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Logical Dependence and Independence in the *Tractatus*

Fabrice Correia

1. Introduction

A central Tractarian thesis is that of the logical independence of the elementary propositions, the atoms. The aim of this paper is threefold. First, I will make precise what the thesis is, and show that Wittgenstein endorses it (section 1). Second, I will identify a concept of logical dependence in terms of which logical independence can be defined (section 2). And finally, I will show that these notions of logical dependence and independence fail to capture certain explanatory links between propositions, and I will then try to refine the previous definitions in order to end up with concepts which do capture such links (section 3). Part of these matters have been dealt with by (Simons, 1981). But as I will show, Simons' investigation is mistaken on independence, and—as a consequence—incomplete on dependence.

2. Independence

The aim is here to make precise sense of the claim that some given propositions are *logically independent*, and to show that Wittgenstein held that the atoms are independent in this sense.

Some preliminary definitions will be useful. Where S is a set of propositions, we define a *state–function* on S as a function taking each element of S into itself or its negation; and we shall say that a set of propositions Σ is a *state–description*[1] on S iff it is the image of S by some state–function on S, i.e. iff there is a state–function ψ on S such that $\Sigma=\{\psi(p):p\in S\}$. We shall use $\Sigma \nabla S$ to mean that Σ is a state–description on S. Note that by definition, there is one and only one state–description on $\varnothing$, namely $\varnothing$ itself. We take as primitive the notions of logical possibility ($\Diamond$) and logical necessity ([]), and we assume they are interdefinable in the usual way. Where S is a set of propositions, T(S) is taken to mean that all the members of S are true (so that in particular, it is trivially the case that $T(\varnothing)$); and where p is a proposition, T(p) will be short for T({p}).

With all this in hands, we define the following notions of independence (S is any set of propositions, and n any integer):[2]

D1 the members of S are independent iff_{df} $\forall\Sigma \nabla S \ \Diamond T(\Sigma)$;

D2 the members of S are n–independent iff_{df} $\forall T \subseteq S$ $(\# T = n \Rightarrow$ the members of T are

[1]The phrase is Carnap's, but my definition differs from his.

[2]For reasons of formal homogeneity, we shall admit that S may be empty or contain only one element—though of course when we say that some things are independent in some given respect, we have in mind a collection of at least *two* things. For the same reason, we do not exclude n=0 or n=1, though 0–independence and 1–independence are somehow uninteresting cases. The members of any set of propositions are 0–independent in any case, and the 1–independence of some propositions means that each of them is contingent: each can be true and can be false.

independent);

D3 the members of S are finitely independent iff$_{df}$ for every integer n, the members of S are n–independent.

It is easy to see that m–independence entails n–independence if $m \geq n$. In case $m < n$, we may have m–independence without n–independence. Assume for instance that p, q and r are independent propositions. Then p, q and $r \vee (\neg p \wedge \neg q)$ are 2–independent, but not 3–independent.

Independence entails finite independence, but the converse does not hold unless certain conditions on the propositions concerned are imposed. For e.g. assume that we have an infinite number of names a_0, a_1, ..., a_i, ..., and consider the set S constituted by (1) the proposition 'there is a finite number of ϕs', (2) for each integer i, the proposition 'a_i is ϕ', and (3) for all integers i andj such that $i \neq j$, the proposition '$a_i \neq a_j$'. Then the members of S are finitely independent, but not independent *tout court*. Among the conditions on a set S of propositions which make the finite independence of its members entail their independence is finiteness. Another one which will be of interest to us is that all the members of S be atomic.[3]

For any integer n, the number of distributions of truth–values over any given n distinct atomic propositions is the number of state–descriptions drawn from these propositions, namely 2^n. Thus, saying that the number of logically possible distributions of truth–values over them is 2^n is equivalent to saying that every distribution of truth–values over them is logically possible—which by our definition means that these propositions are independent. In the *Tractatus*, Wittgenstein claims that the number of logically possible distributions of truth–values among any n distinct elementary propositions is 2^n:

> With regard to the existence of n atomic facts there are $K_n = \sum_{\nu=0}^{n} \binom{n}{\nu}$ possibilities.
>
> It is possible for all combinations of atomic facts to exist; and the others not to exist (4.27).
>
> To these combinations correspond the same number of possibilities of the truth—and falsehood—of n elementary propositions (4.28).

Therefore, Wittgenstein endorses the claim that any n distinct atoms are logically independent, and thus that the atoms are finitely independent. So given that for the atoms finite independence is equivalent to independence, Wittgenstein holds that the atoms are independent *tout court*.

As a conclusion to this section, let me note that (Simons, 1981) attributes to Wittgenstein the thesis that the atoms are pairwise independent, that is, 2–independent, and claims that this is *the* logical independence thesis of the*Tractatus*. According to the previous discussion, thus, Simons is right on the first claim, not on the second. For Wittgenstein not only accepts that the atoms are pairwise independent: as we just saw, he endorses the (much) stronger claim that the atoms are independent *tout court*.

3. Dependence

The question we are facing now is that of identifying a concept of logical dependence, in terms of which logical independence may be defined. The idea will be that some given propositions are independent iff

[3]The fact that finite independence does not entails independence is qualified by logicians as a "non–compactness" fact. First order logic is compact: for sets of first–order propositions (in particular, for sets of atomic sentences), finite independence entails independence *tout court*. In our example, it is the quantifier 'there are finitely many'—which is not first–order definable—which creates problems. It is not clear to me whether Wittgenstein accepts such quantifiers in his "meaningful" language.

none of them depends on the others. Logical dependence will be defined in terms of some concept of partial logical determination. My approach is quite different from that of (Simons, 1981), and in an important sense more complete: for Simons only deals with pairwise independence and related concepts of 1–1 dependence.

Thus, let us first define *logical determination*. Propositions p_1, p_2, ... will be said to logically determine proposition q in case each truth–value assignment to p_1, p_2, ... "fixes" the truth–value of q. This idea can be made more precise as follows (as before S is any set of propositions, and p is any proposition):

D4 the members of S determine p $\text{iff}_{\text{df}}\ \forall\, \Sigma\ \nabla\ S$, not both $\Diamond\ T(\Sigma \cup \{p\})$ and $\Diamond\ T(\Sigma \cup \{\neg\, p\})$.

In case $S=\varnothing$, 'the members of S determine p' means that p is necessarily true or necessarily false.

There are certainly many ways *partial* logical determination could then be defined. Here is one which I shall adopt:

D5 the members of S partly determine p $\text{iff}_{\text{df}}\ \exists \Sigma\ \nabla\ S$, not both $\Diamond\ T(\Sigma \cup \{p\})$ and $\Diamond\ T(\Sigma \cup \{\neg\, p\})$.

That is, propositions p_1, p_2, ... partly determine proposition q in case some truth–value assignments to p_1, p_2, ... logically fix the truth–value of q. Of course, determination is stronger than partial determination, and there may be cases of partial determination without determination *tout court*: for p and q independent, $p \wedge q$ only *partly* determine p.

In some use of the verb 'to depend', something depends of some other things iff the latter at least partly determine the former (suppose for instance I decide to go to a party with Manu and plan to spend the whole night there if she does, without planning anything for the case she does not; then, my decision about my staying the whole night or not is partly determined by her decision on the same topic; and we can say, for that reason, that my decision depends on her's). *Logical dependence* is then defined as the converse of partial logical determination:

D6 p depends on the members of S iff_{df} the members of S partly determine p,

and we naturally put:

D7 p is independent from the members of S iff_{df} p does not depend on the members of S.

One can then verify that for every set of propositions S, the members of S are independent iff every member of S is independent from the others. In case $S \neq \varnothing$, this is also equivalent to: some member of S is independent from the others.[1] We thus have what we were looking for.

4. Refinements

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[1](1) *Proof that the members of S are independent $\Rightarrow$ each member of S is independent from the others*. Suppose some member q of S is dependent on S-{q}. This means that some state–description Σ on S-{q} is such that not–$\Diamond\ T(\Sigma \cup \{q\})$ or not–$\Diamond\ T(\Sigma \cup \{\neg\, q\})$. Suppose not–$\Diamond\ T(\Sigma \cup \{q\})$, and let Σ' be $\Sigma \cup \{q\}$ (the case where not–$\Diamond\ T(\Sigma \cup \{\neg\, q\})$ leads to the same conclusion). Since Σ' is a state–description on S, this means that the members of S are not independent. (2) *Proof that some member of S is independent from the others $\Rightarrow$ the members of S are independent*. Suppose that the members of S are not independent, i.e. that some state–description Σ on S is such that not–$\Diamond\ T(\Sigma)$. Then in particular, $S \neq \varnothing$. Let q be in S, and let Σ' be any state–description on S-{q} such that $\Sigma' \subseteq \Sigma$ (there are such state–descriptions). Then either $\Sigma=\Sigma' \cup \{q\}$ or $\Sigma=\Sigma' \cup \{\neg\, q\}$. In any case, we see that q is not independent from S-{q}.

Perhaps the above notions of independence, dependence and determination capture important logical relationships between propositions, and in particular some relationships Wittgenstein himself took to be important. But it is my view that they fail to capture some other interesting logical ties, which we would actually also express by using the *vocabulary* of dependence and determination.

In one sense of the verb "to determine", by 'the truth–value of p determines the truth–value of q', we intend to express two things: first that once the truth–value of p is given, there is no room for the truth–value of q to vary; and second that whichever truth–value q may have, it has it in virtue of the fact that p has the truth–value it has. Call the first aspect the *fixing component* of determination, and the second its *explanation component*. Logical determination as defined above captures the fixing component of determination. But it is easy to see that it does not capture its explanation component. In fact, consider the following consequences of the above definition of determination (p and q are any propositions):

- p determines $q \vee \neg q$;
- p determines $q \wedge \neg q$;
- $p \wedge p$ determines p;
- $\neg\neg p$ determines p.

Where q is any proposition, it should be clear that it is not the case that for every proposition p, $q \vee \neg q$ has the truth–value it has in virtue of the fact that p has the truth–value it has; some would even be inclined to say that the truth of $q \vee \neg q$ is not to be explained by the truth of any proposition whatsoever. Similar considerations hold of the case of $q \wedge \neg q$. All the same, we do not want to say that proposition p is, say, true in virtue of the fact that $\neg\neg p$ or $p \wedge p$ is true; in fact, we are inclined to say the opposite, viz. that $\neg\neg p$ and $p \wedge p$, if true, are true because p is.

These considerations call for defining a concept of logical determination with both a fixing and an explanation component. There are plausibly several ways to do so. The one I shall sketch here is in terms of a primitive explanation predicate $\blacklozenge$: where S is a set of propositions and p a proposition, $S \blacklozenge p$ expresses that p is true in virtue of the fact that (some of) the members of S are all true. We understand $\blacklozenge$ so that it satisfies the following axioms:

A1 $[] (S \blacklozenge p \Rightarrow T(S) \text{ and } T(p))$;

A2 $\Diamond (S \blacklozenge p) \Rightarrow [](T(S) \Rightarrow S \blacklozenge p)$.

We then define the new concept of determination, *determination'*, by stating that the members of S determine' q iff in any logically possible circumstance, q has the truth–value it has in virtue of the fact that (some of) the members of S have the truth–values they have. This may be formally rendered as follows:

D8 the members of S determine' q iff_{df} $[] ((T(q) \Rightarrow \exists\Sigma \nabla S \;\; \Sigma \blacklozenge q) \text{ and } (T(\neg q) \Rightarrow \exists\Sigma \nabla S \;\; \Sigma \blacklozenge \neg q))$.

It is easy to check that:[1]

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[1] Using the fact that for every proposition q, $[](T(q) \text{ or } T(\neg q))$ and $[](T(q) \Rightarrow \text{not–}T(\neg q))$, we can prove by basic modal reasoning that the members of S determine' q iff $[] (\exists\Sigma \nabla S (\Sigma \blacklozenge q \text{ or } \Sigma \blacklozenge \neg q))$. Now, using the fact that for every set of propositions S, $[](\exists!\Sigma \nabla S \; T(\Sigma))$ and axiom (A2), we can prove that $[] (\exists\Sigma \nabla S (\Sigma \blacklozenge q \text{ or } \Sigma \blacklozenge \neg q))$ iff $\forall \Sigma \nabla S \;\; [](T(\Sigma) \Rightarrow \Sigma \blacklozenge q \text{ or } \Sigma \blacklozenge \neg q)$.

T1 the members of S determine' q iff $\forall\Sigma \nabla S [](T(\Sigma) \Rightarrow \Sigma \blacklozenge q \text{ or } \Sigma \blacklozenge \neg q)$.

One can then prove that determination' entails determination.[2]

Of course, the new concept of determination escapes the problems met by the old one. We cannot prove that any tautology or contradiction is logically determined' by any proposition whatsoever. And moreover, under the assumption that it is logically impossible for a tautology or a contradiction to owe its truth–value to the fact that these or those propositions have such and such truth–values, one can prove that tautologies and contradictions are logically undetermined'. All the same, we cannot prove that $p \wedge p$ determines p and $\neg\neg p$ determines p for any arbitrary proposition p. Actually, under the plausible assumption that proposition p never owes its truth–value to that of $p \wedge p$ or $\neg\neg p$, it follows that p is logically determined' neither by $p \wedge p$ nor by $\neg\neg p$.

Partial determination' can be defined by stating that the members of S partly determine' q iff in some logically possible circumstance, q has the truth–value it has in virtue of the fact that (some of) the members of S have the truth–values they have. The formal rendering is then:

D9 the members of S partly determine' q iff$_{df}$ $\lozenge ((T(q) \Rightarrow \exists\Sigma \nabla S\ \Sigma \blacklozenge q) \text{ and } (T(\neg q) \Rightarrow \exists\Sigma \nabla S\ \Sigma \blacklozenge \neg q))$.

It is easy to check that:

T2 the members of S partly determine' q $\Rightarrow \exists\Sigma \nabla S [](T(\Sigma) \Rightarrow \Sigma \blacklozenge q \text{ or } \Sigma \blacklozenge \neg q)$.

One can then also prove that partial determination' entails partial determination.

Finally, we may define dependence', the independence' of a proposition from the members of a set of propositions and the independence' of the members of a set of propositions in terms of partial determination', in the same way as before. Independence' will then turn out to be weaker than independence; and so, admitting that the atoms are independent will commit one to the view that they are independent' as well.

References

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[2]Suppose that $[](T(\Sigma) \Rightarrow \Sigma \blacklozenge q \text{ or } \Sigma \blacklozenge \neg q)$. Then either $\lozenge T(\Sigma)$ or not. If not, then trivially it is not the case that both $\lozenge T(\Sigma\cup\{q\})$ and $\lozenge T(\Sigma\cup\{\neg q\})$. Suppose now that $\lozenge T(\Sigma)$. Then $\lozenge (\Sigma \blacklozenge q \text{ or } \Sigma \blacklozenge \neg q)$, and so $\lozenge (\Sigma \blacklozenge q)$ or $\lozenge (\Sigma \blacklozenge \neg q)$. So from axioms (A1) and (A2), $[](T(\Sigma) \Rightarrow T(q))$ or $[](T(\Sigma) \Rightarrow T(\neg q))$, that is to say, it is not the case that both $\lozenge T(\Sigma\cup\{q\})$ and $\lozenge T(\Sigma\cup\{\neg q\})$.