

Interpreting Arithmetic: Russell on Applicability and Wittgenstein on Surveyability

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What Russell and Frege do is to make connexions between English and German words “all”, “or”, “and”, etc. and numerical statements. This clears up a few points. But that we should actually then say, “ $3,000,000 + 4,000,000 = 7,000,000$ ”, does not follow from this.

Ludwig Wittgenstein

Our perceptual powers are limited. Indeed, most people can't tell at a glance, without counting, if there are 15 or 16 strokes in this figure:



This fact accounts for the various numerical notations introduced throughout history, as even stroke notations never go further than 3 or 4 (for example in Roman notation, I, II, III, with IV eventually replacing IIII). Arguments that rely on this fact, I shall dub ‘surveyability’ arguments. Such arguments are generally considered irrelevant to foundations of mathematics. They are on occasion deployed, because of the appeal to human epistemic limitations, in order to argue for ‘strict finitism’, a foundational standpoint which is usually considered as extremely narrow and unable to issue in an account of any substantial part of mathematics. One finds ‘surveyability’ arguments in Wittgenstein’s ill-understood *Remarks on the*

Foundations of Mathematics (Wittgenstein 1983) and, unsurprisingly, Wittgenstein's philosophy of mathematics has been described as a form of 'strict finitism'. In this paper, I shall not be concerned at all with these issues. I shall only make two claims: first, that 'surveyability' arguments have *bona fide* uses that have nothing to do with such 'foundational' issues and, secondly, that Wittgenstein's use of 'surveyability' against a particular pair of claims by Russell – see (6) and (7) below – should be counted among those *bona fide* uses. It thus undermines two theses that were originally perceived as part and parcel of logicism.

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That there are *bona fide* uses of 'surveyability' arguments can only be argued by showing one such use and one can find it in Alan Turing's celebrated 1936 paper 'On Computable Numbers, with an Application to the *Entscheidungsproblem*' (Turing 1936), in which he provided a negative answer to the decision problem for the predicate calculus, one of the great limitation results of the 1930s. In order to see where the 'surveyability' argument is invoked and why it has importance, I must recall some basic steps leading to his proof.¹ The decision problem was to find or to show that one cannot find an 'effective' method by which one can decide, for any formula expressible in the predicate calculus, if it is provable or not in it. The problematic notion in this formulation of the problem is that of 'effective method', as it is not 'formal'. Therefore, the first step towards a solution of the problem is to replace it by a formal notion that can then be used to formulate a mathematical proof. So Turing's strategy consisted first in providing in its stead a formal predicate, which he called simply 'computable'; nowadays, we speak of 'computability by a Turing machine'. This is known as Turing's Thesis (TT), which is not Church's Thesis but close to it: 'every effective method for computing the values of a number-theoretic function can be computed by a Turing machine'.

¹ My account owes a lot to careful studies of Turing's 1936 paper by Robin Gandy and Wilfried Sieg (Gandy 1988) (Sieg 1994, 2001).

What Turing showed was not a formal equivalence between the two notions. Indeed, since the notion of 'effective' method is not formal, its identification with a formal predicate such as 'computable by a Turing machine' cannot rely on formal arguments. Incidentally, this shows the non-eliminability of philosophical arguments in these matters, a fact recognized by Turing:

All arguments which can be given are bound to be, fundamentally, appeals to intuition, and for this reason rather unsatisfactory mathematically. (Turing 1936, 249)

The surveyability argument occurs at this stage, when Turing sets up his predicate 'computable' simply by describing what he called 'computing machines' (Turing 1936, 231sq.), i.e., Turing machines. One should note that, since our modern digital computing machines did not exist in 1936, Turing could not have referred to them when using the word 'computer'; in fact he clearly referred to a human being, albeit an idealized one. In order to avoid confusion, I shall use the term 'computer' to refer to the human being and retain 'computer' for machines.² So Turing's 'computer' is in fact an idealized human being, a 'computer'.³

Turing provides three (types of) arguments in support of TT but only the first one is of interest to us (Turing 1936, 249-252). Here, Turing analyses what happens when a computer applies an 'effective method', breaking down the steps in their most elementary form, in order to arrive at " 'simple operations' which are so elementary that it is not easy to imagine them further divided" (Turing 1936, 250). In his analysis, Turing spoke of a (finite) set of 'states of mind' of the computer (working on a tape divided in squares), on which he imposed (strict) finiteness conditions, on the number of symbols and squares observed at any one moment. The 'simple operations' of the computer are (roughly defined) as follows: the computer can change at most one symbol at a time in an observed square; it can move to another square

² In this, I merely follow a convention initiated in (Gandy 1988).

³ Incidentally, when Wittgenstein wrote that Turing machines are "humans who calculate" (Wittgenstein 1980, § 1096), he was thus right.

but only within a bounded distance. Furthermore, any ‘simple operation’ depends only on the current state of mind of the computer and the symbol observed in the square, and Turing further required that from any given state and observed symbol, there is at most one operation that can be performed. Once these ‘simple operations’ of the computer are clearly defined, Turing could carry on with his strategy to solve the *Entscheidungsproblem* by arguing that one could construct a computer (a Turing machine) “to do the work of this comput[o]r” (Turing 1936, 215). In particular, this is done by identifying the ‘states of mind’ of the computer with ‘configurations’ of the computer.

It is in one of the above steps, i.e., when imposing finiteness conditions, that Turing uses a ‘surveyability’ argument concerning the number of symbols printed on the squares of the tape:

The differences from our point of view between the single and compound symbols is that the compound symbols, if they are too lengthy, cannot be observed at one glance. This is in accordance with experience. We cannot tell at a glance whether 9999999999999999 and 9999999999999999 are the same. [...] We may suppose that there is a bound B to the number of symbols or squares which the computer can observe at one moment. If he wishes to observe more, he must use successive observations. We will also suppose that the number of states of mind which need be taken into account is finite. The reasons for this are of the same character as those which restrict the number of symbols. If we admitted an infinity of states of mind, some of them will be ‘arbitrarily close’ and will be confused. (Turing 1936, 250)

Now, I have never heard of any rejection of Turing’s analyses or of this use of a ‘surveyability’ argument, *et pour cause...* One should note that the point Turing is making has nothing to do with arguing for ‘strict finitism’ or some such thing. And this was the point of telling my story: to see that ‘surveyability’ arguments can have *bona fide* use without presupposing some presumably unpalatable views about mathematics.

As I pointed out, one finds uses of ‘surveyability’ arguments in Wittgenstein, e.g., in Part III of the posthumously published *Remarks on the Foundations of Mathematics*. Here is one example:



Is this pattern a proof of $27 + 16 = 43$, because one reaches ‘27’ if one counts the strokes on the left-hand side, ‘16’ on the right-hand side, and ‘43’ when one counts the whole row?

Where is the queerness of calling the pattern the proof of this proposition? It lies in the kind of way this proof is to be reproduced or known again; in its not having any characteristic visual shape? (Wittgenstein 1983, III, § 11)

Embedded here is a ‘surveyability’ argument, since Wittgenstein uses the fact that one cannot tell the equivalence of

(S) + =

and

(A) $27 + 16 = 43$

merely by looking; one cannot even tell what the result in (S) is without counting the strokes.

Part III of the *Remarks on the Foundations of Mathematics* was written in 1939-40, roughly a year after Wittgenstein met Turing but one finds ‘surveyability’ arguments invoked as early as 1929, i.e., the earliest surviving material dating from Wittgenstein return to philosophy. There is thus no reason to think here of an influence of Turing on Wittgenstein or vice-versa. Incidentally, a ‘surveyability’ argument occurs during Wittgenstein’s 1939 *Lectures on the Foundations of Mathematics* that were attended by Turing (Wittgenstein 1976, 258-259); I shall come back to it.

So ‘surveyability’ arguments are ubiquitous in Wittgenstein’s writings and one should beware of lumping them all together, as well as one should try and avoid too simplistic an account of Wittgenstein’s multi-faceted discussion of these issues, e.g., under the heading ‘strict finitism’. For example, the occurrence in 1929 is used

against the claim, usually but falsely attributed to his own *Tractatus Logico-Philosophicus*, that arithmetical propositions are tautologies:

The correctness of an arithmetical proposition is never expressed by a proposition's being a tautology. In the Russellian way of expressing it, the proposition $3 + 4 = 7$ for example can be represented in the following manner:

$$(E3x)\varphi x . (E4x)\psi x . \sim(Ex)\varphi x . \psi x \rightarrow: (E7x). \varphi x \vee \psi x$$

Now one might think that the proof of this equation consisted in this: that the proposition written down was a tautology. But in order to be able to write down this proposition, I have to *know* that $3 + 4 = 7$. The whole tautology is an application and not a proof of arithmetic. (Wittgenstein 1979, 35)

Note here that the argument deployed is slightly more complicated. It is implied here that the Russellian equivalent of $27 + 16 = 43$ is the following formula, involving numerical quantifiers:

$$(R) [\exists!_{27}x (Gx) \wedge \exists!_{16}x (Hx) \wedge \forall x \neg(Gx \wedge Hx)] \rightarrow \exists!_{43}x (Gx \vee Hx)$$

Wittgenstein rejects here the foundational principle that '(A) because of (R)' because the only way to know that (R) is precisely by applying (A), i.e., counting the number of variables in the unabbreviated version of (R), i.e., in something such as (S) which is poorly hidden by the mechanism of numerical quantifiers; here one may run a 'surveyability' argument. As Hao Wang put it, we are able to see that (R) is a theorem of logic

[...] only because we are able to see that a corresponding arithmetic proposition is true, not the other way round. (Wang 1961, 335)

How this use of a 'surveyability' argument is to work against the claim that arithmetical propositions are tautologies is well worth further investigation but I would like to focus here on one particular use of this argument, against a peculiar pair of theses, (6)-(7) below,

first propounded by Russell. The attack against Russell occurs in this passage:

The *application* of the calculation must take care of itself. And that is what is correct about 'formalism'.

The reduction of arithmetic to symbolic logic is supposed to shew the point of application of arithmetic, as it were the attachment by means of which it is plugged in to its application. As if someone were shewn, first a trumpet without a mouthpiece – and then the mouthpiece, which shows how a trumpet is used, brought into contact with the human body. But the attachment which Russell gives us is on the one hand too narrow, on the other hand too wide; too general and too special. The calculation takes care of its own application.

(Wittgenstein 1983 III, § 4)

I shall first explain what is the thesis that Wittgenstein attacked and then argue that Wittgenstein's use of a 'surveyability' argument should be listed among the *bona fide* cases.

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If one is to ask about prospects for logicism today, one must ask first what is it that one wishes to revive or, at least, ponder about. My claim will be that Wittgenstein's argument demolishes for good one part of logicism, namely Russell's theses that I have alluded to. Of course, it may be that these theses were farfetched to begin with or that the logicist project remains essentially untouched as a result of this particular critique. But these are questions into which I shall not enter here; one cannot deal with every aspect of this question within the narrow compass of a single paper.

The heart of logicism is the following theses:

- (1) The concepts of arithmetic can be derived from logical concepts through explicit definitions.
- (2) The theorems of arithmetic can be derived from logical axioms through purely logical deduction.

(I limit myself, for the sake of simplicity, to the arithmetical case.) These can be found in, e.g., Carnap's contribution to the Königsberg

symposium on the foundations of mathematics (1931), from which they were lifted (Carnap 1983, 41). Taken together, they amount to the following:

- (3) There exists some formal system of logic such that arithmetic can be generated from it.

It is not immediately clear what would be gained from a vindication of (3). Frege spoke of ‘fruitful definitions’ (Frege 1980, § 88) but it is debatable to which extent (3) is *arithmetically* ‘fruitful’. One could also draw like Frege some inferences about the nature of arithmetical propositions, e.g., that there are not ‘synthetic a priori’, etc. But these are purely philosophical conclusions. Russell spoke instead of the ‘meaning and justification’ of arithmetic (Russell 1919, 194), i.e., of providing foundations or providing grounds for the acceptance of arithmetic. As a matter of fact, when Wittgenstein presented in his 1939 lectures a ‘surveyability’ argument in order to deny the idea that arithmetic is based on logic, Turing picked it up and, in the course of the discussion, made exactly that point:

Turing: Russell’s definitions show us the *point* of having these ideas of addition and finite cardinals and so on.

Wittgenstein: Yes – and it is just that that I want to deny. (Wittgenstein 1976, 262)

For that reason, logicism cannot be limited to (3). It must also have, at least in Russell’s mind, an ‘epistemological’ side, which can be given by the following pair of theses, as formulated by Mark Steiner in *Mathematical Knowledge* (Steiner 1975, 25):

- (4) It is sufficient to understand proofs written in this system in order to know all the truths of arithmetic that we know.
- (5) It is possible for us actually to come to know arithmetical truths in the way suggested in (4) by constructing logical proofs of them.

In his book, Steiner discusses Wittgenstein’s use of a ‘surveyability’ argument, inasmuch as it can be brought to bear upon these theses; I

shall not go over the points he raised.⁴ I would like instead to point out that this is not the end of the story. Russell framed another pair of claims in the process of arguing for logicism:

- (6) Interpretation within the logical system, in the way suggested by (1), provide a definite meaning to the basic arithmetical concepts.
- (7) This interpretation allows for the applications of arithmetic.

These claims are seldom discussed. I shall first explain what they amount to in Russell's mind and then, after some brief remarks about their historical importance, proceed to show how Wittgenstein's use of 'surveyability' undermines them.

After listing Peano's axioms, in the first chapter of *Introduction to Mathematical Philosophy*, Russell pointed out that the primitive concepts they contain, '0', 'number', and 'successor' are "capable of an infinite number of different interpretations, all of which will satisfy the five primitive propositions" (Russell 1919, 7). Given one such interpretation, one obtains a series which is called by Russell a "progression" and which will, like the series of natural numbers, satisfy these axioms:

In fact, given any series

$$x_0, x_1, x_2, x_3, \dots, x_n, \dots$$

which is endless, contains no repetitions, has a beginning, and has no terms that cannot be reached from the beginning in a finite number of steps, we have a set of terms verifying Peano's axioms.

(Russell 1919, 7-8)

There are indeed an infinity of such progressions, since it suffices for example to pick any natural number other than 0 as the beginning, e.g., 99, to form one. Now, Russell's argument is that "there is nothing" in Peano Arithmetic "to enable us to distinguish between [...] different interpretations of his primitive ideas" but

This point, that "0" and "number" and "successor" cannot be defined by means of Peano's five axioms, but must be independently understood, is important. We want our numbers not merely to verify

⁴ See (Steiner 1975, 41-54) and (Marion 1998, 228-236).

mathematical formulae, but to apply in the right way to common objects. We want to have ten fingers and two eyes and one nose. A system in which “1” meant 100, and “2” meant “101”, and so on, might be all right for pure mathematics, but would not suit daily life. We want “0” and “number” and “successor” to have meanings which will give us the right allowance of fingers and eyes and nose. We have already some knowledge (though not sufficiently articulate or analytic) of what we mean by “1” and “2” and so on, and our use of numbers in arithmetic must conform to this knowledge.
(Russell 1919, 9)

A little further, this point is reiterated:

(...) we want our numbers to be such as can be used for counting common objects, and this requires that our numbers should have a *definite* meaning, not merely that they should have certain formal properties. This definite meaning is defined by the logical theory of arithmetic. (Russell 1919, 10)

These are the passages where Russell expressed (6) and (7). They indicate clearly the very purpose of the system of *Principia Mathematica*: it was set up as an *interpretation* of Peano Arithmetic which, in Turing’s terms, shows us “the *point* of having these ideas of addition and finite cardinals and so on” or, more precisely, to provide a definite meaning to its primitive terms, such that one could recover ordinary applications of arithmetic. And it is at these views of Russell that Wittgenstein takes aim in the passage from Part III, § 4 of the *Remarks on the Foundations of Mathematics* that I quoted earlier.

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Before moving on to Wittgenstein, I would like to point out, with some brief historical remarks about Frege and Carnap, that applicability is a recurring theme within logicism: logicists saw it as a proof of the superiority of their approach that it accounts for applications, while other approaches – especially formalism – fail. It is well-known, for example, that Frege claimed in Part III of his *Grundgesetze der Arithmetik* that “it is applicability alone which elevates arithmetic from a game to the rank of a science” (Frege 1998, § 91). According to him, formalists could not account for the

applicability of mathematics because they held that mathematical formulas express no thoughts:

[...] for an arithmetic with no thought as its content will also be without possibility of application. Why can no application be made of a configuration of chess pieces? Obviously, because it expresses no thought. If it did so and every chess move conforming to the rules corresponded to a transition from one thought to another, applications of chess would also be conceivable. Why can arithmetical equations be applied? Only because they express thoughts. How could we possibly apply an equation which expressed nothing and was nothing more than a group of figures, to be transformed into another group of figures in accordance to certain rules? (Frege 1998, § 91)

Frege's point is admittedly different from Russell's and it involves ideas that are typically his. Indeed, Frege thought it necessary for a mathematical formula to be applied that it expresses a thought, because he believed the application of a mathematical theorem to be an instance of a deductive inference and that such inferences are only possible from (true) thoughts: it is thus senseless to speak of inferring to the truth of a conclusion from something that was not a thought.

I shall make only one brief point in relation to Frege's critique of formalism. It has been answered by Friedrich Waismann in his 1936 *Einführung in das mathematische Denken*:

What, then, has to be added, in order for a mathematical equation to express a thought-content? Application, and nothing more. It is mathematics when the equation is used for the transition from one proposition to another; otherwise it is game. To say that a move in chess expresses no thought is hasty; for it wholly depends on us. [...] 'Because a chess move expresses no thought, one cannot apply it'. Would it not be correct to say that because we have not provided an application for it, the chess move does not express a thought? (Waismann 1951, 240)

This critique has been in turn criticized by Sir Michael Dummett (Dummett 1991 259-261), who puts the following gloss on it:

Waismann denies that we first confer a sense on the proposition, and then, in the light of that sense, make various applications of it: rather, we make the application, and *thereby* give it a sense – a truly Wittgensteinian idea. (Dummett 1991, 260)

I do not wish to discuss here Dummett's argument against Waismann; I simply wish to point out that, although Waismann's position may be described with some justice as 'Wittgensteinian', he does not make any appeal to a 'surveyability' argument and Wittgenstein's use of the latter does not presuppose or imply that it is the application that gives to a proposition its sense.

With the current fascination for his neo-Kantian background⁵, one is likely to forget the fact that Carnap was, early on, a staunch logicist and that he was influenced by Russell on key points⁶. Indeed, Carnap's *Abriß der Logistik* (Carnap 1929) is probably the last major logicist work, incorporating advances made by Ramsey in his 1925 masterpiece on 'The Foundations of Mathematics' (Ramsey 1990, 164-224). The recent publication of manuscripts from the late 1920s, *Untersuchungen zur allgemeinen Axiomatik* (Carnap 2000) allows us better to understand his early philosophy.⁷ I would like simply to point out that Russell's (6) and (7) are everywhere in evidence in Carnap's work during that period. Already in 'Eigentliche und uneigentliche Begriffe' (Carnap 1927), he had argued that formal concepts such as Peano's '0', 'number', and 'successor' are *uneigentliche* and in need of explicit definitions of the kind provided by Russell. The very project of *The Logical Structure of the World* was but a generalization of (6) and (7) to the whole "conceptual system of unified science". Indeed, it was meant to supplement the work of axiomatization by providing an interpretation of the primitives (ultimately into a domain of 'basic objects') through a 'constructional system':

A theory is *axiomatized* when all statements of the theory are arranged in the form of a deductive system whose basis is formed by the axioms, and when all concepts of the theory are arranged in the form of a constructional system whose basis is formed by the fundamental concepts. So far, much attention has been paid to the first task, namely, the deduction of statements from axioms, than to the

⁵ For example (Richardson 1998) or (Awodey & Klein 2004).

⁶ Of course, Frege's lectures in 1910-1914 played a fundamental role. Carnap's student notes are now published (Reck & Awodey 2004). But my point here is simply to emphasize the role of (6) and (7) in Carnap's thinking.

⁷ See the excellent study (Awodey & Carus 2001).

methodology of the systematic construction of concepts. The latter is to be our present concern and is to be applied to the conceptual system of unified science. (Carnap 1967, § 2)

During the discussion at Königsberg, Carnap expressed theses (6) and (7) in so many words:

1. For all mathematical signs there are one or more interpretation, as a matter of fact these are purely logical.
2. If an axiomatic system is consistent, then we replace in each mathematical formula its mathematical signs by the logical interpretation thus found (more precisely, one among many different interpretations), and it becomes a tautology (a sentence with general validity).
3. If an axiomatic system is complete (in the sense given by Hilbert: no formula which is not derivable can be added without contradiction), then the logical interpretation is unique; each sign has exactly one interpretation, and the formalist construction is transformed in a logical construction. (Hahn *et al.* 1931, 141)

These results depended, in Carnap's mind, on a proof that he did not (and could not) possess, that of his *Gabelbarkeitssatz* (Carnap 2000, 133). This theorem would have shown that any consistent axiomatic system is complete if and only if it is categorical (in Carnap's words, 'monomorphic'). With this result, Carnap had hoped, during these years, to provide the standard interpretation to Peano Arithmetic and thus to integrate it into his constructional project. (Gödel's results showed that categoricity could not be proved at first-order level, thus ruining Carnap's plans.)

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Wittgenstein was thus reacting against what was perceived at the time as an essential part of the logicist doctrine. How does his argument, as above, work against it? We may first note that it does not address the manner by which the system of *Principia Mathematica* is supposed to select the series of natural numbers among an infinity of progressions, e.g., how one cleverly defines in that book '0', '1', '2', and so on. Wittgenstein does not criticize the system of *Principia Mathematica*, which is left standing, untouched, but rather some claims made on its

behalf by Russell, (6) and (7)⁸. We should rather note that, in expressing these, Russell admits that we “have already some knowledge (though not sufficiently articulate or analytic) of what we mean by “1” and “2” and so on, and our use of numbers in arithmetic must conform to this knowledge” (Russell 1919, 9). So his position seems to be that we have first knowledge of some arithmetical truths, e.g., $27 + 16 = 43$ or (A), and that we then ‘go formal’, so to speak, and learn that (A) is a ‘consequence’ of some other formal sentence. Here, the claim would be that, thanks to the interpretation of Peano Arithmetic effected in *Principia Mathematica*, we are now in a position to claim that (R) is the ‘ground’ for (A), alternatively, that (A) is the ‘consequence’ of (R). In other words, an ‘explicativist’ epistemology is presupposed by Russell⁹. So we have a ‘why’ claim of the form:

(8) (A), (A) because of (R)

However, this won’t do. One may simply appeal here to Aristotle’s discussion in *Posterior Analytics*, Book I, section 13, of the demonstration that ‘the planets are near’. As Aristotle points out, one can demonstrate this from the facts that ‘the planets do not twinkle’ and that ‘what does not twinkle is near’. But, as he points out,

(...) this deduction is not of the reason why but of the fact: for it is not because they do not twinkle that they are near, but because they are near that they do not twinkle. (Barnes 1995, vol. 1, 127)

In other words, according to Aristotle one has merely asserted here that the planets are near and not that they do not twinkle *because* they are near, for the simple reason that it is because they do not twinkle

⁸ This claim is repeated in many places, e.g., “If Russell’s calculus is to be merely an *auxiliary* calculus, dealing with ‘if’s and ‘then’s, etc., – then it is all right. But that is not what it is meant to be”; “The Russellian method is just one method, like many of these other methods”; “We might say that Russell’s method is perfectly all right, but neither is more *fundamental*” (Wittgenstein 1976, 261, 262 & 263).

⁹ About this explicativist epistemology, see (Mancosu 2000) and for a critique of it, for which this passage is deeply indebted, (Dubucs & Lapointe 2003).

that we *know* that they are near. Now Wittgenstein's argument as expounded above is to the effect that:

(9) We know that (R) because we already know that (A)

This is so, not because of the fact that we happen to know (A) before discovering that it is a 'consequence' of (R) but because it is impossible to know (R) without already knowing that (A). It is not just that (R) is less easily cognizable than (A) – a prerequisite here –, it is not even cognizable at all without it. And the reason for this is but another use of the 'surveyability' argument, as we saw. To paraphrase Aristotle, Wittgenstein's argument is thus that Russell can merely assert that (R) and not that '(A) because of (R)', for the simple reason that it is because of (A) that we *know* that (R). It is an application of (A) that allows us to *know* that (R) and, if anything it is because of (A) that (R); it is thus to delude oneself to assume (8) in order to claim (6) and (7).

Textual evidence for this is easy to find in Wittgenstein's corpus, for example in the midst of his discussion with Turing in 1939 lectures:

The idea that there is a science, namely logic, on which mathematics rests. I want to say it in no way rests on logic. And the fact that you can make logical formulae agree with it, in no way shows that it rests on logic. [...] We have normal ways of finding whether the numbers on both sides of the implication sign are the same [Wittgenstein is referring here to an example similar to (R) in this paper]. And this does not depend at all on Russell's principles; on the contrary, they depend on it. If we didn't have such ways of comparing the different sides, we shouldn't know what to call a tautology.
(Wittgenstein 1976, 260)

I should emphasize again that this use of a 'surveyability' argument does not imply an overall view of mathematics founded on epistemic limitations such as strict finitism (an easy way out for those wishing to dismiss Wittgenstein's critique), it is, like Turing's use that I presented at the beginning of this paper, an appeal to an undeniable fact, it is *bona fide*. As a matter of fact, in his discussion with Wittgenstein, Turing never denies the plausibility of his use of the

'surveyability' argument, although he is sceptical about the claims Wittgenstein derives from it, because he obviously adheres to the 'explicativism' on which Russell's position relies. But 'explicativism' is itself based on a misunderstanding, as I just pointed out. That this is so is quite another matter but the mere expression of one's prejudices (in favour of 'explicativism') is not a refutation, an argument is needed. One will have noticed from the last quotation that, at least in Wittgenstein's mind, his rejection of (8) implies, through a rejection of (6) and (7), a rejection of (3), i.e., logicism is refuted. This may be too brash a claim, as much as Russell was, to begin with, too brash to claim (6) and (7). But this is quite another matter that cannot be dealt with in this paper. One thing seems more certain. It is that logicism, if it is to make sense for us, should be dissociated from claims such as (6) or (7).

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