K-THEORY AND K-HOMOLOGY FOR SEMI-DIRECT PRODUCTS OF $\mathbb{Z}^2$ BY $\mathbb{Z}$

THÈSE

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Résumé

Dans ce travail, étant donné un produit semi-direct de $\mathbb{Z}^2$ par $\mathbb{Z}$, nous étudions d'une part les groupes de K-théorie de sa C^*-algèbre associée et d'autre part les groupes de K-homologie géométrique de son espace classifiant. Plus concrètement, nous déterminons ces groupes et, dans certains cas, des générateurs explicites en fonction des coefficients de la matrice entière décrivant le produit semi-direct.

Pour cela, nous utilisons la suite exacte de Pimsner et Voiculescu en K-théorie et, pour la partie concernant la K-homologie, nous démontrons l'existence d'une suite exacte à six termes associée à un tore d'application.

Mots clefs : K-théorie, K-homologie, produit semi-direct, tore d'application.

Keywords : K-theory, K-homology, semi-direct product, mapping torus.

Contents

Introduction

Historical account

The starting point of K-theory was given by Alexander Grothendieck in the 1950's in order to solve difficult problems in algebraic geometry. More precisely, he introduces this notion to formulate the Grothendieck-Riemann-Roch theorem on smooth and quasi-projective scheme over a field. The letter " K " comes from the word " Klasse " in german (the mother tongue of Grothendieck) which means " class ". This idea quickly invaded other areas of mathematics, as number theory, topology and functional analysis.

In 1959, Jean-Pierre Serre initiated the algebraic K-theory (or K-theory for rings) by applying Grothendieck's construction to finitely generated projective modules over a ring. This enabled him to prove a weak form of his conjecture, which states that every finitely generated projective module over a polynomial ring is free.

At the same time, Michael Atiyah and Friedrich Hirzebruch uses Grothendieck's idea for complex vector bundles over a topological space. This topological K-theory had a central role in the famous index theorem from Atiyah and Singer. Moreover, it led to a K-theory for C^*-algebras, discussed in this thesis, which is of main importance in the subject called " noncommutative geometry " by Alain Connes, in particular in the celebrated conjecture he made with Paul Baum.

The Baum-Connes conjecture arises from work of Gennadi Kasparov and Alexander Mishchenko on the Novikov higher signature conjecture, ideas of Alain Connes in foliation theory and Paul Baum's geometric description of K-homology theory. The original conjecture was about Lie groups acting on smooth manifolds and can be found in [4].

An ancestor of the Baum-Connes conjecture is the Connes-Kasparov conjecture, asserting that the additive group of the representation ring of the maximal compact subgroup in a connected Lie group is isomorphic, via a definite map, to the K-theory of the reduced C^*-algebra of the Lie group. Kasparov showed later that those two conjectures are equivalent for connected Lie groups.

Substantially, the Baum-Connes conjecture states that, given a locally compact group G, the G-equivariant K-homology of the universal G-space for proper actions of G is isomorphic, via an index map, to the K-theory of the reduced C^*-algebra of G. The former group is often called the topological or geometrical side, while the latter is referred to as the analytical side of the conjecture.

Due to the huge consequences a proof of this conjecture will have in various fields of mathematics, it has become one of the most famous unresolved problem over the past thirty years and led to develop in particular the operator algebra and geometric group theories.

The conjecture is true, and essentially trivial, for compact groups. The first non-trivial cases of the conjecture were settled by Pimsner and Voiculescu, who investigated free and free abelian groups, as well as by Connes and Kasparov, who independently considered simply connected solvable groups. These results actually preceded the current formulation of the conjecture.

As other breakthroughs in this domain, we also mention that the Baum-Connes conjecture is true for amenable or Gromov hyperbolic groups, as well as for groups satisfying the Haagerup property.

With the work of Kasparov on the index of invariant elliptic differential operators and the creation of a bivariant KK-theory, the conjecture took quickly another form, presented in [5]. This bivariant functor encompasses both K-theory for C^*-algebras and K-homology, which is actually the theory that is dual to K-theory. The key idea is that any element of $KK(A,B)$, where A and B are C^*-algebras, gives raise to a map between the K-theory groups of A and B in a natural way. Moreover, a " composition product " is expected from $KK(A,B)\times KK(B,C)$ to $KK(A,C)$, corresponding to the compositions of the induced maps in K-theory.

So the question that naturally comes here is : why not stick to the set of homomorphism from the K-theory group of A to the one of B ? An answer can be that we want elliptic differential operators to give maps between K-theory groups. However, the way to get this necessarily goes through a " composition formula " at the level of operators, and hence through KK-theory.

As we said before, the Baum-Connes conjecture was originally set in a different framework, not involving KK-theory, but the geometric description of K-homology, defined by Baum and Douglas in [6]. The advantage of this way of thinking is that the elements forming this theory are more concrete and easier to understand.

Indeed, a geometric K-cycle over a locally compact space is composed of a compact manifold, a complex vector bundle over it and a map linking the manifold to the space. Moreover, the technical aspect of KK-theory can be avoided in several fundamental results. For example, the connecting homomorphisms of the long exact sequence of a pair are simply given by restricting a cycle to the boundary of the manifold.

However, the equivalence relation defining the geometric K-homology groups seems to be more complicated to deal with than those defining KK-theory. In particular, computing the geometric K-homology of a point is not as easy as one could expect. Moreover, we still don't know if applying a sequence of equivalence relations can be reduced in two or three steps. This is a major obstruction in proving that geometric K-homology is actually a homology theory, without appealing to KK-theory, nor to the duality with K-theory.

Despite those facts, this is the way choosen in this thesis to think about K-homology.

Presentation of the results

At the origin of this thesis, Alain Valette, my advisor, wanted me to make explicit the Baum-Connes conjecture for the Heisenberg group, for which we know the conjecture true. More precisely, since this group can be described as a semi-direct product of $\mathbb{Z}^2$ by $\mathbb{Z}$, it has an associated six-terms exact sequence in K-theory, by a result of Pimsner and Voiculescu, and, dually, another one in K-homology. The goal was to find concrete generators for all the groups involved, and to show, by explicit computations, that the Baum-Connes assembly maps intertwine the two sequences.

Thus, the first step was to study the K-theory groups of the C^*-algebra of the Heisenberg group. For this, the paper [2] of Anderson and Paschke has been a great help for me. However, I was quickly led to slightly change their generators of the K-theory groups for generators better adapted to my computations.

Doing this, it became possible to generalize my computations to any semi-direct product of $\mathbb{Z}^2$ by $\mathbb{Z}$. More precisely, given an automorphism of $\mathbb{Z}^2$ described by a matrix $\alpha \in GL_2(\mathbb{Z})$, I computed the K-theory groups of the C^*-algebra associated to $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$, with concrete generators given as functions of the coefficients of α.

The following summarizes the computations in chapter 3.

Theorem. *Let α be an automorphism of $\mathbb{Z}^2$ and G be the associated semi-direct product $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$. Let us assume that $\alpha \neq \mathrm{id}_{\mathbb{Z}^2}$.*

1) *If the determinant of α is 1, then the K-theory groups of the C^*-algebra of G are given in the following table :*

	$\mathrm{tr}(\alpha) = 2$	$\mathrm{tr}(\alpha) = 1, 3$	$\mathrm{tr}(\alpha) \neq 1, 2, 3$
$K_0(C^*(G))$	$\mathbb{Z}^3$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$K_1(C^*(G))$	$\mathbb{Z}^3 \oplus \mathbb{Z}_h$	$\mathbb{Z}^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_{h_1} \oplus \mathbb{Z}_{h_2}$

for explicit $h, h_1, h_2 \in \mathbb{N}^*$.

2) *If the determinant of α is -1, then the K-theory groups of the C^*-algebra of G are given in the following table :*

	$\mathrm{tr}(\alpha) = 0$	$\mathrm{tr}(\alpha) = -1, 1$	$\mathrm{tr}(\alpha) \neq -1, 0, 1$
$K_0(C^*(G))$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$
$K_1(C^*(G))$	$\mathbb{Z}^2 \oplus \mathbb{Z}_h$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_{h_1} \oplus \mathbb{Z}_{h_2}$

for explicit $h, h_1, h_2 \in \mathbb{N}^*$.

Note that the Heisenberg group falls under the case $\det(\alpha) = 1$ and $\operatorname{tr}(\alpha) = 2$, with $h = 1$.

The second step was to compute the K-homology of the classifying space for the Heisenberg group, which is a mapping torus of $\mathbb{T}^2$. For this, I adopted Baum's geometric point of view. Since it seemed that a geometric proof of the existence of a six-terms exact sequence associated to a mapping torus was absent of the litterature, I began with this.

Theorem. *Let G be a compact Lie group, X be a proper, G-compact G-CW-complex and α be a G-equivariant homeomorphism of X. Then we have a natural six-terms exact sequence :*

$$\begin{tikzcd}
K_0^G(X) \arrow[r, "\operatorname{id} - \alpha_*^{-1}"] & K_0^G(X) \arrow[r, "i_*"] & K_0^G(M_{X,\alpha}) \arrow[d] \\
K_1^G(M_{X,\alpha}) \arrow[u] & K_1^G(X) \arrow[l, "i_*"'] & K_1^G(X), \arrow[l, "\operatorname{id} - \alpha_*^{-1}"']
\end{tikzcd}$$

where $M_{X,\alpha}$ is the mapping torus of α, namely the quotient of $X \times [0, 1]$ by the equivalence relation $(x, 0) \sim (\alpha(x), 1)$ for every $x \in X$.

To prove this, a Bott isomorphism in geometric K-homology was needed. Its existence was a well-known fact, thanks to KK-theory, but here I was able to find a purely geometric proof.

I resumed with the computations of this six-term exact sequence for the classifying space of the Heisenberg group and was again able to generalize this and compute the K-homology groups of any semi-direct product of $\mathbb{Z}^2$ by $\mathbb{Z}$. So in chapter 4 we prove the following result.

Theorem. *Let α be an automorphism of $\mathbb{Z}^2$ and G be the associated semi-direct product $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$. Let us assume that $\alpha \neq \operatorname{id}_{\mathbb{Z}^2}$.*

1) *If the determinant of α is 1, then the K-homology groups of the classifying space for G are given in the following table :*

	$\operatorname{tr}(\alpha) = 2$	$\operatorname{tr}(\alpha) = 1, 3$	$\operatorname{tr}(\alpha) \neq 1, 2, 3$
$K_0(BG)$	$\mathbb{Z}^3$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$K_1(BG)$	$\mathbb{Z}^3 \oplus \mathbb{Z}_h$	$\mathbb{Z}^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_{h_1} \oplus \mathbb{Z}_{h_2}$

for explicit $h, h_1, h_2 \in \mathbb{N}^*$.

2) *If the determinant of α is -1, then the K-theory groups of the classifying space for G are given in the following table :*

	$\operatorname{tr}(\alpha) = 0$	$\operatorname{tr}(\alpha) = -1, 1$	$\operatorname{tr}(\alpha) \neq -1, 0, 1$
$K_0(BG)$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$
$K_1(BG)$	$\mathbb{Z}^2 \oplus \mathbb{Z}_h$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_{h_1} \oplus \mathbb{Z}_{h_2}$

for explicit $h, h_1, h_2 \in \mathbb{N}^*$.

It was to be expected that the results are the same for the K-homology of BG and for the K-theory of $C^*(G)$, since we know the Baum-Connes conjecture true for the groups under consideration.

Organisation of the thesis

This thesis has four main parts. In the first one, we explain the Baum-Connes conjecture in its present form, giving also an overview on K- and KK-theory. We end this chapter 1 by introducing the results of Pimsner and Voiculescu obtained in [45], in particular their six-terms exact sequence in K-theory associated to a semi-direct product of a group by $\mathbb{Z}$.

The second chapter is devoted to define the geometric K-homology. An important part of it consists in giving an introduction to Clifford algebras and $Spin^c$-manifolds, and a lot of constructions that seem to be missing in the litterature are given. Finally, we present our proofs of the Bott isomorphism and the six-terms exact sequence in K-homology associated to a mapping torus.

In chapter 3, we show the explicit computations that are done to describe the K-theory of $C^*(\mathbb{Z}^2 \rtimes \mathbb{Z})$, as well as some consequences for lattices in real Heisenberg and SOL groups.

In the final chapter 4, we describe completely the six-terms exact sequence in K-homology associated to the mapping torus of $\mathbb{Z}^2$ by $\mathbb{Z}$.

Notations

In all this work, the following notations will be used.

$*$	The space consisting of a single point.
A^+	The unitization of a C^*-algebra A.
$A \rtimes G$	The full crossed product of a G-C^*-algebra A and a locally compact, Hausdorff and second countable group G.
$A \rtimes_r G$	The reduced crossed product of a G-C^*-algebra A and a locally compact, Hausdorff and second countable group G.
$\mathrm{Aut}(G)$	The group of continuous automorphism of a topological group G.
B_Q	The bilinear form associated to a quadratic form Q.
BG	The classifying space of a group G.
$\mathbb{C}$	The field of complex numbers, given by $\mathbb{R} \oplus i\mathbb{R}$.
$C^*(G)$	The full group C^*-algebra of a locally compact, Hausdorff and second countable group G.
$C^*_r(G)$	The reduced group C^*-algebra of a locally compact, Hausdorff and second countable group G.
$C_c(G)$	The convolution algebra of all compactly supported continuous functions from a locally compact, Hausdorff and second countable group G to $\mathbb{C}$.
$C_c(G, A)$	The convolution algebra of all compactly supported continuous functions from a locally compact, Hausdorff and second countable group G to a G-C^*-algebra A.
$C(Q)$	The Clifford algebra of a quadratic form Q.
$C(X)$	The C^*-algebra of all continuous functions from a compact space X to $\mathbb{C}$.
$C^\infty(X)$	The algebra of all infinitely derivable complex functions on a compact space X.
$C_0(X)$	For a locally compact space X, the C^*-algebra composed by those continuous functions $f : X \longrightarrow \mathbb{C}$ such that for all $\varepsilon > 0$, there exists a compact subset $K \subseteq X$ satisfying $\lvert f(x)\rvert < \varepsilon$ for every $x \in X \setminus K$.
$C(X, A)$	The C^*-algebra of all continuous functions from a compact space X to a C^*-algebra A.

$C_0(X, A)$	For a locally compact space X and a C^*-algebra A, the C^*-algebra composed by those continuous functions $f : X \longrightarrow A$ such that for all $\varepsilon > 0$, there exists a compact subset $K \subseteq X$ satisfying $\|\|f(x)\|\| < \varepsilon$ for every $x \in X \setminus K$.
$Cl(n)$	For $n \in \mathbb{N}$, the Clifford algebra associated to the standard negative definite real quadratic form over $\mathbb{R}^n$.
$Cl'(n)$	For $n \in \mathbb{N}$, the Clifford algebra associated to the standard positive definite real quadratic form over $\mathbb{R}^n$.
$Cl^c(n)$	For $n \in \mathbb{N}$, the Clifford algebra associated to the standard complex quadratic form over $\mathbb{R}^n$.
∂M	The boundary of a manifold with boundary M.
Δ_n	For $n \in \mathbb{N}$, the complex vector space of n-spinors, namely, the space $\mathbb{C}^{2^k}$, where $k \in \mathbb{N}$ is such that $n \in \{2k, 2k+1\}$.
EG	The total space of the universal G-bundle over BG.
$\underline{E}G$	The universal G-space for proper actions of a locally compact, Hausdorff and second countable group G.
$\mathrm{End}(V)$	The endomorphism of a vector space or an algebra V.
$\Gamma(E)$	The set of continuous sections of a bundle E.
$\Gamma^\infty(E)$	The set of infinitely differentiable sections of a smooth bundle E.
$\widehat{G}$	The Pontryagin dual of a locally compact abelian group, namely, the locally compact space $\mathrm{Hom}(G, \mathbb{T})$.
$GL_n(R)$	For a ring R and a positive integer n, the multiplicative group of all invertible $n \times n$ matrices with coefficients in R.
$GL_\infty(R)$	For a ring R, the colimit of all the $GL_n(R)$, namely, the reunion $\bigcup\limits_{n \in \mathbb{N}^*} GL_n(R)$.
$\mathbb{H}$	The real quaternion algebra, given by $\mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$.
$H_k(X)$	The k-th homology group of a space X with coefficients in $\mathbb{Z}$, where $k \in \mathbb{N}$.
$H^k(X)$	The k-th cohomology group of a space X with coefficients in $\mathbb{Z}$, where $k \in \mathbb{N}$.
$H_k(X, R)$	The k-th homology group of a space X with coefficients in a ring R, where $k \in \mathbb{N}$.
$H^k(X, R)$	The k-th cohomology group of a space X with coefficients in a ring R, where $k \in \mathbb{N}$.
$\mathrm{Hom}(G_1, G_2)$	The set of continuous homomorphisms between two locally compact groups G_1 and G_2.
ι	The inclusion map of $\mathbb{T}$ in $\mathbb{C}$.
$\jmath_G$	For a locally compact, Hausdorff and second countable group, the descent homomorphism from $KK_j^G(A, B)$ to $KK_j(A \rtimes_r G, B \rtimes_r G)$, where A, B are G-C^*-algebras.
$\mathcal{K}$	The C^*-algebra of compact operators on the Hilbert space $\ell^2(\mathbb{N})$.

$\mathcal{K}(H)$	The sub-C^*-algebra of $\mathcal{L}(H)$ formed by the compact operators on a Hilbert space H.
$K_j(A)$	The K-theory groups of a C^*-algebra A, where $j \in \mathbb{N}$.
$K^0(X)$	The topological K-theory of a locally compact space X.
$K_j^G(X)$	The G-equivariant geometric K-homology groups of a G-space X, where G is a locally compact, Hausdorff and second countable group and $j \in \{0,1\}$.
$K_j^G(X,Y)$	The G-equivariant geometric K-homology groups of a pair (X,Y) of G-spaces, where G is a locally compact, Hausdorff and second countable group and $j \in \{0,1\}$.
$KK_j^G(A,B)$	The G-equivariant KK-theory groups of a pair (A,B) of G-C^*-algebras, where G is a locally compact, Hausdorff and second countable group and $j \in \{0,1\}$.
l_x	The left support projection of an element x in a C^*-algebra A.
$\ell^\infty(X)$	The space of bounded functions from a space X to $\mathbb{C}$.
$\mathcal{L}(H)$	The C^*-algebra of bounded linear operators on a Hilbert space H.
$\mathcal{L}_A(H)$	The C^*-algebra of linear maps $T : H \longrightarrow H$ for which there exists a map $T^* : H \longrightarrow H$ such that $\langle T(\xi), \eta\rangle = \langle \xi, T^*(\eta)\rangle$ for every $\xi, \eta \in H$, where H is a Hilbert C^*-module over a C^*-algebra A.
$L^2(G,A)$	The Hilbert C^*-module of square integrable functions from a locally compact, Hausdorff and second countable group G endowed with the Haar measure to a G-C^*-algebra A.
$L^2(G)$	The Hilbert space of square integrable functions from a locally compact, Hausdorff and second countable group G endowed with the Haar measure to $\mathbb{C}$.
μ_j^G	For $j = 0, 1$, the Baum-Connes assembly map for a locally compact, Hausdorff and second countable group G.
$\mu_j^{G,A}$	For $j = 0, 1$ and a locally compact, Hausdorff and second countable group G, the Baum-Connes assembly map with coefficients in a G-C^*-algebra A.
$M_{X,\alpha}$	The mapping torus of an automorphism of a topological space X, namely, the quotient of $X \times [0,1]$ by the equivalence relation $(x, 0) \sim (\alpha(x), 1)$ for all $x \in X$.
$\mathcal{M}_n(A)$	For a C^*-algebra A and a positive integer n, the C^*-algebra of all $n \times n$ matrices with coefficients in A.
$\mathcal{M}_\infty(A)$	For a C^*-algebra A, the colimit of all the $\mathcal{M}_n(A)$, namely, the reunion $\displaystyle\bigcup_{n\in\mathbb{N}^*} \mathcal{M}_n(A)$.
$\mathbb{N}$	The monoid of non negative integers.
$\mathbb{N}^*$	The semi-group of positive integers.
π_A	The *-homomorphism from A^+ to $\mathbb{C}$ defined by $\pi_A(x + \lambda) = x$ for every $x \in A$ and $\lambda \in \mathbb{C}$.

$\pi_k(X, x_0)$	For $k \in \mathbb{N}$, the group of homotopy classes of pointed continous maps from S^k to a pointed space (X, x_0).
$\mathcal{P}_n(A)$	For a C^*-algebra A and a positive integer n, the set of all projections in the C^*-algebra $\mathcal{M}_n(A)$.
$\mathcal{P}_\infty(A)$	For a C^*-algebra A, the colimit of all the $\mathcal{P}_n(A)$, namely, the reunion $\bigcup\limits_{n \in \mathbb{N}^*} \mathcal{P}_n(A)$.
$Pin(n)$	For a positive integer n, the subgroup of $(Cl(n))^\times$ generated by elements in S^{n-1}.
r_x	The right support projection of an element x in a C^*-algebra A.
$\mathbb{R}$	The field of real numbers.
$\mathbb{R}^*$	The group of non-zero real numbers.
$\mathbb{R}_+$	The monoid of non-negative real numbers.
$\mathbb{R}_+^*$	The semi-group of positive real numbers.
$R^\times$	The group of invertible elements of a unital ring R.
$RK_j^G(X)$	The G-equivariant K-homology with G-compact supports of a proper G-space X, where G is a locally compact, Hausdorff and second countable group.
$RK_j^G(X, A)$	The G-equivariant K-homology with G-compact supports and coefficients in a G-C^*-algebra A of a proper G-space X, where G is a locally compact, Hausdorff and second countable group.
S^n	For a non negative integer n, the unit sphere in $\mathbb{R}^{n+1}$, namely, the space $\{x \in \mathbb{R}^{n+1} \mid \|\|x\|\| = 1\}$. When a pointed space structure is needed, S^n will be endowed with the basepoint $(1, 0, ..., 0)$.
SA	The suspension of a C^*-algebra A, namely, the C^*-algebra of functions $f \in C([0,1], A)$ satisfying $f(0) = 0 = f(1)$.
$SO(M)$	The bundle of positively oriented orthonormal frames of the tangent bundle of an oriented Riemannian manifold with boundary M.
$SO(n)$	For a positive integer n, the group of orthogonal $n \times n$ real matrices with determinant 1.
$SO(V)$	The bundle of positively oriented orthonormal frames of an oriented Euclidean vector bundle V.
$Spin(n)$	For a positive integer n, the subgroup of $(Cl(n))^\times$ of all products of an even number of elements in S^{n-1}.
$Spin^c(n)$	For a positive integer n, the subgroup of $(Cl^c(n))^\times$ generated by $Spin(n)$ and $\mathbb{T}$.
$\mathbb{T}$	The topological group of complex numbers of modulus 1.
$\mathcal{T}_{A,\alpha}$	The Toeplitz algebra of a couple (A, α), where A is a C^*-algebra and α is an automorphism of A.

$\mathcal{T}_\alpha(A)$	The mapping torus of an automorphism α of a C^*-algebra A, namely, the C^*-algebra composed by all continuous functions f from $[0,1]$ to A satisfying $f(0) = \alpha(f(1))$.
TM	The tangent bundle of a manifold with boundary M.
T^*M	The cotangent bundle of a manifold with boundary M.
$\mathcal{U}_n(A)$	For a C^*-algebra A and a positive integer n, the multiplicative group of all unitary in the C^*-algebra $\mathcal{M}_n(A)$.
$\mathcal{U}_\infty(A)$	For a C^*-algebra A, the colimit of all the $\mathcal{U}_n(A)$, namely, the reunion $\bigcup\limits_{n\in\mathbb{N}^*} \mathcal{U}_n(A)$.
$V(A)$	The abelian monoid of equivalence classes of projections in $\mathcal{M}_\infty(A)$, where A is a C^*-algebra.
$\mathrm{Vect}_{\mathbb{C}}(X)$	The abelian monoid of isomorphism classes of complex vector bundles over a compact space X.
X^+	The one-point compactification, of Alexandroff compactification, of a locally compact space X.
$\mathbb{Z}_n$	For a positive integer n, the quotient of the group $\mathbb{Z}$ by the subgroup $n\mathbb{Z}$.
$\mathcal{Z}(G)$	The center of a group G.

Chapter 1

Introduction to the Baum-Connes conjecture

This first part will be devoted to lay the framework in which the results of this thesis take place. An important effort has been produced to make this part accessible to mathematicians who do not work in this field. In the case where more detailed informations is needed, one may consult the excellent introductory book of Wegge-Olsen on K-theory and C^*-algebra [55], the work of Valette [54] or the well-known article from Baum, Connes and Higson [5] stating the conjecture named after the first two authors.

1.1 K-theory for C^*-algebras

We start by presenting briefly what are the K-theory groups of a C^*-algebra and give some fundamental facts about them.

1.1.1 Review on C^*-algebras

We recall that a C^*-algebra is a complex Banach algebra A with an involution - that is, a conjugate-linear map $^* : A \longrightarrow A$ such that $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in A$ - satisfying what we call the C^*-identity :

$$||x^*x|| = ||x||^2, \quad \forall x \in A.$$

Here are some basic examples of C^*-algebras.

- The complex numbers with the involution given by the conjugation.
- If H is an Hilbert space, then the bounded linear operators on it with the usual operator norm and the involution given by taking the adjoint is a C^*-algebra, denoted by $\mathcal{L}(H)$. It has a norm- and *-closed two-sided ideal given by the compact operators, $\mathcal{K}(H)$. The latter is therefore also a C^*-algebra.

In fact, the Gelfand-Naimark theorem states that every C^*-algebra can be seen as a sub-C^*-algebra of a $\mathcal{L}(H)$ for some Hilbert space H.

- Let X be a locally compact topological space and consider $C_0(X)$, the set of continuous maps from X to $\mathbb{C}$ vanishing at infinity. With pointwise sum, product and conjugation, together with the supremum norm, it gives a C^*-algebra.

  By the Gelfand representation, every commutative C^*-algebra is of this form.

- If A is a C^*-algebra, then $\mathcal{M}_n(A)$, with usual sum and multiplication, is also a C^*-algebra for $n \in \mathbb{N}^*$. Involution is given by applying the involution of A coordinatewise on the transposed matrix, i.e.,

  $$(x_{ij})^* := (x_{ji}^*)$$

  for every $(x_{ij}) \in \mathcal{M}_n(A)$.

Furthermore, a *-homomorphism $\phi : A \longrightarrow B$ between two C^*-algebras is a linear mapping such that $\phi(xy) = \phi(x)\phi(y)$ and $\phi(x^*) = (\phi(x))^*$, for all $x, y \in A$. The C^*-identity forces such a map to be norm decreasing, and hence continuous. Thus, a *-homomorphism that is injective and surjective is necessarily isometric and its inverse is a *-homomorphism too.

Proposition 1.1.1. *The C^*-algebras and *-homomorphisms form a category.*

There is an important remark to make here : a C^*-algebra can be unital (that is, with an identity element for the multiplication) or not (for example $\mathcal{K}(H)$ if H is infinite dimensional or $C_0(X)$ if X is non-compact). This fact, apparently unimportant, will complicate our work by requiring to set two different definitions for the K-theory of a C^*-algebra, as we will see later.

Nevertheless, a C^*-algebra can always be embedded canonically in a unital one. Indeed, let A be a C^*-algebra and form the direct sum of complex vector spaces $A \oplus \mathbb{C}$, with elements denoted by $x + \lambda$, where $x \in A$ and $\lambda \in \mathbb{C}$. On this set we have a product defined by

$$(x + \lambda)(y + \mu) := xy + \mu x + \lambda y + \lambda\mu$$

for all $x, y \in A$ and $\lambda, \mu \in \mathbb{C}$. Thus $A \oplus \mathbb{C}$ becomes a unital complex algebra, with unit $0_A + 1_{\mathbb{C}}$. It can be fitted with an involution simply by considering the sum of the involution on A and the conjugation on $\mathbb{C}$. For the norm on $A \oplus \mathbb{C}$, if A is non unital, we consider the following one :

$$||x + \lambda|| := \sup_{||y||_A = 1} ||xy + \lambda y||_A, \quad \forall x \in A,\ \lambda \in \mathbb{C}.$$

On the other hand, if A is unital, we give $A \oplus \mathbb{C}$ the norm

$$||x + \lambda|| := \max\{||x + \lambda 1_A||_A, |\lambda|\}, \quad \forall x \in A,\ \lambda \in \mathbb{C}.$$

Thus we obtain a unital C^*-algebra, called the unitization of A and written A^+.

As an example, we can show that the unitization of $C_0(X)$, for a locally compact space X, is $C(X^+)$, where X^+ is the one-point compactification of X. This illustrates this straightforward observation : a C^*-algebra is not always isomorphic to its unitization, even a unital one.

For later use, given a C^*-algebra A, we introduce the notation π_A for the *-homomorphism from A^+ to $\mathbb{C}$ defined by $\pi_A(x+\lambda)=\lambda$ for every $x\in A$ and $\lambda\in\mathbb{C}$.

Definition 1.1.2. *Let A be a C^*-algebra and $x\in A$.*

a) *The element x is called an idempotent if $x^2=x$. It is called self-adjoint if $x^*=x$.*

b) *A projection in A is a self-adjoint idempotent.*

c) *We say that x is positive if there exists $y\in A$ such that $x=y^*y$. It is then automatically self-adjoint.*

d) *If A is unital, we call x an isometry if $x^*x=1$ and a unitary if, moreover, $xx^*=1$.*

Considering two C^*-algebras A and B, we can provide their algebraic tensor product with some norms making it a C^*-algebra. For example, the one called maximal is given by

$$||\sum_{k=1}^{n} x_k\otimes y_k|| := \sup_R ||\sum_{k=1}^{n}\rho_A(x_k)\rho_B(y_k)||, \quad \forall x_k\in A,\, y_k\in B,$$

where R is the set of all pairs of representations ρ_A of A and ρ_B of B on the same Hilbert space satisfying

$$\rho_A(x)\rho_B(y)=\rho_B(y)\rho_A(x)$$

for every $x\in A$ and $y\in B$. In general, there exists more that one suitable norm. However, if one of A and B has the property that we call nuclearity, then the maximal norm is the unique one making the algebraic tensor product of A and B a C^*-algebra. For detailed explanation, we refer to Wegge-Olsen's book [55]. We only mention here that commutative C^*-algebras are nuclear.

1.1.2 The K-theory group K_0

Let A be a unital C^*-algebra. We denote by $\mathcal{M}_\infty(A)$ the colimit of the direct system formed by all the $\mathcal{M}_n(A)$ and the *-homomorphisms

$$\begin{aligned}\mathcal{M}_n(A) &\longrightarrow \mathcal{M}_{n+1}(A)\\ M &\longmapsto \begin{pmatrix} M & 0\\ 0 & 0\end{pmatrix},\end{aligned}$$

for $n\in\mathbb{N}^*$. Likewise, we form $\mathcal{P}_\infty(A)$ from the sets $\mathcal{P}_n(A)$ of all projections in $\mathcal{M}_n(A)$, for $n\in\mathbb{N}^*$.

The relation

$$p \sim q \quad \Longleftrightarrow \quad \exists x \in \mathcal{M}_\infty(A) \text{ such that } x^*x = p \text{ and } xx^* = q$$

defines an equivalence relation on $\mathcal{P}_\infty(A)$. Let's denote by $V(A)$ the set of equivalence classes. An addition can be defined on it by

$$[p] + [q] := [p \oplus q], \quad \forall p, q \in \mathcal{P}_\infty(A),$$

where $p \oplus q$ is the diagonal matrix $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$, turning $V(A)$ into an abelian monoid.

Lemma 1.1.3. *Let A be a unital C^*-algebra and $p, q \in \mathcal{P}_\infty(A)$ be such that $pq = 0$. Then $p + q \in \mathcal{P}_\infty(A)$ and $[p] + [q] = [p + q]$.*

Proof. We begin by noticing that $qp = q^*p^* = (pq)^* = 0$. Hence

$$(p + q)^2 = p^2 + pq + qp + q^2 = p + q.$$

Moreover,

$$(p + q)^* = p^* + q^* = p + q,$$

proving the first part of the lemma. On the other hand, defining $x := \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}$, we obtain

$$\begin{aligned} x^*x &= \begin{pmatrix} p & 0 \\ q & 0 \end{pmatrix} \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} p^2 & pq \\ qp & q^2 \end{pmatrix} \\ &= \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = p \oplus q \end{aligned}$$

and

$$\begin{aligned} xx^* &= \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p & 0 \\ q & 0 \end{pmatrix} = \begin{pmatrix} p^2 + q^2 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} p + q & 0 \\ 0 & 0 \end{pmatrix} = (p + q) \oplus 0. \end{aligned}$$

□

Definition 1.1.4. *If G is a semi-group in which the property*

$$x + z = y + z \quad \Longrightarrow \quad x = y$$

holds, for all $x, y, z \in G$, then we say that G has the cancellation property.

Definition 1.1.5. *Let A be a unital C^*-algebra. Then $K_0(A)$ is the Grothendieck completion of $V(A)$.*

Thus, $K_0(A)$ is composed of formal differences of classes in $V(A)$ satisfying

$$[p]-[q]=[p']-[q'] \iff \exists r\in\mathcal{P}_\infty(A) \text{ such that } [p\oplus q'\oplus r]=[p'\oplus q\oplus r]$$

for all $p,q,p',q'\in\mathcal{P}_\infty(A)$. If $V(A)$ has the cancellation property, then this equivalence becomes :

$$[p]-[q]=[p']-[q'] \iff [p\oplus q']=[p'\oplus q],$$

for all $p,q,p',q'\in\mathcal{P}_\infty(A)$.

If $\phi : A\longrightarrow B$ is a *-homomorphism between two unital C^*-algebras, then it induces a homomorphism ϕ_* between $K_0(A)$ and $K_0(B)$ by setting

$$\phi_*([(p_{ij})]-[(q_{ij})]) := [(\phi(p_{ij}))]-[(\phi(q_{ij}))]$$

for all $(p_{ij}),(q_{ij})\in\mathcal{P}_\infty(A)$. This allows us to define the K-theory group K_0 for a non-unital C^*-algebra.

Definition 1.1.6. *Let A be a C^*-algebra. We set*

$$K_0(A) := \ker\left((\pi_A)_* : K_0(A^+)\longrightarrow K_0(\mathbb{C})\right),$$

where $K_0(A^+)$ and $K_0(\mathbb{C})$ are defined according to 1.1.5.

Proposition 1.1.7. *Definitions 1.1.5 and 1.1.6 coincide for any unital C^*-algebra A.*

Furthermore, if $\phi : A\longrightarrow B$ is a *-homomorphism between two C^*-algebras, then it induces a *-homomorphism $\phi^+ : A^+\longrightarrow B^+$ by setting $\phi^+(x+\lambda) := \phi(x)+\lambda$ for all $x\in A$ and $\lambda\in\mathbb{C}$. Since the diagram

$$\begin{tikzcd} A^+ \arrow[rr, "\phi^+"] \arrow[dr, "\pi_A"'] & & B^+ \arrow[dl, "\pi_B"] \\ & \mathbb{C} & \end{tikzcd}$$

commutes, it turns out that ϕ induces a homomorphism ϕ_* from $K_0(A)$ to $K_0(B)$, consistently with the one previously defined in the unital case. Thus, K_0 is a covariant functor from the category of C^*-algebras and *-homomorphisms to the category of abelian groups and homomorphisms.

It may seem very strange that we have another definition for a non-unital C^*-algebra, since we don't need a unit to define projections. In fact, this is inspired by what we get in topological K-theory, as those two theories are closely related.

Theorem 1.1.8. *(Swan, 1962). Let X be a compact space. Then we have an isomorphism of monoids*

$$V(C(X))\cong \mathrm{Vect}_{\mathbb{C}}(X),$$

where $\mathrm{Vect}_{\mathbb{C}}(X)$ is formed by the isomorphism classes of complex vector bundles over X.

Applying completion, we find that $K_0(C(X))$ is isomorphic to $K^0(X)$, the topological K-theory of X.

Now, for a locally compact space X, the group $K^0(X)$ is defined to be the reduced topological K-theory of the one-point compactification X^+, namely

$$K^0(X) = \ker\left(i^* : K^0(X^+) \longrightarrow K^0(*)\right),$$

where $i : * \longrightarrow X$ is the adjunction of a basepoint in X. Since we still want to have a Swan isomorphism in that case, it explains our definition of K_0. In fact, a non-unital C^*-algebra has not enough projections, whereas its unitization has too much of them.

Let us explain how does the map giving the isomorphism in theorem 1.1.8 work. Let E be a complex vector bundle over a compact space X. We form $\Gamma(E)$, the set of continuous sections of the bundle ; it is a finitely generated projective module over $C(X)$. Hence there exists a $C(X)$-module Q and some $n \in \mathbb{N}^*$ such that

$$\Gamma(E) \oplus Q \cong C(X)^n.$$

In short, $\Gamma(E)$ can be seen as a submodule of $C(X)^n$. The projection on this submodule gives a projection in $\mathcal{M}_n(C(X))$ and therefore a class in $V(C(X))$.

- Let us consider the class in $K^0(X)$ of the one-dimensional trivial vector bundle over X. The continuous sections of $X \times \mathbb{C}$ are exactly the continuous functions from X to $\mathbb{C}$, namely, it is $C(X)$ seen as a free module over itself. The projection on it is then the identity. Hence $[X \times \mathbb{C}]$ corresponds to the class in $K_0(C(X))$ represented by the constant function equals to 1.
- Applied to $X = *$, we obtain $K_0(\mathbb{C}) \cong K_0(C(*)) \cong K^0(*) \cong \mathbb{Z}$. A generator can be given by taking $[1]$.
- If $X = \mathbb{T}$, we get $K_0(C(\mathbb{T})) \cong K^0(\mathbb{T}) \cong \mathbb{Z}$, generated by $[1]$, the class of the map being everywhere 1.

Furthermore, Swan's theorem allows us to prove the next proposition that we will need thereafter. We are indebted to Ulrich Suter for this very nice proof.

Proposition 1.1.9. *The semi-group $V(C(\mathbb{T}^2))$ has the cancellation property.*

Proof. In this proof, since we will only work with arcwise connected spaces and since only the isomorphism classes of their homotopy groups will be relevant, we can skip the use of basepoints.

We begin by recalling a theorem that is well-known to topologists. Let X, Y be two arcwise connected topological spaces and $g : X \longrightarrow Y$ be a continuous map between them. Suppose that there is some $m \in \mathbb{N}$ such that the induced map

$$g_* : \pi_k(X) \longrightarrow \pi_k(Y)$$

is an isomorphism for $k < m$ and a surjection for $k = m$. Then, for any CW-complex P satisfying $\dim(P) \leq m$, the map

$$\begin{aligned} g_* : [P, X] &\longrightarrow [P, Y] \\ [f] &\longmapsto [g \circ f] \end{aligned}$$

is surjective, where $[P, X]$, resp. $[P, Y]$, denotes the homotopy classes of continuous maps from P to X, resp. Y (see [51]).

Let E be a n-dimensional complex vector bundle over $\mathbb{T}^2$, where $n \in \mathbb{N}^*$. By the classification of vector bundles, it corresponds to a continuous map f from $\mathbb{T}^2$ to $B\mathcal{U}_n(\mathbb{C})$, the classifying space of the unitary group of complex $n \times n$ matrices. Moreover, the map

$$B\mathcal{U}_1(\mathbb{C}) \longrightarrow B\mathcal{U}_n(\mathbb{C})$$

coming from the inclusion $\mathcal{U}_1(\mathbb{C}) \subseteq \mathcal{U}_n(\mathbb{C})$ is a fibration, with fiber isomorphic to $\mathcal{U}_n(\mathbb{C})/\mathcal{U}_1(\mathbb{C})$. Since this quotient space is 2-connected, the long exact sequence associated to this fibration implies that the induced map in homotopy

$$\pi_k(B\mathcal{U}_1(\mathbb{C})) \longrightarrow \pi_k(B\mathcal{U}_n(\mathbb{C}))$$

is an isomorphism for $k \in \{0, 1, 2\}$. Since $\mathbb{T}^2$ is a 2-dimensional CW-complex, the result stated above gives the existence of a lifting for f, that is, a continuous map $\tilde{f}$ from $\mathbb{T}^2$ to $B\mathcal{U}_1(\mathbb{C})$ such that

$$\begin{tikzcd} & B\mathcal{U}_1(\mathbb{C}) \arrow[d] \\ \mathbb{T}^2 \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B\mathcal{U}_n(\mathbb{C}) \end{tikzcd}$$

commutes. This factorization of f implies an isomorphism between E and a direct sum $E' \oplus \mathbb{C}^{n-1}$, with E' a one-dimensional vector bundle over $\mathbb{T}^2$. However, such bundles are, up to isomorphism, completely classified by their first Chern class, which lives in $H^2(\mathbb{T}^2) \cong \mathbb{Z}$. Thus, a class in $\mathrm{Vect}_{\mathbb{C}}(\mathbb{T}^2)$ is entirely determined by the first Chern class and the dimension of any representative. By additivity of those two invariants, we get that the semi-group of the non-zero elements of $\mathrm{Vect}_{\mathbb{C}}(\mathbb{T}^2)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{N}^*$, which has the cancellation property. We conclude by theorem 1.1.8. $\square$

1.1.3 Higher K-theory groups

For a unital C^*-algebra A, we denote by $GL_\infty(A)$ the colimit of the $GL_n(A)$ and the inclusions

$$\begin{aligned} GL_n(A) &\longrightarrow GL_{n+1}(A) \\ M &\longmapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

for $n \in \mathbb{N}^*$. We define similarly the colimit $\mathcal{U}_\infty(A)$ of all the groups $\mathcal{U}_n(A)$ formed by the unitaries in $\mathcal{M}_n(A)$, for $n \in \mathbb{N}^*$. We can prove that $\mathcal{U}_\infty(A)$ is a deformation retract of $GL_\infty(A)$.

Definition 1.1.10. *Let A be a C^*-algebra and $j \in \mathbb{N}^*$. We define*

$$K_j(A) := \pi_{j-1}(GL_\infty(A^+), 1_{A^+}) = \pi_{j-1}(\mathcal{U}_\infty(A^+), 1_{A^+}).$$

We can show that $K_1(A)$ can be endowed with a group structure by defining

$$[u] \cdot [v] := [uv]$$

for all $u, v \in \mathcal{U}_\infty(A^+)$. Moreover, the group obtained is abelian. Indeed, since

$$\begin{pmatrix} \cos(\frac{\pi}{2}t) & -\sin(\frac{\pi}{2}t) \\ \sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix}, \quad t \in [0,1],$$

defines a path in $\mathcal{U}_2(\mathbb{C})$ from the identity to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we get, for every $u, v \in \mathcal{U}_\infty(A^+)$,

$$\begin{aligned} [u] \cdot [v] &= \left[\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\right] \cdot \left[\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\right] \cdot \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] \cdot \left[\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\right] \cdot \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\right] \cdot \left[\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}\right] = \left[\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\right] \end{aligned}$$

and

$$\begin{aligned} [v] \cdot [u] &= \left[\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\right] \cdot \left[\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] \cdot \left[\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\right] \cdot \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \cdot \left[\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}\right] \cdot \left[\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\right]. \end{aligned}$$

Proposition 1.1.11. *If A is a unital C^*-algebra, then*

$$K_j(A) \cong \pi_{j-1}(GL_\infty(A), 1_A) = \pi_{j-1}(\mathcal{U}_\infty(A), 1_A)$$

for all $j \in \mathbb{N}^*$.

As an example, we will check that $GL_\infty(\mathbb{C})$ is path-connected and thus that $K_1(\mathbb{C}) = 0$. Let $M \in GL_n(\mathbb{C})$ and consider the set $\{(1-z)M + zI_n \mid z \in \mathbb{C}\}$. By a linear algebra argument (using characteristic polynomials), this set intersects the singular matrices of $\mathcal{M}_n(\mathbb{C})$ in at most n points. Since $\mathbb{C}$ remains path-connected when we remove n elements from it, we can find a path c in $\mathbb{C}$ from 0 to 1 such that

$$(1-c(t))M + c(t)I_n \in GL_n(\mathbb{C})$$

for all $t \in [0,1]$, giving a path in $GL_n(\mathbb{C})$ from M to I_n.

Let $j \in \mathbb{N}^*$. A *-homomorphism $\phi : A \longrightarrow B$ between two C^*-algebras gives raise in the natural way to a homomorphism ϕ_* from $K_j(A)$ to $K_j(B)$. Hence K_j is a covariant functor from the category of C^*-algebras and *-homomorphisms to the category of abelian groups and homomorphisms.

1.1.4 Fundamental results

In this part we state some standard and important facts concerning K-theory for C^*-algebras. For proofs of those results, we refer to [55] or [54].

Definition 1.1.12. *For a C^*-algebra A, we define its suspension SA to be*

$$SA := \{f \in C([0,1], A) \mid f(0) = 0 = f(1)\}.$$

It is a closed two-sided ideal of $C([0,1], A)$ and hence is a C^*-algebra in itself.

Theorem 1.1.13. *Let A be a C^*-algebra. Then there exists a natural isomorphism θ from $K_1(A)$ to $K_0(SA)$.*

Theorem 1.1.14. *(Bott). Let A be a C^*-algebra. Then there exists a natural isomorphism β from $K_0(A)$ to $K_1(SA)$.*

Combining those two theorems and the fact that

$$\pi_j(GL_\infty(A^+), 1_{A^+}) \cong \pi_{j-1}(GL_\infty((SA)^+), 1_{(SA)^+}),$$

for all $j \in \mathbb{N}^*$, leads us to this fundamental result.

Corollary 1.1.15. *(Bott periodicity). Let A be a C^*-algebra and $j \in \mathbb{N}$. Then*

$$K_j(A) \cong K_{j+2}(A).$$

Hence there are in fact only two different K-theory groups associated to a C^*-algebra, given by K_0 and K_1.

For the sequel we introduce the notation $\mathcal{K} := \mathcal{K}(\ell^2(\mathbb{N}))$. Considering the standard basis $(e_i)_{i\in\mathbb{N}}$ of $\ell^2(\mathbb{N})$, we associate a basis $(e_{ij})_{i,j\in\mathbb{N}}$ of $\mathcal{K}$ by defining $e_{ij}(e_k) := \delta_{ik}e_j$, for all $k \in \mathbb{N}$. Thus we define, for a C^*-algebra A, the *-homomorphism

$$\begin{aligned} e : A &\longrightarrow A \otimes \mathcal{K} \\ x &\longmapsto x \otimes e_{0,0}. \end{aligned}$$

Theorem 1.1.16. *(Stabilization). The induced map* $e_* : K_j(A) \longrightarrow K_j(A \otimes \mathcal{K})$ *is a natural isomorphism for* $j = 0, 1$.

Theorem 1.1.17. *For* $j = 0, 1$, *the functor* K_j *is homotopy invariant : if* ϕ *and* ψ *are homotopic *-homomorphisms, then* $\phi_* = \psi_*$.

Theorem 1.1.18. *Let* $0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$ *be a short exact sequence of* C^**-algebras and *-homomorphisms. Then there is a six-terms exact sequence*

$$\begin{tikzcd}
K_0(A) \arrow[r, "\phi_*"] & K_0(B) \arrow[r, "\psi_*"] & K_0(C) \arrow[d, "\partial"] \\
K_1(C) \arrow[u, "\partial"] & K_1(B) \arrow[l, "\psi_*"'] & K_1(A) \arrow[l, "\phi_*"']
\end{tikzcd}$$

that is natural in the following sense : if we are given another short exact sequence

$$0 \longrightarrow A' \xrightarrow{\phi'} B' \xrightarrow{\psi'} C' \longrightarrow 0$$

and a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\phi"] \arrow[d, "\rho_1"'] & B \arrow[r, "\psi"] \arrow[d, "\rho_2"] & C \arrow[r] \arrow[d, "\rho_3"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\phi'"] & B' \arrow[r, "\psi'"] & C' \arrow[r] & 0
\end{tikzcd}$$

in the category of C^**-algebras, then the following diagram commutes :*

$$\begin{tikzcd}
 & K_0(A') \arrow[rr, "\phi'_*"] & & K_0(B') \arrow[rr, "\psi'_*"] & & K_0(C') \arrow[dd, "\partial'"] \\
K_0(A) \arrow[ur, "(\rho_1)_*"] \arrow[rr, "\phi_*"] & & K_0(B) \arrow[ur, "(\rho_2)_*"'] \arrow[rr, "\psi_*"] & & K_0(C) \arrow[ur, "(\rho_3)_*"'] \arrow[dd, "\partial"] & \\
 & K_1(C') \arrow[uu, "\partial'"] & & K_1(B') \arrow[ll, "\psi'_*"] & & K_1(A'). \arrow[ll, "\phi'_*"] \\
K_1(C) \arrow[uu, "\partial"] \arrow[ur, "(\rho_1)_*"] & & K_1(B) \arrow[ll, "\psi_*"'] \arrow[ur, "(\rho_2)_*"'] & & K_1(A) \arrow[ll, "\phi_*"'] \arrow[ur, "(\rho_3)_*"'] &
\end{tikzcd}$$

Looking at the suspension of $\mathbb{C}$, which is

$$S\mathbb{C} = \{f : [0,1] \longrightarrow \mathbb{C} \mid f(0) = 0 = f(1)\},$$

we can see it as the sub-C^*-algebra of $C(\mathbb{T})$ formed by all functions vanishing at 1. Hence we get the short exact sequence

$$0 \longrightarrow S\mathbb{C} \xrightarrow{i} C(\mathbb{T}) \xrightarrow{e_1} \mathbb{C} \longrightarrow 0,$$

where e_1 is the evaluation at 1. Then we obtain the following six-terms exact sequence :

$$\begin{tikzcd}
K_0(S\mathbb{C}) \arrow[r, "i_*"] & K_0(C(\mathbb{T})) \arrow[r, "(e_1)_*"] & K_0(\mathbb{C}) \arrow[d, "\partial"] \\
K_1(\mathbb{C}) \arrow[u, "\partial"] & K_1(C(\mathbb{T})) \arrow[l, "(e_1)_*"] & K_1(S\mathbb{C}). \arrow[l, "i_*"]
\end{tikzcd}$$

We have already shown on page 20 that $K_0(\mathbb{C})$ and $K_0(C(\mathbb{T}))$ are both isomorphic to $\mathbb{Z}$, generated by [1]. Moreover, by theorems 1.1.13 and 1.1.14, we have

$$K_0(S\mathbb{C}) \cong K_1(\mathbb{C}) = 0$$

and

$$K_1(S\mathbb{C}) \cong K_0(\mathbb{C}) \cong \mathbb{Z}.$$

Thus the six-terms exact sequence reduced to this one :

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(e_1)_*} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{i_*} K_1(C(\mathbb{T})) \longrightarrow 0.$$

Since $(e_1)_*$ sends obviously [1] on [1], we get that this map is the identity. Hence ∂ is the zero map. It follows that i_* must be an isomorphism, proving that

$$K_1(C(\mathbb{T})) \cong \mathbb{Z}.$$

An explicit generator can by given by the inclusion function from $\mathbb{T}$ into $\mathbb{C}$, that we will denote in the sequel by ι.

Furthermore, we can easily generalize those facts.

Proposition 1.1.19. *Let X be a non-empty compact space. Then*

$$K_0(C(X \times \mathbb{T})) \cong K_0(C(X)) \oplus K_1(C(X)) \cong K_1(C(X \times \mathbb{T})).$$

Proof. We first remark that the suspension of $C(X)$ is composed by all continuous functions from $[0,1]$ to $C(X)$ vanishing at 0 and 1. This C^*-algebra is then isomorphic to the sub-C^*-algebra of $C(X \times \mathbb{T})$ formed by the functions vanishing at $(x, 1)$, for every $x \in X$. Thus we get the following short exact sequence :

$$0 \longrightarrow SC(X) \xrightarrow{i} C(X \times \mathbb{T}) \xrightarrow{\psi} C(X) \longrightarrow 0,$$

where $(\psi(f))(x) := f(x,1)$ for all $f \in C(X \times \mathbb{T})$ and $x \in X$. This sequence has a section $\sigma : C(X) \longrightarrow C(X \times \mathbb{T})$ given by

$$(\sigma(f))(x, \lambda) := f(x)$$

for every $f \in C(X \times \mathbb{T})$, $x \in X$ and $\lambda \in \mathbb{T}$. Therefore, in K-theory, the induced map ψ_* is surjective, since

$$\psi_* \circ \sigma_* = (\psi \circ \sigma)_* = \mathrm{id}_* = \mathrm{id}\,.$$

Thus the six-terms exact sequence

$$\begin{tikzcd}
K_0(SC(X)) \arrow[r, "i_*"] & K_0(C(X \times \mathbb{T})) \arrow[r, "\psi_*"] & K_0(C(X)) \arrow[d, "\partial"] \\
K_1(C(X)) \arrow[u, "\partial"] & K_1(C(X \times \mathbb{T})) \arrow[l, "\psi_*"] & K_1(SC(X)) \arrow[l, "i_*"]
\end{tikzcd}$$

associated to the short exact one above has vertical homomorphisms being both zero, giving, for $j = 0, 1$,

$$0 \longrightarrow K_j(SC(X)) \xrightarrow{i_*} K_j(C(X \times \mathbb{T})) \xrightarrow{\psi_*} K_j(C(X)) \longrightarrow 0.$$

Thanks to σ_*, this sequence of abelian groups splits and we therefore obtain the result by theorems 1.1.13 and 1.1.14. □

By induction, we deduce from this proposition the K-theory groups of all the torus :

$$K_j(C(\mathbb{T}^n)) \cong \mathbb{Z}^{2^{n-1}}$$

for $j = 0, 1$ and all $n \in \mathbb{N}^*$.

1.2 Equivariant KK-theory

Introduced by Gennadi Kasparov in the early 80's, the KK-theory is a bivariant functor on the category of C^*-algebras generalizing both K-theory and K-homology. He was inspired by the Fredholm module concept created by Atiyah for his celebrated index theorem.

In this part, G will always denote a locally compact, Hausdorff and second countable group.

1.2.1 Definitions and functoriality

Definition 1.2.1. *Let A be a C^*-algebra. A Hilbert C^*-module over A is a right A-module H provided with an A-valued scalar product - namely a map $\langle .,. \rangle : H \times H \longrightarrow A$ satisfying the conditions*

- $\langle \xi + \xi', \eta \rangle = \langle \xi, \eta \rangle + \langle \xi', \eta \rangle$,
- $\langle \xi, \eta x \rangle = \langle \xi, \eta \rangle x$,
- $\langle \xi, \eta \rangle = \langle \eta, \xi \rangle^*$,
- $\langle \xi, \xi \rangle$ *is a positive element of A and is zero if and only if* $\xi = 0$,

for every $\xi, \xi', \eta \in H$ and $x \in A$ - and such that H is complete with respect to the norm given by

$$||\xi|| := ||\langle \xi, \xi \rangle||^{1/2}$$

for all $\xi \in H$.

This notion generalizes the concept of Hilbert space, since such a space is a Hilbert C^*-module over $\mathbb{C}$. Moreover, if H is a Hilbert space, then $H \otimes_{\mathbb{C}} A$ is a Hilbert C^*-module over A by defining

$$\langle \xi \otimes x, \eta \otimes y \rangle_A := \langle \xi, \eta \rangle_H \otimes x^* y$$

for every $\xi, \eta \in H$ and $x, y \in A$.

Definition 1.2.2. *Let H be a Hilbert C^*-module over a C^*-algebra A.*

a) *We denote by $\mathcal{L}_A(H)$ the set of linear operators $T : H \longrightarrow H$ having an adjoint, that is, a map $T^* : H \longrightarrow H$ satisfying*

$$\langle T(\xi), \eta \rangle = \langle \xi, T^*(\eta) \rangle$$

for every $\xi, \eta \in H$. Such maps are automatically A-linear and continuous. This gives an involution on $\mathcal{L}_A(H)$ which becomes, together with the operator norm, a C^*-algebra.

b) *An operator $T \in \mathcal{L}_A(H)$ is of finite rank (in the sense of Hilbert C^*-modules) if it is a linear combination of operators of the form $T_{\xi',\xi}$, where*

$$\begin{aligned} T_{\xi',\xi} : H &\longrightarrow H \\ \eta &\longmapsto \xi' \langle \xi, \eta \rangle, \end{aligned}$$

for $\xi, \xi' \in H$.

c) *We call an operator $T \in \mathcal{L}_A(H)$ compact (in the sense of Hilbert C^*-modules) if it is a norm limit of operators of finite rank.*

If $A = \mathbb{C}$, all these definitions coincide with the usual notions of bounded, finite rank and compact operators on Hilbert spaces.

Definition 1.2.3. *Let A and B be G-C^*-algebras, that is, C^*-algebras with an action of G by $*$-automorphisms, and (H, π, F) be a triple where*

- *H is a Hilbert C^*-module over B endowed with a representation ρ of G that is unitary, i.e., such that*
$$\langle \rho_g(\xi), \rho_g(\eta) \rangle = g \cdot \langle \xi, \eta \rangle$$
for every $g \in G$ and $\xi, \eta \in H$,
- *π is a representation of A on H, namely a $*$-homomorphism from A to $\mathcal{L}_B(H)$, which is covariant in the sense that*
$$\rho_g \pi_x \rho_{g^{-1}} = \pi_{g \cdot x}$$
for every $g \in G$ and $x \in A$,
- *F is an operator in $\mathcal{L}_B(H)$.*

Then (H, π, F) is called a G-equivariant Kasparov cycle over (A, B) if the map

$$\begin{aligned} G &\longrightarrow \mathcal{L}_B(H) \\ g &\longmapsto \rho_g F \rho_{g^{-1}} \end{aligned}$$

is strongly continuous and the four operators

$$\pi_x(F^2 - \mathrm{id}_H), \quad \pi_x(F^* - F), \quad [\pi_x, F], \quad [\rho_g, F]$$

are compact for every $g \in G$ and $x \in A$. Moreover, the cycle is called even if H is $\mathbb{Z}_2$-graded and if ρ_g and π_x preserve the graduation for every $g \in G$ and $x \in A$, whereas F reverses it. Otherwise the cycle is called odd.

If the context is clear, a G-equivariant Kasparov cycle will be abbreviated in Kasparov cycle, or just cycle.

Given two such cycles, we can form their direct sum by taking the direct sum of their underlying Hilbert C^*-modules, representations and operators. If the cycles are even, their sum will be too.

- Let $\phi : A \longrightarrow B$ be a $*$-homomorphism between two G-C^*-algebras. Then B is obviously a Hilbert C^*-module over itself and we can provide it with the representation of G coming from the action. Moreover ϕ yields a representation of A on B by multiplication operators. Hence $(B, \phi, 0)$ is an odd cycle over (A, B).
- In the same conditions as above, $(H \oplus 0, \phi \oplus 0, 0)$ gives an even cycle over (A, B).

- The fundamental example is the following one. Assume G is trivial and let M be a compact manifold, E and E' be vector bundles over M and
$$D : \Gamma^\infty(E) \longrightarrow \Gamma^\infty(E')$$
be a symmetric elliptic differential operator of order 1. Then we consider the $\mathbb{Z}_2$-graded Hilbert space
$$H := L^2(M, E) \oplus L^2(M, E'),$$
the *-homomorphism $\pi : C(M) \longrightarrow \mathcal{L}(H)$ given by pointwise multiplication operators and the graduation-reversing operator
$$F := \begin{pmatrix} 0 & \chi(D)^* \\ \chi(D) & 0 \end{pmatrix}$$
given by functional calculus, with $\chi : \mathbb{R} \longrightarrow [-1, 1]$ an odd smooth function being positive on $\mathbb{R}^+_*$ and such that
$$\lim_{x\to\infty} \chi(x) = 1.$$
Thus we can see that the triple (H, π, F) is an even cycle over $(C(M), \mathbb{C})$, as it is proved in [23].

Definition 1.2.4. *Let A, B be G-C^*-algebras and (H, π, F), (H', π', F') be two cycles over (A, B) of the same parity.*

a) *The cycle (H, π, F) is called degenerate if the four operators required to be compact in definition 1.2.3 are zero for every $g \in G$ and $x \in A$.*

b) *The cycles (H, π, F) and (H', π', F') are said to be homotopic if $H = H'$ (as well as their unitary representation of G), $\pi = \pi'$ and there exists a norm-continuous path of operators $(F_t)_{t\in[0,1]} \subseteq \mathcal{L}_B(H)$ such that*
$$F_0 = F, \qquad F_1 = F'$$
and for every $t \in [0, 1]$, the triple (H, π, F_t) is a cycle over (A, B) of the same parity.

c) *The cycles (H, π, F) and (H', π', F') are called unitarily isomorphic if there exists a continuous, bijective and A-linear map $\varphi : H \longrightarrow H'$ which preserves the scalar product, as well as the graduation in the even case, and such that the diagrams*

$$\begin{tikzcd} H \arrow[r, "\rho_g"] \arrow[d, "\varphi"'] & H \arrow[d, "\varphi"] \\ H' \arrow[r, "\rho'_g"] & H' \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "\pi_x"] \arrow[d, "\varphi"'] & H \arrow[d, "\varphi"] \\ H' \arrow[r, "\pi'_x"] & H' \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "F"] \arrow[d, "\varphi"'] & H \arrow[d, "\varphi"] \\ H' \arrow[r, "F'"] & H' \end{tikzcd}$$

commute, for every $g \in G$ and $x \in A$.

d) *The cycles* (H, π, F) *and* (H', π', F') *are equivalent if there exists two degenerate cycles* (H_0, π_0, F_0) *and* (H'_0, π'_0, F'_0) *of the same parity such that*

$$(H, \pi, F) \oplus (H_0, \pi_0, F_0) \quad \textit{and} \quad (H', \pi', F') \oplus (H'_0, \pi'_0, F'_0)$$

are homotopic, up to unitary isomorphism. This is an equivalence relation.

Proposition 1.2.5. *Let* A, B *be* G*-*C^**-algebras. Then the set of equivalence classes of even cycles over* (A, B) *and the one of odd cycles are both abelian groups for the law coming from the direct sum. They are denoted respectively by* $KK_0^G(A, B)$ *and* $KK_1^G(A, B)$ *and are called the* G*-equivariant* KK*-groups of* (A, B).

If G is the trivial group, then we write KK_j instead of KK_j^G, for $j = 0, 1$.

It should be clear from the defining equivalence relation that degenerate cycles represent the zero element of the groups. Now let $(H_0 \oplus H_1, \pi_0 \oplus \pi_1, F)$ be an even Kasparov cycle over a pair of G-C^*-algebras (A, B), where F is given by

$$F = \begin{pmatrix} 0 & F_{10} \\ F_{01} & 0 \end{pmatrix}$$

respectively to the graduation of the Hilbert C^*-module. Then the inverse of its class is simply given by reversing the graduation, namely it is $[H_1 \oplus H_0, \pi_1 \oplus \pi_0, F']$, where

$$F' = \begin{pmatrix} 0 & F_{01} \\ F_{10} & 0 \end{pmatrix}.$$

Indeed, by summing the two Kasparov cycles, we obtain

$$((H_0 \oplus H_1) \oplus (H_1 \oplus H_0), (\pi_0 \oplus \pi_1) \oplus (\pi_1 \oplus \pi_0), F \oplus F'),$$

with

$$F \oplus F' = \begin{pmatrix} 0 & \begin{pmatrix} F_{10} & 0 \\ 0 & F_{01} \end{pmatrix} \\ \begin{pmatrix} F_{01} & 0 \\ 0 & F_{10} \end{pmatrix} & 0 \end{pmatrix}.$$

This cycle is homotopic to a degenerate one via the path of operators given by

$$F_t = \begin{pmatrix} 0 & \begin{pmatrix} \cos(\frac{\pi}{2}t)F_{10} & \sin(\frac{\pi}{2}t) \\ -\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t)F_{01} \end{pmatrix} \\ \begin{pmatrix} \cos(\frac{\pi}{2}t)F_{01} & -\sin(\frac{\pi}{2}t) \\ \sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t)F_{10} \end{pmatrix} & 0 \end{pmatrix},$$

$t \in [0, 1]$. In the non-graded case, the inverse of a class $[H, \pi, F]$ is given by $[H, \pi, -F]$. An homotopy between the sum of the two underlying cycles and a degenerate one can be given by the path

$$\begin{pmatrix} \cos(\frac{\pi}{2}t)F & \sin(\frac{\pi}{2}t) \\ \sin(\frac{\pi}{2}t) & -\cos(\frac{\pi}{2}t)F \end{pmatrix},$$

$t \in [0,1]$.

Let A, B and C be G-C^*-algebras and (H, π, F) be a cycle over (A, B). Given a G-equivariant *-homomorphism $\phi : C \longrightarrow A$, we can consider the triple $(H, \pi\phi, F)$. It is a G-equivariant Kasparov cycle over (C, B) of the same parity as (H, π, F) and thus ϕ induces a homomorphism

$$\phi^* : KK_j^G(A, B) \longrightarrow KK_j^G(C, B)$$

for $j = 0, 1$. Similarly, given a G-equivariant *-homomorphism $\phi : B \longrightarrow C$, we can provide $H \otimes_B C$ with a structure of a Hilbert C^*-module over C in a natural way. Hence

$$(H \otimes_B C, \pi \otimes \mathrm{id}_C, F \otimes \mathrm{id}_C)$$

is a cycle over (A, C), with the same parity as (H, π, F). This gives a homomorphism

$$\phi_* : KK_j^G(A, B) \longrightarrow KK_j^G(A, C)$$

for $j = 0, 1$. It follows that KK_j^G is a bifunctor from the category of G-C^*-algebras and G-equivariant *-homomorphisms to the category of abelian groups and homomorphisms, for $j = 0, 1$. This bifunctor is contravariant in the first variable and covariant in the second.

1.2.2 Fundamental results

Definition 1.2.6. *Let A be a G-C^*-algebra. The analytic G-equivariant K-homology of A is the $\mathbb{Z}_2$-graded group $KK_0^G(A, \mathbb{C}) \oplus KK_1^G(A, \mathbb{C})$.*

This gives a functor from the category of G-C^*-algebras to the category of $\mathbb{Z}_2$-graded abelian groups which is additive, meaning that for two G-C^*-algebras A and B,

$$KK_j^G(A, \mathbb{C}) \oplus KK_j^G(B, \mathbb{C}) \cong KK_j^G(A \oplus B, \mathbb{C})$$

with $j = 0, 1$. The map giving the isomorphism is obtained by adding the cycles. More precisely, given Kasparov cycles (H, π, F) and (H', π', F') over $(A, \mathbb{C})$ and $(B, \mathbb{C})$ respectively, we form the cycle $(H \oplus H', \pi \oplus \pi', F \oplus F')$ with

$$\begin{aligned} \pi \oplus \pi' : A \oplus B &\longrightarrow \mathcal{L}(H) \oplus \mathcal{L}(H') \subseteq \mathcal{L}(H \oplus H') \\ (x, y) &\longmapsto \pi(x) \oplus \pi'(y) \end{aligned}$$

If the cycles are even, then the degree zero (resp. one) part of $H \oplus H'$ will be the direct sum of the degree zero (resp. one) part of H and H'.

Theorem 1.2.7. *(Bott isomorphism). Let A be a G-C^*-algebra and $j \in \{0,1\}$. Then we have an isomorphism*

$$KK_j^G(SA, \mathbb{C}) \cong KK_{1-j}^G(A, \mathbb{C}),$$

where SA is the suspension of A.

A complete proof of this fact can be found in [23].

Theorem 1.2.8. *Let A be a C^*-algebra. Then*

$$KK_j(\mathbb{C}, A) \cong K_j(A)$$

for $j = 0, 1$.

- Applied to $A = \mathbb{C}$, the previous theorem gives the analytic K-homology of the complex numbers :

  $$KK_0(\mathbb{C}, \mathbb{C}) \cong K_i(\mathbb{C}) \cong \mathbb{Z}$$

  and

  $$KK_1(\mathbb{C}, \mathbb{C}) \cong K_1(\mathbb{C}) \cong 0.$$

  Nevertheless, a direct proof can be found in [10], where it is moreover shown that the class of the even Kasparov cycle $(\mathbb{C} \oplus 0, \mathrm{id}_{\mathbb{C}} \oplus 0, 0)$ is a generator for $KK_0(\mathbb{C}, \mathbb{C})$.

- We can show that the analytic K-homology groups of $C(\mathbb{T})$ are both isomorphic to $\mathbb{Z}$, that is,

  $$KK_0(C(\mathbb{T}), \mathbb{C}) \cong \mathbb{Z} \cong KK_1(C(\mathbb{T}), \mathbb{C}),$$

  for example by using the Chern character (see theorem 4.2.3). On the one hand, the group $KK_0(C(\mathbb{T}), \mathbb{C})$ is generated by $[\mathbb{C} \oplus 0, e_1 \oplus 0, 0]$, where

  $$e_1 : C(\mathbb{T}) \longrightarrow \mathbb{C}$$

  is the *-homomorphism given by the evaluation on $1 \in \mathbb{T}$. On the other hand, following [40], a generator of $KK_1(C(\mathbb{T}), \mathbb{C})$ is described by the class of the Kasparov cycle given by the Hilbert space $L^2(\mathbb{T})$, the representation of $C(\mathbb{T})$ on $L^2(\mathbb{T})$ given by pointwise multiplication and the operator F on $L^2(\mathbb{T})$ given by

  $$F(\iota^n) = sgn(n)\iota^n, \quad \forall n \in \mathbb{Z},$$

  on the basis $\{\iota^n \mid n \in \mathbb{Z}\}$ of $L^2(\mathbb{T})$, where sgn is the sign function.

Theorem 1.2.8 explains how KK-theory includes K-theory. It is also clear that this bivariant theory generalize analytic K-theory as well, since we have defined it as a particular case of KK-theory. However, analytic K-homology was originally defined in a completely different manner, as the homology theory dual to Atiyah-Hirzebruch K-theory. This is for example the point of view adopted by Higson and Roe in their book [23], where they also proved the equivalence between the two definitions.

Theorem 1.2.9. *Let* $0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$ *be a short exact sequence of separable* G*-*C^**-algebras and* G*-equivariant *-homomorphisms. Assume that* B *is nuclear. Then, for any separable* G*-*C^**-algebra* D*, there exists two six-terms exact sequences*

$$\begin{tikzcd}
KK_0^G(D,A) \arrow[r, "\phi_*"] & KK_0^G(D,B) \arrow[r, "\psi_*"] & KK_0^G(D,C) \arrow[d, "\partial"] \\
KK_1^G(D,C) \arrow[u, "\partial"] & KK_1^G(D,B) \arrow[l, "\psi_*"] & KK_1^G(D,A) \arrow[l, "\phi_*"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
KK_0^G(C,D) \arrow[r, "\psi^*"] & KK_0^G(B,D) \arrow[r, "\phi^*"] & KK_0^G(A,D) \arrow[d, "\partial"] \\
KK_1^G(A,D) \arrow[u, "\partial"] & KK_1^G(B,D) \arrow[l, "\phi^*"] & KK_1^G(C,D) \arrow[l, "\psi^*"]
\end{tikzcd}$$

which are natural in the same sense as in 1.1.18. Moreover, if $D = \mathbb{C}$*, then the first of these two sequences coincide with the one in* K*-theory.*

As one can see for example in [10] or [23], the hypothesis of the nuclearity of B can be weakened in assuming the short exact sequence

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

to be semi-split, meaning that the map

$$\psi^+ : B^+ \longrightarrow C^+$$

admits a completely positive section.

Theorem 1.2.10. *(Kasparov [26]). Let* A*,* B *and* C *be separable* G*-*C^**-algebras. Then there exists a bilinear coupling*

$$\begin{aligned} KK_i^G(A,B) \times KK_j^G(B,C) &\longrightarrow KK_{i+j}^G(A,C) \\ (\gamma,\delta) &\longmapsto \gamma \otimes_B \delta, \end{aligned}$$

where $i,j \in \{0,1\}$ *and* $i+j$ *has to be taken modulo* 2*, such that :*

- *the coupling is contravariant in* A *and covariant in* C*,*
- *if* D *is another separable* C^**-algebra, then*

  $$(\gamma \otimes_B \delta) \otimes_C \zeta = \gamma \otimes_B (\delta \otimes_C \zeta)$$

  for all $\gamma \in KK_i^G(A,B)$*,* $\delta \in KK_j^G(B,C)$ *and* $\zeta \in KK_k^G(C,D)$*,*

- *if* $B = A$ *and* $\gamma = [A, \mathrm{id}_A, 0]$, *then*
  $$\gamma \otimes_A \delta = \delta$$
  for all $\delta \in KK_j^G(A, C)$ *and, similarly, if* $B = C$ *and* $\delta = [B, \mathrm{id}_B, 0]$, *then*
  $$\gamma \otimes_C \delta = \gamma$$
  for all $\gamma \in KK_i^G(A, C)$.

This defines an associative product on the KK*-theory groups of all pairs of separable* G*-*C^**-algebras, called the Kasparov product.*

The proof of this theorem, and therefore the construction of the Kasparov product, is really hard and technical. We can find it for example in [26] or [10]. Nevertheless, there are some cases in which the computation of this product is simplified.

- Kasparov product with a class of a cycle coming from a G-equivariant *-homomorphism is just applying the functoriality of KK^G.
- Let A, B be separable G-C^*-algebras and $[p] \in K_0(A) \cong KK_0(\mathbb{C}, A)$. The projection p induces a *-homomorphism $\overline{p} : \mathbb{C} \longrightarrow A$ by setting $\overline{p}(1) = p$. Then the product of $[p]$ with a class $[H, \pi, F] \in KK_j(A, B)$ is the class of the cycle
  $$[H, \pi\overline{p}, F] \in KK_j(\mathbb{C}, B) \cong K_j(B)$$
  for $j = 0, 1$.

We end this section by stating another property of the equivariant KK-theory, although we will explain what are the C^*-algebras involved here only in the next section.

Theorem 1.2.11. *(Kasparov [26]). Let* A, B *be* G*-*C^**-algebras. Then for* $j = 0, 1$, *there exists a descent homomorphism*

$$\jmath_G : KK_j^G(A, B) \longrightarrow KK_j(A \rtimes_r G, B \rtimes_r G).$$

Moreover, if A *and* B *are separable, this homomorphism commutes with the Kasparov product.*

1.3 Baum-Connes Conjecture

For this section, let G be a locally compact, Hausdorff and second countable topological group. For simplicity in the notations, we assume G unimodular (for example discrete, abelian, compact or simple).

1.3.1 Group C^*-algebras, proper G-spaces and crossed products

Considering the convolution product on $C_c(G)$, the $\mathbb{C}$-vector space of continuous complex functions with compact support, it gives an algebra over $\mathbb{C}$. We can endow it with the involution defined by

$$f^*(g) := \overline{f(g^{-1})}$$

for every $f \in C_c(G)$ and $g \in G$. Now let $C_c(G)$ acts on $L^2(G)$, the Hilbert space of square integrable functions on G with the left invariant Haar measure, by extending the left regular representation of G, namely by setting

$$(f \cdot \xi)(g) = \int_G f(h)\xi(h^{-1}g)dh$$

for all $f \in C_c(G)$, $\xi \in L^2(G)$ and $g \in G$. Thus the action is simply given by the convolution product in $L^2(G)$ and this gives a faithful representation of $C_c(G)$ by bounded operators.

Definition 1.3.1. *The reduced group C^*-algebra $C_r^*(G)$ is the one obtained by completing $C_c(G)$ with respect to the operator norm given by the left regular representation on $L^2(G)$.*

If G is abelian, the Fourier transform provides an isomorphism of C^*-algebras between $C_r^*(G)$ and $C_0(\widehat{G})$, where $\widehat{G}$ is the Pontryagin dual of G, namely $\widehat{G} := \mathrm{Hom}(G, \mathbb{T})$. Concretely, the map giving the isomorphism is induced by sending an $f \in C_c(G)$ to the continuous function :

$$\begin{aligned} \widehat{G} &\longrightarrow \mathbb{C} \\ \chi &\longmapsto \int_G f(g)\chi(g)dg. \end{aligned}$$

Definition 1.3.2. *The full group C^*-algebra is the completion of $C_c(G)$ with respect to the norm*

$$||f|| := \sup_{\rho} ||\rho(f)||,$$

where ρ runs through all non-degenerate representations of $C_c(G)$ on a Hilbert space.

The full C^*-algebra of G is characterized by the following universal property : every *-homomorphism from $C_c(G)$ to $\mathcal{L}(H)$, where H is any Hilbert space, extends to a *-homomorphism from $C^*(G)$ to $\mathcal{L}(H)$.

Since the left regular representation of $C_c(G)$ on the Hilbert space $L^2(G)$ is non-degenerate, we get that the identity on $C_c(G)$ gives rise to a *-homomorphism $C^*(G) \longrightarrow C_r^*(G)$.

Theorem 1.3.3. *The canonical *-homomorphism from $C^*(G)$ to $C_r^*(G)$ is an isomorphism if and only if the group G is amenable.*

The right-hand side of the Baum-Connes conjecture, or the analytical side, will be the K-theory groups of the reduced C^*-algebra of G. To define the left-hand (or topological) side, we set that G-spaces are topological metrizable spaces endowed with a continuous action of G

such that the orbit space is also metrizable. A G-map between two G-spaces is a G-equivariant continuous map between the two spaces. Two G-maps are said homotopic if there exists a G-equivariant homotopy between them, with G acting trivially on $[0,1]$.

Definition 1.3.4. *A G-space X is called proper if, for every $x \in X$, there exists a triple (V, H, f) such that :*

- *V is an open neighborhood of x satisfying $G \cdot V \subseteq V$,*
- *H is a compact subgroup of G,*
- *f is a G-map from V to the homogeneous space G/H.*

If the G-space X is also locally compact, then it is proper if and only if the map

$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (g \cdot x, x) \end{aligned}$$

is proper, meaning that the preimage of a compact subset of $X \times X$ is compact in $G \times X$. This is a result proved by Chabert, Echterhoff and Meyer in [14].

Definition 1.3.5. *A proper G-space X is universal for proper actions of G if for every proper G-space Y there exists a G-map $Y \longrightarrow X$ unique up to G-homotopy.*

Proposition 1.3.6. *There exists a universal G-space for proper actions of G. It is unique up to G-homotopy equivalence and we write it $\underline{E}G$.*

The universal G-space for proper actions of G will often by shortened in universal G-space.

- If G is a compact group, then every G-space is trivially proper. Hence the space consisting of a single point is universal.
- In contrast, if G has no other compact subgroup than the trivial one, then a universal G-space is given by EG, the universal covering of the classifying space for G. This applies for example if G is discrete and torsion-free.
- If G is discrete, then the set

  $$\left\{ f : G \longrightarrow [0,1] \mid f \text{ has finite support and } \sum_{g \in G} f(g) = 1 \right\},$$

  endowed with the supremum norm and the action of G coming from the left translation on itself, is a universal G-space.
- If G is a Lie group with a finite number of connected components, then we can choose G/K as a universal G-space, where K is a maximal compact subgroup of G.

Proposition 1.3.7. *Suppose G discrete. Let X be a proper G-space for which the action of G is also free. Then*

$$KK_j^G(C_0(X),\mathbb{C}) \cong KK_j(C_0(X/G),\mathbb{C})$$

for $j=0,1$.

A detailed proof of this proposition can be found in [40]. Nevertheless, an analogue of it in the non-discrete case is in general not true.

Corollary 1.3.8. *If G is discrete and torsion-free, then, for $j=0,1$,*

$$KK_j^G(C_0(\underline{E}G),\mathbb{C}) \cong KK_j(C_0(BG),\mathbb{C}),$$

where BG is the classifying space for G.

A subspace Y of a G-space X is called G-compact if it is G-invariant and if Y/G is compact.

Definition 1.3.9. *Let X be a proper G-space. The G-equivariant K-homology with G-compact supports of X is*

$$RK_j^G(X) := \operatorname{colim} KK_j^G(C_0(Y),\mathbb{C}), \quad j=0,1,$$

where the colimit is the one of the direct system formed by the G-compact subspaces Y of X, ordered by the inclusion, and the induced homomorphisms in KK-theory.

It is clear from the definition that if X is itself G-compact, then its G-equivariant K-homology with G-compact supports reduces to its analytic G-equivariant K-homology. Together with 1.3.8, it gives that, if G is discrete, torsion-free and with compact classifying space,

$$RK_j^G(\underline{E}G) = KK_j(C_0(BG),\mathbb{C})$$

for $j=0,1$.

To define the Baum-Connes assembly map that will link the K-theory of the reduced C^*-algebra of G with the G-equivariant K-homology with G-compact supports of $\underline{E}G$, an intermediate group is needed. For this, let us generalize the full and reduced group C^*-algebra concept.

Let A be a G-C^*-algebra. We provide $C_c(G,A)$, the set of continuous functions from G to A with compact support, with the twisted convolution given by

$$(f * f')(g) = \int_G f(h)(h \cdot f'(h^{-1}g))dh$$

for every $f, f' \in C_c(G,A)$, $g \in G$, and the involution defined by

$$f^*(g) = (g \cdot f(g^{-1}))^*$$

for every $f \in C_c(G,A)$ and $g \in G$.

The left regular representation of $C_c(G,A)$ on $L^2(G,A)$ is the one given by

$$(f\cdot\xi)(g)=\int_G (g^{-1}\cdot f(h))\xi(h^{-1}g)dh$$

for $f\in C_c(G,A)$, $\xi\in L^2(G,A)$ and $g\in G$. This gives a representation by adjointable operators.

Definition 1.3.10. *Let A be a G-C^*-algebra.*

1) *The reduced crossed product of A by G, written $A\rtimes_r G$, is the C^*-algebra obtained by completing $C_c(G,A)$ with respect to the operator norm on the Hilbert A-module $L^2(G,A)$ given by the left regular representation.*

2) *The full crossed product of A by G, written $A\rtimes G$, is the C^*-algebra obtained by completing $C_c(G,A)$ with respect to the norm*

$$||f||:=\sup_\rho ||\rho(f)||,$$

where ρ ranges over all non-degenerate representations of $C_c(G,A)$ on some Hilbert C^*-module H over A.

Once again, if the group G is amenable, then the full and reduced crossed products constructions coincide.

It appears clearly that those crossed products constructions generalize the reduced and full C^*-algebras, that is, taking $A=\mathbb{C}$ and seeing it as a G-C^*-algebra by considering the trivial action of G, we obtain

$$\mathbb{C}\rtimes_r G=C_r^*(G)\quad\text{and}\quad\mathbb{C}\rtimes G=C^*(G).$$

Furthermore, assuming that G is a semi-direct product $G=N\rtimes H$, we get that the action of H on N induces a H-C^*-algebra structure on $C^*(N)$ by first defining an action of H on $C_c(N)$ by setting

$$(h\cdot f)(n):=f(h^{-1}\cdot n)$$

for all $h\in H$, $f\in C_c(N)$, $n\in N$, and then completing it into an action of H on $C^*(N)$.

Proposition 1.3.11. *If G is a semi-direct product $G=N\rtimes H$, then*

$$C_r^*(N)\rtimes_r H\cong C_r^*(G)\quad\textit{and}\quad C^*(N)\rtimes H\cong C^*(G).$$

If we suppose G discrete and if we are given a G-C^*-algebra A, then $A\rtimes_r G$ and $A\rtimes G$ can be seen as the completions in the corresponding norm given above of the algebra defined by the finite formal sums of the form

$$\sum_{i=1}^n x_i u_{g_i},\quad x_i\in A,\ g_i\in G,$$

where to every $g \in G$ we have associated an abstract unitary u_g in a way in which

$$u_g u_{g'} = u_{gg'} \quad \text{and} \quad u_g x u_g^* = g \cdot x$$

holds for all $g, g' \in G$ and $x \in A$. For example, if $G = \mathbb{Z}$, then its action on A is given by a single automorphism $\alpha \in \mathrm{Aut}(A)$. Thus, as a C^*-algebra, the crossed product $A \rtimes \mathbb{Z}$ is generated by A and a single abstract unitary u such that

$$uxu^* = \alpha(x)$$

for every $x \in A$. In this special case, to emphasize the dependence on the action α, we will sometimes denote $A \rtimes_\alpha \mathbb{Z}$ for the crossed product.

1.3.2 Statement of the conjecture

Let X be a proper and G-compact G-space. Following [54] we can find a real valued function $f_X \in C_c(X)$ such that $f_X \geq 0$ and

$$\int_G f_X(gx)dg = 1.$$

Defining

$$(p_X(g))(x) := \sqrt{f_X(x) f_X(g^{-1}x)}$$

for $g \in G$ and $x \in X$, we get a well-defined projection in the convolution algebra $C_c(G, C_0(X))$, and therefore a class in $K_0(C_0(X) \rtimes_r G) \cong KK_0(\mathbb{C}, C_0(X) \rtimes_r G)$. Indeed, for every $g \in G$ and $x \in X$, we have

$$\begin{aligned}
(p_X^*(g))(x) &= (g \cdot p_X(g^{-1}))^*(x) = (g \cdot p_X(g^{-1}))(x) \\
&= (p_X(g^{-1}))(g^{-1}x) = \sqrt{f_X(g^{-1}x) f_X(gg^{-1}x)} \\
&= \sqrt{f_X(x) f_X(g^{-1}x)} = (p_X(g))(x)
\end{aligned}$$

and

$$\begin{aligned}
((p_X * p_X)(g))(x) &= \left(\int_G p_X(h)(h \cdot p_X(h^{-1}g))dh \right)(x) \\
&= \int_G (p_X(h)(h \cdot p_X(h^{-1}g)))(x)dh \\
&= \int_G (p_X(h))(x) \cdot (p_X(h^{-1}g))(h^{-1}x)dh \\
&= \int_G \sqrt{f_X(x) f_X(h^{-1}x)} \sqrt{f_X(h^{-1}x) f_X(g^{-1}hh^{-1}x)} dh \\
&= \sqrt{f_X(x) f_X(g^{-1}x)} \int_G f_X(h^{-1}x)dh \\
&= \sqrt{f_X(x) f_X(g^{-1}x)} = (p_X(g))(x).
\end{aligned}$$

If X is a G-compact subspace of $\underline{E}G$, we can compose the descent homomorphism and the Kasparov product with $[p_X]$ to obtain a map $\mu^G_{j,X}$ for $j = 0, 1$:

$$KK^G_j(C_0(X), \mathbb{C}) \xrightarrow{\jmath_G} KK_j(C_0(X) \rtimes_r G, C^*_r(G)) \xrightarrow{[p_X]} KK_j(\mathbb{C}, C^*_r(G)).$$

By the universal property of the colimit, we obtain then the so-called Baum-Connes assembly map for G :

$$\mu^G_j : RK^G_i(\underline{E}G) \longrightarrow K_j(C^*_r(G)),$$

where $j = 0, 1$.

Conjecture 1. *(Baum-Connes, 1982). The Baum-Connes assembly map is an isomorphism for any locally compact, Hausdorff and second countable group.*

There is a huge class of groups for which the Baum-Connes conjecture has been proved. This included :

- finite groups,
- abelian groups,
- free groups (Pimsner-Voiculescu [46]),
- discrete subgroups of amenable, connected Lie groups (Kasparov [27]),
- discrete subgroups of $SO(n, 1)$ and $SU(n, 1)$ (Kasparov [28] & Julg-Kasparov [24]),
- groups with the Haagerup property (Higson-Kasparov [21]),
- Gromov hyperbolic groups acting properly and cocompactly on a CAT(0) space and hence cocompact lattices of the simple Lie groups $SO(n, 1)$, $SU(n, 1)$, $Sp(n, 1)$ and $F_4(-20)$ (V. Lafforgue [30]),
- cocompact lattices of $SL_3(F)$, where F is $\mathbb{R}$, $\mathbb{C}$ or a p-adic field (V. Lafforgue [31]),
- one-relator groups (Beguin-Bettaieb-Valette [3]),
- subgroups of Gromov hyperbolic groups (Mineyev-Yu [38]).
- almost connected groups (Chabert-Echterhoff-Nest [15]),

It is however interesting to notice that the conjecture is still not known for $SL_n(\mathbb{Z})$ if $n \geq 3$.

Moreover, injectivity of the assembly map has been proved for discrete subgroups of real Lie groups having a finite number of connected components (Kasparov [26]), groups acting on euclidean buildings (Kasparov-Skandalis [29]), and countable groups admitting an amenable action on a compact space (Higson [20]). The injectivity of the Baum-Connes map has a number of consequences in topology, for example the Novikov and the Gromov-Lawson-Rosenberg conjectures.

Conjecture 2. *(Novikov, 1970). Higher signatures are oriented homotopy invariants for closed oriented manifolds.*

Theorem 1.3.12. *(Mishchenko [39]). The Novikov conjecture holds for closed oriented manifolds with fundamental group for which the Baum-Connes assembly map is injective.*

Conjecture 3. *(Gromov-Lawson-Rosenberg, 1983). Let M be a closed spin manifold endowed with a metric of positive scalar curvature. Then all higher $\widehat{A}$-genera of M do vanish.*

Theorem 1.3.13. *(Rosenberg [49]). If the Baum-Connes assembly map for $\pi_1(M)$ is injective, then conjecture 3 is true for M.*

Furthermore, surjectivity has its applications rather in analysis, for example the conjecture of idempotents.

Conjecture 4. *(Kaplansky-Kadison, 1949). If G is a torsion-free and discrete group, then $C_r^*(G)$ has no idempotent other than 0 and 1.*

Theorem 1.3.14. *(Baum-Connes-Higson [5]). If G is a torsion-free and discrete group such that the Baum-Connes assembly map is surjective, then the Kaplansky-Kadison conjecture holds for G.*

Conjecture 1 can be extended to a more general one, called the Baum-Connes conjecture with coefficients. Let A be a G-C^*-algebra. We define the G-equivariant K-homology with G-compact supports and with coefficients in A of $\underline{E}G$ to be the colimit of the direct system formed by the groups

$$KK_j^G(C_0(X), A), \quad j = 0, 1,$$

where X runs over all the G-compact subspaces of $\underline{E}G$ ordered by the inclusion. We denote it by

$$RK_j^G(\underline{E}G, A), \qquad j = 0, 1.$$

Thus, taking the colimit over all G-compact subspaces $X \subseteq \underline{E}G$ of the composition

$$KK_j^G(C_0(X), A) \xrightarrow{j_G} KK_j(C_0(X) \rtimes_r G, A \rtimes_r G) \xrightarrow{[p_X]} KK_j(\mathbb{C}, A \rtimes_r G),$$

we obtain a homomorphism $\mu_j^{G,A} : RK_j^G(C_0(\underline{E}G), A) \longrightarrow K_j(A \rtimes_r G)$.

Conjecture 5. *(Baum-Connes with coefficients, 1982). Let A be a G-C^*-algebra. Then $\mu_j^{G,A}$ is an isomorphism for $j = 0, 1$.*

Taking $A = \mathbb{C}$ with the trivial action of G, we find the initial Baum-Connes conjecture.

In 1997, Hervé Oyono-Oyono proved conjecture 5 for discrete countable groups acting simplicialy on a tree whose vertices and edges stabilizators satisfy the conjecture with coefficients (see [41]). Two years later, he also proved this conjecture for groups G fitting into a short exact sequence

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$$

such that H and every subgroup of G containing F as a finite index subgroup satisfy the Baum-Connes conjecture with coefficients. This result was published in [42].

Furthermore, a result of Chabert and Echterhoff in [13] shows that conjecture 5 do pass to subgroups, whereas it is still not know for conjecture 1. Moreover, the results obtained by Higson and Kasparov in [21] showed the conjecture with coefficients for groups having Haagerup property. Recently, V. Lafforgue has announced a proof of this conjecture for Gromov hyperbolic groups (see [32]).

However, in 2001, the paper [22] written by Higson, V. Lafforgue and Skandalis wiped out any hope to prove this conjecture. Indeed, they showed that finite generated groups whose Cayley graph contains arbitrarily large expanders (existence of such groups are insured by Gromov) fails to verify conjecture 5.

Nevertheless it is still interesting to find the groups or the C^*-algebras for which this conjecture holds. For example, it is not known if the conjecture with commutative coefficients is stable under direct products. Similarly, as for the one without coefficients, it is an open question whether groups having a finite index subgroup satisfying the conjecture with coefficients also verify this conjecture or not.

1.4 Pimsner-Voiculescu exact sequence

We end this chapter by presenting the main tool that will be used in the K-theory computations of chapter 3.

In this section, A will denote a unital C^*-algebra and $\alpha : \mathbb{Z} \longrightarrow \mathrm{Aut}(A)$ will be an action of $\mathbb{Z}$ on A. Since $\mathbb{Z}$ is amenable, the reduced and full crossed products of A by $\mathbb{Z}$ coincide. Moreover, by the argument developed on page 39, the resulting crossed product $A \rtimes_\alpha \mathbb{Z}$ is generated by A and an abstract unitary u such that $\alpha(x) = uxu^*$ for all $x \in A$.

Let S be a non-unitary isometry (for example the right shift operator on $\ell^2(\mathbb{N})$) and $C^*(S)$ be the C^*-algebra generated by S. It is nuclear and unital, with unit denoted by I. Since all C^*-algebras generated by a single non-unitary isometry are isomorphic, there is no ambiguity about the choice of such a S. Also, we set $T := I - SS^*$.

Definition 1.4.1. *The Toeplitz algebra for the couple (A, α) is the sub-C^*-algebra of $(A \rtimes_\alpha \mathbb{Z}) \otimes C^*(S)$ generated by $A \otimes I$ and $u \otimes S$. We denote it by $\mathcal{T}_{A,\alpha}$ or simply by $\mathcal{T}$ whenever no confusion is possible.*

There is a *-homomorphism $\varphi : \mathcal{K} \longrightarrow C^*(S)$ given by $\varphi(e_{ij}) = S^iTS^{*j}$. The image of φ is the closed two-sided ideal of $C^*(S)$ generated by T. The quotient C^*-algebra $C^*(S)/\operatorname{im}(\varphi)$ is then isomorphic to $C(\mathbb{T})$, with isomorphism induced by the *-homomorphism $\psi : C^*(S) \longrightarrow C(\mathbb{T})$ associating to S the inclusion $\iota : \mathbb{T} \longrightarrow \mathbb{C}$. In brief, we get the short exact sequence :

$$0 \longrightarrow \mathcal{K} \xrightarrow{\varphi} C^*(S) \xrightarrow{\psi} C(\mathbb{T}) \longrightarrow 0.$$

Tensoring by $A \rtimes_\alpha \mathbb{Z}$ (what is unambiguous since $\mathcal{K}$, $C^*(S)$ and $C(\mathbb{T})$ are nuclear) and defining

$$\begin{aligned} \tilde{\varphi} : A \otimes \mathcal{K} &\longrightarrow \mathcal{T} \\ x \otimes e_{ij} &\longmapsto u^i x u^{*j} \otimes \varphi(e_{ij}), \end{aligned}$$

Pimsner and Voiculescu showed in [45] that

$$\operatorname{im}(\tilde{\varphi}) = ((A \rtimes_\alpha \mathbb{Z}) \otimes \varphi(\mathcal{K})) \cap \mathcal{T}$$

and deduced the short exact sequence

$$0 \longrightarrow A \otimes \mathcal{K} \xrightarrow{\tilde{\varphi}} \mathcal{T} \xrightarrow{\tilde{\psi}} A \rtimes_\alpha \mathbb{Z} \longrightarrow 0,$$

since the image of $\tilde{\psi} = (\mathrm{id} \otimes \psi)|_{\mathcal{T}}$ is the sub-C^*-algebra of $(A \rtimes_\alpha \mathbb{Z}) \otimes C(\mathbb{T})$ generated by $A \otimes 1$ and $u \otimes \iota$, which is isomorphic to $A \rtimes_\alpha \mathbb{Z}$. We call this sequence the Toeplitz extension for (A, α).

Lemma 1.4.2. *The *-homomorphism*

$$\begin{aligned} d : A &\longrightarrow \mathcal{T} \\ x &\longmapsto x \otimes I \end{aligned}$$

induces a natural isomorphism in K-theory.

Theorem 1.4.3. *(Pimsner-Voiculescu [45]). Under the isomorphisms induced by $d : A \longrightarrow \mathcal{T}$ and $e : A \longrightarrow A \otimes \mathcal{K}$, the Toeplitz extension for the couple (A, α) induces a six-terms exact sequence :*

$$\begin{tikzcd} K_0(A) \arrow[r, "\mathrm{id} - \alpha_*^{-1}"] & K_0(A) \arrow[r, "i_*"] & K_0(A \rtimes_\alpha \mathbb{Z}) \arrow[d, "\partial"] \\ K_1(A \rtimes_\alpha \mathbb{Z}) \arrow[u, "\partial"] & K_1(A) \arrow[l, "i_*"'] & K_1(A). \arrow[l, "\mathrm{id} - \alpha_*^{-1}"'] \end{tikzcd}$$

To say one word about naturality let A' be another C^*-algebra, α' be an automorphism of A' and $\phi : A \longrightarrow A'$ be a *-homomorphism such that

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\phi"'] & A \arrow[d, "\phi"] \\ A' \arrow[r, "\alpha'"'] & A' \end{tikzcd}$$

commutes. Then ϕ extends to a *-homomorphism

$$\tilde{\phi} : A \rtimes_\alpha \mathbb{Z} \longrightarrow A' \rtimes_{\alpha'} \mathbb{Z}.$$

Moreover, the restriction of $\tilde{\phi} \otimes \mathrm{id}_{C^*(S)}$ to $\mathcal{T}_{A,\alpha}$ has its image included in $\mathcal{T}_{A',\alpha'}$.

Thus, we can prove that the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \otimes \mathcal{K} \arrow[r, "\tilde{\varphi}"] \arrow[d, "\phi\otimes\mathrm{id}_{\mathcal{K}}"'] & \mathcal{T}_{A,\alpha} \arrow[r, "\tilde{\psi}"] \arrow[d, "\tilde{\phi}\otimes\mathrm{id}_{C^*(S)}"] & A \rtimes_\alpha \mathbb{Z} \arrow[r] \arrow[d, "\tilde{\phi}"] & 0 \\
0 \arrow[r] & A' \otimes \mathcal{K} \arrow[r, "\tilde{\varphi}'"] & \mathcal{T}_{A',\alpha'} \arrow[r, "\tilde{\psi}'"] & A' \rtimes_{\alpha'} \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

commutes. Since e and d induces isomorphisms in K-theory that are natural, theorem 1.1.18 implies the naturality of the Pimsner-Voiculescu exact sequence.

Since we will make some concrete computations afterwards, we now explain how the vertical arrows in theorem 1.4.3 work. Consider first the left-hand connecting *-homomorphism

$$\partial : K_1(A \rtimes_\alpha \mathbb{Z}) \longrightarrow K_0(A),$$

which is called the index map. For a class in $K_1(A \rtimes_\alpha \mathbb{Z})$ given by $x \in \mathcal{U}_n(A \rtimes_\alpha \mathbb{Z})$, choose a $x' \in \mathcal{U}_k(A \rtimes_\alpha \mathbb{Z})$ such that $x \oplus x'$ lives in the same connected component of $\mathcal{U}_\infty(A \rtimes_\alpha \mathbb{Z})$ as the identity matrix I_{n+k}, that is,

$$[x \oplus x'] = [I_{n+k}]$$

holds in $K_1(A \rtimes_\alpha \mathbb{Z})$. For example, one can take $x' = x^*$, since $x \oplus x^*$ is linked to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ via the path in $\mathcal{U}_{2n}(A \rtimes_\alpha \mathbb{Z})$ given by

$$\begin{pmatrix} \cos(\frac{\pi}{2}t)x & -\sin(\frac{\pi}{2}t) \\ \sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t)x^* \end{pmatrix}, \quad t \in [0,1].$$

and since the path

$$\begin{pmatrix} \sin(\frac{\pi}{2}t) & -\cos(\frac{\pi}{2}t) \\ \cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \end{pmatrix}, \quad t \in [0,1],$$

connects $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to the identity matrix I_{2n}. After that, find a $y \in \mathcal{U}_{n+k}(\mathcal{T}_{A,\alpha})$ such that

$$\tilde{\psi}(y) = x \oplus x'.$$

Denoting by p_n the diagonal matrix in $\mathcal{M}_{n+k}(\mathcal{T}_{A,\alpha})$ whose first n diagonal elements are $1 \otimes I$ and last k are 0, we get

$$yp_ny^* - p_n \in \ker(\tilde{\psi}) = \mathrm{im}(\tilde{\varphi}).$$

Therefore, by the injectivity of $\tilde{\varphi}$, there exists a unique $z \in \mathcal{M}_{n+k}(A \otimes \mathcal{K})$ such that

$$\tilde{\varphi}(z) = yp_ny^* - p_n.$$

Let us define $P_n := I_n \oplus 0_k \in \mathcal{M}_{n+k}(\mathbb{C})$. On the one hand, we have

$$\begin{aligned}\tilde{\varphi}^+((z+P_n)^*) &= (\tilde{\varphi}(z)+P_n)^* \\ &= (yp_ny^* - p_n + P_n)^* \\ &= yp_ny^* - p_n + P_n \\ &= \tilde{\varphi}(z) + P_n \\ &= \tilde{\varphi}^+(z+P_n).\end{aligned}$$

On the other hand, noticing that the equalities $P_nM = p_nM$ and $MP_n = Mp_n$ hold in $\mathcal{M}_{n+k}(\mathcal{T}^+_{A,\alpha})$ for all $M \in \mathcal{M}_{n+k}(\mathcal{T}_{A,\alpha})$, we have

$$\begin{aligned}\tilde{\varphi}^+((z+P_n)^2) &= (\tilde{\varphi}(z)+P_n)^2 \\ &= (yp_ny^* - p_n + P_n)^2 \\ &= yp_ny^* + (P_n-p_n)^2 + yp_ny^*(P_n-p_n) + (P_n-p_n)yp_ny^* \\ &= yp_ny^* + P_n - 2p_n + p_n + yp_ny^*(p_n-p_n) + (p_n-p_n)yp_ny^* \\ &= yp_ny^* - p_n + P_n \\ &= \tilde{\varphi}(z) + P_n \\ &= \tilde{\varphi}^+(z+P_n).\end{aligned}$$

Together with the injectivity of $\tilde{\varphi}^+$, this proves that the element $z + P_n$ is a projection in $\mathcal{M}_{n+k}((A \otimes \mathcal{K})^+)$. Then we have

$$\partial([x]) = [z+P_n] - [P_n].$$

The map $\partial : K_0(A \rtimes_\alpha \mathbb{Z}) \longrightarrow K_1(A)$, called the exponential map, is connected to the suspension of the index map via the isomorphisms of theorems 1.1.13 and 1.1.14 in a commutative diagram :

$$\begin{tikzcd} K_0(A \rtimes_\alpha \mathbb{Z}) \arrow[r, "\partial"] \arrow[d, "\beta"'] & K_1(A) \arrow[d, "\theta"] \\ K_1(S(A \rtimes_\alpha \mathbb{Z})) \arrow[r, "S\partial"'] & K_0(SA). \end{tikzcd}$$

One way to give the image of a class $[p] \in K_0(A \rtimes_\alpha \mathbb{Z})$ concretely is to find a self-adjoint preimage $y \in \mathcal{T}$ of p for $\tilde{\psi}$. Then $\exp(2\pi i y)$ is a unitary in $\ker(\tilde{\psi}) = \operatorname{im}(\tilde{\varphi})$, hence lifts uniquely to a unitary $x \in (A \otimes \mathcal{K})^+$. Thus we have

$$\partial([p]) = [x].$$

However, Pimsner and Voiculescu showed in [45] that for projections in $A \rtimes_\alpha \mathbb{Z}$ of a special kind, we can compute the image of their classes in a different way that we explain now, as it will be crucial for us.

Definition 1.4.4. *A Rieffel projection in $A \rtimes_\alpha \mathbb{Z}$ is a self-adjoint idempotent of the form $u^*x_1^* + x_0 + x_1u$, where $x_0, x_1 \in A$.*

We immediately deduce from the self-adjointness condition that x_0 is self-adjoint, and from the idempotence the relations

- $x_0 = x_0^2 + \alpha^{-1}(x_1^*x_1) + x_1x_1^*$,
- $x_1 = x_0x_1 + x_1\alpha(x_0)$,
- $0 = \alpha^{-1}(x_1)x_1$.

In fact, those conditions caracterise the Rieffel projections : a self-adjoint element $x_0 \in A$ and an element $x_1 \in A$ satisfying the three relations above define a Rieffel projection in $A \rtimes_\alpha \mathbb{Z}$.

We recall that the universal representation of A, denoted by $\pi_u : A \longrightarrow \mathcal{L}(H_u)$, is the direct sum of all irreducible representations of A. Furthermore, the commutant of a subset $S \subseteq \mathcal{L}(H_u)$ is the set of all bounded operators commuting with every element of S.

Definition 1.4.5. *The enveloping von Neumann algebra of A is the bicommutant - that is, the commutant of the commutant - of A in $\mathcal{L}(H_u)$.*

In fact, as Banach spaces, the bicommutant of A identifies with its bidual (cf [44]).

Definition 1.4.6. *The left support projection of an element $x \in A$ is the element of $\mathcal{L}(H_u)$ defined by the projection on the closure of* $\operatorname{im} x$; *it is denoted by l_x. The right support projection of x is the projection on* $(\ker x)^\perp$; *it is denoted by r_x.*

Left and right support projections live in the enveloping von Neumann algebra of A. Furthermore, we have $l_x x = x$ and $x r_x = x$ for every $x \in A$. Moreover, if x is a self-adjoint element of A, then $l_x = r_x$.

In the special case $A = C(X)$, where X is a compact space, we can see $\ell^\infty(X)$, the space of bounded functions on X, as a topological direct summand of the enveloping von Neumann algebra of A (see [25]). Furthermore, the left support projection of a function $f \in C(X)$ is the characteristic function on its support. It follows that if $A = \mathcal{M}_n(C(X))$ and $f \in A$, the left support projection of f is the element in $\mathcal{M}_n(\ell^\infty(X))$ whose value in $x \in X$ is the projection on the image of $f(x)$ seen as an endomorphism of $\mathbb{C}^n$.

Proposition 1.4.7. *(Pimsner-Voiculescu [45]). Let $p = u^*x_1^* + x_0 + x_1u \in A \times_\alpha \mathbb{Z}$ be a Rieffel projection. Then the unitary* $\exp(2\pi i x_0 l_{x_1})$ *is in A and the homomorphism*

$$\partial : K_0(A \times_\alpha \mathbb{Z}) \longrightarrow K_1(A)$$

maps $[p]$ *to* $[\exp(2\pi i x_0 l_{x_1})]$.

To be more complete, we should mention that there is another way to obtain the Pimsner-Voiculescu exact sequence. It is given by Paschke in [43], following a theorem of Connes. Although we will not use this result, we find that this alternative approach is interesting and we give here its guideline. Let us consider the mapping torus of the pair (A, α), namely the C^*-algebra

$$T_\alpha(A) := \{f \in C([0,1], A) \mid f(0) = \alpha(f(1))\}.$$

Now look at $e_0 : T_\alpha(A) \longrightarrow A$, the evaluation at 0. This *-homomorphism is clearly surjective and its kernel is SA, the suspension of A. But for $j = 0, 1$, the K-theory group $K_j(SA)$ identifies with $K_{j-1}(A)$ via theorems 1.1.13 and 1.1.14. Hence the short exact sequence

$$0 \longrightarrow SA \longrightarrow T_\alpha(A) \xrightarrow{e_0} A \longrightarrow 0$$

induces the following six-terms exact sequence :

$$\begin{tikzcd}
K_1(A) \arrow[r, "\rho"] & K_0(T_\alpha(A)) \arrow[r, "(e_0)_*"] & K_0(A) \arrow[d] \\
K_1(A) \arrow[u] & K_1(T_\alpha(A)) \arrow[l, "(e_0)_*"'] & K_0(A). \arrow[l, "\rho"']
\end{tikzcd}$$

Lemma 1.4.8. *The vertical *-homomorphisms in the six-terms exact sequence above are both* $\mathrm{id} - \alpha_*^{-1}$.

Theorem 1.4.9. *(Connes [16]). For $j = 0, 1$, there is an isomorphism*

$$\gamma : K_j(T_\alpha(A)) \longrightarrow K_{j-1}(A \rtimes_\alpha \mathbb{Z})$$

such that $\gamma\rho = i_*$ and $\partial\gamma = (e_0)_*$.

Then the following diagram with exact rows is commutative for $j = 0, 1$:

$$\begin{tikzcd}
K_j(A) \arrow[r, "\mathrm{id} - \alpha_*^{-1}"] \arrow[d, equal] & K_j(A) \arrow[r, "\rho"] \arrow[d, equal] & K_{j-1}(T_\alpha(A)) \arrow[r, "(e_0)_*"] \arrow[d, "\gamma"] & K_{j-1}(A) \arrow[r, "\mathrm{id} - \alpha_*^{-1}"] \arrow[d, equal] & K_{j-1}(A) \arrow[d, equal] \\
K_j(A) \arrow[r, "\mathrm{id} - \alpha_*^{-1}"'] & K_j(A) \arrow[r, "i_*"'] & K_j(A \rtimes_\alpha \mathbb{Z}) \arrow[r, "\partial"'] & K_{j-1}(A) \arrow[r, "\mathrm{id} - \alpha_*^{-1}"'] & K_{j-1}(A).
\end{tikzcd}$$

It follows that the six-terms exact sequence given by Paschke is exactly the same as the Pimsner-Voiculescu one, up to suitable isomomorphisms.

Chapter 2

The geometric description of K-homology

In this chapter we explain the geometrical point of view for the K-homology of a topological space. This approach is due to Baum and Douglas (see [6]) and is the original framework in which the Baum-Connes conjecture has been described.

2.1 Clifford algebras

This section is mostly based on Friedrich's book [18].

2.1.1 The Clifford algebra of a quadratic form

Definition 2.1.1. *Let K be a field, V be a vector space over it and $Q : V \longrightarrow K$ be a quadratic form. A Clifford algebra for Q is a unital and associative K-algebra $C(Q)$ together with a linear map $j : V \longrightarrow C(Q)$ such that*

$$j(v)^2 = Q(v) \cdot 1_{C(Q)}, \quad \forall v \in V,$$

satisfying the following universal property : for every unital and associative K-algebra A and every linear map $i : V \longrightarrow A$ such that $i(v)^2 = Q(v) \cdot 1_A$ for all $v \in V$, there exists a unique morphism of algebras $\varphi : C(Q) \longrightarrow A$ making the diagram

$$\begin{tikzcd}
V \arrow[r, "j"] \arrow[dr, "i"'] & C(Q) \arrow[d, "\varphi"] \\
 & A
\end{tikzcd}$$

commutative.

Proposition 2.1.2. *Let K be a field, V be a K-vector space and Q be a quadratic form on V. Consider the graded tensor algebra of V,*

$$T(V) := K \oplus \bigoplus_{n \in \mathbb{N}^*} \bigotimes_{i=1}^{n} V,$$

and denote by $I(Q)$ its two-sided ideal generated by the set $\{v \otimes v - Q(v) \mid v \in V\}$. Then

$$T(V)/I(Q)$$

is a Clifford algebra for Q, with $j : V \longrightarrow T(V)/I(Q)$ being given by the identification of V with the degree one part of $T(V)$.

It follows directly from the universal property that a Clifford algebra for Q is unique up to unique isomorphism of algebras, namely, if $C(Q)$ and $C(Q)'$ are Clifford algebras for Q with underlying maps j and j', then there exists a unique isomorphism of algebras $\varphi : C(Q) \longrightarrow C(Q)'$ such that

$$\begin{tikzcd}
V \arrow[r, "j"] \arrow[dr, "j'"'] & C(Q) \arrow[d, "\varphi"] \\
 & C(Q)'
\end{tikzcd}$$

commutes. Thus we will constantly speak about *the* Clifford algebra for the quadratic form Q, although it is formally incorrect, and will see it as the quotient of the tensor algebra of V given by the proposition above. Moreover, to lighten the notations, an element in $C(V)$ will be denoted as an element in $T(V)$ and the product law will be written $\cdot$ instead of $\otimes$.

Corollary 2.1.3. *The map $j : V \longrightarrow C(Q)$ is injective and its image generates multiplicatively $C(Q)$.*

Considering the trivial quadratic form $Q = 0$, we get that $C(Q)$ is isomorphic as K-algebra to $\Lambda(V)$, the exterior algebra over V. In fact, for any quadratic form Q on V, we have a vector space isomorphism between $C(Q)$ and $\Lambda(V)$, natural if the characteristic of K is not 2, but this isomorphism does not imply a K-algebra isomorphism in general.

Proposition 2.1.4. *Let V be a vector space over a field K and Q_1, Q_2 be quadratic forms on V. Then*

$$C(Q_1 \oplus Q_2) \cong C(Q_1) \otimes C(Q_2)$$

as K-algebras.

Let K be a field of characteristic different from 2 and Q be a quadratic form on a K-vector space V. Then Q induces a bilinear form B_Q on the same space V by defining

$$B_Q(v_1, v_2) := \frac{1}{2} \left(Q(v_1 + v_2) - Q(v_1) - Q(v_2) \right)$$

for all $v_1, v_2 \in V$.

We note that if $v_1, v_2 \in V$ are such that $B_Q(v_1, v_2) = 0$, then the equality

$$v_1 \cdot v_2 = -v_2 \cdot v_1$$

holds in $C(Q)$. Indeed, we have

$$\begin{aligned}(v_1+v_2)\cdot(v_1+v_2) &= Q(v_1+v_2) = B_Q(v_1+v_2, v_1+v_2)\\ &= B_Q(v_1,v_1) + 2B_Q(v_1,v_2) + B_Q(v_2,v_2)\\ &= Q(v_1) + Q(v_2)\end{aligned}$$

and

$$\begin{aligned}(v_1+v_2)\cdot(v_1+v_2) &= v_1\cdot v_1 + v_1\cdot v_2 + v_2\cdot v_1 + v_2\cdot v_2\\ &= Q(v_1) + v_1\cdot v_2 + v_2\cdot v_1 + Q(v_2).\end{aligned}$$

Theorem 2.1.5. *Let K be a field of characteristic different from two, V be a n-dimensional vector space over K and $Q : V \longrightarrow K$ be a quadratic form. Then there exists a basis $\{v_1, ..., v_n\}$ of V in which the matrix of the bilinear form associated to Q is of the form*

$$B_Q = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix},$$

where $\lambda_1, ..., \lambda_n \in K$. Moreover, if $K = \mathbb{R}$, then the λ_i can be choosen to be in $\{-1, 0, 1\}$, and if $K = \mathbb{C}$, then they can be in $\{0, 1\}$.

Corollary 2.1.6. *The K-vector space $C(Q)$ is 2^n-dimensional, where $n = \dim(V)$. A basis can be given by*

$$\{1\} \cup \{v_{i_1} \cdot ... \cdot v_{i_s} \mid 1 \leq s \leq n, 1 \leq i_1 < ... < i_s \leq n\},$$

where $\{v_1, ..., v_n\}$ is a basis of V.

2.1.2 Real and complex Clifford algebras

Definition 2.1.7. *Let $n \in \mathbb{N}$. The real n-dimensional negative definite Clifford algebra, denoted by $Cl(n)$, is the one associated to the quadratic form on $\mathbb{R}^n$ given by*

$$Q(x_1, ..., x_n) := -x_1^2 - ... - x_n^2$$

for all $(x_1, ..., x_n) \in \mathbb{R}^n$.

- For $n = 1$, the $\mathbb{R}$-algebra $Cl(1)$ is isomorphic to $\mathbb{C}$ via the map given by

$$\begin{aligned} Cl(1) &\longrightarrow \mathbb{C} \\ e_1 &\longmapsto i \end{aligned}$$

  where e_1 is the basis element 1 of the vector space $\mathbb{R}$. Indeed, the relation $i^2 = -1 = Q(e_1)$ is verified.

- For $n = 2$, the Clifford algebra $Cl(2)$ is isomorphic to $\mathbb{H}$, the real quaternion algebra, via the homomorphism induced from

$$\begin{aligned} Cl(2) &\longrightarrow \mathbb{H} \\ e_1 &\longmapsto i \\ e_2 &\longmapsto j \end{aligned}$$

  where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$. We therefore obtain that $e_1 \cdot e_2$ is sent on k. Moreover, the relations $i^2 = -1$, $j^2 = -1$ and $ij = k = -ji$ are satisfied.

Definition 2.1.8. *Let $n \in \mathbb{N}$. The real n-dimensional positive definite Clifford algebra, denoted by $Cl'(n)$, is the one associated to the quadratic form on $\mathbb{R}^n$ given by*

$$Q(x_1, ..., x_n) := x_1^2 + ... + x_n^2$$

for all $(x_1, ..., x_n) \in \mathbb{R}^n$.

- We have $Cl'(1)$ isomorphic to the $\mathbb{R}$-algebra $\mathbb{R} \oplus \mathbb{R}$, with e_1 seen in it as the element $(1, -1)$.

- For $n = 2$, the real Clifford algebra $Cl'(2)$ is isomorphic to $\mathcal{M}_2(\mathbb{R})$ via the homomorphism given by

$$\begin{aligned} Cl(2) &\longrightarrow \mathcal{M}_2(\mathbb{R}) \\ e_1 &\longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ e_2 &\longmapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

  where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$. Indeed, we have the relations

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = I_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^2$$

and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Definition 2.1.9. *Let $n \in \mathbb{N}$. The complex n-dimensional Clifford algebra, denoted by $Cl^c(n)$, is the one associated to the quadratic form on $\mathbb{C}^n$ defined by*

$$Q(z_1, ..., z_n) := z_1^2 + ... + z_n^2$$

for all $(z_1, ..., z_n) \in \mathbb{C}^n$.

Proposition 2.1.10. *Let Q be a quadratic form over a real vector space V. We define on the complexification $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ of V the complex quadratic form $Q_{\mathbb{C}}$ given by*

$$Q_{\mathbb{C}}(v \otimes z) := Q(v)z^2$$

for every $v \in V$ and $z \in \mathbb{C}$. Then

$$C(Q_{\mathbb{C}}) \cong C(Q) \otimes_{\mathbb{R}} \mathbb{C}$$

as $\mathbb{C}$-algebras.

It follows that complexifications of positive and negative definite real n-dimensional Clifford algebras are both $Cl^c(n)$. The isomorphisms are given by

$$\begin{array}{rcl} Cl(n) \otimes_{\mathbb{R}} \mathbb{C} & \longrightarrow & Cl^c(n) \\ e_j \otimes 1 & \longmapsto & if_j \end{array} \quad \text{and} \quad \begin{array}{rcl} Cl'(n) \otimes_{\mathbb{R}} \mathbb{C} & \longrightarrow & Cl^c(n) \\ e_j \otimes 1 & \longmapsto & f_j \end{array}$$

where $\{e_j\}_{j=1}^n$ and $\{f_j\}_{j=1}^n$ are the canonical basis of $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively.

Proposition 2.1.11. *We have the $\mathbb{R}$-algebras isomorphisms*

$$Cl(n+2) \cong Cl'(n) \otimes_{\mathbb{R}} Cl(2) \quad \text{and} \quad Cl'(n+2) \cong Cl(n) \otimes_{\mathbb{R}} Cl'(2)$$

for all $n \in \mathbb{N}$.

We deduce the sequence of isomorphisms :

$$\begin{aligned} Cl^c(n+2) &\cong Cl'(n+2) \otimes_{\mathbb{R}} \mathbb{C} \\ &\cong (Cl(n) \otimes_{\mathbb{R}} Cl'(2)) \otimes_{\mathbb{R}} \mathbb{C} \\ &= (Cl(n) \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (Cl'(2) \otimes_{\mathbb{R}} \mathbb{C}) \\ &\cong Cl^c(n) \otimes_{\mathbb{C}} \mathcal{M}_2(\mathbb{C}). \end{aligned}$$

Together with the fact that $Cl^c(1) \cong Cl'(1) \otimes_{\mathbb{R}} \mathbb{C} \cong (\mathbb{R} \oplus \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$, we obtain the next fundamental result.

Corollary 2.1.12. *Let $n \in \mathbb{N}$.*

1) *If $n = 2k$, then*

$$Cl^c(n) \cong \bigotimes_{i=1}^{k} \mathcal{M}_2(\mathbb{C}) \cong \mathrm{End}(\mathbb{C}^{2^k}).$$

1) *If $n = 2k+1$, then*

$$Cl^c(n) \cong \bigotimes_{i=1}^{k} \mathcal{M}_2(\mathbb{C}) \oplus \bigotimes_{i=1}^{k} \mathcal{M}_2(\mathbb{C}) \cong \mathrm{End}(\mathbb{C}^{2^k}) \oplus \mathrm{End}(\mathbb{C}^{2^k}).$$

Let $k \in \mathbb{N}$. For $n \in \{2k, 2k+1\}$, we call $\mathbb{C}^{2^k}$ the vector space of n-spinors and write it Δ_n. Moreover, the spinorial representation of dimension n is the representation of $Cl^c(n)$ on Δ_n given by the isomorphism with $\mathrm{End}(\mathbb{C}^{2^k})$ if n is even, and by the isomorphism with $\mathrm{End}(\mathbb{C}^{2^k}) \oplus \mathrm{End}(\mathbb{C}^{2^k})$ followed by the projection on the first factor if n is odd. Hence Δ_n is a module over $Cl^c(n)$.

2.1.3 The groups $Spin(n)$ and $Spin^c(n)$

Let $n \in \mathbb{N}^*$. An element $x \in \mathbb{R}^n \setminus \{0\}$ is invertible in $Cl(n)$, since

$$x \cdot \frac{-1}{||x||^2} x = \frac{-1}{||x||^2} x \cdot x = \frac{1}{||x||^2} ||x||^2 = 1.$$

If x is in S^{n-1}, we note that its inverse is $-x$. The same argument shows that a non-zero element in $\mathbb{C}^n$ is invertible in $Cl^c(n)$.

Definition 2.1.13. *Let $n \in \mathbb{N}^*$.*

a) *The group $Pin(n)$ is the subgroup of $(Cl(n))^\times$ generated by the elements of S^{n-1}. Specifically, it is formed by products*

$$x_1 \cdot \ldots \cdot x_m$$

such that $m \in \mathbb{N}$ and $x_i \in S^{n-1}$, for all $i \in \{1, \ldots, m\}$.

b) *The group $Spin(n)$ is the subgroup of $Pin(n)$ composed by the products of an even numbers of elements in S^{n-1}.*

c) *The group $Spin^c(n)$ is the subgroup of $(Cl^c(n))^\times$ generated by $Spin(n)$ and $\mathbb{T} \subseteq \mathbb{C}$.*

Since $Spin(n) \cap \mathbb{T} = \{1, -1\}$, we remark that

$$Spin^c(n) \cong (Spin(n) \times \mathbb{T}) / \{(1,1), (-1,-1)\}.$$

The topology on the group $Spin(n)$ is given by restricting the spinorial representation to $Spin(n) \subseteq Cl(n) \subseteq Cl^c(n)$. It gives a faithful representation on Δ_n and therefore an injective map $Spin(n) \longrightarrow \mathcal{M}_{2^k}(\mathbb{C})$, where $k \in \mathbb{N}$ is such that $n \in \{2k, 2k+1\}$. Then we endow $Spin(n)$ with the topology induced by the one on $\mathcal{M}_{2^k}(\mathbb{C})$.

By convention, we define $Pin(0)$, $Spin(0)$ and $Spin^c(0)$ to be the trivial group.

Proposition 2.1.14. *Let $n \in \mathbb{N}$. The spinorial representation of $Spin(n)$ is irreducible if n is odd and decomposes into the direct sum of two irreducible subrepresentations if n is even.*

If $n = 2k$, with $k \in \mathbb{N}^*$, the two vector subspaces giving the decomposition are the eigenspaces of the linear transformation induced by the central element

$$i^k n \cdot e_1 \cdot ... \cdot e_n \in \mathcal{Z}(Spin(n)),$$

where $\{e_1, ..., e_n\}$ is the standard basis of $\mathbb{R}^n$. For $n = 0$, we have $Spin(0) = \{1\}$ and, by convention, we decompose $\Delta_0 = \mathbb{C}$ into the direct sum $\mathbb{C} \oplus 0$.

- As we have seen before, $Cl(1)$ is isomorphic to $\mathbb{C}$. Hence $Pin(1)$ is the group formed by products of i and $-i$, thus it is $\{1, -1, i, -i\}$. Therefore,

  $$Spin(1) = \{1, -1\} \cong \mathbb{Z}_2$$

  By the spinorial representation, 1 acts on $\Delta_1 = \mathbb{C}$ as the identity and -1 as $-\operatorname{id}_{\mathbb{C}}$. Furthermore, we get

  $$Spin^c(1) = \mathbb{T}.$$

- S^1 is viewed in $Cl(2) \cong \mathbb{H}$ as the subset

  $$\{\cos(\theta)i + \sin(\theta)j \mid \theta \in [0, 2\pi]\}.$$

  Since for $\theta, \theta' \in [0, 2\pi]$ we have

  $$\begin{aligned}(\cos(\theta)i + \sin(\theta)j) \cdot (\cos(\theta')i + \sin(\theta')j) =& \\ &= -\cos(\theta)\cos(\theta') + \cos(\theta)\sin(\theta')k - \sin(\theta)\cos(\theta')k - \sin(\theta)\sin(\theta') \\ &= -\cos(\theta - \theta') - \sin(\theta - \theta')k \\ &= \cos(\pi + \theta - \theta') + \sin(\pi + \theta - \theta')k\end{aligned}$$

  we obtain that

  $$Spin(2) = \{\cos(\vartheta) + \sin(\vartheta)k \mid \vartheta \in [0, 2\pi]\} \cong \mathbb{T}.$$

  By the sequence of isomorphisms

  $$Cl(2) \otimes_{\mathbb{R}} \mathbb{C} \cong Cl^c(2) \cong Cl'(2) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathcal{M}_2(\mathbb{R}) \otimes \mathbb{C} \cong \mathcal{M}_2(\mathbb{C})$$

  the element $(e_1 \cdot e_2) \otimes 1$ is sent on

  $$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence the spinorial representation of $Spin(2)$ maps the element given by a $\vartheta \in [0, 2\pi]$ on the linear transformation of $\Delta_2 = \mathbb{C}^2$ given by

$$\begin{pmatrix} \cos(\vartheta) & 0 \\ 0 & \cos(\vartheta) \end{pmatrix} + \begin{pmatrix} 0 & -\sin(\vartheta) \\ \sin(\vartheta) & 0 \end{pmatrix} = \begin{pmatrix} \cos(\vartheta) & -\sin(\vartheta) \\ \sin(\vartheta) & \cos(\vartheta) \end{pmatrix}.$$

Furthermore we have

$$Spin^c(2) = (\mathbb{T} \times \mathbb{T})/\{(1,1), (-1,-1)\}.$$

For $n \in \mathbb{N}^*$, we define $\gamma : Cl(n) \longrightarrow Cl(n)$ to be the linear map given by

$$\gamma(x) = x, \qquad \forall x \in \mathbb{R}^n$$

and by

$$\gamma(x \cdot y) = \gamma(y)\gamma(x), \qquad \forall x, y \in Cl(n).$$

If $n \geq 2$, we can prove that for $x \in S^{n-1}$ and $y \in \mathbb{R}^n$, the element $x \cdot y \cdot x$ corresponds exactly to the reflection of y in the hyperplane perpendicular to the vector x. Furthermore, if $n = 1$, then

$$i \cdot y \cdot i = -y = (-i) \cdot y \cdot (-i)$$

for every $y \in \mathbb{R}$.

Proposition 2.1.15. *Let $n \in \mathbb{N}^*$. Defining $(\lambda(x))(y) := x \cdot y \cdot \gamma(x)$ for $x \in Pin(n)$ and $y \in \mathbb{R}^n$, we get a continuous and surjective map*

$$\lambda : Pin(n) \longrightarrow O(n).$$

Moreover,

- $\ker(\lambda) = \{1, -1\}$,
- $\lambda^{-1}(SO(n)) = Spin(n)$,
- *$Spin(n)$ is connected if $n \geq 2$,*
- *$Spin(n)$ is simply connected and $\lambda|_{Spin(n)}$ is the universal covering of $SO(n)$ if $n \geq 3$.*

For $n = 0$, we set $SO(0) = 1$ and $\lambda = \mathrm{id}$.

By its properties, λ gives raise to a map

$$\begin{aligned} Spin^c(n) &\longrightarrow SO(n) \\ [x, y] &\longmapsto \lambda(x) \end{aligned}$$

that is well-defined, continuous and surjective. We will denote this map also by λ.

- For $n = 0, 1$, we have $SO(n) = 1$ and therefore λ is the trivial map on $Spin(n)$ and on $Spin^c(n)$.
- For $n = 2$, we have shown above that $Spin(2)$ is formed by products

$$(\cos(\theta)i + \sin(\theta)j) \cdot (\cos(\theta')i + \sin(\theta')j) = \cos(\pi + \theta - \theta') + \sin(\pi + \theta - \theta')k$$

with $\theta, \theta' \in [0, 2\pi]$. Thus we obtain, for $x_1 = \cos(\theta)i + \sin(\theta)j$ and $x_2 = \cos(\theta')i + \sin(\theta')j$,

$$\begin{aligned}(\lambda(x_1 \cdot x_2))(i) &= x_1 \cdot x_2 \cdot i \cdot x_2 \cdot x_1 \\ &= (\cos(\pi + \theta - \theta') + \sin(\pi + \theta - \theta')k) \cdot i \cdot \\ &\quad \cdot (\cos(\pi + \theta' - \theta) + \sin(\pi + \theta' - \theta)k) \\ &= \frac{1}{2}\left(1 + \cos(2\theta - 2\theta')\right) i + \frac{1}{2}\sin(2\theta - 2\theta')j + \\ &\quad + \frac{1}{2}\sin(2\theta - 2\theta')j + \frac{1}{2}\left(-1 + \cos(2\theta - 2\theta')\right) i \\ &= \cos(2\theta - 2\theta')i + \sin(2\theta - 2\theta')j\end{aligned}$$

and similarly

$$(\lambda(x_1 \cdot x_2))(j) = \cos(2\theta - 2\theta')j - \sin(2\theta - 2\theta')i.$$

Hence, since $\mathbb{R}^2$ is seen in $Cl^c(2) \cong \mathbb{H}$ as the vector subspace generated by i and j, if an element $x \in Spin(2) \cong \mathbb{T}$ is given by an angle $\vartheta \in [0, 2\pi]$, then $\lambda(x)$ is the rotation by an angle of 2ϑ. Therefore, identifying $SO(2)$ with $\mathbb{T}$ in the usual way, the map λ becomes :

$$\begin{aligned}\lambda : \mathbb{T} &\longrightarrow \mathbb{T} \\ x &\longmapsto x^2.\end{aligned}$$

2.2 Equivariant $Spin^c$ structures

In this section, G will always denote a locally compact, Hausdorff and second countable group, M will be a smooth connected proper G-manifold with boundary and V will be a vector bundle over M, of rank $n \in \mathbb{N}$, which is G-equivariant, meaning that V is endowed with an action of G such that any $g \in G$ sends the fiber over a $x \in M$ linearly on the fiber over $g \cdot x$.

If V is provided with G-invariant orientation and Euclidean structure, then we will denote by $SO(V)$ the bundle over M whose fiber over x is formed by the positively oriented orthonormal basis of the fiber of V over x, for every $x \in M$. It is a G-equivariant principal $SO(n)$-bundle.

If $n = 0$, then we set $SO(V) := M$.

2.2.1 G-$Spin^c$ structures on vector spaces

Definition 2.2.1. *A G-$Spin^c$ structure on V is the following data :*

- *a G-invariant orientation on V and an inner product on its fibers turning it into a G-equivariant Euclidean vector bundle,*
- *a G-equivariant principal $Spin^c(n)$-bundle P over M and a G-equivariant map Λ from P to $SO(V)$ such that*

$$\begin{tikzcd}
P \times Spin^c(n) \arrow[r] \arrow[dd, "\Lambda\times\lambda"'] & P \arrow[dd, "\Lambda"] \arrow[dr] & \\
 & & M \\
SO(V) \times SO(n) \arrow[r] & SO(V) \arrow[ur] &
\end{tikzcd}$$

commutes, where the horizontal arrows are given by the corresponding action.

If V is already endowed with an orientation and an Euclidean structure, then a G-$Spin^c$ structure on V consists only of the second point.

In the case of the definition above, and with the same notations, we can form the bundle

$$S := P \times_{Spin^c(n)} \Delta_n,$$

with $Spin^c(n)$ acting on Δ_n via the spinorial representation. Then S is a G-equivariant complex vector bundle over M, called the spinor bundle associated to the G-$Spin^c$ structure on V. If n is even, proposition 2.1.14 says that the spinorial representation of $Spin(n)$ decomposes into two irreducible subrepresentations, inducing a $\mathbb{Z}_2$-graduation on S. Moreover we have

$$V = P \times_{Spin^c(n)} \mathbb{R}^n,$$

with $Spin^c(n)$ acting on $\mathbb{R}^n$ via Λ and the usual action of $SO(n)$ on $\mathbb{R}^n$. Thus the action of $\mathbb{R}^n \subseteq Cl(n) \subseteq Cl^c(n)$ on Δ_n given by the spinorial representation induces a bundle morphism

$$V \otimes S \longrightarrow S$$

This map from $V \otimes S$ to S is called the Clifford multiplication.

Suppose V is endowed with a G-$Spin^c$ structure. Then it has an orientation and therefore an opposite one. Let us write $-V$ for V taken with this opposite orientation and the same Riemannian metric. We are looking for a G-$Spin^c$ structure on $-V$ induced from the one on V. For this, let us denote by P and Λ the bundle and map giving the G-$Spin^c$ structure on V.

Assume first that n is odd. Then

$$\begin{aligned} \tau : SO(V) &\longrightarrow SO(-V) \\ (E_1, ..., E_n) &\longmapsto (-E_1, ..., -E_n) \end{aligned}$$

is an isomorphism of principal G-equivariant $SO(n)$-bundles. Moreover we have, for elements $(E_1, ..., E_n) \in SO(V)$ and $(m_{ij}) \in SO(n)$:

$$\begin{aligned}
\tau((E_1, ..., E_n) \cdot (m_{ij})) &= \tau\left(\sum_{i=1}^{n} m_{i1}E_i, ..., \sum_{i=1}^{n} m_{in}E_i\right) \\
&= \left(-\sum_{i=1}^{n} m_{i1}E_i, ..., -\sum_{i=1}^{n} m_{in}E_i\right) \\
&= \left(\sum_{i=1}^{n} m_{i1}(-E_i), ..., \sum_{i=1}^{n} m_{in}(-E_i)\right) \\
&= (-E_1, ..., -E_n) \cdot (m_{ij}) \\
&= \tau(E_1, ..., E_n) \cdot (m_{ij}).
\end{aligned}$$

Thus the diagram

$$\begin{tikzcd}
P \times Spin^c(n) \arrow[r] \arrow[d, "\tau\Lambda\times\lambda"'] & P \arrow[d, "\tau\Lambda"] \arrow[dr] & \\
 & & M \\
SO(V) \times SO(n) \arrow[r] & SO(V) \arrow[ur] &
\end{tikzcd}$$

commutes, since

$$\tau\Lambda(p \cdot x) = \tau(\Lambda(p) \cdot \lambda(x)) = \tau(\Lambda(p)) \cdot \lambda(x)$$

for every $p \in P$ and $x \in Spin^c(n)$. We get then a G-$Spin^c$ structure on $-V$ simply by considering the same principal $Spin^c(n)$-bundle P and the map $\tau\Lambda : P \longrightarrow SO(-V)$.

Assume now that n is even and non-zero. Let us define the following isomorphism of bundles :

$$\begin{aligned}
\tau : SO(V) &\longrightarrow SO(-V) \\
(E_1, E_2, ..., E_n) &\longmapsto (E_1, -E_2, ..., -E_n).
\end{aligned}$$

We will consider the G-equivariant principal $Spin^c(n)$-bundle P, but this time with action given by

$$p \cdot_\tau x := p \cdot (e_1 \cdot x \cdot (-e_1))$$

for $p \in P$ and $x \in Spin^c(n)$. By the remark preceding proposition 2.1.15, the isometry $\lambda(e_1)$ corresponds to the reflection of $\mathbb{R}^n$ relative to the hyperplane $x_1 = 0$, as well as $\lambda(-e_1)$. Hence

$$\lambda(e_1) = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} = \lambda(-e_1)$$

and therefore

$$\begin{aligned}\lambda(e_1)(m_{ij})\lambda(-e_1) &= \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} (m_{ij}) \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \\ &= \begin{pmatrix} m_{11} & -m_{12} & \dots & -m_{1n} \\ -m_{21} & & & \\ \vdots & & m_{ij} & \\ -m_{n1} & & & \end{pmatrix}\end{aligned}$$

for every $(m_{ij}) \in SO(n)$. Thus

$$\begin{aligned}&\tau((E_1, ..., E_n) \cdot (\lambda(e_1)(m_{ij})\lambda(-e_1))) = \\ &= \tau\left(m_{11}E_1 - \sum_{i=2}^{n} m_{i1}E_i, -m_{12}E_1 + \sum_{i=2}^{n} m_{i2}E_i, ..., -m_{1n}E_1 + \sum_{i=2}^{n} m_{in}E_i\right) \\ &= \left(m_{11}E_1 - \sum_{i=2}^{n} m_{i1}E_i, m_{12}E_1 - \sum_{i=2}^{n} m_{i2}E_i, ..., m_{1n}E_1 - \sum_{i=2}^{n} m_{in}E_i\right) \\ &= \left(m_{11}E_1 + \sum_{i=2}^{n} m_{i1}(-E_i), m_{12}E_1 + \sum_{i=2}^{n} m_{i2}(-E_i), ..., m_{1n}E_1 + \sum_{i=2}^{n} m_{in}(-E_i)\right) \\ &= (E_1, -E_2, ..., -E_n) \cdot (m_{ij}) \\ &= \tau(E_1, ..., E_n) \cdot (m_{ij})\end{aligned}$$

for every $(E_1, ..., E_n) \in SO(V)$ and $(m_{ij}) \in SO(n)$, implying

$$\tau\Lambda(p \cdot_\tau x) = \tau\Lambda(p \cdot (e_1 \cdot x \cdot (-e_1))) = \tau(\Lambda(p) \cdot \lambda(e_1)\lambda(x)\lambda(-e_1)) = \tau(\Lambda(p)) \cdot \lambda(x)$$

for every $p \in P$ and $x \in Spin^c(n)$. Thereby we get the commutative diagram

$$\begin{tikzcd} P \times Spin^c(n) \arrow[r] \arrow[d, "\tau\Lambda \times \lambda"'] & P \arrow[d, "\tau\Lambda"] \arrow[dr] & \\ & & M \\ SO(V) \times SO(n) \arrow[r] & SO(V) \arrow[ur] & \end{tikzcd}$$

giving a *G-Spin*^{c} structure on $-V$.

Definition 2.2.2. *Let V be provided with a G-Spin^c structure. We call opposite G-Spin^c structure the one on $-V$ obtained by the constructions above.*

Definition 2.2.3. *Assume V is endowed with two G-$Spin^c$ structures, with underlying bundles P_1, P_2 and covering maps Λ_1, Λ_2. We say that the two structures are equivalent if there exists a G- and $Spin^c(n)$-equivariant homeomorphism f from P_1 to P_2 such that*

$$\begin{tikzcd}
P_1 \arrow[rr, "f"] \arrow[dr, "\Lambda_1"'] & & P_2 \arrow[dl, "\Lambda_2"] \\
& SO(V) &
\end{tikzcd}$$

commutes.

2.2.2 G-$Spin^c$ structures on manifolds

Considering $V = TM$, we will write $SO(M)$ instead of $SO(TM)$, provided that M is oriented and with a Riemannian metric. Furthermore, we say G-$Spin^c$ structure on M for a G-$Spin^c$ structure on TM and we call M a G-$Spin^c$-manifold if it is endowed with a G-$Spin^c$ structure on itself.

By convention, we say that the empty manifold is G-$Spin^c$.

Moreover, if G is the trivial group, then we will speak of $Spin^c$ structure and $Spin^c$-manifold.

As an example, we give a $Spin^c$ structure on $\mathbb{T}$. First give it the orientation coming from the basis of the tangent space at $e^{2\pi it} \in \mathbb{T}$ given by the path

$$\begin{aligned}]0,1[&\longrightarrow \mathbb{T} \\ s &\longmapsto \exp\left(2\pi i(t+s-\frac{1}{2})\right) \end{aligned}$$

for every $t \in [0,1]$. Then endow $\mathbb{T}$ with the Riemannian metric induced by the restriction of the usual metric on $\mathbb{R}^2$. The bundle $SO(\mathbb{T})$ is then isomorphic to $\mathbb{T}$. Therefore the principal $Spin^c(1)$-bundle given by $\mathbb{T}\times\mathbb{T}$, seen as a bundle over $\mathbb{T}$ by the projection on the first coordinate and provided with the action of $Spin^c(1) \cong \mathbb{T}$ given fiberwise by the usual multiplication in $\mathbb{T}$, defines a $Spin^c$ structure on $\mathbb{T}$.

Moreover, this is the unique one on the circle, up to equivalence, as the next theorem and the fact that $H^2(\mathbb{T},\mathbb{Z}) = \{0\}$ shows.

Theorem 2.2.4. *Let M be an oriented Riemannian manifold with boundary. Then M admits at least one $Spin^c$ structure if and only if its second Stiefel-Whitney class $w_2(M) \in H^2(M,\mathbb{Z}_2)$ is the reduction modulo 2 of an element in $H^2(M,\mathbb{Z})$. In this case, the number of equivalence classes of $Spin^c$ structure is given by $H^2(M,\mathbb{Z})$.*

In the description of the $Spin^c$-structure on the circle made above, we note that it is simply given by the trivial principal $Spin^c(1)$-bundle over the tangent space of $\mathbb{T}$. This process can be generalized if the tangent bundle is itself trivial.

Proposition 2.2.5. *Assume M is oriented, with a Riemannian metric and with trivial tangent bundle $TM = M \times \mathbb{R}^n$. Then the trivial principal $Spin^c(n)$-bundle $P = M \times Spin^c(n)$ and the G-map*

$$\mathrm{id} \times \lambda : P \longrightarrow M \times SO(n) = SO(M)$$

define a G-$Spin^c$ structure on M.

Proof. We only have to check that the diagram

$$\begin{tikzcd}
M \times Spin^c(n) \times Spin^c(n) \arrow[r] \arrow[d, "\mathrm{id} \times \lambda \times \lambda"'] & M \times Spin^c(n) \arrow[d, "\mathrm{id} \times \lambda"] \arrow[dr] & \\
& & M \\
M \times SO(n) \times SO(n) \arrow[r] & M \times SO(n) \arrow[ur] &
\end{tikzcd}$$

commutes. But this is clear since λ is a group homomorphism. □

Thus, for example, the torus $\mathbb{T}^2$ is a $Spin^c$-manifold. Indeed, its tangent bundle is trivial and we can provide it with the Riemannian metric and orientation coming from those on $\mathbb{T}$ given above and the inclusion $SO(\mathbb{T}) \times SO(\mathbb{T}) \subseteq SO(\mathbb{T} \times \mathbb{T}) = SO(\mathbb{T}^2)$.

Suppose M is provided with a G-$Spin^c$ structure. If N is a G-submanifold with boundary of M of the same dimension, then it inherits a natural G-$Spin^c$ structure simply by restricting the G-equivariant principal $Spin^c(n)$-bundle and the G-map.

Proposition 2.2.6. *Any G-$Spin^c$ structure on M induces a G-$Spin^c$ structure on ∂M.*

Proof. We assume the boundary of M non-empty, and therefore $n \geq 1$. Fixing a G-invariant inward pointing unitary vector field χ normal to ∂M, we obtain a G-invariant orientation and a Riemannian metric on ∂M induced from those of M. Moreover, this gives us a G-map

$$\begin{aligned} j : SO(\partial M) &\longrightarrow SO(M)|_{\partial M} \\ (E_1, ..., E_{n-1}) &\longmapsto (E_1, ..., E_{n-1}, \chi). \end{aligned}$$

Now let us denote by P and $\Lambda : P \longrightarrow SO(M)$ the G-equivariant principal bundle and the G-map given by the G-$Spin^c$ structure on M. By the pullback construction, we obtain a G-equivariant bundle

$$j^*(P|_{\partial M}) =: P_{\partial M}$$

over ∂M. An element of $P_{\partial M}$ is given by a couple $(E, p) \in SO(\partial M) \times P|_{\partial M}$ satisfying the equality $j(E) = \Lambda(p)$. Moreover, the standard inclusion of $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$ induces an inclusion of S^{n-2} in S^{n-1} and then an inclusion of $Spin(n-1)$ in $Spin(n)$, which gives an injective map

$$i : Spin^c(n-1) \longrightarrow Spin^c(n).$$

As we will show in the next lemma, we get a commutative diagram

$$\begin{tikzcd}
Spin^c(n-1) \arrow[r, "i"] \arrow[d, "\lambda"'] & Spin^c(n) \arrow[d, "\lambda"] \\
SO(n-1) \arrow[r, "i'"'] & SO(n),
\end{tikzcd}$$

where i' is the canonical embedding of $SO(n-1)$ in $SO(n)$. Therefore, for every $(E,p) \in P_{\partial M}$ and $x \in Spin^c(n-1)$, since

$$\begin{aligned}
\Lambda(p \cdot i(x)) = \Lambda(p) \cdot \lambda(i(x)) &= j(E) \cdot i'(\lambda(x)) \\
&= j(E \cdot \lambda(x))
\end{aligned}$$

we get that $(E \cdot \lambda(x), p \cdot i(x))$ is an element of $P_{\partial M}$. This gives a right action of $Spin^c(n-1)$ on $P_{\partial M}$, endowing the latter with a structure of principal $Spin^c(n-1)$-bundle over ∂M. By defining $\Lambda_{\partial M} : P_{\partial M} \longrightarrow SO(\partial M)$ to be the projection on the first factor, we get

$$\begin{aligned}
\Lambda_{\partial M}((E,p) \cdot x) &= \Lambda_{\partial M}(E \cdot \lambda(x), p \cdot i(x)) \\
&= E \cdot \lambda(x) = \Lambda_{\partial M}(E,p) \cdot \lambda(x)
\end{aligned}$$

for all $(E,p) \in P_{\partial M}$ and $x \in Spin^c(n-1)$, meaning that the diagram

$$\begin{tikzcd}
P_{\partial M} \times Spin^c(n-1) \arrow[r] \arrow[dd, "\Lambda_{\partial M} \times \lambda"'] & P_{\partial M} \arrow[dd, "\Lambda_{\partial M}"] \arrow[dr] & \\
 & & \partial M \\
SO(\partial M) \times SO(n-1) \arrow[r] & SO(\partial M) \arrow[ur] &
\end{tikzcd}$$

commutes. ☐

Lemma 2.2.7. *For every $k \in \mathbb{N}$, we have a commutative diagram*

$$\begin{tikzcd}
Spin^c(k) \arrow[r, "i"] \arrow[d, "\lambda"'] & Spin^c(k+1) \arrow[d, "\lambda"] \\
SO(k) \arrow[r, "i'"'] & SO(k+1)
\end{tikzcd}$$

where the map i comes from the natural embedding of $\mathbb{R}^k$ in $\mathbb{R}^{k+1}$ and the map i' is the restriction of the natural inclusion of $GL_k(\mathbb{R})$ in $GL_{k+1}(\mathbb{R})$.

Proof. Let us write $\{e_1, ..., e_k\}$ and $\{e'_1, ..., e'_{k+1}\}$ for the usual basis of $\mathbb{R}^k$ and $\mathbb{R}^{k+1}$ respectively. Let $x \in S^k$. Define $i(x) \in S^{k+1}$ to be the image of x via the natural embedding of $\mathbb{R}^k$ into $\mathbb{R}^{k+1}$. Then the hyperplane perpendicular to $i(x)$ is the direct sum of the inclusion of the hyperplane in $\mathbb{R}^k$ perpendicular to x with the subspace generated by e'_{k+1}. Therefore,

$$\lambda(i(x)) = \begin{pmatrix} \lambda(x) & 0 \\ 0 & 1 \end{pmatrix} = i'(\lambda(x)).$$

Hence the diagram

$$\begin{tikzcd} Spin(k) \arrow[r, "i"] \arrow[d, "\lambda"'] & Spin(k+1) \arrow[d, "\lambda"] \\ SO(k) \arrow[r, "i'"'] & SO(k+1) \end{tikzcd}$$

commutes, inducing the desired one. ☐

For the next result, we warmly thank Nicolas Ginoux for his detailed explanations.

Proposition 2.2.8. *Suppose that the boundary of M is empty. Let G act trivially on $[0,1]$. Then any G-$Spin^c$ structure on M induces a G-$Spin^c$ structure on $M \times [0,1]$.*

Proof. Let P and Λ be a bundle and a map giving a G-$Spin^c$ structure on M. We define

$$P' := (P \times [0,1]) \times_{Spin^c(n)} Spin^c(n+1)$$

with $Spin^c(n)$ acting on $Spin^c(n+1)$ via the inclusion above and trivially on $[0,1]$. It is a G-equivariant principal $Spin^c(n+1)$-bundle over $M \times [0,1]$ via the map

$$\begin{aligned} P' &\longrightarrow M \times [0,1] \\ [(p,t),x] &\longmapsto (\pi(p),t), \end{aligned}$$

where π stands for the projection map from P to M. We also define

$$\begin{aligned} \Lambda' : P' &\longrightarrow SO(M \times [0,1]) \\ [(p,t),x] &\longmapsto j(\Lambda(p),t) \cdot \lambda(x), \end{aligned}$$

where $j : SO(M) \times [0,1] \longrightarrow SO(M \times [0,1])$ is given by completing an element of $SO(M)$ with the unit tangent vector given by the usual derivation on $[0,1]$. Since

$$\begin{aligned} \Lambda'([(p,t)\cdot y, x]) &= \Lambda'([(p\cdot y,t),x]) = j(\Lambda(p\cdot y),t)\cdot\lambda(x) \\ &= j((\Lambda(p),t)\cdot\lambda(y))\cdot\lambda(x) = j(\Lambda(p),t)\cdot\lambda(y)\cdot\lambda(x) \\ &= j(\Lambda(p),t)\cdot\lambda(yx) = \Lambda'([(p,t),yx]) \end{aligned}$$

for all $p \in P$, $t \in [0,1]$, $x \in Spin^c(n+1)$ and $y \in Spin^c(n)$, the map Λ' is well-defined. Moreover,

$$\begin{aligned} \Lambda'([(p,t),x]\cdot y) &= \Lambda'([(p,t),xy]) = j(\Lambda(p),t)\cdot\lambda(xy) \\ &= j(\Lambda(p),t)\cdot\lambda(x)\lambda(y) = \Lambda'([(p,t),x])\cdot\lambda(y) \end{aligned}$$

for every $p \in P$, $t \in [0,1]$ and $x, y \in Spin^c(n+1)$. Thus the diagram

$$\begin{tikzcd}
P' \times Spin^c(n+1) \arrow[rr] \arrow[d, "\Lambda' \times \lambda"'] & & P' \arrow[d, "\Lambda'"] \arrow[r] & M \times [0,1] \\
SO(M \times [0,1]) \times SO(n+1) \arrow[rr] & & SO(M \times [0,1]) \arrow[ru] &
\end{tikzcd}$$

commutes, giving a G-$Spin^c$ structure on $M \times [0,1]$. $\square$

Taking the boundary of $M \times [0,1]$, it gives two copies of M, the one endowed with the original G-$Spin^c$ structure and the other with the opposite structure. Indeed, the right derivation on $[0,1]$ gives a inward pointing unitary vector field χ on $M \times \{0\}$. Moreover, the G-map

$$\begin{aligned} SO(M) &\longrightarrow SO(M \times [0,1])|_{M \times \{0\}} \\ (E_1, ..., E_n) &\longmapsto (E_1, ..., E_n, \chi) \end{aligned}$$

corresponds exactly to the restriction of the map

$$j : SO(M) \times [0,1] \longrightarrow SO(M \times [0,1])$$

given in the previous proof to $SO(M) \times \{0\}$. Furthermore, the restriction of the bundle P' to $M \times \{0\}$ is

$$P \times_{Spin^c(n)} Spin^c(n+1).$$

Therefore, the pullback of $P'|_{M \times \{0\}}$ along $j|_{SO(M) \times \{0\}}$ gives the principal $Spin^c(n)$-bundle composed by pairs $(E, [p,x])$ with $E \in SO(M)$, $p \in P$, $x \in Spin^c(n+1)$ and such that

$$(E, \chi) = (\Lambda(p), \chi) \cdot \lambda(x).$$

Hence, given such a pair, we deduce that $\lambda(x)$ should be the inclusion of an element of $SO(n)$, and therefore that x is in fact the image of an element $x' \in Spin^c(n)$ via the inclusion i of lemma 2.2.7. Thus

$$(E, \chi) = (\Lambda(p), \chi) \cdot \lambda(i(x')) = (\Lambda(p) \cdot \lambda(x'), \chi).$$

It follows that $E = \Lambda(p) \cdot \lambda(x')$. Thus, the principal $Spin^c(n)$-bundle we are looking for is simply

$$P \times_{Spin^c(n)} Spin^c(n) = P.$$

Considering $M \times \{1\}$, the inward pointing vector field is the opposite of the one given by the left derivation on $[0,1]$. Thus the orientation induced on this copy of M is the opposite of the one given by proposition 2.2.6. Moreover, we can show that the principal $Spin^c(n)$-bundle on $M \times \{1\}$ induced by P' is also P, but with action of $Spin^c$ twisted as explained before definition 2.2.2.

2.3 Equivariant K-homology

For the end of the chapter, we fix a locally compact, Hausdorff and second countable topological group G.

2.3.1 Definitions and homotopy invariance

Definition 2.3.1. *Let X be a G-space and Y be a closed G-invariant subspace of X. A G-equivariant K-cycle over (X, Y) is a triple (M, E, φ), where :*

- *M is a smooth G-compact proper G-Spin^c-manifold with boundary, in the sense that each of its connected components is endowed with a G-Spin^c structure,*
- *E is a G-equivariant complex vector bundle over M,*
- *φ is a G-map from M to X such that $\varphi(\partial M) \subseteq Y$.*

Moreover, we say that the G-equivariant K-cycle is even, resp. odd, if every connected component of M is even dimensional, resp. odd dimensional.

If the context is clear, we will just say K-cycle instead of G-equivariant K-cycle.

The basic example is, if X is itself a G-compact, proper *G-Spin^c*-manifold with boundary, taking $M = X$, $E = M \times \mathbb{C}^k$ for any $k \in \mathbb{N}$ (with trivial action of G on $\mathbb{C}^k$), and $\varphi = \mathrm{id}_M$. It gives a K-cycle over $(X, \partial X)$.

Furthermore, we will say K-cycle over X for K-cycle over $(X, \emptyset)$. It follows from the third condition of the definition that the manifold composing any K-cycle over X is necessarily without boundary.

Definition 2.3.2. *Let X be a G-space and $Y \subseteq X$ be a closed and G-invariant subspace of X. Suppose (M_1, E_1, φ_1) and (M_2, E_2, φ_2) are two K-cycles over (X, Y) with G-Spin^c structures over M_1 and M_2 given by G-equivariant principal bundles P_1 and P_2, respectively. Then (M_1, E_1, φ_1) and (M_2, E_2, φ_2) are isomorphic if there exists a G-equivariant diffeomorphism $f : M_1 \longrightarrow M_2$ such that :*

- *$P_1 \cong f^*(P_2)$ as G-equivariant principal Spin$^c(n)$-bundles over M_1, where n stands for $\dim(M_1) = \dim(M_2)$,*
- *$E_1 \cong f^*(E_2)$ as G-equivariant vector bundles over M_1,*
- *$\varphi_1 = \varphi_2 \circ f$.*

For example, if M_2 is the same manifold with boundary as M_1 with a *G-Spin^c* structure that is different but isomorphic, in the sense of 2.2.3, to the one on M_1, then the K-cycles (M_1, E_1, φ_1) and (M_2, E_1, φ_1) are isomorphic.

We remark that isomorphic K-cycles should have the same parity.

Definition 2.3.3. *Let X be a G-space and Y be a closed G-invariant subspace of X. A bordism for the pair (X, Y) consists of :*

- *a smooth, G-compact, proper G-Spin^c-manifold with boundary L,*
- *a G-equivariant complex vector bundle F over L,*
- *a G-map $\psi : L \longrightarrow X$,*
- *a smooth and G-invariant function $h : \partial L \longrightarrow \mathbb{R}$ for which 1 and -1 are regular values and such that $\psi(h^{-1}([-1, 1])) \subseteq Y$.*

The last condition implies that $M_+ := h^{-1}([1, \infty[)$ and $M_- := h^{-1}(] - \infty, -1])$ are G-invariant submanifolds with boundary of ∂L, with the same dimension as it (see for example [11] or [19]). Thus, by proposition 2.2.6 and the remark immediately preceding it, they are *G-Spin*^{c}-manifold and, therefore, $(M_+, F|_{M_+}, \psi|_{M_+})$ and $(M_-, F|_{M_-}, \psi|_{M_-})$ are two K-cycles over (X, Y). We say then that the first is bordant to the opposite of the second.

Bordant K-cycles have necessarily the same parity.

Lemma 2.3.4. *Let (M, E, φ_1) and (M, E, φ_2) be two K-cycles over a G-space X. If φ_1 is G-homotopic to φ_2 then the two K-cycles are bordant.*

Proof. Fix a G-homotopy $H : M \times [0, 1] \longrightarrow X$ from φ_1 to φ_2. Since M has empty boundary, then $L := M \times [0, 1]$ is a manifold with boundary, that inherits a *G-Spin*^{c} structure by proposition 2.2.8. Together with the equivariant vector bundle $F := E \times [0, 1]$, the G-map

$$\begin{aligned} \psi : L = M \times [0, 1] &\longrightarrow X \\ (m, t) &\longmapsto H(m, t) \end{aligned}$$

and the smooth G-invariant function

$$\begin{aligned} h : \partial L = M \times \{0, 1\} &\longrightarrow \mathbb{R} \\ (m, 0) &\longmapsto 7 \\ (m, 1) &\longmapsto -7, \end{aligned}$$

it gives a bordism between (M, E, φ_1) and $(-M, E, \varphi_2)$. □

It remains one more operation to introduce. Let X be a G-space, Y be a closed G-invariant subspace of X and (M, E, φ) be a K-cycle over (X, Y). Assume moreover that we are given an even dimensional real vector bundle V over M, endowed with a *G-Spin*^{c} structure. By an argument similar as the one used in proposition 2.2.8, we can provide $V \oplus \mathbb{R}$ with a *G-Spin*^{c} structure in a canonical way. Restricting this structure to the disk bundle of $V \oplus \mathbb{R}$ and then considering its boundary M_V, the sphere bundle of $V \oplus \mathbb{R}$, proposition 2.2.6 gives a *G-Spin*^{c} structure on M_V. By proposition 2.1.14, the spinor bundle associated to this structure is graded. Let us denote by S_V its even-graded part, and by S_V^* the dual bundle of S_V.

Definition 2.3.5. *Keeping the same notations as above, the modification of* (M, E, φ) *by the even dimensional vector bundle* V *is the triple*

$$(M_V, S_V^* \otimes \pi_V^*(E), \varphi \circ \pi_V),$$

where $\pi_V : M_V \longrightarrow M$ *is the restriction of the bundle projection from* $V \oplus \mathbb{R}$ *to* M*. It is a* K*-cycle over* (X, Y)*.*

The vector bundle modification operation preserves parity of K-cycles.

Definition 2.3.6. *Let* X *be a* G*-space and* Y *be a closed* G*-invariant subspace of* X*. On the class of all* G*-equivariant* K*-cycles over* (X, Y)*, we consider the equivalence relation generated by isomorphism and those three elementary relations :*

- *if* (M, E_1, φ) *and* (M, E_2, φ) *are two* K*-cycles with the same underlying manifold with boundary and map, then*

$$(M \amalg M, E_1 \amalg E_2, \varphi \amalg \varphi) \sim (M, E_1 \oplus E_2, \varphi),$$

- *if* (M_1, E_1, φ_1) *and* (M_2, E_2, φ_2) *are bordant* K*-cycles, then*

$$(M_1, E_1, \varphi_1) \sim (M_2, E_2, \varphi_2),$$

- *if* (M, E, φ) *is a* K*-cycle and* V *is an even dimensional* G*-vector bundle endowed with a* G*-Spin*^{c} *structure, then*

$$(M, E, \varphi) \sim (M_V, S_V^* \otimes \pi_V^*(E), \varphi \circ \pi_V).$$

We write $K_0^G(X, Y)$ *for the set of equivalence classes of even* G*-equivariant* K*-cycles over* (X, Y) *and* $K_1^G(X, Y)$ *for the set formed by the odd ones.*

On those sets, we can define a binary operation by considering disjoint union of the different elements of the K-cycles :

$$[M_1, E_1, \varphi_1] + [M_2, E_2, \varphi_2] := [M_1 \amalg M_2, E_1 \amalg E_2, \varphi_1 \amalg \varphi_2]$$

for every $[M_1, E_1, \varphi_1], [M_2, E_2, \varphi_2] \in K_j^G(X, Y)$, where $j = 0, 1$. This operation is clearly associative and commutative.

Proposition 2.3.7. *For any pair consisting of a* G*-space* X *and a closed* G*-invariant subspace* Y*, the sets* $K_0^G(X, Y)$ *and* $K_1^G(X, Y)$ *are abelian groups for the law coming from disjoint union of* K*-cycles. They form the geometric* G*-equivariant* K*-homology of the pair* (X, Y)*.*

The zero element is represented by the K-cycle $(\emptyset, \emptyset, \emptyset)$ which is, by convention, both even and odd.

The inverse of a class $[M, E, \varphi]$ is given by $[-M, E, \varphi]$, where $-M$ is M taken with the opposite G-$Spin^c$ structure on each of its connected components.

If G is the trivial group, then we will skip the superscript " G " in the notations for the K-homology groups. On the other hand, if $Y = \emptyset$, then we will write $K_j^G(X)$ instead of $K_j^G(X, Y)$, for $j = 0, 1$.

Let (X, Y) and (X', Y') be two pairs, each of them formed by a G-space and a closed G-invariant subspace. Let $f : X \longrightarrow X'$ be a G-map such that $f(Y) \subseteq Y'$. Then f induces a homomorphism

$$f_* : K_j^G(X, Y) \longrightarrow K_j^G(X', Y')$$

simply by setting $f_*([M, E, \varphi]) := [M, E, f \circ \varphi]$ for every class $[M, E, \varphi] \in K_j^G(X, Y)$ and for $j = 0, 1$. This turns geometric G-equivariant K-homology into a covariant functor from pairs of G-space and closed G-invariant subspace to abelian groups.

Proposition 2.3.8. *Let X, X' be G-spaces and f_1, f_2 be G-homotopic G-maps from X to X'. Then the induced homomorphisms*

$$(f_1)_*, (f_2)_* : K_j^G(X) \longrightarrow K_j^G(X')$$

are equal, for $j = 0, 1$.

The proof is a straightforward application of lemma 2.3.4. We deduce the homotopy invariance of geometric G-equivariant K-homology on G-spaces.

Corollary 2.3.9. *Let X, X' be G-spaces. If there exists a G-homotopy equivalence f from X to X', then it induces an isomorphism*

$$f_* : K_j^G(X) \longrightarrow K_j^G(X')$$

for $j = 0, 1$.

2.3.2 Link with the analytic equivariant K-homology

This part concerns the main theorem of this chapter, connecting the geometric and analytic equivariant K-homology. Although it was known from long time ago, it has been concretely established only in the last five years, by works of Baum, Higson, Oyono-Oyono and Schick.

Theorem 2.3.10. *Suppose G is a compact Lie group. Let X be a proper, G-compact G-CW-complex and $Y \subseteq X$ be a closed and G-invariant subcomplex. Then we have a natural isomorphism*

$$\nu_j : K_j^G(X, Y) \longrightarrow KK_j^G(C_0(X \setminus Y), \mathbb{C})$$

for $j = 0, 1$.

A complete proof of this theorem can be found in [9], or in [7] in the case where G is trivial. We explain here how the isomorphism is constructed. Let (M,E,φ) be a K-cycle over (X,Y), with M connected. Assume first that M is odd-dimensional. Since it is a Riemannian manifold with boundary, M has a unique torsion-free Riemannian connection on its tangent bundle, called the Levi-Civita connection and written

$$\nabla^{LC} : \Gamma^\infty(TM) \longrightarrow \Gamma^\infty(TM \otimes T^*M).$$

Considering the tensor product of the spinor bundle S associated to the *G-Spin*^{c} structure on M and the vector bundle E, we can show that ∇^{LC} can be lifted to $S \otimes E$, namely, there exists a connection

$$\nabla : \Gamma^\infty(S \otimes E) \longrightarrow \Gamma^\infty(S \otimes E \otimes T^*M)$$

such that

$$\nabla((\chi \cdot f_S) \otimes f_E)) = \chi \cdot \nabla(f_S \otimes f_E) + (\nabla^{LC}(\chi) \cdot f_S) \otimes f_E$$

for all $\chi \in \Gamma^\infty(TM)$, $f_S \in \Gamma^\infty(S)$, $f_E \in \Gamma^\infty(E)$, where the multiplications are each time coming from the Clifford one, applied pointwise. Forming the composition

$$\Gamma^\infty(S \otimes E) \xrightarrow{\nabla} \Gamma^\infty(S \otimes E \otimes T^*M) \cong \Gamma^\infty(S \otimes TM \otimes E) \longrightarrow \Gamma^\infty(S \otimes E),$$

where the last map comes from the Clifford multiplication, we get a symmetric elliptic first order differential operator, as it is shown in [18]. Let us call D_E this operator. Thus, given an odd smooth function $\chi : \mathbb{R} \longrightarrow [-1,1]$ such that $\chi(\mathbb{R}_+^*) \subseteq \mathbb{R}_+^*$ and $\lim\limits_{x\to\infty} \chi(x) = 1$ and setting

$$F_E := \chi(D_E),$$

we obtain a bounded operator on $L^2(M, S \otimes E)$ which satisfies, together with the action π of $C_0(M)$ on $L^2(M, S\otimes E)$ given by pointwise multiplication, the conditions of an equivariant Kasparov cycle over $(C_0(M), \mathbb{C})$. Then

$$\begin{aligned} \nu_1 : K_1^G(X,Y) &\longrightarrow KK_1^G(C_0(X \setminus Y), \mathbb{C}) \\ [M,E,\varphi] &\longmapsto (\varphi^*|_{C_0(X\setminus Y)})([L^2(M, S\otimes E), \pi, F_E]). \end{aligned}$$

The homomorphism ν_0 is constructed exactly in the same way, the only difference being that if M is even-dimensional, then the spinor bundle S is $\mathbb{Z}_2$-graded, thanks to proposition 2.1.14. This gives a corresponding graduation on $L^2(M, S \otimes E)$, which is preserved by π and reversed by F_E.

- If $M = *$, then its *Spin*^{c} structure is given by the principal *Spin*$^c(0)$-bundle $*$. Thus its spinor bundle is $S = * \times (\mathbb{C} \oplus 0) = \mathbb{C} \oplus 0$. Considering the trivial vector bundle $E = \mathbb{C}$ over $*$, we get $S \otimes E = \mathbb{C} \oplus 0$. But, since $T^*(*) = 0$, we have that ∇ is the zero map, as well as D_E. Hence F_E is the zero operator on $L^2(*, \mathbb{C}\oplus 0) = \mathbb{C} \oplus 0$. Therefore, given any proper G-compact G-CW-complex X, we have

$$\nu_0([*, \mathbb{C}, \varphi : * \to X]) = \varphi^*([\mathbb{C}, \mathrm{id}_\mathbb{C} \oplus 0, 0]).$$

In particular, the class $[*, \mathbb{C}, \mathrm{id}_*]$ generates the K-homology group

$$K_0(*) \cong K_0(C(*)) \cong K_0(\mathbb{C}) = \mathbb{Z}.$$

- We have from the previous chapter that the analytic K-homology groups of $C(\mathbb{T})$ are both isomorphic to $\mathbb{Z}$. Moreover, a generator of $KK_0(C(\mathbb{T}), \mathbb{C})$ is given by $[\mathbb{C}, e_1 \oplus 0, 0]$, with e_1 being the evalution at 1, whereas $KK_1(C(\mathbb{T}), \mathbb{C})$ is generated by $[L^2(\mathbb{T}), \pi, F]$, where π is the representation of $C(\mathbb{T})$ on $L^2(\mathbb{T})$ given by multiplication operators and $F(\iota^n) = sgn(n)\iota^n$, for all $n \in \mathbb{Z}$. Defining

$$\delta_1'' := [*, \mathbb{C}, * \to \{1\} \subseteq \mathbb{T}] \in K_0(\mathbb{T}),$$

the point above induces $\nu_0(\delta_1'') = [\mathbb{C}, e_1 \oplus 0, 0]$. On the other hand, the class

$$\gamma_1'' := [\mathbb{T}, \mathbb{T} \times \mathbb{C}, \mathrm{id}_\mathbb{T}] \in K_1(\mathbb{T})$$

is sent by ν_1 on the class of the Kasparov cycle $(L^2(\mathbb{T}), \pi, F)$. Indeed, the spinor bundle of $\mathbb{T}$ is $S = \mathbb{T} \times \mathbb{C}$. Hence $S \otimes E = \mathbb{T} \times \mathbb{C}$. A standard result in Riemannian geometry shows that, under the identifications

$$\Gamma^\infty(T(\mathbb{T})) \cong C^\infty(\mathbb{T}) \cong \Gamma^\infty(T(\mathbb{T}) \otimes T^*(\mathbb{T}))$$

the connection ∇^{LC} is simply given by the usual derivative. Thus, the connection ∇ is also $\frac{d}{dt}$. To identify the Clifford multiplication, we recall that $\mathbb{R}$ is seen in $Cl(1) \cong \mathbb{C}$ as the subspace $i\mathbb{R}$. Via the isomorphism $Cl(1) \otimes_\mathbb{R} \mathbb{C} \cong Cl^c(1)$, the element $i \otimes 1$ is sent on if_1, where f_1 is the canonical basis element of the vector space $\mathbb{C}$. But the isomorphisms

$$Cl^c(1) \cong Cl'(1) \otimes_\mathbb{R} \mathbb{C} \cong (\mathbb{R} \oplus \mathbb{R}) \otimes_\mathbb{R} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$$

map if_1 successively on $e_1 \otimes i$, on $(1, -1) \otimes i$ and on $(i, -i)$. Thus the basis element $e_1 \in \mathbb{R}$ acts on $\mathbb{C}$ via the spinorial representation as the multiplication by i. Therefore the Clifford multiplication is given by pointwise multiplication by i and then

$$D_E = i\frac{d}{dt}.$$

Choosing a smooth odd function $\chi : \mathbb{R} \longrightarrow [-1, 1]$ positive on $\mathbb{R}_+^*$ and such that $\chi(t) = 1$ for every $t \geq 1$, we obtain that

$$F_E(\iota^n) = (\chi(D_E))(\iota^n) = sgn(n)\iota^n = F(\iota^n).$$

> In the end, we get that δ''_1 and γ''_1 are generators for the groups $K_0(\mathbb{T})$ and $K_1(\mathbb{T})$ respectively, which are both isomorphic to $\mathbb{Z}$.

Another important result of [9] and [7] is that, if we consider the *-homomorphism from $C_0(X \setminus Y)$ to $C_0(X)$ obtained by extending a function on $X \setminus Y$ by setting zero on every point of Y (which is well-defined since Y is closed) and the map from $C_0(X)$ to $C_0(Y)$ obtained by restriction, we get the short exact sequence

$$0 \longrightarrow C_0(X \setminus Y) \longrightarrow C_0(X) \longrightarrow C_0(Y) \longrightarrow 0.$$

By theorem 1.2.9, this induces the following six-terms exact sequence in analytic equivariant K-homology :

$$\begin{tikzcd}
KK_0^G(C_0(Y),\mathbb{C}) \arrow[r] & KK_0^G(C_0(X),\mathbb{C}) \arrow[r] & KK_0^G(C_0(X \setminus Y),\mathbb{C}) \arrow[d] \\
KK_1^G(C_0(X \setminus Y),\mathbb{C}) \arrow[u] & KK_1^G(C_0(X),\mathbb{C}) \arrow[l] & KK_1^G(C_0(Y),\mathbb{C}). \arrow[l]
\end{tikzcd}$$

Then, under the isomorphisms of theorem 2.3.10 applied to the pairs $(Y, \emptyset)$, $(X, \emptyset)$ and (X, Y), this six-terms exact sequence becomes

$$\begin{tikzcd}
K_0^G(Y) \arrow[r, "i_*"] & K_0^G(X) \arrow[r, "i'_*"] & K_0^G(X,Y) \arrow[d, "\partial"] \\
K_1^G(X,Y) \arrow[u, "\partial"] & K_1^G(X) \arrow[l, "i'_*"] & K_1^G(Y), \arrow[l, "i_*"]
\end{tikzcd}$$

where i and i' are the obvious G-maps coming from the inclusions $Y \subseteq X$ and $\emptyset \subseteq Y$ and the vertical homomorphisms are given by

$$\partial([M, E, \varphi]) = [\partial M, E|_{\partial M}, \varphi|_{\partial M}]$$

for every $[M, E, \varphi] \in K_j^G(X, Y)$, with $j = 0, 1$.

We also mention an analogue of theorem 2.3.10 in the discrete case. A proof is presented in [8].

Theorem 2.3.11. *Suppose G is a discrete group and let X be a proper, G-compact G-CW-complex. Then*

$$K_j^G(X) \cong KK_j^G(C_0(X), \mathbb{C})$$

for $j = 0, 1$.

2.3.3 Bott isomorphism

From now on, we will write I for the closed unit interval $[0,1]$ with trivial G-action, $\mathring{I}$ for its interior and ∂I for its boundary.

Let X be a proper G-space. The suspension of the G-C^*-algebra $C_0(X)$ is given by :

$$\begin{aligned} SC_0(X) &= \{f \in C(I, C_0(X)) \mid f(0) = 0 = f(1)\} \\ &\cong \{f \in C_0(X \times I) \mid f(x,0) = 0 = f(x,1), \forall x \in X\} \\ &\cong C_0(X \times \mathring{I}). \end{aligned}$$

Hence the Bott isomorphism 1.2.7 in analytic equivariant K-homology gives that

$$KK_j^G(C_0(X \times \mathring{I}), \mathbb{C}) \cong KK_{1-j}^G(C_0(X), \mathbb{C})$$

for $j = 0, 1$. We deduce from theorem 2.3.10 that, if the corresponding assumptions on G and X are satisfied, we also have a Bott isomorphism in geometric equivariant K-homology. However, a direct isomorphism can be given. To this end, let us define

$$\begin{aligned} \partial_0 : K_j^G(X \times I, X \times \partial I) &\longrightarrow K_{1-j}^G(X \times \{0\}) \cong K_{1-j}^G(X) \\ [M, E, \varphi] &\longmapsto [\partial_0 M := \varphi^{-1}(X \times \{0\}) \cap \partial M, E|_{\partial_0 M}, \varphi|_{\partial_0 M}] \end{aligned}$$

for $j = 0, 1$.

Proposition 2.3.12. *(Bott isomorphism). Suppose G is a compact Lie group. Let X be a proper, G-compact G-CW-complex. Then*

$$\partial_0 : K_j^G(X \times I, X \times \partial I) \longrightarrow K_{1-j}^G(X)$$

is an isomorphism, for $j = 0, 1$.

Proof. First we note that $X \times I$ and $X \times \partial I$ are G-homotopy equivalent to X and $X \amalg X$ respectively. Furthermore, we have $K_j^G(X \amalg X) \cong K_j^G(X) \times K_j^G(X)$ in an obvious way, for $j = 0, 1$. Hence, by corollary 2.3.9, the six-terms exact sequence for the pair $(X \times I, X \times \partial I)$ becomes :

$$\begin{tikzcd} K_0^G(X) \times K_0^G(X) \arrow[r, "i_*"] & K_0^G(X) \arrow[r, "i'_*"] & K_0^G(X \times I, X \times \partial I) \arrow[d, "\partial"] \\ K_1^G(X \times I, X \times \partial I) \arrow[u, "\partial"] & K_1^G(X) \arrow[l, "i'_*"'] & K_1^G(X) \times K_1^G(X). \arrow[l, "i_*"'] \end{tikzcd}$$

The map i'_* comes from the inclusion i' of the pair $(X, \emptyset)$ in $(X \times I, X \times \partial I)$ given by $i'(x) = (x, 0)$, for every $x \in X$. Thus, for a $j \in \{0, 1\}$ and a $[M, E, \varphi] \in K_j^G(X)$, the class

$$i'_*([M, E, \varphi]) = [M, E, i' \circ \varphi]$$

is the zero element of $K_j^G(X \times I, X \times \partial I)$. Indeed, the K-cycle $(M, E, i' \circ \varphi)$ is bordant to the zero K-cycle via the bordism

- $L := M \times I$, with G-$Spin^c$ structure given by proposition 2.2.8,
- $F := E \times I$,
- $\psi : L = M \times I \longrightarrow X \times I$ defined by $\psi(m,t) = (\varphi(m), t)$,
- $h : \partial L = M \times \partial I \longrightarrow \mathbb{R}$ defined by $h(m,0) = 7$ and $h(m,1) = 0$,

since we have :

- $\psi(h^{-1}([-1,1])) = \psi(M \times \{1\}) \subseteq X \times \{1\} \subseteq X \times \partial I$,
- $(M_+ = h^{-1}([1,\infty[), F|_{M_+}, \psi|_{M_+}) = (M, E, i' \circ \varphi)$,
- $(M_- = h^{-1}(]-\infty,-1]), F|_{M_-}, \psi|_{M_-}) = (\emptyset, \emptyset, \emptyset)$.

Thereby, for $j = 0,1$, we obtain the short exact sequence :

$$0 \longrightarrow K_j^G(X \times I, X \times \partial I) \xrightarrow{\partial} K_{1-j}^G(X) \times K_{1-j}^G(X) \xrightarrow{i_*} K_{1-j}^G(X) \longrightarrow 0.$$

For the sequel, fix a $j \in \{0,1\}$ and define

$$\begin{aligned} \partial_1 : K_j^G(X \times I, X \times \partial I) &\longrightarrow K_{1-j}^G(X) \\ [M, E, \varphi]) &\longmapsto [\partial_1 M := \varphi^{-1}(X \times \{1\}) \cap \partial M, E|_{\partial_1 M}, \varphi|_{\partial_1 M}]. \end{aligned}$$

Then $\partial = \partial_0 \times \partial_1$. Moreover we can see that $\partial_0 = -\partial_1$, since a K-cycle over $(X \times I, X \times \partial I)$ gives itself a bordism between its two images by ∂_0 and ∂_1. Finally, the homomorphism

$$i_* : K_{1-j}^G(X) \times K_{1-j}^G(X) \longrightarrow K_{1-j}^G(X)$$

comes from the inclusion of $X \times \partial I$ in $X \times I$ and the homotopy equivalences given previously, and so is simply defined by $i_*(\gamma_1, \gamma_2) = \gamma_1 + \gamma_2$ for every $\gamma_1, \gamma_2 \in K_{1-j}^G(X)$. On the other hand, we have the short exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & K_{1-j}^G(X) & \longrightarrow & K_{1-j}^G(X) \times K_{1-j}^G(X) & \longrightarrow & K_{1-j}^G(X) & \longrightarrow 0 \\ & \gamma & \longmapsto & (\gamma, -\gamma) & & & \\ & & & (\gamma_1, \gamma_2) & \longmapsto & \gamma_1 + \gamma_2 & \end{array}$$

and, denoting A_j^G for $K_j^G(X \times I, X \times \partial I)$, the commutative diagram with exact rows :

$$\begin{tikzcd}
0 \arrow[r] & K_{1-j}^G(X) \arrow[r] & K_{1-j}^G(X) \times K_{1-j}^G(X) \arrow[r] & K_{1-j}^G(X) \arrow[r] & 0 \\
0 \arrow[r] & A_j^G \arrow[r, "\partial"'] \arrow[u, "\partial_0"] & K_{1-j}^G(X) \times K_{1-j}^G(X) \arrow[r, "i_*"'] \arrow[u, equal] & K_{1-j}^G(X) \arrow[r] \arrow[u, equal] & 0.
\end{tikzcd}$$

By the five lemma, we get that ∂_0 is an isomorphism. □

An explicit inverse to ∂_0 can be constructed in the following way. Let (M, E, φ) be a K-cycle over X. Since M is without boundary, proposition 2.2.8 provided a *G-Spin*^{c} structure on the smooth G-compact proper G-manifold with boundary $M \times I$. Moreover, $E \times I$ is a G-equivariant complex vector bundle over $M \times I$ and $\varphi \times \mathrm{id}_I$ is a G-map from $M \times I$ to $X \times I$ satisfying

$$(\varphi \times \mathrm{id}_I)(\partial(M \times I)) = (\varphi \times \mathrm{id}_I)(M \times \partial I) \subseteq X \times \partial I.$$

Hence $(M \times I, E \times I, \varphi \times \mathrm{id}_I)$ gives a K-cycle over $(X \times I, X \times \partial I)$. In addition, the classe of this K-cycle is mapped on $[M, E, \varphi]$ by the homomorphism ∂_0, since the *G-Spin*^{c} structure on $\partial_0(M \times I) = M \times \{0\}$ is the same as the one on M. Thus

$$\begin{aligned} K^G_{1-j}(X) &\longrightarrow K^G_j(X \times I, X \times \partial I) \\ [M, E, \varphi] &\longmapsto [M \times I, E \times I, \varphi \times \mathrm{id}_I] \end{aligned}$$

defines an inverse to ∂_0, for $j = 0, 1$.

2.3.4 Six-terms exact sequence associated to a mapping torus

Let X be a G-space endowed with an action of $\mathbb{Z}$ given by a G-equivariant homeomorphism $\alpha : X \longrightarrow X$. We define the mapping torus $M_{X,\alpha}$ of α to be the cylinder $X \times I$ quotiented by the equivalence relation

$$(x, 0) \sim (\alpha(x), 1), \quad \forall x \in X.$$

We provide it the structure of a G-space with the action given by $g \cdot [x, t] := [g \cdot x, t]$, for all $x \in X$, $t \in I$. This is well-defined since α is G-equivariant.

Moreover, X can be seen as a closed G-invariant subspace of $M_{X,\alpha}$ via the injective map :

$$\begin{aligned} i : X &\longrightarrow M_{X,\alpha} \\ x &\longmapsto [x, 0]. \end{aligned}$$

Lemma 2.3.13. *Assume G is a compact Lie group. Let X be a proper, G-compact G-CW-complex and α be a G-equivariant homeomorphism of X. Then the quotient map q from $X \times I$ to $M_{X,\alpha}$ gives raise to the commutative diagram :*

$$\begin{tikzcd}
K^G_j(X \times I, X \times \partial I) \arrow[d, "q_*"'] \arrow[dr, "\nu_j"] & \\
 & KK^G_j(C_0(X \times \mathring{I}), \mathbb{C}) \\
K^G_j(M_{X,\alpha}, X) \arrow[ur, "\nu_j"'] &
\end{tikzcd}$$

for $j = 0, 1$. In particular, the map q_* is an isomorphism.

Proof. Let (M,E,φ) be a K-cycle over $(X\times I, X\times\partial I)$. Following the description on page 69 and taking the same notations, we get

$$\nu_j([M,E,\varphi]) = (\varphi^*|_{C_0(X\times\mathring{I})})([L^2(M,S\otimes E),\pi,F_E])$$

and

$$\nu_j(q_*([M,E,\varphi])) = \nu_j([M,E,q\circ\varphi]) = ((q\circ\varphi)^*|_{C_0(X\times\mathring{I})})([L^2(M,S\otimes E),\pi,F_E])$$

for $j=0,1$. But it is clear that the restriction of $(q\circ\varphi)^*$ to $C_0(X\times\mathring{I})$ is exactly $\varphi^*|_{C_0(X\times\mathring{I})}$, since $q\circ\varphi$ coincide with φ on $X\times\mathring{I}$. □

Theorem 2.3.14. *Suppose G is a compact Lie group. Let X be a proper, G-compact G-CW-complex and α be a G-equivariant homeomorphism of X. Then we have a natural six-terms exact sequence :*

$$\begin{tikzcd}
K_0^G(X) \arrow[r, "\mathrm{id}-\alpha_*^{-1}"] & K_0^G(X) \arrow[r, "i_*"] & K_0^G(M_{X,\alpha}) \arrow[d] \\
K_1^G(M_{X,\alpha}) \arrow[u] & K_1^G(X) \arrow[l, "i_*"] & K_1^G(X). \arrow[l, "\mathrm{id}-\alpha_*^{-1}"]
\end{tikzcd}$$

The vertical homomorphisms are given by the composition $\partial_0\circ(q_*)^{-1}\circ i'_*$, where i' is the inclusion map from the pair $(M_{X,\alpha},\emptyset)$ into $(M_{X,\alpha},X)$.

Proof. Let us look at the six-terms exact sequence associated to the pair $(M_{X,\alpha},X)$:

$$\begin{tikzcd}
K_0^G(X) \arrow[r, "i_*"] & K_0^G(M_{X,\alpha}) \arrow[r, "i'_*"] & K_0^G(M_{X,\alpha},X) \arrow[d, "\partial"] \\
K_1^G(M_{X,\alpha},X) \arrow[u, "\partial"] & K_1^G(M_{X,\alpha}) \arrow[l, "i'_*"] & K_1^G(X). \arrow[l, "i_*"]
\end{tikzcd}$$

By lemma 2.3.13 and proposition 2.3.12 we deduce that the composition

$$K_j^G(M_{X,\alpha},X) \overset{(q_*)^{-1}}{\longrightarrow} K_j^G(X\times I, X\times\partial I) \overset{\partial_0}{\longrightarrow} K_{j-1}^G(X)$$

is an isomorphism for $j=0,1$. It remains us to check that $\partial\circ q_*\circ(\partial_0)^{-1}$ is $\mathrm{id}-\alpha_*^{-1}$. Let (M,E,φ) be a K-cycle over X. By the remark following proposition 2.3.12,

$$(\partial_0)^{-1}([M,E,\varphi]) = [M\times I, E\times I, \varphi\times_{\mathrm{id}_I}].$$

On the other hand, $q_*([M\times I,E\times I,\varphi\times_{\mathrm{id}_I}]) = [M\times I, E\times I, q\circ(\varphi\times_{\mathrm{id}_I})]$. Therefore,

$$\begin{aligned}
(\partial\circ q_*\circ(\partial_0)^{-1})([M,E,\varphi]) &= \partial([M\times I,E\times I,q\circ(\varphi\times_{\mathrm{id}_I})]) \\
&= [M\times\partial I, E\times\partial I, q\circ(\varphi\times\mathrm{id}_{\partial I})] \\
&= [M,E,\varphi] - [M,E,\alpha^{-1}\circ\varphi]
\end{aligned}$$

since $M \times I$ induces a G-*Spin*^{c} structure on $M \times \{1\}$ that is the opposite of the original one on M and

$$\begin{aligned}(q \circ (\varphi \times \mathrm{id}_{\partial I}))(x,1) &= q(\varphi(x),1)\\ &= [\varphi(x),1]\\ &= [\alpha^{-1}(\varphi(x)),0]\\ &= i(\alpha^{-1}(\varphi(x)))\end{aligned}$$

for all $x \in M$.

□

Chapter 3

The K-theory of the group C^*-algebra of a semi-direct product $\mathbb{Z}^2 \rtimes \mathbb{Z}$

In this chapter we compute the K-theory groups of the C^*-algebra of any semi-direct product of $\mathbb{Z}^2$ by $\mathbb{Z}$. The calculation will be explicit, meaning that we will be able to find concrete generators, in terms of the coefficients of the matrix in $GL_2(\mathbb{Z})$ giving the semi-direct structure. The main tool will be the six-terms exact sequence given by Pimsner and Voiculescu and explained in the first chapter. Thus, we begin by looking for generators of the C^*-algebra of $\mathbb{Z}^2$, which is nothing else that $C(\mathbb{T}^2)$, the continuous complex functions on the torus.

3.1 The K-theory of the continuous fonctions on the torus

As a corollary of proposition 1.1.19, we have already obtained that the K-theory groups of $C(\mathbb{T}^2)$ are both $\mathbb{Z}^2$. Here we give a new and non-traditional way to compute them, by using the results of Pimsner and Voiculescu and the isomorphisms

$$C(\mathbb{T}^2) \cong C^*(\mathbb{Z}^2) \cong C^*(\mathbb{Z}) \rtimes_\tau \mathbb{Z} \cong C(\mathbb{T}) \rtimes_\tau \mathbb{Z},$$

where τ denotes the trivial action of $\mathbb{Z}$.

Let us recall that $\iota \in C(\mathbb{T})$ denotes the inclusion of $\mathbb{T}$ into $\mathbb{C}$ and define $\pi_i \in C(\mathbb{T}^2)$ to be the i-th coordinate function, $i = 1, 2$. Thus, as C^*-algebras, $C(\mathbb{T})$ is generated by ι and $C(\mathbb{T}^2)$ is generated by π_1 and π_2.

Moreover, thinking of $C(\mathbb{T}^2)$ as the crossed product described above, $C(\mathbb{T})$ is seen in it as the sub-C^*-algebra generated by π_2 and π_1 represents the unitary giving the action τ. Indeed,

$$\pi_1 \pi_2 \pi_1^* = \pi_1 \pi_1^* \pi_2 = \pi_2 = \tau(\pi_2).$$

Also, the homomorphism $\mathrm{id} - \tau_*^{-1}$ in the six-terms exact sequence associated to this crossed product is zero both in K_0 and K_1 and then we obtain two short exact sequences :

$$0 \longrightarrow K_0(C(\mathbb{T})) \xrightarrow{i_*} K_0(C(\mathbb{T}^2)) \xrightarrow{\partial} K_1(C(\mathbb{T})) \longrightarrow 0$$

and

$$0 \longrightarrow K_1(C(\mathbb{T})) \xrightarrow{i_*} K_1(C(\mathbb{T}^2)) \xrightarrow{\partial} K_0(C(\mathbb{T})) \longrightarrow 0.$$

We know from chapter 1 that the K-theory groups of $C(\mathbb{T})$ are both free abelian of rank 1 and have generators given by $[1]$ and $[\iota]$ respectively. Thus

$$K_0(C(\mathbb{T}^2)) \cong \mathbb{Z}^2,$$

generated by $[1] = i_*([1])$ and a preimage of $[\iota]$ by ∂, and

$$K_1(C(\mathbb{T}^2)) \cong \mathbb{Z}^2,$$

generated by $[\pi_2] = i_*([\iota])$ and a preimage of $[1]$ by ∂.

Proposition 3.1.1. *The homomorphism $\partial : K_1(C(\mathbb{T}^2)) \longrightarrow K_0(C(\mathbb{T}))$ maps $[\pi_1]$ on $-[1]$.*

Proof. We follow the steps proposed on page 44, with the same notations. The Toeplitz extension associated to the crossed product $C(\mathbb{T}) \rtimes_\tau \mathbb{Z}$ is given by

$$0 \longrightarrow C(\mathbb{T}) \otimes \mathcal{K} \xrightarrow{\tilde{\varphi}} \mathcal{T}_{C(\mathbb{T}),\tau} \xrightarrow{\tilde{\psi}} C(\mathbb{T}^2) \longrightarrow 0$$

with $\tilde{\varphi}(\iota^n \otimes e_{i,j}) = \pi_1^{i-j}\pi_2^n$ for every $n \in \mathbb{Z}$, $i, j \in \mathbb{N}$, and

$$\tilde{\psi}(\pi_1 \otimes S) = \pi_1, \qquad \tilde{\psi}(\pi_2 \otimes I) = \pi_2.$$

First we have to find a preimage by $\tilde{\psi}$ of $\begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_1^* \end{pmatrix}$. The matrix

$$M := \begin{pmatrix} \pi_1 \otimes S & 1 \otimes T \\ 0 & \pi_1^* \otimes S^* \end{pmatrix}$$

is one, since

$$\begin{aligned} \tilde{\psi}(1 \otimes T) &= \tilde{\psi}(1 \otimes I) - \tilde{\psi}(1 \otimes SS^*) \\ &= \tilde{\psi}(1 \otimes I) - \tilde{\psi}(\pi_1\pi_1^* \otimes SS^*) \\ &= \tilde{\psi}(1 \otimes I) - \tilde{\psi}(\pi_1 \otimes S)\tilde{\psi}(\pi_1^* \otimes S^*) \\ &= 1 - \pi_1\pi_1^* = 0. \end{aligned}$$

Moreover, $M \in \mathcal{U}_2(\mathcal{T}_{C(\mathbb{T}),\tau})$ since

$$\begin{aligned} MM^* &= \begin{pmatrix} \pi_1 \otimes S & 1 \otimes T \\ 0 & \pi_1^* \otimes S^* \end{pmatrix} \begin{pmatrix} \pi_1^* \otimes S^* & 0 \\ 1 \otimes T & \pi_1 \otimes S \end{pmatrix} \\ &= \begin{pmatrix} 1 \otimes SS^* + 1 \otimes T & \pi_1 \otimes 0 \\ \pi_1^* \otimes 0 & 1 \otimes I \end{pmatrix} \\ &= \begin{pmatrix} 1 \otimes I & 0 \\ 0 & 1 \otimes I \end{pmatrix} \end{aligned}$$

and, similarly,

$$M^*M = \begin{pmatrix} 1 \otimes I & \pi_1^* \otimes 0 \\ \pi_1 \otimes 0 & 1 \otimes T + 1 \otimes SS^* \end{pmatrix} = \begin{pmatrix} 1 \otimes I & 0 \\ 0 & 1 \otimes I \end{pmatrix}.$$

On the other hand,

$$\begin{aligned} Mp_1M^* &= \begin{pmatrix} \pi_1 \otimes S & 1 \otimes T \\ 0 & \pi_1^* \otimes S^* \end{pmatrix} \begin{pmatrix} 1 \otimes I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \pi_1^* \otimes S^* & 0 \\ 1 \otimes T & \pi_1 \otimes S \end{pmatrix} \\ &= \begin{pmatrix} \pi_1 \otimes S & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \pi_1^* \otimes S^* & 0 \\ 1 \otimes T & \pi_1 \otimes S \end{pmatrix} \\ &= \begin{pmatrix} 1 \otimes SS^* & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence $Mp_1M^* - p_1 = (1 \otimes SS^* - 1 \otimes I) \oplus 0 = (-1 \otimes T) \oplus 0$ which has the preimage by $\tilde{\varphi}$ given by

$$z := (-1 \otimes e_{0,0}) \oplus 0 \in \mathcal{M}_2(C(\mathbb{T}) \otimes \mathcal{K}).$$

Then, by lemma 1.1.3,

$$\partial([\pi_1]) = [z + P_1] - [P_1] = -[-z]$$

since $(z+P_1)(-z) = -z^2 - z = 0$ and $z + P_1 + (-z) = P_1$. But $[-z] = [1 \otimes e_{0,0}]$ corresponds to $[1]$ via the isomorphism $K_0(C(\mathbb{T}) \otimes \mathcal{K}) \cong K_0(C(\mathbb{T}))$ of theorem 1.1.16. Hence $\partial([\pi_1]) = -[1]$. □

Corollary 3.1.2. *The group $K_1(C(\mathbb{T}^2)) \cong \mathbb{Z}^2$ is generated by $[\pi_1]$ and $[\pi_2]$.*

For $t \in [0,1]$, let us define $X_0(t)$ to be the matrix

$$\begin{pmatrix} \cos^4(\frac{\pi}{2}t) + \sin^4(\frac{\pi}{2}t) & \cos^3(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t) - \cos(\frac{\pi}{2}t)\sin^3(\frac{\pi}{2}t) \\ \cos^3(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t) - \cos(\frac{\pi}{2}t)\sin^3(\frac{\pi}{2}t) & 2\cos^2(\frac{\pi}{2}t)\sin^2(\frac{\pi}{2}t) \end{pmatrix}$$

and $X_1(t)$ to be

$$\cos(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t) \begin{pmatrix} \cos(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t) & \sin^2(\frac{\pi}{2}t) \\ -\cos^2(\frac{\pi}{2}t) & -\cos(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t) \end{pmatrix}.$$

Since $X_0(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = X_0(1)$ and $X_1(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = X_1(1)$, we get that X_0 and X_1 can be seen as continuous functions from $\mathbb{T}$ to $\mathcal{M}_2(\mathbb{C})$, i.e., as elements of $\mathcal{M}_2(C(\mathbb{T}))$. Clearly X_0 is self-adjoint and by simple computations we can prove that the three relations characterizing a Rieffel projection are satisfied by X_0 and X_1. Thus, defining

$$P(y, e^{2\pi i t}) := \begin{pmatrix} \overline{y} & 0 \\ 0 & \overline{y} \end{pmatrix} X_1^*(t) + X_0(t) + X_1(t) \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}$$

for $y \in \mathbb{T}$ and $t \in [0,1]$, we get a Rieffel projection $P \in \mathcal{M}_2(C(\mathbb{T}^2))$. We can remark that, for all $t \in [0,1]$,

$$\begin{aligned} X_0(t) &= \frac{1}{2} \begin{pmatrix} 2 - \sin^2(\pi t) & \sin(\pi t)\cos(\pi t) \\ \sin(\pi t)\cos(\pi t) & \sin^2(\pi t) \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 3 + \cos(2\pi t) & \sin(2\pi t) \\ \sin(2\pi t) & 1 - \cos(2\pi t) \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} X_1(t) &= \frac{\sin(\pi t)}{4} \begin{pmatrix} \sin(\pi t) & 1 - \cos(\pi t) \\ -1 - \cos(\pi t) & -\sin(\pi t) \end{pmatrix} \\ &= \frac{1}{8} \begin{pmatrix} 1 - \cos(2\pi t) & \sqrt{2(1 - \cos(2\pi t))} - \sin(2\pi t) \\ -\sqrt{2(1 - \cos(2\pi t))} - \sin(2\pi t) & -1 + \cos(2\pi t) \end{pmatrix}. \end{aligned}$$

Proposition 3.1.3. *The homomorphism $\partial : K_0(C(\mathbb{T}^2)) \longrightarrow K_1(C(\mathbb{T}))$ maps $[P]$ on $[\iota]$.*

Proof. The enveloping von Neumann algebra of $\mathcal{M}_2(C(\mathbb{T}))$ contains $\mathcal{M}_2(\ell^\infty(\mathbb{T}))$ and the left support projection of X_1 is defined by $l_{X_1}(t) = \mathrm{proj}_{\mathrm{im}(X_1(t))}$. But the subspace $\mathrm{im}(X_1(t))$ is clearly generated by the normed vector $\begin{pmatrix} \sin(\frac{\pi}{2}t) \\ -\cos(\frac{\pi}{2}t) \end{pmatrix}$ if $t \neq 0, 1$ and is $\{0\}$ if $t = 0, 1$. Therefore,

$$l_{X_1}(t) = \begin{pmatrix} \sin^2(\frac{\pi}{2}t) & -\cos(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t) \\ -\cos(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t) & \cos^2(\frac{\pi}{2}t) \end{pmatrix}$$

if $t \neq 0, 1$, whereas $l_{X_1}(0) = 0 = l_{X_1}(1)$. For all $t \in [0,1]$, let us define $L(t) \in \mathcal{M}_2(\mathbb{C})$ by

$$L(t) := \begin{pmatrix} \sin^2(\frac{\pi}{2}t) & -\cos(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t) \\ -\cos(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t) & \cos^2(\frac{\pi}{2}t) \end{pmatrix}.$$

Then, for all $t \in [0,1]$, a direct computation shows that $L(t)$ is an idempotent and

$$X_0(t)L(t) = \sin^2(\frac{\pi}{2}t)L(t).$$

Moreover, for every $t \in [0,1]$, we get $\exp(2\pi i X_0(t)L(t)) = \exp(2\pi i X_0(t) l_{X_1}(t))$.

Indeed, $L(t) = l_{X_1}(t)$ if $t \neq 0, 1$,

$$\begin{aligned} X_0(0)L(0) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= 0 \\ &= X_0(0) l_{X_1}(0) \end{aligned}$$

and

$$\begin{aligned} \exp(2\pi i X_0(1)L(1)) &= \exp\left(2\pi i \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= \exp\left(2\pi i \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= \sum_{n \geq 0} \frac{(2\pi i)^n}{n!} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^n \\ &= I_2 + \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= I_2 + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (\exp(2\pi i) - 1) \\ &= I_2 \\ &= \exp(0) \\ &= \exp(2\pi i X_0(t) l_{X_1}(t)). \end{aligned}$$

Let $c : [0,1] \longrightarrow GL_2(C(\mathbb{T}))$ defined by $c_r(t) = \exp\left(2\pi i \sin^2(\frac{\pi}{2}t) L(rt)\right)$, for every $r, t \in [0,1]$. It is well-defined, since for $t = 0$ we have

$$c_s(0) = \exp(0) = I_2$$

and for $t = 1$ we obtain

$$\begin{aligned} \exp(2\pi i L(r)) &= \sum_{n \geq 0} \frac{(2\pi i)^n}{n!} L(r)^n \\ &= I_2 + \sum_{n \geq 1} \frac{(2\pi i)^n}{n!} L(r) \\ &= I_2 + L(r)(\exp(2\pi i) - 1) \\ &= I_2 \end{aligned}$$

for all $r \in [0,1]$. Thus, exp, sin and L being continuous, c is a continuous path in $GL_2(C(\mathbb{T}))$.

In K_1, it induces

$$\begin{aligned}
[\exp(2\pi i X_0 l_{X_1})] &= [\exp(2\pi i X_0 L)] \\
&= [\exp(2\pi i s^2 L)] \\
&= [c_1] \\
&= [c_0] \\
&= \left[\exp\left(2\pi i s^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)\right] \\
&= [\exp(2\pi i s^2)],
\end{aligned}$$

where $s(t) := \sin(\frac{\pi}{2}t)$ for all $t \in [0,1]$. Thus we get $\partial([P]) = [\exp(2\pi i s^2)]$. But the map $c' : [0,1] \longrightarrow C(\mathbb{T})$ given by

$$c'_r(t) := \exp\left(2\pi i\left(rt + (1-r)\sin^2(\frac{\pi}{2}t)\right)\right)$$

provides a continuous path in the invertible elements of $C(\mathbb{T})$ linking $\exp(2\pi i s^2)$ with the inclusion ι. Hence $\partial([P]) = [\iota]$. □

Corollary 3.1.4. *The group $K_0(C(\mathbb{T}^2)) \cong \mathbb{Z}^2$ is generated by* $[1]$ *and* $[P]$.

3.2 The K-theory of $C^*(\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z})$

Let G be a semi-direct product of $\mathbb{Z}^2$ by $\mathbb{Z}$. Concretely, $G = \mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$, where α belongs to $\mathrm{Aut}(\mathbb{Z}^2) = GL_2(\mathbb{Z})$. Let us introduce the notation

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for the rest of the chapter. The map α extends to an automorphism of $C^*(\mathbb{Z}^2)$ as explained on page 38. This gives, for $f \in C_c(\mathbb{Z}^2)$ and $\chi \in \widehat{\mathbb{Z}^2}$,

$$\int_{\mathbb{Z}^2} (\alpha \cdot f)(g)\chi(g)dg = \int_{\mathbb{Z}^2} f(\alpha^{-1}(g))\chi(g)dg = \int_{\mathbb{Z}^2} f(g)\chi(\alpha(g))dg.$$

Thus the action on $C(\widehat{\mathbb{Z}^2})$ corresponding to α via the Fourier transform is given by

$$(\alpha \cdot f)(\chi) = f(\alpha \cdot \chi),$$

where $(\alpha \cdot \chi)(g) := \chi(\alpha(g))$ for all $g \in \mathbb{Z}^2$. Now the identification $\mathbb{T}^2 \cong \widehat{\mathbb{Z}^2}$ is provided by defining

$$\chi_{(t,s)}(m,n) := e^{2\pi i(mt+ns)}$$

for every $(e^{2\pi it}, e^{2\pi is}) \in \mathbb{T}^2$ and $(m, n) \in \mathbb{Z}^2$. Thus

$$\begin{aligned}(\alpha \cdot \chi_{(t,s)})(m,n) &= \chi_{(t,s)}(\alpha(m,n)) \\ &= \chi_{(t,s)}(am+bn, cm+dn) \\ &= \exp(2\pi i(amt + bnt + cms + dns)) \\ &= \exp(2\pi i((at+cs)m + (bt+ds)n)) \\ &= \chi_{(at+cs,bt+ds)}(m,n)\end{aligned}$$

for all $(e^{2\pi it}, e^{2\pi is}) \in \mathbb{T}^2$, $(m, n) \in \mathbb{Z}^2$. We finally obtain that α gives raise to the automorphism of $C(\mathbb{T}^2)$ coming from the homeomorphism

$$\begin{aligned}\mathbb{T}^2 &\longrightarrow \mathbb{T}^2 \\ (x,y) &\longmapsto (x^a y^c, x^b y^d),\end{aligned}$$

that we will also denote by α. Therefore we get

$$C^*(G) \cong C^*(\mathbb{Z}^2) \rtimes_\alpha \mathbb{Z} \cong C(\mathbb{T}^2) \rtimes_\alpha \mathbb{Z},$$

generated by three unitaries u, v, w satisfying the relations

$$uvu^* = \alpha(v) = v^a w^c, \qquad uwu^* = \alpha(w) = v^b w^d$$

and

$$vw = wv.$$

By Pimsner and Voiculescu, we have the six-terms exact sequence

$$\begin{tikzcd} K_0(C(\mathbb{T}^2)) \arrow[r, "\mathrm{id} - \alpha_*^{-1}"] & K_0(C(\mathbb{T}^2)) \arrow[r, "i_*"] & K_0(C^*(G)) \arrow[d, "\partial"] \\ K_1(C^*(G)) \arrow[u, "\partial"] & K_1(C(\mathbb{T}^2)) \arrow[l, "i_*"'] & K_1(C(\mathbb{T}^2)), \arrow[l, "\mathrm{id} - \alpha_*^{-1}"'] \end{tikzcd} \qquad (\star)$$

where i is the inclusion of $C(\mathbb{T}^2)$ in $C^*(G)$ given by

$$i(\pi_1) = v \quad \text{and} \quad i(\pi_2) = w.$$

The aim is to make this sequence completely explicit, whatever the action α is. Before that, we have to state some preliminary facts that will make our task easier.

Let x' be a unitary in $C(\mathbb{T}^2)$. Suppose that x' is fixed by α. Hence, defining $x := i(x')$, we have $uxu^* = x$. Then we get a surjection of $C(\mathbb{T}^2)$ onto $C^*(u, x)$, the sub-C^*-algebra of $C^*(G)$

generated by u and x, given by :

$$\begin{aligned} C(\mathbb{T}^2) &\longrightarrow C^*(u,x) \\ \pi_1 &\longmapsto u \\ \pi_2 &\longmapsto x. \end{aligned}$$

Let $P_{u,x}$ be the image of P by the extension of this surjection at the level of matrices. The naturality of the Pimsner-Voiculescu sequence and proposition 3.1.3 immediately give the next fact.

Corollary 3.2.1. *The map* $\partial : K_0(C^*(G)) \longrightarrow K_1(C(\mathbb{T}^2))$ *sends* $[P_{u,x}]$ *on* $[x']$.

Similarly, with proposition 3.1.1, we obtain the following result.

Corollary 3.2.2. *The map* $\partial : K_1(C^*(G)) \longrightarrow K_0(C(\mathbb{T}^2))$ *sends* $[u]$ *on* $-[1]$.

By Swan's theorem 1.1.8 and the Chern character, we have a link between K-theory and cohomology. Since the effect induced in cohomology by the action α is easier to understand, we will use this connection thereafter. Let us start by identifying the image of our two generators for $K_0(C(\mathbb{T}^2))$ under Swan's isomorphism. To do this, we write E_P for the vector subbundle of $\mathbb{T}^2 \times \mathbb{C}^2$ whose fiber over $x \in \mathbb{T}^2$ is given by the image of $P(x)$.

Lemma 3.2.3. *The isomorphism* $K_0(C(\mathbb{T}^2)) \cong K^0(\mathbb{T}^2)$, *maps* $[1]$ *on the class of the trivial bundle of rank* 1 *and maps* $[P]$ *on* $[E_P]$.

Proof. We have already proved the first statement in chapter 1. For the second, following the description of Swan's isomorphism given on page 20, we have to identify the continuous sections of E_P. But $\Gamma(E_P)$ is the sub-$C(\mathbb{T}^2)$-module of $\Gamma(\mathbb{T}^2 \times \mathbb{C}^2) \cong C(\mathbb{T}^2)^2$ given by the functions whose image at $x \in \mathbb{T}^2$ lies in $\mathrm{im}(P(x)) \subseteq \mathbb{C}^2$. Hence the projection on this submodule is given by P itself. □

The next lemma, and its proof, has been suggested to us by Alain Valette. Before stating it, we note that $\mathbb{T}^2$ has its integer cohomology groups all torsion-free. Therefore, the image of the Chern character for the torus lies in its integer cohomology.

Lemma 3.2.4. *There exists isomorphisms* $H^0(\mathbb{T}^2) \cong \mathbb{Z} \cong H^2(\mathbb{T}^2)$ *under which the Chern character* $ch : K^0(\mathbb{T}^2) \longrightarrow H^0(\mathbb{T}^2) \oplus H^2(\mathbb{T}^2)$ *maps* $[\mathbb{T}^2 \times \mathbb{C}]$ *to* $(1,0)$ *and* $[E_P]$ *to* $(1,1)$.

Proof. Let us consider the commutative diagram

$$\begin{tikzcd} K^0(*) \arrow[r, "p^*"] \arrow[d, "ch=ch_0"'] & K^0(\mathbb{T}^2) \arrow[d, "ch=ch_0\oplus ch_2"] \\ H^0(*) \arrow[r, "p^*"'] & H^0(\mathbb{T}^2) \oplus H^2(\mathbb{T}^2), \end{tikzcd}$$

where p stands for the projection of $\mathbb{T}^2$ onto $*$. Since all vector bundles over a point are trivial, we have a canonical isomorphism

$$\begin{aligned} K^0(*) &\longrightarrow \mathbb{Z} \\ [E]-[F] &\longmapsto \dim(E)-\dim(F). \end{aligned}$$

Let us identify $H^0(*)$ and $H^0(\mathbb{T}^2)$ with $\mathbb{Z}$ in the canonical way. Then the Chern character for the point and the map induced by p in H^0 both become the identity on $\mathbb{Z}$. Hence, under the isomorphism $H^0(\mathbb{T}^2) \cong \mathbb{Z}$, the map $ch_0 : K^0(\mathbb{T}^2) \longrightarrow H^0(\mathbb{T}^2)$ becomes

$$\begin{aligned} ch_0 : K^0(\mathbb{T}^2) &\longrightarrow \mathbb{Z} \\ [E]-[F] &\longmapsto \dim(E)-\dim(F). \end{aligned}$$

Thus it sends the class of the trivial rank-one vector bundle on 1, as well as $[E_P]$. Indeed, the fibers of E_P are one-dimensional, since

$$P(1,1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Furthermore, since $[\mathbb{T}^2 \times \mathbb{C}]$ is the image of $[* \times \mathbb{C}]$ by p^*, whose image is $H^0(\mathbb{T}^2)$, the commutative diagram above implies that $ch_2([\mathbb{T}^2 \times \mathbb{C}])$ is the zero element of $H^2(\mathbb{T}^2)$. Thus, as $[\mathbb{T}^2 \times \mathbb{C}]$ and $[E_P]$ generates $K^0(\mathbb{T}^2)$, the map ch_2 should send $[E_P]$ on a generator of $H^2(\mathbb{T}^2)$. Then we choose the isomorphism $H^2(\mathbb{T}^2) \cong \mathbb{Z}$ such that, composed with ch_2, the element $[E_P]$ is sent on 1. $\square$

3.2.1 The case $\det(\alpha) = 1$

Here we assume that α has determinant 1, meaning that $ad - bc = 1$.

Proposition 3.2.5. *The homomorphism* $\alpha_*^{-1} : K_0(C(\mathbb{T}^2)) \longrightarrow K_0(C(\mathbb{T}^2))$ *is the identity.*

Proof. By an argument similar to the one expained on page 84, the *-homomorphism induced by α^{-1} on $C(\mathbb{T}^2) \cong C^*(\mathbb{Z}^2)$ is exactly the map induced by the homeomorphism

$$\begin{aligned} \alpha^{-1} : \mathbb{T}^2 &\longrightarrow \mathbb{T}^2 \\ (x_1, x_2) &\longmapsto (x_1^d x_2^{-c}, x_1^{-b} x_2^a). \end{aligned}$$

Since $da - (-c)(-b) = ad - bc = \det(\alpha) = 1$, this homeomorphism has to be orientation-preserving. Hence, in cohomology, the fundamental class of $\mathbb{T}^2$ is fixed by α^{-1} and then α^{-1} induces the identity on the even cohomology groups of the torus. Therefore, by naturality of the Chern character, we have $\alpha_*^{-1} = \mathrm{id}_{K_0(C(\mathbb{T}^2))}$. $\square$

This proposition implies that the homomorphism induced in K_0 by $\mathrm{id} - \alpha^{-1}$ is zero. Thus, we have $[\alpha^{-1}(P)] = [P]$. Together with the fact that $V(C(\mathbb{T}^2))$ has the cancellation property by proposition 1.1.9, it induces the existence of a $D'_\alpha \in \mathcal{M}_2(C(\mathbb{T}^2))$ such that ${D'_\alpha}^* D'_\alpha = P$ and $D'_\alpha {D'_\alpha}^* = \alpha^{-1}(P)$. Let us introduce the following notations :

$$D_\alpha := i(D'_\alpha), \qquad P_{v,w} := i(P)$$

and

$$Q_\alpha := D^*_\alpha(u^* \oplus u^*)P_{v,w} + I_2 - P_{v,w}.$$

Lemma 3.2.6. *The element* Q_α *is a unitary in* $\mathcal{M}_2(C^*(G))$.

Proof. By definition of D_α we obtain that

$$\begin{aligned} P_{v,w}(u \oplus u)D_\alpha &= (u \oplus u)\alpha^{-1}(P_{v,w})D_\alpha \\ &= (u \oplus u)D_\alpha D^*_\alpha D_\alpha \\ &= (u \oplus u)D_\alpha P_{v,w}. \end{aligned}$$

Therefore $P_{v,w}$ commutes with $(u \oplus u)D_\alpha$, as well as with its adjoint. Hence

$$\begin{aligned} Q_\alpha Q^*_\alpha = &\ (D^*_\alpha(u^* \oplus u^*)P_{v,w} + I_2 - P_{v,w})\,(P_{v,w}(u \oplus u)D_\alpha + I_2 - P_{v,w}) \\ = &\ D^*_\alpha(u^* \oplus u^*)P_{v,w}(u \oplus u)D_\alpha + (I_2 - P_{v,w})\,(P_{v,w}(u \oplus u)D_\alpha) + \\ &\ + (D^*_\alpha(u^* \oplus u^*)P_{v,w})\,(I_2 - P_{v,w}) + (I_2 - P_{v,w})^2 \\ = &\ P_{v,w} + 0 + 0 + I_2 - P_{v,w} = I_2 \end{aligned}$$

and

$$\begin{aligned} Q^*_\alpha Q_\alpha = &\ (P_{v,w}(u \oplus u)D_\alpha + I_2 - P_{v,w})\,(D^*_\alpha(u^* \oplus u^*)P_{v,w} + I_2 - P_{v,w}) \\ = &\ P_{v,w} + (I_2 - P_{v,w})\,(P_{v,w}D^*_\alpha(u^* \oplus u^*)) + \\ &\ + ((u \oplus u)D_\alpha P_{v,w})\,(I_2 - P_{v,w}) + (I_2 - P_{v,w})^2 \\ = &\ P_{v,w} + 0 + 0 + I_2 - P_{v,w} = I_2. \end{aligned}$$

□

Proposition 3.2.7. *The homomorphism* $\partial : K_1(C^*(G)) \longrightarrow K_0(C(\mathbb{T}^2))$ *maps* $[Q_\alpha]$ *to* $[P]$.

Proof. Once again, we use the development outlined on page 44. Here the exact sequence of the Toeplitz extension associated to the crossed product considered is

$$0 \longrightarrow C(\mathbb{T}^2) \otimes \mathcal{K} \xrightarrow{\tilde{\varphi}} \mathcal{T}_{C(\mathbb{T}^2),\alpha} \xrightarrow{\tilde{\psi}} C(\mathbb{T}^2) \times_\alpha \mathbb{Z} \longrightarrow 0.$$

A preimage of $Q_\alpha \oplus Q^*_\alpha$ by $\tilde{\psi}$ is given by the matrix M defined by

$$\begin{pmatrix} D^*_\alpha(u^* \oplus u^*)P_{v,w} \otimes S^* + (I_2 - P_{v,w}) \otimes I & 0 \\ P_{v,w} \otimes T & P_{v,w}(u \oplus u)D_\alpha \otimes S + (I_2 - P_{v,w}) \otimes I \end{pmatrix}$$

since

$$\begin{aligned}\tilde{\psi}(P_{v,w} \otimes T) &= \tilde{\psi}((P_{v,w} \otimes I) - (P_{v,w} \otimes S)(P_{v,w} \otimes S^*)) \\ &= \tilde{\psi}(P_{v,w} \otimes I) - \tilde{\psi}(P_{v,w} \otimes S)\tilde{\psi}(P_{v,w} \otimes S^*) \\ &= P_{v,w} - P_{v,w}^2 \\ &= 0.\end{aligned}$$

By direct computation, we can show that M is a unitary in $\mathcal{M}_4(\mathcal{T}_{C(\mathbb{T}^2),\alpha})$. Moreover, recalling that $p_2 = (1 \otimes I) \oplus (1 \otimes I) \oplus 0 \oplus 0 \in \mathcal{M}_4(\mathcal{T}_{C(\mathbb{T}^2),\alpha})$, we have

$$\begin{aligned}Mp_2M^* &= \begin{pmatrix} D_\alpha^*(u^* \oplus u^*)P_{v,w} \otimes S^* + (I_2 - P_{v,w}) \otimes I & 0 \\ P_{v,w} \otimes T & 0 \end{pmatrix} \cdot M^* \\ &= \begin{pmatrix} Q_\alpha Q_\alpha^* \otimes I & D_\alpha^*(u^* \oplus u^*)P_{v,w} \otimes 0 + 0 \otimes T \\ P_{v,w}(u \oplus u)D_\alpha \otimes 0 + 0 \otimes T & (P_{v,w} \otimes T)^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \otimes I & 0 \\ 0 & P_{v,w} \otimes T \end{pmatrix}\end{aligned}$$

implying $Mp_2M^* - p_2 = 0 \oplus (P_{v,w} \otimes T)$. A preimage by $\tilde{\varphi}$ of the latter is provided by $0 \oplus (P \otimes e_{0,0})$. Thus, with $P_2 = I_2 \oplus 0_2 \in \mathcal{M}_4(\mathbb{C})$,

$$\partial([Q_\alpha]) = [P_2 \oplus (P \otimes e_{0,0})] - [P_2] = [P_2] + [P \otimes e_{0,0}] - [P_2] = [P \otimes e_{0,0}]$$

which is sent on $[P]$ by the isomorphism $K_0(C(\mathbb{T}^2) \otimes \mathcal{K}) \cong K_0(C(\mathbb{T}^2))$ of theorem 1.1.16. □

Coming back to the Pimsner-Voiculescu sequence $(\star)$, since $\mathrm{id} - \alpha^{-1}$ induces the zero map in K_0, we get the following exact sequence :

$$0 \longrightarrow K_0(C(\mathbb{T}^2)) \xrightarrow{i_*} K_0(C^*(G)) \xrightarrow{\partial} \mathrm{im}(\partial) \longrightarrow 0.$$

Being a subgroup of $K_1(C(\mathbb{T}^2))$, which is free, $\mathrm{im}(\partial)$ is also free. Hence the sequence splits, giving

$$K_0(C^*(G)) \cong K_0(C(\mathbb{T}^2)) \oplus \mathrm{im}(\partial) \qquad (\heartsuit)$$

generated by $[1] = i_*([1])$, $[P_{v,w}] = i_*([P])$ and preimage by ∂ of generators of $\ker(\mathrm{id} - \alpha_*^{-1}) = \mathrm{im}(\partial)$, given by corollary 3.2.1. Similarly, $(\star)$ induces the short exact sequence

$$0 \longrightarrow \ker(\partial) \longrightarrow K_1(C^*(G)) \xrightarrow{\partial} K_0(C(\mathbb{T}^2)) \longrightarrow 0$$

which implies

$$K_1(C^*(G)) \cong \ker(\partial) \oplus K_0(C(\mathbb{T}^2)). \qquad (\diamondsuit)$$

The summand in $K_1(C^*(G))$ corresponding to $K_0(C(\mathbb{T}^2))$ is generated by $[u]$ and $[Q_\alpha]$, as showed by 3.2.2 and 3.2.7. Moreover, by exactness of the Pimsner-Voiculescu sequence $(\star)$ and the first isomorphism theorem,

$$\ker(\partial) = \operatorname{im}(i_*) \cong K_1(C(\mathbb{T}^2))/\ker(i_*) = K_1(C(\mathbb{T}^2))/\operatorname{im}(\mathrm{id} - \alpha_*^{-1}).$$

Thus, to go further, we need to compute the kernel and the image of the map induced by $\mathrm{id} - \alpha^{-1}$ in K_1. But, in the basis $\{[\pi_1], [\pi_2]\}$ of $K_1(\mathbb{T}^2)$, this homomorphism is exactly described by the matrix $I_2 - \alpha^{-1}$. Hence, as

$$\begin{aligned}\det(I_2 - \alpha^{-1}) = \det \begin{pmatrix} 1-d & b \\ c & 1-a \end{pmatrix} &= (1-d)(1-a) - bc \\ &= 1 - a - d + ad - bc = 1 - \operatorname{tr}(\alpha) + \det(\alpha) \\ &= 2 - \operatorname{tr}(\alpha),\end{aligned}$$

we will make several cases, depending on the trace of α.

The case $\alpha = I_2$

Let us first suppose that α is the identity matrix and therefore that $G = \mathbb{Z}^3$. Thus $C^*(G)$ is the free commutative C^*-algebra generated by the three unitaries u, v and w. Hence

$$C^*(G) \cong C(\mathbb{T}^3).$$

Moreover, the *-homomorphism $\mathrm{id} - \alpha_*^{-1} : C(\mathbb{T}^2) \longrightarrow C(\mathbb{T}^2)$ is zero and we immediately have the result by (♡) and (◇).

Proposition 3.2.8. *The K-theory groups of $C(\mathbb{T}^3)$ are both $\mathbb{Z}^4$. Generators for $K_0(C(\mathbb{T}^3))$ are given by $[1]$, $[P_{v,w}]$, $[P_{u,v}]$ and $[P_{u,w}]$. On the other hand, generators for $K_1(C(\mathbb{T}^3))$ are given by $[Q_\alpha]$, $[u]$, $[v]$ and $[w]$.*

To get a concrete Q_α, we can choose in this case the one corresponding to $D'_\alpha = P$, since

$$P^*P = P^2 = P \quad \text{and} \quad PP^* = P^2 = P = \alpha^{-1}(P).$$

The case $\operatorname{tr}(\alpha) = 2$ and $\alpha \neq I_2$

If $b = 0$ and $a + d = \operatorname{tr}(\alpha) = 2$, we deduce from $ad = \det(\alpha) = 1$ that

$$\alpha = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \quad \text{and} \quad I_2 - \alpha^{-1} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix},$$

with $c \neq 0$ if $\alpha \neq I_2$. Hence, in K_1, the homomorphism $\mathrm{id} - \alpha_*^{-1}$ maps $[\pi_1]$ to $c[\pi_2]$ and $[\pi_2]$ to 0. It induces that its kernel and image are both isomorphic to $\mathbb{Z}$, generated by $[\pi_2]$ and $c[\pi_2]$ respectively. Thus (♡) and (◇) give the next proposition.

Proposition 3.2.9. *Assume $b = 0$ and $c \neq 0$.*

1) *The group $K_0(C^*(G))$ is isomorphic to $\mathbb{Z}^3$. Generators are given by $[1]$, $[P_{v,w}]$ and $[P_{u,w}]$.*

2) *The group $K_1(C^*(G))$ is isomorphic to $\mathbb{Z}^3 \oplus \mathbb{Z}_{|c|}$. Generators are provided by the torsion-free elements $[Q_\alpha]$, $[u]$ and $[v]$, together with the class $[w]$, which is of order $|c|$.*

To identify the element Q_α, let us define

$$B(y,t) := \begin{pmatrix} \cos(\frac{\pi}{2}t) & -\sin(\frac{\pi}{2}t) \\ \sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix}$$

for $y \in \mathbb{T}$ and $t \in [0,1]$. Although B is not an element of $C(\mathbb{T}^2)$, since $B(y,0) \neq B(y,1)$ for every $y \in \mathbb{T}$, the element $D'_\alpha := B(\pi_2^c \oplus 1)B^*P$ is one, since

$$B(y,0)(\pi_2^c(y,e^0) \oplus 1)B^*(y,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1^c & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned} B(y,1)(\pi_2^c(y,e^{2\pi i}) \oplus 1)B^*(y,1) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1^c & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Moreover, noticing that $B^*B = I_2 = BB^*$, we get

$${D'_\alpha}^* D'_\alpha = P^*B(\pi_2^{*c} \oplus 1)B^*B(\pi_2^c \oplus 1)B^*P = PP = P$$

and

$$\begin{aligned} D'_\alpha {D'_\alpha}^* &= B(\pi_2^c \oplus 1)B^*PP^*B(\pi_2^{*c} \oplus 1)B^* \\ &= B(\pi_2^c \oplus 1)B^*PB(\pi_2^{*c} \oplus 1)B^* \\ &= B(\pi_2^c \oplus 1)B^*B(\pi_1^* \oplus 1)B^*p_1B(\pi_1 \oplus 1)B^*B(\pi_2^{*c} \oplus 1)B^* \\ &= B(\pi_2^c\pi_1^* \oplus 1)B^*p_1B(\pi_1\pi_2^{*c} \oplus 1)B^* \\ &= B(\alpha^{-1}(\pi_1^*) \oplus 1)B^*p_1B(\alpha^{-1}(\pi_1) \oplus 1)B^* \\ &= \alpha^{-1}\left(B(\pi_1^* \oplus 1)B^*p_1B(\pi_1 \oplus 1)B^*\right) \\ &= \alpha^{-1}(P), \end{aligned}$$

since a direct computation shows that $P = B(\pi_1^* \oplus 1)B^*p_1B(\pi_1 \oplus 1)B^*$. Thus we can choose

$$Q_\alpha = D_\alpha^*(u^* \oplus u^*)P_{v,w} + I_2 - P_{v,w}.$$

It remains to treat the case where $b \neq 0$. From $\mathrm{tr}(\alpha) = 2$ we deduce $d = 2 - a$, and from $\det(\alpha) = 1$ we get

$$c = \frac{ad-1}{b} = \frac{a(2-a)-1}{b} = \frac{2a-a^2-1}{b} = -\frac{(1-a)^2}{b}.$$

Thus $I_2 - \alpha^{-1} = \begin{pmatrix} -(1-a) & b \\ -\frac{(1-a)^2}{b} & 1-a \end{pmatrix}$. Then we obtain that

$$(\mathrm{id} - \alpha_*^{-1})([\pi_1]) = -(1-a)[\pi_1] - \frac{(1-a)^2}{b}[\pi_2]$$

and

$$(\mathrm{id} - \alpha_*^{-1})([\pi_2]) = b[\pi_1] + (1-a)[\pi_2].$$

Let $a', b' \in \mathbb{Z}$ be such that $\frac{1-a}{b} = \frac{1-a'}{b'}$ and $\gcd(1-a', b') = 1$. By Bézout, there exists $m, n \in \mathbb{Z}$ satisfying

$$m(1-a') + nb' = 1.$$

We also set $g := \gcd(1-a, b)$.

Lemma 3.2.10. *The homomorphism* $\mathrm{id} - \alpha_*^{-1} : K_1(C(\mathbb{T}^2)) \longrightarrow K_1(C(\mathbb{T}^2))$ *has infinite cyclic kernel, generated by* $b'[\pi_1] + (1-a')[\pi_2]$.

Proof. From $\frac{1-a}{b} = \frac{1-a'}{b'}$ we get $b'(1-a) = b(1-a')$ and $\frac{b'(1-a)^2}{b} = (1-a')(1-a)$. Therefore

$$\begin{aligned}
&(\mathrm{id} - \alpha_*^{-1})(b'[\pi_1] + (1-a')[\pi_2]) = \\
&\quad = b'\left(-(1-a)[\pi_1] - \frac{(1-a)^2}{b}[\pi_2]\right) + (1-a')\left(b[\pi_1] + (1-a)[\pi_2]\right) \\
&\quad = (-b'(1-a) + b(1-a'))[\pi_1] + \left(-\frac{b'(1-a)^2}{b} + (1-a')(1-a)\right)[\pi_2] \\
&\quad = 0
\end{aligned}$$

and then $\langle b'[\pi_1] + (1-a')[\pi_2]\rangle \subseteq \ker(\mathrm{id} - \alpha_*^{-1})$. Conversely, let $k_1[\pi_1] + k_2[\pi_2]$ be an element in $\ker(\mathrm{id} - \alpha_*^{-1})$. Then

$$\begin{aligned}
0 &= (\mathrm{id} - \alpha_*^{-1})(k_1[\pi_1] + k_2[\pi_2]) \\
&= k_1\left(-(1-a)[\pi_1] - \frac{(1-a)^2}{b}[\pi_2]\right) + k_2(b[\pi_1] + (1-a)[\pi_2]) \\
&= (-k_1(1-a) + k_2 b)[\pi_1] + \left(-k_1\frac{(1-a)^2}{b} + k_2(1-a)\right)[\pi_2] \\
&= (-k_1(1-a) + k_2 b)[\pi_1] + \frac{1-a}{b}(-k_1(1-a) + k_2 b)[\pi_2].
\end{aligned}$$

We deduce that $k_1(1-a) = k_2 b$ and therefore, dividing by g, that $k_1(1-a') = k_2 b'$. Hence

$$\begin{aligned} k_1 &= k_1(m(1-a') + nb') = mk_1(1-a') + nk_1 b' \\ &= mk_2 b' + nk_1 b' = (mk_2 + nk_1)b' \end{aligned}$$

and similarly

$$\begin{aligned} k_2 &= k_2(m(1-a') + nb') = mk_2(1-a') + nk_2 b' \\ &= mk_2(1-a') + nk_1(1-a') = (mk_2 + nk_1)(1-a'). \end{aligned}$$

Thus $k_1[\pi_1] + k_2[\pi_2] = (mk_2 + nk_1)(b'[\pi_1] + (1-a')[\pi_2])$. □

Combined with (♡), it immediately implies the following.

Proposition 3.2.11. *The group $K_0(C^*(G))$ is isomorphic to $\mathbb{Z}^3$. Generators are given by* $[1]$, $[P_{v,w}]$ *and* $[P_{u,v^{b'}w^{1-a'}}]$.

Lemma 3.2.12. *The image of* $\operatorname{id} - \alpha_*^{-1} : K_1(C(\mathbb{T}^2)) \longrightarrow K_1(C(\mathbb{T}^2))$ *is isomorphic to* $\mathbb{Z}$ *and generated by* $g[\pi_1] + \frac{g(1-a)}{b}[\pi_2]$.

Proof. The computation

$$\begin{aligned} &(\operatorname{id} - \alpha_*^{-1})(-m[\pi_1] + n[\pi_2]) = \\ &\quad - m(-(1-a)[\pi_1] - \frac{(1-a)^2}{b}[\pi_2]) + n(b[\pi_1] + (1-a)[\pi_2]) \\ &\quad = (m(1-a) + nb)[\pi_1] + \left(m\frac{(1-a)^2}{b} + n(1-a)\right)[\pi_2] \\ &\quad = (m(1-a) + nb)[\pi_1] + (m(1-a) + nb)\frac{1-a}{b}[\pi_2] \\ &\quad = g[\pi_1] + \frac{g(1-a)}{b}[\pi_2] \end{aligned}$$

proves that $\langle g[\pi_1] + \frac{g(1-a)}{b}[\pi_2]\rangle$ is a subgroup of $\operatorname{im}(\operatorname{id} - \alpha_*^{-1})$. On the other hand, the image of $\operatorname{id} - \alpha_*^{-1}$ is generated by $-(1-a)[\pi_1] - \frac{(1-a)^2}{b}[\pi_2]$ and $b[\pi_1] + (1-a)[\pi_2]$. But

$$-(1-a)[\pi_1] - \frac{(1-a)^2}{b}[\pi_2] = -(1-a')\left(g[\pi_1] + \frac{g(1-a)}{b}[\pi_2]\right)$$

and

$$b[\pi_1] + (1-a)[\pi_2] = b'\left(g[\pi_1] + \frac{g(1-a)}{b}[\pi_2]\right).$$ □

To make the group $K_1(C^*(G))$ explicit, let us define $h := \gcd(g, \frac{g(1-a)}{b})$. By Bézout, there exists $r, s \in \mathbb{Z}$ such that $rg + s\frac{g(1-a)}{b} = h$.

Proposition 3.2.13. *The group $K_1(C^*(G))$ is isomorphic to $\mathbb{Z}^3 \oplus \mathbb{Z}_h$. Generators are given by $[Q_\alpha]$, $[u]$, $-s[v] + r[w]$, whose order is infinite, and $\frac{g}{h}[v] + \frac{g(1-a)}{hb}[w]$, whose order is h.*

Proof. For this proof, let us write g' for the integer $\frac{g(1-a)}{b}$. We recall that the kernel of the map $\partial : K_1(C^*(G)) \longrightarrow K_0(C(\mathbb{T}^2))$ is isomorphic to $K_1(C(\mathbb{T}^2))/\operatorname{im}(\mathrm{id} - \alpha_*^{-1})$. Let us consider the group homomorphism

$$\begin{aligned} \varphi : K_1(C(\mathbb{T}^2)) &\longrightarrow \mathbb{Z} \oplus \mathbb{Z}_h \\ k_1[\pi_1] + k_2[\pi_2] &\longrightarrow \left(\frac{-k_1 g' + k_2 g}{h}, [k_1 r + k_2 s] \right). \end{aligned}$$

It is well-defined, since h divides both g and g'. Moreover, it is surjective since the image of $-s[\pi_1] + r[\pi_2]$ is

$$\left(\frac{sg' + rg}{h}, [-sr + rs] \right) = (1, [0])$$

and the image of $\frac{g}{h}[\pi_1] + \frac{g'}{h}[\pi_2]$ is

$$\left(\frac{-\frac{gg'}{h} + \frac{g'g}{h}}{h}, \left[\frac{rg}{h} + \frac{sg'}{h} \right] \right) = (0, [1]).$$

Furthermore, an element $k_1[\pi_1] + k_2[\pi_2]$ of $K_1(C(\mathbb{T}^2))$ is in the kernel of φ if and only if

$$-k_1 g' + k_2 g = 0 \quad \text{and} \quad k_1 r + k_2 s \in h\mathbb{Z}.$$

The first condition is equivalent to

$$k_2 = \frac{k_1 g'}{g}$$

since $g \neq 0$. Then the second condition is equivalent to

$$k_1 h = k_1 rg + k_1 sg' \in hg\mathbb{Z},$$

that is, $k_1 \in g\mathbb{Z}$. This implies that $\ker(\varphi)$ is the subgroup of $K_1(C(\mathbb{T}^2))$ generated by the element $g[\pi_1] + g'[\pi_2]$, which is exactly the image of $\mathrm{id} - \alpha_*^{-1}$, by lemma 3.2.12. Therefore, by the first isomorphism theorem,

$$\ker(\partial) \cong K_1(C(\mathbb{T}^2))/\operatorname{im}(\mathrm{id} - \alpha_*^{-1}) \cong \mathbb{Z} \oplus \mathbb{Z}_h.$$

The beginning of the proof gives that $-s[v] + r[w]$ and $\frac{g}{h}[v] + \frac{g'}{h}[w]$ generate $\ker(\partial)$, the second being of order h. We conclude by $(\diamondsuit)$. □

The case $\operatorname{tr}(\alpha) \in \{1, 3\}$

This is the case requiring the less effort. Indeed, if $\operatorname{tr}(\alpha)$ equals 1 or 3, then

$$\det(I_2 - \alpha^{-1}) = 2 - \operatorname{tr}(\alpha) \in \{-1, 1\}.$$

Hence the map $\mathrm{id} - \alpha_*^{-1}$ is an isomorphism from $K_1(C(\mathbb{T}^2))$ onto itself. Then the Pimsner-Voiculescu sequence $(\star)$ reads as follows :

$$\begin{tikzcd}
K_0(C(\mathbb{T}^2)) \arrow[r, "0"] & K_0(C(\mathbb{T}^2)) \arrow[r, "i_*"] & K_0(C^*(G)) \arrow[d, "\partial"] \\
K_1(C^*(G)) \arrow[u, "\partial"] & K_1(C(\mathbb{T}^2)) \arrow[l, "i_*"'] & K_1(C(\mathbb{T}^2)). \arrow[l, "\cong"']
\end{tikzcd}$$

By exactness, we deduce that i_* is the zero map in K_1, as well as the right-hand vertical homomorphism. Thus, the map i_* in K_0 and the left-hand vertical homomorphism are both preceded and followed by zero maps, and hence are isomorphisms.

Proposition 3.2.14. *The K-theory groups of $C^*(G)$ are both isomorphic to $\mathbb{Z}^2$. The group $K_0(C^*(G))$ is generated by $[1]$ and $[P_{v,w}]$, whereas $K_1(C^*(G))$ is generated by $[u]$ and $[Q_\alpha]$.*

The case $\mathrm{tr}(\alpha) \notin \{1, 2, 3\}$

In this case, $I_2 - \alpha_*^{-1}$ has nonzero determinant, and hence is injective. Then, by exactness of the Pimsner-Voiculescu sequence $(\star)$, the connecting homomorphism

$$\partial : K_0(C^*(G)) \longrightarrow K_1(C(\mathbb{T}^2))$$

is the zero map, and therefore $i : C(\mathbb{T}^2) \longrightarrow C^*(G)$ induces an isomorphism in K_0.

Proposition 3.2.15. *The group $K_0(C^*(G))$ is isomorphic to $\mathbb{Z}^2$, generated by $[1]$ and $[P_{v,w}]$.*

On the other hand, $\mathrm{id} - \alpha_*^{-1}$ is non-surjective, since $\det(I_2 - \alpha^{-1}) \notin \{-1, 1\}$. Even more : its image has rank two in the $\mathbb{Z}$-module $K_0(C(\mathbb{T}^2)) \cong \mathbb{Z}^2$. This forces $K_1(C(\mathbb{T}^2))/\operatorname{im}(\mathrm{id} - \alpha_*^{-1})$ to be isomorphic to a direct sum of two cyclic groups of finite order. Let us give more explanations.

Lemma 3.2.16. *Applying elementary operations on its rows and columns, we can transform $I_2 - \alpha^{-1}$ into a diagonal matrix $h_1 \oplus h_2$, with $h_1, h_2 \in \mathbb{N}^*$. Moreover, the equality $h_1 h_2 = |2 - \mathrm{tr}(\alpha)|$ holds.*

Proof. We recall that $I_2 - \alpha^{-1} = \begin{pmatrix} 1-d & b \\ c & 1-a \end{pmatrix}$. Let us define $g_1 := \gcd(1-d, b)$. By Bézout, there exists $m, n \in \mathbb{Z}$ such that

$$m(1-d) + nb = g_1.$$

Thus the matrix $M_1 := \begin{pmatrix} m & \frac{-b}{g_1} \\ n & \frac{1-d}{g_1} \end{pmatrix}$ lives in $GL_2(\mathbb{Z})$ and satisfies

$$(I_2 - \alpha^{-1}) M_1 = \begin{pmatrix} g_1 & 0 \\ c_1 & a_1 \end{pmatrix}$$

with $a_1, c_1 \in \mathbb{Z}$.

If g_1 divides c_1, left multiplying by $\begin{pmatrix} 1 & 0 \\ \frac{-c_1}{g_1} & 1 \end{pmatrix} \in GL_2(\mathbb{Z})$ provides a diagonal matrix. Otherwise, using an argument similar as before on the rows, we find an $M_2 \in GL_2(\mathbb{Z})$ such that

$$M_2 \begin{pmatrix} g_1 & 0 \\ c_1 & a_1 \end{pmatrix} = \begin{pmatrix} g_2 & b_2 \\ 0 & a_2 \end{pmatrix}$$

with $a_2, b_2 \in \mathbb{Z}$ and $g_2 = \gcd(g_1, c_1)$. Since g_1 does not divide c_1, we have $g_2 < g_1$. Since those g_j are greatest common divisors, they are positive. Thus, resuming with this process, it should ends at some step with a diagonal matrix. In brief, we have proved that there exists $M, M' \in GL_2(\mathbb{Z})$ such that

$$M(I_2 - \alpha^{-1})M'$$

is diagonal. If one of the coefficients is negative, we can transform it into its abolute value by multiplying the matrix by the diagonal one $-1 \oplus 1$ or $1 \oplus -1$. We end this proof by recalling the linear algebra fact that $GL_2(\mathbb{Z})$ is multiplicatively generated by elementary matrices, whose determinant is 1 or -1, together with the equality $\det(I_2 - \alpha^{-1}) = 2 - \operatorname{tr}(\alpha)$. □

Proposition 3.2.17. *With the same notations as in the previous lemma, the group $K_1(C^*(G))$ is isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}_{h_1} \oplus \mathbb{Z}_{h_2}$. Generators for the torsion-free summand are given by $[u]$ and $[Q_\alpha]$.*

Proof. Right-multiplying the matrix $I_2 - \alpha_*^{-1}$ by an elementary one only consists to change the basis of the image. On the other hand, left-multiplying by an invertible matrix defines an isomorphism φ of $\mathbb{Z}^2 \cong K_1(C(\mathbb{T}^2))$ and we have

$$K_1(C(\mathbb{T}^2))/\operatorname{im}(\operatorname{id} - \alpha_*^{-1}) \cong \mathbb{Z}^2/\operatorname{im}(\varphi(\operatorname{id} - \alpha_*^{-1})).$$

Thus previous lemma and $(\diamondsuit)$ give

$$K_1(C^*(G)) \cong \mathbb{Z}^2 \oplus (\mathbb{Z}^2/\langle (h_1, 0), (0, h_2)\rangle) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_{h_1} \oplus \mathbb{Z}_{h_2}.$$ □

As an example, we can treat the case where $b = 0$. Since $\operatorname{tr}(\alpha^{-1}) \neq 2$, we deduce that $\alpha^{-1} = \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix}$ and then

$$I_2 - \alpha^{-1} = \begin{pmatrix} 2 & 0 \\ c & 2 \end{pmatrix}.$$

Thus, if c is even, adding $-\frac{c}{2}$ times the second column to the first one gives the diagonal matrix $2 \oplus 2$ and therefore

$$K_1(C^*(G)) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

On the other hand, if c is odd, we get, by successive applications of the steps of lemma 3.2.16

proof,

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ c & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{c-1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

and therefore $K_1(C^*(G)) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_4$.

3.2.2 The case $\det(\alpha) = -1$

For this section, let us assume that α has determinant -1.

Proposition 3.2.18. *The homomorphism* $\alpha_*^{-1} : K_0(C(\mathbb{T}^2)) \longrightarrow K_0(C(\mathbb{T}^2))$ *maps* $[1]$ *to itself and* $[P]$ *to* $2[1] - [P]$.

Proof. The map α^{-1} gives raise to an automorphism of $C(\mathbb{T}^2)$ that is induced by an orientation-reversing homeomorphism of $\mathbb{T}^2$. Thus, in cohomology, α^{-1} induces the identity on $H^0(\mathbb{T}^2)$ and the map obtained by multiplication with -1 in $H^2(\mathbb{T}^2)$. Consequently, with the same identifications as in lemma 3.2.4,

$$ch(\alpha_*^{-1}([\mathbb{T}^2 \times \mathbb{C}])) = (1, 0) = ch([\mathbb{T}^2 \times \mathbb{C}])$$

and

$$\begin{aligned} ch(\alpha_*^{-1}([E_P])) &= (1, -1) = 2(1, 0) - (1, 1) \\ &= 2\, ch([\mathbb{T}^2 \times \mathbb{C}]) - ch([E_P]). \end{aligned}$$

□

This proposition implies that, in K_0, the image of $\mathrm{id} - \alpha_*^{-1}$ is infinite cyclic, generated by $2([P] - [1])$. Hence,

$$K_0(C(\mathbb{T}^2)) / \operatorname{im}(\mathrm{id} - \alpha_*^{-1}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$$

generated by $[1]$ and $[P] - [1]$, the latter being of order 2. Thus, since

$$\ker(\partial) = \operatorname{im}(i_*) \cong K_0(C(\mathbb{T}^2)) / \ker(i_*) = K_0(C(\mathbb{T}^2)) / \operatorname{im}(\mathrm{id} - \alpha_*^{-1}),$$

the Pimsner-Voiculescu sequence $(\star)$ induces the short exact sequence

$$0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}_2 \longrightarrow K_0(C^*(G)) \xrightarrow{\partial} \operatorname{im}(\partial) \longrightarrow 0.$$

Therefore,

$$K_0(C^*(G)) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \operatorname{im}(\partial) \qquad (\spadesuit)$$

with generators given by $[1]$, $[P_{v,w}] - [1]$ and preimages by ∂ of generating elements of $\operatorname{im}(\partial) = \ker(\mathrm{id} - \alpha_*^{-1})$, given by 3.2.1. On the other hand, proposition 3.2.18 shows that the image of $\partial : K_1(C^*(G)) \longrightarrow K_0(C(\mathbb{T}^2))$, which is equal to the kernel of $\mathrm{id} - \alpha_*^{-1}$, is generated by $[1]$. Thus $(\star)$ induces

$$0 \longrightarrow \ker(\partial) \longrightarrow K_1(C^*(G)) \xrightarrow{\partial} \langle [1] \rangle \cong \mathbb{Z} \longrightarrow 0.$$

Hence, by splitness of this sequence and corollary 3.2.2,

$$K_1(C^*(G)) \cong \mathbb{Z} \oplus \ker(\partial) \cong \langle [u] \rangle \oplus \ker(\partial) \tag{$\clubsuit$}$$

with, as before,

$$\ker(\partial) = \operatorname{im}(i_*) \cong K_0(C(\mathbb{T}^2))/\ker(i_*) = K_0(C(\mathbb{T}^2))/\operatorname{im}(\operatorname{id} - \alpha_*^{-1}).$$

Therefore, once again, we are led to find the kernel and image of the homomorphism induced by $\operatorname{id} - \alpha_*^{-1}$ in K_1. Different cases will again arise and depend on the trace of α, but will be slightly different than in the previous section, since

$$\begin{aligned} \det(I_2 - \alpha^{-1}) = \det \begin{pmatrix} 1+d & -b \\ -c & 1+a \end{pmatrix} &= (1+d)(1+a) - bc \\ &= 1 + a + d + ad - bc = 1 + \operatorname{tr}(\alpha) + \det(\alpha), \\ &= \operatorname{tr}(\alpha). \end{aligned}$$

The case $\operatorname{tr}(\alpha) = 0$

If we suppose $\operatorname{tr}(\alpha) = 0$, then $d = -a$. Consequently,

$$-1 = \det(\alpha) = ad - bc = -a^2 - bc.$$

Assume first that $b = 0$, hence $a^2 = 1$. If $a = 1$, then

$$\alpha = \begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix} = \alpha^{-1}$$

and therefore

$$I_2 - \alpha^{-1} = \begin{pmatrix} 0 & 0 \\ -c & 2 \end{pmatrix}.$$

Thus $\operatorname{id} - \alpha_*^{-1}$ sends $[\pi_1]$ on $-c[\pi_2]$ and $[\pi_2]$ on $2[\pi_2]$.

Proposition 3.2.19. *Suppose $a = 1$ and $b = 0$.*

1) *If c is even, then*
$$K_0(C^*(G)) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2 \cong K_1(C^*(G)).$$
Generators are provided by $[P_{u,vw^{\frac{c}{2}}}]$, $[1]$ and $[P_{v,w}] - [1]$ for $K_0(C^*(G))$ and by $[u]$, $[v]$ and the order-two element $[w]$ for $K_1(C^*(G))$.

2) *If c is odd, then $K_0(C^*(G)) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2$, generated by $[P_{u,v^2w^c}]$, $[1]$ and $[P_{v,w}] - [1]$, and $K_1(C^*(G)) \cong \mathbb{Z}^2 \cong \langle [u] \rangle \oplus \langle [v] \rangle$.*

Proof. From (♠) and (♣), it remains to identify the kernel and image of $\operatorname{id} - \alpha_*^{-1}$ and find generators for them. If c is even, it is obvious that $\operatorname{im}(\operatorname{id} - \alpha_*^{-1}) = \langle 2[\pi_2] \rangle$. Hence

$$K_1(C(\mathbb{T}^2))/\operatorname{im}(\operatorname{id} - \alpha_*^{-1}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$$

generated by $[\pi_1]$ and the order-two element $[\pi_2]$. On the other hand, $[\pi_1] + \frac{c}{2}[\pi_2]$ lies in the kernel of $\operatorname{id} - \alpha_*^{-1}$. But, if $k_1[\pi_1] + k_2[\pi_2]$ is sent on 0, then

$$0 = ck_1[\pi_2] + 2k_2[\pi_2] = (-ck_1 + 2k_2)[\pi_2] = 2(-\frac{c}{2}k_1 + k_2)[\pi_2]$$

and therefore $k_2 = \frac{c}{2}k_1$. Hence

$$\ker(\operatorname{id} - \alpha_*^{-1}) = \langle [\pi_1] + \frac{c}{2}[\pi_2] \rangle,$$

proving the first part. If c is odd, then a process similar as in the proof of lemmas 3.2.10 and 3.2.12 shows

$$\ker(\operatorname{id} - \alpha_*^{-1}) = \langle 2[\pi_1] + c[\pi_2] \rangle$$

and

$$\operatorname{im}(\operatorname{id} - \alpha_*^{-1}) = \langle [\pi_2] \rangle. \qquad \square$$

Furthermore, if $b = 0$ and $a = -1$, we get

$$\alpha = \begin{pmatrix} -1 & 0 \\ c & 1 \end{pmatrix} = \alpha^{-1}$$

and therefore

$$I_2 - \alpha^{-1} = \begin{pmatrix} 2 & 0 \\ -c & 0 \end{pmatrix}.$$

Thus $\ker(\operatorname{id} - \alpha^{-1}) = \langle [\pi_2] \rangle \cong \mathbb{Z}$ and $\operatorname{im}(\operatorname{id} - \alpha_*^{-1}) = \langle 2[\pi_1] - c[\pi_2] \rangle \cong \mathbb{Z}$. By identifying the quotient as in the proof of proposition 3.2.13, we deduce the following result.

Proposition 3.2.20. *Assume $a = -1$ and $b = 0$.*

1) *If c is even, then*

$$K_0(C^*(G)) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2 \cong K_1(C^*(G)).$$

Generators are provided by $[P_{u,w}]$, $[1]$ and $[P_{v,w}] - [1]$ for $K_0(C^*(G))$ and by $[u]$, $[w]$ and the order-two element $[v] - \frac{c}{2}[w]$ for $K_1(C^*(G))$.

2) *If c is odd, then*

$$K_0(C^*(G)) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2,$$

generated by $[P_{u,w}]$, $[1]$ and $[P_{v,w}] - [1]$, and

$$K_1(C^*(G)) \cong \mathbb{Z}^2 \cong \langle [u] \rangle \oplus \langle -[v] + \frac{c+1}{2}[w] \rangle.$$

Assume now that $b \neq 0$. Hence $c = \frac{ad+1}{b} = \frac{-a^2}{b}$, since $\operatorname{tr}(\alpha) = 0$. Thus

$$\alpha = \begin{pmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{pmatrix} = \alpha^{-1}$$

and then

$$I_2 - \alpha^{-1} = \begin{pmatrix} 1-a & -b \\ \frac{-(1-a^2)}{b} & 1+a \end{pmatrix}.$$

Let us write

$$g := \gcd(1-a, b), \qquad 1-a' := \frac{1-a}{g}, \qquad b' := \frac{b}{g}$$

and

$$h := \gcd\left(g, \frac{-g(1+a)}{b}\right).$$

By Bézout, we have $m, n, r, s \in \mathbb{Z}$ such that

$$m(1-a) + nb = g \quad \text{and} \quad rg + s\frac{-g(1+a)}{b} = h.$$

Following the same arguments as in lemmas 3.2.10, 3.2.12 and proposition 3.2.13, we obtain the next results.

Lemma 3.2.21. 1) *The homomorphism* $\operatorname{id} - \alpha_*^{-1} : K_1(C(\mathbb{T}^2)) \longrightarrow K_1(C(\mathbb{T}^2))$ *has infinite cyclic kernel, generated by* $b'[\pi_1] + (1-a')[\pi_2]$.

2) *The image of* $\operatorname{id} - \alpha_*^{-1} : K_1(C(\mathbb{T}^2)) \longrightarrow K_1(C(\mathbb{T}^2))$ *is isomorphic to* $\mathbb{Z}$ *and generated by* $g[\pi_1] + \frac{-g(1+a)}{b}[\pi_2]$.

Proposition 3.2.22. 1) *The group* $K_0(C^*(G))$ *is isomorphic to* $\mathbb{Z}^2 \oplus \mathbb{Z}_2$. *Generators are given by* $[P_{u,v^{b'}w^{1-a'}}]$, $[1]$ *and* $[P_{v,w}] - [1]$.

2) *The group* $K_1(C^*(G))$ *is isomorphic to* $\mathbb{Z}^2 \oplus \mathbb{Z}_h$. *Generators are given by the torsion-free elements* $[u]$, $-s[v] + r[w]$ *and the element* $\frac{g}{h}[v] + \frac{-g(1+a)}{hb}[w]$, *whose order is* h.

The case $\operatorname{tr}(\alpha) \in \{-1, 1\}$

In this case, since $\det(I_2 - \alpha^{-1}) = \operatorname{tr}(\alpha)$, we have that $\operatorname{id} - \alpha^{-1}$ is an isomorphism in K_1. Then (♠) and (♣) give the result.

Proposition 3.2.23. *The group* $K_0(C^*(G))$ *is isomorphic to* $\mathbb{Z} \oplus \mathbb{Z}_2$ *and generated by* $[1]$ *and* $[P_{v,w}] - [1]$. *Moreover,* $K_1(C^*(G)) = \langle [u] \rangle \cong \mathbb{Z}$.

The case $\operatorname{tr}(\alpha) \notin \{-1, 0, 1\}$

Once again, we can lean on what has been done in the last part of the case $\det(\alpha) = 1$. By elementary operations on the rows and columns of $I_2 - \alpha^{-1}$, we can turn it into a diagonal matrix $h_1 \oplus h_2$, with $h_1, h_2 \in \mathbb{N}^*$ and

$$h_1 h_2 = |\det(I_2 - \alpha^{-1})| = |\operatorname{tr}(\alpha)|.$$

Therefore, together with (♠) and (♣), we obtain this last result.

Proposition 3.2.24. *The group $K_0(C^*(G))$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$ and generated by $[1]$ and $[P_{v,w}] - [1]$, whereas $K_1(C^*(G)) \cong \mathbb{Z} \oplus \mathbb{Z}_{h_1} \oplus \mathbb{Z}_{h_2}$.*

3.3 Some applications

In this final part of chapter 3, we give some consequences of the computations we have made. They all result from discussion with Alain Valette, to whom we are grateful.

3.3.1 Applications to traces of the C^*-algebra of the discrete Heisenberg group

Let us consider the subgroup of $GL_3(\mathbb{Z})$ formed by matrices of the type :

$$\begin{pmatrix} 1 & n & k \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix}, \quad n, m, k \in \mathbb{Z}.$$

It is called the discrete Heisenberg group and we will denote it by H in the sequel. Defining

$$u := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad w := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we obtain three generators for H, subject to the relations

$$uv = wvu, \quad uw = wu \quad \text{and} \quad vw = wv.$$

Thus v and w generate a group isomorphic to $\mathbb{Z}^2$, on which $\mathbb{Z}$ acts by conjugation by u :

$$uvu^{-1} = wv, \quad uwu^{-1} = w.$$

Hence H is the semi-direct product $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$, with $\alpha \in GL_2(\mathbb{Z})$ being given by

$$\alpha = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

By the previous part, and keeping the same notations, we have $K_0(C^*(H))$ isomorphic to $\mathbb{Z}^3$, generated by $[1]$, $[P_{v,w}]$ and $[P_{u,w}]$, and $K_1(C^*(H))$ also isomorphic to $\mathbb{Z}^3$, generated by $[u]$, $[v]$

and $[Q_\alpha]$. Thus we find back the results of Anderson and Paschke obtained in [2], but with slightly different generators.

Since $\mathcal{Z}(H)$, the center of H, is isomorphic to $\mathbb{Z}$ and generated by w, the C^*-algebra $C^*(\mathcal{Z}(H))$ is isomorphic to $C(S^1)$. Thus, recalling that $C^*(H)$ is a completion of $C_c(H)$, we can define

$$\begin{aligned}\tau : C^*(H) &\longrightarrow C(S^1)\\ f &\longmapsto f|_{\mathcal{Z}(H)}.\end{aligned}$$

This map satisfies the tracial properties of linearity and

$$\begin{aligned}(fg)(z) &= \sum_{h\in H} f(h)g(h^{-1}z) = \sum_{h\in H} f(zh^{-1})g(h)\\ &= \sum_{h\in H} g(h)f(h^{-1}z) = (gf)(z)\end{aligned}$$

for every $z \in \mathcal{Z}(H)$. Hence it induces a homomorphism

$$\begin{aligned}\tau_* : K_0(C^*(H)) &\longrightarrow C(S^1)\\ [(p_{ij})] &\longmapsto \sum_i \tau(p_{ii}).\end{aligned}$$

Proposition 3.3.1. *The image of the map τ_* defined above is formed by the constant functions on S^1 with value in $\mathbb{Z}$.*

Proof. It is clear that $\tau_*([1]) = \tau(1)$ is the function being everywhere 1. Furthermore, writing ϕ for the isomorphism between $C^*(v,w)$ and $C(\mathbb{T}^2)$, we have

$$\begin{aligned}\tau_*([P_{v,w}]) &= \tau_*([\phi^{-1}(P)]) = \tau(\phi^{-1}(P_{11})) + \tau(\phi^{-1}(P_{22}))\\ &= \tau(\phi^{-1}(P_{11}+P_{22})) = \tau(\phi(\pi_1^* \operatorname{tr}(X_1^*) + \operatorname{tr}(X_0) + \operatorname{tr}(X_1)\pi_1))\\ &= \tau(v^*\phi^{-1}(\operatorname{tr}(X_1^*))) + \tau(\phi^{-1}(\operatorname{tr}(X_0))) + \tau(\phi^{-1}(\operatorname{tr}(X_1))v)\\ &= \tau(\phi^{-1}(\operatorname{tr}(X_0))) = \tau(\phi^{-1}(1))\\ &= \tau(1) = 1,\end{aligned}$$

since for all $t \in [0,1]$,

$$\begin{aligned}\operatorname{tr}(X_0(t)) &= \cos^4(\frac{\pi}{2}t) + \sin^4(\frac{\pi}{2}t) + 2\cos^2(\frac{\pi}{2}t)\sin^2(\frac{\pi}{2}t)\\ &= \Big(\cos^2(\frac{\pi}{2}t) + \sin^2(\frac{\pi}{2}t)\Big)^2\\ &= 1.\end{aligned}$$

A similar computation gives $\tau_*([P_{u,w}]) = 1$. ☐

For $\theta \in \mathbb{R} \setminus \mathbb{Q}$, we define the noncommutative torus A_θ to be the C^*-algebra generated by two unitaries x and y satisfying the single relation

$$xy = e^{2\pi i\theta} yx.$$

Those C^*-algebras are of great interest and are the simplest examples of noncommutative C^*-algebras. Moreover, they have a unique trace $\tau_\theta : A_\theta \longrightarrow \mathbb{C}$ given by

$$\tau_\theta(x^n y^m) = \begin{cases} 1 & \text{if } n = 0 = m \\ 0 & \text{otherwise} \end{cases}$$

for every $n, m \in \mathbb{Z}$.

Proposition 3.3.2. *The group $K_0(A_\theta)$ is isomorphic to $\mathbb{Z}^2$. Moreover, the homomorphism $(\tau_\theta)_* : K_0(A_\theta) \longrightarrow \mathbb{C}$ is injective and its image is $\mathbb{Z} \oplus \theta\mathbb{Z}$.*

A proof of these facts can be found in [17]. Writing π_θ for the *-homomorphism defined by

$$\begin{aligned} \pi_\theta : C^*(H) &\longrightarrow A_\theta \\ u &\longmapsto x \\ v &\longmapsto y \\ w &\longmapsto e^{2\pi i\theta} \end{aligned}$$

we obtain a commutative diagram

$$\begin{tikzcd} C^*(H) \arrow[r, "\pi_\theta"] \arrow[d, "\tau"'] & A_\theta \arrow[d, "\tau_\theta"] \\ C(S^1) \arrow[r, "e_\theta"'] & \mathbb{C} \end{tikzcd}$$

where e_θ is the map given by evaluating a function in $C(S^1)$ at $e^{2\pi i\theta}$. This commutative diagram induces the following one :

$$\begin{tikzcd} K_0(C^*(H)) \arrow[r, "(\pi_\theta)_*"] \arrow[d, "\tau_*"'] & K_0(A_\theta) \arrow[d, "(\tau_\theta)_*"] \\ C(S^1) \arrow[r, "e_\theta"'] & \mathbb{C}. \end{tikzcd}$$

Corollary 3.3.3. *The image of $(\pi_\theta)_*$ is $\langle [1] \rangle \cong \mathbb{Z}$.*

Proof. Since τ_* has image formed by constant functions equal to an integer, we get $\mathrm{im}(e_\theta \circ \tau_*) = \mathbb{Z} \subseteq \mathbb{C}$. Thus, by injectivity of $(\tau_\theta)_*$ and the fact that it maps [1] to 1, we should have the image of $(\pi_\theta)_*$ being the subgroup of $K_0(A_\theta)$ generated by [1]. $\square$

This allows us to give a new proof of a result in [2], stating that if κ is the trace on $C^*(H)$ that extends the characteristic function of the identity element in H, then the image of its induced map

$$\kappa_* : K_0(C^*(H)) \longrightarrow \mathbb{R}$$

is exactly $\mathbb{Z}$. Indeed, with the notations above, we have

$$\kappa = e_0 \circ \tau.$$

3.3.2 Applications to lattices in the real Heisenberg and SOL groups

Let SOL be the connected solvable real Lie group defined by the semi-direct product $\mathbb{R}^2 \rtimes \mathbb{R}$, where an element $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ via the matrix

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Lemma 3.3.4. *A group G is isomorphic to a lattice in SOL if and only if it is isomorphic to a semi-direct product $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$, where $\alpha \in GL_2(\mathbb{Z})$ is such that* $\det(\alpha) = 1$ *and* $\mathrm{tr}(\alpha) > 2$.

Proof. To start with the proof, let us mention the following result of [47] : if H is a connected solvable Lie group and N its maximum connected normal nilpotent Lie subgroup, then, for any lattice G in H, the subgroup $G \cap N$ is a lattice in N.

The maximum connected normal nilpotent Lie subgroup of SOL is $\mathbb{R}^2$. Thus, if G is a lattice in SOL, the result above gives that $G \cap \mathbb{R}^2$ is a lattice in $\mathbb{R}^2$. Therefore $G \cap \mathbb{R}^2$ is isomorphic to $\mathbb{Z}^2$. Moreover, the quotient $G/(G \cap \mathbb{R}^2)$ is a lattice in SOL$/\mathbb{R}^2$, which is isomorphic to $\mathbb{R}$, and hence is isomorphic to $\mathbb{Z}$. Thus we get the split short exact sequence

$$0 \longrightarrow G \cap \mathbb{R}^2 \cong \mathbb{Z}^2 \longrightarrow G \longrightarrow G/(G \cap \mathbb{R}^2) \cong \mathbb{Z} \longrightarrow 0,$$

inducing that G is a semi-direct product of $\mathbb{Z}^2$ by $\mathbb{Z}$. To identify the matrix giving the action of $\mathbb{Z}$, let us fix two vectors (v_1, v_2) and (w_1, w_2) in $\mathbb{R}^2$ generating the lattice $G \cap \mathbb{R}^2$ and a generator $t_0 \in \mathbb{R} \cong \mathrm{SOL}/\mathbb{R}^2$ of the lattice $G/(G \cap \mathbb{R}^2)$. Then, since α is the automorphism $e^{t_0} \oplus e^{-t_0}$ of $\mathbb{R}^2$ expressed in the basis $\{(v_1, v_2), (w_1, w_2)\}$, we get

$$\alpha = \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}^{-1} \begin{pmatrix} e^{t_0} & 0 \\ 0 & e^{-t_0} \end{pmatrix} \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}$$

Therefore, since the determinant and the trace of a matrix is invariant by conjugation, we get $\det(\alpha) = e^{t_0} e^{-t_0} = 1$ and $\mathrm{tr}(\alpha) = e^{t_0} + e^{-t_0} > 2$.

Conversely, let G be a semi-direct product $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$ with $\det(\alpha) = 1$ and $\mathrm{tr}(\alpha) > 2$. Then α, seen as an automorphism of $\mathbb{R}^2$, has necessarily two different eigenvalues, since its characteristic polynomial is

$$\lambda^2 - \mathrm{tr}(\alpha)\lambda + \det(\alpha).$$

Hence α is conjugate to a diagonal matrix $e^{t_0} \oplus e^{-t_0}$ for some $t_0 \in \mathbb{R}$. More precisely, there exists a $M \in GL_2(\mathbb{R})$ such that

$$M\alpha M^{-1} = e^{t_0} \oplus e^{-t_0}.$$

Thus we get a map

$$\begin{aligned} \phi : G \cong \mathbb{Z}^2 \rtimes_\alpha \mathbb{Z} &\longrightarrow \mathrm{SOL} = \mathbb{R}^2 \rtimes \mathbb{R} \\ (v, n) &\longmapsto (Mv, nt_0) \end{aligned}$$

which is indeed a homomorphism, since for $v_1, v_2 \in \mathbb{R}^2$ and $n_1, n_2 \in \mathbb{Z}$ we get

$$\begin{aligned} \phi(v_1, n_1)\phi(v_2, n_2) &= (Mv_1, n_1t_0)(Mv_2, n_2t_0) \\ &= (Mv_1 + (e^{n_1t_0} \oplus e^{-n_1t_0})Mv_2, n_1t_0 + n_2t_0) \\ &= (Mv_1 + M\alpha^{n_1}M^{-1}Mv_2, (n_1 + n_2)t_0) \\ &= (M(v_1 + \alpha^{n_1}v_2), (n_1 + n_2)t_0) \\ &= \phi(v_1 + \alpha^{n_1}v_2, n_1 + n_2) \\ &= \phi((v_1, n_1)(v_2, n_2)). \end{aligned}$$

Moreover ϕ is injective and therefore allows us to see G as a lattice in SOL. □

By the computations done previously in the chapter and this result, we obtain the K-theory of the group C^*-algebra of any lattice in SOL.

Let $H(\mathbb{R})$ be the real Heisenberg group, that is, the group of real matrices of the form

$$\begin{pmatrix} 1 & r & q \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}, \quad q, r, s \in \mathbb{R}.$$

This group is a simply connected nilpotent three-dimensional Lie group and can be expressed as the semi-direct product $\mathbb{R}^2 \rtimes \mathbb{R}$, with action being given by

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathrm{Aut}(\mathbb{R}^2) \\ t &\longmapsto \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}. \end{aligned}$$

Furthermore, the center of $H(\mathbb{R})$ is given by the elements

$$\begin{pmatrix} 1 & 0 & q \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q \in \mathbb{R}$$

and hence is isomorphic to $\mathbb{R}$.

Moreover, the projection

$$\begin{aligned} p : H(\mathbb{R}) &\longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} 1 & r & q \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} &\longmapsto (x, y) \end{aligned}$$

is a well-defined homomorphism and gives raise to the short exact sequence

$$0 \longrightarrow \mathcal{Z}(H(\mathbb{R})) \longrightarrow H(\mathbb{R}) \xrightarrow{p} \mathbb{R}^2 \longrightarrow 0.$$

We also note that $\mathcal{Z}(H(\mathbb{R}))$ coincide with $[H(\mathbb{R}), H(\mathbb{R})]$, the commutator subgroup of $H(\mathbb{R})$.

Lemma 3.3.5. *A group G is isomorphic to a lattice in $H(\mathbb{R})$ if and only if it is isomorphic to a semi-direct product $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$, where $\alpha \in GL_2(\mathbb{Z}) \setminus \{I_2\}$ is such that* $\det(\alpha) = 1$ *and* $\operatorname{tr}(\alpha) = 2$.

Proof. Let us first state a result of [47] : a subgroup of a simply connected nilpotent Lie group is Zariski dense if and only if it is cocompact.

Let G be a lattice in $H(\mathbb{R})$. Then, by Zariski density of G, an element $g \in \mathcal{Z}(G)$ should also commute with every elements of $H(\mathbb{R})$. Thus $\mathcal{Z}(G) = \mathcal{Z}(H(\mathbb{R})) \cap G$. Hence the short exact sequence above restricts to the following one :

$$0 \longrightarrow \mathcal{Z}(G) \longrightarrow G \xrightarrow{p|_G} p(G) \longrightarrow 0.$$

By $\mathcal{Z}(G) = \mathcal{Z}(H(\mathbb{R})) \cap G$ and $\mathcal{Z}(H(\mathbb{R})) \cong \mathbb{R}$, we deduce that $\mathcal{Z}(G)$ is isomorphic to a lattice in $\mathbb{R}$, and therefore to $\mathbb{Z}$. Let us choose a generating element $z \in \mathcal{Z}(G)$. Furthermore, the image by p of the lattice G is a lattice in $p(H(\mathbb{R})) = \mathbb{R}^2$. Thus $p(G) \cong \mathbb{Z}^2$. Let v_1 and v_2 be two elements generating $p(G)$. Since $p|_G$ is surjective, there exists $x, y \in G$ such that

$$p(x) = v_1 \quad \text{and} \quad p(y) = v_2.$$

Since z is central in G, we have $\langle y, z \rangle \cong \mathbb{Z}^2$. Moreover $xzx^{-1} = z$ and

$$xyx^{-1} = [x, y]y = z^c y$$

for some $c \in \mathbb{Z}$. Indeed, the commutator $[x, y]$ lives in $[H(\mathbb{R}), H(\mathbb{R})] \cap G$ and, as we have noticed before,

$$[H(\mathbb{R}), H(\mathbb{R})] \cap G = \mathcal{Z}(H(\mathbb{R})) \cap G = \mathcal{Z}(G) = \langle z \rangle.$$

If c was zero, then G would be abelian and therefore, by Zariski density, $H(\mathbb{R})$ would also be abelian. But this is clearly not the case. Finally, we get $G \cong \mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$, with

$$\alpha = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \quad \text{and} \quad c \in \mathbb{Z} \setminus \{0\}.$$

Conversely, let $G = \mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$ be a semi-direct product, with α different from I_2 and such that $\det(\alpha) = 1$ and $\mathrm{tr}(\alpha) = 2$. Since the characteristic polynomial of α is

$$\lambda^2 - \mathrm{tr}(\alpha)\lambda + \det(\alpha) = \lambda^2 - 2\lambda + 1 = (1-\lambda)^2,$$

we have, by Cayley-Hamilton, that $(I_2 - \alpha)^2$ is zero. Hence there exists a non-zero element $(x, y) \in \mathbb{Z}^2$ lying in the kernel of $I_2 - \alpha$. Dividing by their greatest common divisor if necessary, we can assume x and y to be prime to each other. By Bézout, there exists $m, n \in \mathbb{Z}$ such that

$$mx + ny = 1.$$

Thus, defining $M := \begin{pmatrix} y & -x \\ m & n \end{pmatrix}$, we get

$$M(I_2 - \alpha)M^{-1} = M(I_2 - \alpha)\begin{pmatrix} n & x \\ -m & y \end{pmatrix} = M\begin{pmatrix} k' & 0 \\ c' & 0 \end{pmatrix} = \begin{pmatrix} k & 0 \\ c & 0 \end{pmatrix}$$

for some $c, c', k, k' \in \mathbb{Z}$. Since the trace is invariant by conjugation and

$$\mathrm{tr}(I_2 - \alpha) = \mathrm{tr}(I_2) - \mathrm{tr}(\alpha) = 0,$$

we deduce that $k = 0$. Furthermore $c \neq 0$ since $\alpha \neq I_2$. It follows that

$$\begin{aligned} \phi : G = \mathbb{Z}^2 \rtimes_\alpha \mathbb{Z} &\longrightarrow H(\mathbb{R}) = \mathbb{R}^2 \rtimes \mathbb{R} \\ (v, n) &\longmapsto (Mv, nd) \end{aligned}$$

in an injective homomorphism that allows us to see G as a lattice in $H(\mathbb{R})$. □

Thus, we have computed earlier in the chapter the K-theory of the group C^*-algebra of any lattice in the real Heisenberg group.

Chapter 4

The K-homology of the classifying space of a semi-direct product $\mathbb{Z}^2 \rtimes \mathbb{Z}$

Given a semi-direct product of $\mathbb{Z}^2$ by $\mathbb{Z}$, we have computed in the previous chapter the K-theory of its C^*-algebra and given concrete generators. Now we will look after the other side of the Baum-Connes conjecture, which is the K-homology of its classifying space, since the group is discrete, torsion-free and has compact classifying space. For this, we will adopt the geometric point of view described in chapter 2 and will deeply use the six-terms exact sequence given by theorem 2.3.14. Therefore, we begin by calculating the K-homology groups of the classifying space of $\mathbb{Z}^2$, namely the torus $\mathbb{T}^2$.

For all this chapter, I will denote the closed interval $[0, 1]$ and ∂I will be $\{0, 1\}$. Moreover, when considered as compact manifold with boundary, I will be oriented from 0 to 1 and endowed with the *Spin*^{c} structure given by 2.2.5, as well as the circle $\mathbb{T}$ and the torus $\mathbb{T}^2$.

4.1 The K-homology of the torus

The space $\mathbb{T}^2$ can be seen as the mapping torus of the homeomorphism of $\mathbb{T}$ given by the identity, that is

$$\mathbb{T}^2 \cong M_{\mathbb{T},\mathrm{id}} = \mathbb{T} \times I/\sim$$

with the equivalence relation given by $(x, 0) \sim (x, 1)$ for all $x \in \mathbb{T}$. We recall that we will see $\mathbb{T}$ in $\mathbb{T}^2$ via the injective map $i_1 : \mathbb{T} \longrightarrow \mathbb{T} \times \mathbb{T} = \mathbb{T}^2$ given by

$$i_1(x) = (x, 1)$$

for every $x \in \mathbb{T}$. From theorem 2.3.14, we deduce the six-terms exact sequence

$$\begin{tikzcd}
K_0(\mathbb{T}) \arrow[r, "0"] & K_0(\mathbb{T}) \arrow[r, "(i_1)_*"] & K_0(\mathbb{T}_2) \arrow[d, "\partial"] \\
K_1(\mathbb{T}^2) \arrow[u, "\partial"] & K_1(\mathbb{T}) \arrow[l, "(i_1)_*"'] & K_1(\mathbb{T}). \arrow[l, "0"']
\end{tikzcd}$$

Since we have from chapter 2 that $K_0(\mathbb{T}) \cong \mathbb{Z} \cong K_1(\mathbb{T})$, we obtain a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow K_j(\mathbb{T}^2) \longrightarrow \mathbb{Z} \longrightarrow 0$$

for $j = 0, 1$. As $\mathbb{Z}$ is free abelian, we get

$$K_0(\mathbb{T}^2) \cong \mathbb{Z}^2 \cong K_1(\mathbb{T}^2).$$

To get concrete generators, let us recall that $K_0(\mathbb{T})$ is generated by $\delta_1'' = [*, \mathbb{C}, * \to \{1\} \subseteq \mathbb{T}]$ and $K_1(\mathbb{T})$ by the class $\gamma_1'' = [\mathbb{T}, \mathbb{T} \times \mathbb{C}, \mathrm{id}_{\mathbb{T}}]$. Hence $K_1(\mathbb{T}^2)$ is generated by

$$\gamma_1' := i_*(\gamma_1'') = [\mathbb{T}, \mathbb{T} \times \mathbb{C}, i_1]$$

and a preimage of δ_1'' by ∂. We show that one is given by the class

$$\gamma_2' := [\mathbb{T}, \mathbb{T} \times \mathbb{C}, i_2],$$

with i_2 being the inclusion of $\mathbb{T}$ in the second factor of $\mathbb{T} \times \mathbb{T} = \mathbb{T}^2$.

Lemma 4.1.1. *There exists a bordism in* $K_1(\mathbb{T}^2, \mathbb{T})$ *between* $(\mathbb{T}, \mathbb{T} \times \mathbb{C}, i_2)$ *and the opposite of* $(I, I \times \mathbb{C}, i_2 \circ \exp')$, *where* $\exp'(t) := e^{2\pi i t}$ *for every* $t \in I$.

Proof. First we define

$$\begin{aligned} f : \mathbb{T} \times I &\longrightarrow \mathbb{T} \\ (e^{2\pi i t}, s) &\longmapsto \begin{cases} \exp\left(\pi i \frac{4t-s}{2-s}\right) & \text{if } t \in [\frac{s}{4}, 1 - \frac{s}{4}] \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

It is clearly well-defined and continuous, since for $s \in I$ we get, for $t = \frac{s}{4}$,

$$f(e^{2\pi i t}, s) = \exp\left(\pi i \frac{s - s}{2 - s}\right) = \exp(0) = 1$$

and, for $t = 1 - \frac{s}{4}$,

$$f(e^{2\pi i t}, s) = \exp\left(\pi i \frac{4 - 2s}{2 - s}\right) = \exp(2\pi i) = 1.$$

Moreover, we note that f restricted to $\mathbb{T} \cong \mathbb{T} \times \{0\} \subseteq \mathbb{T} \times I$ is exactly the identity on $\mathbb{T}$. Now let us consider the bordism given by :

- the smooth compact manifold with boundary $L := \mathbb{T} \times I$ with *Spin*^{c} structure defined as in proposition 2.2.8,
- the trivial complex vector bundle $F := L \times \mathbb{C}$ over L,
- the continuous map $\psi : L \longrightarrow \mathbb{T}^2$ given by $\psi(x) = (1, f(x))$ for every $x \in L$,

- the smooth map $h : \partial L = \mathbb{T} \times \{0,1\} \longrightarrow \mathbb{R}$ given by $h(e^{2\pi it},0) = -7$ and $h(e^{2\pi it},1) = 1-\cos(2\pi t)$ for all $t \in I$.

It is really a bordism in $K_1(\mathbb{T}^2,\mathbb{T})$ since

$$h^{-1}([-1,1]) = \left\{(e^{2\pi it},1) \mid t \in [0,\frac{1}{4}] \cup [\frac{3}{4},1]\right\}$$

is mapped by ψ on $\{(1,1)\} \subseteq \mathbb{T} \times \{1\}$. Then we have

$$M_- := h^{-1}(]-\infty,-1]) = \mathbb{T} \times \{0\} \cong \mathbb{T},$$

$F|_{M_-} = \mathbb{T} \times \mathbb{C}$ and $\psi|_{M_-} = \mathrm{id}_\mathbb{T}$, since $f(e^{2\pi it},0) = e^{2\pi it}$ for all $t \in [0,1]$. On the other hand,

$$M_+ := h^{-1}([1,\infty[) = \left\{(e^{2\pi it},1) \mid t \in [\frac{1}{4},\frac{3}{4}]\right\},$$

$F|_{M_+} = M_+ \times \mathbb{C}$ and $\psi|_{M_+}(e^{2\pi it},1) = (1,e^{\pi i(4t-1)})$, for every $t \in [\frac{1}{4},\frac{3}{4}]$, give raise to a K-cycle that is isomorphic, in the sense of definition 2.3.2, to the opposite of $(I, I \times \mathbb{C}, i_2 \circ \exp')$ via the diffeomorphism

$$\begin{aligned} I &\longrightarrow M_+ \\ t &\longmapsto \exp\left(2\pi i \frac{2t+1}{4}\right), \end{aligned}$$

where I is taken with its opposite orientation. □

Proposition 4.1.2. *The homomorphism $\partial : K_1(\mathbb{T}^2) \longrightarrow K_0(\mathbb{T})$ maps γ_2' on δ_1''.*

Proof. We know from theorem 2.3.14 that the map ∂ is given by the composition

$$\partial_0 \circ (\mathrm{id} \times \exp'_*)^{-1} \circ i'_*,$$

where i' is the inclusion of $(\mathbb{T}^2,\emptyset)$ in $(\mathbb{T}^2,\mathbb{T})$ and $\mathrm{id} \times \exp'$ is the standard quotient map from $\mathbb{T} \times [0,1]$ to $\mathbb{T} \times \mathbb{T} = \mathbb{T}^2$. Thus the proposition can be reformulated

$$i'_*(\gamma_2') = ((\mathrm{id} \times \exp'_*) \circ (\partial_0)^{-1})(\delta_1'').$$

Here $i'_*(\gamma_2')$ is simply $[\mathbb{T}, \mathbb{T} \times \mathbb{C}, i_2]$ seen in $K_1(\mathbb{T}^2,\mathbb{T})$. Moreover,

$$(\partial_0)^{-1}(\delta_1'') = [I, I \times \mathbb{C}, \{1\} \times \mathrm{id}_I] \in K_1(\mathbb{T} \times I, \mathbb{T} \times \partial I).$$

Hence

$$((\mathrm{id} \times \exp'_*) \circ (\partial_0)^{-1})(\delta_1'') = [I, I \times \mathbb{C}, \{1\} \times \exp'] = [I, I \times \mathbb{C}, i_2 \circ \exp'].$$

But, by the previous lemma, this class is equal to $[\mathbb{T}, \mathbb{T} \times \mathbb{C}, i_2]$ in $K_1(\mathbb{T}^2,\mathbb{T})$. □

Corollary 4.1.3. *The group $K_1(\mathbb{T}^2) \cong \mathbb{Z}^2$ is generated by γ_1' and γ_2'.*

Furthermore, we deduce from the exactness of the six-terms sequence on page 109 that $K_0(\mathbb{T}^2) \cong \mathbb{Z}^2$ is generated by

$$\delta_1' := i_*(\delta_1'') = [*, \mathbb{C}, * \to \{(1,1)\} \subseteq \mathbb{T}^2]$$

and a preimage of γ_1'' via ∂.

Proposition 4.1.4. *The map* $\partial : K_0(\mathbb{T}^2) \longrightarrow K_1(\mathbb{T})$ *sends* $\delta_2' := [\mathbb{T}^2, \mathbb{T}^2 \times \mathbb{C}, \mathrm{id}_{\mathbb{T}^2}]$ *on* γ_1''.

Proof. Here we can procede exactly as in the proof of the previous proposition, the only difference being, informally, taking the product with $\mathbb{T}$ in all the K-cycles considered. More explicitely, the class $i_*'(\delta_2')$ is $[\mathbb{T}^2, \mathbb{T}^2 \times \mathbb{C}, \mathrm{id}_{\mathbb{T}^2}]$ seen in $K_0(\mathbb{T}^2, \mathbb{T})$. Moreover,

$$(\partial_0)^{-1}(\gamma_1'') = [\mathbb{T} \times I, \mathbb{T} \times I \times \mathbb{C}, \mathrm{id}_{\mathbb{T}\times I}] \in K_0(\mathbb{T} \times I, \mathbb{T} \times \partial I).$$

Hence

$$((\mathrm{id} \times \exp_*') \circ (\partial_0)^{-1})(\gamma_1'') = [\mathbb{T} \times I, \mathbb{T} \times I \times \mathbb{C}, \mathrm{id}_{\mathbb{T}} \times \exp'].$$

But, keeping the same notations as in the proof of lemma 4.1.1, the bordism given by the manifold with boundary $\mathbb{T} \times L$, the vector bundle $\mathbb{T} \times F$, the continuous map $\mathrm{id}_{\mathbb{T}} \times f$ from $\mathbb{T} \times L$ to $\mathbb{T}^2$ and the smooth map

$$\begin{aligned} \mathbb{T} \times \partial L &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto h(y) \end{aligned}$$

induces that, in $K_0(\mathbb{T} \times I, \mathbb{T} \times \partial I)$, the class $[\mathbb{T}^2, \mathbb{T}^2 \times \mathbb{C}, \mathrm{id}_{\mathbb{T}^2}]$ equals the one of a K-cycle isomorphic to $(\mathbb{T} \times I, \mathbb{T} \times I \times \mathbb{C}, \mathrm{id}_{\mathbb{T}} \times \exp')$. □

Corollary 4.1.5. *The group* $K_0(\mathbb{T}^2) \cong \mathbb{Z}^2$ *is generated by* δ_1' *and* δ_2'.

4.2 The K-homology of $B(\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z})$

Let G be a semi-direct product of $\mathbb{Z}^2$ by $\mathbb{Z}$. As in the previous chapter, we will write $G = \mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$, with

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}).$$

It is known that for a semi-direct product $N \rtimes H$, the classifying space is

$$B(N \rtimes H) = BN \times_H EH,$$

where BN is the classifying space for N and EH is the universal covering of the classifying space for H. In our case, we get $\mathbb{T}^2 \times_{\mathbb{Z}} \mathbb{R}$, with action of $\mathbb{Z}$ on $\mathbb{R}$ obtained by translation and action of $\mathbb{Z}$ on $\mathbb{T}^2$ induced by α.

More precisely, the automorphism α gives raise to a homeomorphism of $B\mathbb{Z}^2 = \mathbb{T}^2$ which is the classifying map for the principal $\mathbb{Z}^2$-bundle $\mathbb{R}^2 \times_{\mathbb{Z}^2} \mathbb{Z}^2$, with right action of the fiber simply given by additivity in $\mathbb{Z}^2$. We claim that this map is

$$\begin{aligned} \alpha : \mathbb{T}^2 &\longrightarrow \mathbb{T}^2 \\ (x,y) &\longmapsto (x^a y^b, x^c y^d). \end{aligned}$$

Indeed, writing $\alpha^*(\mathbb{R}^2)$ for the pullback of the bundle $(\exp')^2 : \mathbb{R}^2 \longrightarrow \mathbb{T}^2$ by α, namely

$$\alpha^*(\mathbb{R}^2) = \{(x,y,t,s) \in \mathbb{T}^2 \times \mathbb{R}^2 \mid \alpha(x,y) = (e^{2\pi i t}, e^{2\pi i s})\},$$

we have an isomorphism of principal $\mathbb{Z}^2$-bundles given by :

$$\begin{aligned} f : \mathbb{R}^2 \times_{\mathbb{Z}^2} \mathbb{Z}^2 &\longrightarrow \alpha^*(\mathbb{R}^2) \\ [(t,s),(m,n)] &\longmapsto (e^{2\pi i t}, e^{2\pi i s}, at+bs+m, ct+ds+n). \end{aligned}$$

This map is well-defined, since

$$\begin{aligned} f([(t,s),\alpha(m',n')+(m,n)]) &= f([(t,s),(am'+bn'+m, cm'+dn'+n)]) \\ &= (e^{2\pi i t}, e^{2\pi i s}, at+bs+am'+bn'+m, ct+ds+cm'+dn'+n) \\ &= (e^{2\pi i (t+m')}, e^{2\pi i (s+n')}, a(t+m')+b(s+n')+m, c(t+m')+d(s+n')+n) \\ &= f([(t+m', s+n'),(m,n)]) \end{aligned}$$

and

$$\begin{aligned} \alpha(e^{2\pi i t}, e^{2\pi i s}) &= (\exp'(at+bs), \exp'(ct+ds)) \\ &= (\exp'(at+bs+m), \exp'(ct+ds+n)) \\ &= (\exp' \times \exp')(at+bs+m, ct+ds+n), \end{aligned}$$

as well as $\mathbb{Z}^2$-equivariant, since

$$\begin{aligned} f([(t,s),(m,n)] \cdot (m',n')) &= f([(t,s),(m+m', n+n')]) \\ &= (e^{2\pi i t}, e^{2\pi i s}, at+bs+m+m', ct+ds+n+n') \\ &= (e^{2\pi i t}, e^{2\pi i s}, at+bs+m, ct+ds+n) \cdot (m',n') \\ &= f([(t,s),(m,n)]) \cdot (m',n'). \end{aligned}$$

Furthermore, the space $\mathbb{T}^2 \times_{\mathbb{Z}} \mathbb{R}$ is clearly homeomorphic to

$$\mathbb{T}^2 \times I / \sim$$

with equivalence relation given by $(x,0) \sim (\alpha(x),1)$ for all $x \in \mathbb{T}^2$.

We recognize here the mapping torus of α, and therefore

$$BG = M_{\mathbb{T}^2,\alpha}.$$

By theorem 2.3.14 we have a six-terms exact sequence in geometric K-homology :

$$\begin{tikzcd}
K_0(\mathbb{T}^2) \arrow[r, "\mathrm{id} - \alpha_*^{-1}"] & K_0(\mathbb{T}^2) \arrow[r, "i_*"] & K_0(BG) \arrow[d, "\partial"] \\
K_1(BG) \arrow[u, "\partial"] & K_1(\mathbb{T}^2) \arrow[l, "i_*"'] & K_1(\mathbb{T}^2). \arrow[l, "\mathrm{id} - \alpha_*^{-1}"']
\end{tikzcd} \tag{$\star$}$$

As for the K-theory of $C^*(G)$, we will make this sequence completely explicit, by finding generators and computing their images. First we explain some general facts.

Assume that $\varphi : \mathbb{T} \longrightarrow \mathbb{T}^2$ is a smooth loop fixed by α, namely

$$\alpha(\varphi(x)) = \varphi(x), \quad \forall x \in \mathbb{T}.$$

Then, if φ is injective, the subspace T_φ of $M_{\mathbb{T}^2,\alpha}$ given by classes $[\varphi(x), t]$, with $x \in \mathbb{T}$ and $t \in I$, is homeomorphic to the torus $\mathbb{T}^2$ via

$$\begin{aligned} T_\varphi &\longrightarrow \mathbb{T}^2 \\ [\varphi(x), t] &\longmapsto (x, e^{2\pi i t}). \end{aligned}$$

Let us define $\delta_\varphi := [\mathbb{T}^2, \mathbb{T}^2 \times \mathbb{C}, \mathbb{T}^2 \cong T_\varphi \subseteq M_{\mathbb{T}^2,\alpha}]$. By naturality of the six-terms exact sequence of theorem 2.3.14 and proposition 4.1.4, we obtain the next result.

Corollary 4.2.1. *The homomorphism* $\partial : K_0(BG) \longrightarrow K_1(\mathbb{T}^2)$ *maps the class* δ_φ *on the element* $[\mathbb{T}, \mathbb{T} \times \mathbb{C}, \varphi]$.

Let us introduce the notation $\gamma_3 := [\mathbb{T}, \mathbb{T} \times \mathbb{C}, i_3] \in K_1(BG)$, with

$$\begin{aligned} i_3 : \mathbb{T} &\longrightarrow BG \\ e^{2\pi i t} &\longmapsto [1, 1, t]. \end{aligned}$$

The map i_3 is well-defined, since $\alpha(1, 1) = (1, 1)$. Then naturality of the six-terms exact sequence of 2.3.14 and proposition 4.1.2 have the following consequence.

Corollary 4.2.2. *The homomorphism* $\partial : K_1(BG) \longrightarrow K_0(\mathbb{T}^2)$ *maps* $[\mathbb{T}, \mathbb{T} \times \mathbb{C}, i_3]$ *on* δ'_1.

As for the computations in K-theory, we will need a link with the topological world. This time, we will use the Connes-Chern character that links the K-homology with the homology of a space.

Proposition 4.2.3. *Let* X *be a connected CW-complex of dimension less or equal to* 3*. Then there are isomorphisms*

$$ch_0 \oplus ch_2 : K_0(X) \longrightarrow H_0(X) \oplus H_2(X)$$

and

$$ch_1 \oplus ch_3 : K_1(X) \longrightarrow H_1(X) \oplus H_3(X)$$

which are compatible with the Connes-Chern character and natural if $\dim(X) \leq 2$.

This result is proved in Matthey's paper [36]. An other highlight of this article is the contruction of right-inverses

$$\beta_j : H_j(X) \longrightarrow K_{j \bmod 2}(X)$$

of ch_j, for $j = 0, 1, 2$. We will not give here the construction of those three maps, but will nevertheless state some useful facts caracterising them.

Since X is connected, the projection of X onto a point $*$ yields an isomorphism $H_0(X) \cong H_0(*)$. Moreover, we know that we have an isomorphism

$$H_0(*) \cong K_0(*)$$

by mapping the fundamental class to $[*, \mathbb{C}, \mathrm{id}_*]$. Then any adjunction of a basepoint in X provides a homomorphism $K_0(*) \longrightarrow K_0(X)$. The compositions of those three maps defines

$$\beta_0 : H_0(X) \longrightarrow K_0(X).$$

Let $\varphi : \mathbb{T} \longrightarrow X$ be a loop in X. On the one hand it defines a K-homology class $[\mathbb{T}, \mathbb{T} \times \mathbb{C}, \varphi]$ in $K_1(X)$ and therefore we get a homomorphism

$$\kappa : \pi_1(X) \longrightarrow K_1(X)$$

which is well-defined by lemma 2.3.4. On the other hand, it gives raise to a homomorphism

$$\varphi_* : H_1(\mathbb{T}) \longrightarrow H_1(X).$$

Then we have $\beta_1(\varphi_*([\mathbb{T}])) = \kappa([\varphi])$, where $[\mathbb{T}] \in H_1(\mathbb{T})$ is the fundamental class of the circle.

Finally, we recall from [52] that for any $x \in H_2(X)$, there exists a surface Σ_x, of genus at least 1, and a continuous map

$$f_x : \Sigma_x \longrightarrow X$$

such that $(f_x)_*$ sends the fundamental class $[\Sigma_x] \in H_2(\Sigma_x)$ on x. Then

$$\beta_2(x) = [\Sigma_x, \Sigma_x \times \mathbb{C}, f_x].$$

Proposition 4.2.4. *In $K_1(\mathbb{T}^2)$, we have the equality*

$$[\mathbb{T}, \mathbb{T} \times \mathbb{C}, i_1^{k_1} i_2^{k_2}] = k_1 \gamma_1' + k_2 \gamma_2'$$

for every $k_1, k_2 \in \mathbb{Z}$.

Proof. First we note that the loop

$$\begin{aligned} i_1^{k_1} i_2^{k_2} : \mathbb{T} &\longrightarrow \mathbb{T}^2 \\ x &\longmapsto (x^{k_1}, x^{k_2}) \end{aligned}$$

is clearly homotopic to the concatenation of k_1 loops i_1 and k_2 loops i_2. Hence

$$[i_1^{k_1} i_2^{k_2}] = k_1[i_1] + k_2[i_2]$$

in $\pi_1(\mathbb{T}^2)$. Thus, since κ is a homomorphism,

$$\begin{aligned} [\mathbb{T}, \mathbb{T} \times \mathbb{C}, i_1^{k_1} i_2^{k_2}] &= \kappa([i_1^{k_1} i_2^{k_2}]) = \kappa(k_1[i_1] + k_2[i_2]) \\ &= k_1\kappa([i_1]) + k_2\kappa([i_2]) = k_1[\mathbb{T}, \mathbb{T} \times \mathbb{C}, i_1] + k_2[\mathbb{T}, \mathbb{T} \times \mathbb{C}, i_2] \\ &= k_1\gamma_1' + k_2\gamma_2' \end{aligned}$$

□

4.2.1 The case $\det(\alpha) = 1$

Let us assume that $ad - bc = \det(\alpha) = 1$.

Proposition 4.2.5. *The homomorphism* $\mathrm{id} - \alpha_*^{-1} : K_0(\mathbb{T}^2) \longrightarrow K_0(\mathbb{T}^2)$ *is the zero map.*

Proof. Since α^{-1} has positive determinant, it induces an orientation-preserving homeomorphism on $\mathbb{T}^2 = B\mathbb{Z}^2$. Then α_*^{-1} is the identity on the even homology of $\mathbb{T}^2$. Since the torus is a two-dimensional CW-complex, the isomorphism

$$ch_0 \oplus ch_2 : K_0(\mathbb{T}^2) \longrightarrow H_0(\mathbb{T}^2) \oplus H_2(\mathbb{T}^2)$$

is natural. Therefore, $\alpha_*^{-1} : K_0(\mathbb{T}^2) \longrightarrow K_0(\mathbb{T}^2)$ is the identity. □

The next lemma is a consequence of a result of Steenrod, asserting that every three-dimensional orientable manifold is parallelizable. However, we give here a direct proof for our case, for which we thank Ryan Budney.

Lemma 4.2.6. *The mapping torus $M_{\mathbb{T}^2,\alpha}$ is a parallelizable smooth compact manifold.*

Proof. First we note that the differential of the homeomorphism α at a point $x \in \mathbb{T}^2$ is given by α itself, but taken as a matrix in $GL_2(\mathbb{R})$. Furthermore, since α lies in the same connected components of $GL_2(\mathbb{R})$ as the identity matrix, we have a path $c : [0, 1] \longrightarrow GL_2(\mathbb{R})$ such that

$c(1) = \alpha$ and $c(0) = I_2$. Thus the assignment

$$\begin{aligned} \mathbb{R}^2 \times [0,1] &\longrightarrow GL_3(\mathbb{R}) \\ (x,y,t) &\longmapsto \begin{pmatrix} c(t) & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

provides a trivialization of $\mathbb{R}^2 \times [0,1]$ that induces a trivialization of $\mathbb{T}^2 \times [0,1]$, since it is invariant by the usual action of $\mathbb{Z}^2$ on $\mathbb{R}^2$. Moreover, by the remark of the beginning of the proof, this trivialization passes to the quotient by the equivalence relation given by $(x,0) \sim (\alpha(x),1)$, for all $x \in \mathbb{T}^2$. Thus the mapping torus $M_{\mathbb{T}^2,\alpha}$ has trivial tangent bundle. □

This lemma and proposition 2.2.5 implies that $M_{\mathbb{T}^2,\alpha}$ is a *Spin*^{c}-manifold, with *Spin*^{c} structure given by the trivial principal *Spin*$^c(3)$-bundle. Hence the triple $(M_{\mathbb{T}^2,\alpha}, M_{\mathbb{T}^2,\alpha} \times \mathbb{C}, \mathrm{id}_{M_{\mathbb{T}^2,\alpha}})$ is a well-defined odd K-cycle over BG. We will write γ_4 for its K-homology class.

Proposition 4.2.7. *The homomorphism* $\partial : K_1(BG) \longrightarrow K_0(\mathbb{T}^2)$ *maps the class* γ_4 *on* δ_2'.

Proof. We recall that $\partial = \partial_0 \circ (q_*)^{-1} \circ i'_*$, where q is the quotient map from $\mathbb{T}^2 \times I$ to $M_{\mathbb{T}^2,\alpha} = BG$ and i' is the inclusion of the pair $(M_{\mathbb{T}^2,\alpha}, \emptyset)$ into $(M_{\mathbb{T}^2,\alpha}, \mathbb{T}^2)$. Thus the equality $\partial(\gamma_4) = \delta_2'$ is equivalent to

$$i'_*(\gamma_4) = (q_* \circ (\partial_0)^{-1})(\delta_2').$$

The class on the right of the equality is $[\mathbb{T}^2 \times I, \mathbb{T}^2 \times I \times \mathbb{C}, q]$. Using the same idea as in lemma 4.1.1, we can show that this element of $K_1(M_{\mathbb{T}^2,\alpha}, \mathbb{T}^2)$ is the same as $i'_*(\gamma_4)$ via the bordism given by the manifold with boundary $M_{\mathbb{T}^2,\alpha} \times I$, the vector bundle $M_{\mathbb{T}^2,\alpha} \times I \times \mathbb{C}$, the continuous map

$$\begin{aligned} \psi : M_{\mathbb{T}^2,\alpha} \times I &\longrightarrow M_{\mathbb{T}^2,\alpha} \\ ([x,t],s) &\longmapsto \begin{cases} \left[x, \frac{4t-s}{4-2s}\right] & \text{if } t \in \left[\frac{s}{4}, 1-\frac{s}{4}\right] \\ [x,0] & \text{if } t < \frac{s}{4} \\ [x,1] & \text{if } t > 1-\frac{s}{4} \end{cases} \end{aligned}$$

and the smooth function

$$\begin{aligned} h : M_{\mathbb{T}^2,\alpha} \times \partial I &\longrightarrow \mathbb{R} \\ ([x,t],0) &\longmapsto -7 \\ ([x,t],1) &\longmapsto 1-\cos(2\pi t). \end{aligned}$$

□

Since $\mathrm{id} - \alpha_*^{-1}$ is the zero map in K_0, the sequence $(\underline{\star})$ induces the short exact sequence

$$0 \longrightarrow K_0(\mathbb{T}^2) \xrightarrow{i_*} K_0(BG) \xrightarrow{\partial} \mathrm{im}(\partial) \longrightarrow 0$$

and therefore

$$K_0(BG) \cong K_0(\mathbb{T}^2) \oplus \mathrm{im}(\partial). \qquad (\underline{\heartsuit})$$

Generators are given by

$$\delta_1 := [*, \mathbb{C}, * \to \{[(1,1),0]\} \subseteq BG]$$

and

$$\delta_2 := [\mathbb{T}^2, \mathbb{T}^2 \times \mathbb{C}, i],$$

since they are images by i_* of the generators δ_1' and δ_2' of $K_0(\mathbb{T}^2)$, together with preimage by ∂ of generators of $\ker(\mathrm{id} - \alpha_*^{-1}) = \mathrm{im}(\partial)$, which will be found using corollary 4.2.1. Similarly, ($\underline{\star}$) gives raise to the short exact sequence

$$0 \longrightarrow \ker \partial \longrightarrow K_1(BG) \xrightarrow{\partial} K_0(\mathbb{T}^2) \longrightarrow 0$$

which implies

$$K_1(BG) = \ker(\partial) \oplus K_0(\mathbb{T}^2). \tag{$\underline{\diamondsuit}$}$$

Corollary 4.2.2 and proposition 4.2.7 shows that γ_3 and γ_4 generate the summand corresponding to $K_0(\mathbb{T}^2)$. To find generators of $\ker(\partial)$, we will first identify the image of the map induced by $\mathrm{id} - \alpha^{-1}$ in K_1, since

$$\ker(\partial) = \mathrm{im}(i_*) \cong K_1(\mathbb{T}^2)/\ker(i_*) = K_1(\mathbb{T}^2)/\,\mathrm{im}(\mathrm{id} - \alpha_*^{-1}).$$

Lemma 4.2.8. *In the basis $\{\gamma_1', \gamma_2'\}$ of $K_1(\mathbb{T}^2) \cong \mathbb{Z}^2$, the homomorphism $\mathrm{id} - \alpha_*^{-1}$ is given precisely by the matrix $I_2 - \alpha^{-1}$.*

Proof. The class γ_1' is sent by α_*^{-1} to the class $[\mathbb{T}, \mathbb{T} \times \mathbb{C}, \alpha^{-1} \circ i_1]$, with

$$\begin{aligned} \alpha^{-1} \circ i_1 : \mathbb{T} &\longrightarrow \mathbb{T}^2 \\ x &\longmapsto (x^d, x^{-c}). \end{aligned}$$

By proposition 4.2.4, this class is exactly $d\gamma_1' - c\gamma_2'$. Thus

$$(\mathrm{id} - \alpha_*^{-1})(\gamma_1') = \gamma_1' - d\gamma_1' + c\gamma_2' = (1-d)\gamma_1' + c\gamma_2'.$$

In the same way we get

$$(\mathrm{id} - \alpha_*^{-1})(\gamma_2') = \gamma_2' + b\gamma_1' - a\gamma_2' = b\gamma_1' + (1-a)\gamma_2'. \qquad \square$$

Also, recalling that i_1 and i_2 are the two canonical injections of $\mathbb{T}$ into $\mathbb{T} \times \mathbb{T} = \mathbb{T}^2$, we define

$$\gamma_1 := i_*(\gamma_1') = [\mathbb{T}, \mathbb{T} \times \mathbb{C}, i \circ i_1]$$

and

$$\gamma_2 := i_*(\gamma_2') = [\mathbb{T}, \mathbb{T} \times \mathbb{C}, i \circ i_2],$$

which are two classes in $K_1(BG)$.

The case $\alpha = I_2$

First we assume that α is the identity matrix. Thus the mapping torus of α is $\mathbb{T}^3$, the classifying space of $\mathbb{Z}^3 = \mathbb{Z}^2 \rtimes_{\mathrm{id}} \mathbb{Z}$. Moreover, the induced map $\mathrm{id} - \alpha_*^{-1}$ in K_1 is the zero map.

Proposition 4.2.9. *The K-homology groups of $\mathbb{T}^3$ are both $\mathbb{Z}^4$. Generators for $K_0(\mathbb{T}^3)$ are given by δ_1, δ_2, δ_{i_1} and δ_{i_2}. On the other hand $\{\gamma_j\}_{j=1}^4$ is a generating set for $K_1(\mathbb{T}^3)$.*

Proof. It is clear that i_1 and i_2 are injective loops in $\mathbb{T}^2$. Moreover, by corollary 4.2.1, the class δ_{i_j} is mapped by ∂ on $[\mathbb{T}, \mathbb{T} \times \mathbb{C}, i_j] = \gamma'_j \in K_1(\mathbb{T}^2)$ for $j = 1, 2$. But γ'_1 and γ'_2 generate $K_1(\mathbb{T}^2) = \mathrm{im}(\partial)$. Thus we can conclude the part of the proposition concerning $K_0(\mathbb{T}^3)$ by $(\underline{\heartsuit})$. For the other part, since $\mathrm{id} - \alpha_*^{-1}$ is zero in K_1, we get that i_* is an isomorphism between $K_1(\mathbb{T}^2)$ and $\ker(\partial)$. This and $(\underline{\diamondsuit})$ end the proof. $\square$

The case $\mathrm{tr}(\alpha) = 2$ and $\alpha \neq I_2$

Now suppose that $\alpha \neq I_2$ and $b = 0$. We deduce from $a + d = \mathrm{tr}(\alpha) = 2$ that $d = 2 - a$. Together with $ad = \det(\alpha) = 1$ it gives that

$$\alpha = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \quad \text{and} \quad I_2 - \alpha = \begin{pmatrix} 0 & 0 \\ -c & 0 \end{pmatrix}$$

with $c \neq 0$. By proposition 4.2.8, the kernel of $\mathrm{id} - \alpha^{-1}$ is generated by γ'_1, whereas its image is generated by $c\gamma'_2$. Therefore $(\underline{\heartsuit})$ and $(\underline{\diamondsuit})$ give the next result.

Proposition 4.2.10. *Assume $b = 0$ and $c \neq 0$.*

1) *The group $K_0(BG)$ is isomorphic to $\mathbb{Z}^3$ and generated by δ_1, δ_2 and δ_{i_1}.*

2) *The group $K_1(BG)$ is isomorphic to $\mathbb{Z}^3 \oplus \mathbb{Z}_{|c|}$. Generators are provided by the torsion-free elements γ_3, γ_4 and γ_1, together with the class γ_2, whose order is $|c|$.*

Finally, if $b \neq 0$, then

$$c = \frac{ad - 1}{b} = \frac{a(2 - a) - 1}{b} = -\frac{(1 - a)^2}{b}.$$

Thus

$$I_2 - \alpha^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 - a & -b \\ \frac{(1-a)^2}{b} & a \end{pmatrix} = \begin{pmatrix} -(1 - a) & b \\ -\frac{(1-a)^2}{b} & 1 - a \end{pmatrix},$$

Let $a', b' \in \mathbb{Z}$ be such that $\frac{1-a}{b} = \frac{1-a'}{b'}$ and $\gcd(1 - a', b') = 1$. By Bézout, there exists $m, n \in \mathbb{Z}$ satisfying

$$m(1 - a') + nb' = 1.$$

We also set $g := \gcd(1 - a, b)$. Computations analogous to lemmas 3.2.10 and 3.2.12, replacing $[\pi_1]$ and $[\pi_2]$ by γ'_1 and γ'_2 respectively, give the next result.

Lemma 4.2.11. 1) *The homomorphism* $\mathrm{id} - \alpha_*^{-1} : K_1(\mathbb{T}^2) \longrightarrow K_1(\mathbb{T}^2)$ *has infinite cyclic kernel, generated by* $b'\gamma_1' + (1-a')\gamma_2'$.

2) *The image of* $\mathrm{id} - \alpha_*^{-1} : K_1(\mathbb{T}^2) \longrightarrow K_1(\mathbb{T}^2)$ *is isomorphic to* $\mathbb{Z}$ *and generated by the element* $g\gamma_1' + \frac{g(1-a)}{b}\gamma_2'$.

Together with $(\underline{\heartsuit})$ and $(\underline{\diamondsuit})$ implies the following proposition.

Proposition 4.2.12. 1) *The group* $K_0(BG)$ *is isomorphic to* $\mathbb{Z}^3$. *Generators are given by* δ_1, δ_2 *and* $\delta_{i_1^{b'} i_2^{1-a'}}$, *for every* $x \in \mathbb{T}$.

2) *The group* $K_1(BG)$ *is isomorphic to* $\mathbb{Z}^3 \oplus \mathbb{Z}_h$. *Generators are given by* γ_3, γ_4, $-s\gamma_1 + r\gamma_2$, *whose order is infinite, and* $\frac{g}{h}\gamma_1 + \frac{g(1-a)}{hb}\gamma_2$, *whose order is* h.

Proof. First we note that the loop $i_1^{b'} i_2^{1-a'}$ is injective. Indeed, if $e^{2\pi it}, e^{2\pi is} \in \mathbb{T}$ are such that

$$(e^{2\pi itb'}, e^{2\pi it(1-a')}) = (e^{2\pi isb'}, e^{2\pi is(1-a')}),$$

then $(t-s)b' = k_1$ and $(t-s)(1-a') = k_2$ for some $k_1, k_2 \in \mathbb{Z}$. Since $b' \neq 0$, we deduce that $(t-s) = \frac{k_1}{b'}$ and therefore

$$\frac{k_1}{b'}(1-a') = k_2 \quad \Longleftrightarrow \quad \frac{(1-a')}{b'} = \frac{k_2}{k_1}.$$

Since $\frac{(1-a')}{b'}$ is a reduced fraction, there is a $l \in \mathbb{Z}$ such that $k_2 = l(1-a')$ and $k_1 = lb'$. Thus

$$(t-s) = \frac{k_1}{b'} = \frac{lb'}{b'} = l$$

and $e^{2\pi it}$ is the same element as $e^{2\pi is}$. By proposition 4.2.1, the map ∂ sends the class $\delta_{i_1^{b'} i_2^{1-a'}}$ on $[\mathbb{T}, \mathbb{T} \times \mathbb{C}, i_1^{b'} i_2^{1-a'}]$. But, by proposition 4.2.4,

$$[\mathbb{T}, \mathbb{T} \times \mathbb{C}, i_1^{b'} i_2^{1-a'}] = b'\gamma_1' + (1-a')\gamma_2',$$

and then generates $\ker(\mathrm{id} - \alpha_*^{-1})$, thanks to the previous lemma. The second part of the proposition arises from an argument similar as in proposition 3.2.13 proof, together with the previous lemma and 4.2.1. □

The case $\mathrm{tr}(\alpha) \in \{1, 3\}$

If $\mathrm{tr}(\alpha)$ equals 1 or 3, then $\mathrm{id} - \alpha_*^{-1}$ is an isomorphism from $K_1(\mathbb{T}^2)$ onto itself, since

$$\det(I_2 - \alpha^{-1}) = 2 - \mathrm{tr}(\alpha) \in \{-1, 1\}.$$

Then the six-terms exact sequence ($\underline{\star}$) reads as follows :

$$\begin{tikzcd}
K_0(\mathbb{T}^2) \arrow[r, "0"] & K_0(\mathbb{T}^2) \arrow[r, "i_*"] & K_0(BG) \arrow[d, "\partial"] \\
K_1(BG) \arrow[u, "\partial"] & K_1(\mathbb{T}^2) \arrow[l, "i_*"'] & K_1(\mathbb{T}^2). \arrow[l, "\cong"']
\end{tikzcd}$$

By exactness, we deduce that i_* is the zero map in K_1, as well as the right-hand vertical homomorphism. Thus, the map i_* in K_0 and the left-hand vertical homomorphism are both preceded and followed by zero maps, and hence are isomorphisms.

Proposition 4.2.13. *The K-homology groups of BG are both isomorphic to $\mathbb{Z}^2$. The group $K_0(BG)$ is generated by δ_1 and δ_2, whereas $K_1(BG)$ is generated by γ_3 and γ_4.*

The case $\operatorname{tr}(\alpha) \notin \{1, 2, 3\}$

In this case, $I_2 - \alpha_*^{-1}$ is injective, since it has nonzero determinant. Then, by exactness of the six-terms sequence ($\underline{\star}$), the homomorphism $\partial : K_0(BG) \longrightarrow K_1(\mathbb{T}^2)$ is the zero map, and therefore $i : \mathbb{T}^2 \longrightarrow BG$ induces an isomorphism in K_0.

Proposition 4.2.14. *The group $K_0(B(\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}))$ is isomorphic to $\mathbb{Z}^2$, generated by δ_1 and δ_2.*

On the other hand, by $\det(I_2 - \alpha^{-1}) \notin \{-1, 1\}$ and its injectivity, the map $\mathrm{id} - \alpha_*^{-1}$ is not surjective. Moreover, its image has rank two in the $\mathbb{Z}$-module $K_0(\mathbb{T}^2) \cong \mathbb{Z}^2$ and then $K_1(\mathbb{T}^2)/\operatorname{im}(\mathrm{id} - \alpha_*^{-1})$ has to be isomorphic to a direct sum of two cyclic groups of finite order. We recall from lemma 3.2.16 that we can transform $I_2 - \alpha^{-1}$ into a diagonal matrix $h_1 \oplus h_2$ only by using elementary operations on its rows and columns. Moreover, we can choose $h_1, h_2 \in \mathbb{N}^*$ and such that $h_1 h_2 = |\det(I_2 - \alpha_*^{-1})|$. Then we obtain the result by ($\underline{\diamondsuit}$).

Proposition 4.2.15. *The group $K_1(BG)$ is isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}_{h_1} \oplus \mathbb{Z}_{h_2}$. Generators for the torsion-free summand are given by γ_3 and γ_4.*

As in K-theory, if we explicit the case $b = 0$, we find that $\alpha^{-1} = \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix}$ and then

$$I_2 - \alpha^{-1} = \begin{pmatrix} 2 & 0 \\ c & 2 \end{pmatrix}.$$

Thus, if c is even,

$$K_1(BG) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

and, if c is odd,

$$K_1(BG) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_4.$$

4.2.2 The case $\det(\alpha) = -1$

For this section, let us assume that α has determinant -1.

Proposition 4.2.16. *The homomorphism* $\operatorname{id} - \alpha_*^{-1} : K_0(\mathbb{T}^2) \longrightarrow K_0(\mathbb{T}^2)$ *maps* δ_1' *to zero and* δ_2' *to* $2\delta_2'$.

Proof. Since $\det(\alpha^{-1}) = \det(\alpha) = -1$, the homeomorphism of $\mathbb{T}^2$ induced by α^{-1} is orientation-reversing. Thus, in homology, α^{-1} induces the identity on $H_0(\mathbb{T}^2)$ and the map obtained by multiplication with -1 in $H_2(\mathbb{T}^2)$. But this class is the same as $\operatorname{id}_*([-\mathbb{T}^2])$, where $-\mathbb{T}^2$ is the torus considered with the opposite orientation and

$$\begin{aligned} \operatorname{id} : -\mathbb{T}^2 &\longrightarrow \mathbb{T}^2 \\ x &\longmapsto x. \end{aligned}$$

Thus, by Matthey's results presented on pages 114 and following,

$$\beta_2(-[\mathbb{T}^2]) = [-\mathbb{T}^2, \mathbb{T}^2 \times \mathbb{C}, \operatorname{id}] = -[\mathbb{T}^2, \mathbb{T}^2 \times \mathbb{C}, \operatorname{id}_{\mathbb{T}^2}] = -\delta_2'. \qquad \square$$

Consequently, the image of $\operatorname{id} - \alpha_*^{-1}$ in K_0 is infinite cyclic, generated by $2\delta_2'$. Hence,

$$K_0(\mathbb{T}^2)/\operatorname{im}(\operatorname{id} - \alpha_*^{-1}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$$

generated by δ_1' and δ_2', the latter being of order 2. Thus, since

$$\ker(\partial) = \operatorname{im}(i_*) \cong K_0(\mathbb{T}^2)/\ker(i_*) = K_0(\mathbb{T}^2)/\operatorname{im}(\operatorname{id} - \alpha_*^{-1}),$$

the six-terms exact sequence $(\underline{\star})$ induces the short one

$$0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}_2 \longrightarrow K_0(BG) \xrightarrow{\partial} \operatorname{im}(\partial) \longrightarrow 0.$$

Therefore,

$$K_0(BG) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \operatorname{im}(\partial) \qquad (\underline{\spadesuit})$$

with generators given by δ_1, δ_2 and preimages by ∂ of elements generating $\operatorname{im}(\partial) = \ker(\operatorname{id} - \alpha_*^{-1})$, given by proposition 4.2.1. On the other hand, by the proposition above, the image of the map $\partial : K_1(BG) \longrightarrow K_0(\mathbb{T}^2)$, which is equal to the kernel of $\operatorname{id} - \alpha_*^{-1}$, is generated by δ_1'. Thus $(\underline{\star})$ induces

$$0 \longrightarrow \ker(\partial) \longrightarrow K_1(BG) \xrightarrow{\partial} \langle \delta_1' \rangle \cong \mathbb{Z} \longrightarrow 0.$$

Hence, by splitness of this sequence and corollary 4.2.2,

$$K_1(BG) \cong \mathbb{Z} \oplus \ker(\partial) \cong \langle \gamma_3 \rangle \oplus \ker(\partial) \qquad (\underline{\clubsuit})$$

again with

$$\ker(\partial) = \operatorname{im}(i_*) \cong K_0(\mathbb{T}^2)/\ker(i_*) = K_0(\mathbb{T}^2)/\operatorname{im}(\operatorname{id} - \alpha_*^{-1}).$$

Let us yet recall that

$$I_2 - \alpha^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -d & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 1+d & -b \\ -c & 1+a \end{pmatrix}.$$

The case $\operatorname{tr}(\alpha) = 0$

If we assume $\operatorname{tr}(\alpha) = 0$, then $d = -a$ and, therefore,

$$-1 = \det(\alpha) = ad - bc = -a^2 - bc.$$

First suppose also that $b = 0$, hence $a^2 = 1$. If $a = 1$, then

$$\alpha = \begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix} = \alpha^{-1}$$

and then

$$I_2 - \alpha^{-1} = \begin{pmatrix} 0 & 0 \\ -c & 2 \end{pmatrix}.$$

Thus $(\operatorname{id} - \alpha_*^{-1})(\gamma_1') = -c\gamma_2'$ and $(\operatorname{id} - \alpha_*^{-1})(\gamma_2') = 2\gamma_2'$

Proposition 4.2.17. *Suppose $a = 1$ and $b = 0$.*

1) *If c is even, then*

$$K_0(BG) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2 \cong K_1(BG),$$

with $\delta_{i_1 i_2^{\frac{c}{2}}}$, δ_1 and δ_2 providing generators for $K_0(BG)$, whereas γ_3, γ_1 and the order-two element γ_2 generates $K_1(BG)$.

2) *If c is odd, then $K_0(BG) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2$, with generators given by the classes $\delta_{i_1^2 i_2^c}$, δ_1 and δ_2 and $K_1(BG) \cong \mathbb{Z}^2 \cong \langle \gamma_3 \rangle \oplus \langle \gamma_1 \rangle$.*

Proof. From $(\underline{\spadesuit})$ and $(\underline{\clubsuit})$, it remains to identify the kernel and image of $\operatorname{id} - \alpha_*^{-1}$ and find generators for them. If c is even, it is obvious that $\operatorname{im}(\operatorname{id} - \alpha_*^{-1}) = \langle 2\gamma_2' \rangle$. Hence

$$K_1(\mathbb{T}^2) / \operatorname{im}(\operatorname{id} - \alpha_*^{-1}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$$

generated by γ_1' and the order-two element γ_2'. On the other hand, $\gamma_1' + \frac{c}{2}\gamma_2'$ lies in the kernel of $\operatorname{id} - \alpha_*^{-1}$. But, if $k_1\gamma_1' + k_2\gamma_2'$ is sent on 0, then

$$0 = ck_1\gamma_2' + 2k_2\gamma_2' = (-ck_1 + 2k_2)\gamma_2' = 2(-\frac{c}{2}k_1 + k_2)\gamma_2'$$

and therefore $k_2 = \frac{c}{2}k_1$. Hence $\ker(\operatorname{id} - \alpha_*^{-1}) = \langle \gamma_1' + \frac{c}{2}\gamma_2' \rangle$ and we end the proof by proposition 4.2.4. If c is odd, then a process similar as in the proof of lemmas 3.2.10 and 3.2.12 shows $\ker(\operatorname{id} - \alpha_*^{-1}) = \langle 2\gamma_1' + c\gamma_2' \rangle$ and $\operatorname{im}(\operatorname{id} - \alpha_*^{-1}) = \langle \gamma_2' \rangle$. □

Furthermore, if $b = 0$ and $a = -1$, we get

$$\alpha = \begin{pmatrix} -1 & 0 \\ c & 1 \end{pmatrix} = \alpha^{-1}$$

and therefore

$$I_2 - \alpha^{-1} = \begin{pmatrix} 2 & 0 \\ -c & 0 \end{pmatrix}.$$

Thus $\ker(\mathrm{id} - \alpha^{-1}) = \langle \gamma_2' \rangle \cong \mathbb{Z}$ and $\mathrm{im}(\mathrm{id} - \alpha_*^{-1}) = \langle 2\gamma_1' - c\gamma_2' \rangle \cong \mathbb{Z}$. By identifying the quotient as in the proof of proposition 3.2.13, we deduce the following result.

Proposition 4.2.18. *Assume $a = -1$ and $b = 0$.*

1) *If c is even, then*

$$K_0(BG) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2 \cong K_1(BG).$$

Generators are provided by δ_{i_2}, δ_1 and δ_2 for $K_0(BG)$ and by γ_3, γ_2 and the order-two element $\gamma_1 - \frac{c}{2}\gamma_2$ for $K_1(BG)$.

2) *If c is odd, then*

$$K_0(BG) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2,$$

generated by δ_{i_2}, δ_1 and δ_2, and

$$K_1(BG) \cong \mathbb{Z}^2 \cong \langle \gamma_3 \rangle \oplus \langle -\gamma_1 + \frac{c+1}{2}\gamma_2 \rangle.$$

Assume now that $b \neq 0$. Hence $c = \frac{ad+1}{d} = \frac{-a^2}{d}$, since $\mathrm{tr}(\alpha) = 0$. Thus

$$\alpha = \begin{pmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{pmatrix} = \alpha^{-1}$$

and then

$$I_2 - \alpha^{-1} = \begin{pmatrix} 1-a & -b \\ \frac{-(1-a^2)}{b} & 1+a \end{pmatrix}.$$

Let us write

$$g := \gcd(1-a, b), \qquad 1 - a' := \frac{1-a}{g}, \qquad b' := \frac{b}{g}$$

and

$$h := \gcd\left(g, \frac{-g(1+a)}{b}\right).$$

By Bézout, we have $m, n, r, s \in \mathbb{Z}$ such that

$$m(1-a) + nb = g \quad \text{and} \quad rg + s\frac{-g(1+a)}{b} = h.$$

Following the same arguments as in lemmas 3.2.10, 3.2.12 and proposition 3.2.13, we obtain the next results.

Lemma 4.2.19. 1) *The homomorphism* $\mathrm{id} - \alpha_*^{-1} : K_1(\mathbb{T}^2) \longrightarrow K_1(\mathbb{T}^2)$ *has infinite cyclic kernel, generated by* $b'\gamma_1' + (1-a')\gamma_2'$.

2) *The image of* $\mathrm{id} - \alpha_*^{-1} : K_1(\mathbb{T}^2) \longrightarrow K_1(\mathbb{T}^2)$ *is isomorphic to* $\mathbb{Z}$ *and generated by the element* $g\gamma_1' + \frac{-g(1+a)}{b}\gamma_2'$.

Proposition 4.2.20. 1) *The group* $K_0(BG)$ *is isomorphic to* $\mathbb{Z}^2 \oplus \mathbb{Z}_2$. *Generators are given by* $\delta_{i_1^{b'} i_2^{1-a'}}$, δ_1 *and* δ_2.

2) *The group* $K_1(BG)$ *is isomorphic to* $\mathbb{Z}^2 \oplus \mathbb{Z}_h$. *Generators are given by the torsion-free elements* γ_3, $-s\gamma_1 + r\gamma_2$ *and the element* $\frac{g}{h}\gamma_1 + \frac{-g(1+a)}{hb}\gamma_2$, *whose order is* h.

The case $\mathrm{tr}(\alpha) \in \{-1, 1\}$

In this case, since $\det(I_2 - \alpha^{-1}) = \mathrm{tr}(\alpha)$, we have that $\mathrm{id} - \alpha^{-1}$ is an isomorphism in K_1. Then $(\underline{\spadesuit})$ and $(\underline{\clubsuit})$ immediately give the result.

Proposition 4.2.21. *The group* $K_0(BG)$ *is isomorphic to* $\mathbb{Z} \oplus \mathbb{Z}_2$ *and generated by* δ_1 *and* δ_2. *Moreover,* $K_1(BG) = \langle \gamma_3 \rangle \cong \mathbb{Z}$.

The case $\mathrm{tr}(\alpha) \notin \{-1, 0, 1\}$

Once again, we can lean on what has been done in the last part of the section 3.2.1. By elementary operations on the rows and columns of $I_2 - \alpha^{-1}$, we can turn it into a diagonal matrix $h_1 \oplus h_2$, with $h_1, h_2 \in \mathbb{N}^*$ and

$$h_1 h_2 = |\det(I_2 - \alpha^{-1})| = |\mathrm{tr}(\alpha)|.$$

Therefore, together with $(\underline{\spadesuit})$ and $(\underline{\clubsuit})$, we obtain this last result.

Proposition 4.2.22. *The group* $K_0(BG)$ *is isomorphic to* $\mathbb{Z} \oplus \mathbb{Z}_2$ *and generated by* δ_1 *and* δ_2, *whereas* $K_1(BG) \cong \mathbb{Z} \oplus \mathbb{Z}_{h_1} \oplus \mathbb{Z}_{h_2}$.

Epilogue

In view of the last two chapters, one could reasonably ask if, for a semi-direct product

$$G := \mathbb{Z}^2 \rtimes_\alpha \mathbb{Z},$$

the Baum-Connes map intertwines the Pimsner-Voiculescu six-terms exact sequence with the one of proposition 2.3.14, namely if the diagram

$$\begin{tikzcd}
 & K_0(C(\mathbb{T}^2)) \arrow[rr, "\mathrm{id}-\alpha_*^{-1}"] & & K_0(C(\mathbb{T}^2)) \arrow[rr, "i_*"] & & K_0(C^*(G)) \arrow[dd, "\partial"] \\
K_0(\mathbb{T}^2) \arrow[ur, "\mu_0^{\mathbb{Z}^2}"] \arrow[rr, "\mathrm{id}-\alpha_*^{-1}"] & & K_0(\mathbb{T}^2) \arrow[ur, "\mu_0^{\mathbb{Z}^2}"] \arrow[rr, "i_*"] & & K_0(BG) \arrow[ur, "\mu_0^G"] \arrow[dd, "\partial"] & \\
 & K_1(C^*(G)) \arrow[uu, "\partial"] & & K_1(C(\mathbb{T}^2)) \arrow[ll, "i_*"] & & K_1(C(\mathbb{T}^2)) \arrow[ll, "\mathrm{id}-\alpha_*^{-1}"] \\
K_1(BG) \arrow[uu, "\partial"] \arrow[ur, "\mu_1^G"] & & K_1(\mathbb{T}^2) \arrow[ll, "i_*"] \arrow[ur, "\mu_1^{\mathbb{Z}^2}"] & & K_1(\mathbb{T}^2) \arrow[ll, "\mathrm{id}-\alpha_*^{-1}"] \arrow[ur, "\mu_1^{\mathbb{Z}^2}"] &
\end{tikzcd}$$

commutes. Thus a direct application of the five lemma would give a new proof that the Baum-Connes conjecture holds for $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$. Unfortunately, we are unable to check the compatibility of the assembly map with respect to the connecting homomorphisms in K-theory and K-homology.

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