^{Z_C}(X)_{\{l\}} := \{[\mathcal{H}, \pi, F] \in K_*^{Z_C}(X) \mid \gamma_C \text{ acts on } \mathcal{H} \text{ by } \omega_C^l\}.$$

(We do *not* claim tbat if $[\mathcal{H}, \pi, F] = [\mathcal{H}', \pi', F'])$ and γ_C acts on $\mathcal{H}$ by ω_C^l, then the same holds for $\mathcal{H}'$.) Similarly, we define

$$K_*^{Z_C}(\underline{E}Z_C)_{\{l\}} := \left\{ [\mathcal{H}, \pi, F] \in K_*^{Z_C}(\underline{E}Z_C) \;\middle|\; \begin{array}{l} [\mathcal{H}, \pi, F] \in K_*^{Z_C}(X)_{\{l\}}, \\ \text{for some } Z_C\text{-compact} \\ \text{subspace } X \text{ of } \underline{E}Z_C \end{array} \right\}.$$

Obviously, $K_*^{Z_C}(X)_{\{l\}}$ and $K_*^{Z_C}(\underline{E}Z_C)_{\{l\}}$ are well-defined subgroups of $K_*^{Z_C}(X)$ and $K_*^{Z_C}(\underline{E}Z_C)$ respectively; $K_*^{Z_C}(X)_{\{l\}}$ is functorial in X (for proper Z_C-equivariant maps), and $K_*^{Z_C}(\underline{E}Z_C)_{\{l\}} = \varinjlim_X K_*^{Z_C}(X)_{\{l\}}$, where the direct limit runs over all Z_C-compact subspaces X of $\underline{E}Z_C$.

Let X be a proper Z_C-compact space with γ_C acting trivially. For $l = 0, \dots, n_C - 1$ and $x = [\mathcal{H}, \pi, F] \in K_*^{Z_C}(X)$, we let $x_{\{l\}} := [\mathcal{H}_l, \pi_l, F_l] \in K_*^{Z_C}(X)_{\{l\}}$, where $\mathcal{H}_l := \{\xi \in \mathcal{H} \mid \gamma_C \cdot \xi = \omega_C^l \cdot \xi\}$ is a Hilbert space acted upon the left by Z_C (since γ_C is central in Z_C), π_l is the restriction to $\mathcal{H}_l$ of the covariant action π of $C_0(X)$ on $\mathcal{H}$, and F_l is the restriction of F to $\mathcal{H}_l$. Let us check that π and F really map $\mathcal{H}_l$ into itself. Let $\xi \in \mathcal{H}_l$ and $f \in C_0(X)$. One has

$$\gamma_C \cdot \pi(f)(\xi) = \pi(f \circ \gamma_C^{-1})(\gamma_C \cdot \xi) = \omega_C^l \cdot \pi(f)(\xi),$$

where the first equality comes from the covariance of π, and the second from the fact that γ_C acts trivially on X. On the other hand, one has

$$\gamma_C \cdot F\xi = F(\gamma_C \cdot \xi) = \omega_C^l \cdot F\xi,$$

where the first equality is consequence of the Z_C-equivariance of F. This proves that $\pi(f)(\xi)$ and $F\xi$ are in $\mathcal{H}_l$, as was to be shown.

The correspondence $x \longmapsto x_{\{l\}}$ is well-defined, since addition of a degenerate cycle to x does not change $x_{\{l\}}$ (if y is degenerate, so is $y_{\{l\}}$), an operator-homotopy on x gives an operator-homotopy on $x_{\{l\}}$ (during the homotopy, the Z_C-action remains the same, and the operator F_t in the homotopy is Z_C-equivariant for any $t \in [0, 1]$), and finally, a unitary equivalence for x does not modify the class of $x_{\{l\}}$.

If $f : X \longrightarrow Y$ is a proper Z_C-equivariant map between Z_C-compact spaces on which γ_C acts trivially, one immediately sees that

$$f_*(x_{\{l\}}) = f_*(x)_{\{l\}}, \text{ for all } x \in K_*^{Z_C}(X).$$

Now the following result is clear.

6.5.1 Lemma. *Let X be a Z_C-compact space on which the element γ_C acts trivially. Then the map*

$$\delta_X : K_*^{Z_C}(X) \longrightarrow \bigoplus_{l=0}^{n_C-1} K_*^{Z_C}(X)_{\{l\}}\,,\ x \longmapsto \sum_{l=0}^{n_C-1} x_{\{l\}}$$

is a well-defined isomorphism, natural for Z_C-compact spaces with trivial action of γ_C. Consequently, for any model for $\underline{E}Z_C$ on which γ_C acts trivially, one has an isomorphism

$$\delta_C : K_*^{Z_C}(\underline{E}Z_C) \xrightarrow{\cong} \bigoplus_{l=0}^{n_C-1} K_*^{Z_C}(\underline{E}Z_C)_{\{l\}}\,,\ x \longmapsto \sum_{l=0}^{n_C-1} x_{\{l\}}\,,$$

where $x \in K_*^{Z_C}(X)$ and $x_{\{l\}} \in K_*^{Z_C}(X)_{\{l\}}$, for some Z_C-compact subspace X of $\underline{E}Z_C$. Moreover this decomposition is natural for Z_C-equivariant homotopy equivalences between such models for $\underline{E}Z_C$.

We define now a map that is very similar to ψ_C (see section 6.2). Consider

$$\begin{aligned} \bar{\psi}_C :\ K_*^{Z_C}(\underline{E}Z_C) \otimes \mathbb{C} &\longrightarrow K_*^{Z_C \times \mathbb{Z}/n_C}(\underline{E}(Z_C \times \mathbb{Z}/n_C)) \otimes \mathbb{C} \\ x \otimes \lambda &\longmapsto \sum_{l=0}^{n_C-1} x[l] \otimes \omega_C^l \cdot \lambda\,, \end{aligned}$$

the only difference with ψ_C being that $\bar{\omega}_C^l$ is replaced by ω_C^l, hence the notation. The main result in the present section is the

6.5.2 Proposition. *Assume that $\underline{E}Z_C$ is a model for the classifying space for proper Z_C-actions on which γ_C acts trivially. Then the map $((p_C)_* \otimes Id) \circ \psi_C$ is explicitly given by*

$$\begin{aligned} K_*^{Z_C}(\underline{E}Z_C) \otimes \mathbb{C} &\longrightarrow K_*^{Z_C}(\underline{E}Z_C) \otimes \mathbb{C} \\ \sum_{l=0}^{n_C-1} x_{\{l\}} \otimes \lambda_l &\longmapsto \sum_{l=0}^{n_C-1} x_{\{l\}} \otimes \bar{\omega}_C^l \cdot \lambda_l\,, \end{aligned}$$

where $K_*^{Z_C}(\underline{E}Z_C)$ is identified with $\bigoplus_{l=0}^{n_C-1} K_*^{Z_C}(\underline{E}Z_C)_{\{l\}}$ via δ_C. It is therefore an isomorphism with inverse $((p_C)_* \otimes Id) \circ \bar{\psi}_C$.

For the proof we will need the following lemma.

6.5.3 Lemma. *Assume that $\underline{E}Z_C$ is a model for the classifying space for proper Z_C-actions on which γ_C acts trivially. Then for $x \in K_*^{Z_C}(\underline{E}Z_C)$, one has*

$$(p_C)_*(x_{\{l\}}[k]) = \delta_{kl} \cdot x_{\{l\}} \in K_*^{Z_C \times \mathbb{Z}/n_C}(\underline{E}(Z_C \times \mathbb{Z}/n_C)),$$

for any $k, l \in \{0, \dots, n_C - 1\}$, where we choose $\underline{E}(Z_C \times \mathbb{Z}/n_C) := \underline{E}Z_C$.

(See section 6.2 for the definition of $y[k]$ for $y \in K_*^{Z_C}(\underline{E}Z_C)$.)

Proof. Let $x \in K_*^{Z_C}(\underline{E}Z_C)$ and let X be a Z_C-compact subspace of $\underline{E}Z_C$ such that $x_{\{l\}} = [\mathcal{H}, \pi, F] \in K_*^{Z_C}(X)$ with γ_C acting by ω_C^l on $\mathcal{H}$. For the definition of the induced homomorphism, $(p_C)_*$, we refer to Valette [107] and use the same notations. On has $N := \operatorname{Ker}(p_C) = \{\gamma_C^s b^{-s} \mid 0 \le s \le n_C - 1\}$, where b is the generator of $\mathbb{Z}/n_C$. We have to consider $\widetilde{X} := N \backslash X = X$ (since b and γ_C act trivially on $\underline{E}Z_C$). For $x_{\{l\}}[k]$, we endow $\pi(C_c(X)) \cdot \mathcal{H}$ with the (non-negative, but not necessarily positive definite) scalar product

$$\langle\langle \xi \mid \eta \rangle\rangle := \sum_{s=0}^{n_C - 1} \langle \xi \mid \gamma_C^s b^{-s} \cdot \eta \rangle_{\mathcal{H}} = \sum_{s=0}^{n_C - 1} \omega_C^{(l-k)s} \cdot \langle \xi \mid \eta \rangle_{\mathcal{H}} = \delta_{kl} \cdot n_C \cdot \langle \xi \mid \eta \rangle_{\mathcal{H}}$$

(for $\xi, \eta \in \mathcal{H}$). The Hilbert space $\widetilde{\mathcal{H}}$ is then, by definition, the Hausdorff completion of $\mathcal{H}$ for this scalar product. Since $\pi(C_c(X)) \cdot \mathcal{H}$ is dense in $\mathcal{H}$ (by our standing assumption that a Kasparov triple is essential), we see that $\widetilde{\mathcal{H}} = \mathcal{H}$ if $k = l$ and is zero otherwise. This shows that $(p_C)_*(x_{\{l\}}[k]) = 0$ if $k \neq l$.

Let us now assume that $k = l$. The group $(Z_C \times \mathbb{Z}/n_C)/N$ acts on $\widetilde{\mathcal{H}}$, and under the identification of Z_C with $(Z_C \times \mathbb{Z}/n_C)/N$ via p_C, this action is precisely the original action of Z_C on $\mathcal{H} = \widetilde{\mathcal{H}}$. Finally, $\tilde{\pi} = \pi$ and $\tilde{F} = F$. We thus have

$$(p_C)_*(x_{\{l\}}[l]) = [\widetilde{\mathcal{H}}, \tilde{\pi}, \tilde{F}] = [\mathcal{H}, \pi, F] = x_{\{l\}} \in K_*^{Z_C}(\underline{E}Z_C).$$

This concludes the proof. □

Proof of proposition 6.5.2. By definition of ψ_C, for $x = \sum_l x_{\{l\}}$, one has

$$\psi_C(x \otimes \lambda) = \sum_{k,l} x_{\{l\}}[k] \otimes \bar{\omega}_C^k \cdot \lambda.$$

The result follows from lemma 6.5.3. □

6.6 Rational injectivity of $K_*(B\Gamma) \longrightarrow K_*^{\Gamma}(\underline{E}\Gamma)$

In this section, we first prove that the canonical map $B\Gamma \longrightarrow \underline{B}\Gamma$ induces an isomorphism in rational homology, and deduce that it also induces a rational isomorphism in K-homology. As a direct application, we prove that the canonical map $K_*(B\Gamma) \longrightarrow K_*^\Gamma(\underline{E\Gamma})$ is rationally injective, by constructing a retraction. This will prove proposition 4.5.3.

Here is an easy technical lemma for the proof of the main result in the present section.

6.6.1 Lemma. *Let H be a finite subgroup of a discrete group Γ. Let R be a commutative ring with unit such that the order $|H|$ of H is a unit. Then the permutation module $R[\Gamma/H]$ is $R\Gamma$-projective.*

Proof. The canonical $R\Gamma$-morphism $R\Gamma \longrightarrow R[\Gamma/H]$ defined by $\gamma \longmapsto \gamma H$ has a splitting given by

$$\gamma H \longmapsto \frac{1}{|H|} \sum_{h \in H} h .$$

This means that $R[\Gamma/H]$ is a direct summand in the $R\Gamma$-free module $R\Gamma$. □

6.6.2 Proposition. *Let Γ be any discrete group. Let R be a commutative ring with unit such that the order of any finite subgroup of Γ is invertible (as for example the rationals $\mathbb{Q}$). Then the usual classifying map $B\Gamma \longrightarrow \underline{B\Gamma}$ (unique up to homotopy) induces an isomorphism*

$$H_*(B\Gamma;\, R) \xrightarrow{\cong} H_*(\underline{B\Gamma};\, R) .$$

In particular, the map $K_*(B\Gamma) \longrightarrow K_*(\underline{B\Gamma})$ is rationally an isomorphism.

Proof. The cellular chain complex of $\underline{E\Gamma} = E_{\mathcal{F}in}(\Gamma)$ yields the exact sequence

$$\ldots \longrightarrow C_1(\underline{E\Gamma};\, R) \xrightarrow{d_1} C_0(\underline{E\Gamma};\, R) \xrightarrow{\varepsilon} R \longrightarrow 0$$

with $C_n(\underline{E\Gamma};\, R) = \left(\bigoplus_{H \in \mathcal{F}} R[\Gamma/H] \right)^{\otimes_R n+1}$ (see the end of section 4.2). By lemma 6.6.1 and by the assumption on R, for any finite subgroup H of Γ, the permutation module $R[\Gamma/H]$ is $R\Gamma$-projective. Since the direct sum and the tensor product of projective modules is projective, the above exact sequence is an $R\Gamma$-projective resolution of the trivial $R\Gamma$-module R. Consequently, the homology of the complex $C_\bullet(\underline{E\Gamma};\, R) \otimes_{R\Gamma} R$ is $H_*(\Gamma;\, R) = H_*(B\Gamma;\, R)$. But one has

$$C_\bullet(\underline{E\Gamma};\, R) \otimes_{R\Gamma} R \cong \Gamma \backslash C_\bullet(\underline{E\Gamma};\, R) \cong C_\bullet(\Gamma \backslash \underline{E\Gamma};\, R) = C_\bullet(\underline{B\Gamma};\, R) .$$

This shows that $H_*(B\Gamma;\, R) \cong H_*(\underline{B\Gamma};\, R)$. From the description of the map

$$E\Gamma = E_{\mathcal{T}}(\Gamma) \longrightarrow \underline{E\Gamma} = E_{\mathcal{F}in}(\Gamma)$$

given at the end of section 4.2, it is clear that this isomorphism is induced by the classifying map $B\Gamma \longrightarrow \underline{B\Gamma}$.

Taking $R = \mathbb{Q}$, the statement about K-homology follows form a direct application of the usual Chern character. □

We thank Ian Leary for suggesting this proof.

From propositions 4.3.4 and 6.6.2, we immediately get the

6.6.3 Corollary. *For any discrete group* Γ *there is a commutative diagram*

$$\begin{array}{ccc} K_*^{\Gamma}(E\Gamma)\otimes\mathbb{C} & \longrightarrow & K_*^{\Gamma}(\underline{E\Gamma})\otimes\mathbb{C} \\ \cong\big\downarrow & & \big\downarrow \\ K_*(B\Gamma)\otimes\mathbb{C} & \xrightarrow{\cong} & K_*(\underline{B\Gamma})\otimes\mathbb{C} \end{array}$$

In particular, the canonical map $K_*(B\Gamma) \longrightarrow K_*^{\Gamma}(\underline{E\Gamma})$ *is rationally injective, with the composition*

$$K_*^{\Gamma}(\underline{E\Gamma})\otimes\mathbb{C} \twoheadrightarrow K_*(\underline{B\Gamma})\otimes\mathbb{C} \xrightarrow{\cong} K_*(B\Gamma)\otimes\mathbb{C}$$

as rational retraction.

6.7 Application to the delocalized equivariant Chern character

In this section, we show how one can define the delocalized equivariant Chern character ch_*^{Γ} if conjecture 6.2.1 holds. This is of interest in view of the remark of section 4.6.

If conjecture 6.2.1 is true, in other words if the map ϕ is an isomorphism, one can define the delocalized equivariant Chern character ch_*^{Γ} as the composition

$$\begin{array}{ccccc} K_*^{\Gamma}(\underline{E\Gamma}) & \xrightarrow{c} & K_*^{\Gamma}(\underline{E\Gamma})\otimes\mathbb{C} & \underset{\cong}{\xrightarrow{\phi^{-1}}} & \bigoplus_{C\in(\Gamma)^{ell}} K_*(BZ_C)\otimes\mathbb{C} \\ {\scriptstyle ch_*^{\Gamma}}\Big\downarrow & & & & \cong\Big\downarrow{\scriptstyle \bigoplus_C ch\otimes Id} \\ H_*(\Gamma; F\Gamma) & & \underset{\cong}{\xleftarrow{\quad\text{Shapiro}\quad}} & & \bigoplus_{C\in(\Gamma)^{ell}} H_*(Z_C;\mathbb{C}) \end{array}$$

where c is the complexification map, and ch is the usual Chern character "à la Atiyah" in K-homology. This would be obviously an isomorphism after tensoring with $\mathbb{C}$. To make another choice for the representative elements of the elliptic conjugacy classes amounts to making conjugations on Z_C by an element of Γ and, under the Shapiro isomorphism, we claim that this induces the identity on $H_*(\Gamma; F\Gamma)$, as we now prove. First, one has a canonical decomposition of Γ-module

$$F\Gamma = \bigoplus_{C \in \langle\Gamma\rangle^{ell}} \mathbb{C}[C] \cong \bigoplus_{C \in \langle\Gamma\rangle^{ell}} \mathbb{C}\Gamma \otimes_{\mathbb{C}Z_C} \mathbb{C} \cong \bigoplus_{C \in \langle\Gamma\rangle^{ell}} \mathrm{Ind}_{Z_C}^{\Gamma}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C},$$

where $\mathbb{C}[C]$ is the complex vector space freely generated by the elements of C, and is equipped with the action of Γ by conjugation. The first isomorphism is given by

$$\mathbb{C}[C] \xrightarrow{\cong} \mathbb{C}\Gamma \otimes_{\mathbb{C}Z_C} \mathbb{C}, \sum_{\gamma \in C} \lambda_\gamma \cdot \gamma \longmapsto \sum_{\gamma \in C} \lambda_\gamma \cdot \gamma \otimes 1,$$

with inverse

$$\sum_{\gamma \in \Gamma} \lambda_\gamma \cdot \gamma \otimes \lambda = \sum_{D \in \langle\Gamma\rangle^{ell}} \sum_{\gamma \in D} \lambda_\gamma \cdot \gamma \otimes \lambda \longmapsto \sum_{\gamma \in C} \lambda\lambda_\gamma \cdot \gamma .$$

The Shapiro isomorphism is the composition

$$\begin{array}{ccccc} \bigoplus_{C \in \langle\Gamma\rangle^{ell}} H_*(Z_C; \mathbb{C}) & \xrightarrow{(i_C)_*} & \bigoplus_{C \in \langle\Gamma\rangle^{ell}} H_*(Z_C; \mathbb{C}[C]) & \xrightarrow{(j_C)_*} & \bigoplus_{C \in \langle\Gamma\rangle^{ell}} H_*(\Gamma; \mathbb{C}[C]) \\ & & & & \downarrow \cong \\ & & & & H_*(\Gamma; F\Gamma) \end{array}$$

where $i_C : \mathbb{C} \hookrightarrow \mathbb{C}[C]$ is the map $\lambda \longmapsto \lambda \cdot \gamma_C$, and j_C is the inclusion of Z_C in Γ (see [25], pp. 73, 79 and 80). By prop. III.8.1 in [25], the conjugation by an element of Γ (acting at the same time on Γ and on the Γ-module $\mathbb{C}[C]$) induces the identity on $H_*(\Gamma; \mathbb{C}[C])$. This shows that ch_*^Γ is well-defined (i.e. independent of tbese choices).

We claim that under the hypothesis that ϕ is an isomorphism, ch_*^Γ is natural for group homomorphisms, where the functoriality of $H_*(-; F(-))$ is defined as follows: for a group homomorphism $\varphi : \Gamma_1 \longrightarrow \Gamma_2$, the map $\varphi_* = H_*(\varphi; F\varphi)$ is given by

$$\varphi_* : H_*(\Gamma_1; F\Gamma_1) \xrightarrow{H_*(\Gamma_1;\, F\varphi)} H_*(\Gamma_1; F\Gamma_2) \xrightarrow{H_*(\varphi;\, F\Gamma_2)} H_*(\Gamma_2; F\Gamma_2).$$

It suffices to show that the composition $\phi \circ (\bigoplus_C (ch \otimes Id)^{-1}) \circ S^{-1}$ is natural, where S is the Shapiro isomorphism. Since a conjugation by an element of Γ_2 induces the identity on $K_*^{\Gamma_2}(\underline{E}\Gamma_2)$, the map $\phi \circ (\bigoplus_C (ch \otimes Id)^{-1}) \circ S^{-1}$ is independent of the choices of representative elements in the elliptic conjugacy classes. The Chern cbaracter ch being natural, the only difficulty lies in this choice. Indeed, φ maps an elliptic conjugacy class C_1 of Γ_1 into an elliptic conjugacy class D of Γ_2, and may map another such class C_2 into D. We must therefore make "compatible choices".

The idea is to choose representatives γ_D for Γ_2, and then, for $C \in \langle\Gamma_1\rangle^{ell}$ such that $\varphi(C) \subseteq D$, choose $\gamma_C \in C \cap \varphi^{-1}(\gamma_D)$. Clearly, the Shapiro isomorphism and the map ϕ are "natural for these compatible choices". This establishes the naturality.

We have just proved the following result.

6.7.1 Proposition. *If conjecture 6.2.1 holds, in other words if the map ϕ is an isomorphism for any countable discrete group Γ, then the composition*

$$ch_*^\Gamma := \text{Shapiro} \circ (\bigoplus_C ch \otimes Id) \circ \phi^{-1} \circ c : K_*^\Gamma(\underline{E}\Gamma) \longrightarrow H_*(\Gamma; F\Gamma)$$

is a well-defined natural homomorphism, and an isomorphism after tensoring with the complex numbers $\mathbb{C}$. Moreover, the diagram

$$\begin{array}{ccc} K_*(B\Gamma) & \longrightarrow & K_*^\Gamma(\underline{E}\Gamma) \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch_*^\Gamma} \\ H_*(\Gamma; \mathbb{C}) & \overset{i_*}{\hookrightarrow} & H_*(\Gamma; F\Gamma) \end{array}$$

commutes, where i_* is induced by $i : \mathbb{C} \hookrightarrow F\Gamma$, $\lambda \longmapsto \lambda \cdot e$.

6.7.2 Remark. *The first diagram in the present section also makes conjecture 6.2.1 plausible. Indeed, if there exists a delocalized equivariant Chern character ch_*^Γ that is an isomorphism after tensoring with $\mathbb{C}$, then it shows that there is an (abstract) isomorphism*

$$K_*^\Gamma(\underline{E}\Gamma) \otimes \mathbb{C} \cong \bigoplus_{C \in \langle\Gamma\rangle^{ell}} K_*(BZ_C) \otimes \mathbb{C}.$$

What conjecture 6.2.1 does, is to propose an explicit isomorphism, independent of such a Chern character.

Chapter 7

The delocalization in Hochschild and in cyclic homology, and an application in algebraic K-theory

We prove that, for any group Γ, the assembly maps in Hochschild homology and in cyclic homology share, with the rational Baum-Connes assembly map, the decomposition property that we have called delocalization in chapter 6. More precisely, they admit an expression in terms of simpler assembly maps (localized at the identity of Γ), by means of spectral projectors associated to finite order elements in Γ. We prove that the same holds for the assembly maps in periodic cyclic homology and in negative cyclic homology. As an application of the delocalization in Hochschild homology, we prove that for any ring A such that $\bar{\mathbb{Q}}\Gamma \subseteq A \subseteq \mathbb{C}\Gamma$, there is an injective assembly map $H_*(\Gamma; F\Gamma) \hookrightarrow K_*^{alg}(A) \otimes_{\mathbb{Z}} \mathbb{C}$, where $F\Gamma$ is the $\mathbb{C}$-module freely generated by the torsion elements of Γ, with the action of Γ by conjugation.

7.1 Introduction

Given a discrete group Γ and $R\Gamma$ its group algebra over a commutative ring R with unit, there are well-known decompositions of the Hochschild homology $HH_*(R\Gamma)$ and of the cyclic homology $HC_*(R\Gamma)$ of the R-algebra $R\Gamma$ as a direct sum, running over the conjugacy classes in Γ, of usual homology groups of sub-quotients of Γ and with coefficients in R (in the case of cyclic homology, under the assumption that $R \supseteq \mathbb{Q}$: this a celebrated theorem of Burghelea [27]). These decompositions are given by explicit isomorphisms, called assembly maps. They map the corresponding direct sum of usual homology groups into the Hochschild (resp. cyclic) homology of $R\Gamma$. By restriction to the direct summand associated to the conjugacy class $\{e\}$

of the identity, we get the *localized* assembly maps.

In this chapter, we show bow one can express the "full" assembly maps for Γ in terms of localized ones corresponding to centralizers of finite order elements (also called elliptic elements) in Γ: This is what we call the *delocalization property for the assembly maps.* The formulas involved in these decompositions put on stage the spectral projectors in the group algebra $R\Gamma$ associated to the elliptic elements of Γ. This program is achieved in the case where R contains the field of algebraic numbers $\bar{\mathbb{Q}}$, or even just the extension $\mathbb{Q}_0$ of $\mathbb{Q}$ obtained by adjunction of all roots of unity. For Hochschild homology, we give in fact the minimal hypothesis on R (for a given group Γ) so that the delocalization works: $R\Gamma$ should contain spectral projectors associated to the finite order elements in Γ. In the same vein, for cyclic homology, we can prove the delocalization under the same hypothesis with the extra requirement that R should contain the rationals (this is needed in Burghelea's theorem). As an application of the delocalization property for Hochschild homology, we prove that for any ring A such that $\mathbb{Q}_0\Gamma \subseteq A \subseteq \mathbb{C}\Gamma$, there is an injective assembly map $H_*(\Gamma; F\Gamma) \hookrightarrow K_*^{alg}(A) \otimes_{\mathbb{Z}} \mathbb{C}$, where $F\Gamma$ is the $\mathbb{C}$-module generated by the torsion elements of Γ, on which Γ acts by conjugation.

This chapter is organized as follows. In section 7.2, we introduce the notations we will use throughout the notes, and we recall briefly the theorems expressing $HH_*(R\Gamma)$ and $HC_*(R\Gamma)$ in terms of usual group homology. We also explain precisely what we mean in this context by "full assembly map" and "localized assembly map". The spectral projectors associated to elliptic elements in a group and their main properties is the subject of section 7.3. We review the construction of the assembly map in Hochschild homology and we state and prove the delocalization property in this case in section 7.4. After a technical lemma, the proof is a simple chase around the definitions. Section 7.5 is devoted to the delocalization for cyclic homology, whose proof is based on the delocalization for Hochschild homology and the Connes ISB exact sequence. We compute (the "elliptic part" of) periodic and negative cyclic homology of $R\Gamma$, and establish the delocalization for these homologies in section 7.6. We discuss a few results on the algebraic K-theory of group-rings in section 7.7. One of them is a direct application of the delocalization in Hochschild homology.

7.2 Preliminaries on the assembly maps

Let Γ be a discrete group and $R\Gamma$ its group algebra over a fixed commutative ring R with unit. Let $\langle\Gamma\rangle$ be the set of conjugacy classes of Γ, partitioned as

$$\langle\Gamma\rangle = \langle\Gamma\rangle^{ell} \amalg \langle\Gamma\rangle^{\infty} ,$$

where $\langle\Gamma\rangle^{ell}$ is the (non-empty) set of conjugacy classes of elliptic elements of Γ, i.e. of finite order, and $\langle\Gamma\rangle^{\infty}$ is the (possibly empty) set of conjugacy classes of hyperbolic elements, i.e. of infinite order. Let $\{\gamma_C\}_{C\in\langle\Gamma\rangle}$ be a once and for all chosen set of representatives of the conjugacy classes, i.e. $\gamma_C \in C$. Let us denote $Z_\Gamma(\gamma_C)$, the centralizer of γ_C in Γ, by Z_C. (There is a slight abuse of notation since Z_C does not only depend on the class C, but on its chosen representative γ_C. Changing the representative amounts to taking a conjugate of Z_C.) Notice that $Z_{\{e\}} = \Gamma$. For any hyperbolic conjugacy class D, we let Q_D be the quotient group $Z_\Gamma(\gamma_D)/<\gamma_D>$, which is also defined up to conjugation.

Given a *unital* associative algebra A over R, we denote the Hochschild homology of A by $HH_*(A)$, and its cyclic homology by $HC_*(A)$ (without any particular mention of R). For the definitions of these theories, the reader is referred (for example) to [72] or to [111]. We consider them as being $\mathbb{N}$-graded.

Recall that the cyclic homology of the algebra R itself is given by

$$HC_n(R) \cong \begin{cases} 0 & \text{for } n \text{ odd} \\ R & \text{for } n \text{ even} \end{cases}$$

as is easily proved (see 9.6.14 in [111]).

We now state the two fundamental results on these homologies for group algebras. The first is contained in Cartan-Eilenberg [31], and is also a direct consequence of a theorem of MacLane and of the Shapiro lemma (we come back to this in section 7.3). The second has been obtained by Dan Burghelea [27] in 1985.

7.2.1 Theorem. (Cartan-Eilenberg; MacLane-Shapiro) *Let Γ be a discrete group, and R be any commutative ring with unit. Then, there is an isomorphism*

$$HH_*(R\Gamma) \cong \bigoplus_{C\in\langle\Gamma\rangle} H_*(Z_C;\, R).$$

7.2.2 Theorem. (Burghelea) *Let Γ be a discrete group, and R be a commutative ring containing the rationals $\mathbb{Q}$. Then, there is an isomorphism*

$$HC_*(R\Gamma) \cong \bigoplus_{C\in\langle\Gamma\rangle^{ell}} (H_*(Z_C;\, R) \otimes_R HC_*(R)) \oplus \bigoplus_{D\in\langle\Gamma\rangle^{\infty}} H_*(Q_D;\, R).$$

The tensor product occurring in the statement of the theorem is a *graded* tensor product (over R), i.e. the term of degree $2n \geq 0$ of $H_*(Z_C;\, R) \otimes_R HC_*(R)$ is

$$H_0(Z_C;\, R) \oplus H_2(Z_C;\, R) \oplus \ldots \oplus H_{2n}(Z_C;\, R)$$

and the term of degree $2n+1 \geq 1$ is

$$H_1(Z_C;\, R) \oplus H_3(Z_C;\, R) \oplus \ldots \oplus H_{2n+1}(Z_C;\, R).$$

In fact, there are explicit graded homomorphisms, called *assembly maps*, realizing the isomorphisms of the theorems, and mapping the right-hand side onto the left-hand side. We denote them by $\hat{\theta}^{\Gamma}_{*}$ for Hochschild homology, and by $\hat{\nu}^{\Gamma}_{*}$ for cyclic homology. In particular, by restricting $\hat{\theta}^{\Gamma}_{*}$ and $\hat{\nu}^{\Gamma}_{*}$ to the direct summands corresponding to elliptic conjugacy classes, we get an assembly map in Hochschild homology

$$\theta^{\Gamma}_{*} : \bigoplus_{C \in \langle\Gamma\rangle^{ell}} H_{*}(Z_C; R) \hookrightarrow HH_{*}(R\Gamma)$$

and an assembly map in cyclic homology (for $R \supseteq \mathbb{Q}$)

$$\nu^{\Gamma}_{*} : \bigoplus_{C \in \langle\Gamma\rangle^{ell}} H_{*}(Z_C; R) \otimes_R HC_{*}(R) \hookrightarrow HC_{*}(R\Gamma),$$

for any discrete group Γ. By further restricting to the summand corresponding to the trivial element e of Γ, a procedure we call *localization*, we get the localized assembly map in Hochschild homology

$$\theta^{\Gamma, loc}_{*} : H_{*}(\Gamma; R) \hookrightarrow HH_{*}(R\Gamma)$$

and the localized assembly map in cyclic homology (for $R \supseteq \mathbb{Q}$)

$$\nu^{\Gamma, loc}_{*} : H_{*}(\Gamma; R) \otimes_R HC_{*}(R) \hookrightarrow HC_{*}(R\Gamma).$$

We refer to the maps θ^{Γ}_{*} and ν^{Γ}_{*} as the "full" assembly maps (in contrast with their localized counterparts), although they are just restrictions to the elliptic part of the corresponding direct sum decomposition. We denote the image of θ^{Γ}_{*} by $HH^{ell}_{*}(R\Gamma)$ and that of ν^{Γ}_{*} by $HC^{ell}_{*}(R\Gamma)$; these are direct summands in $HH_{*}(R\Gamma)$ and in $HC_{*}(R\Gamma)$ respectively. We will later on see that $HH^{ell}_{*}(R\Gamma)$ and $HC^{ell}_{*}(R\Gamma)$ are independent of the chosen set of representatives $\{\gamma_C \mid C \in \langle\Gamma\rangle^{ell}\}$, as they should be.

Motivated by the delocalization theorem 6.3.1 for the Baum-Connes assembly map, our first goal in this chapter is to show how to construct the assembly maps θ^{Γ}_{*} and ν^{Γ}_{*} out of localized versions. More precisely, we will prove that one can "glue together" the localized assembly maps $\{\theta^{Z_C, loc}_{*}\}_{C \in \langle\Gamma\rangle^{ell}}$ (resp. $\{\nu^{Z_C, loc}_{*}\}_{C \in \langle\Gamma\rangle^{ell}}$) in an appropriate manner to get the "full" assembly map θ^{Γ}_{*} (resp. ν^{Γ}_{*}). As already mentioned, this will be achieved in the case where R contains the field extension $\mathbb{Q}_0$ of $\mathbb{Q}$ by all roots of unity (and even in more general situations). It is important to notice that the way we "glue together" the localized assembly maps here is exactly the same as for the Baum-Connes assembly map (both cases are however independent).

For the proof of the delocalization in Hochschild homology, we will need an explicit description of the assembly map θ^{Γ}_{*}. For this reason, in section 7.4, following Weibel [111], we will construct explicitly the map $\hat{\theta}^{\Gamma}_{*}$ and, as a by-product, we will sketch the proof of theorem 7.2.1.

7.3 The spectral projectors of elliptic elements

In section 5.3, we have considered the spectral projectors

$$P_l^{(C)} := \frac{1}{n_C} \sum_{s=0}^{n_C-1} (\omega_C^l \gamma_C)^s \in \mathbb{C}\Gamma,$$

(where $l \in \mathbb{Z}$) associated to an the elliptic element $\gamma_C \in C$. These projectors take values in smaller group algebras than $\mathbb{C}\Gamma$.

7.3.1 Definition. *For a conjugacy class $C \in \langle\Gamma\rangle^{ell}$, we let $n_C < \infty$ be the order of γ_C, and $\omega_C := e^{2\pi i/n_C} \in \mathbb{C}$. Let $\mathbb{Z}_\Gamma$ be the ring extension of the integers $\mathbb{Z}$ by the set*

$$\left\{ \frac{\omega_C}{n_C} \,\middle|\, C \in \langle\Gamma\rangle^{ell} \right\}.$$

Similarly, we denote by $\mathbb{Q}_\Gamma$ the field extension of $\mathbb{Q}$ by the set $\{\omega_C | C \in \langle\Gamma\rangle^{ell}\}$. In order to unify some results for all groups, we also let $\mathbb{Q}_0$ denote the extension of $\mathbb{Q}$ by all roots of unity in $\mathbb{C}$.

Notice that the field $\mathbb{Q}_0$ is strictly contained in the field $\bar{\mathbb{Q}}$ of algebraic numbers.

It is clear that $P_l^{(C)} \in \mathbb{Z}_\Gamma\Gamma$, for any $l \in \mathbb{Z}$ and $C \in \langle\Gamma\rangle^{ell}$. For the most important properties (in our context) of the spectral projectors, we refer to proposition 5.3.3. As an immediate consequence of the latter, we get the

7.3.2 Lemma. *Let R be a unital commutative ring containing $\mathbb{Z}_\Gamma$ (or simply $\mathbb{Z}_{Z_C}$). For any of the elliptic elements γ_C of Γ, one has a decomposition of the group algebra RZ_C as a direct sum of R-algebras with unit*

$$RZ_C = \bigoplus_{l=0}^{n_C-1} P_l^{(C)} \cdot RZ_C,$$

the unit of $P_l^{(C)} \cdot RZ_C$ being $P_l^{(C)}$.

7.4 The delocalization in Hochschild homology

In this section, we precisely state the theorem on the delocalization property for the assembly map in Hochschild homology. We then carefully review the construction of this assembly map, prove a technical lemma and finally the theorem (by a direct computation).

Given a commutative ring R with unit, recall that we have the "full" assembly map

$$\theta_*^\Gamma : \bigoplus_{C \in \langle\Gamma\rangle^{ell}} H_*(Z_C; R) \hookrightarrow HH_*(R\Gamma)$$

for the group Γ, and its localized counterpart $\theta_*^{\Gamma, loc} : H_*(\Gamma; R) \hookrightarrow HH_*(R\Gamma)$, corresponding to the trivial element e in Γ. The same holds for the subgroup Z_C. In particular, we have localized assembly maps

$$\theta_*^{Z_C, loc} : H_*(Z_C; R) \hookrightarrow HH_*(RZ_C),$$

for all elliptic conjugacy classes $C \in \langle\Gamma\rangle^{ell}$.

Before stating the delocalization theorem, we have to recall briefly some definitions and results on Hochschild homology. First, a not necessarily unital homomorphism of *unital* R-algebras $\alpha : A \longrightarrow B$ induces a homomorphism

$$HH_*(A) \longmapsto HH_*(B), \; [a_0 \otimes a_1 \otimes \ldots a_n] \longmapsto [\alpha(a_0) \otimes \alpha(a_1) \otimes \ldots \alpha(a_n)].$$

In particular, for $R \supseteq \mathbb{Z}_\Gamma$, the inclusion $\iota_l^{(C)}$ of $P_l^{(C)} \cdot RZ_C$ in $R\Gamma$ induces a group homomorphism

$$(\iota_l^{(C)})_* : HH_*(P_l^{(C)} \cdot RZ_C) \longrightarrow HH_*(R\Gamma).$$

For the second classical result, let A be a unital algebra that is a direct sum of two unital algebras, say $A = B \oplus C$; then there is an isomorphism

$$\begin{array}{rcl} HH_*(A) = HH_*(B \oplus C) & \xrightarrow{\cong} & HH_*(B) \oplus HH_*(C) \\ [(b_0 \oplus c_0) \otimes \ldots \otimes (b_n \oplus c_n)] & \longmapsto & [b_0 \otimes \ldots \otimes b_n] \oplus [c_0 \otimes \ldots \otimes c_n]. \end{array}$$

More generally, the following well-known lemma holds (see [111]).

7.4.1 Lemma. *Let $(A_i)_{i \in I}$ be a collection of unital R-algebras. Then there exist canonical isomorphisms*

$$\begin{array}{rcl} HH_*\left(\bigoplus_{i \in I} A_i\right) & \xrightarrow{\cong} & \bigoplus_{i \in I} HH_*(A_i) \\ HC_*\left(\bigoplus_{i \in I} A_i\right) & \xrightarrow{\cong} & \bigoplus_{i \in I} HC_*(A_i). \end{array}$$

The former is explicitly given by "component-wise" addition.

Now, we can state our main theorem for Hochschild homology.

7.4.2 Theorem. (Delocalization in Hochschild homology)
For a discrete group Γ, and a unital commutative ring $R \supseteq \mathbb{Z}_\Gamma$, the composition

$$\begin{array}{ccc} \bigoplus_{C \in \langle\Gamma\rangle^{ell}} H_*(Z_C; R) & \xrightarrow{\;\bigoplus_C \theta_*^{Z_C, loc}\;} & \bigoplus_{C \in \langle\Gamma\rangle^{ell}} HH_*(RZ_C) \\ \Big\downarrow \theta_*^{\Gamma} & & \Big\downarrow \cong \\ HH_*(R\Gamma) & \xleftarrow{\;\sum_C \sum_l \bar{\omega}_C^l \cdot (\iota_l^{(C)})_*\;} & \bigoplus_{C \in \langle\Gamma\rangle^{ell}} \bigoplus_{l=0}^{n_C - 1} HH_*(P_l^{(C)} \cdot RZ_C) \end{array}$$

is the assembly map θ_*^{Γ}, in other words, the diagram commutes.

To illustrate the non-triviality of the above theorem, let us point out that the obvious diagram

$$\begin{array}{ccc} H_*(Z_C;R) & \xrightarrow{\theta_*^{Z_C,loc}} & HH_*(RZ_C) \\ & \searrow^{\theta_*^{\Gamma}} & \downarrow (i_C)_* \\ & & HH_*(R\Gamma) \end{array}$$

where i_C is the inclusion of Z_C in Γ, does generally *not* commute, as is easily checked, for example when Γ is the group $\mathbb{Z}/2 = \{\pm 1\}$ and C is the conjugacy class $\{-1\}$. (In fact, for any group Γ and any elliptic conjugacy class $C \neq \{e\}$, it does *not* commute.)

Before the proof of the theorem, we first carefully explain how the assembly map θ_*^{Γ} is defined. We then state and prove a crucial lemma. Finally, the proof will be a direct computation based on the lemma.

For the definitions of a cyclic object $X_\bullet$ (as for example a cyclic set or a cyclic R-module), of the Hochschild complex associated to a cyclic R-module, and of the Hochschild homology $HH_*(M_\bullet)$ of a cyclic R-module $M_\bullet$, we refer the reader to Weibel [111], 9.6.1, 8.2.1 and 9.6.5. Still following Weibel, let us make a few more definitions (see [111], 9.6.2, 9.7.3).

7.4.3 Definition. *i) The cyclic set $B_\bullet\Gamma$ is defined by $B_n\Gamma = \Gamma^n$ (for $n \geq 0$) and*

$$d_i(\gamma_1, \dots, \gamma_n) = \begin{cases} (\gamma_2, \dots, \gamma_n), & \text{if } i = 0 \\ (\gamma_1, \dots, \gamma_i\gamma_{i+1}, \dots, \gamma_n), & \text{if } 1 \leq i \leq n-1 \\ (\gamma_1, \dots, \gamma_{n-1}), & \text{if } i = n \end{cases}$$
$$s_i(\gamma_1, \dots, \gamma_n) = (\gamma_1, \dots, \gamma_i, 1, \gamma_{i+1} \dots, \gamma_n), \text{ for } 0 \leq i \leq n$$
$$t(\gamma_1, \dots, \gamma_n) = ((\gamma_1 \cdots \gamma_n)^{-1}, \gamma_1, \dots, \gamma_{n-1})$$

(Its geometric realization $|B_\bullet\Gamma|$ is the classifying space $B\Gamma$.)

ii) For an element $\gamma \in \Gamma$, denote by C_γ its conjugacy class in Γ and let

$$Z_n(\Gamma, \gamma) = \{(\gamma_0, \gamma_1, \dots, \gamma_n) \in \Gamma^{n+1} \,|\, \gamma_0\gamma_1 \cdots \gamma_n \in C_\gamma\};$$

this defines a cyclic subset $Z_\bullet(\Gamma, \gamma)$ of the cyclic set $Z_\bullet\Gamma$ of tuples of elements of Γ, given by $Z_n\Gamma = \Gamma^{n+1}$ (for $n \geq 0$) and

$$d_i(\gamma_0, \dots, \gamma_n) = \begin{cases} (\gamma_1, \dots, \gamma_{i-1}, \gamma_i\gamma_{i+1}, \gamma_{i+2} \dots, \gamma_n), & \text{if } 0 \leq i \leq n-1 \\ (\gamma_n\gamma_0, \gamma_1, \dots, \gamma_{n-1}), & \text{if } i = n \end{cases}$$
$$s_i(\gamma_0, \dots, \gamma_n) = (\gamma_0, \dots, \gamma_i, 1, \gamma_{i+1} \dots, \gamma_n), \text{ for } 0 \leq i \leq n$$
$$t(\gamma_0, \dots, \gamma_n) = (\gamma_n, \gamma_0, \dots, \gamma_{n-1}).$$

iii) *By applying the functor "free R-module" to* $B_\bullet\Gamma$, $Z_\bullet\Gamma$ *and to* $Z_\bullet(\Gamma, \gamma)$, *one gets a cyclic R-module* $RB_\bullet\Gamma$ *and a cyclic R-submodule* $RZ_\bullet(\Gamma, \gamma)$ *of* $RZ_\bullet\Gamma$.

iv) *We denote by* $HH_*(\Gamma, \gamma)$ *the Hochschild homology of the cyclic R-module* $RZ_\bullet(\Gamma, \gamma)$.

Notice that $Z_\bullet(\Gamma, \gamma)$ as well as $HH_*(\Gamma, \gamma)$ only depend on the conjugacy class C_γ of γ, but *not* on the particular representative γ.

There is an isomorphism of cyclic sets between $Z_\bullet(\Gamma, e)$ and $B_\bullet\Gamma$, obtained by forgetting γ_0. We will need the explicit form of the inverse. It is given by

$$B_n\Gamma \longrightarrow Z_n\Gamma\,,\ (\gamma_1, \ldots, \gamma_n) \longmapsto ((\gamma_1 \cdots \gamma_n)^{-1}, \gamma_1, \ldots, \gamma_n)\,.$$

By definition, the Hochschild complex associated to the cyclic R-module $RZ_\bullet\Gamma$ (resp. $RB_\bullet\Gamma$) is precisely equal to the usual Hochschild complex of the R-algebra $R\Gamma$ (resp. to the usual bar complex of the group Γ, with coefficients in R). Consequently, one has

$$HH_*(RZ_\bullet\Gamma) = HH_*(R\Gamma)$$
$$HH_*(B_\bullet\Gamma) = H_*(B\Gamma;\, R) = H_*(\Gamma;\, R)\,.$$

7.4.4 Remark. *It is obvious that one has the following canonical decomposition of cyclic R-module and induced canonical splitting:*

$$RZ_\bullet\Gamma = \bigoplus_{C\in\langle\Gamma\rangle} RZ_\bullet(\Gamma, \gamma_C) \ \text{ and } \ HH_*(R\Gamma) = \bigoplus_{C\in\langle\Gamma\rangle} HH_*(\Gamma, \gamma_C)\,.$$

We can therefore set, independently of any choice,

$$HH_*^{ell}(R\Gamma) := \bigoplus_{C\in\langle\Gamma\rangle^{ell}} HH_*(\Gamma, \gamma_C)\,.$$

Now, we have all the needed ingredients to recall the construction of the assembly map $\hat{\theta}_*^\Gamma$ itself. We proceed in four steps.

First, proposition 9.7.4 in [111] shows that the inclusion $Z_C = Z_\Gamma(\gamma_C) \hookrightarrow \Gamma$ induces an isomorphism

$$HH_*(Z_C, \gamma_C) \xrightarrow{\cong} HH_*(\Gamma, \gamma_C)\,,$$

for any $C \in \langle\Gamma\rangle$.

Secondly, by the proof of corollary 9.7.5 in [111], if g is any central element in a group G, then the homomorphism of simplicial sets

$$Z_\bullet(G, e) \longrightarrow Z_\bullet(G, g)\,,\ (g_0, g_1, \ldots, g_n) \longmapsto (gg_0, g_1, \ldots, g_n)$$

is an isomorphism, hence

$$H_*(G;\,R) \cong HH_*(G,\,\mathrm{e}) \xrightarrow{\cong} HH_*(G,\,g)\,,$$

where the first isomorphism is due to the fact, mentioned earlier, that $Z_\bullet(G,\,\mathrm{e})$ and $B_\bullet G$ are isomorphic as cyclic sets.

The third step is given by remark 7.4.4:

$$HH_*(R\Gamma) = \bigoplus_{C\in\langle\Gamma\rangle} HH_*(\Gamma,\,\gamma_C)\,.$$

Finally, assembling the first three stages (in disorder), we get a chain of isomorphisms

$$\bigoplus_{C\in\langle\Gamma\rangle} H_*(Z_C;\,R) \xrightarrow{\cong} \bigoplus_{C\in\langle\Gamma\rangle} HH_*(Z_C,\,\mathrm{e}) \xrightarrow{\cong} \bigoplus_{C\in\langle\Gamma\rangle} HH_*(Z_C,\,\gamma_C) \xrightarrow{\cong} \underbrace{\bigoplus_{C\in\langle\Gamma\rangle} HH_*(\Gamma,\,\gamma_C)}_{=\,HH_*(R\Gamma)}$$

whose composition is, by definition, the assembly map $\hat\theta_*^\Gamma$. This shows that the image of θ_*^Γ coincides with what we have denoted by $HH_*^{ell}(R\Gamma)$. Let us now give explicit formulas. First, we consider $H_*(Z_C;\,R)$ as the homology of the bar complex:

$$H_*(Z_C;\,R) = H_*(RB_\bullet Z_C)\,,\ \text{where}\ \ RB_nZ_C = \begin{cases} R\,, & \text{for } n=0\\ RZ_C^{\otimes_R n}\,, & \text{for } n\geq 1\,.\end{cases}$$

By definition, $HH_*(R\Gamma)$ is given by the homology of the Hochschild complex of the R-algebra $R\Gamma$:

$$HH_*(R\Gamma) = H_*(RZ_\bullet\Gamma)\,,\ \text{where}\ \ RZ_n\Gamma = R\Gamma^{\otimes_R n+1}\,,\ \text{for } n\geq 0\,.$$

By chasing along the definition of $\hat\theta_*^\Gamma$ as given above, one gets the formula

$$\begin{aligned}\hat\theta_*^\Gamma : H_*(Z_C;\,R) &\longrightarrow HH_*(R\Gamma)\\ [\gamma_1\otimes\ldots\otimes\gamma_n] &\longmapsto [\gamma_C(\gamma_1\cdots\gamma_n)^{-1}\otimes\gamma_1\otimes\ldots\otimes\gamma_n]\,.\end{aligned}$$

In particular, for any $C\in\langle\Gamma\rangle^{ell}$, we obtain

$$\begin{aligned}\theta_*^{Z_C,loc} : H_*(Z_C;\,R) &\longrightarrow HH_*(RZ_C)\\ [\gamma_1\otimes\ldots\otimes\gamma_n] &\longmapsto [(\gamma_1\cdots\gamma_n)^{-1}\otimes\gamma_1\otimes\ldots\otimes\gamma_n]\,.\end{aligned}$$

This way of defining $\hat\theta_*^\Gamma$ has the considerable advantage of sketching the proof of theorem 7.2.1 at the same time.

As already mentioned, theorem 7.4.2 is also a consequence of a result of MacLane and of the Shapiro lemma. Indeed, as a particular case of proposition 7.4.2 in [72], there is an isomorphism

$$HH_*(R\Gamma) \cong H_*(\Gamma; R\Gamma),$$

called the MacLane isomorphism, where $R\Gamma$ is acted upon by conjugation. By the Shapiro lemma (compare with section 6.7), there is an isomorphism

$$H_*(\Gamma; R\Gamma) \cong \bigoplus_{C\in\langle\Gamma\rangle} H_*(Z_C; R).$$

The composition of both isomorphisms is precisely $(\hat{\theta}_*^\Gamma)^{-1}$.

We now state and prove the crucial technical lemma.

7.4.5 Lemma. *Let R be a commutative ring with unit, and A be an associative unital R-algebra. Let e be a central idempotent in A. Then the maps $f = (f_n)$ and $g = (g_n)$ defined by $f_n, g_n : A^{\otimes_R n+1} \longrightarrow A^{\otimes_R n+1}$ and*

$$f_n(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = ea_0 \otimes ea_1 \otimes \ldots \otimes ea_n$$
$$g_n(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = ea_0 \otimes a_1 \otimes \ldots \otimes a_n$$

are homotopic chain maps of the Hochschild complex $(A^{\otimes_R n+1}, d_n)$ of A. Consequently, for any cycle $a_0 \otimes a_1 \otimes \ldots \otimes a_n$ in $A^{\otimes_R n+1}$, the elements

$$ea_0 \otimes ea_1 \otimes \ldots \otimes ea_n \text{ and } ea_0 \otimes a_1 \otimes \ldots \otimes a_n$$

of $A^{\otimes_R n+1}$ are homotopic cycles, in other words they represent the same class in $HH_n(A)$.

Proof. First, notice that the Hochschild complex is a presimplicial module in the sense of definition 1.0.6 in [72] (see the proof of lemma 1.1.2 in [72]). By using the assumption that e is a central idempotent, it is straightforward to check that f and g are chain maps, and even maps of presimplicial modules (see p. 4 in [72]). It remains therefore to construct a presimplicial homotopy between the maps f and g, i.e. a collection of maps $h_i^{(n)} : A^{\otimes_R n+1} \longrightarrow A^{\otimes_R n+2}$ of R-modules, for all $n \geq 0$ and all $0 \leq i \leq n$, satisfying the set equations of definition 1.0.8 in [72]. (By lemma 1.0.9 in [72], it will follow that $h^{(n)} := \sum_{j=0}^{n}(-1)^j h_j^{(n)}$ is a chain homotopy from f to g.) One can easily check that the maps defined by

$$\begin{aligned}
h_0^{(n)}(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= ea_0 \otimes e \otimes ea_1 \otimes \ldots \otimes ea_n \\
&\vdots \\
h_j^{(n)}(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= ea_0 \otimes a_1 \otimes \ldots \otimes a_j \otimes e \otimes ea_{j+1} \otimes \ldots \otimes ea_n \\
&\vdots \\
h_n^{(n)}(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= ea_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes e
\end{aligned}$$

(where $1 \leq j \leq n-1$) satisfy all the desired properties. This concludes the proof. □

For sake of completeness, let us give another proof of the part of the above lemma saying that $ea_0 \otimes ea_1 \otimes \ldots \otimes ea_n$ and $ea_0 \otimes a_1 \otimes \ldots \otimes a_n$ represent the same class in $HH_n(A)$, whenever $a_0 \otimes a_1 \otimes \ldots \otimes a_n$ is a Hochschild cycle.

Proof. Letting $\bar{A} := A/R \cdot 1$, we consider here Hochschild homology as the homology of the *normalized* Hochschild complex $\overline{C}_n = A \otimes_R \bar{A}^{\otimes_R n}$, where the equivalence between both definitions is given by the canonical morphism of complexes

$$C_n = A \otimes_R A^{\otimes_R n} \twoheadrightarrow \overline{C}_n = A \otimes_R \bar{A}^{\otimes_R n}$$

(see [72], 1.1.14, 1.1.15, 1.6.4 and 1.6.5). With these notations, we have to prove that

$$[ea_0 \otimes \overline{ea}_1 \otimes \overline{ea}_2 \otimes \ldots \otimes \overline{ea}_n] = [ea_0 \otimes \bar{a}_1 \otimes \bar{a}_2 \otimes \ldots \otimes \bar{a}_n] \in H_*(\overline{C}_\bullet).$$

By lemma 2.12, p. 29, in [59], for $n \geq 1$(and obviously for $n = 0$), the composition map

$$HH_n(A) \longrightarrow \Omega^n(A) \twoheadrightarrow \Omega^n_{ab}(A), [a_0 \otimes \bar{a}_1 \otimes \ldots \otimes \bar{a}_n] \longmapsto a_0 da_1 \cdots da_n$$

is a well-defined injection, where $\Omega^*(A)$ is the algebra of non-commutative differential forms on A, and $\Omega^*_{ab}(A) := \Omega^*(A)/[\Omega^*(A), \Omega^*(A)]$ is the quotient of the above by the submodule (not the ideal!) generated by the commutators. We are therefore reduced to showing that

$$ea_0 d(ea_1) d(ea_2) \cdots d(ea_n) = ea_0 da_1 da_2 \cdots da_n \in \Omega^n_{ab}(A),$$

but this is obvious under our hypotheses, because d is a derivation and the equation $ede = 0$ holds in $\Omega^*_{ab}(A)$ (indeed, $de = d(e^2) = ede + de \cdot e = 2ede$, hence $ede = 2ede$ and $ede = 0$). □

We are now in position to prove the delocalization theorem 7.4.2.

Proof of theorem 7.4.2. We keep notations as in the above construction of the map $\hat{\theta}^\Gamma_*$, and we work at the level of the chain complexes. Let C be a fixed elliptic conjugacy class in Γ. Let $x = \lambda\gamma_1 \otimes \ldots \otimes \gamma_n \in RB_n Z_C = RZ_C^{\otimes_R n}$, where $\lambda \in R$. By θ^Γ_*, it maps to

$$\lambda\gamma_C (\gamma_1 \cdots \gamma_n)^{-1} \otimes \gamma_1 \otimes \ldots \otimes \gamma_n \in RZ_n\Gamma = R\Gamma^{\otimes_R n+1}.$$

The map defined by the delocalization process factorizes through the map

$$\sum_{D \in (\Gamma)^{ell}} (j_D)_* : \bigoplus_{D \in (\Gamma)^{ell}} HH_*(RZ_D) \longrightarrow HH_*(R\Gamma)$$

in the obvious way, where j_D denotes the inclusion of RZ_D in $R\Gamma$. The image of x in $HH_*(RZ_C)$ under the corresponding factorization is

$$\lambda \sum_{l=0}^{n_C-1} \bar{\omega}_C^l P_l^{(C)} (\gamma_1 \cdots \gamma_n)^{-1} \otimes P_l^{(C)} \gamma_1 \otimes \ldots \otimes P_l^{(C)} \gamma_n .$$

Since $P_l^{(C)}$ is a central idempotent in the algebra RZ_C, by lemma 7.4.5, this element represents the same class in $HH_*(RZ_C)$ as

$$\lambda \left(\sum_{l=0}^{n_C-1} \bar{\omega}_C^l P_l^{(C)} \right) (\gamma_1 \cdots \gamma_n)^{-1} \otimes \gamma_1 \otimes \ldots \otimes \gamma_n = \lambda \gamma_C (\gamma_1 \cdots \gamma_n)^{-1} \otimes \gamma_1 \otimes \ldots \otimes \gamma_n ,$$

where the equality comes from proposition 5.3.3 iv). By mapping this element via $(j_C)_*$ in $HH_*(R\Gamma)$, we see that both maps coincide on x.

This completes the proof. □

7.4.6 Remark. *i) It is easy to adapt lemma 7.4.5 and its proof so as to see that the assembly map θ_*^Γ and the map obtained from the delocalization process of theorem 7.4.2, both considered as chain maps*

$$\bigoplus_{C \in \langle \Gamma \rangle^{ell}} RB_\bullet Z_C \longrightarrow RZ_\bullet \Gamma,$$

are homotopic, for any unital ring R containing $\mathbb{Z}_\Gamma$.

ii) The delocalization theorem 7.4.2 clearly holds (and the proof is the same) as soon as the ring R is commutative with a unit 1 and satisfies the following properties: for any $C \in \langle \Gamma \rangle^{ell}$, $\frac{1}{n_C} \in R$ and R contains a primitive n_C-th root ω_C of 1 (i.e. the multiplicative group $R^\times$ contains n_C-torsion) satisfying the formulas of lemma 5.3.2 (no uniqueness assumption in neither case). This is equivalent to requiring, for every $C \in \langle \Gamma \rangle^{ell}$, the existence of a unital ring homomorphism $\mathbb{Z}\left[\frac{1}{n_C} e^{2\pi i/n_C}\right] \longrightarrow R$. This is in fact the minimal hypothesis on R because otherwise, the spectral projectors $P_l^{(C)}$ have no meaning in $R\Gamma$ and the statement of the theorem does not make sense. This applies for example to the case $\Gamma = \mathbb{Z}/2$ (or any group only with 2-torsion) and $R = \mathbb{Z}/3$, and to $\Gamma = \mathbb{Z}/3$ (or any group only with 3-torsion) and $R = \mathbb{Z}/7 \times \mathbb{Z}/13$.

7.5 The delocalization in cyclic homology

In this section, we state the theorem on the delocalization property for the Burghelea assembly map in cyclic homology. We prove it by applying the corresponding result for Hochschild homology.

We keep notations as in section 7.4.

Recall that we have the "full" assembly map

$$\nu_*^\Gamma : \bigoplus_{C \in \langle\Gamma\rangle^{ell}} H_*(Z_C; R) \otimes_R HC_*(R) \hookrightarrow HC_*(R\Gamma)$$

for the group Γ, and its localized counterpart

$$\nu_*^{\Gamma, loc} : H_*(\Gamma; R) \otimes_R HC_*(R) \hookrightarrow HC_*(R\Gamma)\,.$$

The same holds for the subgroup Z_C. In particular, we have localized assembly maps

$$\nu_*^{Z_C, loc} : H_*(Z_C; R) \otimes_R HC_*(R) \hookrightarrow HC_*(RZ_C)\,,$$

for all elliptic conjugacy classes $C \in \langle\Gamma\rangle^{ell}$.

Again we denote the (non-unital) inclusion of $P_l^{(C)} \cdot RZ_C$ in $R\Gamma$ by $\iota_l^{(C)}$. By 2.1.3 in [72], it induces a homomorphism

$$(\iota_l^{(C)})_* : HC_*(P_l^{(C)} \cdot RZ_C) \longrightarrow HC_*(R\Gamma)\,.$$

We are now is position to state our main theorem for cyclic homology.

7.5.1 Theorem. (Delocalization in cyclic homology)
Let Γ be a discrete group, and R be a unital commutative ring containing $\mathbb{Q}_\Gamma$. Then the composition

$$\begin{array}{ccc}
\displaystyle\bigoplus_{C \in \langle\Gamma\rangle^{ell}} H_*(Z_C; R) \otimes_R HC_*(R) & \xrightarrow{\;\bigoplus_C \nu_*^{Z_C, loc}\;} & \displaystyle\bigoplus_{C \in \langle\Gamma\rangle^{ell}} HC_*(RZ_C) \\
\Big\downarrow \nu_*^\Gamma & & \Big\downarrow \cong \\
HC_*(R\Gamma) & \xleftarrow{\;\sum_C \sum_l \bar{\omega}_C^l \cdot (\iota_l^{(C)})_*\;} & \displaystyle\bigoplus_{C \in \langle\Gamma\rangle^{ell}} \bigoplus_{l=0}^{n_C - 1} HC_*(P_l^{(C)} \cdot RZ_C)
\end{array}$$

is the Burghelea assembly map ν_*^Γ, in other words, the above diagram commutes.

Proof. According to [27] and to [111], in the Connes ISB exact sequence

$$\ldots \to HH_*(R\Gamma) \xrightarrow{I} HC_*(R\Gamma) \xrightarrow{S} HC_{*-2}(R\Gamma) \xrightarrow{B} HH_{*-1}(R\Gamma) \to \ldots$$

the map B is zero, and, after identification of HH_* and HC_* with direct sums of usual group-homologies via the assembly maps $\hat{\theta}_*^\Gamma$ and $\hat{\nu}_*^\Gamma$ respectively, for all

$C \in \langle\Gamma\rangle^{ell}$, one has two commutative diagrams

$$\begin{array}{ccc} H_n(Z_C;\,R) & \hookrightarrow & \underbrace{H_n(Z_C;\,R) \oplus H_{n-2}(Z_C;\,R) \oplus \dots} \\ \theta_n^\Gamma \big\downarrow & & \big\downarrow \nu_n^\Gamma \\ HH_n(R\Gamma) & \xhookrightarrow{\quad I \quad} & HC_n(R\Gamma) \end{array} \qquad (*)$$

(with the top horizontal map given by the inclusion in the first summand) and, for any $k \geq 0$,

$$\begin{array}{ccc} \underbrace{H_n(Z_C;\,R) \oplus H_{n-2}(Z_C;\,R) \oplus \dots} & \twoheadrightarrow & \underbrace{H_{n-2k}(Z_C;\,R) \oplus H_{n-2k-2}(Z_C;\,R) \oplus \dots} \\ \nu_n^\Gamma \big\downarrow & & \big\downarrow \nu_{n-2k}^\Gamma \\ HC_n(R\Gamma) & \xrightarrow{\quad S^k \quad} & HC_{n-2k}(R\Gamma) \end{array} \qquad (**)$$

(with the top horizontal map given by projection onto the last summands). Since the ISB exact sequence is natural and compatible with the decompositions

$$\begin{aligned} HH_*\left(\textstyle\bigoplus_{i \in I} A_i\right) &\xrightarrow{\cong} \textstyle\bigoplus_{i \in I} HH_*(A_i) \\ HC_*\left(\textstyle\bigoplus_{i \in I} A_i\right) &\xrightarrow{\cong} \textstyle\bigoplus_{i \in I} HC_*(A_i), \end{aligned}$$

one has, for all $C \in \langle\Gamma\rangle^{ell}$ and all $n,\, k \geq 0$, the following commutative diagram:

$$\begin{array}{ccccccc} H_n(Z_C;\,R) & \xhookrightarrow{\theta_n^{Z_C,loc}} & HH_n(RZ_C) & \xrightarrow{\cong} & \displaystyle\bigoplus_{l=0}^{n_C-1} HH_n(P_l^{(C)}\cdot RZ_C) & \xrightarrow{\sigma_C} & HH_n(R\Gamma) \\ \Big\| & & \big\downarrow I & & \big\downarrow \oplus I & & \big\downarrow I \\ H_n(Z_C;\,R) & \xhookrightarrow{\nu_n^{Z_C,loc}} & HC_n(RZ_C) & \xrightarrow{\cong} & \displaystyle\bigoplus_{l=0}^{n_C-1} HC_n(P_l^{(C)}\cdot RZ_C) & \xrightarrow{\sigma_C} & HC_n(R\Gamma) \\ \Big\| & & \big\uparrow S^k & & \big\uparrow \oplus S^k & & \big\uparrow S^k \\ H_n(Z_C;\,R) & \xhookrightarrow{\nu_{n+2k}^{Z_C,loc}} & HC_{n+2k}(RZ_C) & \xrightarrow{\cong} & \displaystyle\bigoplus_{l=0}^{n_C-1} HC_{n+2k}(P_l^{(C)}\cdot RZ_C) & \xrightarrow{\sigma_C} & HC_{n+2k}(R\Gamma) \end{array}$$

where σ_C denotes at the same time $\displaystyle\sum_{l=0}^{n_C-1} \bar{\omega}_C^l \cdot (\iota_l^{(C)})_*$ on the level of HH_* and of HC_*.

We have to prove that for every $k \geq 0$ (and every $n \geq 0$), the bottom composition in this diagram coincides with the restriction of the Burghelea assembly map ν_{n+2k}^Γ.

By the delocalization theorem 7.4.2 in Hochschild homology, the composition in the first row is the assembly map

$$\theta_n^\Gamma : H_n(Z_C; R) \hookrightarrow HH_n(R\Gamma),$$

and thanks to diagram $(*)$ above, the result holds for $k = 0$.

For the general case ($k \geq 0$), thanks to diagram $(**)$, one sees that it is enough to show that if ψ denotes the composition in the third row, then

$$\psi(H_n(Z_C; R)) \cap \operatorname{Ker}(S^k : HC_{n+2k}(R\Gamma) \to HC_n(R\Gamma)) = \{0\}$$

(so that $\psi(H_n(Z_C; R))$ injects in $HC_n(R\Gamma)$). But this follows readily from the fact that the composition in the second row is an injection (as a consequence of what has just been said for the case $k = 0$). □

It is probably also possible to prove the theorem directly by chasing around the definitions that allow to build the assembly map in cyclic homology, alike we did it in the case of Hochschild homology. We have chosen another option here, that is to say to appeal to the corresponding result for Hochschild homology and to exploit the deep interplay between both theories (embodied by the Connes ISB exact sequence).

7.5.2 Remark. *By 9.7.3 in [111], $HC_*(R\Gamma)$ is equal to $HC_*(RZ_\bullet\Gamma)$, the cyclic homology of the cyclic R-module $RZ_\bullet\Gamma$ (see [111], 9.6.7). It follows from remark 7.4.4 and from the proof of theorem 9.7.9 in [111] that one can set unambiguously*

$$HC_*^{ell}(R\Gamma) := \bigoplus_{C \in \langle\Gamma\rangle^{ell}} HC_*(RZ_\bullet(\Gamma, \gamma_C)),$$

and that it coincides precisely with the image of ν_*^Γ.

7.6 The delocalization in periodic and in negative cyclic homology

We compute the "elliptic part" of periodic and negative cyclic homology of $R\Gamma$, and we prove that the corresponding assembly maps both satisfy the delocalization property.

Let R be a unital commutative ring. For the definitions of periodic cyclic homology and negative cyclic homology of a unital associative R-algebra and of a cyclic R-module, we refer to Weibel [111] 9.6.17. We write HP_* for periodic cyclic homology and HN_* for negative cyclic homology. Notice that HP_* is 2-periodic. It follows from the definitions that HP_* and HN_* are additive in the sense that they commute with finite direct sums (i.e. the result corresponding to lemma 7.4.1 holds for

I finite). They are also functorial for non-unital morphisms between unital R-algebras.

Let Γ be a discrete group. The splitting given in remark 7.4.4 gives rise to canonical decompositions (see 7.4.6 in [72])

$$HP_*(R\Gamma) = \bigoplus_{C\in\langle\Gamma\rangle} HP_*(RZ_\bullet(\Gamma,\gamma_C)) \text{ and } HN_*(R\Gamma) = \bigoplus_{C\in\langle\Gamma\rangle} HN_*(RZ_\bullet(\Gamma,\gamma_C)).$$

As we did it for Hochschild and for cyclic homology, we set

$$HP_*^{ell}(R\Gamma) := \bigoplus_{C\in\langle\Gamma\rangle^{ell}} HP_*(RZ_\bullet(\Gamma,\gamma_C)) \text{ and } HN_*^{ell}(R\Gamma) := \bigoplus_{C\in\langle\Gamma\rangle^{ell}} HN_*(RZ_\bullet(\Gamma,\gamma_C)).$$

It is clear from 9.6.17 in [111] that there is a short exact sequence

$$0 \longrightarrow \varprojlim_p{}^1 HC^{ell}_{n+2p+1}(R\Gamma) \longrightarrow HP^{ell}_n(R\Gamma) \longrightarrow \varprojlim_p HC^{ell}_{n+2p}(R\Gamma) \longrightarrow 0$$

for any $n \in \mathbb{Z}$, where the inverse limit is taken with respect to the operator $S : HC^{ell}_q \longrightarrow HC^{ell}_{q-2}$. As mentioned in the proof of theorem 7.5.1, the maps S are surjective. Therefore, the inverse system $\{HC^{ell}_q; S\}$ satisfies the Mittag-Leffler condition, and in particular, the $\lim^1$ term vanishes (see [111] 3.5.6 and 3.5.7). Consequently, we find

$$HP^{ell}_*(R\Gamma) \cong \varprojlim_p HC^{ell}_{*+2p}(R\Gamma).$$

For a discrete group G and $n \in \mathbb{Z}$, let us define

$$H^{\Pi}_n(G;R) := \begin{cases} \prod_{j\geq 0} H_{2j}(G;R), & \text{if } n \text{ is even} \\ \prod_{j\geq 0} H_{2j+1}(G;R), & \text{if } n \text{ is odd.} \end{cases}$$

From the above considerations together with theorem 7.2.2 and the properties of S mentioned in the proof of theorem 7.5.1, we deduce the

7.6.1 Theorem. *Let Γ be a discrete group and R be a unital commutative ring containing the rationals $\mathbb{Q}$. Then there is an injective assembly map*

$$\sigma^{\Gamma}_* : \bigoplus_{C\in\langle\Gamma\rangle^{ell}} H^{\Pi}_*(Z_C;R) \hookrightarrow HP_*(R\Gamma),$$

whose image is $HP^{ell}_*(R\Gamma)$.

By restricting to the direct summand corresponding to the identity, we obtain the localized assembly map

$$\sigma_*^{\Gamma, loc} : H_*^{\Pi}(\Gamma; R) \hookrightarrow HP_*(R\Gamma).$$

From theorem 7.5.1 and the third diagram in its proof, we immediately obtain the corresponding delocalization result.

7.6.2 Theorem. (Delocalization in periodic cyclic homology)
Let Γ be a discrete group, and R be a unital commutative ring containing $\mathbb{Q}_\Gamma$. Then the composition

$$\begin{array}{ccc}
\bigoplus_{C \in \langle\Gamma\rangle^{ell}} H_*^{\Pi}(Z_C; R) & \xrightarrow{\ \bigoplus_C \sigma_*^{Z_C, loc}\ } & \bigoplus_{C \in \langle\Gamma\rangle^{ell}} HP_*(RZ_C) \\
\Big\downarrow{\scriptstyle \sigma_*^{\Gamma}} & & \Big\downarrow{\scriptstyle \cong} \\
HP_*(R\Gamma) & \xleftarrow{\ \sum_C \sum_l \bar{\omega}_C^l \cdot (\iota_l^{(C)})_*\ } & \bigoplus_{C \in \langle\Gamma\rangle^{ell}} \bigoplus_{l=0}^{n_C - 1} HP_*(P_l^{(C)} \cdot RZ_C)
\end{array}$$

is the assembly map σ_*^{Γ}, in other words, the above diagram commutes.

For negative cyclic homology, by [111] 9.6.17, there is long exact sequence

$$\ldots \xrightarrow{I} HP_{n+1}^{ell}(R\Gamma) \xrightarrow{S} HC_{n-1}^{ell}(R\Gamma) \xrightarrow{B} HN_n^{ell}(R\Gamma) \xrightarrow{I} HP_n^{ell}(R\Gamma) \xrightarrow{S} \ldots$$

Since S is surjective, B is zero and I is an injection. If for a discrete group G and $n \in \mathbb{Z}$, we define

$$H_n^{\Pi+}(G; R) := \prod_{j \geq 0} H_{n+2j}(G; R),$$

we have just proved the

7.6.3 Theorem. *Let Γ be a discrete group and R be a unital commutative ring containing the rationals $\mathbb{Q}$. Then there is an injective assembly map*

$$\tau_*^{\Gamma} : \bigoplus_{C \in \langle\Gamma\rangle^{ell}} H_*^{\Pi+}(Z_C; R) \hookrightarrow HN_*(R\Gamma),$$

whose image is $HN_*^{ell}(R\Gamma)$.

By restricting to the direct summand corresponding to the identity, we obtain the localized assembly map

$$\tau_*^{\Gamma, loc} : H_*^{\Pi+}(\Gamma; R) \hookrightarrow HN_*(R\Gamma).$$

The long exact sequence ([111] 9.6.17)

$$\dots \xrightarrow{I} HP_{n+1}(R\Gamma) \xrightarrow{S} HC_{n-1}(R\Gamma) \xrightarrow{B} HN_n(R\Gamma) \xrightarrow{I} HP_n(R\Gamma) \xrightarrow{S} \dots$$

and the one corresponding to the elliptic part are functorial (for non-unital morphisms) and are compatible with the additivity of the functors HC_*, HP_* and HN_*. Therefore, from theorems 7.5.1 and 7.6.2, we can deduce the following delocalization result.

7.6.4 Theorem. (Delocalization in negative cyclic homology)
Let Γ be a discrete group, and R be a unital commutative ring containing $\mathbb{Q}_\Gamma$. Then the composition

$$\begin{array}{ccc}
\bigoplus_{C\in\langle\Gamma\rangle^{ell}} H_*^{\Pi_+}(Z_C; R) & \xhookrightarrow{\bigoplus_C \tau_*^{Z_C, loc}} & \bigoplus_{C\in\langle\Gamma\rangle^{ell}} HN_*(RZ_C) \\
\cap\ \Big\downarrow \tau_*^{\Gamma} & & \Big\downarrow \cong \\
HN_*(R\Gamma) & \xleftarrow{\sum_C \sum_l \bar{\omega}_C^l \cdot (\iota_l^{(C)})_*} & \bigoplus_{C\in\langle\Gamma\rangle^{ell}} \bigoplus_{l=0}^{n_C-1} HN_*(P_l^{(C)} \cdot RZ_C)
\end{array}$$

is the assembly map τ_*^{Γ}, in other words, the above diagram commutes.

As by-product, we have just proved that both Connes ISB exact sequences (for Hochschild homology and for negative cyclic homology) are compatible with the five corresponding assembly maps and with the five delocalization properties. The same holds for the large diagram intertwining both sequences (see Weibel [111], 9.6.17).

7.7 An application in algebraic K-theory

In this section, we collect some known results relating the homology groups $H_i(\Gamma; R)$ with the algebraic K-theory groups $K_i^{alg}(R\Gamma)$. Moreover, as an application of the delocalization in Hochschild homology we prove the following theorem:

7.7.1 Theorem. *Let Γ be a discrete group, and let A be a ring equipped with maps $\mathbb{Z}_\Gamma\Gamma \xrightarrow{f} A \xrightarrow{g} \mathbb{C}\Gamma$ such that $g \circ f$ is the inclusion (as for example $R\Gamma$ for any ring such that $\mathbb{Z}_\Gamma \subseteq R \subseteq \mathbb{C}$). Then there is an injective assembly map*

$$\tilde{\alpha}_*^A : HH_*^{ell}(\mathbb{C}\Gamma) \hookrightarrow K_*^{alg}(A) \otimes_{\mathbb{Z}} \mathbb{C},$$

where $HH_*^{ell}(\mathbb{C}\Gamma) \cong \bigoplus_{C\in\langle\Gamma\rangle^{ell}} H_*(Z_C;\, \mathbb{C}) \cong H_*(\Gamma;\, F\Gamma)$, *with* $F\Gamma$ *the* $\mathbb{C}$*-module generated by the finite order elements in* Γ, *acted upon by conjugation. If* $\mathbb{Q}_0$ *denotes the field extension of the rationals by all roots of unity, then the map* $\tilde{\alpha}_*^{\mathbb{Q}_0\Gamma}$ *is natural.*

The assembly map $\tilde{\alpha}_*^A$ will be explicitly constructed in the proof of the theorem, as the delocalization of a classical assembly map in algebraic K-theory. We will give explicit formulas for $\tilde{\alpha}_0^A$ and $\tilde{\alpha}_1^A$ at the end of the section.

7.7.2 Remark. *For any group* G, *the map* $H_*(G;\, \mathbb{Q}_\Gamma) \longrightarrow H_*(G;\, \mathbb{C})$ *is clearly injective. The assembly map of theorem 7.7.5 being defined over* $\mathbb{Q}$ *(in place of* $\mathbb{C}$*), we deduce that, under the hypotheses of theorem 7.7.1, there is an injective assembly map*

$$HH_*^{ell}(\mathbb{Q}_\Gamma\Gamma) \hookrightarrow K_*^{alg}(A) \otimes_{\mathbf{Z}} \mathbb{Q}_\Gamma\,,$$

where $HH_*^{ell}(\mathbb{Q}_\Gamma\Gamma) \cong \bigoplus_{C\in\langle\Gamma\rangle^{ell}} H_*(Z_C;\, \mathbb{Q}_\Gamma) \cong H_*(\Gamma;\, F_\Gamma\Gamma)$, *with* $F_\Gamma\Gamma$ *the* $\mathbb{Q}_\Gamma$*-module generated by the finite order elements in* Γ, *acted upon by conjugation.*

Before starting the proof of theorem 7.7.1, we consider three related results (only the third of which is necessary for the proof; the first two illustrate the situation in degree 0 and 1).

Let R be any unital commutative ring. The assembly map in algebraic K-theory, in degree 0 and 1, is defined by (see [71])

$$\begin{aligned} \alpha_0^R &: H_0(\Gamma;\, R) \longrightarrow K_0^{alg}(R\Gamma) \otimes_{\mathbf{Z}} R\,, & \lambda &\longmapsto [1] \otimes \lambda \\ \alpha_1^R &: H_1(\Gamma;\, R) \longrightarrow K_1^{alg}(R\Gamma) \otimes_{\mathbf{Z}} R\,, & \gamma^{ab} \otimes \lambda &\longmapsto [\gamma] \otimes \lambda\,, \end{aligned}$$

where $H_0(\Gamma;\, R)$ is identified with R, $[1]$ is the K-theory class of the idempotent (1×1)-matrix 1, $H_1(\Gamma;\, R)$ is identified with $\Gamma^{ab} \otimes_{\mathbf{Z}} R$, γ^{ab} represents the group element γ in Γ^{ab}, and $[\gamma]$ is the K-theory class of the invertible (1×1)-matrix γ. These maps are clearly well-defined and natural in the group Γ. When $R = \mathbb{Z}$, we write α_i for $\alpha_i^{\mathbf{Z}}$, for $i = 0$ and 1.

Let $C^*\Gamma$ be the maximal C^*-algebra of the group Γ, and $C_r^*\Gamma$ its reduced C^*-algebra. The first proposition is an easy consequence of the definitions and of the results in [12].

7.7.3 Proposition. *Let* Γ *be a discrete group, and* A *be a ring equipped with two maps* $\mathbb{Z}\Gamma \xrightarrow{f} A \xrightarrow{g} C_r^*\Gamma$ *such that* $g \circ f$ *is the inclusion (as for example* $C^*\Gamma$, $C_r^*\Gamma$, *or* $R\Gamma$ *for any ring such that* $\mathbb{Z} \subseteq R \subseteq \mathbb{C}$*). Then the assembly map*

$$f_* \circ \alpha_i : H_i(\Gamma;\, \mathbb{Z}) \longrightarrow K_i^{alg}(A)$$

is split-injective for $i = 0$ and rationally injective for $i = 1$; it is injective for $i = 1$ whenever g factorizes through $C^*\Gamma$. Moreover, for any unital commutative ring R, the assembly map

$$\alpha_i^R : H_i(\Gamma; R) \longrightarrow K_i^{alg}(R\Gamma) \otimes_{\mathbb{Z}} R,$$

is a split-injection for $i = 0$ and 1.

Proof. Let us first show that $H_0(\Gamma; \mathbb{Z}) \longrightarrow K_0^{alg}(A)$ is split-injective. It suffices to prove it for $A = C_r^*\Gamma$. The canonical trace $\tau : A \longrightarrow \mathbb{C}$ induces the splitting we are looking for. In [40] (and also in [12]), it is proved that the composition

$$H_1(\Gamma; \mathbb{Z}) \xrightarrow{(g \circ f)_* \circ \alpha_1} K_1^{alg}(C_r^*\Gamma) \xrightarrow{\rho_1} K_1(C_r^*\Gamma)$$

is rationally injective, where ρ_1 is the canonical map. It follows that $f_* \circ \alpha_1$ is rationally injective for any A.

Let q denote the quotient map from Γ to its abelianization Γ^{ab}. By functoriality of the maximal C^*-algebra in the group, and by naturality of the maps α_1 and $\iota_\Gamma : \mathbb{Z}\Gamma \hookrightarrow C^*\Gamma$, one has the commutative diagram

$$\begin{array}{ccc} H_1(\Gamma; \mathbb{Z}) & \xrightarrow{(\iota_\Gamma)_* \circ \alpha_1} & K_1^{alg}(C^*\Gamma) \\ {\scriptstyle q_*}\downarrow\cong & & \downarrow{\scriptstyle q_*} \\ H_1(\Gamma^{ab}; \mathbb{Z}) & \xrightarrow{(\iota_{\Gamma^{ab}})_* \circ \alpha_1} & K_1^{alg}(C^*\Gamma^{ab}) \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle \det} \\ \Gamma^{ab} & \hookrightarrow & \mathrm{GL}_1(C^*\Gamma^{ab}) \end{array}$$

This shows that $f_* \circ \alpha_1$ is injective whenever g factorizes through $C^*\Gamma$.

Let B be a *unital* associative R-algebra, where R is a commutative ring with unit. The Dennis trace map $\mathrm{Dtr}_* : K_*^{alg}(B) \longrightarrow HH_*(B)$ (Hochschild homology as an R-algebra), in degree 0 and 1, is given by

$$\mathrm{Dtr}_0 : K_0^{alg}(B) \longrightarrow HH_0(B), \ [x] \longmapsto [x]$$
$$\mathrm{Dtr}_1 : K_1^{alg}(B) \longrightarrow HH_1(B), \ [u] \longmapsto [u^{-1} \otimes u]$$

for (1×1)-matrices x and u (see [72] or [87]). Since the map $\theta_*^{\Gamma, loc}$ is defined, at the chain level, by

$$R\Gamma^{\otimes_R n} \longrightarrow R\Gamma^{\otimes_R n+1}, \ \lambda\gamma_1 \otimes \ldots \otimes \gamma_n \longmapsto \lambda(\gamma_1 \cdots \gamma_n)^{-1} \otimes \gamma_1 \otimes \ldots \otimes \gamma_n,$$

it is straightforward to check that the triangle

$$\begin{array}{ccc} H_i(\Gamma; R) & \xrightarrow{\alpha_i^R} & K_i^{alg}(R\Gamma) \otimes_{\mathbb{Z}} R \\ & {\scriptstyle \theta_i^{\Gamma, loc}} \searrow & \downarrow{\scriptstyle \mathrm{Dtr}_i \otimes Id_R} \\ & & HH_i(R\Gamma) \end{array}$$

commutes, for $i = 0$ and 1. Since $\theta_*^{\Gamma,loc}$ is split-injective (thm. 7.2.1), so are the maps α_0^R and α_1^R. $\square$

The second result follows readily from theorem 5.7.7 and the identification of $H_*(\Gamma; F\Gamma)$ with $HH_*^{ell}(\mathbb{C}\Gamma)$, which follows from theorem 7.2.1 and the Shapiro lemma (see section 6.7).

7.7.4 Proposition. *Let Γ be a discrete group, and let A be a ring equipped with two maps $\mathbb{Z}_\Gamma\Gamma \xrightarrow{f} A \xrightarrow{g} \ell^1\Gamma$ such that $g \circ f$ is the inclusion (as for example $\ell^1\Gamma$, or $R\Gamma$ for any ring such that $\mathbb{Z}_\Gamma \subseteq R \subseteq \mathbb{C}$). Then there is an injective assembly map*

$$\tilde{\alpha}_0^A : HH_0^{ell}(\mathbb{C}\Gamma) \hookrightarrow K_0^{alg}(A) \otimes_{\mathbf{Z}} \mathbb{C}.$$

Apart from the delocalization property for the assembly map in Hochschild homology, the main ingredient for the proof of theorem 7.7.1 is the following deep theorem (see [87], thm. 6.3.24):

7.7.5 Theorem. (Rosenberg) *For any group Γ, there is a natural assembly map ρ_*^Γ in algebraic K-theory fitting in the commutative diagram*

$$\begin{array}{ccccc} H_*(\Gamma;\mathbb{C}) & \xrightarrow{\rho_*^\Gamma} & K_*^{alg}(\mathbb{Z}\Gamma)\otimes_{\mathbf{Z}}\mathbb{C} & \xrightarrow{\iota_*\otimes Id} & K_*^{alg}(\mathbb{C}\Gamma)\otimes_{\mathbf{Z}}\mathbb{C} \\ & {\scriptstyle\theta_*^{\Gamma,loc}}\searrow & & \swarrow{\scriptstyle \mathrm{Dtr}_*\otimes Id} & \\ & & HH_*(\mathbb{C}\Gamma) & & \end{array}$$

where $\iota : \mathbb{Z}\Gamma \hookrightarrow \mathbb{C}\Gamma$ is the inclusion and $\mathrm{Dtr}_* : K_*^{alg}(\mathbb{C}\Gamma) \longrightarrow HH_*(\mathbb{C}\Gamma)$ is the Dennis trace map. Since $\theta_*^{\Gamma,loc}$ is injective, so are ρ_*^Γ and $(\iota_* \otimes Id) \circ \rho_*^\Gamma$.

For sake of clarity, let us quickly recall the construction of the assembly map ρ_*^Γ arising in the above theorem. Let $\mathbf{K}\mathbb{Z}$ denote the algebraic K-theory spectrum. It is an Ω-spectrum whose homotopy groups are given by

$$\pi_n(\mathbf{K}\mathbb{Z}) \cong K_n^{alg}(\mathbb{Z}), \ \forall n \in \mathbb{Z}.$$

(Since spectra have negative homotopy groups, this definitely includes negative K-theory. However, $K_n^{alg}(\mathbb{Z}) = 0$ for all $n < 0$, since $\mathbb{Z}$ is a regular ring (see [87], 3.1.2(4) and 3.3.1).) Associated to this spectrum, there is a homology theory on the category of CW-complexes, defined by

$$h_*(X; \mathbf{K}\mathbb{Z}) := \pi_*(X_+ \wedge \mathbf{K}\mathbb{Z}),$$

where X_+ is X with an extra base-point added. In particular, for the one-point space, one has $h_*(pt; \mathbf{K}\mathbb{Z}) \cong K_*^{alg}(\mathbb{Z})$. In [71], Loday has defined, at the level of spectra, an assembly map

$$\lambda_*^\Gamma(\mathbb{Z}) : h_*(B\Gamma; \mathbf{K}\mathbb{Z}) \longrightarrow K_*^{alg}(\mathbb{Z}\Gamma)$$

(see prop. 4.1.1 in [71]). There is a map of spectra $\mathbf{K}\mathbb{Z} \longrightarrow \mathbf{H}\mathbb{Z}$, where $\mathbf{H}\mathbb{Z}$ is the usual Eilenberg-MacLane spectrum (defining integral homology). In turns out (see [87]) that the induced map $h_*(X; \mathbf{K}\mathbb{Z}) \longrightarrow H_*(X; \mathbb{Z})$ is a rational surjection, for any CW-complex X. Denoting by ψ_*^X the corresponding natural splitting (over $\mathbb{C}$), by definition, one has

$$\rho_*^\Gamma = (\lambda_*^\Gamma(\mathbb{Z}) \otimes Id_{\mathbb{C}}) \circ \psi_*^{B\Gamma}.$$

There is a less abstract way to see this map. Indeed, the definition of ρ_0^Γ is clear. For $k \geq 1$, ρ_k^Γ is given by the composition

$$H_k(\Gamma; \mathbb{C}) \xrightarrow{i_*} H_*(\mathrm{GL}(\mathbb{Z}\Gamma); \mathbb{C}) \twoheadrightarrow \mathrm{Prim}(H_*(\mathrm{GL}(\mathbb{Z}\Gamma); \mathbb{C})) \cong \bigoplus_{n\geq 1} K_n^{alg}(\mathbb{Z}\Gamma) \otimes_{\mathbb{Z}} \mathbb{C},$$

where i_* is induced by the inclusion $i : \Gamma \hookrightarrow \mathrm{GL}_1(\mathbb{Z}\Gamma)$, $\mathrm{Prim}(H_*(\mathrm{GL}(\mathbb{Z}\Gamma); \mathbb{C}))$ denotes the primitive part of the Hopf algebra $H_*(\mathrm{GL}(\mathbb{Z}\Gamma); \mathbb{C})$, and the isomorphism is given by cor. 11.2.12 in [72]. (We thank Jean-Louis Loday for pointing out to us this construction.)

Recall that the Dennis trace map $\mathrm{Dtr}_* : K_*^{alg}(A) \longrightarrow HH_*(A)$ (Hochschild homology over R) is defined for *unital* associative algebras A over a unital commutative ring R. It is natural for *unital* R-algebra morphisms (see [72], 11.4.3 and 8.4.5). The following lemma is crucial for our main theorem.

7.7.6 Lemma. *For any unital commutative ring R, the Dennis trace map Dtr_* is natural for non-unital R-algebra morphisms between unital R-algebras.*

Proof. Let A be a unital associative R-algebra. Let A_+ denote the "unitalization" of A over R, i.e. A_+ is the direct sum $R \oplus A$ as an R-module and is endowed with the product $A_+ \times A_+ \longrightarrow A_+$, $((\lambda, a), (\mu, b)) \longmapsto (\lambda\mu, \lambda b + a\mu + ab)$. It is a unital associative R-algebra, with unit $(1, 0)$, and there is an isomorphism of unital R-algebras $A_+ \xrightarrow{\cong} R \oplus A$, $(\lambda, a) \longmapsto (\lambda, \lambda \cdot 1_A + a)$, where 1_A is the unit of A. A non-unital R-algebra morphism $\varphi : A \longrightarrow B$ induces a unital R-algebra morphism $\varphi_+ : A_+ \longrightarrow B_+$, $(\lambda, a) \longmapsto (\lambda, \varphi(a))$. The lemma follows from the facts that Dtr_* is natural for unital morphisms and that there is an isomorphism $F_*(A) \cong \mathrm{Coker}(F_*(R) \longrightarrow F_*(A_+))$, that is natural for non-unital morphisms, where F_* can denote either functor K_*^{alg} or HH_* on the category of R-algebras with unit (this follows from additivity of both functors: see [72], 1.4.1). □

We are now in position to prove theorem 7.7.1.

Proof of theorem 7.7.1. For the complex group algebra $\mathbb{C}\Gamma$, by commutativity in the diagram of theorem 7.7.5, by naturality of the Dennis trace map Dtr_* (in the sense of lemma 7.7.6), and by "compatibility of Dtr_* with direct sums", there is a commutative diagram

$$\begin{array}{ccccc}
 & & H_*(\Gamma;\, F\Gamma) & & \\
 & & \Big\downarrow \cong & & \\
 & & \bigoplus_{C\in\langle\Gamma\rangle^{ell}} H_*(Z_C;\,\mathbb{C}) & & \\
 & \swarrow \bigoplus_C \rho_*^{Z_C,\mathbb{C}Z_C} & & \searrow \theta_*^{Z_C,loc} & \\
\bigoplus_{C\in\langle\Gamma\rangle^{ell}} K_*^{alg}(\mathbb{C}Z_C)\otimes_{\mathbf{Z}}\mathbb{C} & & \xrightarrow{\bigoplus_C \mathrm{Dtr}_*\otimes Id} & & \bigoplus_{C\in\langle\Gamma\rangle^{ell}} HH_*(\mathbb{C}Z_C) \\
\Big\downarrow \cong & & & & \Big\downarrow \cong \\
\bigoplus_{C\in\langle\Gamma\rangle^{ell}} \bigoplus_{l=0}^{n_C-1} K_*^{alg}(P_l^{(C)}\cdot\mathbb{C}Z_C)\otimes_{\mathbf{Z}}\mathbb{C} & & \xrightarrow{\bigoplus_{C,l} \mathrm{Dtr}_*\otimes Id} & & \bigoplus_{C\in\langle\Gamma\rangle^{ell}} \bigoplus_{l=0}^{n_C-1} HH_*(P_l^{(C)}\cdot\mathbb{C}Z_C) \\
\Big\downarrow \sum_{C,l}(\iota_l^{(C)})_*\otimes\bar{\omega}_C^l & & & & \Big\downarrow \sum_{C,l}\bar{\omega}_C^l\cdot(\iota_l^{(C)})_* \\
K_*^{alg}(\mathbb{C}\Gamma)\otimes_{\mathbf{Z}}\mathbb{C} & & \xrightarrow{\mathrm{Dtr}_*\otimes Id} & & HH_*(\mathbb{C}\Gamma)
\end{array}$$

with $\iota_l^{(C)} : P_l^{(C)}\cdot\mathbb{C}Z_C \hookrightarrow \mathbb{C}\Gamma$ denoting the (non-unital) inclusion. The assembly map $\tilde{\alpha}_*^{\mathbb{C}\Gamma}$ is defined as the left-hand side composition. By the delocalization in Hochschild homology (theorem 7.4.2), the right-hand composition is the "full" assembly map θ_*^{Γ}. Theorem 7.2.1 implies that it is injective. Consequently, the homomorphism $\tilde{\alpha}_*^{\mathbb{C}\Gamma} : HH_*^{ell}(\mathbb{C}\Gamma) \longrightarrow K_*^{alg}(\mathbb{C}\Gamma)\otimes_{\mathbf{Z}}\mathbb{C}$ is injective.

Let $i : \mathbb{Z}Z_C \hookrightarrow \mathbb{Z}_\Gamma Z_C$ be the inclusion. The assembly map $\tilde{\alpha}_*^{\mathbb{Z}_\Gamma\Gamma}$ is defined by the left-hand side composition in the above diagram, with $(i_*\otimes Id)\circ\rho_*^{Z_C}$, $\mathbb{Z}_\Gamma Z_C$ and $\mathbb{Z}_\Gamma\Gamma$ in place of $(\iota_*\otimes Id)\circ\rho_*^{Z_C}$, $\mathbb{C}Z_C$ and $\mathbb{C}\Gamma$ respectively. One has clearly $\tilde{\alpha}_*^{\mathbb{C}\Gamma} = (j_*\otimes Id)\circ\tilde{\alpha}_*^{\mathbb{Z}_\Gamma\Gamma}$, where j is the inclusion of $\mathbb{Z}_\Gamma\Gamma$ in $\mathbb{C}\Gamma$. For the algebra A of the statement, the assembly map $\tilde{\alpha}_*^A$ is defined by

$$\tilde{\alpha}_*^A := (f_*\otimes Id)\circ\tilde{\alpha}_*^{\mathbb{Z}_\Gamma\Gamma}.$$

It follows readily that $\tilde{\alpha}_*^A$ is injective for any A.

As already mentioned, the isomorphisms

$$HH_*^{ell}(\mathbb{C}\Gamma) \cong \bigoplus_{C\in\langle\Gamma\rangle^{ell}} H_*(Z_C;\,\mathbb{C}) \cong H_*(\Gamma;\, F\Gamma)$$

follow from theorem 7.2.1 and the Shapiro lemma (see section 6.7).

The naturality of $\tilde{\alpha}_*^{\mathbb{Q}_0\Gamma}$ follows from the fact the groups Z_C are defined up to conjugation by an element of Γ, and that such a conjugation induces the identity on $K_*^{alg}(\mathbb{Q}_0\Gamma)$. (The necessity of taking the group-algebra $\mathbb{Q}_0\Gamma$ here, is that $\mathbb{Q}_0$ is the smallest ring-extension R of the integers $\mathbb{Z}$ such that for any group G, RG contains all the spectral projectors associated to finite order elements in G.) □

Let us now write explicit formulas for the assembly maps $\tilde{\alpha}_0^A$ and $\tilde{\alpha}_1^A$. From the construction (in the above proof) of these maps, one easily finds

$$\tilde{\alpha}_0^A : H_0(Z_C;\,\mathbb{C}) \longrightarrow K_0^{alg}(A)\otimes_{\mathbb{Z}}\mathbb{C},\ 1 \longmapsto \sum_{l=0}^{n_C-1}[P_l^{(C)}]\otimes\bar{\omega}_C^l\,,$$

where $H_0(Z_C;\,\mathbb{C})$ is identified with $\mathbb{C}$, and similarly (with the help of lemma 5.6.3)

$$\tilde{\alpha}_1^A : H_1(Z_C;\,\mathbb{C}) \longrightarrow K_1^{alg}(A)\otimes_{\mathbb{Z}}\mathbb{C},\ \gamma^{ab}\otimes 1 \longmapsto \sum_{l=0}^{n_C-1}[P_l^{(C)}\gamma + 1 - P_l^{(C)}]\otimes\bar{\omega}_C^l\,,$$

where $H_1(Z_C;\,\mathbb{C})$ is identified with $Z_C^{ab}\otimes_{\mathbb{Z}}\mathbb{C}$, and γ^{ab} is the class in Z_C^{ab} of $\gamma\in Z_C$.

With the help of theorem 3.3.6, one could give a more or less explicit description of $\tilde{\alpha}_2^A$ (see also sections 8.2 and 8.3 on this point).

7.7.7 Remark. *The question raises whether the composition*

$$\tilde{\alpha}_*^{C_r^*\Gamma} : H_*(\Gamma;\,F\Gamma) \xrightarrow{\tilde{\alpha}_*^{\mathbb{C}\Gamma}} K_*^{alg}(\mathbb{C}\Gamma)\otimes_{\mathbb{Z}}\mathbb{C} \xrightarrow{incl_*} K_*^{alg}(C_r^*\Gamma)\otimes_{\mathbb{Z}}\mathbb{C}$$

is injective, and similarly for other Banach group algebras such as $\ell^1\Gamma$ and $C^*\Gamma$. For the degree 0, theorem 5.7.7 says that the map $\tilde{\alpha}_0^{\ell^1\Gamma}$ is injective. Another question is: Can one describe $K_*^{alg}(R\Gamma)\otimes_{\mathbb{Z}}\mathbb{C}$ completely in homological terms (depending on Γ)? Of course, such an expression is impossible without accepting $K_*^{alg}(R)\otimes_{\mathbb{Z}}\mathbb{C}$ as a building block (as the case of the trivial group shows). Recently, Wolfgang Lück has given, in [66], a very satisfactory piece of answer: If the rational Farrell-Jones conjecture (with respect to the family of finite subgroups) holds for Γ, then, for any $n\in\mathbb{Z}$, there is a suitable assembly map

$$\bigoplus_{p+q=n} H_p(\Gamma;\,F\Gamma)\otimes_{\mathbb{Z}} K_q^{alg}(\mathbb{C}) \longrightarrow K_n^{alg}(\mathbb{C}\Gamma)\otimes_{\mathbb{Z}}\mathbb{C}\,,$$

that is an isomorphism. (Notice that $K_q^{alg}(\mathbb{C}) = 0$ for $q<0$, since $\mathbb{C}$ is a regular ring (see [87], 3.1.2(4) and 3.3.1).) The Farrell-Jones conjecture is still an open problem, and the question remains open for other ground rings R than $\mathbb{C}$. What theorem 7.7.1 does is to prove injectivity of the above map on the summand corresponding to $q=0$, independently of this conjecture.

Chapter 8

Algebraic K-theory in low degree and the Baum-Connes assembly map

We prove that the Baum-Connes assembly map in "low homological degree" factorizes through the algebraic K-theory of suitable group rings. In homological degree 2, this answers a question of N. Higson and P. Julg. As a direct application, we prove that if a group Γ is torsion-free and satisfies the Baum-Connes conjecture, then the homology group $H_1(\Gamma;\mathbb{Z})$ injects in $K_1(C_r^*\Gamma)$ and in $K_1^{alg}(A)$ for any ring A such that $\mathbb{Z}\Gamma \subseteq A \subseteq C_r^*(\Gamma)$. We also prove that if Γ satisfies the Baum-Connes conjecture and if $B\Gamma$ is of dimension ≤ 4, then $H_2(\Gamma;\mathbb{Z})$ injects in $K_0(C_r^*\Gamma)$ and in $K_2^{alg}(A)/\Delta_2$, where A is as before, and Δ_2 is generated by the Steinberg symbols $\{\gamma, \gamma\}$, for $\gamma \in \Gamma$. Moreover, we give a "delocalized" version of these results, that takes the torsion in Γ into account: For any countable group Γ, let $F\Gamma$ be the $\mathbb{C}$-module freely generated by the set of torsion elements in Γ, acted upon by conjugation. If Γ satisfies the rational Baum-Connes conjecture, then there is an injection of $H_i(\Gamma; F\Gamma)$ in $K_i^{alg}(A)\otimes\mathbb{C}$ for any ring A such that $\bar{\mathbb{Q}}\Gamma \subseteq A \subseteq C_r^*\Gamma$, for $i = 0, 1$ and 2. This is a joint work with Hervé Oyono-Oyono.

8.1 Introduction and statement of the theorems

Given a countable discrete group Γ, let $\underline{E\Gamma}$ be a universal example for proper actions of Γ, and $\lambda_\Gamma : C^*\Gamma \longrightarrow C_r^*\Gamma$ be the canonical map between the maximal and the reduced C^*-algebras of Γ. The Baum-Connes conjecture for the group Γ is the statement that the analytical assembly map (or Baum-Connes assembly map)

$$\mu_*^\Gamma : K_*^\Gamma(\underline{E\Gamma}) \xrightarrow{\tilde{\mu}_*^\Gamma} K_*(C^*\Gamma) \xrightarrow{(\lambda_\Gamma)_*} K_*(C_r^*\Gamma)$$

is an isomorphism (see chapter 4 for more details). There is a canonical map $K_*(B\Gamma) \longrightarrow K_*^\Gamma(\underline{E\Gamma})$, which is rationally injective; it is even an isomorphism for torsion-free groups. Recall that by Bott periodicity, all these K-theories and K-homologies are 2-periodic. The strong Novikov conjecture says that the composition

$$\tilde{\beta}_*^\Gamma : K_*(B\Gamma) \longrightarrow K_*^\Gamma(\underline{E\Gamma}) \xrightarrow{\tilde{\mu}_*^\Gamma} K_*(C^*\Gamma)$$

is a rational injection. We write β_*^Γ for the composition $(\lambda_\Gamma)_* \circ \tilde{\beta}_*^\Gamma$.

Let $F\Gamma$ be the $\mathbb{C}$-module freely generated by the set of torsion elements in Γ, acted upon by conjugation. Motivated by the fact that there exists a delocalized equivariant Chern character

$$ch_*^\Gamma : K_*^\Gamma(\underline{E\Gamma}) \longrightarrow H_*(\Gamma; F\Gamma)$$

that is an isomorphism after tensoring with $\mathbb{C}$, it is proved in chapter 5 that there is a commutative diagram

$$\begin{array}{ccccc}
\beta_j^\Gamma : K_j(B\Gamma) & \xrightarrow{\tilde{\beta}_j^\Gamma} & K_j(C^*\Gamma) & \xrightarrow{(\lambda_\Gamma)_*} & K_j(C_r^*\Gamma) \\
& {\scriptstyle \beta_t^{(i),loc}} \nwarrow & \uparrow {\scriptstyle \tilde{\beta}_a^{(i),loc}} & \nearrow {\scriptstyle \beta_a^{(i),loc}} & \\
& & H_i(\Gamma; \mathbb{Z}) & &
\end{array}$$

and its "delocalized counterpart" in the sense of chapters 5, 6 and 7, namely

$$\begin{array}{ccccc}
\mu_j^\Gamma \otimes Id : K_j^\Gamma(\underline{E\Gamma}) \otimes \mathbb{C} & \xrightarrow{\tilde{\mu}_j^\Gamma \otimes Id} & K_j(C^*\Gamma) \otimes \mathbb{C} & \xrightarrow{(\lambda_\Gamma)_* \otimes Id} & K_j(C_r^*\Gamma) \otimes \mathbb{C} \\
& {\scriptstyle \beta_t^{(i)}} \nwarrow & \uparrow {\scriptstyle \tilde{\beta}_a^{(i)}} & \nearrow {\scriptstyle \beta_a^{(i)}} & \\
& & H_i(\Gamma; F\Gamma) & &
\end{array}$$

for $i = 0$, 1 and 2, where $j \in \{0, 1\}$ is the reduction modulo 2 of i. Moreover, the maps $\tilde{\beta}_a^{(i),loc}$ and $\tilde{\beta}_t^{(i),loc}$ factorize through $K_j(\ell^1\Gamma)$ and $K_j(\ell^1\Gamma) \otimes \mathbb{C}$ respectively. If an element x of $K_j(B\Gamma)$ (resp. $K_j^\Gamma(\underline{E\Gamma}) \otimes \mathbb{C}$) is in the image of $\beta_t^{(i),loc}$ (resp. $\beta_t^{(i)}$) as in the above diagrams, we say that x is of homological degree i. When there is no risk of confusion, we write β_t^{loc} for $\beta_t^{(i),loc}$ and similarly for the other maps.

We show that β_a^{loc} and β_a factorize through the algebraic K-theory of suitable group rings. Before giving the precise statements, we have to introduce some more notation. First, as in chapter 7, we let $\mathbb{Z}_\Gamma$ be the ring extension of the integers by the set

$$\left\{ \frac{e^{2\pi i/n}}{n} \;\middle|\; \text{there exists } \gamma \in \Gamma \text{ of order } n \right\} .$$

For any group Γ, $\mathbb{Z}_\Gamma$ is a subring of the field extension $\mathbb{Q}_0$ of the rationals by all roots of unity, itself contained in the field $\bar{\mathbb{Q}}$ of algebraic numbers.

Given a unital ring A containing the group Γ in its group of invertibles $\mathrm{GL}_1(A)$, we let

$$\Delta_2 = \Delta_2(A) := \left\langle \{\gamma, \gamma\} \in K_2^{alg}(A) \,\middle|\, \gamma \in \Gamma \right\rangle ,$$

the subgroup of $K_2^{alg}(A)$ generated by the Steinberg symbols $\{\gamma, \gamma\}$ (see [71], p. 359). In order to unify the forthcoming statements, for $i = 0$ and 1, we denote by $\Delta_i = \Delta_i(A)$ the trivial subgroup of $K_i^{alg}(A)$.

Finally recall that for any unital Banach algebra A, there is a canonical $\mathbb{N}$-graded map $\phi_*^A : K_*^{alg}(A) \longrightarrow K_*(A)$. Indeed, by definition, one has $K_0^{alg}(A) = K_0(A)$, and for any $n \geq 1$,

$$K_n^{alg}(A) = \pi_n(B\mathrm{GL}^\delta(A)^+) \text{ and } K_n(A) = \pi_n(B\mathrm{GL}^{top}(A)),$$

where $\mathrm{GL}^\delta(A)$ and $\mathrm{GL}^{top}(A)$ stand for the group $\mathrm{GL}(A)$ with the discrete topology and the usual direct limit topology, respectively. In the sequel, we simply write $\mathrm{GL}(A)$ for $\mathrm{GL}^{top}(A)$. The map ϕ_n^A is obtained by applying the functor $\pi_n(B(-)^+)$ to the continuous map $Id : \mathrm{GL}^\delta(A) \longrightarrow \mathrm{GL}(A)$. (The $+$-construction is performed with respect to the perfect subgroups $E(A)$ and $\{[\mathbb{I}]\}$ of $\mathrm{GL}^\delta(A) = \pi_1(B\mathrm{GL}^\delta(A))$ and $\pi_0(\mathrm{GL}(A)) = \pi_1(B\mathrm{GL}(A))$ respectively.)

Now, we are in position to state our first two theorems.

8.1.1 Theorem. *Let Γ be a countable discrete group. Then, for $i = 0, 1$ and 2, there is a natural map $\beta_{alg}^{loc} = \beta_{alg}^{(i), loc} : H_i(\Gamma; \mathbb{Z}) \longrightarrow K_i^{alg}(\mathbb{Z}\Gamma)/\Delta_i$ fitting in the commutative diagram*

$$\begin{array}{ccc}
K_j(B\Gamma) & \xrightarrow{\quad\beta_*^\Gamma\quad} & K_j(C_r^*\Gamma) \\
\Big\uparrow & & \cong\Big\uparrow \mathscr{B}^{-1} \\
\beta_i^{loc}\Big| & & K_i(C_r^*\Gamma) \\
\Big| & & \Big\uparrow \phi_i^{C_r^*\Gamma} \\
H_i(\Gamma; \mathbb{Z}) & \xrightarrow{\beta_{alg}^{loc}} K_i^{alg}(\mathbb{Z}\Gamma)/\Delta_i \xrightarrow{\iota_*} & K_i^{alg}(C_r^*\Gamma)/\Delta_i
\end{array}$$

where $j \in \{0, 1\}$ is the reduction modulo 2 of i, $\mathscr{B}$ is the Bott periodicity isomorphism, and $\iota : \mathbb{Z}\Gamma \hookrightarrow C_r^*\Gamma$ is the inclusion. The corresponding statements hold for $\ell^1\Gamma$ and $C^*\Gamma$ in place of $C_r^*\Gamma$. Moreover, in degree 2, the map β_{alg}^{loc} factorizes through the direct summand $\widetilde{K}_2^{alg}(\mathbb{Z}\Gamma)/\Delta_2$ of $K_2^{alg}(\mathbb{Z}\Gamma)/\Delta_2$.

As we will see in the proof, the cases $i = 0$ and 1 are easy. The factorization of β_a^{loc} through $K_2^{alg}(\mathbb{Z}\Gamma)/\Delta_2$ is a question posed by Nigel Higson and Pierre Julg. In this case, the idea is to reduce to the situation for the single group $\Gamma = \mathbb{Z}^2$. The proof is then a long explicit computation for this group.

The second theorem is the delocalized version of the first one, in the sense of chapters 5, 6 and 7.

8.1.2 Theorem. *Let Γ be a countable discrete group. Then, for $i = 0$, 1 and 2, there is a map $\beta_{alg} = \beta_{alg}^{(i)} : H_i(\Gamma;\, F\Gamma) \longrightarrow K_i^{alg}(\mathbb{Z}_\Gamma\Gamma)\otimes\mathbb{C}$ whose composition with the map $K_i^{alg}(\mathbb{Z}_\Gamma\Gamma)\otimes\mathbb{C} \longrightarrow K_i^{alg}(\mathbb{Q}_0\Gamma)\otimes\mathbb{C}$ is natural in Γ, and fitting in the commutative diagram*

$$
\begin{array}{ccccc}
K_j^\Gamma(\underline{E}\Gamma)\otimes\mathbb{C} & & \xrightarrow{\quad \mu_*^\Gamma\otimes Id \quad} & & K_j(C_r^*\Gamma)\otimes\mathbb{C} \\
\Big\uparrow & & & & \Big\uparrow{\scriptstyle \cong}\ \mathscr{B}^{-1}\otimes Id \\
\beta_t & & & & K_i(C_r^*\Gamma)\otimes\mathbb{C} \\
\Big| & & & & \Big\uparrow \phi_i^{C_r^*\Gamma}\otimes Id \\
H_i(\Gamma;\, F\Gamma) & \xrightarrow{\beta_{alg}} & K_i^{alg}(\mathbb{Z}_\Gamma\Gamma)\otimes\mathbb{C} & \xrightarrow{\iota_*\otimes Id} & K_i^{alg}(C_r^*\Gamma)\otimes\mathbb{C}
\end{array}
$$

where $j \in \{0,\, 1\}$ is the reduction modulo 2 of i, $\mathscr{B}$ is the Bott periodicity isomorphism, and $\iota : \mathbb{Z}_\Gamma\Gamma \hookrightarrow C_r^*\Gamma$ is the inclusion. The corresponding statements hold for $\ell^1\Gamma$ and $C^*\Gamma$ in place of $C_r^*\Gamma$.

Recall that by the Shapiro lemma (see section 6.7) and theorem 7.2.1, one has isomorphisms

$$
H_*(\Gamma;\, F\Gamma) \cong \bigoplus_{C\in\langle\Gamma\rangle^{ell}} H_*(Z_C;\, \mathbb{C}) \cong HH_*^{ell}(\mathbb{C}\Gamma)\,,
$$

where $\langle\Gamma\rangle^{ell}$ is the set of elliptic conjugacy classes in Γ, i.e. conjugacy classes of finite order elements, and for such a class C, $Z_C = Z_\Gamma(\gamma_C)$ is the centralizer in Γ of a chosen representative γ_C in C (the subgroup Z_C of Γ is uniquely defined up to a conjugation); $HH^{ell}(\mathbb{C}\Gamma)$ is the elliptic part of the Hochschild homology of the complex group algebra $\mathbb{C}\Gamma$ of Γ (see also section 7.4).

We will also explain (after Loday [71]) that, for $i = 2$, β_{alg}^{loc} factorizes through $K_2^{alg}(\mathbb{Z}\Gamma)$ if and only if $\Delta_2(\mathbb{Z}\Gamma)$ vanishes, which in turn is equivalent to requiring $H_1(\Gamma;\, \mathbb{Z}/2) = \Gamma^{ab}\otimes\mathbb{Z}/2$ to be trivial (this is for example the case when Γ is perfect or even if it is finitely generated and has no 2-torsion in its abelianization).

The significance for the appearance of the group ring $\mathbb{Z}_\Gamma\Gamma$ is that the earlier mentioned delocalization process puts on stage the spectral projectors associated to the finite order elements in Γ; they live in the complex group algebra $\mathbb{C}\Gamma$, however, they can be lifted in $\mathbb{Z}_\Gamma\Gamma$, but no smaller subring of $\mathbb{C}\Gamma$ contains them all. In this sense, it is the minimal possible ring in the above statement.

We are mainly interested in the algebraic K-theory of group rings. By a group ring of Γ, we mean here a ring A such that $\mathbb{Z}\Gamma \subseteq A \subseteq C^*\Gamma$ or such that $\mathbb{Z}\Gamma \subseteq A \subseteq C^*_r\Gamma$.

As a direct application of theorems 8.1.1 and 8.1.2 and of the main results of chapter 3, we can prove the next two theorems.

8.1.3 Theorem. *Let Γ is a countable discrete group. Let A be a ring equipped with two maps $\mathbb{Z}\Gamma \xrightarrow{f} A \xrightarrow{g} C^*_r\Gamma$ such that the composition $g \circ f$ is the inclusion. Then, the composition*

$$f_* \circ \beta_{alg}^{(i),loc} : H_i(\Gamma;\, \mathbb{Z}) \longrightarrow K_i^{alg}(A)/\Delta_i\,,$$

is injective for $i = 0$, rationally injective for $i = 1$, and if the Novikov assembly map β_0^Γ is rationally injective, then $f_* \circ \beta_{alg}^{(i),loc}$ is also rationally injective for $i = 2$. Moreover, if β_*^Γ is injective (as for example if Γ is torsion-free and the Baum-Connes assembly map μ_*^Γ is injective), then the composition $f_* \circ \beta_{alg}^{(i),loc}$ and the map $\beta_a^{(i)} : H_i(\Gamma;\, \mathbb{Z}) \longrightarrow K_i(C^*_r\Gamma)$ are injective in the following two cases:

i) For $i = 0$ and 1, without further assumption on Γ.

ii) For $i = 2$, if Γ has its reduced integral homology concentrated in even degree, except possibly for H_1 and H_3, as for example if there is a model for the classifying space $B\Gamma$ that is a CW-complex of dimension ≤ 4.

The next theorem is a delocalized version of the preceding one.

8.1.4 Theorem. *Let Γ is a countable discrete group. Let A be a ring equipped with two maps $\mathbb{Z}_\Gamma\Gamma \xrightarrow{f} A \xrightarrow{g} C^*_r\Gamma$ such that the composition $g \circ f$ is the inclusion. If the Baum-Connes assembly map μ_*^Γ is rationally injective, then the composition*

$$(f_* \otimes Id) \circ \beta_{alg}^{(i)} : H_i(\Gamma;\, F\Gamma) \longrightarrow K_i^{alg}(A) \otimes \mathbb{C}$$

is injective, for $i = 0$, 1 and 2.

The Baum-Connes assembly map is injective for discrete subgroups of real Lie groups with finitely many connected components, for discrete subgroups of p-adic groups, and for Gromov hyperbolic groups (as for example fundamental groups of compact Riemannian manifolds of negative scalar curvature). The Baum-Connes assembly map is an isomorphism for finite groups, abelian groups, and more generally, for groups with the Haagerup property (as for example amenable groups, free groups, surface-groups, knot groups and affine Coxeter groups). It is also an isomorphism for one-relator groups, for discrete subgroups of real or complex Lorentz groups (i.e. $SO(n,\, 1)$ and $SU(n,\, 1)$), and for co-compact lattices in $Sp(n,\, 1)$, in $F_4(-20)$ or in $SL_3(F)$, where F is $\mathbb{R}$, $\mathbb{C}$ or a p-adic field. See section 4.8 for more examples (and for references).

The latter two theorems have to be compared to the known results on the analogue in algebraic K-theory of the strong Novikov conjecture: Given a (discrete) group Γ, the assembly map in algebraic K-theory has been constructed by Loday in [71] as a map

$$\theta_* : h_*(B\Gamma; \mathbf{K}\mathbb{Z}) \longrightarrow K_*^{alg}(\mathbb{Z}\Gamma),$$

where the left-hand side is the extraordinary homology theory associated to the algebraic K-theory spectrum $\mathbf{K}\mathbb{Z}$. (See also section 7.7, where θ_*, denoted by $\lambda_*^\Gamma(\mathbb{Z})$ as in [72], is discussed in some more details.) It is conjectured that θ_* is an isomorphism for torsion-free groups ("K-theory isomorphism conjecture"), and a rational injection for any group ("K-theory Novikov conjecture"): see for example [88]. Rationally, the assembly map becomes (after identification of the "rationalization" of the spectrum $\mathbf{K}\mathbb{Z}$ with a product of Eilenberg-MacLane spectra) a sequence of maps

$$\theta_n^{\mathbb{Q}} : H_n(\Gamma; \mathbb{Q}) \oplus \bigoplus_{i\geq 0} H_{n-4i-1}(\Gamma; \mathbb{Q}) \longrightarrow K_n^{alg}(\mathbb{Z}\Gamma) \otimes \mathbb{Q},$$

for all $n \geq 0$. The fundamental result was obtained in 1991 by M. Bökstedt, W. C. Hsiang and I. Madsen (see [14] and [72]): They have proved that for any group Γ whose integral homology is finitely generated in each degree (and even for a larger class of groups), the assembly map θ_* is rationally injective. Farrell and Jones [42] have established the K-theory isomorphism conjecture rationally for all (not necessarily torsion-free) co-compact lattices in connected Lie groups. We finally mention that Carlsson and Pedersen ([28] and [29]) have shown split-injectivity of θ_* for groups Γ with a model for $B\Gamma$ that is a closed smooth manifold with negative curvature (and in more general situations).

It turns out that the maps β_{alg}^{loc} and $\theta_* : h_*(B\Gamma; \mathbf{K}\mathbb{Z}) \longrightarrow K_*^{alg}(\mathbb{Z}\Gamma)$ coincide in degree 0 and 1 (up to sign); in degree 2, β_{alg}^{loc} is a (well-known) quotient map of θ_* (see section 8.3 for precise statements).

Another comparison is with theorem 7.7.1: under the same hypotheses as in theorem 8.1.4, it implies that the assembly map

$$\tilde{\alpha}_i^A : H_i(\Gamma; F\Gamma) \longrightarrow K_i^{alg}(A) \otimes \mathbb{C}$$

is injective for any $i \geq 0$, provided that g factorizes through $\mathbb{C}\Gamma$. It will be clear from the definition that $\tilde{\alpha}_i^{\mathbb{Z}_\Gamma\Gamma} = \beta_{alg}^{(i)}$, for $i = 0$ and 1. We show in section 8.3 that the same holds for $i = 2$. This leads naturally to the following conjecture.

8.1.5 Conjecture. *For any countable discrete group Γ, and any $i = 0$, 1 or 2, the map $\iota_* \circ \beta_{alg}^{(i)} : H_i(\Gamma; F\Gamma) \longrightarrow K_i^{alg}(C_r^*\Gamma) \otimes \mathbb{C}$ is injective, where $\iota : \mathbb{Z}_\Gamma\Gamma \hookrightarrow C_r^*\Gamma$ is the inclusion.*

This conjecture would follow from the Baum-Connes conjecture and from theorem 8.1.4.

Considering the delocalized equivariant Chern character ch_*^Γ (as constructed in section 6.7 under the hypothesis that conjecture 6.2.1 holds), we also ask the following question.

8.1.6 Question. *Let Γ be a countable discrete group. Let ι denote the inclusion of $\mathbb{Z}_\Gamma\Gamma$ in $C_r^*\Gamma$. Is it true that for any $i \geq 0$, the diagram*

$$\begin{array}{ccccc}
K_j^\Gamma(\underline{E\Gamma}) \otimes \mathbb{C} & & \xrightarrow{\mu_j^\Gamma \otimes Id} & & K_j(C_r^*\Gamma) \otimes \mathbb{C} \\
\Big\downarrow ch_i^\Gamma \otimes Id & & & & \cong \Big\uparrow \mathscr{B}^{-1} \otimes Id \\
 & & & & K_i(C_r^*\Gamma) \otimes \mathbb{C} \\
 & & & & \Big\uparrow \phi_i^{C_r^*\Gamma} \otimes Id \\
H_i(\Gamma; F\Gamma) & \xrightarrow{\tilde{\alpha}_i^{\mathbb{Z}_\Gamma\Gamma}} & K_i^{alg}(\mathbb{Z}_\Gamma\Gamma) \otimes \mathbb{C} & \xrightarrow{\iota_* \otimes Id} & K_i^{alg}(C_r^*\Gamma) \otimes \mathbb{C} \\
 & \searrow \tilde{\alpha}_i^{\mathbb{C}\Gamma} & \Big\downarrow & \nearrow incl_* & \\
 & & K_i^{alg}(\mathbb{C}\Gamma) \otimes \mathbb{C} & &
\end{array}$$

commutes? (Here again, $j \in \{0, 1\}$ is the reduction modulo 2 of i, and $\mathscr{B}$ is the Bott periodicity isomorphism.) If this is true and if the rational Baum-Connes conjecture holds, then the composition $(\iota_* \otimes Id) \circ \tilde{\alpha}_*^{\mathbb{Z}_\Gamma\Gamma}$ is injective, and the map

$$\bigoplus_{i \geq 0} (\mathscr{B}^{-1} \circ \phi_i^{C_r^*\Gamma} \circ \iota_*) \otimes Id : K_i^{alg}(\mathbb{Z}_\Gamma\Gamma) \otimes \mathbb{C} \longrightarrow K_*(C_r^*\Gamma) \otimes \mathbb{C}$$

is surjective (where $K_*(C_r^*\Gamma)$ is considered as being $\mathbb{Z}/2$-graded).

In section 8.2, we construct the maps β_{alg}^{loc} and β_{alg}. We compare, in section 8.3, β_{alg}^{loc} with the Loday assembly map in algebraic K-theory, and with the map $\tilde{\alpha}_*^{\mathbb{Z}_\Gamma\Gamma}$. In section 8.4, we give a description of the Bott generator in $\widetilde{K}_2(C(\mathbb{T}^2))$ as an element of $\pi_1(SU_2(C(\mathbb{T}^2)))$ represented by a differentiable map. This is needed in the proof of theorem 8.1.1, that is contained in section 8.5, together with other useful explicit computations on the map $K_2^{alg}(\mathbb{Z}[\mathbb{Z}^2]) \longrightarrow K_2(C_r^*\mathbb{Z}^2)$. The final section 8.6 contains the proofs of the other main theorems, namely 8.1.2, 8.1.3 and 8.1.4.

8.2 The maps β_{alg}^{loc} and β_{alg}

In this section, we define the maps β_{alg}^{loc} and their delocalized counterparts β_{alg}.

In degree 0, $H_0(\Gamma; \mathbb{Z}) = \mathbb{Z}$, and we define

$$\beta_{alg}^{loc} : H_0(\Gamma; \mathbb{Z}) \longrightarrow K_0(\mathbb{Z}\Gamma),\ 1 \longmapsto [1],$$

where $[1]$ is the K-theory class of the unit (viewed as an idempotent (1×1)-matrix over $\mathbb{Z}\Gamma$). For $j=1$, one has a canonical and natural isomorphism $H_1(\Gamma;\mathbb{Z})\cong\Gamma^{ab}$, the abelianization of Γ. We write γ^{ab} for the class in Γ^{ab} of an element $\gamma\in\Gamma$. By definition, one has

$$\beta^{loc}_{alg}: H_1(\Gamma;\mathbb{Z})\longrightarrow K_1(\mathbb{Z}\Gamma),\ \gamma^{ab}\longmapsto -[\gamma]=[\gamma^{-1}],$$

where $[\gamma^{\pm 1}]$ is the K-theory class of $\gamma^{\pm 1}$ (viewed as an invertible (1×1)-matrix over $\mathbb{Z}\Gamma$). It is clear that the compositions of these maps with the homomorphism induced by the inclusion $\mathbb{Z}\Gamma\hookrightarrow\ell^1\Gamma$ are the maps $\hat{\beta}^{loc}_a$ defined in section 5.2.

Let us pass to degree 2. We begin by discussing the surface-groups. Let Σ_g denote a closed and canonically oriented surface of genus g, and let $\Gamma_g:=\pi_1(\Sigma_g)$. Recall that $\Sigma_g=B\Gamma_g$, and that Γ_g admits the presentation

$$\Gamma_g=\left\langle a_1,\ldots,a_g,b_1,\ldots,b_g\ \middle|\ \prod_{j=1}^{g}[a_j,b_j]\right\rangle.$$

In the sequel, we always consider Γ_g as this *presented group* and Σ_g is merely considered as a model for $B\Gamma_g$. Therefore, Γ_g is the quotient of the free group F_{2g} on $2g$ generators

$$\{\tilde{a}_1,\ldots,\tilde{a}_g,\tilde{b}_1,\ldots,\tilde{b}_g\}$$

by the normal subgroup R_{2g} generated (as a normal subgroup of F_{2g}) by the element $\tilde{c}_g:=\prod_{j=1}^{g}[\tilde{a}_j,\tilde{b}_j]$. We denote by $[\Sigma_g]$ the fundamental class in $H_2(\Gamma_g;\mathbb{Z})\cong\mathbb{Z}$. The Hopf formula gives an isomorphism

$$H_2(\Gamma_g;\mathbb{Z})\cong(R_{2g}\cap[F_{2g},F_{2g}])/[F_{2g},R_{2g}],$$

and by Loday [71] (proof of lem. 2.2.4), the generator $[\Sigma_g]$ corresponds to the coset $\tilde{c}_g^{-1}\cdot[F_{2g},R_{2g}]$ (note the inverse sign). Consider the following product of Steinberg symbols:

$$z_g:=\prod_{j=1}^{g}\{a_j,b_j\}\in K_2^{alg}(\mathbb{Z}\Gamma_g).$$

There is a slight abuse of terminology here, since the element $\{a_j,b_j\}$ is not in $K_2^{alg}(\mathbb{Z}\Gamma_g)$ (unless $g=1$); it is however a well-defined element in the Steinberg group $\mathrm{St}(\mathbb{Z}\Gamma_g)$, and the above product z_g is definitely in

$$K_2^{alg}(\mathbb{Z}\Gamma_g)=\mathcal{Z}(\mathrm{St}(\mathbb{Z}\Gamma_g))=\mathrm{Ker}(\mathrm{St}(\mathbb{Z}\Gamma_g)\twoheadrightarrow E(\mathbb{Z}\Gamma_g)).$$

We have to discuss briefly reduced algebraic K-groups. For a unital ring A, let

$$\widetilde{K}_2^{alg}(A):=\mathrm{Coker}(K_2^{alg}(\mathbb{Z})\xrightarrow{j_*}K_2^{alg}(A)),$$

where $j : \mathbb{Z} \longrightarrow A$ is the map taking 1 to the unit of A. This is a functor for unital rings (and unital morphisms). If A is augmented, i.e. if there exists a ring homomorphism $\varepsilon : A \longrightarrow \mathbb{Z}$ such that $\varepsilon \circ j = Id_{\mathbb{Z}}$, then there is a canonical splitting

$$K_2^{alg}(A) \cong K_2^{alg}(\mathbb{Z}) \oplus \widetilde{K}_2^{alg}(A) \cong \mathbb{Z}/2 \oplus \widetilde{K}_2^{alg}(A),$$

where $\widetilde{K}_2^{alg}(A)$ is identified with the subgroup $\mathrm{Ker}(\varepsilon_*)$ of $K_2^{alg}(A)$. This holds in particular for the group ring $\mathbb{Z}\Gamma$ of any group Γ. Indeed, it is equipped with a canonical augmentation defined by $\varepsilon(\sum_\gamma \lambda_\gamma \cdot \gamma) := \sum_\gamma \lambda_\gamma$. We simply write $K_2^{alg}(\mathbb{Z}\Gamma) = K_2^{alg}(\mathbb{Z}) \oplus \widetilde{K}_2^{alg}(\mathbb{Z}\Gamma)$ (this decomposition is natural in Γ). For an element $\gamma \in \Gamma$, one has $\varepsilon_*(\{\gamma, \gamma\}) = \{1, 1\} = 1$ in $K_2^{alg}(\mathbb{Z}) = \{\pm 1\}$ (we consider $K_2(A)$ as a subgroup of $\mathrm{St}(A)$ and write it multiplicatively). Consequently, one has $\Delta_2(\mathbb{Z}\Gamma) \subseteq \widetilde{K}_2^{alg}(\mathbb{Z}\Gamma)$ (see the introduction for the definition of $\Delta_2(\mathbb{Z}\Gamma)$). In particular, we have a canonical and natural decomposition

$$K_2^{alg}(\mathbb{Z}\Gamma)/\Delta_2(\mathbb{Z}\Gamma) = K_2^{alg}(\mathbb{Z}) \oplus (\widetilde{K}_2^{alg}(\mathbb{Z}\Gamma)/\Delta_2(\mathbb{Z}\Gamma)).$$

For the surface-group Γ_g, by naturality of the Steinberg symbols, one has clearly $\varepsilon_*(z_g) = \{1, 1\}^g = 1$ in $K_2^{alg}(\mathbb{Z})$. This means that $z_g \in \widetilde{K}_2^{alg}(\mathbb{Z}\Gamma_g)$. We now define

$$\bar{z}_g := \text{class of } z_g \text{ in } \widetilde{K}_2^{alg}(\mathbb{Z}\Gamma_g)/\Delta_2(\mathbb{Z}\Gamma_g).$$

Now, let Γ denote an arbitrary group. By theorem 3.3.6 (see also lemma 2.2.4 in Loday [71], Thom's theorem 5.2.1 and Zimmermann's theorem 5.2.2), any homology class $x \in H_2(B\Gamma; \mathbb{Z})$ is "Steenrod-representable". In other words, there exists a surface Σ_g of genus $g \geq 1$ and a continuous pointed map $f : \Sigma_g \longrightarrow B\Gamma$ such that $x = f_*([M])$. In this case, we write $x = [\Sigma_g, f]$, and we define

$$\begin{aligned} \beta_{alg}^{(2),loc} : H_2(\Gamma; \mathbb{Z}) &\longrightarrow \widetilde{K}_2^{alg}(\mathbb{Z}\Gamma)/\Delta_2(\mathbb{Z}\Gamma) \subset K_2^{alg}(\mathbb{Z}\Gamma)/\Delta_2(\mathbb{Z}\Gamma) \\ [\Sigma_g, f] &\longmapsto \pi_1(f)_*(\bar{z}_g), \end{aligned}$$

where $\pi_1(f)_* : \widetilde{K}_2^{alg}(\mathbb{Z}\Gamma_g)/\Delta_2(\mathbb{Z}\Gamma_g) \longrightarrow \widetilde{K}_2^{alg}(\mathbb{Z}\Gamma)/\Delta_2(\mathbb{Z}\Gamma)$ is induced by $\pi_1(f)$.

Notice that this map is precisely the one defined by Loday in [71] and denoted there by $\lambda''(\Gamma)$. Indeed, this follows from his proposition 4.3.1 (naturality), his lemma 4.3.4 (explicit computation for Γ_g) and his lemma 2.2.4 ("universality of the groups $\{\Gamma_g\}_{g \geq 1}$ for the second homology"). He shows in particular that $\lambda''(\Gamma)$ is a well-defined and natural homomorphism, hence so is $\beta_{alg}^{(2),loc}$. Let us however prove directly that it is a natural homomorphism (under the assumption that it is well-defined), and single out the difficulties in proving that it is well-defined.

The naturality of $\beta_{alg}^{(2),loc}$ is clear from the description of the functoriality of the $\Omega(\Gamma)$ appearing in Zimmermann's theorem 5.2.2 (as we described it after the statement). The fact that $\beta_{alg}^{(2),loc}$ is a homomorphism poses no particular difficulty. Indeed, let

$x_1 = [\Sigma_{g_1}, f_1]$ and $x_2 = [\Sigma_{g_2}, f_2]$ be two classes in $H_2(\Gamma; \mathbb{Z})$. The sum $x_1 + x_2$ corresponds to $[\Sigma_g, f]$, where $g = g_1 + g_2$, $\Sigma_g = \Sigma_{g_1} \# \Sigma_{g_2}$ is the oriented connected sum, and $f = f_1 \# f_2$. For $k = 1$ and 2, let $A_k = \{a_i^{(k)}, b_i^{(k)} \mid 1 \le i \le g_k\}$ be the set of prescribed generators of Γ_{g_k}. The group Γ_g is "canonically generated" by the set $A_1 \coprod A_2$, and $\pi_1(f) : \Gamma_g \longrightarrow \Gamma$ is clearly given by

$$a_i^{(k)} \longmapsto \pi_1(f_k)(a_i^{(k)}) \text{ and } b_i^{(k)} \longmapsto \pi_1(f_k)(b_i^{(k)}).$$

Consequently, one has

$$(\pi_1(f))_*(z_g) = (\pi_1(f_1))_*(z_{g_1}) \cdot (\pi_1(f_2))_*(z_{g_2}) \in K_2^{alg}(\mathbb{Z}\Gamma).$$

This proves additivity of $\beta_{alg}^{(2),loc}$.

Thanks to theorem 5.2.2, in order to prove that β_{alg}^{loc} is well-defined, it is enough to prove that for any $g, \tilde{g} \ge 1$, and any orientation-preserving homeomorphism h of Σ_g, one has

$$(\pi_1(p))_*(\bar{z}_{g+\tilde{g}}) = \bar{z}_g \text{ and } \pi_1(h)_*(\bar{z}_g) = \bar{z}_g,$$

where $p : \Sigma_{g+\tilde{g}} = \Sigma_g \# \Sigma_{\tilde{g}} \twoheadrightarrow \Sigma_g$ is the canonical projection (defined up to homotopy). The first equality is clear since the homomorphism $\pi_1(p) : \Gamma_{g+\tilde{g}} \longrightarrow \Gamma_g$ is given by

$$\pi_1(p)(a_i) = \begin{cases} a_i, & \text{if } i \le g \\ e, & \text{otherwise} \end{cases} \qquad \pi_1(p)(b_i) = \begin{cases} b_i, & \text{if } i \le g \\ e, & \text{otherwise.} \end{cases}$$

(One even has $(\pi_1(p))_*(z_{g+\tilde{g}}) = z_g$, without factoring out $\Delta_2(\mathbb{Z}\Gamma_g)$.) The second equality poses difficulties. In order to illustrate the situation (and the necessity of factoring out the subgroup Δ_2), we consider the case of an orientation preserving homeomorphism of the 2-torus $\Sigma_1 = \mathbb{T}^2$. It is given, at the level of the fundamental group $\mathbb{Z}^2$, by a matrix

$$A = \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

The ring $\mathbb{Z}[\mathbb{Z}^2]$ being commutative, by means of the standard relations for the Steinberg symbols (skew-symmetry and bilinearity), one easily computes that

$$\begin{aligned} z_1 = \{a, b\} &\longmapsto \{a, a\}^{n_1+n_3} \cdot \{a, b\}^{n_1 n_4 - n_2 n_3} \cdot \{b, b\}^{n_2+n_4} \\ &= \{a, a\}^{n_1+n_3} \cdot z_1 \cdot \{b, b\}^{n_2+n_4}. \end{aligned}$$

As we will see in section 8.3, $\{a, a\}$ and $\{b, b\}$ are non-zero in $K_2^{alg}(\mathbb{Z}[\mathbb{Z}^2])$. This means that we cannot avoid factoring out Δ_2 in this case.

For the general case of an orientation-preserving homeomorphism h of Σ_g, let $\mathrm{Aut}_0(\Gamma_g)$ be the group of automorphisms of Γ_g inducing the identity on $H_2(\Gamma_g; \mathbb{Z})$, as for example $\alpha := \pi_1(h)$ or any conjugation. By the Dehn-Nielsen theorem (see

[85], p. 391), one has an isomorphism $\sigma : \mathrm{Aut}_0(\Gamma_g)/\mathrm{Inn}(\Gamma_g) \xrightarrow{\cong} \mathrm{Map}_g$, where Map_g is the mapping class group of Σ_g (also called the group of homeotopies of Σ_g), i.e. the quotient of the group of orientation-preserving homeomorphisms of Σ_g by the normal subgroup of such homeomorphisms that are homotopic to the identity. By the theorem of Lickorish (see [69] and [70]), the group Map_g is generated by the image under σ of $3g-1$ Dehn twists λ_i, μ_i and ν_k where $i = 1, \dots, g$ and $k = 1, \dots, g-1$ (by Humphries, one could just take μ_1 and μ_2 among the μ_i's, see [53]). Consequently, the group $\mathrm{Aut}_0(\Gamma_g)$ is generated by the conjugations by the generators a_i and b_i of Γ_g, and by the $3g-1$ Dehn twists λ_i, μ_i and ν_k. By Morita's computations (see [79], lem. 4.4), one has

$$\begin{aligned}
\lambda_i(b_i) &= b_i a_i \\
\mu_i(a_i) &= a_i b_i^{-1} \\
\nu_k(a_k) &= a_k b_k^{-1} a_{k+1} b_{k+1} a_{k+1}^{-1} \\
\nu_k(a_{k+1}) &= a_{k+1} b_{k+1}^{-1} a_{k+1}^{-1} b_k a_{k+1} \\
\nu_k(b_k) &= a_{k+1} b_{k+1}^{-1} a_{k+1}^{-1} b_k a_{k+1} b_{k+1} a_{k+1}^{-1}
\end{aligned}$$

and all the other generators a_l and b_l of Γ_g are left fixed by these maps. Therefore, since a conjugation by an element of Γ_g induces the identity on $K_2^{alg}(\mathbb{Z}\Gamma_g)$, showing that $\beta_{alg}^{(2),loc}$ is well-defined amounts to prove that for α denoting any of the automorphisms λ_i, μ_i or ν_k of Γ_g, one has

$$\alpha_* : K_2^{alg}(\mathbb{Z}\Gamma_g)/\Delta_2 \longrightarrow K_2^{alg}(\mathbb{Z}\Gamma_g)/\Delta_2\,,\ \bar{z}_g \longmapsto \bar{z}_g\,.$$

Unfortunately, without appealing to Loday's result [71] (or to Waldhausen's papers [109]), we do not know how to prove this directly. It would of course be interesting to compute directly the image in $K_2^{alg}(\mathbb{Z}\Gamma_g)$ of z_g itself.

We pass now to the definition of the maps $\beta_{alg}^{(i)}$.

We keep notations as in section 5.3, and for a conjugacy class $C \in \langle\Gamma\rangle^{ell}$, we denote by $\iota_l^{(C)} : P_l^{(C)} \cdot \mathbb{Z}_\Gamma Z_C \hookrightarrow \mathbb{Z}_\Gamma \Gamma$ and by $j_C : \mathbb{Z}Z_C \hookrightarrow \mathbb{Z}_\Gamma Z_C$ the inclusions. For $i = 0, 1$ and 2, the maps $\beta_{alg}^{(i)}$ (for the group Γ) are defined as the composition

$$\begin{array}{ccc}
H_{\le 2}(\Gamma; F\Gamma) & & \\
\cong \big\downarrow & & \\
\displaystyle\bigoplus_{C\in\langle\Gamma\rangle^{ell}} H_{\le 2}(Z_C; \mathbb{C}) & \xrightarrow{\ \bigoplus_C ((j_C)_* \circ \beta_{alg}^{Z_C, loc}) \otimes Id\ } & \displaystyle\bigoplus_{C\in\langle\Gamma\rangle^{ell}} K_*^{alg}(\mathbb{Z}_\Gamma Z_C) \otimes \mathbb{C} \\
 & & \big\downarrow \cong \\
K_*^{alg}(\mathbb{Z}_\Gamma \Gamma) \otimes \mathbb{C} & \xleftarrow{\ \sum_C \sum_l (\iota_l^{(C)})_* \otimes \bar{\omega}_C^l\ } & \displaystyle\bigoplus_{C\in\langle\Gamma\rangle^{ell}} \bigoplus_{l=0}^{n_C-1} K_*^{alg}(P_l^{(C)} \cdot \mathbb{Z}_\Gamma Z_C) \otimes \mathbb{C}
\end{array}$$

where $\beta_{alg}^{Z_C, loc}$ denotes the map β_{alg}^{loc} for the group Z_C. (The functoriality of K_2^{alg} for non-unital morphisms between unital rings is described as in the proof of lemma 7.7.6.)

Let us now give explicit formulas for these maps restricted to a direct summand $H_i(Z_C;\,\mathbb{C})$ of $H_i(\Gamma;\,F\Gamma)$. From the definition of these maps, one easily finds

$$\beta^{(0)}_{alg}: H_0(Z_C;\,\mathbb{C}) \longrightarrow K^{alg}_0(\mathbb{Z}_\Gamma\Gamma)\otimes\mathbb{C},\ 1 \longmapsto \sum_{l=0}^{n_C-1}[P^{(C)}_l]\otimes\bar{\omega}^l_C\,,$$

where $H_0(Z_C;\,\mathbb{C})$ is identified with $\mathbb{C}$, and similarly (with the help of lemma 5.6.3)

$$\beta^{(1)}_{alg}: H_1(Z_C;\,\mathbb{C}) \longrightarrow K^{alg}_1(\mathbb{Z}_\Gamma\Gamma)\otimes\mathbb{C},\ \gamma^{ab}\otimes 1 \longmapsto \sum_{l=0}^{n_C-1}[P^{(C)}_l\gamma^{-1}+1-P^{(C)}_l]\otimes\bar{\omega}^l_C\,,$$

where $H_1(Z_C;\,\mathbb{C})$ is identified with $Z^{ab}_C\otimes\mathbb{C}$, and γ^{ab} is the class in Z^{ab}_C of $\gamma\in Z_C$. For $i=2$, using "Steenrod-representation" (see theorem 3.3.6) and writing K^{alg}_2 multiplicatively, we find

$$\begin{aligned}\beta^{(2)}_{alg}: H_2(Z_C;\,\mathbb{C}) &\longrightarrow K^{alg}_2(\mathbb{Z}_\Gamma\Gamma)\otimes\mathbb{C}\\ [\Sigma_g,\,f]\otimes 1 &\longmapsto \sum_{l=0}^{n_C-1}(\iota^{(C)}_l)_*\circ(\pi_1(f))_*(\bar{z}_g)\otimes\bar{\omega}^l_C\\ &= \prod_{j=1}^{g}\left[\left\{\sum_{l=0}^{n_C-1}P^{(C)}_l\pi_1(f)(a_j),\ \sum_{l=0}^{n_C-1}P^{(C)}_l\pi_1(f)(b_j)\right\}\right]\otimes\bar{\omega}^l_C\,,\end{aligned}$$

where $(\pi_1(f))_*: K^{alg}_2(\mathbb{Z}_\Gamma\Gamma_g)\longrightarrow K^{alg}_2(\mathbb{Z}_\Gamma Z_C)$ is induced by $\pi_1(f)$, and $\bar{z}_g$ is considered, in the obvious way, as an element of $K^{alg}_2(\mathbb{Z}_\Gamma\Gamma_g)/\Delta_2$; the equality comes from a direct computation of $(\iota^{(C)}_l)_*$.

8.3 Comparison of the assembly maps $\tilde{\alpha}^{\mathbb{Z}_\Gamma\Gamma}_*$ and θ_* with β^{loc}_{alg}

We discuss rapidly the relationship between the assembly map defined by Loday in [71] and the map β^{loc}_{alg}. We also explain that, in degree 2, one cannot avoid quotienting K^{alg}_2 by Δ_2. For details, the reader is referred to [71].

Let $\mathbf{K}\mathbb{Z}$ denote the algebraic K-theory spectrum. It is an Ω-spectrum whose homotopy groups are given by

$$\pi_n(\mathbf{K}\mathbb{Z})\cong K^{alg}_n(\mathbb{Z})\,,\ \forall n\in\mathbb{Z}.$$

(Since spectra have negative homotopy groups, this definitely includes negative K-theory. However, $K_n^{alg}(\mathbb{Z}) = 0$ for all $n < 0$, since $\mathbb{Z}$ is a regular ring (see [87], 3.1.2(4) and 3.3.1).)

Associated to this spectrum, there is a homology theory on the category of CW-complexes, defined by

$$h_*(X;\, \mathbf{K}\mathbb{Z}) := \pi_*(X_+ \wedge \mathbf{K}\mathbb{Z}),$$

where X_+ is X with an extra base-point added. In particular, for the one-point space, one has $h_*(pt;\, \mathbf{K}\mathbb{Z}) \cong K_*^{alg}(\mathbb{Z})$. Notice that $h_n(X;\, \mathbf{K}\mathbb{Z}) = 0$ for any X and any $n < 0$. The reduced homology theory associated to h_* is defined, for X pointed, by

$$\widetilde{h}_*(X;\, \mathbf{K}\mathbb{Z}) := \pi_*(X \wedge \mathbf{K}\mathbb{Z}),$$

and one has a canonical splitting $h_*(X;\, \mathbf{K}\mathbb{Z}) = K_*^{alg}(\mathbb{Z}) \oplus \widetilde{h}_*(X;\, \mathbf{K}\mathbb{Z})$.

The assembly map in algebraic K-theory is a natural map

$$\theta_* : h_*(B\Gamma;\, \mathbf{K}\mathbb{Z}) \longrightarrow K_*^{alg}(\mathbb{Z}\Gamma),$$

for any group Γ. It is defined at the level of spectra on the summand $\widetilde{h}_*(B\Gamma;\, \mathbf{K}\mathbb{Z})$, and on the summand $K_*^{alg}(\mathbb{Z})$ as the map j_* induced by the inclusion $j : \mathbb{Z} \hookrightarrow \mathbb{Z}\Gamma$.

In degree 0, one has $h_0(B\Gamma;\, \mathbf{K}\mathbb{Z}) \cong K_0^{alg}(\mathbb{Z}) \cong \mathbb{Z}$, and under this identification, it is obvious to check that the maps $\beta_{alg}^{(0),loc}$ and θ_0 coincide.

In degree 1, the Atiyah-Hirzebruch spectral sequence for the homology theory $h_*(-;\, \mathbf{K}\mathbb{Z})$ shows that there is a canonical a natural isomorphism

$$\widetilde{h}_1(B\Gamma;\, \mathbf{K}\mathbb{Z}) \cong H_1(B\Gamma;\, K_0^{alg}(\mathbb{Z})) \cong H_1(\Gamma;\, \mathbb{Z})$$

(see 4.2.1 in [71]). By lemma 4.1.2 in [71], by naturality of θ_1 and by "universality of the group $\mathbb{Z}$ for the first homology", one has

$$\beta_{alg}^{(1),loc} = -\theta_{1\,|\widetilde{h}_1(B\Gamma;\mathbf{K}\mathbb{Z})}.$$

In degree 2, already in the definition of $\beta_{alg}^{(2),loc}$, we have applied results from [71]. In particular, to show that it is well-defined and a homomorphism, we have exploited the equality between $\beta_{alg}^{(2),loc}$ and Loday's map $\lambda''(\Gamma)$. One can therefore expect a close relationship between $\beta_{alg}^{(2),loc}$ and θ_2. Indeed, one has a decomposition $h_2(B\Gamma;\, \mathbf{K}\mathbb{Z}) = K_2^{alg}(\mathbb{Z}) \oplus \widetilde{h}_2(B\Gamma;\, \mathbf{K}\mathbb{Z})$, and the Atiyah-Hirzebruch spectral sequence yields the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & H_1(\Gamma;\, K_1^{alg}(\mathbb{Z})) & \xrightarrow{\chi} & \widetilde{h}_2(B\Gamma;\, \mathbf{K}\mathbb{Z}) & \longrightarrow & H_2(\Gamma;\, \mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda'} & & \downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \beta_{alg}^{loc}} & & \\
0 & \longrightarrow & \Delta_2 & \longrightarrow & \widetilde{K}_2^{alg}(\mathbb{Z}\Gamma) & \longrightarrow & \widetilde{K}_2^{alg}(\mathbb{Z}\Gamma)/\Delta_2 & \longrightarrow & 0
\end{array}$$

(see 4.3.3 in [72]), hence the leitmotiv "in degree 2, β_{alg}^{loc} is a well-known quotient-map of θ_2".

By lemma 4.3.2 in [71], the image of $\theta_2 \circ \chi$ is precisely Δ_2. It is therefore necessary to factor out Δ_2 in order to define a quotient map of θ_2 with domain equal to $H_2(\Gamma;\, \mathbb{Z})$. This implies that β_{alg}^{loc} factorizes through $K_2^{alg}(\mathbb{Z}\Gamma)$ if and only if $\Delta_2 = 0$.

As mentioned in the introduction, an important feature of the K-theory spectrum $\mathbf{K}\mathbb{Z}$ is that rationally, it splits as a product of Eilenberg-MacLane spectra. This allows proving that $\beta_{alg}^{(i)}$ coincides with the map $\bar{\alpha}_i^{\mathbf{Z}_\Gamma \Gamma}$ of section 7.7, for $i = 0, 1$ and 2.

8.3.1 Proposition. *For any discrete group Γ, the maps $\beta_{alg}^{(i)}$ and $\bar{\alpha}_i^{\mathbf{Z}_\Gamma \Gamma}$ coincide for $i = 0$, 1 and 2.*

Proof. For $i = 0$ and 1, this is clear. For $i = 2$, the delocalization process being the same in the construction of both maps, it suffices to show that $\beta_{alg}^{(2),\,loc} \otimes Id_{\mathbb{C}}$ is the same as the map ρ_2^Γ of theorem 7.7.5. This follows from the above commutative diagram and from the definition of the map ρ_*^Γ in [87]: ρ_*^Γ is the composition

$$H_*(\Gamma;\, \mathbb{C}) \hookrightarrow h_*(B\Gamma;\, \mathbf{K}\mathbb{Z}) \otimes \mathbb{C} \xrightarrow{\theta_* \otimes Id} K_*^{alg}(\mathbb{Z}\Gamma) \otimes \mathbb{C}.$$

This completes the proof. □

Example. Let us illustrate the situation by the surface-groups Γ_g (with $g \geq 1$). By 4.3.3 in [71], in this case, the first row in the above diagram is a split short exact sequence, and all the vertical maps are isomorphisms. In particular, one has

$$\widetilde{h}_2(B\Gamma_g;\, \mathbf{K}\mathbb{Z}) \cong H_1(\Gamma_g;\, \mathbb{Z}/2) \oplus H_2(\Gamma_g;\, \mathbb{Z}) \cong (\mathbb{Z}/2)^{2g} \oplus \mathbb{Z}$$

and $\Delta_2 = \{\{a_1,\, a_1\},\, \{b_1,\, b_1\},\, \ldots,\, \{a_g,\, a_g\},\, \{b_g,\, b_g\}\} \cong (\mathbb{Z}/2)^{2g}$. Moreover, the map $\beta_{alg}^{(2),\,loc}$ is an isomorphism onto $\widetilde{K}_2^{alg}(\mathbb{Z}\Gamma_g)/\Delta_2$, and one has

$$K_2^{alg}(\mathbb{Z}\Gamma_g) = \underbrace{\mathrm{Im}(K_2^{alg}(\mathbb{Z}))}_{\cong \mathbb{Z}/2} \oplus \underbrace{\Delta_2}_{\cong (\mathbb{Z}/2)^{2g}} \oplus \underbrace{\mathbb{Z} \cdot z_g}_{\cong \mathbb{Z}} \cong (\mathbb{Z}/2)^{2g+1} \oplus \mathbb{Z}.$$

For the group $\Gamma_1 = \mathbb{Z}^2$, this gives $K_2^{alg}(\mathbb{Z}[a,\, a^{-1},\, b,\, b^{-1}]) \cong (\mathbb{Z}/2)^3 \oplus \mathbb{Z}$, with $\Delta_2 = \{\{a,\, a\},\, \{b,\, b\}\} \cong (\mathbb{Z}/2)^2$.

To end this section, we mention the Whitehead groups, that are rather concerned with surjectivity of θ_*. They are defined by

$$Wh_n(\Gamma) := \begin{cases} \widetilde{K}_0(\mathbb{Z}\Gamma)\,, & \text{if } n = 0 \\ K_1(\mathbb{Z}\Gamma)/(\pm\Gamma)\,, & \text{if } n = 1 \\ K_2(\mathbb{Z}\Gamma)/(W(\pm\Gamma) \cap K_2(\mathbb{Z}\Gamma))\,, & \text{if } n = 2 \end{cases}$$

where $W(\pm\Gamma)$ is the subgroup of $\mathrm{St}(\mathbb{Z}\Gamma)$ generated by the elements $w_{ij}(\pm\gamma)$, with $\gamma \in \Gamma$ (where $w_{ij}(u) := x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u)$ for $u \in \mathrm{GL}_1(\mathbb{Z}\Gamma)$). Loday has shown that

$$Wh_n(\Gamma) \cong \mathrm{Coker}(\theta_n : h_n(B\Gamma; \mathbf{K}\mathbb{Z}) \longrightarrow K_n(\mathbb{Z}\Gamma))\,, \text{ for } n = 0,\, 1,\, 2$$

(see 4.2.1 and 4.2.2 in [71]). Moreover, Waldhausen has proved in [109] that $Wh_2(\Gamma)$ vanishes if Γ is in one of the following families of groups:

- free groups, free abelian groups, and poly-$\mathbb{Z}$-groups;
- fundamental groups of 2-manifolds different from the real projective plane;
- torsion-free one-relator groups;
- fundamental groups of sub-manifolds of S^3;
- iterated amalgamated products and iterated HNN-extensions of free groups;
- and subgroups of the families above.

8.4 The Bott generator in $\pi_1(SU_2(C(\mathbb{T}^2)))$

In the proof of theorem 8.1.1, we will need an explicit description of the Bott generator $\hat{\delta}$ of $\widetilde{K}_2(C(\mathbb{T}^2)) \cong \pi_1(SU(C(\mathbb{T}^2))) \cong \mathbb{Z}$. More precisely, we will need to see it as an element of $\pi_1(SU_2(C(\mathbb{T}^2))) \cong \mathbb{Z}$, in other words as the homotopy class of a map

$$f : S^1 \times \mathbb{T}^2 \longrightarrow SU(2)\,.$$

We realize f as a differentiable map, and compute its degree. (The degree gives the usual isomorphism $\pi_1(SU_2(C(\mathbb{T}^2))) \xrightarrow{\cong} \mathbb{Z}$.)

Let us first recall the Bott periodicity isomorphism and the suspension isomorpbism (see [16] for details). Let A be a Banach algebra (unital or not), and let I denote the open interval $]0,\, 1[$. The suspension of A is defined by

$$SA := C_0(I,\, A) \cong C_0(I)\hat{\otimes}A\,,$$

i.e. the Banach algebra of continuous functions from I to A that vanish at infinity. (The tensor product "$\hat{\otimes}$" is the completed projective tensor product.) Notice that $S\mathbb{C} = C_0(I)$ and $SC_0(I) \cong C_0(I^2)$.

The Bott periodicity isomorphism is given by

$$K_0(A) \xrightarrow{\cong} K_1(SA)\,,\ [p] \longmapsto \left[t \mapsto e^{2\pi i t\cdot p} = e^{2\pi i t} \cdot p + \mathbb{1} - p\right]\,.$$

If A is a C^*-algebra, the latter element is unitary. The suspension isomorphism is given by

$$\sigma : K_1(A) \xrightarrow{\cong} K_0(SA)\,,\ [u] \longmapsto \left[t \mapsto R_t \cdot P_n \cdot R_t^{-1} - P_n\right]\,,$$

where $u \in \mathrm{GL}_n(A)$, $P_n := \mathrm{Diag}(\mathbb{1}_n, \mathbb{O}_n)$, and R_t is a homotopy (i.e. a path) in $\mathrm{GL}_{2n}(A)$ from $\mathbb{1}_{2n}$ to the matrix $\mathrm{Diag}(u, u^{-1})$ which, by the Whitehead lemma, belongs to the arc component of $\mathbb{1}_{2n}$ in $\mathrm{GL}_{2n}(A)$. The suspension isomorphism is independent of the chosen homotopy, and we can for example take

$$\begin{aligned} R_t &:= \begin{pmatrix} u & \mathbb{O}_n \\ \mathbb{O}_n & \mathbb{1}_n \end{pmatrix} \cdot \begin{pmatrix} C\cdot\mathbb{1}_n & S\cdot\mathbb{1}_n \\ -S\cdot\mathbb{1}_n & C\cdot\mathbb{1}_n \end{pmatrix} \cdot \begin{pmatrix} u^{-1} & \mathbb{O}_n \\ \mathbb{O}_n & \mathbb{1}_n \end{pmatrix} \cdot \begin{pmatrix} C\cdot\mathbb{1}_n & -S\cdot\mathbb{1}_n \\ S\cdot\mathbb{1}_n & C\cdot\mathbb{1}_n \end{pmatrix} \\ &= \begin{pmatrix} C^2\cdot\mathbb{1}_n + S^2\cdot u & CS\cdot(u-\mathbb{1}_n) \\ CS\cdot(\mathbb{1}_n - u^{-1}) & C^2\cdot\mathbb{1}_n + S^2\cdot u^{-1} \end{pmatrix} \end{aligned}$$

where $C = C(t) := \cos(\pi t/2)$ and $S = S(t) := \sin(\pi t/2)$. If A is a C^*-algebra and if u is unitary, then this path R_t is unitary.

By definition, $K_2(A) = \pi_1(\mathrm{GL}(A))$, and there is a canonical isomorphism

$$K_1(SA) \xrightarrow{\cong} \pi_1(\mathrm{GL}(A)) = K_2(A)\,,\ [t \mapsto u(t)] \longmapsto \left[e^{2\pi i t} \mapsto u(t)\right]\,.$$

If A is a C^*-algebra, we can replace $\mathrm{GL}(A)$ by $U(A)$, the infinite unitary group over A.

Viewing the 2-torus $\mathbb{T}^2 = B\mathbb{Z}^2$ as a quotient of the square $[0, 1] \times [0, 1]$, yields an injection $C_0(I^2) \hookrightarrow C(\mathbb{T}^2)$. It induces a split-injection of $K_0(C_0(I^2))$ into $K_0(C(\mathbb{T}^2)) \cong K_0(C_r^*\mathbb{Z}^2)$. By composing Bott and suspension isomorphisms and this split-injection, we get a sequence of maps

$$\begin{aligned} \mathbb{Z} \cong \quad K_0(\mathbb{C}) \quad &\cong\ K_1(C_0(I)) \cong K_0(C_0(I^2)) \\ \hookrightarrow K_0(C(\mathbb{T}^2)) &\cong K_1(SC(\mathbb{T}^2)) \cong K_2(C(\mathbb{T}^2))\,. \end{aligned}$$

The Bott generator in $K_0(\mathbb{C})$ is given by the (1×1)-matrix 1. In $K_1(C_0(I))$, it maps to $u(y) = \mathrm{e}^{2\pi i y}$. In $K_0(C_0(I^2))$, $u(y)$ maps to $R_x(y) \cdot P \cdot R_x(y)^* - P$, where $P = P_1 = \mathrm{Diag}(1, 0)$ and

$$\begin{aligned} R_x(y) &:= \begin{pmatrix} \mathrm{e}^{2\pi i y} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} C(x) & S(x) \\ -S(x) & C(x) \end{pmatrix} \cdot \begin{pmatrix} e^{-2\pi i y} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} C(x) & -S(x) \\ S(x) & C(x) \end{pmatrix} \\ &= \begin{pmatrix} C^2 + e^{2\pi i y}S^2 & (\mathrm{e}^{2\pi i y} - 1)CS \\ (1 - \mathrm{e}^{-2\pi i y})CS & C^2 + \mathrm{e}^{-2\pi i y}S^2 \end{pmatrix} \end{aligned}$$

Finally, the Bott generator of $K_0(\mathbb{C})$ maps, in the group $K_1(SC(\mathbb{T}^2))$, to the unitary

$$\begin{aligned} G(t, x, y) &:= \left(\mathrm{e}^{2\pi i t} \cdot Q_x(y) + \mathbb{1}_2 - Q_x(y)\right) \cdot \left(\mathrm{e}^{-2\pi i t} \cdot P + \mathbb{1}_2 - P\right) \\ &= \left(\mathrm{e}^{2\pi i t} \cdot Q_x(y) + \mathbb{1} - Q_x(y)\right) \cdot \begin{pmatrix} e^{-2\pi i t} & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

where $Q_x(y) := R_x(y) \cdot P \cdot R_x(y)^*$ and $t, x, y \in [0, 1[$. This gives a description of the Bott generator $\hat{\delta}$ as a differentiable map

$$f : S^1 \times \mathbb{T}^2 \longrightarrow SU(2), \ \left(e^{2\pi it}, e^{2\pi ix}, e^{2\pi iy}\right) \longmapsto G(t, x, y).$$

We would now like to compute the degree of this map (this will characterize completely the Bott generator among such differentiable maps). We give $S^1 \times \mathbb{T}^2$ the orientation determined by the canonical orientation of $[0, 1[^3$ and the parameterization $(t, x, y) \longmapsto \left(e^{2\pi it}, e^{2\pi ix}, e^{2\pi iy}\right)$. On the other hand, we chose the following "standard" oriented basis of $SU(2)$ at $\mathbb{I}_2$:

$$\left(X_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, X_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, X_3 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right)$$

(these matrices correspond to the quaternionic units i, j and k respectively). The associated basis at $-\mathbb{I}_2$ is $(-X_1, -X_2, -X_3)$.

8.4.1 Lemma. *The equation $f(z, u, v) = -\mathbb{I}_2$ has a unique solution in the domain $S^1 \times \mathbb{T}^2$ given by $(-1, -1, -1)$. At this point, f is regular and orientation-reversing. Consequently, f is of degree -1.*

Proof. One has $G(t, x, y) = -\mathbb{I}_2$ if and only if

$$e^{2\pi it}Q_x(y) + (\mathbb{I}_2 - Q_x(y)) = -\left(e^{2\pi it}P + \mathbb{I}_2 - P\right).$$

The eigenvalues of the left-hand side are $e^{2\pi it}$ and 1, associated to eigenvectors in the image of $Q_x(y)$ and of $\mathbb{I}_2 - Q_x(y)$, respectively. The eigenvalues of the right-hand side are $-e^{2\pi it}$ and -1, associated to eigenvectors in the image of P and of $\mathbb{I}_2 - P$, respectively. We must therefore have $e^{2\pi it} = -1$, i.e. $t = 1/2$, and the corresponding eigenvectors

$$R_x(y) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} C^2 + e^{2\pi iy}S^2 \\ (1 - e^{-2\pi iy})CS \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

must be colinear. We get $e^{2\pi iy} = -1$, i.e. $y = 1/2$, and $x = 1/2$ as unique solution. Finally, for the jacobian matrix, we compute

$$\begin{aligned}
\frac{\partial G}{\partial t}(1/2, 1/2, 1/2) &= 2\pi \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\
\frac{\partial G}{\partial x}(1/2, 1/2, 1/2) &= 2\pi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
\frac{\partial G}{\partial y}(1/2, 1/2, 1/2) &= 2\pi \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}
\end{aligned}$$

In the basis $(-X_1, -X_2, -X_3)$ of the tangent space at $-\mathbb{I}_2$, the jacobian matrix is

$$2\pi \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It is invertible and of negative determinant, as was to be shown. □

8.5 Proof of theorem 8.1.1

In this section, we prove theorem 8.1.1. By commutativity of the first diagram in the introduction, this amounts to showing the equality

$$\mathscr{B}^{-1} \circ \phi_i^{C^*\Gamma} \circ \bar{\iota}_* \circ \beta_{alg}^{(i),loc} = \tilde{\beta}_a^{(i),loc},$$

where $\bar{\iota}$ is the inclusion of $\mathbb{Z}\Gamma$ in $C^*\Gamma$ (and similarly for $\ell^1\Gamma$). In degree $i = 0$ and 1, after we have described the map ϕ_i^A, the result will follow readily from the definitions of $\tilde{\beta}_a^{(i),loc}$ and of $\beta_{alg}^{(i),loc}$. We will therefore focus on the case of degree $i = 2$. We will first explain why one can reduce the proof to the case of the group $\Gamma = \mathbb{Z}^2$. We will then describe explicitly the map ϕ_2^A. Next, we will state and prove a few technical results (dealing with the case of $\mathbb{Z}^2$, and proving that the diagram in the statement of the theorem is well-defined), and we will finally arrive to the main proof.

By definition of the maps $\tilde{\beta}_a^{(2),loc}$ and of $\beta_{alg}^{(2),loc}$, we can first reduce the proof to the case of the surface-groups $\Gamma_g = \pi_1(\Sigma_g)$, with $g \geq 1$. Now, consider the group homomorphism

$$\varphi : \Gamma_g \longrightarrow \mathbb{Z}^2, \; a_1 \mapsto a, \; b_1 \mapsto b, \; a_2, b_2, \ldots, a_g, b_g \mapsto 1,$$

where $a_1, b_1, \ldots, a_g, b_g$ (resp, a and b) are the prescribed generators of Γ_g (resp. $\mathbb{Z}^2$, denoted multiplicatively). This map induces injections (and even isomorphisms)

$$\varphi_* : H_2(\Gamma_g; \mathbb{Z}) \hookrightarrow H_2(\mathbb{Z}^2; \mathbb{Z})$$

(see [72], proof of lemma 2.2.4),

$$\varphi_* : K_0(\ell^1\Gamma_g) \hookrightarrow K_0(\ell^1\mathbb{Z}^2) \text{ and } \varphi_* : K_0(C^*\Gamma_g) \hookrightarrow K_0(C^*\mathbb{Z}^2).$$

To see the latter, note that by [13], the map $\tilde{\beta}_a^{(2),loc}$ is a natural isomorphism between $H_2(\Gamma; \mathbb{Z})$ and $\widetilde{K}_0(C^*\Gamma)$, for one-relator groups, as for example Γ_g and $\mathbb{Z}^2 = \Gamma_1$. This means that injectivity of the former map implies injectivity of the latter. Since the surface-groups have the Haagerup property, they satisfy the Baum-Connes and the Bost conjecture, and they are K-amenable (see [13]). In particular, the inclusion $\ell^1\Gamma_g \hookrightarrow C^*\Gamma_g$ induces a natural isomorphism $K_*(\ell^1\Gamma_g) \cong K_*(C^*\Gamma_g)$. This proves injectivity of the middle map.

As a consequence, one can reduce the proof to the case of the single group $\mathbb{Z}^2$ (since "everything is detected through the homomorphism φ").

Let A be a Banach algebra with unit. Let us now describe explicitly the canonical map

$$\phi_i^A : K_i^{alg}(A) \longrightarrow K_i(A),$$

where as before, $K_*(A)$ is the usual 2-periodic topological K-theory of A. We need an explicit description in degree 0, 1 and 2.

In degree 0, ϕ_0^A is simply the identity, and in degree 1 it is the map

$$K_1^{alg}(A) = \mathrm{GL}(A)^{ab} \longrightarrow K_1(A) = \mathrm{GL}(A)/\,\mathrm{GL}(A)_0\,,\ x^{ab} \longmapsto [x]$$

where x^{ab} is the class of the matrix $x \in \mathrm{GL}(A)$, and $\mathrm{GL}(A)_0$ is the identity component of the topological group $\mathrm{GL}(A)$.

We pass to degree 2. Let $\mathrm{St}(A)$ be the infinite Steinberg group of A, and $\widetilde{\mathrm{GL}}(A)$ be the universal covering space of the topological group $\mathrm{GL}(A)$ of "infinite matrices" over A. As usual, we see the group $\widetilde{\mathrm{GL}}(A)$ as the set of homotopy classes (rel. to $\{0, 1\}$) of paths in $\mathrm{GL}(A)$ (parameterized by $\tau \in [0, 1]$) emanating from $\mathbb{1}$, with pointwise multiplication, and the projection $\widetilde{\mathrm{GL}}(A) \longrightarrow \mathrm{GL}(A)$ is given by evaluation at $\tau = 1$, and has its kernel equal to $\pi_1(\mathrm{GL}(A)) = K_2(A)$. Consider the map $\mathrm{St}(A) \longrightarrow \widetilde{\mathrm{GL}}(A)$ defined on the canonical generators of $\mathrm{St}(A)$ by

$$\psi : x_{ij}(a) \longmapsto [t \mapsto e_{ij}(t \cdot a)],$$

where $a \in A$, t ranges over $[0, 1]$, and the above brackets designate a homotopy class. One can easily check that the image of the $x_{ij}(a)$'s satisfy all the defining relations of $\mathrm{St}(A)$, consequently, the map ψ is a well-defined homomorphism. Now, the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_2^{alg}(A) & \longrightarrow & \mathrm{St}(A) & \longrightarrow & E(A) & \longrightarrow & 0 \\
 & & {\scriptstyle \phi_2^A}\downarrow & & {\scriptstyle \psi}\downarrow & & \downarrow & & \\
0 & \longrightarrow & K_2(A) & \longrightarrow & \widetilde{\mathrm{GL}}(A) & \longrightarrow & \mathrm{GL}(A) & \longrightarrow & 0
\end{array}$$

commutes. Therefore, by restriction, ψ induces the canonical homomorphism ϕ_2^A we are looking for; explicitly

$$\begin{gathered}
\phi_2^A : K_2^{alg}(A) \longrightarrow K_2(A) = \pi_1(\mathrm{GL}(A)) \\
\textstyle\prod_s x_{i_s j_s}(a_s) \longmapsto \left[e^{2\pi i t} \mapsto \prod_s x_{i_s j_s}(t \cdot a_s)\right]
\end{gathered}$$

Let us now describe explicitly the image of the Steinberg symbol $\{u, v\} \in K_2^{alg}(A)$, of two commuting invertibles $u, v \in \mathrm{GL}_1(A)$, in $K_2(A) = \pi_1(\mathrm{GL}(A))$ under ϕ_2^A.

For $u \in \mathrm{GL}_1(A)$, the image via ψ in $\widetilde{\mathrm{GL}}(A)$ of the element

$$x_{12}(u)x_{21}(-u^{-1})x_{12}(u)x_{12}(1)x_{21}(-1)x_{12}(1) \in \mathrm{St}(A),$$

which occurs in the definition of a Steinberg symbol $\{u, .\}$, is given by the path (in $\mathrm{GL}_2(A) \subset \mathrm{GL}(A)$)

$$\begin{pmatrix} (1-t^2)^2 + u(2-t^2)t^2 & (u-1)t(2-t^2)(1-t^2) \\ (1-u^{-1})(1-t^2)t & u^{-1}t^2(2-t^2) + (1-t^2)^2 \end{pmatrix}$$

For any $t \in [0, 1]$, the path of matrices

$$\begin{pmatrix} (1-t^2)^2 + u(2-t^2)t^2 & (u-1)t(2-t^2)^{1-s/2}(1-t^2) \\ (1-u^{-1})(1-t^2)t(2-t^2)^{s/2} & u^{-1}t^2(2-t^2) + (1-t^2)^2 \end{pmatrix}$$

(parameterized by $s \in [0, 1]$) realizes a homotopy in $\mathrm{GL}_2(A)$ between the above matrix and the matrix

$$\begin{pmatrix} (1-t^2)^2 + u(2-t^2)t^2 & (u-1)t(2-t^2)^{1/2}(1-t^2) \\ (1-u^{-1})(1-t^2)t(2-t^2)^{1/2} & u^{-1}t^2(2-t^2) + (1-t^2)^2 \end{pmatrix}$$

that is unitary if A is a C^*-algebra and if u is unitary. Now, by performing a "linear interpolation" between the function $\cos(\pi t/2)$ (resp. $\sin(\pi t/2)$) and $1-t^2$ (resp. $t(2-t^2)^{1/2}$), we can replace the former functions by the latter without changing the homotopy class, and we see that the above path of matrices is homotopic to the path given by

$$R(t, u) := \begin{pmatrix} \cos^2(\pi t/2) + u\sin^2(\pi t/2) & (u-1)\cos(\pi t/2)\sin(\pi t/2) \\ (1-u^{-1})\cos(\pi t/2)\sin(\pi t/2) & \cos^2(\pi t/2) + u^{-1}\sin^2(\pi t/2) \end{pmatrix}$$

It is unitary if A is a C^*-algebra and if u is unitary. In order to simplify the formulas involved, we write $C = C(t) = \cos(\pi t/2)$ and $S = S(t) = \sin(\pi t/2)$. Note that one has

$$R(t, u) = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} C(t) & S(t) \\ -S(t) & C(t) \end{pmatrix} \cdot \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} C(t) & -S(t) \\ S(t) & C(t) \end{pmatrix}$$

Consequently, if u and v are commuting elements in $\mathrm{GL}_1(A)$, the image under ϕ_2^A of their Steinberg symbol $\{u, v\}$ is given by

$$t \longmapsto F(t, u, v) := R_{12}(t, u) \cdot R_{13}(t, v) \cdot R_{12}(t, u)^{-1} \cdot R_{13}(t, v)^{-1} \in \mathrm{GL}_3(A),$$

where

$$R_{12}(t, u) := \begin{pmatrix} R(u, t) & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} \in \mathrm{GL}_3(A)$$

and $R_{13}(t, v)$ is obtained from $R_{12}(t, v)$ by switching the second and third coordinates. If A is a C^*-algebra and if u and v are commuting unitaries, then $F(t, u, v)$ is in $U_3(A)$ (and even in $SU_3(A)$ if A is commutative).

Let us now apply this to the case where $u = \gamma$ and $v = \gamma'$ are two commuting elements of a group Γ, and are considered as invertibles in $A = \ell^1\Gamma$ or as unitaries in $A = C^*\Gamma$ or $C^*_r\Gamma$. Consider the morphisms of Banach algebras

$$\hat{f}_{\gamma,\gamma'} : \ell^1\mathbb{Z}^2 \longrightarrow \ell^1\Gamma, \quad \tilde{f}_{\gamma,\gamma'} : C^*\mathbb{Z}^2 \longrightarrow C^*\Gamma \text{ and } f_{\gamma,\gamma'} : C^*_r\mathbb{Z}^2 \longrightarrow C^*_r\Gamma$$

all defined by $a \mapsto \gamma$ and $b \mapsto \gamma'$, where a and b are the prescribed generators of $\mathbb{Z}^2$.

Recall that in section 8.4, we have explicitly determined the Bott generator $\hat{\delta}$ of $K_2(C(\mathbb{T}^2)) \cong K_2(C^*\mathbb{Z}^2)$. By example iv) in section 4.5, the inclusion of $\ell^1\mathbb{Z}^n$ in $C^*\mathbb{Z}^n = C^*_r\mathbb{Z}^n$ induces an isomorphism $K_*(\ell^1\mathbb{Z}^n) \cong K_*(C^*_r\mathbb{Z}^n)$. We also denote by $\hat{\delta}$ the corresponding image in $K_2(\ell^1\mathbb{Z}^2)$ of the Bott generator.

8.5.1 Proposition. *i) Let a and b be the prescribed generators of the group $\mathbb{Z}^2$. Then the image of the Steinberg symbol $\{a, b\}$ by the homomorphism $K_2^{alg}(\mathbb{Z}[\mathbb{Z}^2]) \longrightarrow K_2(C^*\mathbb{Z}^2)$ is $\hat{\delta}$. Similarly, the image of $\{a, b\}$ in $K_2(\ell^1\mathbb{Z}^2)$ is also $\hat{\delta}$.*

ii) Let γ and γ' be commuting elements in the group Γ. The image of the Steinberg symbol $\{\gamma, \gamma'\}$ under the map $K_2^{alg}(\mathbb{Z}\Gamma) \longrightarrow K_2(C^*_r\Gamma)$ is $(f_{\gamma,\gamma'})_*(\hat{\delta})$. Similarly, the image of $\{\gamma, \gamma'\}$ in $K_2(\ell^1\Gamma)$ and in $K_2(C^*\Gamma)$ is $(\hat{f}_{\gamma,\gamma'})_*(\hat{\delta})$ and $(\tilde{f}_{\gamma,\gamma'})_*(\hat{\delta})$ respectively.

Part i) of this proposition is stated ("up to sign" and without proof) by Milnor in his book [75] (see p. 66 therein).

Proof. From now on, we identify $C^*\mathbb{Z}^2$ with $C(\mathbb{T}^2)$, the continuous functions on the torus $\mathbb{T}^2 = S^1 \times S^1$ (via the usual Fourier transform). We denote a point in the torus by (u, v). Therefore, the generators a and b of the group $\mathbb{Z}^2$ correspond to the unitary functions $a(u, v) = u$ and $b(u, v) = v$, that we simply denote by u and v respectively. The image of $\{a, b\}$ in $K_2(C^*\mathbb{Z}^2) \cong \pi_1(U(C(\mathbb{T}^2)))$ is given by the (pointed) homotopy class of the map

$$t \longmapsto F(t, u, v) = R_{12}(t, u) \cdot R_{13}(t, v) \cdot R_{12}(t, u)^{-1} \cdot R_{13}(t, v)^{-1}.$$

One can easily check that $R_{12}(t, u)^{-1} \cdot R_{13}(t, v)^{-1} - R_{13}(t, v)^{-1} \cdot R_{12}(t, u)^{-1}$, which is equal to

$$\begin{pmatrix} 0 & (u-1)(1-\bar{v})CS^3 & (\bar{u}-1)(v-1)CS^3 \\ (\bar{u}-1)(1-\bar{v})CS^3 & 0 & (\bar{u}-1)(1-v)C^2S^2 \\ (\bar{u}-1)(\bar{v}-1)CS^3 & (u-1)(\bar{v}-1)C^2S^2 & 0 \end{pmatrix},$$

is a singular matrix with the normed vector

$$e_1(t, u, v) := \frac{1}{\sqrt{1+S^2}}(uvC, vS, uS) \in \mathbb{C}^3$$

in its kernel. Therefore, $e_3(t, u, v)$ is a normed eigenvector of the matrix $F(t, u, v)$ associated to the eigenvalue 1. If we moreover set

$$\begin{array}{rcl} e_2(t, u, v) & := & (-\bar{v}S, \bar{u}\bar{v}C, 0) \\ e_3(t, u, v) & := & \frac{1}{\sqrt{1+S^2}}(-vCS, -\bar{u}vS^2, 1), \end{array}$$

we get an orthonormal basis (e_1, e_2, e_3) of $\mathbb{C}^3$ (for each $(t, u, v) \in [0, 1] \times \mathbb{T}^2$). Let $P(t, u, v)$ be the matrix of $S\acute{U}(3)$ with columns $e_1(t, u, v)$, $e_2(t, u, v)$ and $e_3(t, u, v)$. Then the map

$$(s, t) \longmapsto P(st, u, v) \cdot P(t, u, v)^* \cdot F(t, u, v) \cdot P(t, u, v) \cdot P(st, u, v)^*$$

gives a (pointed) homotopy between $t \mapsto F(t, u, v)$ and a map of the type

$$t \longmapsto \left(\begin{array}{c|c} 1 & 0 \quad 0 \\ \hline 0 & \\ 0 & F_0(t, u, v) \end{array}\right)$$

where $F_0(t, u, v)$ lies in $SU(2)$, and satisfies

$$F_0(0, u, v) = F_0(1, u, v) = F_0(t, 1, v) = F_0(t, u, 1) = \mathbb{1}_2 .$$

By means of the identification $[0, 1] \times \mathbb{T}^2/\{0, 1\} \times \mathbb{T}^2 = S^1 \times \mathbb{T}^2$, we get, thanks to the above relations, a continuous map

$$f_0 : S^1 \times \mathbb{T}^2 \longrightarrow SU(2), \ \left(e^{2\pi it}, u, v\right) \longmapsto F_0(t, u, v),$$

that is differentiable outside of the subspace $(S^1 \times S^1 \times \{1\}) \cup (S^1 \times \{1\} \times S^1)$. So, to check that f_0 represents the Bott generator $\hat{\delta}$, we just have to compare the degree of f_0 with the degree of the map f of section 8.4 representing $\hat{\delta}$. We chose the orientation for $S^1 \times \mathbb{T}^2$ and for $SU(2)$ as in lemma 8.4.1.

8.5.2 Lemma. *The inverse image in $S^1 \times \mathbb{T}^2$ of $-\mathbb{1}_2$ by f_0 is*

$$\left\{\left(\frac{1}{\pi}\arccos(1-\sqrt{2}), -1, -1\right)\right\}$$

and f_0 is regular and orientation reversing at that point. Consequently, f_0 is of degree -1, and $\{a, b\}$ maps in $K_2(C(\mathbb{T}^2))$ to the Bott generator.

Proof. One has $F_0(t, u, v) = -\mathbb{1}_2$ if and only if -1 is an eigenvalue of $F(t, u, v)$ of multiplicity 2, and so if and only if

$$M(t, u, v) := R_{12}(t, u)^{-1} \cdot R_{13}(t, v)^{-1} + R_{13}(t, v)^{-1} \cdot R_{12}(t, u)^{-1}$$

is of rank 1. One easily computes that $M(t, u, v)$ equals

$$\begin{pmatrix} 2(C^2+\bar{u}S^2)(C^2+\bar{v}S^2) & (u-1)CS(1+C^2+\bar{v}S^2) & (v-1)CS(1+C^2+\bar{u}S^2) \\ (1-\bar{u})CS(1+C^2+\bar{v}S^2) & 2(C^2+uS^2) & (\bar{u}-1)(1-v)C^2S^2 \\ (1-\bar{v})CS(1+C^2+\bar{u}S^2) & (u-1)(1-\bar{v})C^2S^2 & 2(C^2+vS^2) \end{pmatrix}$$

The minor given by the (2×2)-determinant of the first and third components of the first and second columns is equal to

$$\bar{v}(\bar{u}-1)(\bar{v}-1)C^2S^2(1-u-v-uv-2(u-1)(v-1)C^2+(u-1)(v-1)C^4).$$

We have $u \neq 1$, $v \neq 1$, $t \neq 0$ and $t \neq 1$ (otherwise F_0 is the identity). Therefore, if M is of rank 1, one has

$$1-u-v-uv-2(u-1)(v-1)Y+(u-1)(v-1)Y^2=0,$$

where $Y := C^2 \geq 0$. The discriminant of this equation is

$$\Delta := 8uv(u-1)(v-1) = 8(u_0^2-1/4)(v_0^2-1/4),$$

where $u_0 := u-1/2$ and $v_0 := v-1/2$. Clearly, Δ is ≥ 0 if and only if u_0^2 and v_0^2 are conjugate, which amounts to u and v being conjugate (the case $u_0 = -\bar{v}_0$ being excluded by the fact that $u, v \in S^1$). Therefore, this equation has a real solution if and only if u and v are conjugate, and in this case, if $u = e^{2\pi i x}$, with $x \in [0, 1[$, one finds

$$C^2 = \cos^2(\pi t/2) = 1 - \frac{1}{\sqrt{2}\sin(\pi x)}.$$

Let $t_0 = t_0(x) \in [0, 1[$ be the unique solution (for a given x) of this equation in the variable t. The (2, 3)-minor of the matrix $M(t, u, \bar{u})$ is

$$M_{2,3} := 4(C^2+uS^2)(C^2+\bar{u}S^2)-(u-1)^2(\bar{u}-1)^2C^4S^4.$$

By evaluating at $t = t_0(x)$, we can make the substitutions $C^2 = 1 - \frac{1}{\sqrt{2}\sin(\pi x)}$ and $S^2 = 1 - C^2$, and we get $M_{2,3} = 8\cos^2(\pi x)$. We see that if M has rank one, then

$$x = y = \frac{1}{2} \text{ and } \cos^2(\pi t_0/2) = 1 - \frac{1}{\sqrt{2}},$$

where we parameterize v as $v = e^{2\pi i y}$, with $y \in [0, 1[$. This means that $u = v = -1$ and

$$t = t_0(1/2) = \frac{1}{\pi}\arccos(1-\sqrt{2}) \in [0, 1[.$$

For these values, and these values only, $F = -\mathbb{I}_3$. Now we have to compute the jacobian matrix of f_0 at that point. With the help of Mathematica, we have found

$$\begin{aligned} \frac{\partial f_0}{\partial t}(t_0, 1/2, 1/2) &= 2\pi\sqrt{\tfrac{4(4+3\sqrt{2})}{10+7\sqrt{2}}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \frac{\partial f_0}{\partial x}(t_0, 1/2, 1/2) &= 2\pi\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\ \frac{\partial f_0}{\partial y}(t_0, 1/2, 1/2) &= 2\pi\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \end{aligned}$$

In the oriented basis $(-X_1, -X_2, -X_3)$ of $SU(2)$ at $-\mathbb{1}_2$ (see the proof of lemma 8.4.1), the jacobian matrix is given by

$$2\pi \begin{pmatrix} 0 & 1 & 0 \\ -\sqrt{\frac{4(4+3\sqrt{2})}{10+7\sqrt{2}}} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

It is obviously invertible with negative determinant, as was the map representing $\hat{\delta}$ in section 8.4. This concludes the proof of lemma 8.5.2. □

Evidently, lemma 8.5.2 proves part i) of proposition 8.5.1 (the statement for $\ell^1\mathbb{Z}^2$ now being clear). Part ii) is a direct consequence of i) and of the naturality of the map $K_2^{alg} \longrightarrow K_2$, applied to $\hat{f}_{\gamma,\gamma'}$ ($\ell^1(-)$ is a functor).

This completes the proof of proposition 8.5.1. □

The following lemma shows that the diagram involved in the statement of theorem 8.1.1 is well-defined.

8.5.3 Lemma. *Let γ be an element of a group Γ. The image of the Steinberg symbol $\{\gamma, \gamma\}$ in $K_2(C_r^*\Gamma)$ is trivial. In particular, the map $K_2^{alg}(\mathbb{Z}\Gamma) \longrightarrow K_2(C_r^*\Gamma)$ takes the subgroup Δ_2 to zero. The same holds for $\ell^1\Gamma$ and $C^*\Gamma$ in place of $C_r^*\Gamma$.*

Proof. There is a C^*-algebra morphism $f : C^*\mathbb{Z} \longrightarrow C^*\Gamma$ mapping the prescribed generator a of the group $\mathbb{Z}$ to γ. Therefore, by naturality of the Steinberg symbol, it suffices to check that $\{a, a\}$ is zero in $K_2(C^*\mathbb{Z})$. But the inclusion of a base-point in the circle $S^1 = B\mathbb{Z}$ induces isomorphisms $K_2(C^*\mathbb{Z}) \cong K_2(C(S^1)) \cong K_2(\mathbb{C}) \cong \mathbb{Z}$. Now it follows readily from the basic properties of the Steinberg symbols that $\{a, a\}$ is of order at most 2; it is therefore trivial. (Another way to see this is to notice that $F(t, 1, 1) = \mathbb{1}_3$ for any t (see the proof of proposition 8.5.1).) The proof for $C_r^*\Gamma$ is exactly the same, because $C_r^*\mathbb{Z} = C^*\mathbb{Z}$. For $\ell^1\Gamma$, we have seen that the Wiener lemma and the density theorem imply that $K_2(\ell^1\mathbb{Z}) \cong K_2(C_r^*\mathbb{Z})$. We can therefore conclude as before. □

We are in position give the proof of theorem 8.1.1.

Proof of Theorem 8.1.1. We first prove the theorem for the maximal C^*-algebra. As already mentioned, in degree $i = 0$, the result is clear. In degree $i = 1$, by naturality, we can restrict to the group $\mathbb{Z}$. From example iii) in section 4.5, this is clear.

In degree $i = 2$, since the diagram of the statement is well-defined (lemma 8.5.3), one can reduce the proof to the case of the single group $\mathbb{Z}^2$ (as is explained at the beginning of the present section). By definition, $\tilde{\beta}_a^{(2),loc}$ maps the orientation class $[\mathbb{T}^2] \in H_2(\mathbb{Z}^2; \mathbb{Z})$ to $\tilde{\beta}_0^{\mathbb{Z}^2}([\mathbb{T}^2]_K)$ in $K_0(C^*\mathbb{Z}^2) \cong K_0(C(\mathbb{T}^2))$, where $[\mathbb{T}^2]_K \in K_0(\mathbb{T}^2)$ is the canonical generator, and where $\tilde{\beta}_0^{\mathbb{Z}^2}$ is the Novikov assembly map for the group $\mathbb{Z}^2$. As is well-known, $\tilde{\beta}_0^{\mathbb{Z}^2}$ is an isomorphism and it takes $[\mathbb{T}^2]_K$ to the Bott generator $\hat{\delta}$ (see example iii) in section 4.5,). On the other hand, also by definition, $\beta_{alg}^{(2),loc}$ maps $[\mathbb{T}^2]$ to $\bar{z}_1$, the class of $z_1 = \{a, b\}$ in $K_2^{alg}(\mathbb{Z}[\mathbb{Z}^2])/\Delta_2$. So, the result follows from proposition 8.5.1 i).

For the algebras $C_r^*\Gamma$ and $\ell^1\Gamma$, the result follows readily. □

8.6 Proofs of theorems 8.1.2, 8.1.3 and 8.1.4

We first prove theorem 8.1.2 as an easy consequence of theorem 8.1.1 and of the delocalization. Finally, theorems 8.1.3 and 8.1.4 are established by directly applying theorems 8.1.1 and 8.1.2, and some results of chapters 3 and 5.

Proof of theorem 8.1.2. The proof that $H_i(\Gamma;\, F\Gamma) \longrightarrow K_i^{alg}(\mathbb{Q}_0\Gamma)\otimes\mathbb{C}$ is natural, for $0 \le i \le 2$, is the same as the proof made in chapter 5 for the map

$$\hat{\beta}_a^{(i)} : H_i(\Gamma;\, F\Gamma) \longrightarrow K_i(\ell^1\Gamma).$$

Let us just mention that the key ingredient is lemma 5.3.5.

Since each element in Δ_2 is a product of elements of order at most 2 (because $\{\gamma, \gamma\}^2 = 1$ in $K_2^{alg}(A)$), the map $K_i^{alg}(A) \otimes \mathbb{C} \longrightarrow (K_i^{alg}(A)/\Delta_2) \otimes \mathbb{C}$ is an isomorphism. This shows that the diagram in the statement is well-defined. As for the localized case (i.e. the proof of theorem 8.1.1), we just have to prove the equality

$$((\mathscr{B}^{-1} \circ \phi_i^{C^*\Gamma} \circ \tilde{\iota}_*) \otimes Id) \circ \beta_{alg}^{(2)} = \tilde{\beta}_a^{(i)}$$

(and similarly for $\ell^1\Gamma$). Since the delocalization process is the same for the construction of $\beta_{alg}^{(i)}$ and of $\tilde{\beta}_a^{(i)}$, the result follows from theorem 8.1.1. □

We pass to theorem 8.1.3.

Proof of theorem 8.1.3. The result follows from theorem 8.1.1, from the fact that $\beta_t^{(i),loc} = \beta_i^{B\Gamma}$ is always rationally injective (see proposition 5.2.3), and from the injectivity of $\beta_1^{\Gamma} \circ \beta_t^{(1),loc} = \beta_a^{(1),loc}$ (see theorem 5.1.1). Injectivity in cases i) and ii) follows from theorem 3.7.1 and proposition 3.7.3. □

Finally, here comes the proof of theorem 8.1.4.

Proof of theorem 8.1.4. If $\mu_*^{\Gamma} \otimes Id$ is injective, since $\beta_t^{(i)}$ is rationally injective for $i = 0$, 1 and 2 (see chapter 5), it follows from theorem 8.1.2 that $(f_* \otimes Id) \circ \beta_{alg}^{(i)}$ is also injective. □

Chapter 9

Comparison of the product structures in algebraic and topological K-theory

We answer by the positive to an open question of Milnor on the compatibility, in low degree, of the product structures in algebraic and in topological K-theory of unital Banach algebras.

9.1 Statement of the theorem and definition of the product structures in K-theory

As an application of the computations made in section 8.5 for the proof of theorem 8.1.1, we prove the following result.

9.1.1 Theorem. *Let A and B be two unital Banach algebras. Then the diagram*

$$\begin{array}{ccc} K_i^{alg}(A) \otimes K_j^{alg}(B) & \xrightarrow{\star} & K_{i+j}^{alg}(A \otimes B) \\ \phi_i \otimes \phi_j \downarrow & & \downarrow \hat{\phi}_{i+j} \\ K_i(A) \otimes K_j(B) & \xrightarrow{\cup} & K_{i+j}(A \hat{\otimes} B) \end{array}$$

commutes for $i,\ j \geq 0$ satisfying $i + j \leq 2$. In other words, the product structures in algebraic and in topological K-theory of unital Banach algebras are compatible in total degree ≤ 2.

Let us explain the notations. For a unital Banach algebras A, the algebraic and topological K-theories are defined by

$$K_i^{alg}(A) := \pi_i(B\mathrm{GL}^\delta(A)^+) \text{ and } K_i(A) := \pi_{i-1}(\mathrm{GL}(A)) \cong \pi_i(B\mathrm{GL}(A)),$$

where $\mathrm{GL}^\delta(A)$ stands for the group $\mathrm{GL}(A)$ made discrete, and $i \geq 1$. (The definition of algebraic K-theory makes sense for any unital ring.) The canonical map

$$B(Id) : B\mathrm{GL}^\delta(A) \longrightarrow B\mathrm{GL}(A)$$

induces at the level of fundamental groups a map taking $E(A) \subseteq \mathrm{GL}^\delta(A)$ to zero, since $\pi_1(B\mathrm{GL}(A)) = \pi_0(\mathrm{GL}(A)) = \mathrm{GL}(A)/\mathrm{GL}(A)_0$, where $\mathrm{GL}(A)_0$ is the component of the identity, and since $E(R) \subseteq \mathrm{GL}(A)_0$ (see prop. 3.4.1 in [16]). Consequently, $B(Id)$ induces a map $B(Id)^+ : B\mathrm{GL}^\delta(A)^+ \longrightarrow B\mathrm{GL}(A)$. This allows for defining a canonical and natural map for any $i \geq 1$, namely

$$\phi_i^A = \phi_i := \pi_i(B(Id)^+) : K_i^{alg}(A) \longrightarrow K_i(A).$$

For $i = 0$, we define ϕ_0^A as the identity of $K_0^{alg}(A) = K_0(A)$. We have explicitly described ϕ_i for $i \leq 2$ in section 8.5.

For two unital rings A and B, the external product in algebraic K-theory (see [71]) is denoted by

$$K_i^{alg}(A) \otimes K_j^{alg}(B) \xrightarrow{\star} K_{i+j}^{alg}(A \otimes_{\mathbf{Z}} B).$$

As noticed by Loday in [71], the product he defines at the level of spectra coincides, in total degree $i + j \leq 2$, with the product defined (case by case) by Milnor *only up to sign.* More precisely, both definitions coincide, except for $i = j = 1$, where Loday's product is minus Milnor's product (see prop. 2.2.3 in [71]). We make the following choice for the sign:

The product $\star$ for $i = j = 1$ is Milnor's product, i.e. the Steinberg symbol.

The internal product is defined for A commutative by composing the external product with the homomorphism $K_{i+j}^{alg}(A \otimes_{\mathbf{Z}} A) \longrightarrow K_{i+j}^{alg}(A)$, induced by the product map $\mu : A \otimes_{\mathbf{Z}} A \longrightarrow A$ (which is an ring homomorphism, precisely because A is commutative).

Let $A\hat{\otimes}B$ be the completed projective tensor product (over $\mathbb{C}$) of two unital Banach algebras A and B. The equality of functors $K_0^{alg} = K_0$ and the suspension isomorphism uniquely define the external cup product

$$K_i(A) \otimes K_j(B) \xrightarrow{\cup} K_{i+j}(A\hat{\otimes}B),$$

in topological K-theory, by requiring commutativity in the diagram

$$\begin{array}{ccccc}
K_i(A) \otimes K_j(B) & & \overset{\cup}{\dashrightarrow} & & K_{i+j}(A\hat{\otimes}B) \\
\cong\downarrow & & & & \downarrow\cong \\
K_0(S^iA) \otimes K_0(S^jB) & \xrightarrow{\star} & K_0(S^iA \otimes_{\mathbf{Z}} S^jB) & \xrightarrow{\iota_*} & K_0(S^{i+j}(A\hat{\otimes}B))
\end{array}$$

where we write $S^0A := A$, $S^iA := S(S^{i-1}A) \cong A\hat{\otimes}C_0(\mathbb{R}^i)$ for $i \geq 1$, and ι is the composition $S^iA \otimes_{\mathbf{Z}} S^jB \longrightarrow S^iA \otimes_{\mathbf{C}} S^jB \hookrightarrow S^iA\hat{\otimes}S^jB \cong S^{i+j}(A\hat{\otimes}B)$. As in the algebraic case, the internal product is defined for A commutative by composing with the homomorphism $K_{i+j}(A\hat{\otimes}A) \longrightarrow K_{i+j}(A)$, induced by the "completed product map" $\hat{\mu} : A\hat{\otimes}A \longrightarrow A$.

Finally, for $i \geq 0$, $\hat{\phi}_i$ denotes the composition

$$K_i^{alg}(A \otimes_{\mathbf{Z}} B) \longrightarrow K_i^{alg}(A \otimes_{\mathbf{C}} B) \longrightarrow K_i^{alg}(A\hat{\otimes}B) \xrightarrow{\phi_i} K_i(A\hat{\otimes}B).$$

(Notice that ι_* in the above diagram is just $\hat{\phi}_0$.)

This makes all the notations used in theorem 9.1.1 meaningful. The proof is subdivided in five parts, one for each of the forthcoming sections.

Before stating an important corollary of theorem 9.1.1, for a compact (Hausdorff) space X, we let

$$\theta_* : K_*(C(X)) \xrightarrow{\cong} K^{-*}(X)$$

be the Swan-Serre isomorphism, where $C(X)$ is the commutative unital C^*-algebra of continuous complex valued functions on X, with the norm of uniform convergence.

9.1.2 Corollary. *For a compact space X, the diagram*

$$\begin{array}{ccc}
K_i^{alg}(C(X)) \otimes K_j^{alg}(C(X)) & \xrightarrow{\star} & K_{i+j}^{alg}(C(X)) \\
{\scriptstyle \phi_i \otimes \phi_j}\downarrow & & \downarrow{\scriptstyle \phi_{i+j}} \\
K_i(C(X)) \otimes K_j(C(X)) & \xrightarrow{\cup} & K_{i+j}(C(X)) \\
{\scriptstyle \theta_i \otimes \theta_j}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \theta_{i+j}} \\
K^{-i}(X) \otimes K^{-j}(X) & \xrightarrow{\cup} & K^{-(i+j)}(X)
\end{array}$$

commutes, for $i, j \geq 0$ satisfying $i + j \leq 2$, where the bottom horizontal map is the usual cup product in K-theory.

Proof. The product $\mu : C(X) \otimes_{\mathbf{Z}} C(X) \longrightarrow A$ yields a commutative diagram

$$\begin{array}{ccc}
K_{i+j}^{alg}(C(X) \otimes_{\mathbf{Z}} C(X)) & \xrightarrow{K_{i+j}^{alg}(\mu)} & K_{i+j}^{alg}(C(X)) \\
{\scriptstyle \hat{\phi}_{i+j}}\downarrow & & \downarrow{\scriptstyle \phi_{i+j}} \\
K_{i+j}(C(X)\hat{\otimes}C(X)) & \xrightarrow{K_{i+j}(\hat{\mu})} & K_{i+j}(C(X))
\end{array}$$

Consequently, commutativity of the upper square follows from theorem 9.1.1. The bottom square commutes, since the Swan-Serre isomorphism is an ring isomorphism. Indeed, the products

$$K_0(C(X)) \otimes K_0(C(X)) \longrightarrow K_0(C(X)) \text{ and } K^0(X) \otimes K^0(X) \longrightarrow K^0(X)$$

are both given by the tensor product (of vector bundles for the former, and of finitely generated projective $C(X)$-modules for the latter). This implies commutativity for $i = j = 0$. The other cases follow readily from compatibility of both products with suspensions (it is well-known that $C(SX) \cong SC(X)$, canonically and naturally, where SX is the unreduced suspension of X). ☐

Corollary 9.1.2 was an open question in Milnor's book [75] (see p. 67).

Many thanks to Hervé Oyono-Oyono for helpful discussions.

This chapter contains only the proof of theorem 9.1.1, and is organized as follows. In section 9.2, we show how one can reduce the proof to the case of (1×1)-matrices over a commutative Banach algebra. We discuss the cases $i+j \leq 1$ (resp. $i = 0$ and $j = 2$) in section 9.3 (resp. 9.5) by writing explicit formulas for the products and by proving theorem 9.1.1 for (1×1)-matrices over a commutative Banach algebra. In section 9.4, we consider the most difficult case, namely $i = j = 1$, by applying results of chapter 8 (dealing with the C^*-algebra $C^*\mathbb{Z}^2 \cong C(\mathbb{T}^2)$). Finally, the proof of theorem 9.1.1 is completed in section 9.6.

9.2 General reductions

In this section, we explain how we can more or less reduce the proof of theorem 9.1.1 to the case of (1×1)-matrices over a commutative Banach algebra.

Let A be a unital ring and $n \geq 1$. We set

$$P(A) := \{x \in M(A) \,|\, x = x^2\} \qquad P_n(A) := \{x \in M_n(A) \,|\, x = x^2\} \hookrightarrow P(A)$$
$$K_{0,n}(A) := \{z \in K_0(A) = \mathcal{G}(P(A)/\sim) \,|\, \exists x,\, y \in P_n(A) \text{ such that } z = [x] - [y]\},$$

where $\sim$ is the usual equivalence relation, and $\mathcal{G}(-)$ stands for the Grothendieck construction (i.e. the group completion). Notice that any z in $K_0(A)$ belongs to $K_{0,n}(A)$ for some $n \geq 1$. Similarly, we define

$$K_{1,n}^{alg}(A) := \operatorname{Im}(\mathrm{GL}_n(A) \longrightarrow K_1^{alg}(A) = \mathrm{GL}(A)/[\mathrm{GL}(A), \mathrm{GL}(A)])$$
$$K_{1,n}(A) := \operatorname{Im}(\mathrm{GL}_n(A) \longrightarrow K_1(A) = \mathrm{GL}(A)/\,\mathrm{GL}(A)_0),$$

where in the latter case, A is assumed to be a unital Banach algebra. In this case, we denote the canonical map $K_{1,n}^{alg}(A) \longrightarrow K_{1,n}(A)$ by $\phi_{1,n}$ (it is the restriction of ϕ_1), and we notice that the suspension isomorphism induces (by restriction) a map

$$\sigma : K_{1,n}(A) \longrightarrow K_{0,2n}(SA)$$

(see section 8.4 for the definition of σ).

Before starting with the first lemma, let us recall that we have (non canonical) isomorphisms

$$M_n(A) \otimes_{\mathbf{Z}} M_m(B) \cong M_{nm}(A \otimes_{\mathbf{Z}} B)$$
$$M_n(A) \hat{\otimes} M_m(B) \cong M_{nm}(A \hat{\otimes} B)$$

that, by Morita invariance of K-theory, induce canonical isomorphisms

$$K_*^{alg}(M_n(A) \otimes_{\mathbf{Z}} M_m(B)) \cong K_*^{alg}(A \otimes_{\mathbf{Z}} B)$$
$$K_*(M_n(A) \hat{\otimes} M_m(B)) \cong K_*(A \hat{\otimes} B).$$

The following lemma says that the product in algebraic K-theory is compatible with Morita invariance. It is a direct consequence of the definition of the product in algebraic K-theory (see Milnor [75], pp. 27, 51 and 67).

9.2.1 Lemma. *Let A and B be two unital rings, and $n, m \geq 1$. Then, for any $i, j \in \{0, 1\}$, the composition*

$$\begin{array}{ccc} K_{i,n}^{alg}(A) \times K_{j,m}^{alg}(B) & \xrightarrow{\cong} & K_{i,1}^{alg}(M_n(A)) \times K_{j,1}^{alg}(M_m(B)) \\ \star \downarrow & & \downarrow \star \\ K_{i+j}^{alg}(A \otimes_{\mathbf{Z}} B) & \xleftarrow{\cong} & K_{i+j}^{alg}(M_n(A) \otimes_{\mathbf{Z}} M_m(B)) \end{array}$$

is the product $\star$ in algebraic K-theory, in other words the above diagram commutes. Similarly, the composition

$$\begin{array}{ccc} K_{0,n}^{alg}(A) \times K_2^{alg}(B) & \xrightarrow{\cong} & K_{0,1}^{alg}(M_n(A)) \times K_2^{alg}(B) \\ \star \downarrow & & \downarrow \star \\ K_2^{alg}(A \otimes_{\mathbf{Z}} B) & \xleftarrow{\cong} & K_2^{alg}(M_n(A) \otimes_{\mathbf{Z}} B) \end{array}$$

is the product $\star$ in algebraic K-theory, in other words the above diagram commutes.

Now, we show that the same holds for the product in topological K-theory.

9.2.2 Lemma. *Let A and B be two unital Banach algebras. Then for any $i, j \geq 0$, the composition*

$$\begin{array}{ccc} K_{i,n}(A) \times K_{j,m}(B) & \xrightarrow{\cong} & K_{i,1}(M_n(A)) \times K_{j,1}(M_m(B)) \\ \cup \downarrow & & \downarrow \cup \\ K_{i+j}(A \hat{\otimes} B) & \xleftarrow{\cong} & K_{i+j}(M_n(A) \hat{\otimes} M_m(B)) \end{array}$$

coincides with the cup product, in other words the above diagram commutes.

Proof. By Bott periodicity (and its obvious compatibility with the product), we can assume that $i,\ j \in \{0,\ 1\}$. Let us deal with the case $i = j = 1$ (the other cases being similar, but even simpler). Consider the diagram

$$\begin{array}{ccccc}
K_{1,n}(A) \times K_{1,m}(B) & \xrightarrow{\cong} & K_{1,1}(M_n(A)) \times K_{1,1}(M_m(B)) & \xrightarrow{\cup} \\
\downarrow{\scriptstyle \sigma\times\sigma} & & \downarrow{\scriptstyle \sigma\times\sigma} & \\
K_{0,2n}(SA) \times K_{0,2m}(SB) & \xrightarrow{\cong} & K_{0,2}(M_n(SA)) \times K_{0,2}(M_m(SB)) & \xrightarrow{\star}
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{\cup} & K_2(M_n(A)\hat{\otimes} M_m(B)) & \xrightarrow{\cong} & K_2(A\hat{\otimes} B) \\
& \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
\xrightarrow{\star} & K_0(M_n(SA)\hat{\otimes} M_m(SB)) & \xrightarrow{\cong} & K_0(SA\hat{\otimes} SB)
\end{array}$$

(We identify $M_n(SA)$ with $SM_n(A)$ in the obvious way.) The first square clearly commutes, so does the second (by definition of the product), and also the third (because the suspension isomorphism is compatible with Morita invariance, as is easily checked). Since the bottom composition is the product on $K_0 \times K_0$ (by lemma 9.2.1), the top composition has to be the product defined on $K_1 \times K_1$, and we are done. □

Both these lemmas illustrate what we mean by "reduction to the case of (1×1)-matrices". The forthcoming lemma explains the further reduction to the commutative case (at least for i and j different from 2).

9.2.3 Lemma. *Let A and B be two unital Banach algebras, and let $0 \le i,\ j \le 1$. Given $x \in K^{alg}_{i,1}(A)$ and $y \in K^{alg}_{j,1}(B)$, let $C := \overline{\langle x\hat{\otimes}1,\ 1\hat{\otimes}y\rangle}$ be the unital Banach sub-algebra of $A\hat{\otimes}B$ generated by $x\hat{\otimes}1$ and $1\hat{\otimes}y$, and let i be the inclusion of C in $A\hat{\otimes}B$. Then C is a commutative unital Banach algebra and the formulas*

$$j_*(x \star y) = i_*((x\hat{\otimes}1) \star (1\hat{\otimes}y)) \in K^{alg}_{i+j}(A\hat{\otimes}B)$$
$$\phi_{i,1}(x) \cup \phi_{j,1}(y) = i_*(\phi_{i,1}(x\hat{\otimes}1) \cup \phi_{j,1}(1\hat{\otimes}y)) \in K_{i+j}(A\hat{\otimes}B).$$

hold, where $j : A \otimes_{\mathbf{Z}} B \longrightarrow A\hat{\otimes}B$ is the canonical map.

Proof. Let $C_x := \overline{\langle x\hat{\otimes}1\rangle}$ and $C_y := \overline{\langle 1\hat{\otimes}y\rangle}$. Consider the maps

$$i_x : C_x \longrightarrow A,\ x\hat{\otimes}1 \longmapsto x \ \text{ and }\ i_y : C_y \longrightarrow B,\ 1\hat{\otimes}y \longmapsto y.$$

One has $C = C_x \hat{\otimes} C_y$ and $i = i_x \hat{\otimes} i_y$. The canonical map $C_x \otimes_{\mathbf{Z}} C_y \longrightarrow C$ is denoted by ι. It is clear that

$$(i_x)_*(x \hat{\otimes} 1) = x \in K_{i,1}^{alg}(A) \text{ and } (i_y)_*(1 \hat{\otimes} y) = y \in K_{j,1}^{alg}(B),$$

and similarly

$$(i_x)_*(\phi_{i,1}(x \hat{\otimes} 1)) = \phi_{i,1}(x) \in K_{i,1}(A) \text{ and } (i_y)_*(\phi_{i,1}(1 \hat{\otimes} y)) = \phi_{j,1}(y) \in K_{j,1}(B).$$

By naturality of the external $\star$-product, we get

$$x \star y = (i_x)_*(x \hat{\otimes} 1) \star (i_y)_*(1 \hat{\otimes} y) = (i_x \otimes i_y)_*((x \hat{\otimes} 1) \star (1 \hat{\otimes} y)) \in K_{i+j}^{alg}(A \otimes_{\mathbf{Z}} B).$$

Applying the map j_* to this equality, the identity $j \circ (i_x \otimes i_y) = i \circ \iota$ and naturality of the $\star$-product yields

$$j_*(x \star y) = i_* \circ \iota_*((x \hat{\otimes} 1) \star (1 \hat{\otimes} y)) = i_*((x \hat{\otimes} 1) \star (1 \hat{\otimes} y)).$$

Similarly, by naturality of the external $\cup$-product, we get

$$\begin{aligned} \phi_{i,1}(x) \cup \phi_{j,1}(y) &= (i_x)_*(\phi_{i,1}(x \hat{\otimes} 1)) \cup (i_y)_*(\phi_{j,1}(1 \hat{\otimes} y)) \\ &= (i_x \hat{\otimes} i_y)_*(\phi_{i,1}(x \hat{\otimes} 1) \cup \phi_{j,1}(1 \hat{\otimes} y)). \end{aligned}$$

This completes the proof. □

9.3 The cases $i + j \leq 1$

In this section, we prove the following lemma.

9.3.1 Lemma. *Let A be a commutative unital Banach algebra, and let $n \geq 1$. Then, for $j = 0$ or 1, the diagram*

$$\begin{array}{ccc} K_{0,1}^{alg}(A) \times K_{j,n}^{alg}(A) & \xrightarrow{\star} & K_j^{alg}(A) \\ {\scriptstyle \phi_{0,1} \times \phi_{j,n}} \downarrow & & \downarrow {\scriptstyle \phi_j} \\ K_{0,1}(A) \times K_{j,n}(A) & \xrightarrow{\cup} & K_j(A) \end{array}$$

commutes.

Before starting the proof, we give explicit formulas for the product $\star$ in the cases considered in the lemma. We assume that A is commutative. For $i = j = 0$, it is clear that

$$K_{0,1}^{alg}(A) \times K_{0,n}^{alg}(A) \xrightarrow{\star} K_0^{alg}(A), \ (x, y) \longmapsto xy.$$

For $i = 0$ and $j = 1$, following the definition given on page 27 of Milnor [75], one easily checks that

$$K^{alg}_{0,1}(A) \times K^{alg}_{1,n}(A) \xrightarrow{\star} K^{alg}_1(A)\,,\ (x,\, y) \longmapsto xy + (1-x)\mathbb{1}_n\,.$$

(The inverse of the $(n \times n)$-matrix $xy + (1-x)\mathbb{1}_n$ is $xy^{-1} + (1-x)\mathbb{1}_n$.)

This generalizes to the case of two unital rings A and B to give

$$\star : K_{0,1}(A) \times K_{1,n}(B) \longrightarrow K_1(A \otimes_{\mathbf{Z}} B)\,,\ (x,\, y) \longmapsto x \otimes y + (1-x) \otimes \mathbb{1}_n\,.$$

Proof of lemma 9.3.1. For $j = 0$, there is nothing to prove. For $j = 1$, on the one hand, for $x \in K^{alg}_{0,1}(A)$ and $y \in K^{alg}_{1,n}(A)$, we have

$$\phi_1(x \star y) = xy + (1-x)\mathbb{1}_n \in K_1(A)\,.$$

By the suspension isomorphism (see section 8.4), this element maps to

$$\left(t \mapsto R_t(xy + (1-x)\mathbb{1}_n) \cdot Q_n \cdot R_t(xy + (1-x)\mathbb{1}_n)^{-1} - Q_n\right)$$

in $K_0(SA)$, where $Q_n := \mathrm{Diag}(\mathbb{1}_n,\, \mathbb{O}_n)$, and for $u \in \mathrm{GL}_n(A)$, we define

$$\begin{aligned} R_t(u) &:= \begin{pmatrix} u & \mathbb{O}_n \\ \mathbb{O}_n & \mathbb{1}_n \end{pmatrix} \cdot \begin{pmatrix} C \cdot \mathbb{1}_n & S \cdot \mathbb{1}_n \\ -S \cdot \mathbb{1}_n & C \cdot \mathbb{1}_n \end{pmatrix} \cdot \begin{pmatrix} u & \mathbb{O}_n \\ \mathbb{O}_n & \mathbb{1}_n \end{pmatrix} \cdot \begin{pmatrix} C \cdot \mathbb{1}_n & -S \cdot \mathbb{1}_n \\ S \cdot \mathbb{1}_n & C \cdot \mathbb{1}_n \end{pmatrix} \\ &= \begin{pmatrix} C^2 \cdot \mathbb{1}_n + S^2 \cdot u & CS \cdot (u - \mathbb{1}_n) \\ CS \cdot (\mathbb{1}_n - u^{-1}) & C^2 \cdot \mathbb{1}_n + S^2 \cdot u^{-1} \end{pmatrix} \in \mathrm{GL}_{2n}(A)\,, \end{aligned}$$

with $C = C(t) := \cos(\pi t/2)$ and $S = S(t) := \sin(\pi t/2)$. On the other hand, by definition of the cup product, the image of $\phi_0(x) \cup \phi_1(y)$ in $K_0(SA)$ under the suspension isomorphism is

$$\left(t \mapsto x \cdot R_t(y) \cdot Q_n \cdot R_t(y)^{-1} - x \cdot Q_n\right)\,.$$

A direct computation shows that both matrices are equal (and not just equivalent). This completes the proof. □

As a consequence of the proof and of the explicit description of the isomorphism $K_1(A) \cong K_0(SA)$ given in section 8.4, we find that for any commutative unital Banach algebra A,

$$\cup : K_{0,1}(A) \times K_{1,n}(A) \longrightarrow K_1(A)\,,\ (x,\, y) \longmapsto xy + (1-x)\mathbb{1}_n\,.$$

This generalizes to the case of two unital Banach algebras A and B to give

$$\cup : K_{0,1}(A) \times K_{1,n}(B) \longrightarrow K_1(A \hat{\otimes} B)\,,\ (x,\, y) \longmapsto x \hat{\otimes} y + (1-x)\hat{\otimes}\mathbb{1}_n\,.$$

9.4 The case $i = j = 1$

In this section, we prove the following lemma, that is the most difficult step towards the proof of theorem 9.1.1 (the difficulty is not conspicuous here, since it is almost completely contained in the lengthy computations of chapter 8).

9.4.1 Lemma. *Let A be a commutative unital Banach algebra. Then the diagram*

$$\begin{array}{ccc} K^{alg}_{1,1}(A) \times K^{alg}_{1,1}(A) & \xrightarrow{\star} & K^{alg}_2(A) \\ {\scriptstyle \phi_{1,1} \times \phi_{1,1}} \downarrow & & \downarrow {\scriptstyle \phi_2} \\ K_{1,1}(A) \times K_{1,1}(A) & \xrightarrow{\cup} & K_2(A) \end{array}$$

commutes.

Proof. The lemma is a consequence of the computations made in section 8.5 for the proof of theorem 8.1.1. Indeed, we have proved this result for the particular Banach algebra $C^*\mathbb{Z}^2 \cong C(\mathbb{T}^2)$ and for the product $a \star b$, where a and b are the prescribed generators of $\mathbb{Z}^2$, viewed as unitaries in $C^*\mathbb{Z}^2$ (since $a \cup b$ is well-known to be the Bott element $\hat{\delta}$ of $K_0(C^*\mathbb{Z}^2) \cong K^0(\mathbb{T}^2)$). Now, we claim that by naturality and by classical results on the K-theory of commutative Banach algebras, the general case follows. To prove this, we first consider the sub-algebra

$$\mathcal{A}_\rho := \left\{ (\lambda_n)_{n \in \mathbf{Z}} \in \mathbb{C}^{\mathbf{Z}} \;\middle|\; \sum_{n \in \mathbf{Z}} \rho^{|n|} \cdot |\lambda_n| < \infty \right\}$$

of $\ell^1\mathbb{Z}$, where $\rho \geq 1$ is a real number. In other words, $\mathcal{A}_\rho$ is the completion of the algebra $\mathbb{C}[\mathbb{Z}]$ for the norm

$$\left\| \sum_{n \in \mathbf{Z}} \lambda_n \cdot a^n \right\|_\rho := \sum_{n \in \mathbf{Z}} \rho^{|n|} \cdot |\lambda_n| ,$$

where a is the prescribed generator of the group $\mathbb{Z}$. Consequently, $\mathcal{A}_\rho$ is a unital Banach algebra for this norm.

The Banach algebra $\mathcal{A}_\rho$ has the following "universal property": Given $x \in \mathrm{GL}_1(A)$, where A is any unital Banach algebra, one has $1 = ||1||_A \leq ||x||_A \cdot ||x^{-1}||_A$, therefore $\rho_x := \max\{||x^{-1}||_A, ||x||_A\}$ is ≥ 1, and the inequalities

$$\left\| \sum_{n \in \mathbf{Z}} \lambda_n \cdot x^n \right\|_A \leq \sum_{n<0} |\lambda_n| \cdot ||x^{-1}||_A^{|n|} + \sum_{n \geq 0} |\lambda_n| \cdot ||x||_A^n \leq \left\| \sum_{n \in \mathbf{Z}} \lambda_n \cdot a^n \right\|_{\rho_x}$$

imply that the algebra map $\nu_x : \mathbb{C}[\mathbb{Z}] \longrightarrow A$, $a \longmapsto x$ extends uniquely to a unital Banach algebra morphism $\bar{\nu}_x : \mathcal{A}_{\rho_x} \longrightarrow A$. Applying this result twice, by the universal property of the projective tensor product of Banach algebras, given $x, y \in \mathrm{GL}_1(A)$, we obtain a unital Banacb algebra morphism

$$\bar{\nu}_{x,y} : \mathcal{A}_{\rho_x}\hat{\otimes}\mathcal{A}_{\rho_y} \longrightarrow A\,, \xi \otimes \eta \longmapsto \bar{\nu}_x(\xi)\cdot\bar{\nu}_y(\eta)\,.$$

It is clear that $\bar{\nu}_{x,y}(a) = x$ and $\bar{\nu}_{x,y}(b) = y$, where a and b designate the prescribed generators of $\mathbb{Z}^2$, considered as elements of $\mathrm{GL}_1(\mathcal{A}_{\rho_x}\hat{\otimes}\mathcal{A}_{\rho_y})$ via

$$\mathbb{Z}[\mathbb{Z}^2] \cong \mathbb{Z}[\mathbb{Z}] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}] \hookrightarrow \mathcal{A}_{\rho_x}\hat{\otimes}\mathcal{A}_{\rho_y}\,.$$

In our context, the second important feature of the algebra $\mathcal{A}_\rho$ is that it is dense in $\ell^1\mathbb{Z}$ and that the inclusions

$$\mathcal{A}_\rho \hookrightarrow \ell^1\mathbb{Z} \hookrightarrow C^*\mathbb{Z}$$

induce isomorphisms in analytical theory, for any $\rho \geq 1$. For the second inclusion, this follows from the Wiener lemma and the density theorem, and the first is a consequence of the Oka principle in K-theory established by Bost in [19] (see thm. 1.1.1 and ex. 1.1.3 therein). This also follows from a theorem of Arens, Eidlin and Novodvorskii: Let B be a *commutative* unital Banach algebra, and let $\mathrm{Spec}(B)$ be its spectrum (it is a compact Hausdorff space). Then, the Gelfand transform

$$\mathscr{G} : B \longrightarrow C(\mathrm{Spec}(B))$$

is a natural map and induces an isomorphism in analytical K-theory (see [19], thm. 1.3.2). It is clear that $\mathrm{Spec}(\ell^1\mathbb{Z})$ identifies with the unit circle S^1 and is included in $\mathrm{Spec}(\mathcal{A}_\rho)$, that correspondingly identifies with the closed crown with radii ρ^{-1} and ρ. This inclusion is a homotopy equivalence, hence the isomorphism $\mathscr{G}_* : K_*(\mathcal{A}_\rho) \cong K_*(\ell^1\mathbb{Z})$.

Similarly, the inclusions $\mathcal{A}_{\rho_x}\hat{\otimes}\mathcal{A}_{\rho_y} \hookrightarrow \ell^1\mathbb{Z}\hat{\otimes}\ell^1\mathbb{Z} \cong \ell^1\mathbb{Z}^2 \hookrightarrow C^*\mathbb{Z}^2$ induce isomorphisms

$$K_*(\mathcal{A}_{\rho_x}\hat{\otimes}\mathcal{A}_{\rho_y}) \xrightarrow{\cong} K_*(\ell^1\mathbb{Z}^2) \xrightarrow{\cong} K_*(C^*\mathbb{Z}^2)\,,$$

since for two commutative unital Banach algebras B_1 and B_2, there is a canonical homeomorphism

$$\mathrm{Spec}(B_1\hat{\otimes}B_2) \cong \mathrm{Spec}(B_1) \times \mathrm{Spec}(B_2)$$

(see [49], prop. IV.1.20).

We can deduce by naturality of the $\star$-product, of the cup product, and of the maps $\phi_{1,1}$ and ϕ_2 that

$$\phi_{1,1}^{\mathcal{A}_{\rho_x}\hat{\otimes}\mathcal{A}_{\rho_y}}(a) \cup \phi_{1,1}^{\mathcal{A}_{\rho_x}\hat{\otimes}\mathcal{A}_{\rho_y}}(b) = \phi_2^{\mathcal{A}_{\rho_x}\hat{\otimes}\mathcal{A}_{\rho_y}}(\{a, b\})\,.$$

By the same argument, it follows that $\phi_{1,1}^A(x) \cup \phi_{1,1}^A(y) = \phi_2^A(\{x, y\})$, and this completes the proof. □

We thank Paul Jolissaint for pointing out a problem in a previous proof, and Nigel Higson for suggesting to use the Banach algebra $\mathcal{A}_\rho$ and for indicating Bost's article [19].

9.5 The case $i = 0$ and $j = 2$

In this section, we prove the following lemma.

9.5.1 Lemma. *For two unital Banach algebras A and B, the diagram*

$$\begin{array}{ccc} K_{0,1}^{alg}(A) \times K_2^{alg}(B) & \xrightarrow{\star} & K_2^{alg}(A \otimes_{\mathbb{Z}} B) \\ \phi_{0,1}\times\phi_2 \downarrow & & \downarrow \hat{\phi}_2 \\ K_{0,1}(A) \times K_2(B) & \xrightarrow{\cup} & K_2(A \hat{\otimes} B) \end{array}$$

commutes.

Before starting the proof, we give explicit formulas for the corresponding products in algebraic and in topological K-theory. Following the definition given by Milnor (see [75], p. 67), one easily checks that for $x \in K_{0,1}^{alg}(A)$ (A a commutative ring), the product

$$x \star (-) : K_2^{alg}(A) \longrightarrow K_2^{alg}(A)\,,\; y \longmapsto x \star y$$

is given by the automorphism $(\gamma_x)_*$ of $H_2(E(R);\mathbb{Z}) \cong K_2^{alg}(A)$ induced by the map

$$\gamma_x : E(A) \longrightarrow E(A)\,,\; E_n(A) \ni X \longmapsto x \cdot X + (1-x)\cdot \mathbb{1}_n\,.$$

We need to express the map $(\gamma_x)_*$ explicitly on $K_2^{alg}(A)$ considered as the kernel of the universal central extension

$$0 \longrightarrow K_2^{alg}(A) \longrightarrow \mathrm{St}(A) \xrightarrow{\varphi} E(A) \longrightarrow 0\,.$$

Let $X = \prod_s e_{i_s j_s}(a_s) \in E_n(A)$ (a finite product of elementary matrices). Since $x = x^2$, one has clearly

$$x \cdot X + (1-x)\cdot \mathbb{1}_n = \prod_s (x \cdot e_{i_s j_s}(a_s) + (1-x)\cdot \mathbb{1}_n) = \prod_s e_{i_s j_s}(x a_s)\,.$$

This means that the map γ_x is simply given by $e_{ij}(a) \longmapsto e_{ij}(xa)$. We can therefore lift this map to $\mathrm{St}(A)$ by defining

$$\tilde{\gamma}_x : \mathrm{St}(A) \longrightarrow \mathrm{St}(A)\,,\; x_{ij}(a) \longmapsto x_{ij}(xa)\,.$$

We obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_2^{alg}(A) & \longrightarrow & \mathrm{St}(A) & \xrightarrow{\varphi} & E(A) & \longrightarrow & 0 \\ & & (\gamma_x)_* \downarrow & & \tilde{\gamma}_x \downarrow & & \downarrow \gamma_x & & \\ 0 & \longrightarrow & K_2^{alg}(A) & \longrightarrow & \mathrm{St}(A) & \xrightarrow{\varphi} & E(A) & \longrightarrow & 0 \end{array}$$

This shows that $(\gamma_x)_* = \bar{\gamma}_{x|K_2^{alg}(A)}$, and gives a satisfactory description of the product $\star$ in question, namely

$$\star : K_{0,1}^{alg}(A) \times K_2^{alg}(A) \longrightarrow K_2^{alg}(A)\,,\ \left(x,\, \prod_s x_{i_s j_s}(a_s)\right) \longmapsto \prod_s x_{i_s j_s}(x a_s)\,.$$

For A and B two unital rings, tbis generalizes to give

$$\star : K_{0,1}^{alg}(A) \times K_2^{alg}(B) \longrightarrow K_2^{alg}(A \otimes_{\mathbf{Z}} B)\,,\ \left(x,\, \prod_s x_{i_s j_s}(b_s)\right) \longmapsto \prod_s x_{i_s j_s}(x \otimes b_s)\,.$$

Now, for a unital commutative Banach algebra, we would like to describe the product $\cup : K_{0,1}(A) \times K_2(A) \longrightarrow K_2(A)$. First, observe that by definition of the cup product and naturality of the suspension isomorphism, the diagram

$$\begin{array}{ccccc}
K_{0,1}(A) \times K_2(A) & \xrightarrow{\ \cup\ } & K_2(A\hat{\otimes}A) & \xrightarrow{K_2(\hat{\mu})} & K_2(A) \\
\cong\downarrow & & \cong\downarrow & & \downarrow\cong \\
K_{0,1}(A) \times K_1(SA) & \xrightarrow{\ \cup\ } & K_1(S(A\hat{\otimes}A)) & \xrightarrow{K_1(S\hat{\mu})} & K_1(SA) \\
\cong\downarrow & & \cong\downarrow & & \\
K_{0,1}(A) \times K_0(S^2A) & \xrightarrow{\ \cup\ } & K_0(S^2(A\hat{\otimes}A)) & &
\end{array}$$

commutes, where $S\hat{\mu}$ is induced by $\hat{\mu} : A\hat{\otimes}A \longrightarrow A$ and is explicitly given by

$$S\hat{\mu} : S(A\hat{\otimes}A) \longrightarrow SA\,,\ (t \mapsto a(t)\hat{\otimes}b(t)) \longmapsto (t \mapsto a(t)b(t))\,.$$

The explicit descriptions of the isomorphism $K_1(SA) \cong \pi_1(\mathrm{GL}_A) = K_2(A)$, given in section 8.4, and of the product $K_{0,1} \times K_1 \longrightarrow K_1$, given at the end of section 9.3, allow for computing

$$\begin{gathered}
\cup : K_{0,1}(A) \times K_2(A) \longrightarrow K_2(A) \\
\left(x,\, \left[e^{2\pi i t} \mapsto \textstyle\prod_s e_{i_s j_s}(t \cdot a_s)\right]\right) \longmapsto \left[e^{2\pi i t} \mapsto \textstyle\prod_s e_{i_s j_s}(t \cdot x a_s)\right]\,.
\end{gathered}$$

For two unital Banach algebras A and B, this generalizes to give

$$\begin{gathered}
\cup : K_{0,1}(A) \times K_2(B) \longrightarrow K_2(A\hat{\otimes}B) \\
\left(x,\, \left[e^{2\pi i t} \mapsto \textstyle\prod_s e_{i_s j_s}(t \cdot b_s)\right]\right) \longmapsto \left[e^{2\pi i t} \mapsto \textstyle\prod_s e_{i_s j_s}(t \cdot x\hat{\otimes}b_s)\right]\,.
\end{gathered}$$

We are now is position to prove lemma 9.5.1.

Proof of lemma 9.5.1. In section 8.5, we have explicitly described the map ϕ_2. For $x \in K_{0,1}^{alg}(A)$ and $y = \prod_s x_{i_s j_s}(b_s) \in K_2^{alg}(B)$, this, together with the above considerations, yields $\phi_2(y) = [e^{2\pi i t} \mapsto \prod_s e_{i_s j_s}(t \cdot b_s)]$ and also

$$\begin{array}{ccccc} K_2^{alg}(A \otimes_{\mathbf{Z}} B) & \longrightarrow & K_2^{alg}(A \hat{\otimes} B) & \xrightarrow{\phi_2} & K_2(A \hat{\otimes} B) \\ \underbrace{\prod_s x_{i_s j_s}(x \otimes b_s)}_{=x \star y} & \longmapsto & \prod_s x_{i_s j_s}(x \hat{\otimes} b_s) & \longmapsto & \underbrace{[e^{2\pi i t} \mapsto \prod_s e_{i_s j_s}(t \cdot x \hat{\otimes} b_s)]}_{=x \star \phi_2(y)} \end{array} .$$

This completes the proof. □

9.6 Proof of theorem 9.1.1

By applying all the preceding lemmas, we prove theorem 9.1.1.

Proof of theorem 9.1.1. We can obviously assume that $i \leq j$. For $j \neq 2$, consider the diagram

$$\begin{array}{ccccc} K_{i,n}^{alg}(A) \times K_{j,m}^{alg}(B) & \xrightarrow{\cong} & K_{i,1}^{alg}(M_n(A)) \times K_{j,1}^{alg}(M_m(B)) & \xrightarrow{\star} & \\ \downarrow{\scriptstyle \phi_{i,n} \times \phi_{j,n}} & & \downarrow{\scriptstyle \phi_{i,1} \times \phi_{j,1}} & & \\ K_{i,n}(A) \times K_{j,m}(B) & \xrightarrow{\cong} & K_{i,1}(M_n(A)) \times K_{j,1}(M_m(B)) & \xrightarrow{\cup} & \end{array}$$

$$\begin{array}{ccccc} \xrightarrow{\star} & K_{i+j}^{alg}(M_n(A) \otimes_{\mathbf{Z}} M_m(B)) & \xrightarrow{\cong} & K_{i+j}(A \otimes_{\mathbf{Z}} B) \\ & \downarrow{\scriptstyle \hat{\phi}_{i+j}} & & \downarrow{\scriptstyle \hat{\phi}_{i+j}} \\ \xrightarrow{\cup} & K_{i+j}(M_n(A) \hat{\otimes} M_m(B)) & \xrightarrow{\cong} & K_{i+j}(A \hat{\otimes} B) \end{array}$$

It is clear that the first and third squares commute. By combining naturality of ϕ_{i+j} with lemmas 9.3.1, 9.4.1 and 9.2.3, one sees that so does the middle one. We have seen in lemma 9.2.1 that the top composition is the product in algebraic K-theory, and the bottom composition is the cup product by lemma 9.2.2. This proves the result for $j \neq 2$.

For $j = 2$ (and $i = 0$), by lemma 9.5.1, the diagram

$$\begin{array}{ccccc} K_{0,n}^{alg}(A) \times K_2^{alg}(B) & \xrightarrow{\cong} & K_{0,1}^{alg}(M_n(A)) \times K_2^{alg}(B) & \xrightarrow{\star} & \\ \downarrow{\scriptstyle \phi_{0,n} \times \phi_2} & & \downarrow{\scriptstyle \phi_{0,1} \times \phi_2} & & \\ K_{0,n}(A) \times K_2(B) & \xrightarrow{\cong} & K_{0,1}(M_n(A)) \times K_2(B) & \xrightarrow{\cup} & \end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{\ \star\ } & K_2^{alg}(M_n(A)\otimes_{\mathbf{Z}} B) & \xrightarrow{\cong} & K_2(A\otimes_{\mathbf{Z}} B) \\
& \Big\downarrow{\scriptstyle \hat{\phi}_2} & & \Big\downarrow{\scriptstyle \hat{\phi}_2} \\
\xrightarrow{\ \cup\ } & K_2(M_n(A)\hat{\otimes} B) & \xrightarrow{\cong} & K_2(A\hat{\otimes} B)
\end{array}$$

commutes. Again by lemma 9.2.1, the top composition is the product in algebraic K-theory, and by lemma 9.2.2 the bottom composition is the cup product. This proves the result in this case too.

This completes the proof. □

Appendix A

The cones from the rational point of view

We show that after "rationalizing", both the c-cone and the γ-cone coincide, and that they are, as subsets of the positive cone in K-theory with rational coefficients, as large as possible (the statement will be made precise).

For a simply-connected CW-complex Y, we denote by $Y_{\mathbb{Q}}$ the rationalization (or $\mathbb{Q}$-localization) of Y. The rationalization is functorial and provides Y with a canonical map $j_Y : Y \longrightarrow Y_{\mathbb{Q}}$. We denote the application induced by a map $f : Y \longrightarrow Z$ between simply connected CW-complexes by $f_{\mathbb{Q}}$; it satisfies $j_Z \circ f = f_{\mathbb{Q}} \circ j_Y$. The space $Y_{\mathbb{Q}}$ is itself simply-connected, and $(Y_{\mathbb{Q}})_{\mathbb{Q}} = Y_{\mathbb{Q}}$, with $j_{Y_{\mathbb{Q}}} = Id_{Y_{\mathbb{Q}}}$; similarly, $(f_{\mathbb{Q}})_{\mathbb{Q}} = f_{\mathbb{Q}}$. (For details, the reader may refer to [48], chap. VII, or to [92], sect. II.5.)

For a connected finite CW-complex X, the reduced K-theory with rational coefficients may be defined by

$$\widetilde{K}(X; \mathbb{Q}) := [X, BU_{\mathbb{Q}}].$$

This is in fact the term in degree 0 of a 2-periodic reduced cohomology theory on the category of finite CW-complexes. The canonical map $j_{BU} : BU \longrightarrow BU_{\mathbb{Q}}$ induces a coefficient homomorphism $(j_{BU})_* : \widetilde{K}(X) \longrightarrow \widetilde{K}(X; \mathbb{Q})$, and one can show that there is a natural isomorphism $\widetilde{K}(X) \otimes \mathbb{Q} \xrightarrow{\cong} \widetilde{K}(X; \mathbb{Q})$ such that the triangle

$$\begin{array}{ccc} \widetilde{K}(X) & \longrightarrow & \widetilde{K}(X) \otimes \mathbb{Q} \\ & \searrow^{(j_{BU})_*} & \downarrow \cong \\ & & \widetilde{K}(X; \mathbb{Q}) \end{array}$$

commutes (see thm. II.5.4 in [92]). With a slight abuse, for $x \in \widetilde{K}(X)$, we denote $(j_{BU})_*(x) = j_{BU} \circ x$ by $x_{\mathbf{Q}}$. On the other hand, we can consider the sets

$$\mathrm{Vect}_n(X;\mathbb{Q}) := [X, BU(n)_{\mathbf{Q}}] \quad \text{and} \quad \mathrm{Vect}(X;\mathbb{Q}) := \coprod_{n\geq 0} \mathrm{Vect}_n(X;\mathbb{Q}),$$

the latter being equipped with a structure of abelian semigroup in the same way as $\mathrm{Vect}(X)$ is. Exactly in the same vein as one establishes the group isomorphism $\mathcal{G}(\mathrm{Vect}(X)) \cong [X, \mathbb{Z} \times BU]$ (see for example p. 45 in [4]), one shows that $\mathcal{G}(\mathrm{Vect}(X;\mathbb{Q})) \cong [X, \mathbb{Z} \times BU_{\mathbf{Q}}]$ is a group isomorphism. Therefore, we get the following sequence of isomorphisms:

$$\mathcal{G}(\mathrm{Vect}(X;\mathbb{Q})) \cong [X, \mathbb{Z} \times BU_{\mathbf{Q}}] = \mathbb{Z} \oplus \widetilde{K}(X;\mathbb{Q}) \cong \mathbb{Z} \oplus (\widetilde{K}(X) \otimes \mathbb{Q}).$$

From now on, we identify tbese groups.

A.0.1 Remark. *There is another very natural way of defining rational K-theory. Indeed, one can define (in the obvious way) the notion of semi-module over a semi-ring, as for example $\mathrm{Vect}(X)$ over $\mathbb{N}$, and there is a meaningful notion of tensor product of semi-modules over a semi-ring. It is therefore reasonable to define unreduced rational K-theory of a finite CW-complex X as to be the ring*

$$K(X, \mathbb{Q}) := \mathcal{G}(\mathrm{Vect}(X) \otimes_{\mathbf{N}} \mathbb{Q}).$$

It is easy to show that this is the expected ring, in other words, there are canonical ring isomorphisms

$$K(X, \mathbb{Q}) \cong K(X) \otimes_{\mathbf{Z}} \mathbb{Q} \cong \mathbb{Q} \oplus \widetilde{K}(X;\mathbb{Q}).$$

Another choice would be $\mathcal{G}(\mathrm{Vect}(X) \otimes_{\mathbf{N}} \mathbb{Q}_+)$, where $\mathbb{Q}_+$ is the semi-ring of non-negative rationals, but this ring is canonically isomorphic to the previous one. For details about these constructions and results, see appendix B.

A.0.2 Definition. *The positive cone in rational K-theory is the sub-semigroup*

$$K_+(X;\mathbb{Q}) := \mathrm{Im}(\mathrm{Vect}(X;\mathbb{Q}) \longrightarrow \mathbb{Z} \oplus \widetilde{K}(X;\mathbb{Q}))$$

of $\mathbb{Z} \oplus \widetilde{K}(X;\mathbb{Q})$. For $z \in \widetilde{K}(X;\mathbb{Q})$, we define the rational geometric dimension of z as the minimal n such that z lifts as map $X \longrightarrow BU(n)_{\mathbf{Q}}$. We denote it by $\mathrm{g-dim}_{\mathbf{Q}}(z)$.

The connection between the positive cone in rational K-theory and the rational geometric dimension is clear:

$$K_+(X;\mathbb{Q}) = \{(n, z) \in \mathbb{Z} \oplus \widetilde{K}(X;\mathbb{Q}) \mid n \geq \mathrm{g-dim}_{\mathbf{Q}}(z)\}.$$

For $m \geq 2$, let $K(m, \mathbb{Q})$ denote the Eilenberg-MacLane space of type $(m, \mathbb{Q})$. It is a rational space, that is $K(m, \mathbb{Q})_{\mathbf{Q}} = K(m, \mathbb{Q})$ (see chap. VII in [48]). For any $n \geq 0$, the rational Chern classes $\bar{c}_j$ provide us with a commutative diagram

$$\begin{array}{ccc} BU(n)_{\mathbf{Q}} & \xrightarrow[\simeq]{(\bar{c}_1)_{\mathbf{Q}} \times \ldots \times (\bar{c}_n)_{\mathbf{Q}}} & \prod_{k=1}^{n}{}' K(2k, \mathbb{Q}) \\ \downarrow{\scriptstyle (i_n)_{\mathbf{Q}}} & & \downarrow{\scriptstyle j_n} \\ BU_{\mathbf{Q}} & \xrightarrow[\simeq]{(\bar{c}_1)_{\mathbf{Q}} \times \ldots} & \prod_{k \geq 1}{}' K(2k, \mathbb{Q}) \end{array}$$

where $(i_n)_{\mathbf{Q}}$ is induced by the inclusion i_n of $BU(n)$ in BU, $\prod'$ stands for the weak product (the product in the category of CW-complexes), j_n is the inclusion, and the horizontal maps are homotopy equivalences (see [48], chap. VII).

Now the following lemma is clear.

A.0.3 Lemma. *For $x \in \widetilde{K}(X)$, one has* $\mathrm{g-dim}_{\mathbf{Q}}(x_{\mathbf{Q}}) = \bar{\mathrm{c}}\mathrm{-dim}(x)$.

We would now like to define the rational analogues of the γ-cone, of the c-cone and of the $\bar{c}$-cone.

A.0.4 Definition. *The rational positive cone of X (not to be confused with the positive cone in rational K-theory) is the sub-semigroup*

$$K_+^{\mathbf{Q}}(X) = \mathrm{Im}(K_+(X) \longrightarrow \mathbb{Z} \oplus \widetilde{K}(X; \mathbb{Q}))$$

of $\mathbb{Z} \oplus \widetilde{K}(X; \mathbb{Q})$. The rational γ-cone is the sub-semigroup of $\mathbb{Z} \oplus \widetilde{K}(X; \mathbb{Q})$ given by $K_\gamma^{\mathbf{Q}}(X) = \mathrm{Im}(K_\gamma(X) \longrightarrow \mathbb{Z} \oplus \widetilde{K}(X; \mathbb{Q}))$. The rational c-cone $K_c^{\mathbf{Q}}(X)$ and the rational $\bar{c}$-cone $K_{\bar{c}}^{\mathbf{Q}}(X)$ are defined similarly. Finally, we let

$$K_+(j_X; \mathbb{Q}) := \{(n, z) \in K_+(X, \mathbb{Q}) \mid z \in j_X(X)\} \subseteq K_+(X, \mathbb{Q}).$$

Here comes the main result concerning the rational cones.

A.0.5 Proposition. *For a connected finite CW-complex X, one has*

$$K_+^{\mathbf{Q}}(X) \subseteq K_\gamma^{\mathbf{Q}}(X) = K_c^{\mathbf{Q}}(X) = K_{\bar{c}}^{\mathbf{Q}}(X) = K_+(j_X; \mathbb{Q}) \subseteq K_+(X; \mathbb{Q}).$$

Proof. The inclusion $K_+^{\mathbf{Q}}(X) \subseteq K_\gamma^{\mathbf{Q}}(X)$ follows from proposition 1.3.2 iii). The equality $K_\gamma^{\mathbf{Q}}(X) = K_{\bar{c}}^{\mathbf{Q}}(X)$ follows from lemma 1.2.5. The fact that $K_c^{\mathbf{Q}}(X)$ coincides with $K_{\bar{c}}^{\mathbf{Q}}(X)$ is obvious. Finally, the equality $K_{\bar{c}}^{\mathbf{Q}}(X) = K_+(j_X; \mathbb{Q})$ is an immediate consequence of lemma A.0.3. This completes the proof. □

A.0.6 Remark. *In general, the rational positive cone does not coincide with the other rational cones, even for simply connected torsion-free spaces. Indeed, in section 1.9, we show that for* $S^4 \times S^4$, *the positive cone and the* γ*-cone do not coincide. Since* $S^4 \times S^4$ *is torsion-free, this implies that the rational positive cone and* $K_+(j_X; \mathbb{Q})$ *are different.*

Appendix B

Tensor product of semi-modules and Grothendieck construction

We define carefully the notions of monoids, semi-rings, semi-modules over semi-rings, semi-bimodules over semi-rings, Grothendieck construction of semi-bimodules, and tensor products (over a semi-ring) of semi-bimodules. Our goal is to express the Grothendieck construction of a tensor product of two semi-bimodules M and N in terms of the tensor product of the corresponding Grothendieck construction of M and of N.

B.0.7 Definition. *A semi-group M is a non-empty set endowed with a map*

$$M \times M \longrightarrow M\,,\ (x,\, y) \longmapsto x \cdot y = xy\,,$$

called product (or multiplication), that is associative. It is a monoid if moreover there exits a unit element 1. (We do not ask for the existence of inverse elements: this would define a group.) If the product is commutative, then the monoid M is called abelian, and the product xy is often denoted by $x + y$ and called the sum (or addition), and the unit is called the zero element and denoted by 0.

A map $f : M \longrightarrow N$ between two monoids M and N is a monoid homomorphism if $f(xy) = f(x)f(y)$ for any $x,\, y \in M$, and if $f(1) = 1$.

A semi-ring S is an abelian monoid (for an addition $+$) endowed with a map

$$S \times S \longrightarrow S\,,\ (x,\, y) \longmapsto x \cdot y = xy\,,$$

called product (or multiplication), that is associative, and distributive with respect to the addition, and such that $0 \cdot \lambda = \lambda \cdot 0 = 0$ for any $\lambda \in S$. (We do not require the product to be commutative.) The semi-ring is unital if there exists a unit for the product.

A map $f : S \longrightarrow T$ between two semi-rings S and T is a semi-ring homomorphism if it is a monoid homomorphism and if $f(\lambda\mu) = f(\lambda)f(\mu)$ for any $\lambda,\, \mu \in S$

Notice that in a monoid, the unit is unique.

Typical examples of semi-rings that are not rings are given by the naturals $\mathbb{N}$, and the non-negative rationals $\mathbb{Q}_+$.

Exactly in the same way as one defines a module over a ring, one defines a semi-module over a semi-ring.

B.0.8 Definition. *Let S be a semi-ring. A left semi-module M over S (or left S-semi-module) is a monoid endowed with a map*

$$S \times M \longrightarrow M\,,\ (\lambda, x) \longmapsto \lambda \cdot x\,,$$

such that $0 \cdot x = 0$, $\lambda \cdot 0 = 0$, $(\lambda + \mu) \cdot x = (\lambda \cdot x) + (\mu \cdot x)$, $(\lambda\mu) \cdot x = \lambda \cdot (\mu \cdot x)$, $\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$, *for any* λ, $\mu \in S$ *and any* x, $y \in M$.

One defines similarly right semi-modules over S.

Given two left S-semi-modules M and N, a map $f : M \longrightarrow N$ is a left S-semi-module homomorphism if f is additive, i.e. $f(x + x') = f(x) + f(x')$ for any x, $x' \in M$, and if f is homogeneous, i.e. $f(\lambda \cdot x) = \lambda \cdot f(x)$ for any $\lambda \in S$ and $x \in M$.

One defines similarly right S-semi-module homomorphisms.

Given two semi-rings S and T, a semi-bimodule over S and T (or an (S, T)-semi-bimodule) is a monoid M that is at the same time a left semi-module over S and a right semi-module over T, and such that both actions commute, that is $\lambda \cdot (x \cdot \mu) = (\lambda \cdot x) \cdot \mu$ for any $\lambda \in S$, $x \in M$ and $\mu \in T$.

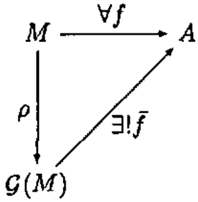
Given two (S, T)-semi-bimodules M and N, a map $f : M \longrightarrow N$ is an (S, T)-semi-bimodule homomorphism if f is at the same time a left S-semi-module homomorphism and a right T-semi-module homomorphism.

Recall that an abelian group is the same as a $(\mathbb{Z}, \mathbb{Z})$-bimodule. Similarly, an abelian monoid is the same as an $(\mathbb{N}, \mathbb{N})$-semi-bimodule. We consider therefore any right S-semi-module as an $(S, \mathbb{N})$-semi-bimodule, and any left S-semi-module as an $(\mathbb{N}, S)$-semi-bimodule. Similarly, a monoid homomorphism is the same thing as an $(\mathbb{N}, \mathbb{N})$-semi-bimodule homomorphism.

The condition $0 \cdot \lambda = \lambda \cdot 0 = 0$ in the definition of a semi-ring S ensures that S is in a canonical way an (S, S)-semi-bimodule.

Let us now review the Grothendieck construction.

B.0.9 Definition. *Let M be an abelian monoid. A Grothendieck construction (or group-completion) for M is an abelian group $\mathcal{G}(M)$ equipped with a monoid homomorphism $\rho : M \longrightarrow \mathcal{G}(M)$ such that the following universal property is satisfied: For any abelian group A and any monoid homomorphism $f : M \longrightarrow A$,* *there exists a unique group homomorphism* $\bar{f} : \mathcal{G}(M) \longrightarrow A$ *such that* $f = \bar{f} \circ \rho$, *in other words, the diagram*



commutes.

It is clear that if a Grothendieck construction for M exists, it is unique up to a unique group isomorphism "compatible with ρ". In the next proposition, we prove the existence of the group completion for any abelian monoid, by providing with an explicit model. We first need a technical lemma, whose proof is obvious.

B.0.10 Lemma. *Let* M *be a monoid, and let* $\sim$ *be a "multiplicative" equivalence relation, i.e. an equivalence relation satisfying*

$$x \sim y \text{ and } x' \sim y' \implies xx' \sim yy' .$$

(If M *is abelian and the operation on* M *is denoted by* $+$, *the equivalence relation is called "additive".) Then the quotient* $M/\sim$ *carries a structure of monoid for the product* $[x] \cdot [y] := [xy]$ *and the unit* $[1]$. *The quotient map* $M \longrightarrow M/\sim$ *is a monoid homomorphism. Moreover, if the monoid* M *is abelian, so is* $M/\sim$.

On the other hand, if $f : M \longrightarrow N$ *is a monoid homomorphism, and if* $f(x) = f(y)$ *whenever* $x \sim y$, *then* f *factors through the homomorphism* $\bar{f} : M/\sim \longrightarrow N$ *given by* $\bar{f}([x]) := f(x)$ *for* $x \in M$.

Notice that the cartesian product of two monoids carries a canonical structure of monoid. It is abelian if and only if both factors are abelian. The cartesian product of two semi-rings is in a canonical way a semi-ring.

A "multiplicative" equivalence relation on M as in the above lemma, as any relation, can be viewed as a subset R of the cartesian product $M \times M$. The "multiplicativity" property simply means that R is a multiplicative subset of $M \times M$ (or equivalently a sub-semi-group).

B.0.11 Proposition. *Let* M *be an abelian monoid. On the cartesian product* $M \times M$, *the equivalence relation*

$$(x, y) \sim (x', y') \iff \exists z \in M \text{ such that } x + y' + z = x' + y + z$$

is additive. The quotient abelian monoid $\mathcal{G}(M) := (M \times M)/\sim$ *is an abelian group: the opposite of* $[x, y]$ *is* $[y, x]$. *Moreover,* $\mathcal{G}(M)$ *together with the map* $\rho : M \longrightarrow \mathcal{G}(M)$, $x \longrightarrow [x, 0]$, *it is a Grothendieck construction for* M.

Proof. The fact that $\mathcal{G}(M)$ is an abelian group is clear.

Given a monoid homomorphism $f : M \longrightarrow A$ as in definition B.0.9, we have to define the group homomorphism $\bar{f}$. It is simply given by $\bar{f}([x, y]) := f(x) - f(y)$.

□

For example, any abelian group is its own group completion, and the monoid $\mathbb{N}$ has $\mathbb{Z}$ as group completion. Another example is $\mathcal{G}(\mathbb{Q}_+) = \mathcal{G}(\mathbb{Q}) = \mathbb{Q}$.

Notice that the universal property for $\mathcal{G}(M)$ also holds when A is any group (not necessarily abelian).

The following two lemmas are straightforward.

B.0.12 Lemma. *Let S be a semi-ring. Then the Grothendieck construction $\mathcal{G}(S)$ carries a canonical ring structure defined by*

$$[\lambda, \mu] \cdot [\lambda', \mu'] := [\lambda\lambda' + \mu\mu', \lambda\mu' + \mu\lambda'],$$

for $\lambda, \lambda', \mu, \mu' \in S$. Moreover, the map $\rho : S \longrightarrow \mathcal{G}(S)$ is a semi-ring homomorphism. If S has a unit 1, then $[1, 0]$ is a unit for $\mathcal{G}(S)$. If S is commutative, so is $\mathcal{G}(S)$.

B.0.13 Lemma. *Let M be a left (resp. right) S-semi-module. Then the group completion $\mathcal{G}(M)$ carries a canonical structure of left (resp. right) $\mathcal{G}(S)$-module given by*

$$[\lambda, \mu] \cdot [x, y] := [\lambda \cdot x + \mu \cdot y, \mu \cdot x + \lambda \cdot y]$$

(resp. $[x, y] \cdot [\lambda, \mu] := [x \cdot \lambda + y \cdot \mu, x \cdot \mu + y \cdot \lambda]$), where $x, y \in M$ and $\lambda, \mu \in S$. Moreover, the canonical map $\rho : M \longrightarrow \mathcal{G}(M)$ is a homomorphism of left (resp. right) S-semi-modules. In particular, if M is an (S, T)-semi-bimodule, then $\mathcal{G}(M)$ carries a canonical structure $(\mathcal{G}(S), \mathcal{G}(T))$-bimodule.

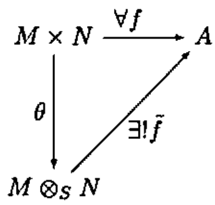
B.0.14 Definition. *Let M be a right S-semi-module, and N be a left S-semi-module. A map $f : M \times N \longrightarrow A$ in an abelian monoid A is S-bilinear, if f is bi-additive (i.e. $f(x+x', y) = f(x, y)+f(x', y)$ and $f(x, y+y') = f(x, y)+f(x, y')$ for any $x, x' \in M$ and $y, y' \in N$), and if*

$$f(x \cdot \lambda, y) = f(x, \lambda \cdot y),$$

for any $x \in M$, $\lambda \in S$, and $y \in N$.

We now would like to define the tensor product of abelian monoids, or more generally of semi-bimodules.

B.0.15 Definition. *Let M be a right S-semi-module, and N be a left S-semi-module. Then, a tensor product of M by N over S is an abelian monoid, denoted by $M \otimes_S N$, equipped with an S-bilinear homomorphism $\theta : M \times N \longrightarrow M \otimes_S N$, and* *satisfying the following universal property: Given any S-bilinear homomorphism $f : M \times N \longrightarrow A$ with values in an abelian monoid A, there exists a unique monoid homomorphism $\bar{f} : M \otimes_S N \longrightarrow A$ such that $f = \bar{f} \circ \theta$, in other words, the diagram*



commutes.

It is clear that if a tensor product exists, it is unique up to a unique isomorphism "compatible with θ". In order to prove the existence of a tensor product, we have to deal with free monoids and free abelian monoids.

B.0.16 Lemma. *Let X be any set (empty or non-empty). The free monoid $F(X)$ on the set X is the set consisting in* 1 *and in the "positive words" $x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$, where $n \in \mathbb{N}^*$, $x_i \in X$ and $\epsilon_i \in \mathbb{N}^*$ for $1 \le i \le n$, and $x_{i+1} \neq x_i$ for $1 \le i < n$. The product is given by concatenation and simplification by the rules $x^n x^m = x^{n+m}$ and $1 \cdot x^n = x^n = x^n \cdot 1$. It satisfies the following universal property: For any map $f : X \longrightarrow N$ in a monoid N, there exists a unique monoid homomorphism $\bar{f} : F(X) \longrightarrow N$ such that $\bar{f}(x) = f(x)$ for any $x \in X$.*

The free abelian monoid $F_{ab}(X)$ on the set X is the quotient of $F(X)$ by the "multiplicative" equivalence relation, viewed as a subset R of $F(X) \times F(X)$,

$$R := \{\text{finite products of elements of the form } (uv, vu)\,,\ u\,, v \in F(X)\}\,.$$

It satisfies the following universal property: For any map $f : X \longrightarrow N$ in an abelian monoid N, there exists a unique monoid homomorphism $\bar{f} : F_{ab}(X) \longrightarrow N$ such that $\bar{f}(x) = f(x)$ for any $x \in X$.

Proof. The case of the free monoid over X is clear. The case of the free abelian monoid over X follows from the first case (by virtue of lemma B.0.10).

□

We now prove the existence of tensor products.

B.0.17 Proposition. *Let M be a right S-semi-module, and N be a left S-semi-module. Consider the free abelian monoid $A := F_{ab}(M \times N)$ on the set $M \times N$, and the following subsets of $A \times A$:*

$$\begin{aligned} B &:= \{\,((x + x', y), (x, y) + (x', y))\,|\, x, x' \in M \text{ and } y \in N\,\} \\ C &:= \{\,((x, y + y'), (x, y) + (x, y'))\,|\, x \in M \text{ and } y, y' \in N\,\} \\ D &:= \{\,((x \cdot \lambda, y), (x, \lambda \cdot y)\,|\, x \in M\,,\ \lambda \in S \text{ and } y \in N\,\} \end{aligned}$$

Then, on the free abelian monoid A, *consider the "additive" equivalence relation* $\sim$, *viewed as a subset* R *of* $A \times A$, *given by*

$$R := \{\text{finite sums of elements of } B,\ C \text{ and } D\}.$$

Then the quotient abelian monoid $M \otimes_S N := F_{ab}(M \times N)/\sim$ *equipped with the map* $\theta : M \times N \longrightarrow M \otimes_S N$, $(x, y) \longmapsto [x, y]$ *is a tensor product of* M *by* N *over* S. *The element* $[x, y] \in M \otimes_S N$ *is denoted by* $x \otimes y$ *(and called an elementary tensor).*

Proof. First, by the universal property of the free abelian monoid, f extends uniquely to a map $\tilde{f} : F_{ab}(M \times N) \longrightarrow A$. Since $\tilde{f}(r) = \tilde{f}(s)$ for any pair (r, s) in the subset R (i.e. whenever $r \sim s$), $\tilde{f}$ factors through a monoid homomorphism $\bar{f} : (F_{ab}(M \times N)/\sim) \longrightarrow A$ (see lemma B.0.10).

Secondly, it is clear that the quotient $F_{ab}(M \times N)/\sim$ is generated, as a monoid, by the elementary tensors $x \otimes y$. Since $\bar{f}(x \otimes y)$ has to be equal to $f(x, y)$, it follows that $\bar{f}$ is unique. □

Notice that for any $x \in M$ and any $y \in N$, the elementary tensors $x \otimes 0$ and $0 \otimes y$ are zero in $M \otimes_S N$.

As in the case of the usual tensor product over a ring, it is clear that the tensor product over a semi-ring is commutative and associative (up to canonical isomorphisms).

B.0.18 Lemma. *Let* S *be a unital semi-ring, and* M *be a left (resp. right) S-semi-module. Then there is a canonical isomorphism*

$$\alpha : S \otimes_S M \xrightarrow{\cong} M,\ \lambda \otimes x \longmapsto \lambda \cdot x$$

(resp. $\beta : M \otimes_S S \xrightarrow{\cong} M$, $x \otimes \lambda \longmapsto x \cdot \lambda$*).*

Proof. The map $S \times M \longrightarrow M$, $(\lambda, x) \longmapsto \lambda \cdot x$ is S-bilinear, therefore, the map α is well defined, with inverse $x \longmapsto 1 \otimes x$. The other case is similar. □

In fact, the above lemma is the only place in these notes where we really need the semi-ring S to satisfy the property $\lambda \cdot 0 = 0 \cdot \lambda = 0$ for any λ in S.

B.0.19 Proposition. *Consider three semi-rings* R, S *and* T. *Let* M *be an* (R, S)*-semi-bimodule, and* N *be an* (S, T)*-semi-bimodule. Then the tensor product* $M \otimes_S N$ *carries a canonical* (R, T)*-semi-bimodule structure. Moreover, the map* θ *is not only* S*-bilinear, but also an left* (R, T)*-semi-bimodule homomorphism.*

Proof. The structures are defined by

$$\lambda \cdot (x \otimes y) := (\lambda \cdot x) \otimes y \text{ and } (x \otimes y) \cdot \mu := x \otimes (y \cdot \mu),$$

for $\lambda \in R$, $x \in M$, $y \in N$, and $\mu \in T$. The rest is clear. □

We finally arrive at the main result. It expresses the group completion of a tensor product of semi-modules in terms of the tensor product of the corresponding "Grothendieck constructed" modules.

B.0.20 Theorem. *Let R, S and T be three semi-rings. Let M be an (R, S)-semi-bimodule, and N be an (S, T)-semi-bimodule. Then there is a canonical isomorphism of $(\mathcal{G}(R), \mathcal{G}(T))$-bimodules*

$$\varphi : \mathcal{G}(M \otimes_S N) \xrightarrow{\cong} \mathcal{G}(M) \otimes_{\mathcal{G}(S)} \mathcal{G}(N).$$

Proof. First, the map

$$f : M \times N \longrightarrow \mathcal{G}(M) \otimes_{\mathcal{G}(S)} \mathcal{G}(N), \ (x, y) \longmapsto [x, 0] \otimes [y, 0].$$

is S-bilinear, therefore, by the universal property of the tensor product over S, it induces a monoid homomorphism $\tilde{f} : M \otimes_S N \longrightarrow \mathcal{G}(M) \otimes_{\mathcal{G}(S)} \mathcal{G}(N)$ given by $\tilde{f}(x \otimes y) = [x, 0] \otimes [y, 0]$. By the universal property of the group completion, $g = \tilde{f}$ induces a homomorphism $\bar{g} = \varphi : \mathcal{G}(M \otimes_S N) \longrightarrow \mathcal{G}(M) \otimes_{\mathcal{G}(S)} \mathcal{G}(N)$ (of abelian groups) given by

$$\varphi([x \otimes y, x' \otimes y']) \longmapsto [x, 0] \otimes [y, 0] - [x', 0] \otimes [y', 0].$$

On the other hand, the map

$$\begin{aligned} h : \mathcal{G}(M) \times \mathcal{G}(N) &\longrightarrow \mathcal{G}(M \otimes_S N) \\ ([x, x'], [y, y']) &\longmapsto [x \otimes y, x \otimes y'] + [x' \otimes y', x' \otimes y] \end{aligned}$$

is S-bilinear. By the universal property of the tensor product (of modules), it induces a group homomorphism $\tilde{h} = \psi : \mathcal{G}(M) \otimes_{\mathcal{G}(S)} \mathcal{G}(N) \longrightarrow \mathcal{G}(M \otimes_S N)$ given by

$$\psi([x, x'] \otimes [y, y']) = [x \otimes y, x \otimes y'] + [x' \otimes y', x' \otimes y].$$

It is easy to check that $\psi \circ \varphi = Id$. On the other hand, one has

$$\begin{aligned} \varphi \circ \psi([x, x'] \otimes [y, y']) &= [x, 0] \otimes [y, 0] - [x, 0] \otimes [y', 0] \\ &\quad + [x', 0] \otimes [y', 0] - [x', 0] \otimes [y, 0] \\ &= [x, 0] \otimes [y, y'] + [0, x'] \otimes [y, y'] \\ &= [x, x'] \otimes [y, y']. \end{aligned}$$

Therefore, $\varphi \circ \psi = Id$, and this proves that φ is a group isomorphism. It is obviously a $(\mathcal{G}(R), \mathcal{G}(T))$-bimodule homomorphism. This completes the proof. □

The following result is an immediate consequence.

B.0.21 Corollary. *Let M be an abelian monoid. Then*

$$\mathcal{G}(M \otimes_{\mathbf{N}} \mathbb{Q}_+) \cong \mathcal{G}(M \otimes_{\mathbf{N}} \mathbb{Q}) \cong \mathcal{G}(M) \otimes_{\mathbf{Z}} \mathbb{Q}.$$

Another consequence is less obvious.

B.0.22 Corollary. *Let R, S and T be three semi-rings. Let M be an (R, S)-semi-bimodule, and N be an (S, T)-semi-bimodule. Assume that M and N are abelian groups. Then M is a $(\mathcal{G}(R), \mathcal{G}(S))$-bimodule, N is a $(\mathcal{G}(S), \mathcal{G}(T))$-bimodule, and one has*

$$M \otimes_S N \cong M \otimes_{\mathcal{G}(S)} N$$

as $(\mathcal{G}(R), \mathcal{G}(T))$-bimodules.

Proof. The first two statements follow directly from lemma B.0.13. By the theorem, for the third statement, it suffices to prove that $M \otimes_S N$ is a group. Given x in M and y in N, we have to construct the opposite $-(x \otimes y)$ of $x \otimes y$ in $M \otimes_S N$. Since $0 \otimes y$ vanishes in $M \otimes_S N$, it is clear that

$$-(x \otimes y) = (-x) \otimes y$$

(which by the way coincides with $x \otimes (-y)$, by uniqueness of inverse elements in a group.) □

Let us single out a particular case.

B.0.23 Corollary. *Let M and N be two abelian groups. Then*

$$M \otimes_{\mathbf{N}} N \cong M \otimes_{\mathbf{Z}} N.$$

As an application to K-theory (see chapter 1), we immediately have the following

B.0.24 Proposition. *For a connected finite CW-complex X, one has canonical and natural ring isomorphisms*

$$\mathcal{G}(\mathrm{Vect}(X) \otimes_{\mathbf{N}} \mathbb{Q}) \cong \mathcal{G}(\mathrm{Vect}(X) \otimes_{\mathbf{N}} \mathbb{Q}_+) \cong K(X) \otimes_{\mathbf{Z}} \mathbb{Q}.$$

Since the Chern character induces a natural isomorphism $K(X) \otimes_{\mathbf{Z}} \mathbb{Q} \cong H^*(X; \mathbb{Q})$, this shows that rational cohomology has a canonical expression in terms of the Grothendieck construction of the abelian monoid $\mathrm{Vect}(X) \otimes_{\mathbf{N}} \mathbb{Q}$ (which is *not* a group). It would be of great interest to have a direct interpretation of this fact.

Appendix C

On the Moore-Postnikov tower of the fibration $BSU(3) \longrightarrow BSU(5)$

We compute the first few stages of the Moore-Postnikov decomposition of the canonical fibration $f : BSU(3) \longrightarrow BSU(5)$. (The reader may refer to [24], p. 501 for the definition of the Moore-Postnikov decomposition of a fibration.) More precisely, we construct a 13-connected map $\tilde{f} : BSU(3) \longrightarrow Z$, i.e. a map inducing an isomorphism on homotopy groups up to degree 13 (and an epimorphism in degree 14), together with a map $g : Z \longrightarrow BSU(5)$ such that $f = g \circ \tilde{f}$ (up to homotopy). As an application, we get some general statements on the geometric dimension of stable classes of complex vector bundles. This applies, for example, to the computation of the positive cone of $S^6 \times S^6$ and of $\mathbb{H}P^3$, the quaternionic projective space of (quaternionic) dimension 3. We end this appendix by computing the c-cone and the γ-cone, and partially determining the positive cone, of the quaternionic projective space $\mathbb{H}P^3$.

C.1 Statement of the main results

In the present section, we first state the main theorem of this chapter. We then give some immediate corollaries. As an application, we re-compute the positive cone of the product $S^6 \times S^6$ without appealing to the deep results on the homotopy groups of spheres that we needed in chapter 1.

Here is the main theorem of this appendix.

C.1.1 Theorem. *Let Y be a connected finite CW-complex of dimension ≤ 12, and let $\xi \in \widetilde{K}(Y)$. Assume that $H^{10}(Y;\mathbb{Z})$ has no 2-torsion, and that the cohomology groups $H^{11}(Y;\mathbb{Z}/4)$ and $H^{11}(Y;\mathbb{Z}/3)$ vanish. Then*

$$c_i(\xi) = 0 \text{ for } i = 1, 4, 5 \text{ and } 6 \quad \Longrightarrow \quad \text{g-dim}(\xi) \leq 3\,.$$

By the universal coefficient theorem (see cor. V.7.2 in [24]), we get the

C.1.2 Corollary. *Let Y be a connected finite CW-complex of dimension ≤ 12, and let $\xi \in \widetilde{K}(Y)$. Assume that $H^{10}(Y;\mathbb{Z})$ has no 2-torsion, that $H^{12}(Y;\mathbb{Z})$ has no 2- and no 3-torsion, and that $H^{11}(Y;\mathbb{Z})$ vanishes. Then*

$$c_i(\xi) = 0 \text{ for } i = 1, 4, 5 \text{ and } 6 \quad \Longrightarrow \quad \text{g-dim}(\xi) \leq 3\,.$$

The second corollary follows readily from the first.

C.1.3 Corollary. *Let $Y = X \cup e^{12}$ where X is a finite connected CW-complex of dimension ≤ 7, and let $\xi \in \widetilde{K}(Y)$. Then*

$$c_1(\xi) = 0 \text{ and } c_6(\xi) = 0 \quad \Longrightarrow \quad \text{g-dim}(\xi) \leq 3\,.$$

This applies to $S^6 \times S^6 = (S^6 \vee S^6) \cup e^{12}$. Together with the computation of the total Chern classes made in section 1.8 of chapter 1, this proves directly the

C.1.4 Theorem. *For the product $S^6 \times S^6$, the positive cone coincides with the c-cone and the γ-cone. The latter is described in theorem 1.7.1.*

The appendix is organized as follows. In section C.2, we review some basic results from homotopy theory, such as cohomology operations, Serre's and Cartan's results on the cohomology of Eilenberg-MacLane spaces, the path-loop fibration and the universal coefficient theorem. We also evaluate some Steenrod squares on the Chern classes. The proof of the main theorem C.1.1 is the subject of section C.3. The idea is to "kill" cohomology classes and compute the cohomology of the corresponding spaces by means of Lerray-Serre spectral sequence arguments. Finally, in section C.3, as another application of the main theorem, we partially compute the positive cone of $\mathbb{H}P^3$.

C.2 Preliminaries in homotopy theory

We review the basic notions and results from homotopy theory that we need. These include mod p Steenrod operations, Bockstein operators, the Wu formula (that computes some Steenrod squares of the Chern classes), Serre's and Cartan's computation of the mod p cohomology of the Eilenberg-MacLane spaces $K(\mathbb{Z}/p^k, n)$ and $K(\mathbb{Z}, n)$, and other related topics.

We start with cohomology operations.

The mod p Steenrod operations are cohomology operations β and $\mathcal{P}^n$ $(n \geq 1)$ acting on the mop p cohomology of CW-complexes. When $p = 2$, $\mathcal{P}^n$ is often denoted by Sq^n (and called a Steenrod square); one has $\beta = Sq^1$. The Steenrod operations are of the following degree:

$$\deg(\beta) = 1 \qquad \deg(\mathcal{P}^n) = 2n(p-1).$$

For any prime p, the Bockstein operator β satisfies $\beta\beta = 0$.

Let us recall some identities among the Steenrod squares (see [80]):

$$\begin{array}{lll} Sq^1Sq^{2n+1} = 0 & Sq^1Sq^{2n} = Sq^{2n+1} & Sq^2Sq^3 = Sq^5 + Sq^4Sq^1 \\ Sq^2Sq^4 = Sq^6 + Sq^5Sq^1 & Sq^3Sq^2 = 0 & \end{array}$$

The standard short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p^k} \mathbb{Z} \longrightarrow \mathbb{Z}/p^k \longrightarrow 0$$

defines a boundary operator in the corresponding long exact coefficient sequence; it is called the Bockstein operator, and is denoted by

$$\beta_{p^k} : H^*(X; \mathbb{Z}/p^k) \longrightarrow H^{*+1}(X; \mathbb{Z}).$$

The mod p reduction will be denoted by

$$\rho_p : H^*(X; \mathbb{Z}) \longrightarrow H^*(X; \mathbb{Z}/p).$$

and the reduction of coefficients from $\mathbb{Z}/p^k$ to $\mathbb{Z}/p$ by

$$\rho_{p^k,p} : H^*(X; \mathbb{Z}/p^k) \longrightarrow H^*(X; \mathbb{Z}/p),$$

One has $\beta_p \circ \rho_p = 0$ (as compose of two successive homomorphism in a short exact sequence!) and also the following property (see [24], p. 363):

$$\rho_p \circ \beta_p = \beta \; (= Sq^1 , \text{ if } p = 2)$$

If there is a non-zero class $x \in H^n(X; \mathbb{Z}/p)$ with $\beta(x) \neq 0$ then $\beta_p(x)$ is a non-zero element in $H^{n+1}(X; \mathbb{Z})$, whose order is p, and there is no element $z \in H^{n+1}(X; \mathbb{Z})$ with $pz = \beta_p(x)$ (otherwise we would have $\beta(x) = p \cdot \rho_p(\beta_p(z)) = 0$). This is one of the reasons why we will almost systematically compute the action of Sq^1 and of β in the mod 2, respectively mod 3, cohomology of the encountered spaces.

Let us also mention the following formula:

$$\beta_p \circ \rho_{p^k,p} = p^{k-1} \cdot \beta_{p^k} : H^*(X; \mathbb{Z}/p^k) \longrightarrow H^{*+1}(X; \mathbb{Z}).$$

Let us consider the standard short exact sequence

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{Z}/p^{k+1} \longrightarrow \mathbb{Z}/p^k \longrightarrow 0.$$

We will denote the corresponding Bockstein operator by

$$\delta_{p^k} : H^*(X; \mathbb{Z}/p^k) \longrightarrow H^{*+1}(X; \mathbb{Z}/p).$$

We will very often need the universal coefficient theorem in cohomology:

$$H^k(X; \mathbb{Z}/n) \cong (H^k(X; \mathbb{Z}) \otimes \mathbb{Z}/n) \oplus \mathrm{Tor}(H^{k+1}(X; \mathbb{Z}), \mathbb{Z}/n),$$

and the fact that, for a finitely generated abelian group $A = \mathbb{Z}^d \oplus F$, where F is a finite abelian group, one has

$$\mathrm{Tor}(A, \mathbb{Z}/n) \cong F \otimes \mathbb{Z}/n$$

(see [24], pp. 278-285).

We now recall Serre's and Cartan's results on the cohomology of Eilenberg-MacLane spaces.

Let p be a fixed prime. A sequence of natural numbers, $I = (a_1, a_2, \dots, a_m)$ is admissible if each a_i is of the form $a_i = 2\lambda_i(p-1) + \varepsilon_i$, with $\varepsilon_i \in \{0, 1\}$, and if $a_i \geq pa_{i+1}$ for all $i = 1, \dots, m-1$. For such a sequence, we define

$$St^I = St^{a_1} \circ \dots \circ St^{a_m},$$

where St^a, for $a = 2\lambda(p-1) + \varepsilon$, is defined by

$$St^a = \begin{cases} Sq^a & \text{if } p = 2 \\ \mathcal{P}^\lambda & \text{if } p \text{ is odd, and if } \varepsilon = 0 \\ \beta \circ \mathcal{P}^\lambda & \text{if } p \text{ is odd, and if } \varepsilon = 1. \end{cases}$$

This operation St^I is of degree $|I| = a_1 + \dots + a_m$, and the excess of I is defined to be

$$\mathrm{e}(I) = \begin{cases} 2a_1 - |I| + 1 & \text{if } p = 2 \\ 2\lambda_1 p + \varepsilon_1 - |I| + 1 & \text{if } p \text{ is odd}. \end{cases}$$

We will also use the notation $Sq^{a_1, \dots, a_m} = St^I$ when $p = 2$.

Here come the Serre and Cartan theorems. We let $n \geq 1$ and $k \geq 2$.

C.2.1 Theorem. *$H^*(K(\mathbb{Z}/p, n); \mathbb{Z}/p)$ is a free graded commutative algebra (over $\mathbb{Z}/p$) on free generators $St^I \iota_n$, where $\iota_n \in H^n(K(\mathbb{Z}/p, n); \mathbb{Z}/p)$ is the fundamental class and I is any admissible sequence with $\mathrm{e}(I) \leq n$.*

C.2.2 Theorem. *$H^*(K(\mathbb{Z}, n); \mathbb{Z}/p)$ is a free graded commutative algebra (over $\mathbb{Z}/p$) on free generators $St^I \iota_n$, where $\iota_n \in H^n(K(\mathbb{Z}, n); \mathbb{Z}/p)$ is the fundamental class and $I = \{a_1, \dots, a_m\}$ is any admissible sequence with $e(I) \leq n$ and with $a_m \geq 2p - 2$.*

Before stating the third of these theorems, let us recall that

$$H^n(K(\mathbb{Z}/p^k, n); \mathbb{Z}/p^k) = \mathbb{Z}\cdot \iota'_n \cong \mathbb{Z}/p^k$$
$$H^n(K(\mathbb{Z}/p^k, n); \mathbb{Z}/p) \ = \mathbb{Z}\cdot \iota_n \cong \mathbb{Z}/p$$

where ι'_n and ι_n are the fundamental classes. Moreover, these two classes are related: $\iota_n = \rho_{p^k,p}(\iota'_n)$.

C.2.3 Theorem. *$H^*(K(\mathbb{Z}/p^k, n); \mathbb{Z}/p)$ is a free graded commutative algebra (over $\mathbb{Z}/p$) on free generators $St^I\iota_n$ and $St^J\delta_{p^k}\iota_n$, where $\iota_n \in H^n(K(\mathbb{Z}/p^k, n); \mathbb{Z}/p)$ and $\iota'_n \in H^n(K(\mathbb{Z}/p^k, n); \mathbb{Z}/p^k)$ are the fundamental classes and I (resp. J) is any admissible sequence with $e(I) \leq n$ (resp. $e(J) \leq n-1$).*

The proofs of these results (and some more details on the cohomology operations and the admissible sequences) can be found in [95] and in [30].

Using these results, the universal coefficient theorem and the remarks concerning the operations Sq^1 and β, one can compute the first few integral cohomology groups of these Eilenberg-MacLane spaces.

We will need a few values of the Steenrod operations on the Chern classes. As is usual, by invoking the splitting principle, we will identify the Chern class c_k for $BU(n)$ with the k^{th} elementary symmetric function on n variables $t_1, \dots, t_n$ of degree 2. For example, $c_1 = t_1 + \dots + t_n$ and $c_n = t_1 \cdots t_n$. (The t_i's are in fact generators of the cohomology of the classifying space of a maximal torus of $U(n)$.) Since these variables t_i are of degree 2 and since $BU(n)$ has no cohomology in odd degree, we have $Sq^2 t_i = t_i^2$, $Sq^m t_i = 0$ for any other positive value of m, and $\mathcal{P}^1 t_i = t_i^3$ for $p = 3$. Moreover this can in principle be used to compute the action of Sq^m and of $\mathcal{P}^1$ ($p = 3$) on the Chern classes, by invoking the Cartan formula for the squaring operation of a product:

$$Sq^m(xy) = \sum_{i=0}^{m} Sq^i x \cdot Sq^{m-i} y$$

(and the analogue for the operations $\mathcal{P}^m$).

We can also make use of the Wu formula (for BU):

$$\begin{aligned} Sq^{2^k} c_i \ &= \ \binom{i-1}{k}\cdot c_{i+k} + \binom{i-2}{k-1}\cdot c_1 c_{i+k-1} + \binom{i-3}{k-2}\cdot c_2 c_{i+k-2} + \dots + \\ &\quad + \binom{i-k}{1}\cdot c_{k-1}c_{i+1} + c_k c_i . \end{aligned}$$

Let us start the needed computations (for BU):

$$
\begin{aligned}
Sq^2c_2 &= \binom{1}{1}c_3 + c_1c_2 = c_3 + c_1c_2\,;\\
Sq^2c_4 &= \binom{3}{1}c_5 + c_1c_4 = c_5 + c_1c_4\,;\\
Sq^2c_6 &= \binom{5}{1}c_7 + c_1c_6 = c_7 + c_1c_6\,;\\
Sq^4c_4 &= \binom{3}{2}c_6 + \binom{2}{1}c_1c_5 + c_2c_4 = c_6 + c_2c_4\,;\\
Sq^4c_5 &= \binom{4}{2}c_7 + \binom{3}{1}c_1c_6 + c_2c_5 = c_1c_6 + c_2c_5\,;\\
Sq^{4,2}c_4 &= Sq^4c_5 + Sq^4(c_1c_4) = Sq^4c_5 + c_1^2Sq^2c_4 + c_1Sq^4c_4\\
&= c_2c_5 + c_1^2c_5 + c_1^3c_4 + c_1c_2c_4\,;\\
Sq^{2,4}c_4 &= Sq^2c_6 + Sq^2(c_2c_4) = Sq^2c_6 + c_2Sq^2c_4 + Sq^2c_2\cdot c_4\\
&= c_7 + c_1c_6 + c_2c_5 + c_3c_4\,;\\
Sq^6c_4 &= Sq^{2,4}c_4 + Sq^{5,1}c_4 = Sq^{2,4}c_4\\
&= c_7 + c_1c_6 + c_2c_5 + c_3c_4\,.
\end{aligned}
$$

For the fifth Chern class, one has the equalities $Sq^2c_5 = \binom{4}{1}c_6 + c_1c_5 = c_1c_5$ and $Sq^4c_5 = \binom{4}{2}c_7 + \binom{3}{1}c_1c_6 + c_2c_5 = c_1c_6 + c_2c_5$.

We now summarize these results when restricted to $BSU(5)$:

$$
\begin{aligned}
Sq^2c_4 &= c_5\\
Sq^4c_4 &= c_2c_4\\
Sq^6c_4 &= c_2c_5 + c_3c_4\\
Sq^{4,2}c_4 &= c_2c_5\\
Sq^{6,2}c_4 &= c_3c_5\\
Sq^2c_5 &= 0\\
Sq^4c_5 &= c_2c_5\,.
\end{aligned}
$$

We now have to compute $\mathcal{P}^1c_n$ and $\mathcal{P}^1c_{n-1}$ for $BU(n)$ $(p=3)$. We find

$$
\begin{aligned}
\mathcal{P}^1c_n &= \sum_j t_1\cdots t_j^3\cdots t_n\\
&= \left(\sum_j t_j^2\right)\cdot(t_1\cdots t_n)\\
&= \left(\left(\sum_j t_j\right)^2 - \sum_j\sum_{k\neq j} t_jt_k\right)\cdot(t_1\cdots t_n)\\
&= (c_1^2 - 2c_2)\cdot c_n = c_1^2c_n + c_2c_n\,.
\end{aligned}
$$

We have $c_{n-1} = \sum_j t_1 \cdots \hat{t}_j \cdots t_n$, where the hat "$\hat{\ }$" means that we omit the corresponding term, thus

$$\begin{aligned}
\mathcal{P}^1 c_{n-1} &= \sum_j \sum_{k \neq j} t_1 \cdots t_k^3 \cdots \hat{t}_j \cdots t_n \\
&= \sum_j \Big((t_1^2 + \ldots + \hat{t}_j^2 + \ldots + t_n^2) \cdot (t_1 \cdots \hat{t}_j \cdots t_n) \Big) \\
&= \left(\sum_j t_j^2 \right) \cdot \left(\sum_j t_1 \cdots \hat{t}_j \cdots t_n \right) - \sum_j t_1 \cdots t_j^2 \cdots t_n \\
&= \left(\left(\sum_j t_j \right)^2 - \sum_j \sum_{k \neq j} t_j t_k \right) \cdot \left(\sum_j t_1 \cdots \hat{t}_j \cdots t_n \right) + \\
&\quad - \left(\sum_j t_j \right) \cdot (t_1 \cdots t_n) \\
&= (c_1^2 - 2c_2) \cdot c_{n-1} - c_1 c_n = c_1^2 c_{n-1} + c_2 c_{n-1} - c_1 c_n \,.
\end{aligned}$$

We recollect this now (for $BSU(5)$ and $p = 3$):

$$\begin{aligned}
\mathcal{P}^1 c_5 &= c_2 c_5 \\
\mathcal{P}^1 c_4 &= c_2 c_4 \,.
\end{aligned}$$

We will use some homotopy theory. Recall that $\Omega K(G, n) \simeq K(G, n-1)$ (we will identify both spaces).

For any pointed space Z one has the path-loop fibration

$$\Omega Z \hookrightarrow PZ \longrightarrow Z \,,$$

where PZ is the set of paths originating at the base-point of Z; this space is contractible.

We particularize to the case where $Z = K(G, n)$, $n \geq 1$. Let $x \in H^n(Y; G)$. We can consider it as a map $x : Y \longrightarrow K(G, n)$. Let us perform the pull-back of the path-loop fibration for $K(G, n)$:

$$\begin{array}{ccc}
K(G, n-1) & = & K(G, n-1) \\
\downarrow & & \downarrow \\
Y_0 & \longrightarrow & PK(G, n) \\
\pi \downarrow & & \downarrow \\
Y & \xrightarrow{\ x\ } & K(G, n)
\end{array}$$

C.2.4 Proposition. *A continuous map $f : X \longrightarrow Y$ lifts to a map $f_0 : X \longrightarrow Y_0$ if and only if the compose map $x \circ f$ is homotopic to zero (and this is equivalent to $f^*(x) = 0$ in $H^n(X;\, G)$).*

Let us finally recall the following classical theorem.

C.2.5 Theorem. *Let $f : X \longrightarrow Y$ be a pointed continuous map between simply-connected pointed CW-complexes of finite type. Let $n \geq 1$. Assume that*

$$f^* : H^k(Y;\, \mathbb{Z}) \longrightarrow H^k(X;\, \mathbb{Z})$$

is an isomorphism for $k = 0, \ldots, n+1$ and a monomorphism for $k = n+2$. Then for any pointed CW-complex K of dimension $\leq n$, the induced map

$$f_* : [K,\, X] \longrightarrow [K,\, Y]$$

is a bijection. More precisely, f is an $(n+1)$-connected map (or $(n+1)$-equivalence).

Before proving it, we state a lemma.

C.2.6 Lemma. *Let A be a finitely generated group. Then*

$$A \cong \mathrm{Ext}(A,\, \mathbb{Z}) \oplus \mathrm{Hom}(A,\, \mathbb{Z}),.$$

Proof. Write $A \cong \mathbb{Z}^d \oplus F$, where F is a finite abelian group (and therefore a finite direct sum of finite cyclic groups $\mathbb{Z}/n_i$). The result follows from the formulas

$$\begin{gathered} \mathrm{Ext}(\mathbb{Z},\, B) = 0 \quad \mathrm{Ext}(\mathbb{Z}/n,\, \mathbb{Z}) \cong \mathbb{Z}/n \\ \mathrm{Hom}(\mathbb{Z},\, B) = B \quad \mathrm{Hom}(\mathbb{Z}/n,\, \mathbb{Z}) = 0 \\ \mathrm{Ext}(B \oplus C,\, D) \cong \mathrm{Ext}(B,\, D) \oplus \mathrm{Ext}(C,\, D) \\ \mathrm{Hom}(B \oplus C,\, D) \cong \mathrm{Hom}(B,\, D) \oplus \mathrm{Hom}(C,\, D) \end{gathered}$$

for any abelian groups B, C and D (see [24], p. 278). □

Proof of Theorem C.2.5. The proof is based on the following universal coefficient theorem (cf. cor. V.7.2 in [24]): For CW-complexes, there is a natural short exact sequence

$$0 \longrightarrow \underbrace{\mathrm{Ext}(H_{n-1}(X;\, \mathbb{Z}),\, \mathbb{Z})}_{\text{finite abelian}} \longrightarrow H^n(X,\, \mathbb{Z}) \longrightarrow \underbrace{\mathrm{Hom}(H_n(X;\, \mathbb{Z}),\, \mathbb{Z})}_{\text{fin. gen. free abelian}} \longrightarrow 0;$$

it splits, but not naturally. Since X is of finite type, the first non-zero term is a finite abelian group, and the last non-zero one is a free abelian group of finite rank (as follows from the formulas cited in the proof of the above lemma). The same properties hold for Y. An easy induction argument shows that the map

$$f_* : H_k(X;\, \mathbb{Z}) \longrightarrow H_k(Y;\, \mathbb{Z})$$

is an isomorphism for $k = 0, \ldots, n+1$. Since X and Y are simply-connected CW-complexes, by the Whitehead theorem (see thm. VII.11.2 in [24]), the map f induces on homotopy groups an isomorphism in degrees $\leq n$, and an epimorphism in degree $n+1$. This precisely means that f in an $(n+1)$-equivalence, and the statement about the CW-complex K follows (see [24], pp. 485-486). This completes the proof. □

We would like to thank Alain Jeanneret for suggesting this proof.

C.3 Proof of the main theorem C.1.1

We first explain the strategy of the proof, and then perform the required lengthy spectral sequence computations.

The cohomology of $BSU(3)$ is the polynomial ring on the Chern classes c_2 and c_3; for $BSU(5)$, it is the polynomial ring on c_2, c_3, c_4 and c_5. On the cohomological level, the canonical fibration $f : BSU(3) \longrightarrow BSU(5)$ is the canonical projection. We are therefore going to "kill" successively the cohomology classes of $BSU(5)$ that are not polynomials in c_2 or c_3 to get spaces X_i closer and closer to $BSU(3)$ until we get a space X_m together with a 13-connected map $f_m : BSU(3) \longrightarrow X_m$. We will consequently obtain a bijection

$$(f_m)_* : [Y, BSU(3)] \xrightarrow{\cong} [Y, X_m],$$

for any CW-complex Y of dimension ≤ 12.

The leitmotiv is to first "kill" the Chern class c_4 and then kill successively, among the classes that remain or appear, one of lowest degree until we have killed everything we have to.

A few words about our conventions: In the spectral sequences, we usually only write down the multiplicative generators (this is justified by the fact that the differentials are derivations and that all elements on the vertical axis will happen to be transgressive). We also neglect the Chern classes c_2 and c_3 because they won't disappear from the cohomology of the spaces X_i. When we state a result such as

$$H^*(X_i;\, R)/(c_2,\, c_3) = S \quad (\text{where } R = \mathbb{Z} \text{ or } \mathbb{Z}/p),$$

we not only mean that the quotient of the ring $H^*(X_i;\, R)$ by the ideal generated by c_2 and c_3 is isomorphic to the ring S, but we also understand that the group S is a direct summand in the group $H^*(X_i;\, R)$.

Let us start the process.

We consider the Chern class c_4 of $BSU(5)$ as a map

$$c_4 : BSU(5) \longrightarrow K(\mathbb{Z},\, 8),$$

and we get the diagram

$$\begin{array}{ccc}
K(\mathbb{Z},\,7) & = & K(\mathbb{Z},\,7) \\
{\scriptstyle i_1}\downarrow & & \downarrow \\
X_1 & \longrightarrow & PK(\mathbb{Z},\,8) \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \mu} \\
BSU(5) & \xrightarrow{c_4} & K(\mathbb{Z},\,8)
\end{array}$$

by performing the pull-back of the pair of maps $(c_4,\, \mu)$.

To compute the needed part of the cohomology of X_1, we use two Lerray-Serre spectral sequences, the one for the path-loop fibration, for which the values of all the differentials are clear, and the one for X_1. There is a morphism of spectral sequences between both (induced by the map c_4) and we use it to compute the value of the transgressive differential on the fundamental class of the fiber. This process will be repeated at each step without further mention.

We will need a few cohomology groups of $K(\mathbb{Z},\,7)$.

$H^*(K(\mathbb{Z},\,7);\,\mathbb{Z}/2)$:

7	8	9	10	11	12	13	14
ι_7	0	$Sq^2\iota_7$	$Sq^3\iota_7$	$Sq^4\iota_7$	$Sq^5\iota_7$	$Sq^6\iota_7,\, Sq^{4,2}\iota_7$	$Sq^7\iota_7,\, Sq^{5,2}\iota_7$

$H^*(K(\mathbb{Z},\,7);\,\mathbb{Z}/3)$:

7	8	9	10	11	12	13	14
ι_7	0	0	0	$\mathcal{P}^1\iota_7$	$\beta\mathcal{P}^1\iota_7$	0	0

$H^*(K(\mathbb{Z},\,7);\,\mathbb{Z}/p)$, $p \geq 5$:

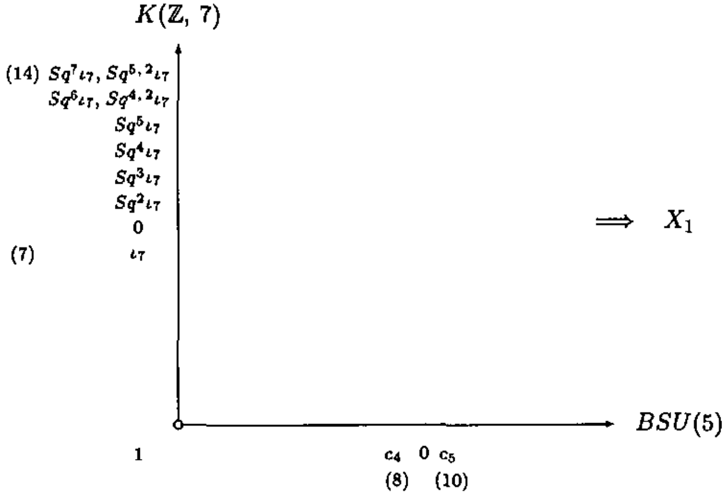
7	8	9	10	11	12	13	14
ι_7	0	0	0	0	0	0	0

$H^*(K(\mathbb{Z},\,7);\,\mathbb{Z})$:

7	8	9	10	11	12	13	14	15
$\mathbb{Z}$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	0
ι_7	0	0	$\beta_2(Sq^2\iota_7)$	0	$\beta_2(Sq^4\iota_7),\, \beta_3(\mathcal{P}^1\iota_7)$	0	$\beta_2(Sq^6\iota_7),\, \beta_2(Sq^{4,2}\iota_7)$	0

Since nothing happens in this range for primes $p \geq 5$, we will only work with coefficients $\mathbb{Z}/2$, $\mathbb{Z}/3$ and $\mathbb{Z}$.

Now the spectral sequence, for $\mathbb{Z}/2$ coefficients, takes the following form:



Since ι_7 is obviously transgressive, Kudo's theorem gives us the following values for the transgressive differentials d_i, denoted d to simplify:

$$\begin{aligned} d(\iota_7) &= c_4 \\ d(Sq^2\iota_7) &= Sq^2 c_4 = c_5 \\ d(Sq^4\iota_7) &= Sq^4 c_4 = 0 \\ d(Sq^6\iota_7) &= Sq^6 c_4 = c_2 c_5 = 0 \\ d(Sq^{4,2}\iota_7) &= Sq^{4,2} c_4 = c_2 c_5 = 0 \end{aligned}$$

and the other values on the generators are 0 for obvious dimensional reasons. (In such relations, when we write for example $c_2c_5 = 0$, we mean that at the corresponding page of the spectral sequence, the product of c_2 by c_5 is zero, and this is the case because c_5 has already been "killed" in a previous page.)

Here is the list of the new generators we get:

$$\begin{aligned} &Sq^3\iota_7 \rightsquigarrow x_{10} \quad (\text{which means } i_1^*(x_{10}) = Sq^3\iota_7) \\ &Sq^4\iota_7 \rightsquigarrow x_{11} \\ &Sq^5\iota_7 = Sq^{1,4}\iota_7 \rightsquigarrow Sq^1 x_{11} \\ &Sq^{4,2}\iota_7 \rightsquigarrow x_{13} \\ &Sq^6\iota_7 \rightsquigarrow x'_{13} \\ &Sq^7\iota_7 = Sq^{1,6}\iota_7 \rightsquigarrow Sq^1 x'_{13} \\ &Sq^{5,2}\iota_7 = Sq^{1,4,2}\iota_7 \rightsquigarrow Sq^1 x_{13}. \end{aligned}$$

In particular, we have $i_1^*(x'_{13}) = Sq^6\iota_7$. On the other hand, we have

$$i_1^*(Sq^2 x_{11}) = Sq^{2,4}\iota_7 = Sq^6\iota_7 + Sq^{5,1}\iota_7 = Sq^6\iota_7.$$

Since i_1^* is injective in odd degrees, we get $x'_{13} = Sq^2 x_{11}$ and

$$Sq^7\iota_7 \rightsquigarrow Sq^1 x'_{13} = Sq^{1,2}x_{11} = Sq^3 x_{11}\,.$$

We will later need a few values of the Steenrod squares on some of these elements.

Now, we compute

$$i_1^*(Sq^2x_{10}) = Sq^{2,3}\iota_7 = Sq^{4,1}\iota_7 + Sq^5\iota_7 = Sq^5\iota_7 = i_1^*(Sq^1x_{11})\,,$$

hence $Sq^2x_{10} \equiv Sq^1x_{11}$ modulo $(c_2,\, c_3)$.

Since c_4 is zero for $BSU(3)$, the inclusion $f : BSU(3) \longrightarrow BSU(5)$ lifts to a map

$$f_1 : BSU(3) \longrightarrow X_1$$

over the map $\pi_1 : X_1 \longrightarrow BSU(5)$. This lifting is not unique, but this will cause us no trouble. Since $f_1^*(x_{10}) = 0$, we have $f_1^*(Sq^2x_{10}) = 0$. Since, in degree 12, $\mathrm{Ker}(f_1^*) = \mathbb{Z}\cdot Sq^1x_{11} \cong \mathbb{Z}/2$, we get $Sq^2x_{10} = Sq^1x_{11}$. We implicitly used the fact that in cohomology the map f^* is an epimorphism with a canonical (homomorphic) cross-section. This property is inherited by f_1.

We also have $i_1^*(Sq^3x_{10}) = i_1^*(Sq^{1,2}x_{10}) = Sq^{1,1}x_{11} = 0$, so $Sq^3x_{10} = 0$.

We will need a few more values. First, we have

$$\begin{aligned} i_1^*(Sq^4x_{10}) &= Sq^{4,3}\iota_7 = Sq^{4,1,2}\iota_7 = Sq^{2,3,2}\iota_7 + Sq^{5,2}\iota_7 \\ &= Sq^{5,2}\iota_7 = Sq^{1,4,2}\iota_7 = i_1^*(Sq^1x_{13})\,. \end{aligned}$$

Since $f_1^*(x_{10}) = 0$, we have $Sq^4x_{10} \in \mathrm{Ker}(f_1^*)$, and so, $Sq^4x_{10} = Sq^1x_{13}$. As a consequence, we compute

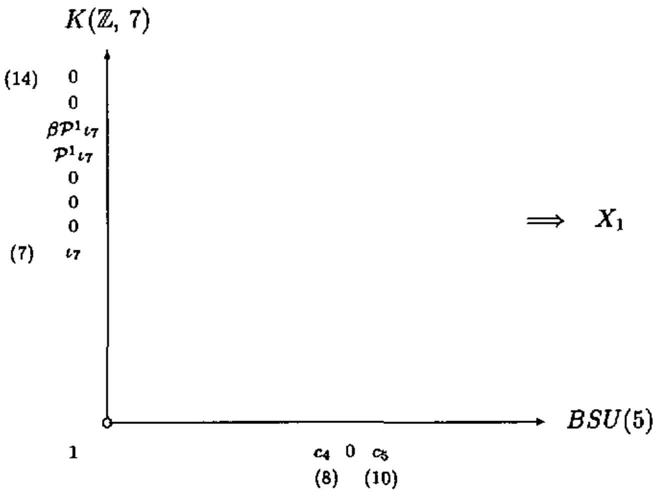
$$Sq^5x_{10} = Sq^{1,4}x_{10} = Sq^{1,1}x_{13} = 0\,.$$

We can now summarize the result we have obtained for the mod 2 cohomology of X_1.

$H^*(X_1;\, \mathbb{Z}/2)/(c_2,\, c_3)$:

	10	11	12	13	14
	x_{10}	x_{11}	Sq^1x_{11}	$Sq^2x_{11},\, x_{13}$	$Sq^3x_{11},\, Sq^1x_{13}$
Sq^1	0	Sq^1x_{11}	0	$Sq^3x_{11},\, Sq^1x_{13}$	0, 0
Sq^2	Sq^1x_{11}	Sq^2x_{11}			
Sq^3	0	Sq^3x_{11}			
Sq^4	Sq^1x_{13}				
Sq^5	0				

The spectral seqnence, for $\mathbb{Z}/3$ coefficients, takes the following form:



By Kudo's theorem, tbe differentials are given by

$$\begin{aligned} d(\iota_7) &= c_4 \\ d(\mathcal{P}^1\iota_7) &= \mathcal{P}^1 c_4 = c_2 c_4 = 0 \\ d(\beta\mathcal{P}^1\iota_7) &= \beta d(\mathcal{P}^1\iota_7) = 0\,. \end{aligned}$$

Here is the list of the new generators we get:

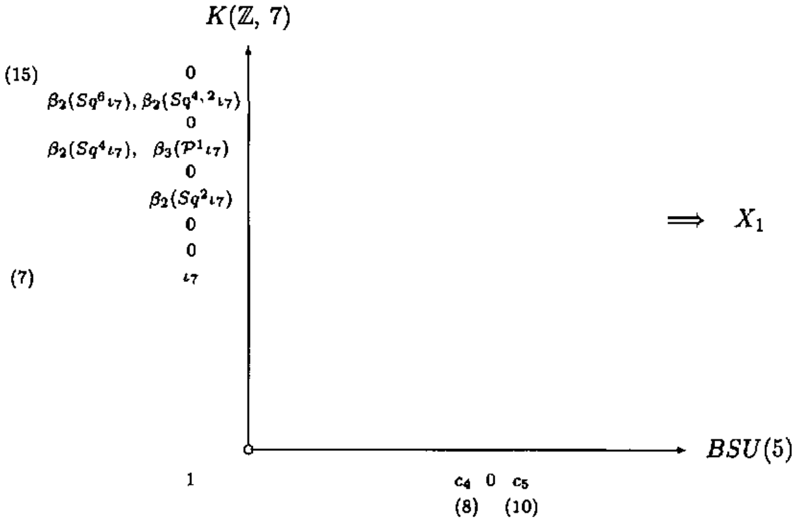
$$\begin{aligned} &\mathcal{P}^1\iota_7 \rightsquigarrow y_{11} \\ &\beta\mathcal{P}^1\iota_7 \rightsquigarrow \beta y_{11}\,. \end{aligned}$$

We can now summarize the result we have obtained for the mod 3 cohomology of X_1.

$H^*(X_1;\, \mathbb{Z}/3)/(c_2,\, c_3)$:

10	11	12	13	14
c_5	y_{11}	βy_{11}	0	0

The spectral sequence, for $\mathbb{Z}$ coefficients, takes the following form:



Obviously, $d(\iota_7) = c_4$ and the other differentials (on the generators) are zero.

Here is the list of the new generators we get:

$$\begin{aligned}
&\beta_2(Sq^4\iota_7) \rightsquigarrow \beta_2(x_{11}) \ \ (\text{because } Sq^2x_{11} \neq 0)\\
&\beta_3(\mathcal{P}^1\iota_7) \rightsquigarrow \beta_3(y_{11}) \ \ (\text{because } \beta y_{11} \neq 0)\\
&\beta_2(Sq^6\iota_7) \rightsquigarrow \beta_2(x'_{13}) = \beta_2(Sq^2x_{11}) \ \ (\text{because } Sq^2x'_{13} = Sq^3x_{11} \neq 0)\\
&\beta_2(Sq^{4,2}\iota_7) \rightsquigarrow \beta_2(x_{13}) \ \ (\text{because } Sq^2x_{13} \neq 0).
\end{aligned}$$

By comparing with the obtained results for $\mathbb{Z}/2$ and $\mathbb{Z}/3$ coefficients, one can easily solve the extension problem (that is to recover the cohomology itself from the corresponding E_∞-page).

$H^*(X_1; \mathbb{Z})/(c_2, c_3)$:

10	11	12	13	14	15
$\mathbb{Z}$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	0
$c_5/2$	0	$\beta_2(x_{11})$, $\beta_3(y_{11})$	0	$\beta_2(Sq^2x_{11})$, $\beta_2(x_{13})$	0

By $c_5/2$, we mean a class z such that $2z = c_5$. Moreover this class satisfies $i_1^*(c_5/2) = \beta_2(Sq^2\iota_7)$ and $\rho_2(c_5/2) = x_{10}$. Since $\rho_3(c_5) = c_5$, we deduce that $\rho(c_5/2) = 2c_5$, the only solution y to the equation $2y = c_5$.

We now "kill" the class $c_5/2$. We consider it as a map $c_5/2 : X_1 \longrightarrow K(\mathbb{Z}, 10)$, and we build the following pull-back:

$$\begin{array}{ccc} K(\mathbb{Z},\,9) & = & K(\mathbb{Z},\,9) \\ {\scriptstyle i_2}\downarrow & & \downarrow \\ X_2 & \longrightarrow & PK(\mathbb{Z},\,10) \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \mu} \\ X_1 & \xrightarrow{c_5/2} & K(\mathbb{Z},\,10) \end{array}$$

To compute the needed part of the cohomology of X_2, we use the Lerray-Serre spectral sequence.

We will need a few cobomology groups of $K(\mathbb{Z},\,9)$.

$H^*(K(\mathbb{Z},\,9);\,\mathbb{Z}/2)$:

9	10	11	12	13	14
ι_9	0	$Sq^2\iota_9$	$Sq^3\iota_9$	$Sq^4\iota_9$	$Sq^5\iota_9$

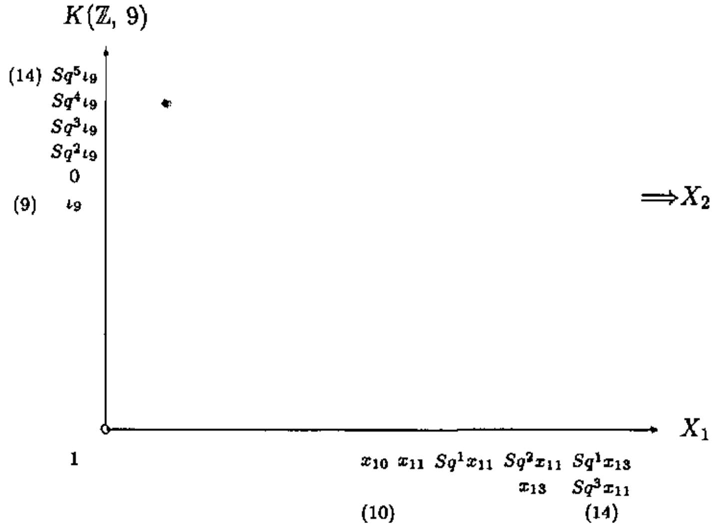
$H^*(K(\mathbb{Z},\,9);\,\mathbb{Z}/3)$:

9	10	11	12	13	14
ι_9	0	0	0	$\mathcal{P}^1\iota_9$	$\beta\mathcal{P}^1\iota_9$

$H^*(K(\mathbb{Z},\,9);\,\mathbb{Z})$:

9	10	11	12	13	14	15
$\mathbb{Z}$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	0
ι_9	0	0	$\beta_2(Sq^2\iota_9)$	0	$\beta_2(Sq^4\iota_9),\ \beta_3(\mathcal{P}^1\iota_9)$	0

The spectral sequence, for $\mathbb{Z}/2$ coefficients, takes the following form:



Since $\rho_2(c_5/2) = x_{10}$, we have $d(\iota_9) = x_{10}$, and by Kudo's theorem, we find

$$\begin{aligned} d(Sq^2\iota_9) &= Sq^2x_{10} = Sq^1x_{11} \\ d(Sq^3\iota_9) &= Sq^3x_{10} = 0 \\ d(Sq^4\iota_9) &= Sq^4x_{10} = Sq^1x_{13} \\ d(Sq^5\iota_9) &= Sq^5x_{10} = 0\,. \end{aligned}$$

Here is the list of the new generators we get:

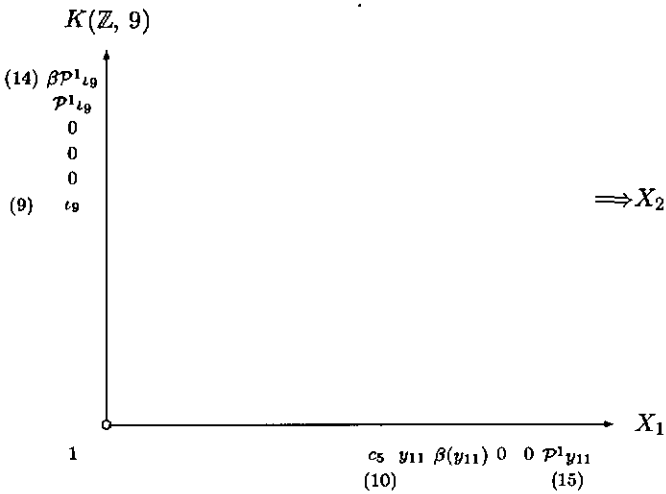
$$\begin{aligned} Sq^3\iota_9 &\rightsquigarrow x_{12} \\ Sq^5\iota_9 &\rightsquigarrow x_{14}\,. \end{aligned}$$

Since the horizontal axis of the E_∞-page is a sub-module of the cohomology, we deduce a few values of the Steenrod squares, as summarized in the following array:

$H^*(X_2;\, \mathbb{Z}/2)/(c_2,\, c_3)$:

	11	12	13	14
	x_{11}	x_{12}	$Sq^2x_{11},\ x_{13}$	$Sq^3x_{11},\ x_{14}$
Sq^1	0		$Sq^3x_{11},\ 0$	
Sq^2	Sq^2x_{11}			
Sq^3	Sq^3x_{11}			

The spectral sequence, for $\mathbb{Z}/3$ coefficients, takes the following form:



Since $\rho_3(c_5/2) = 2c_5$, we find $d(\iota_9) = 2c_5$, and by Kudo's theorem, the differentials are given by

$$d(\mathcal{P}^1\iota_9) = 2\mathcal{P}^1c_5 = 2c_2c_5 = 0$$
$$d(\beta\mathcal{P}^1\iota_9) = \beta d(\mathcal{P}^1\iota_9) = 0\,.$$

Here is the list of the new generators we get:

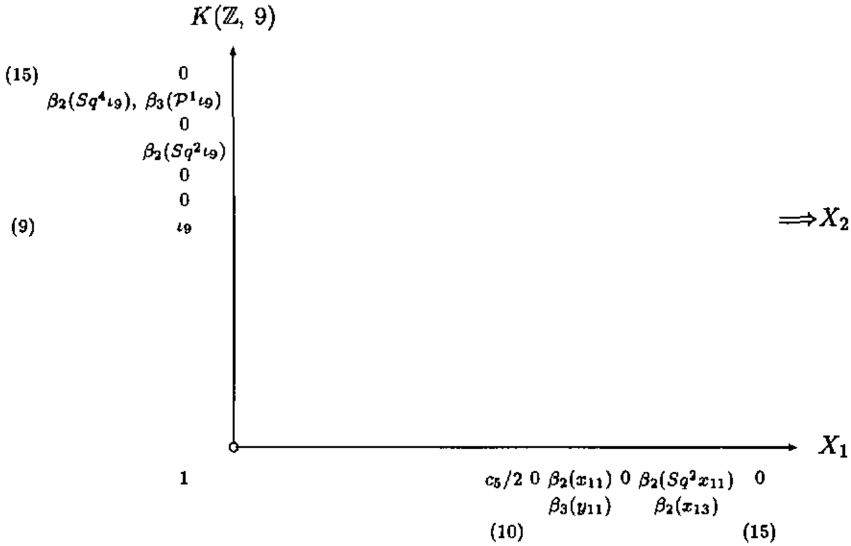
$$\mathcal{P}^1\iota_9 \rightsquigarrow y_{13}$$
$$\beta\mathcal{P}^1\iota_9 \rightsquigarrow \beta y_{13}\,.$$

We can now summarize the result we have obtained for the mod 3 cohomology of X_2.

$H^*(X_2;\, \mathbb{Z}/3)/(c_2,\, c_3)$:

	11	12	13	14
	y_{11}	βy_{11}	y_{13}	βy_{13}
β	βy_{11}	0	βy_{13}	0

The spectral sequence, for $\mathbb{Z}$ coefficients, takes the following form:



We have $d(\iota_9) = c_5/2$, and the other differentials on the generators are zero because X_1 has no cohomology in dimensions 13 and 15.

Here is the list of the new generators we get:

$$\begin{aligned}
&\beta_2(Sq^2\iota_9) \rightsquigarrow z_{12}\\
&\beta_2(Sq^4\iota_9) \rightsquigarrow x_{14}\\
&\beta_3(\mathcal{P}^1\iota_7) \rightsquigarrow \beta_3(y_{13}) \ \ (\text{because } \beta y_{13} \neq 0).
\end{aligned}$$

By comparing with the obtained results for $\mathbb{Z}/2$ and $\mathbb{Z}/3$ coefficients, one can solve the extension problem. Here is the result (to be soon improved!):

$H^*(X_2; \mathbb{Z})/(c_2, c_3)$:

11	12	13	14	15
0	$\mathbb{Z}/4 \oplus \mathbb{Z}/3$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$	0
0	z_{12}, $\beta_3(y_{11})$	0	z'_{14}, z_{14}, $\beta_3(y_{13})$	0

The class z_{12} satisfies $2z_{12} = \beta_2(x_{11})$. Moreover

$$2z_{14} \in \mathbb{Z} \cdot \beta_2(Sq^2x_{11}) \oplus \mathbb{Z} \cdot \beta_2(x_{13}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

and $2z_{14} \neq 0$, thus there are 3 possible values for it. We have

$$\begin{aligned}
&\rho_2(2z_{14}) = 2\rho(z_{14}) = 0\\
&\rho_2 \circ \beta_2(Sq^2x_{11}) = Sq^1Sq^2x_{11} = Sq^3x_{11} \neq 0\\
&\rho_2 \circ \beta_2(x_{13}) = Sq^1x_{13} = 0,
\end{aligned}$$

so we must have $2z_{14} = \beta_2(x_{13})$. This implies that $z'_{14} = \beta_2(Sq^2 x_{11})$.

We now have a result as complete as necessary:

$H^*(X_2; \mathbb{Z})/(c_2, c_3)$:

11	12	13	14	15
0	$\mathbb{Z}/4 \oplus \mathbb{Z}/3$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$	0
0	$\beta_2(x_{11})/2,\ \beta_3(y_{11})$	0	$\beta_2(Sq^2 x_{11}),\ \beta_2(x_{13})/2,\ \beta_3(y_{13})$	0

Since $c_5 = 0$ for $BSU(3)$ and since it has no torsion, it is clear that the map $f_1 : BSU(3) \longrightarrow X_1$ lifts to a map

$$f_2 : BSU(3) \longrightarrow X_2$$

over the map $\pi_2 : X_2 \longrightarrow X_1$. Moreover, in cohomology, f_2^* is an epimorphism with a canonical cross-section.

We "kill" the class $z_{12} = \beta_2(x_{11})/2$, considered as a map $z_{12} : X_2 \longrightarrow K(\mathbb{Z}/4, 11)$ (the range is definitely not $K(\mathbb{Z}/4, 12)$!). We form the pull-back diagram

$$\begin{array}{ccc}
K(\mathbb{Z}/4, 10) & = & K(\mathbb{Z}/4, 10) \\
{\scriptstyle i_3}\downarrow & & \downarrow \\
X_3 & \longrightarrow & PK(\mathbb{Z}/4, 11) \\
{\scriptstyle \pi_3}\downarrow & & \downarrow{\scriptstyle \mu} \\
X_2 & \xrightarrow{z_{12}} & K(\mathbb{Z}/4, 11)
\end{array}$$

To compute the needed part of the cohomology of X_3, we once again use the Lerray-Serre spectral sequence.

We will need a few cohomology groups of $K(\mathbb{Z}/4, 10)$. First, we have

$$H^{10}(K(\mathbb{Z}/4, 10); \mathbb{Z}/4) = \mathbb{Z} \cdot \iota'_{10} \cong \mathbb{Z}/4,$$

where ι'_{10} is the fundamental class. We set

$$\begin{aligned}
\iota_{10} &:= \rho_{4,2}(\iota'_{10}) \in H^{10}(K(\mathbb{Z}/4, 10); \mathbb{Z}/2) \\
\iota_{11} &:= \delta_4(\iota'_{10}) \in H^{11}(K(\mathbb{Z}/4, 10); \mathbb{Z}/2),
\end{aligned}$$

where $\rho_{4,2}$ is the reduction of coefficients induced by the canonical surjection $\mathbb{Z}/4 \longrightarrow \mathbb{Z}/2$. We then have $\beta_2(\iota_{10}) = 2\beta_4(\iota'_{10})$, as already mentioned.

We find the following result:

$H^*(K(\mathbb{Z}/4, 10); \mathbb{Z}/2)$:

10	11	12	13	14
ι_{10}	ι_{11}	$Sq^2\iota_{10}$	$Sq^3\iota_{10},\ Sq^2\iota_{11}$	$Sq^4\iota_{10},\ Sq^3\iota_{11}$

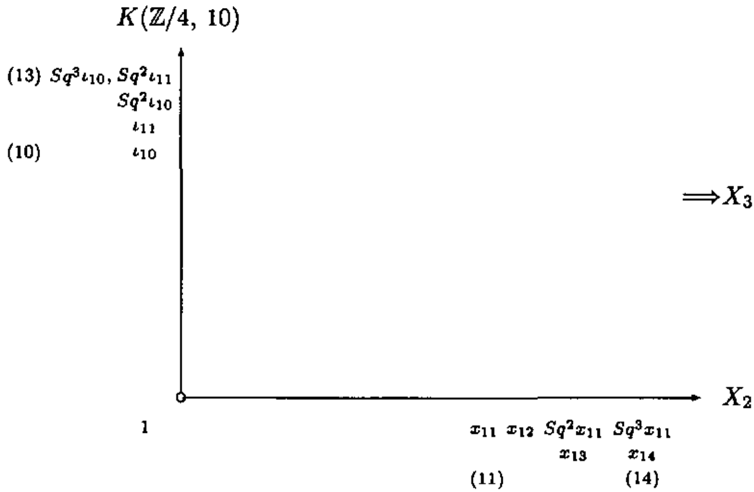
$\widetilde{H}^*(K(\mathbb{Z}/4,\ 10);\ \mathbb{Z}/p) = 0$ for $p \geq 3$.

$H^*(K(\mathbb{Z}/4,\ 10);\ \mathbb{Z})$:

10	11	12	13	14
0	$\mathbb{Z}/4$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
0	$\beta_4(\iota'_{10})$	0	$\beta_2(Sq^2\iota_{10})$	$\beta_2(Sq^2\iota_{11})$

There is nothing to do for $\mathbb{Z}/p$ coefficients when $p \geq 3$. On the other side, both spectral sequences for $\mathbb{Z}/2$ and $\mathbb{Z}$ coefficients have to be studied in parallel.

For $\mathbb{Z}/2$ coefficients, we have



In integral cohomology, the map z_{12}^* sends the class $\beta_4(\iota'_{10})$ to z_{12}, we thus get

$$\begin{aligned}\beta_2(d(\iota_{10})) = d(\beta_2(\iota_{10})) &= d(2\beta_4(\iota'_{10})) \\ &= 2d(\beta_4(\iota'_{10})) = 2z_{12} \\ &= \beta_2(x_{11})\end{aligned}$$

and since the kernel of β_2 in degree 11 is contained in $H^{11}(X_2;\ \mathbb{Z}) = 0$, we finally get $d(\iota_{10}) = x_{11}$.

By looking at the forthcoming spectral sequence with integral coefficients, we find that $d(\iota_{11}) \equiv x_{12}$ modulo $(c_2,\ c_3)$ (otherwise this would lead to a contradiction with the fact that we will independently get $H^{12}(X_3;\ \mathbb{Z})/(c_2,\ c_3) \cong \mathbb{Z}/3$).

Since $BSU(3)$ has no (mod 4) cohomology in dimension 11, the map f_2 lifts to a map $f_3 : BSU(3) \longrightarrow X_3$ over π_3. Moreover f_3^* is a split epimorphism.

Since the horizontal axis of the E_∞-page is the image of π_3^* and since f_2^* is an epimorphism, we must have $d(\iota_{11}) = x_{12}$.

By Kudo's theorem, we find

$$\begin{aligned} d(Sq^2\iota_{10}) &= Sq^2x_{11} \\ d(Sq^3\iota_{10}) &= Sq^3x_{11} \\ d(Sq^2\iota_{11}) &= Sq^2x_{12}\,. \end{aligned}$$

Unfortunately we don't know Sq^2x_{12}. But, again by looking at the spectral sequence over $\mathbb{Z}$, and in particular the fact that $H^{13}(X_3;\,\mathbb{Z})/(c_2,\,c_3) = 0$, we deduce that $d(Sq^2\iota_{11}) \equiv x_{14}$ modulo $(c_2,\,c_3)$. And as before, we can conclude that this is an equality: $d(Sq^2\iota_{11}) = x_{14}$.

We get no new generator.

For the $\mathbb{Z}/2$ cohomology, we find

$H^*(X_3;\,\mathbb{Z}/2)/(c_2,\,c_3)$:

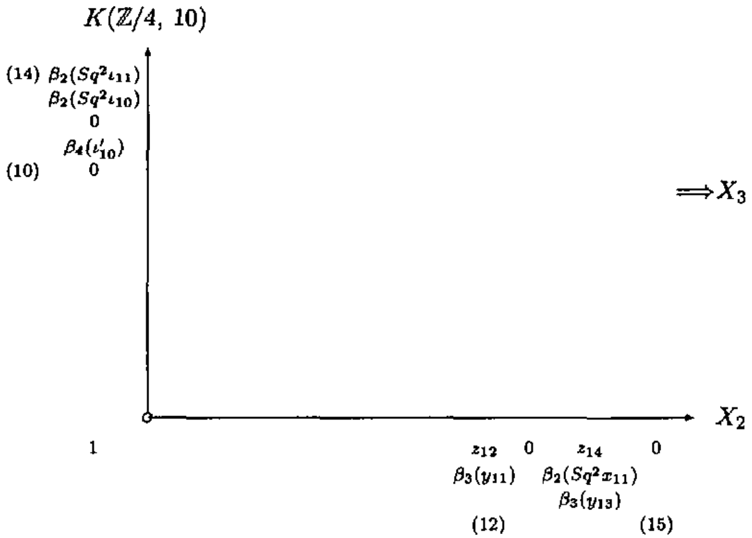
13
x_{13}

For the $\mathbb{Z}/3$ cohomology, we have

$H^*(X_3;\,\mathbb{Z}/3)/(c_2,\,c_3) = H^*(X_2;\,\mathbb{Z}/3)/(c_2,\,c_3)$:

	11	12	13	14
	y_{11}	βy_{11}	y_{13}	βy_{13}
β	βy_{11}	0	βy_{13}	0

Here is the spectral sequence for $\mathbb{Z}$ coefficients:



As already indicated, we have $d(\beta_4(\iota'_{10})) = z_{12}$ and we deduce from the $\mathbb{Z}/2$ coefficients spectral sequence and Kudo's theorem that $d(\beta_2(Sq^2\iota_{10})) = \beta_2(Sq^2 x_{11})$.

We thus get just one new generator: $\beta_2(Sq^2\iota_{11}) \rightsquigarrow z''_{14}$ satisfying $2z_{14''} = z_{14}$.

We now sum up the results:

$H^*(X_3;\, \mathbb{Z})/(c_2,\, c_3)$:

12	13	14
$\mathbb{Z}/3$	0	$\mathbb{Z}/8 \oplus \mathbb{Z}/3$
$\beta_3(y_{11})$	0	$z''_{14},\, \beta_3(y_{13})$

We now want to "kill" the class $\beta_3(y_{11})$. We build the following pull-back:

$$\begin{array}{ccc} K(\mathbb{Z}/3,\, 10) & = & K(\mathbb{Z}/3,\, 10) \\ {\scriptstyle i_4}\downarrow & & \downarrow \\ X_4 & \longrightarrow & PK(\mathbb{Z}/3,\, 11) \\ {\scriptstyle \pi_4}\downarrow & & \downarrow{\scriptstyle \mu} \\ X_3 & \xrightarrow{\beta_3(y_{11})} & K(\mathbb{Z}/3,\, 11) \end{array}$$

To compute the needed part of the cohomology of X_4, we use the Lerray-Serre spectral sequence.

We will need a few cohomology groups of $K(\mathbb{Z}/3,\, 10)$.

$\widetilde{H}^*(K(\mathbb{Z}/3,\, 10);\, \mathbb{Z}/p) = 0$ for $p \neq 3$.

$H^*(K(\mathbb{Z}/3,\, 10);\, \mathbb{Z}/3)$:

10	11	12	13
ι_{10}	$\beta\iota_{10}$	0	0

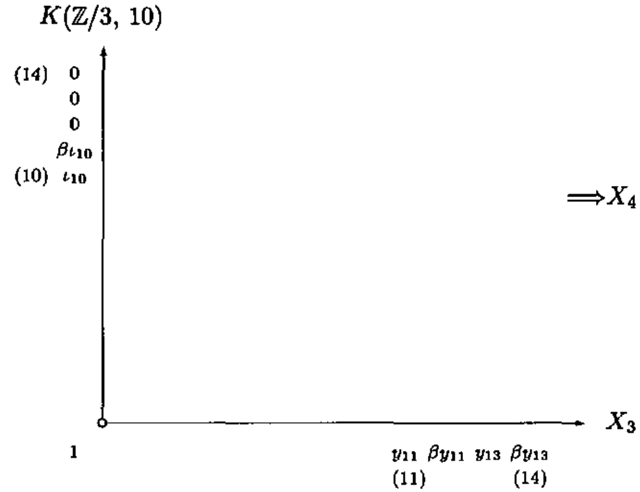
$H^*(K(\mathbb{Z}/3,\, 10);\, \mathbb{Z})$:

10	11	12	13	14
0	$\mathbb{Z}/3$	0	0	0
0	$\beta_3(\iota_{10})$	0	0	0

There is nothing to do for $\mathbb{Z}/p$ coefficients when $p \neq 3$:

$H^*(X_4;\, \mathbb{Z}/p)/(c_2,\, c_3) = H^*(X_3;\, \mathbb{Z}/p)/(c_2,\, c_3)$ for $p \neq 3$.

For $\mathbb{Z}/3$ coefficients, we have



Since $d(\iota_{10}) = y_{11}$, we have $d(\beta\iota_{10}) = \beta y_{11}$.

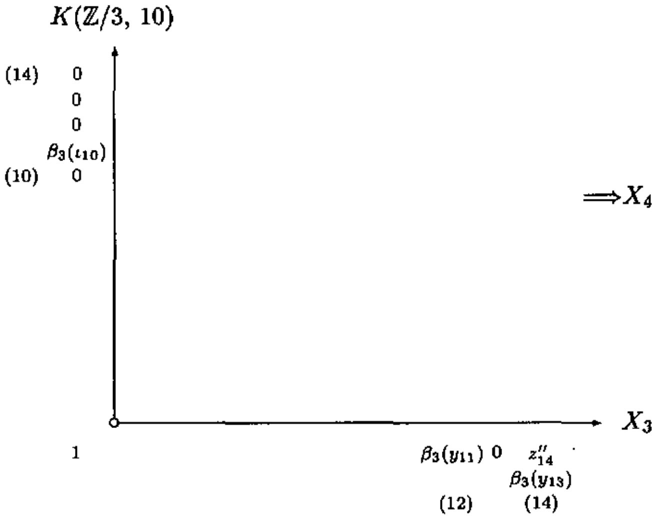
We get *no new generator*.

For the cohomology, we get

$H^*(X_4; \mathbb{Z}/3)/(c_2, c_3)$:

13	14
y_{13}	βy_{13}

Here is the spectral sequence for $\mathbb{Z}$ coefficients:



No new generator is introduced.

For the integral cohomology, we have

$H^*(X_4;\,\mathbb{Z})/(c_2,\,c_3)$:

14
$\mathbb{Z}/8 \oplus \mathbb{Z}/3$
$z''_{14},\ \beta_3(y_{13})$

Again, the map f_3 lifts to a map $f_4 : BSU(3) \longrightarrow X_4$ over π_4, because $BSU(3)$ has no (mod 3) cohomology in dimension 11.

We now want to "kill" the class $\beta_3(y_{13})$. We build the following pull-back:

$$\begin{array}{ccc} K(\mathbb{Z}/3,\,12) & = & K(\mathbb{Z}/3,\,12) \\ {\scriptstyle i_5}\downarrow & & \downarrow \\ X_5 & \longrightarrow & PK(\mathbb{Z}/3,\,13) \\ {\scriptstyle \pi_5}\downarrow & & \downarrow{\scriptstyle \mu} \\ X_4 & \xrightarrow{\beta_3(y_{13})} & K(\mathbb{Z}/3,\,13) \end{array}$$

To compute the needed part of the cohomology of X_5, we use the Lerray-Serre spectral sequence.

We will need a few cohomology groups of $K(\mathbb{Z}/3,\,12)$.

$\widetilde{H}^*(K(\mathbb{Z}/3,\,12);\,\mathbb{Z}/p) = 0$ for $p \neq 3$.

$H^*(K(\mathbb{Z}/3,\,12);\,\mathbb{Z}/3)$:

12	13
ι_{12}	$\beta\iota_{12}$

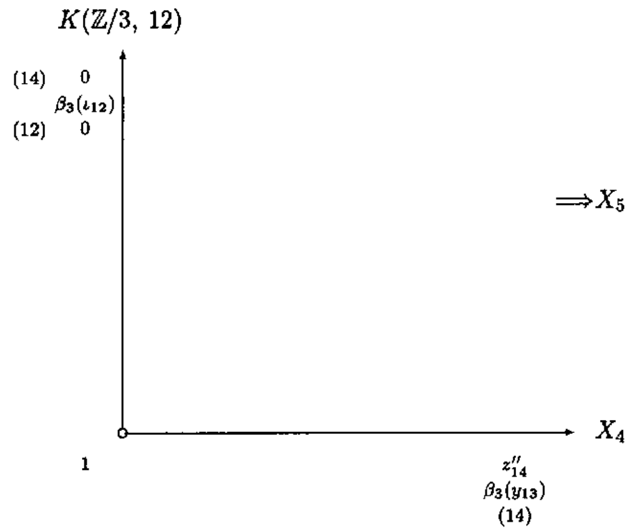
$H^*(K(\mathbb{Z}/3,\,12);\,\mathbb{Z})$:

12	13	14
0	$\mathbb{Z}/3$	0
0	$\beta_3(\iota_{12})$	0

There is nothing to do for $\mathbb{Z}/p$ coefficients when $p \neq 3$:

$H^*(X_5;\,\mathbb{Z}/p)/(c_2,\,c_3) = H^*(X_4;\,\mathbb{Z}/p)/(c_2,\,c_3)$ for $p \neq 3$.

We can from now on just work with the integral coefficients spectral sequence:



It is clear that $d(\beta_3(\iota_{12})) = \beta_3(y_{13})$.

No new generator is introduced.

For the integral cohomology, we have

$H^*(X_5;\, \mathbb{Z})/(c_2,\, c_3)$:

14
$\mathbb{Z}/8$
z''_{14}

Again, the map f_4 lifts to a map $f_5 : BSU(3) \longrightarrow X_5$ over π_5 (because $BSU(3)$ has no (mod 3) cohomology in dimension 13).

We finally "kill" the class z''_{14}, considered as a map $z''_{14} : X_5 \longrightarrow K(\mathbb{Z}/8,\, 13)$. We form the pull-back diagram

$$\begin{array}{ccc} K(\mathbb{Z}/8,\, 12) & = & K(\mathbb{Z}/8,\, 12) \\ {\scriptstyle i_6}\downarrow & & \downarrow \\ X_6 & \longrightarrow & PK(\mathbb{Z}/8,\, 13) \\ {\scriptstyle \pi_6}\downarrow & & \downarrow{\scriptstyle \mu} \\ X_5 & \xrightarrow{\ z''_{14}\ } & K(\mathbb{Z}/8,\, 13) \end{array}$$

We will need a few cohomology groups of $K(\mathbb{Z}/8,\, 12)$. First, we have

$$H^{12}(K(\mathbb{Z}/8,\, 12);\, \mathbb{Z}/4) = \mathbb{Z} \cdot \iota'_{12} \cong \mathbb{Z}/8\,,$$

where ι'_{12} is the fundamental class.

We find the following result:

$H^*(K(\mathbb{Z}/8,\, 12);\, \mathbb{Z}/2)$:

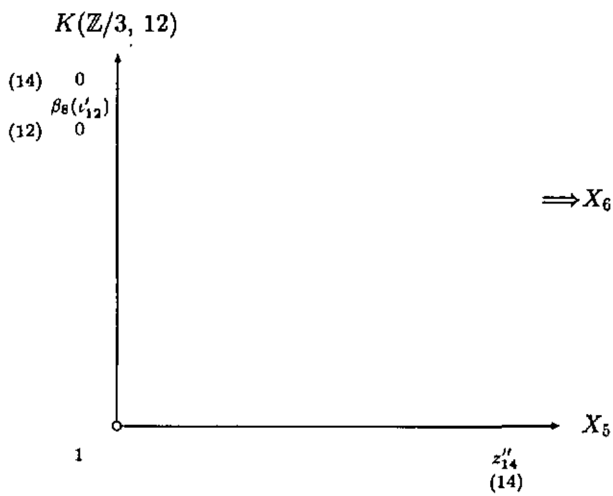
12	13	14
ι_{12}	$\delta_8\iota'_{12}$	$Sq^2\iota_{12}$

$\widetilde{H}^*(K(\mathbb{Z}/8,\, 12);\, \mathbb{Z}/p) = 0$ for $p \geq 3$.

$H^*(K(\mathbb{Z}/8,\, 12);\, \mathbb{Z})$:

12	13	14
0	$\mathbb{Z}/8$	0
0	$\beta_8(\iota'_{12})$	0

Here is the spectral sequence over $\mathbb{Z}$:



It is clear that $d(\beta_8(\iota'_{12})) = z''_{14}$.

No new generator is introduced.

For the integral cohomology, we finally get

$H^*(X_6;\, \mathbb{Z})/(c_2,\, c_3)$:

14
0

Finally, the map f_5 lifts to a map $f_6 : BSU(3) \longrightarrow X_6$ over π_6, because $BSU(3)$ has no (mod 8) cohomology in dimension 13. Moreover, we have the following commutative diagram (still up to homotopy):

$$\begin{array}{ccc} & & X_6 \\ & \overset{f_6}{\nearrow} & \Big\downarrow \pi_1 \circ \ldots \circ \pi_6 \\ BSU(3) & \xrightarrow{\ f\ } & BSU(5) \end{array}$$

From this, we deduce that, in integral cohomology, f_6^* maps c_2 to c_2 and c_3 to c_3, so f_6^* is an isomorphism in degrees ≤ 14. As a consequence of theorem C.2.5, f_6 is a 13-connected map and induces a bijection

$$(f_6)_* : [Y, BSU(3)] \longrightarrow [Y, X_6],$$

for any CW-complex Y of dimension ≤ 12.

Now, let Y and $\xi \in \widetilde{K}(Y)$ be as in the statement of theorem C.1.1. We can consider ξ as a map $Y \longrightarrow BSU$ that lifts to $BSU(6)$. Since $c_6(\xi) = 0$, there is, as a consequence of theorem 1.2.3, a lifting ξ_0 to $BSU(5)$. Since in addition $c_4(\xi) = 0$, ξ_0 lifts to a map ξ_1 to X_1, and the vanishing of c_5, together with the fact that Y has no torsion in H^{10} imply that ξ_1 lifts to a map ξ_2 to X_2. Finally, since Y has no mod 4 and no mod 3 cohomology in dimension 11, and trivial cohomology dimension 13, we get successive liftings ξ_3, ξ_4, ξ_5 and ξ_6 to X_3, X_4, X_5 and X_6 respectively. Now, the map

$$(f_6)_*^{-1} : [Y, X_6] \longrightarrow [Y, BSU(3)]$$

provides us with the desired lifting of ξ to $BSU(3)$.

This completes the proof of theorem C.1.1.

C.4 On the positive cone of $\mathbb{H}P^3$

In this section, after having computed the c-cone of the quaternionic projective space $\mathbb{H}P^3$, as an application of the main theorem, we get precise information on its positive cone.

The projective space $\mathbb{H}P^3$ has a cell structure given by

$$\mathbb{H}P^3 = \mathbb{H}P^2 \cup e^{12} = e^0 \cup e^4 \cup e^8 \cup e^{12},$$

and its integral cohomology ring is $H^*(\mathbb{H}P^3; \mathbb{Z}) = \mathbb{Z}[y]/(y^4)$, where y is in degree 4 (see prop. VI.10.2 in [24]). As a direct consequence of the main theorem C.1.1, we get the

C.4.1 Proposition. *For $\xi \in \widetilde{K}(\mathbb{H}P^3)$, one has*

$$\begin{aligned} c_6(\xi) = 0 \quad &\Longleftrightarrow \quad \text{g}-\dim(\xi) \le 5 \\ c_4(\xi) = c_6(\xi) = 0 \quad &\Longleftrightarrow \quad \text{g}-\dim(\xi) \le 3 \,. \end{aligned}$$

We will soon come back to $\mathbb{H}P^3$, but let us before consider the general quaternionic projective space $\mathbb{H}P^n$. The integral cohomology and K-theory rings are given by

$$H^*(\mathbb{H}P^n, \mathbb{Z}) = \mathbb{Z}[y]/(y^{n+1}) \quad \text{and} \quad K(\mathbb{H}P^n) = \mathbb{Z}[x]/(x^{n+1}) \,,$$

where y is in degree 4, and $x = \zeta - 2$ witb ζ the canonical quaternionic line bundle considered as a complex 2-dimensional bundle (see prop. VI.10.2 in [24], and prop. 4.3.8 in [74]). By [76], pp. 243-244, the total Chern class of ζ is $c(\zeta) = 1 + y$. Of course, y is only determined up to sign. For some consistency with the case of $\mathbb{C}P^2$ and $\mathbb{C}\mathrm{a}P^2$ (that are considered in the chapter 3), we are going to choose y so that $c(\zeta) = 1 - y$. So, the "total Chern polynomial" is $c_{(t)}(\zeta) = 1 - y \cdot t^2$. As in [52] (p. 64), by invoking the splitting principle, one factorizes formally this polynomial as

$$c_{(t)}(\zeta) = (1 + s_1 \cdot t)(1 + s_2 \cdot t) = 1 + (s_1 + s_2) \cdot t + s_1 s_2 \cdot t^2 \,,$$

and we get the two (formal) relations $s_1 + s_2 = 0$ and $s_1 s_2 = -y$. Formula III of theorem 4.4.3 in [52] yields

$$\begin{aligned} c_{(t)}(\zeta^2) &= (1 + 2s_1 \cdot t)(1 + \underbrace{(s_1 + s_2)}_{=0} \cdot t)^2 (1 + 2s_2 \cdot t) \\ &= 1 + 2(s_1 + s_2) \cdot t + 4 s_1 s_2 \cdot t^2 \\ &= 1 - 4y \cdot t^2 \,, \end{aligned}$$

and similarly

$$\begin{aligned} c_{(t)}(\zeta^3) &= (1 + 3s_1 \cdot t)(1 + (s_1 + \underbrace{s_1 + s_2}_{=0}) \cdot t)^3 (1 + (\underbrace{s_1 + s_2}_{=0} + s_2) \cdot t)^3 (1 + 3s_2 \cdot t) \\ &= (1 + 3s_1 \cdot t)(1 + 3s_2 \cdot t)((1 + s_1 \cdot t)(1 + s_2 \cdot t))^3 \\ &= (1 - 9y \cdot t^2)(1 - y \cdot t^2)^3 \\ &= 1 - 12y \cdot t^2 + 30y^2 \cdot t^4 - 28y^3 \cdot t^6 + 9y^4 \cdot t^8 \,. \end{aligned}$$

Therefore, $c(x^2) = c((\zeta - 2)^2) = c(\zeta^2 - 4\zeta + 4) = c(\zeta^2) \cdot c(\zeta)^{-4}$, and one has

$$c(\zeta)^{-4} = \frac{1}{(1 - y)^4} = 1 - \binom{-4}{1} y + \binom{-4}{2} y^2 - + \ldots + (-1)^n \binom{-4}{n} y^n \,.$$

This shows that $c(x) = 1 - y$ and $c(x^2) = 1 + \sum_{j=1}^{n} (-1)^j \left(\binom{-4}{j} + 4\binom{-4}{j-1}\right) y^j$.

Let us quickly consider the case of the complex projective plane $\mathbb{H}P^2$ (i.e. $n = 2$). We find $c(x^2) = 1 - 6y^2$. Now, from the results above and the Newton binomial formula, it is straightforward to compute that

$$c(ax + bx^2) = 1 - ay + \frac{a^2 - a - 12b}{2}\, y^2 .$$

This formula is referred to in section 2.5.

For the projective space $\mathbb{H}P^3$, by the formulas above, one has $c(x) = 1 - y$ and $c(x^2) = 1 - 6y^2 - 20y^3$. From the equality $c(\zeta^3) = 1 - 12y + 30y^2 - 28y^3$, one finds

$$c(x^3) = c((\zeta - 2)^3) = c(\zeta^3)c(\zeta^2)^{-6}c(\zeta)^{12} = 1 - 120y^3 .$$

Let $\xi := ax + bx^2 + lx^3$, where $a, b, l \in \mathbb{Z}$. We have computed that

$$c(\xi) = 1 - ay + \frac{a(a-1) - 12b}{2}\, y^2 - \frac{a^3 - 3a^2 + 2a - 36ab + 120b + 720l}{6}\, y^3 .$$

We see that $c_6(\xi) = 0$ if and only if $a^3 - 3a^2 + 2a - 36ab + 120b + 720l = 0$. On the other hand, $c_4(\xi) = 0$ if and only if $b = a(a-1)/12$, and inserting this condition in the preceding relation yields

$$c_4(\xi) = c_6(\xi) = 0 \quad \Longleftrightarrow \quad b = \frac{a(a-1)}{12} \text{ and } l = \frac{a^3 - 5a^2 + 4a}{360} .$$

If moreover $c_2(\xi) = 0$, we see that $\xi = 0$. Notice that for $a = 0$, $l \neq 0$ and $b = -6l$, $c_6(\xi) = 0$ but $c_4(\xi) \neq 0$, which means that $c{-}\dim(\xi) = 4$. Consequently, theorem 1.2.3 with propositions 1.3.3 and C.4.1 prove the

C.4.2 Theorem. *For the quaternionic projective space $\mathbb{H}P^3$, the K-theory ring is $K(\mathbb{H}P^3) = \mathbb{Z}[x]/(x^4)$, where $x = \zeta - 2$ with ζ the canonical quaternionic line bundle considered as a complex 2-dimensional bundle, the c-cone coincides with the γ-cone and is given, in terms of the c-dimension of $\xi = ax + bx^2 + lx^3$, with $a, b, l \in \mathbb{Z}$, by*

$$c{-}\dim(\xi) = \begin{cases} 0, & \text{if } a = 0,\ b = 0 \text{ and } l = 0 \\ 2, & \text{if } b = \frac{a(a-1)}{12} \neq 0 \text{ and } l = \frac{a^3-5a^2+4a}{360} \\ 4, & \text{if } a^3 - 3a^2 + 2a - 36ab + 120b + 720l = 0 \text{ and } b \neq \frac{a(a-1)}{12} \\ 6, & \text{otherwise} \end{cases}$$

(and all cases occur). Moreover, the positive cone is partially determined by the following information on the geometric dimension function:

$$g{-}\dim(\xi) = \begin{cases} 0, & \text{if } a = 0,\ b = 0 \text{ and } l = 0 \\ 2 \text{ or } 3, & \text{if } b = \frac{a(a-1)}{12} \neq 0 \text{ and } l = \frac{a^3-5a^2+4a}{360} \\ 4 \text{ or } 5, & \text{if } a^3 - 3a^2 + 2a - 36ab + 120b + 720l = 0 \text{ and } b \neq \frac{a(a-1)}{12} \\ 6, & \text{otherwise.} \end{cases}$$

Remark that there is another method than the one used here to compute the total Chern class $c(ax + bx^2 + lx^3)$. It consists in first determining the Chern character $ch(x)$ by means of the formulas relating the Chern character to the Chern classes, and then calculating $ch(ax + bx^2 + lx^3)$ by remembering that ch is a ring homomorphism. The last step is to go back to the Chern classes by means of the formulas we have just alluded to. (See section 2.4 for details.)

Appendix D

Comparison of obstructions for the positive cone

The goal of this appendix is to compare shortly the different "obstructions" (like γ-operations, Chern classes, Whitehead products and the usual primary obstruction from homotopy theory) that are involved in trying to compute the geometric dimension of a given stable class of vector bundles.

The first comparison is between γ-operations and Chern classes. This was the matter of proposition 1.2.2 and of lemma 1.2.5. Theorem 1.2.3 shows that the "top Chern class" is a very efficient tool.

We would now like to compare the Whitehead product and the classical obstructions from homotopy theory. Let us first recall the basics about the "primary obstruction", applied to a particular situation. (We still assume that all spaces and maps are pointed.)

Let $p : E \longrightarrow B$ be a fibration with fiber F. Let us assume that E, B and F are CW-complexes, and that B is simply connected. Consider $f : S^n \longrightarrow Y$ a pointed map, with Y a finite connected CW-complex. Let $X = C_f = Y \cup_f e^{n+1}$ be the mapping cone of f. Given a diagram

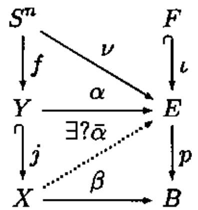
$$\begin{array}{ccc} S^n & & F \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \iota} \\ Y & \xrightarrow{\alpha} & E \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{\beta} & B \end{array}$$

commuting up to homotopy, a necessary and sufficient condition for the existence of a lifting and extension $\bar{\alpha} : X \longrightarrow E$ (that is a map such that $\bar{\alpha} \circ j \simeq \alpha$ and

$p \circ \bar{\alpha} \simeq \beta$) is the vanishing of the primary obstruction

$$\bar{c}^{n+1}_{\alpha,\beta} \in H^{n+1}(X, Y; \pi_n(F)) \cong H^{n+1}(X/Y; \pi_n(F)) \cong \pi_n(F)\,,$$

the latter map is an isomorphism, since X/Y is homeomorphic to the sphere S^{n+1}. For details on this obstruction, see for example thm. VII.14.1 in [24]. Letting $\nu := \alpha \circ f \in \pi_n(E)$, we get the diagram



The following proposition makes the relationship between $\bar{c}^{n+1}_{\alpha,\beta}$ and ν explicit.

D.0.3 Proposition. *In the preceding lifting and extension problem, one has*

$$\nu = \iota_*(\bar{c}^{n+1}_{\alpha,\beta}) \in \pi_n(E)\,,$$

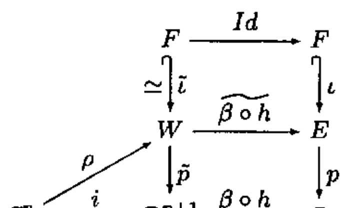
where $\iota_* : \pi_n(F) \longrightarrow \pi_n(E)$ is induced by the inclusion of the fiber.

This proposition illustrates the fact that $\bar{c}^{n+1}_{\alpha,\beta}$ is an obstruction for the whole of the above lifting and extension problem, but that ν is only an obstruction to extending α from Y to X: it is completely independent of β (and of p and B).

Proof. Let D^{n+1} denote the closed $(n+1)$-dimensional unit disk, with boundary S^n. Let $h : D^{n+1} \longrightarrow X$ be the cone on the map f. By definition, $\bar{c}^{n+1}_{\alpha,\beta}$ is constructed as follows: Let

$$W := \{(x, y) \in D^{n+1} \times E \,|\, \beta \circ h(x) = p(y)\}$$

be the pull-back of $(\beta \circ h, p)$. The map $\rho : S^n \longrightarrow W$, $z \longmapsto (z, \alpha \circ h(z))$ fits into the following diagram commuting up to homotopy:



Notice that the map $\tilde{p}$ is a fibration with fiber F, and, since D^{n+1} is contractible, that the inclusion $\tilde{\iota}$ of the fiber is a homotopy equivalence. The map ρ therefore defines an element

$$\rho \in \pi_n(W) \overset{\tilde{\iota}_*}{\cong} \pi_n(F)\,,$$

which is by definition the primary obstruction $\bar{c}^{n+1}_{\alpha,\beta} \in \pi_n(F)$. (Notice that $\bar{c}^{n+1}_{\alpha,\beta}$ really depends on the map β, because so does $\tilde{\iota}$.) Since the above diagram is homotopy-commutative, and since $h|_{S^n} = f : S^n \longrightarrow Y$, one has

$$\iota_*(\iota) = \widetilde{\beta \circ h} \circ \rho : S^n \longrightarrow E\,,\ z \longmapsto \alpha \circ h(z) = \alpha \circ f(z)\,,$$

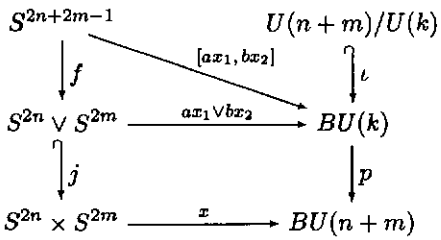
which is precisely the map ν. This completes the proof. □

If we particularize to the case where $X = S^p \times S^q = (S^p \vee S^q) \cup_f e^{p+q}$ and $\alpha = \alpha_1 \vee \alpha_2$, we have $\nu = [\alpha_1, \alpha_2]$, the Whitehead product, and get

$$[\alpha_1, \alpha_2] = i_*(\bar{c}^{n+1}_{\alpha_1 \vee \alpha_2, \beta}) \in \pi_{p+q-1}(E)\,.$$

As in section 1.8, let $n \leq m$ be two positive integers, let $m \leq k < n+m$ and $l \in \mathbb{Z}$. Let x_1 and x_2 be generators of $\pi_{2n}(BU)$ and $\pi_{2m}(BU)$ respectively. Since BU and of $BU(n+m)$ admit CW-decompositions with the same $(2n+2m+1)$-skeleton, i.e. $BU^{[2n+2m+1]} = BU(n+m)^{[2n+2m+1]}$, we can identify $\pi_{2n+2m}(BU(n+m))$ with $\pi_{2n+2m}(BU)$.

Let $x = ax_1 + bx_2 + lx_1x_2 \in \widetilde{K}(S^{2n} \times S^{2m}) = [S^{2n} \times S^{2m}, BU(n+m)]$, where $a, b, l \in \mathbb{Z}$. Let $q : S^{2n} \times S^{2m} \twoheadrightarrow S^{2n} \wedge S^{2m} \cong S^{2n+2m}$ denote the quotient map, and let y be the generator of $\pi_{2n+2m}(BU)$ such that $q^*(y) = x_1x_2$. Consider the diagram



and the following part of the long exact sequence of the fibration p:

$$\pi_{2n+2m}(BU(n+m)) \xrightarrow{\delta} \pi_{2n+2m-1}(U(n+m)/U(k)) \xrightarrow{\iota_*} \pi_{2n+2m-1}(BU(k))\,.$$

Let us denote $\bar{c}^{2n+2m}_{ax_1 \vee bx_2,\, ax_1+bx_2+lx_1x_2}$ simply by $\bar{c}_{a,b,l}(k)$ (with "k" recalling that the obstruction depends on the value of the parameter $k \in \{m, \dots, m+n-1\}$).

D.0.4 Theorem. *Let $1 \leq m \leq n$ and $m \leq k < m+n$. Let x_1, x_2 and y be suitable generators of $\pi_{2n}(BU)$, $\pi_{2m}(BU)$ and $\pi_{2n+2m}(BU)$ respectively. One then has*

$$\iota_*(\bar{c}_{a,b,l}(k)) = [ax_1, bx_2] = ab[x_1, x_2] \in \pi_{2n+2m-1}(BU(k))$$
$$\bar{c}_{a,b,l}(k) - \bar{c}_{a,b,0}(k) \in \mathrm{Ker}(\iota_*) = \mathrm{Im}(\delta) = \mathbb{Z} \cdot \delta(y)\,.$$

The obstruction $\bar{c}_{a,b,l}(k)$ vanishes if and only if $ax_1 + bx_2 + lx_1x_2 \in \widetilde{K}(S^{2n} \times S^{2m})$ has geometric dimension $\leq k$, in other words

$$\bar{c}_{a,b,l}(k) = 0 \iff \text{g-dim}(ax_1 + bx_2 + lx_1x_2) \leq k\,.$$

If $ab \neq 0$, $\bar{c}_{a,b,l}(k)$ only depends on l and on the product ab. For fixed a and b, the Whitehead product $[ax_1, bx_2]$ vanishes in $\pi_{2n+2m-1}(BU(k))$ if and only if there exists a value $l \in \mathbb{Z}$ such that $ax_1 + bx_2 + lx_1x_2 \in \widetilde{K}(S^{2n} \times S^{2m})$ has geometric dimension $\leq k$, in other words

$$\left.\begin{array}{c}[ax_1, bx_2] = 0 \\ \text{in } \pi_{2n+2m-1}(BU(k))\end{array}\right\} \iff \exists l \in \mathbb{Z} \text{ such that } \text{g-dim}(ax_1 + bx_2 + lx_1x_2) \leq k\,.$$

Moreover, under this condition, either $ab = 0$ or ab is a non-zero integral multiple of $(n+m-1)!/((n-1)!\,(m-1)!)$ and then the value of l is unique and given by $l = ab\,(n-1)!\,(m-1)!/(n+m-1)!$. Finally, for for this value of l, one has

$$\bar{c}_{a,b,l}(2n+2m-1) = 0 \in \pi_{2n+2m-1}(S^{2n+2m-1}) \cong \mathbb{Z}\,.$$

Proof. The first formula is a direct consequence of proposition D.0.3 and of the $\mathbb{Z}$-bilinearity of the Whitehead product. It follows that $\bar{c}_{a,b,l}(k) - \bar{c}_{a,b,0}(k)$ is in $\text{Ker}(\iota_*) = \text{Im}(\delta) = \mathbb{Z} \cdot \delta(y)$.

The first equivalence $\bar{c}_{a,b,l}(k) = 0 \iff \text{g-dim}(ax_1 + bx_2 + lx_1x_2) \leq k$ is the fundamental property of the obstruction $\bar{c}_{a,b,l}(k)$.

We have already proved that the geometric dimension of $ax_1 + bx_2 + lx_1x_2$ only depends on l and on the product ab, provided that $ab \neq 0$ (see thm. 1.8.2). (Notice that this result is based on considerations involving K-theory, or more precisely γ-operations (or Chern classes).)

We pass to the equivalence about the vanishing of the Whitehead product $[ax_1, bx_2]$. If $[ax_1, bx_2] = 0$, then there exists an extension $g : S^{2n} \times S^{2m} \longrightarrow BU(k)$ of $ax_1 \vee bx_2$. The composition $p \circ g$ is an element of $\pi_{2n+2m}(BU(n+m))$, which coincides with $\widetilde{K}(S^{2n} \times S^{2m})$. It is therefore of the form $x = ax_1 + bx_2 + lx_1x_2$ for some $l \in \mathbb{Z}$. Consequently, g is not only an extension of $ax_1 \vee bx_2$, but also a lifting of x. Therefore, x has geometric dimension $\leq k$. For the converse, if $\text{g-dim}(x) \leq k$, then, by the first equivalence in the theorem, we have $\bar{c}_{a,b,l}(k) = 0$. The result now follows from proposition D.0.3.

The final statements about the values of ab and l, the uniqueness of l, and the vanishing of $\bar{c}_{a,b,l}(2n+2m-1)$ are immediate consequences of theorems 1.8.1 and 1.8.2.

This completes the proof. □

By lemma 1.4.2, the groups $\pi_{2n+2m-1}(BU(k))$ are finite for $m \leq k < n+m$. It follows from the long exact sequence in homotopy of the fibration p that the map $\delta : \pi_{2n+2m}(BU(n+m)) \longrightarrow \pi_{2n+2m-1}(U(n+m)/U(k))$ is injective, where the former group is $\mathbb{Z} \cdot y \cong \mathbb{Z}$. We therefore have an infinite cyclic subgroup $\mathbb{Z} \cdot \delta(y)$ in $\pi_{2n+2m-1}(U(n+m)/U(k))$. Now, in view of the theorem, we risk the following little conjecture:

D.0.5 Conjecture. *Let $1 \leq m \leq n$ and $m \leq k < m+n$. Let a, b, l be integers. If $[ax_1, bx_2] = 0$ in $\pi_{2n+2m-1}(BU(k))$, then*

$$\bar{c}_{a,b,l}(k) = \bar{c}_{a,b,0}(k) + l \cdot \delta(y) \in \pi_{2n+2m-1}(U(n+m)/U(k)).$$

More precisely, one has

$$\bar{c}_{a,b,l}(k) = \left(l - ab\frac{(n-1)!\,(m-1)!}{(n+m-1)!}\right) \cdot \delta(y) \in \pi_{2n+2m-1}(U(n+m)/U(k)).$$

If $[ax_1, bx_2] \neq 0$ in $\pi_{2n+2m-1}(BU(k))$, then in the group $\pi_{2n+2m-1}(U(n+m)/U(k))$, one has

$$0 \neq \bar{c}_{a,b,l}(k) \notin \mathbb{Z} \cdot \delta(y).$$

Notice that the first statement implies the rest (by virtue of the above theorem).

It would of course be interesting to determine, in the case where $[ax_1, bx_2] \neq 0$, if $\bar{c}_{a,b,l}(k)$ is a torsion element or not, or even to compute its precise order.

Appendix E

A second proof of injectivity of the map β_1^X

We propose a second proof of injectivity of the map $\beta_1^X : H_1(X; \mathbb{Z}) \longrightarrow K_1(X)$ defined in chapter 3, for X any connected CW-complex. The proof is based on the universal coefficient theorem in K-homology and on homology approximations of simply connected CW-complexes. It is of independent interest, in particular because it sheds new light on this result and gives information on the question of injectivity of β_2^X. It also provides with a fifth description of both maps.

E.1 Universal coefficient theorems and homology approximations

In the present proof of injectivity of β_1^X, we need two important tools, namely the universal coefficient theorem for K-homology, and homology approximations of 1-connected CW-complexes. We quickly review these topics in the present section.

The universal coefficient theorem for cohomology, expressing integral cohomology in terms of integral homology, is classical (see for example cor. V.7.2 in [24]). For a *finite* CW-complex X, there is also a (less famous) universal coefficient theorem for homology, stating that for any j, the sequence

$$0 \longrightarrow \mathrm{Ext}(H^{j+1}(X; \mathbb{Z}); \mathbb{Z}) \longrightarrow H_j(X; \mathbb{Z}) \longrightarrow \mathrm{Hom}(H^j(X; \mathbb{Z}); \mathbb{Z}) \longrightarrow 0$$

is exact and natural; it splits, but not naturally (see thm. 5.12 in Spanier [99]). The corresponding results do generally *not* hold for all cohomology theories and corresponding homology theories. However, it turns out that there are such theorems for K-theory, i.e. K-cohomology, and K-homology. As far as we know, it is Zen-Ichi Yosimura [113] who has first established it for K-theory, in 1972. For

K-homology, it was first proved in the framework of the KK-theory of C^*-algebras by Lawrence Brown [26] in 1980. Here are the precise statements.

E.1.1 Theorem. (Universal coefficient theorem for K-theory) *For any connected CW-complex X and $j \in \mathbb{Z}$, there is a short exact sequence*

$$0 \longrightarrow \mathrm{Ext}(K_{j-1}(X); \mathbb{Z}) \longrightarrow K^j(X) \longrightarrow \mathrm{Hom}(K_j(X); \mathbb{Z}) \longrightarrow 0$$

that is natural in X. Moreover, it splits, but not naturally.

(In this theorem, infinite CW-complexes are allowed, and the K-theory is the representable one.)

E.1.2 Theorem. (Universal coefficient theorem for K-homology) *For a connected finite CW-complex X and $j \in \mathbb{Z}$, there is a short exact sequence*

$$0 \longrightarrow \mathrm{Ext}(K^{j+1}(X); \mathbb{Z}) \longrightarrow K_j(X) \longrightarrow \mathrm{Hom}(K^j(X); \mathbb{Z}) \longrightarrow 0$$

that is natural in X. Moreover, it splits, but not naturally.

Notice that in [26], it is stated for X a compact metric space, and for $j = 1$. Since any finite CW-complex X is compact and metrizable, the result holds for X and $j = 1$. The general case follows replacing X by its suspension (and invoking the suspension isomorphism and Bott periodicity).

We pass now to homology approximations. We refer the reader to [51].

E.1.3 Definition. *A 1-connected CW-complex X is normal if it admits a filtration into simply-connected sub-complexes*

$$X_2 \subseteq X_3 \subseteq \ldots \subseteq \bigcup_{n \geq 2} X_n = X\,,$$

such that for each $n \geq 2$, the following holds: First, X_n lies in between the n-skeleton and the $(n+1)$-skeleton of X, i.e. $X^{[n]} \subseteq X_n \subseteq X^{[n+1]}$; second, $H_{>n}(X_n; \mathbb{Z}) = 0$; finally, $(i_n)_* : H_{\leq n}(X_n; \mathbb{Z}) \xrightarrow{\cong} H_{\leq n}(X; \mathbb{Z})$, where $i_n : X_n \hookrightarrow X$ is the inclusion.

We have added to the definition of [51] the condition on the skeletons of X. This is no restriction, as remark 1 on page 57 of [51] explains. The main result in this framework is the following (see thm. 8.2 in [51]).

E.1.4 Theorem. (Eckmann-Hilton [39]) *For any 1-connected CW-complex Y, there exists a normal CW-complex X and a homotopy equivalence $f : X \xrightarrow{\simeq} Y$. The corresponding sub-complex X_n of X is called a homology approximation (or homology decomposition) of Y in degree $\leq n$.*

Notice that by the universal coefficient theorem in cellular cohomology, we also know the cohomology of the X_n's in terms of $H^*(Y;\mathbb{Z}) \overset{f^*}{\cong} H^*(X;\mathbb{Z})$. Omitting the coefficients $\mathbb{Z}$, for any $n \geq 2$, one has

$$H^{\leq n}(X_n) \cong H^{\leq n}(Y), \quad H^{n+1}(X_n) \cong \mathrm{Ext}(H_n(Y);\mathbb{Z}), \quad \text{and} \quad H^{>n+1}(X_n) = 0,$$

where the inverse of the former map is given by $i_n^* \circ f^*$.

These results apply to the suspension of any connected CW-complex.

E.2 Injectivity of β_1^X

We prove the following result:

E.2.1 Theorem. *The map $\beta_1^X : H_1(X;\mathbb{Z}) \longrightarrow K_1(X)$ is injective, for any connected CW-complex X.*

Proof. Let us first assume that X is finite. Let Y be the suspension ΣX of X, and let Z be a homology approximation of Y in degree ≤ 3 (see section E.1). The CW-complex Z is finite, of dimension ≤ 4, and equipped with a map $f : Z \longrightarrow Y$ inducing an isomorphism in integral homology in degree ≤ 3. We will implicitly use the suspension isomorphism in (co)homology and in K-(co)homology. All the coefficients in (co)homology being $\mathbb{Z}$ in the sequel, we omit them. By the results of section E.1, and the fact that Z is simply-connected, we have isomorphisms

$$H_1(Z) = 0, \quad H_2(Z) \cong H_1(X), \quad H_3(Z) \cong H_2(X),$$

and also

$$H^1(Z) = 0, \; H^2(Z) \cong H^1(X), \; H^3(Z) \cong H^2(X), \text{ and } \; H^4(Z) \cong \mathrm{Ext}(H_2(X);\mathbb{Z})$$

(all the other (co)homology groups of Z vanish). By proposition 3.2.4, we have isomorphisms

$$\widetilde{K}_0(Z) \cong H_2(Z) \;\text{ and }\; K_1(Z) \cong H_3(Z),$$

and by proposition 3.4.2 and corollary 3.4.6, there are bijections

$$\widetilde{K}^0(Z) \approx H^2(Z) \oplus SK^0(Z) \;\text{ and }\; K^1(Z) \cong SK^1(Z) \cong H^3(Z),$$

the latter two being group isomorphisms.

The strategy of the proof is the following. We show that the composition

$$\gamma_1^X : H_1(X) \xrightarrow{\cong} H_2(Z) \xrightarrow{\cong} \widetilde{K}_0(Z) \xrightarrow{f_*} \widetilde{K}_0(Y) \xrightarrow{\cong} K_1(X)$$

coincides with β_1^X, and that f_* is injective.

As a first step, we want to prove that $f_* = K_0(f) : K_0(Z) \longrightarrow K_0(Y)$ is injective. By propositions 3.4.2, Y being simply-connected, one has $K^1(Y) \cong SK^1(Y)$. We claim that the map $f^* = K^1(f) : K^1(Y) \longrightarrow K^1(Z)$ is surjective. To prove this, recall that

$$H^*(SU) \cong \Lambda_{\mathbb{Z}}(x_3, x_5, \dots),$$

with x_{2j+1} of degree $2j+1$, considered as a map $x_{2j+1} : SU \longrightarrow K(\mathbb{Z}, 2j+1)$. We also write x_{2j+1} for the composition, on the right, with this map. There is a commutative diagram

$$\begin{array}{ccccccc}
K^1(Y) & \xrightarrow{\cong} & [Y, SU] & \xrightarrow{x_3} & [Y, K(\mathbb{Z}, 3)] & \xrightarrow{\cong} & H^3(Y) \\
{\scriptstyle K^1(f)}\downarrow & & & & & & \cong\downarrow{\scriptstyle f^*} \\
K^1(Z) & \xrightarrow{\cong} & [Z, SU] & \underset{\cong}{\xrightarrow{x_3}} & [Z, K(\mathbb{Z}, 3)] & \xrightarrow{\cong} & H^3(Z)
\end{array}$$

Under the homotopy equivalence $\Omega SU \simeq BU$ given by Bott periodicity, x_3 corresponds to the first Chern class $c_1 : BU \longrightarrow K(\mathbb{Z}, 2)$. So, from proposition 3.4.2, we get $[Y, SU] \cong [X, \Omega SU] \cong [X, BU] \approx H^2(X) \oplus SK^0(X)$. From corollary 3.4.6 and the isomorphisms $[Y, K(\mathbb{Z}, 3)] \cong [X, \Omega K(\mathbb{Z}, 3)] \cong [X, K(\mathbb{Z}, 2)] \cong H^2(X)$, we obtain the commutative diagram

$$\begin{array}{ccccc}
[Y, SU] & \xrightarrow{\cong} & [X, BU] & \xrightarrow{\approx} & H^2(X) \oplus SK^0(X) \\
{\scriptstyle x_3}\downarrow & & {\scriptstyle c_1}\downarrow & & \downarrow{\scriptstyle p_1} \\
[Y, K(\mathbb{Z}, 3)] & \xrightarrow{\cong} & [X, K(\mathbb{Z}, 2)] & \xrightarrow{\cong} & H^2(X)
\end{array}$$

(We have just proved the well-known fact that the first Chern class is a surjective map.) This shows that x_3 is surjective, and the first diagram implies the surjectivity of $K_1(f)$.

Now, consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & & & \widetilde{K}^0(Y) & \xrightarrow{c_1} & H^2(Y) & \longrightarrow & 0 \\
 & & & & {\scriptstyle \widetilde{K}^0(f)}\downarrow & & {\scriptstyle f^*}\downarrow\cong & & \\
0 & \longrightarrow & \mathrm{Ext}(H_2(X); \mathbb{Z}) & \longrightarrow & \widetilde{K}^0(Z) & \xrightarrow{c_1} & H^2(Z) & \longrightarrow & 0 \\
 & & & & {\scriptstyle \varphi}\downarrow & & & & \\
 & & & & \mathbb{Z} & & & &
\end{array}$$

where the second row comes from the identification $H^4(Z) \cong \mathrm{Ext}(H_2(X); \mathbb{Z})$ and proposition 3.4.4; φ is any homomorphism. Since $\mathrm{Hom}(\mathrm{Ext}(H_2(X); \mathbb{Z}); \mathbb{Z}) = 0$, it follows that if φ is non-zero, then so is $\varphi \circ \widetilde{K}^0(f)$. This proves that the map

$$\mathrm{Hom}(\widetilde{K}^0(f); \mathbb{Z}) : \mathrm{Hom}(\widetilde{K}^0(Z); \mathbb{Z}) \longrightarrow \mathrm{Hom}(\widetilde{K}^0(Y); \mathbb{Z}), \ \varphi \longmapsto \varphi \circ \widetilde{K}^0(f)$$

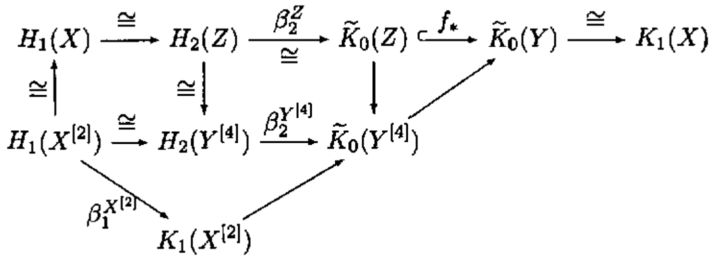
is injective, and similarly for $\mathrm{Hom}(K^0(f);\,\mathbb{Z})$.

By the universal coefficient theorem E.1.2, one has a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}(K^1(Z);\,\mathbb{Z}) & \longrightarrow & K_0(Z) & \longrightarrow & \mathrm{Hom}(K^0(Z);\,\mathbb{Z}) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \mathrm{Ext}(K^1(f);\,\mathbb{Z})} & & \Big\downarrow{\scriptstyle K_0(f)} & & \Big\downarrow{\scriptstyle \mathrm{Hom}(K^0(f);\,\mathbb{Z})} & & \\
0 & \longrightarrow & \mathrm{Ext}(K^1(Y);\,\mathbb{Z}) & \longrightarrow & K_0(Y) & \longrightarrow & \mathrm{Hom}(K^0(Y);\,\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

Since $K^1(f)$ is surjective, $\mathrm{Ext}(K^1(f);\,\mathbb{Z})$ is injective, and by the five-lemma, $K_0(f)$ is also injective.

The conclude the proof, we show that $\gamma_1^X = \beta_1^X$ (for X not necessarily finite). The diagram



commutes. For the left-hand square, this follows from the equality $Y^{[4]} = \Sigma(X^{[3]})$ and the inclusion $Z \subseteq Y^{[4]}$, for the middle square, from the naturality of β_2, and for the bottom triangle, from the construction of β_1 and β_2 and the fact that the Atiyah-Hirzebruch spectral sequence "commutes with suspensions".

Since γ_1^X is given by the first row, and β_1^X by the composition down through $K_1(X^{[2]})$, both maps coincide. This completes the proof for X finite.

The general case follows by the same direct limit argument as in the proof given in section 3.7. □

Remark that if we keep notations as in the above proof and define

$$\gamma_2^X : H_2(X) \xrightarrow{\cong} H_3(Z) \xrightarrow{\cong} K_1(Z) \xrightarrow{f_*} K_1(Y) \xrightarrow{\cong} \widetilde{K}_0(X),$$

then one checks, as for γ_1^X and β_1^X, that γ_2^X coincides with β_2^X. This is valid for any connected CW-complex X (not necessarily finite). This provides with a fifth construction of β_j^X for $j = 1$ and 2. Moreover, by the same procedure as in the proof, one verifies that, for X finite, β_2^X is injective if so is the map

$$\mathrm{Ext}(\widetilde{K}^0(f);\,\mathbb{Z}) : \mathrm{Ext}(\widetilde{K}^0(Z);\,\mathbb{Z}) \longrightarrow \mathrm{Ext}(\widetilde{K}^0(Y);\,\mathbb{Z}).$$

This holds for example if $\widetilde{K}^0(f)$ is surjective.

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