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Upper bounds for Steklov eigenvalues: from graphs and discretization to hypersurfaces of revolution and numerical experiments

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par

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"Les mots ne sont pas choisis au hasard ; s'il en est un qui est déplacé, incorrect, voire injuste, alors trouve-le et rétablis la vérité."

- Votre serviteur.

Résumé

Un des buts de la géométrie spectrale est de comprendre la relation entre la géométrie ou la topologie d'une variété riemannienne et le spectre d'un opérateur différentiel de type laplacien associé à cette variété. Ce concept est résumé dans la célèbre phrase de Mark Kac

"Peut-on entendre la forme d'un tambour ?"

and constitue un sujet actif de recherche fondamentale.

Dans cette thèse on considère le problème de Steklov. Dans une première partie du travail, nous obtenons des bornes supérieures pour les valeurs propres de Steklov $\sigma_k(\Omega, B)$, où (Ω, B) est un sous-graphe d'un graphe hôte Γ. Le procédé utilisé ici consiste à construire une variété M à partir du sous-graphe, de telle façon que nous contrôlons $\sigma_k(M)$, et transférons l'information spectrale au sous-graphe grâce à une méthode appelée discrétisation. Le procédé nous permet de travailler dans deux différentes classes de graphes hôtes : une première classe est composée des graphes de Cayley de groupes à croissance polynomiale, et une seconde classe est composée de graphes de pavage triangulaire du plan hyperbolique. Dans une seconde partie du travail, nous obtenons des bornes supérieures optimales pour la première valeur propre de Steklov d'une hypersurface de révolution M à deux composantes connexes du bord de l'espace euclidien. Pour ce faire, nous comparons $\sigma_1(M)$ avec $\sigma_0^D(A)$ et $\sigma_1^N(A)$, les premières valeurs propres non triviales des problèmes mixtes Steklov-Dirichlet et Steklov-Neumann sur un anneau euclidien A. Afin d'étendre le résultat à toutes les valeurs propres, nous introduisons le concept de longueur critique finie et infinie, ce qui nous amène à faire des expériences numériques qui supportent une conjecture présentée dans le dernier chapitre.

Mots-clés: Géométrie spectrale, problème de Steklov, problème de Steklov discret, bornes supérieures, hypersurfaces de révolution, expériences numériques.

Abstract

One aim of spectral geometry is to understand the relationship between the geometry or topology of a Riemannian manifold and the spectrum of a differential Laplacian-type operator associated to that manifold. This concept is encapsulated in the famous sentence of Mark Kac

"Can one hear the shape of a drum?"

and constitutes an active topic of fundamental research.

In this thesis we consider the Steklov problem. In a first part of the dissertation, we find upper bounds for the Steklov eigenvalues $\sigma_k(\Omega, B)$, where (Ω, B) is a subgraph of a host graph Γ. The procedure used here consists in building a manifold M from the subgraph in such a way that we control $\sigma_k(M)$, and transfer the spectral piece of information to the subgraph thanks to a method called discretization. This procedure allows us to work on two different classes of host graphs: a first class consists of Cayley graphs of polynomial growth groups, and a second class consists of triangle-tiling graphs of the hyperbolic plane. In a second part of the dissertation, we find sharp upper bounds for the first Steklov eigenvalue of a hypersurface of revolution M with two boundary components of the Euclidean space. To do that, we compare $\sigma_1(M)$ with $\sigma_0^D(A)$ and $\sigma_1^N(A)$, the first non trivial eigenvalues of the mixed Steklov-Dirichlet and Steklov-Neumann problems on a Euclidean annulus A. To extend the result to every eigenvalue, we introduce the concept of finite and infinite critical lengths, which makes us perform some numerical experiments that support a conjecture presented in the last chapter.

Keywords: Spectral geometry, Steklov problem, discrete Steklov problem, upper bounds, hypersurfaces of revolution, numerical experiments.

Contents

Introduction

This introduction is intended to define what the mathematical objects that we study in this treatise are, set the context and the notation used all throughout the document and present our principal results. After that, the text takes the form of an article-based thesis. As such, there will be redundancies with the introduction of the articles that are coming. Although the last chapter is not a published nor submitted article, we keep with this format until this point, which sets the end of the dissertation.

The Steklov problem on a manifold with boundary

The classical Steklov problem, on a bounded domain of $\mathbb{R}^n$ and more generally on a manifold with boundary, was first introduced by Vladimir A. Steklov in a talk in a session of the Kharkov Mathematical Society in December 1895, see [26] for a historical discussion about Steklov himself and his work and influence on mathematical society. The study of the problem was later initiated by Weinstock in 1954 [37] and was followed during the years by many papers. We refer to [18] for an overview of the main results discovered before 2014, and [11] for a survey of the contributions made up until 2022.

Let us now define the Steklov problem.

We denote by (M, g) a smooth compact connected Riemannian manifold of dimension $n \geq 2$, with smooth boundary Σ.

We write Δ the Laplace-Beltrami operator acting on functions on M, i.e

$$\begin{aligned} \Delta : C^\infty(M) &\longrightarrow C^\infty(M) \\ f &\longmapsto \Delta f = -\operatorname{div}(\nabla f). \end{aligned}$$

We also write $\frac{\partial}{\partial \nu}$ for the outward normal derivative along Σ.

With this notation, the *Steklov problem* on (M, g) consists in finding all $\sigma \in \mathbb{R}$ such that there exists $0 \neq f : M \longrightarrow \mathbb{R}$, such that

$$\begin{cases} \Delta f = 0 & \text{on } M \\ \frac{\partial f}{\partial \nu} = \sigma f & \text{on } \Sigma. \end{cases}$$

If $\sigma \in \mathbb{R}$ and f satisfy the problem, we say that σ is a *Steklov eigenvalue* on (M, g) and that f is a *Steklov eigenfunction* associated with σ.

It is known [27, Chapter 7] that the Steklov eigenvalues form a sequence

$$0 = \sigma_0 < \sigma_1 \leq \sigma_2 \ldots \nearrow \infty,$$

where each eigenvalue is repeated with its multiplicity, which is finite. The multiplicity of $\sigma_0 = 0$ is equal to the number of connected components of M, which is one in our case. The set of all eigenvalues is known as the *Steklov spectrum* of (M, g).

Let us remark that some authors start numbering the eigenvalues from 1, meaning that for them, the first non trivial eigenvalue is σ_2. All along this dissertation, we use the convention presented here, namely, starting at $\sigma_0 = 0$. Actually, this will be the case for all eigenvalue problems that we will encounter.

A purpose of spectral geometry is to establish connections between the geometric properties of a manifold and its spectrum. We distinguish two major approaches.

1. *Direct problems:* given a manifold, or a set of manifolds satisfying certain geometric conditions, can we say something about their spectra? Can we find lower or upper bounds for their eigenvalues? For example, a famous result is due to Brock, who proves that among smooth bounded domains in $\mathbb{R}^d$ with fixed volume, σ_1 is maximized by the ball, see [5].

2. *Inverse problems:* given the spectrum of a manifold, can we provide information about its geometry? It is known that there exist manifolds which are isospectral but are not isometric, see for instance [11, Section 9]. However, there exist some Steklov isospectral invariants such as the Weyl law, see [11, Section 8]. Therefore, the knowledge of the spectrum gives geometric information, for instance the dimension of the manifold, or the volume of its boundary.

In this thesis, we only consider direct problems. In Chapter 1 and Chapter 2, we focus on finding upper bounds for the Steklov eigenvalues of some graphs, see the next section for relevant definitions. In Chapter 3 and Chapter 4, we find upper bounds for the first Steklov eigenvalue of some particular manifolds which are what we call *hypersurfaces of revolution* in Euclidean space.

In many cases, to estimate the eigenvalues, we do not work directly with the definition of the Steklov problem. Rather, we use a variational characterization of the eigenvalues: introducing the *Rayleigh quotient*

$$R_g(f) = \frac{\int_M |\nabla f|^2 dV_g}{\int_\Sigma f^2 dV_\Sigma},$$

we have

$$\sigma_k(M,g) = \min_{E \in \mathcal{H}_{k+1}} \max_{0 \neq f \in E} \left\{ R_g(f) : \int_\Sigma f^2 dV_\Sigma \neq 0 \right\},$$

where $\mathcal{H}_{k+1}$ is the set of all $(k+1)$-dimensional subspaces of the Sobolev space $H^1(M)$, see [27, Section 7.1.1].

Let us now describe another problem, that comes from the classical Steklov problem but defined on graphs, and that we call *discrete Steklov problem.*

The discrete Steklov problem

The discrete Steklov problem was introduced in the late 2010s independently in the articles [9], [21] and [24]. These articles were quickly followed by many others, such as [19, 22, 30, 31].

From a certain point of view, studying a discrete version of the problem is like considering a smooth manifold without deep local perturbation. Therefore, it is possible to study the discrete problem to get information about the classical problem, and it is possible to transfer some known results about the spectrum of manifolds to graphs *modeled* on them. That is the purpose of the article [9], and Chapter 1 and Chapter 2 of this document. Let us give details on the context.

For an introduction of the concepts about graphs and geometric groups theory that we will use, we refer to [20]. We denote by (V, E) a simple, connected, undirected graph. Then a *graph with boundary* is a triplet (V, E, B), where $B \subset V$ is a non empty subset of vertices called the boundary of the graph. Its complement B^c is called the interior of the graph. Remark that it is possible for the interior to be empty.

In this thesis, all graphs with boundary are finite.

We denote by $\mathbb{R}^V$ the space of all functions $V \longrightarrow \mathbb{R}$, which is isomorphic to the Euclidean space $\mathbb{R}^{|V|}$. Then the *discrete Laplacian* is $\Delta : \mathbb{R}^V \longrightarrow \mathbb{R}^V$, defined for $u \in \mathbb{R}^V$ by

$$\begin{aligned} \Delta u : V &\longrightarrow \mathbb{R} \\ i &\longmapsto \Delta u(i) = \sum_{j\sim i}(u(i) - u(j)), \end{aligned}$$

where $i \sim j \iff \{i, j\} \in E$.

Similarly, the *normal derivative* $\frac{\partial}{\partial \nu} : \mathbb{R}^V \longrightarrow \mathbb{R}^B$ is defined by

$$\begin{aligned} \frac{\partial u}{\partial \nu} : B &\longrightarrow \mathbb{R} \\ i &\longmapsto \frac{\partial u}{\partial \nu}(i) = \sum_{j\sim i}(u(i) - u(j)). \end{aligned}$$

With this setting, the *Steklov problem* on (V, E, B) consists in finding all $\sigma \in \mathbb{R}$ such that there exists $0 \neq u \in \mathbb{R}^V$, such that

$$\begin{cases} \Delta u(i) = 0 & \text{if } i \in B^c \\ \frac{\partial u}{\partial \nu}(i) = \sigma u & \text{if } i \in B. \end{cases}$$

Such a σ is called a *Steklov eigenvalue* of (V, E, B), and the set of all eigenvalues is called the *Steklov spectrum* of the graph.

The Steklov spectrum of (V, E, B) forms a sequence

$$0 = \sigma_0 < \sigma_1 \leq \ldots \leq \sigma_{|B|-1},$$

so there are exactly $|B|$ eigenvalues, counted with multiplicity, and the very first one is $\sigma_0 = 0$, known as the trivial eigenvalue, associated with the space of constant functions.

Denoting

$$q(u) := \sum_{j\sim i}(u(i) - u(j))^2$$

the *Dirichlet energy* of $u \in \mathbb{R}^V$,

$$\|u\|_B^2 := \sum_{j \in B} u(j)^2,$$

and

$$R(u) := \frac{q(u)}{\|u\|_B^2}$$

the *Rayleigh quotient* of u, the eigenvalues satisfy the following Min-Max characterization:

$$\sigma_k = \min_{E \in \mathcal{H}_{k+1}} \max_{u \in E} \left\{ R(u) : \|u\|_B \neq 0 \right\},$$

where $\mathcal{H}_{k+1}$ is the set of all linear subspaces of $\mathbb{R}^V$ of dimension $k+1$.

Even if we can define the Steklov problem on any graph with boundary, we will focus on some particular ones, those who are called *subgraphs* of an infinite *host graph*. Let us present what it is.

Let us denote $\Gamma = (V, E)$ an infinite simple, connected, undirected graph. Let $\Omega \subset V$ be a finite subset of vertices, which are connected for Γ (meaning that for all $i, j \in \Omega$, there is a path joining i and j that consists of vertices of Ω). This choice of Ω induces a graph with boundary $(\bar{\Omega}, E', B)$ that is included in Γ, defined as follows:

1. $B = \{ j \in V \backslash \Omega : \text{ there exists } i \in \Omega \text{ such that } \{i, j\} \in E \}$;
2. $\bar{\Omega} = \Omega \cup B$;
3. $E' = \{ \{i, j\} \in E : i \in \Omega, j \in \bar{\Omega} \}$.

Such a graph is called *subgraph* of the *host graph* Γ.

In Chapter 1, subgraphs will be denoted (Ω, B) while in Chapter 2, they will simply be denoted Ω. This difference is due to the comments made by the different referees of the articles.

As we will see in the results section, some particular choices of host graphs lead to interesting geometric constructions, that allow us to find upper bounds for their subgraphs Steklov eigenvalues.

Main results

We present in this section the main results that we prove in the thesis. We also discuss the questions that naturally arise from our research.

The discrete Steklov problem on polynomial growth Cayley graphs

In the context of the discrete Steklov problem, we define a class of host graphs that we are going to study.

Let G be a finitely generated infinite discrete group, and $S \subset G$ be a symmetric, finite generating set. Let us suppose that the neutral element $e \in G$ does not belong to S. Then the *Cayley graph* $Cay(G, S)$ associated is the graph defined by

1. The set V of vertices is $V = G$;
2. The set of edges is $E := \{\{i, j\} : i, j \in V :$ there exists $s \in S$ such that $j = is\}$.

This graph is then regular (each vertex has degree $|S|$), is infinite since G is, has no loop (since $e \notin S$), no multiple edges, and is undirected since we chose $S = S^{-1}$. Then it can be used as a host graph and we can study the discrete Steklov problem on its subgraphs.

We can then endow $Cay(G, S)$ with the path distance to see it as a metric space, which allows us to define its *growth function* $\mathbb{N} \longrightarrow \mathbb{N}$ by $n \longmapsto |B(n)|$, where $B(n)$ is the ball of radius n centered at $e \in V$.

Then we say that $Cay(G, S)$ is a *polynomial growth* Cayley graph of order $d \geq 1$ if there exists a constant $C \geq 1$ such that for all $n \in \mathbb{N}$, we have

$$C^{-1} \cdot n^d \leq |B(n)| \leq C \cdot n^d.$$

In this context, a first result due to Han and Hua [19] is the following:

Theorem (Han, Hua, 2019)**.** *Let $\mathbb{Z}^d$ be the integer lattice of dimension d. Let (Ω, B) be a subgraph of $\mathbb{Z}^d$. Then we have*

$$\sum_{l=1}^{d} \frac{1}{\sigma_l(\Omega, B)} \geq \bar{C} \cdot |\Omega|^{\frac{1}{d}} - \frac{C'}{|\Omega|},$$

where $\bar{C} = (64d^3\omega_d^{\frac{1}{d}})^{-1}, C' = \frac{1}{32d}$ and ω_d is the volume of the unit ball in $\mathbb{R}^d$.

In this theorem, the choice of host graphs is very restricted as it only concerns the integer lattices $\mathbb{Z}^d$. The proof consists in a subtle construction involving hypercubes of $\mathbb{R}^d$, since we can of course embed the integer lattices in Euclidean spaces.

The result has later then been partially extended by Perrin in [30], who proves the following:

Theorem (Perrin, 2020)**.** *Let $\Gamma = (V, E)$ be a Cayley graph with polynomial growth of order $d \geq 2$. There exists $\tilde{C}(\Gamma) > 0$ such that for any finite subgraph (Ω, B) of Γ, we have*

$$\sigma_1(\Omega, B) \leq \tilde{C}(\Gamma) \cdot \frac{1}{|B|^{\frac{1}{d-1}}}.$$

This time, the class of host graphs considered is much wider, however the result concerns only the first Steklov eigenvalue. The proof is much more accessible since Perrin was able to work directly on the graph, without having to do specific geometric constructions.

With these two results in mind, a natural question arises:

Can we extend our control to every eigenvalue while staying in the same general setting of polynomial growth Cayley graphs?

This question motivated some investigations which lead to our first result:

Theorem (Theorem 1.6)**.** *Let* $\Gamma = Cay(G, S)$ *be a polynomial growth Cayley graph of order* $d \geq 2$*. Let* (Ω, B) *be a subgraph of* Γ*. Then there exists a constant* $C(\Gamma) > 0$ *such that for all* $k < |B|$*, we have*

$$\sigma_k(\Omega, B) \leq C(\Gamma) \cdot \frac{1}{|B|^{\frac{1}{d-1}}} \cdot k^{\frac{d+2}{d}}.$$

This result has an immediate interesting consequence:

Corollary (Corollary 1.7)**.** *Let* Γ *be a polynomial growth Cayley graph of order* $d \geq 2$ *and* $(\Omega_l, B_l)_{l=1}^{\infty}$ *be a sequence of subgraphs of* Γ *such that* $|\Omega_l| \underset{l\to\infty}{\longrightarrow} \infty$*. Fix* $k \in \mathbb{N}$*. Then we have*

$$\sigma_k(\Omega_l, B_l) \underset{l\to\infty}{\longrightarrow} 0.$$

The proof of these results lies within the spectral link that exists between a subgraph of a polynomial growth Cayley graph and a well-chosen domain inside a manifold modeled on the Cayley graph. The spectral link can be precisely defined thanks to [9, Theorem 3 point 4)], and used to give the upper bound of the result above.

The discrete Steklov problem on triangle-tiling graphs

We remark that the previous theorems concern host graphs which have polynomial growth. It is therefore natural to investigate if the growth rate is necessary to find some Steklov upper bounds. This consideration lead He and Hua in [22] to the following:

Theorem (He, Hua, 2022)**.** *Let* $\mathcal{T}$ *be a finite tree with (uniformly) bounded degree* D*. Let* B *be the boundary of the tree, i.e the set of vertices of degree one. Then we have*

$$\sigma_1 \leq \frac{4(D-1)}{|B|}.$$

Higher Steklov eigenvalues are bounded as well: for all $k = 2, \ldots, |B| - 1$*, we have*

$$\sigma_k \leq \frac{8(D-1)^2(k-1)}{|B|}.$$

This result shows that it is possible to find Steklov upper bounds for host graphs which do not possess the polynomial growth rate assumption. Indeed, the finite tree of the theorem can for example be chosen as the subgraph of a free group Cayley graph.

Another natural way to build a host graph with exponential growth rate consists in considering a kind of discretization of the hyperbolic plane. Let us clarify this idea.

We work in the hyperbolic plane, represented here by Poincaré's disc model:

$$\mathbb{H}^2 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

equipped with the Riemannian metric

$$g(x, y) = 4 \cdot \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}.$$

Then, for three integers $p, q, r \geq 2$, the *triangle group* associated is the one presented by

$$T^*(p, q, r) = \langle P, Q, R : P^2 = Q^2 = R^2 = (PQ)^r = (QR)^p = (RP)^q = 1 \rangle.$$

If p, q, r satisfy $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, then there exists a triangle in the hyperbolic plane $\mathbb{H}^2$ with angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$, which is unique up to isometry, see [2, Exercise 7.12]. Moreover, as stated in [29, Theorem 2.8], the images of this triangle under the action of the distinct elements of the group $T^*(p, q, r)$ produces a tessellation of $\mathbb{H}^2$, see Figure 1.

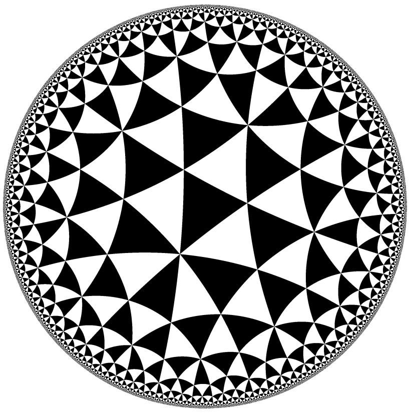

Figure 1: Tessellation of the hyperbolic plane with congruent triangles of angles $\frac{\pi}{3}, \frac{\pi}{3}$ and $\frac{\pi}{4}$, coming from [38].

This construction allows us to define a host graph Γ as follows:

1. The set of vertices V is the set of triangles;
2. Two triangles are called adjacent if they have a common edge.

We call Γ a *triangle-tiling graph*. Because of its connection with $\mathbb{H}^2$, this host graph Γ has an exponential growth rate. Investigating it leads to the following results:

Theorem (Theorem 2.9)**.** *Let Γ be a triangle-tiling graph. Then there exists a constant $C(\Gamma) > 0$ such that for all subgraphs (Ω, B) of Γ and all $k < |B|$, we have*

$$\sigma_k(\Omega) \leq C(\Gamma) \cdot \frac{1}{|B|} \cdot k^2.$$

The proof consists in finding a domain of $\mathbb{H}^2$ that reflects the structural information of the subgraph, then building a manifold from the domain in a way that we can control the spectrum of the manifold and such that we can use [9, Theorem 3 point 4)] to transfer the spectral information to the subgraph.

Once again, the proof of our result shows the deep connection that exists between the discrete Steklov problem on a graph and the classical Steklov problem on a manifold modeled on the graph.

As an immediate consequence of this result, we have

Corollary (Corollary 2.10)**.** *Let* $(\Omega_l, B_l)_{l\geq 1}$ *be a family of subgraphs of* Γ *which satisfy* $|\Omega_l| \underset{l\to\infty}{\longrightarrow} \infty$. *Then for all* $k \in \mathbb{N}$ *fixed,*

$$\sigma_k(\Omega_l, B_l) \underset{l\to\infty}{\longrightarrow} 0.$$

As we can see, some host graphs have this interesting feature that, if (Ω, B) is a subgraph such that $|\Omega|$ is large, then for $k \in \mathbb{N}$ fixed, $\sigma_k(\Omega, B)$ is small.

This observation leads to the following definition: we say that a host graph Γ has the *property (S)* if for each $k \in \mathbb{N}$ and each family $(\Omega_l, B_l)_{l\geq 1}$ of subgraphs of Γ, we have

$$|\Omega_l| \underset{l\to\infty}{\longrightarrow} \infty \implies \sigma_k(\Omega_l, B_l) \underset{l\to\infty}{\longrightarrow} 0.$$

We know many host graphs that do not have property (S), see Appendix 2.B. However, until now, we never encountered two roughly isometric graphs (see Definition 1.29) such that one has (S) but the other does not. This observation leads to open Question 2.36:

If Γ_1 *and* Γ_2 *are roughly isometric host graphs and* Γ_1 *has the property (S), does* Γ_2 *also have the property (S)?*

The Steklov problem on hypersurfaces of revolution

After having worked on the discrete Steklov problem in Chapter 1 and Chapter 2, we focus on the classical Steklov problem on manifolds in Chapter 3 and Chapter 4.

To obtain upper bounds, it is necessary to consider a class of manifolds that satisfy some geometric constraints, see [6, Theorem 1.1]. One of them was investigated by Colbois et Verma in [10]. They considered hypersurfaces of revolution of the Euclidean space with one boundary component, and come to the following:

Theorem (Colbois, Verma, 2021)**.** *Let* $M \subset \mathbb{R}^{n+1}$ *be a hypersurface of revolution with one boundary component isometric to* $\mathbb{S}^{n-1}$ *with* $n \geq 3$. *Then for each* $k \geq 1$, *we have*

$$\sigma_{(k)}(M) < k + n - 2,$$

where $\sigma_{(k)}(M)$ *is the kth distinct Steklov eigenvalue of* M. *Although there exists no equality case within the collection of hypersurfaces of revolution, this upper bound is sharp.*

The result naturally raises the following question:

Can we find an upper bound for hypersurfaces of revolution of the Euclidean space with two boundary components?

Let us define what those hypersurfaces of revolution are precisely.

As usual, we denote by $\mathbb{S}^{n-1}$ the unit sphere of dimension $n-1$, and we fix $L > 0$. Then we consider a map

$$\begin{aligned} \Psi : \mathbb{S}^{n-1} \times [0, L] &\longrightarrow \mathbb{R}^{n+1} \\ (p, r) &\longmapsto h(r)p + z(r)e_{n+1}, \end{aligned}$$

where $h, z : [0, L] \longrightarrow \mathbb{R}$ are smooth and satisfy

$$h(0) = h(L) = 1, \quad h > 0, \quad h'^2 + z'^2 \equiv 1.$$

The image of Ψ is called a *hypersurface of revolution* of the Euclidean space with two boundary components each isometric to $\mathbb{S}^{n-1}$. As explained in [8, Section 3.1], we can see such a hypersurface as the warped product $M := [0, L] \times \mathbb{S}^{n-1}$ endowed with the Riemannian metric g given by

$$g(r, p) = dr^2 + h^2(r)g_0(p),$$

where

1. g_0 is the canonical metric on $\mathbb{S}^{n-1}$;
2. $h : [0, L] \longrightarrow \mathbb{R}_+^*$ is a smooth function satisfying
    a) $h(0) = h(L) = 1$;
    b) $|h'| \leq 1$.

With this setting, we simply say that (M, g) is a *hypersurface of revolution* of $\mathbb{R}^{n+1}$, we say that g is a *metric of revolution* on M, and we say that $L > 0$ is the *meridian length* of M.

We are interested in finding upper bounds for the Steklov eigenvalues of (M, g). Let us state our first result:

Theorem (Theorem 3.2). *Let* $(M = [0, L] \times \mathbb{S}^{n-1}, g_1)$ *be a hypersurface of revolution in Euclidean space with two boundary components each isometric to* $\mathbb{S}^{n-1}$ *and meridian length* L*. We suppose* $n \geq 3$*. Then there exists a metric of revolution* g_2 *on* M *such that for each* $k \geq 1$,

$$\sigma_k(M, g_1) < \sigma_k(M, g_2).$$

This result implies that no metric of revolution on M maximizes the eigenvalues. However, there exists a *degenerated maximizing metric* g^* on M that we can look for and that gives a sharp upper bound $B_n^k(L)$ such that for all metrics of revolution g on M, we have

$$\sigma_k(M, g) < B_n^k(L) = \sigma_k(M, g^*).$$

Finding an explicit expression for $B_n^k(L)$ appears to be difficult, so we start by investigating the case $k = 1$. We get the following results:

Theorem (Theorem 3.3)**.** *Let* $(M = [0, L] \times \mathbb{S}^{n-1}, g)$ *be a hypersurface of revolution in Euclidean space with two boundary components each isometric to* $\mathbb{S}^{n-1}$ *and dimension* $n \geq 3$*. Then the first non trivial Steklov eigenvalue* $\sigma_1(M, g)$ *is bounded above, by a bound that depends only on the dimension n and the meridian length L of M:*

$$\sigma_1(M, g) < B_n(L) := \min\left\{ \frac{(n-2)\,(1+L/2)^{n-2}}{(1+L/2)^{n-2}-1}, \frac{(n-1)\,((1+L/2)^n-1)}{(1+L/2)^n+n-1} \right\}.$$

Moreover, this bound is sharp: for each $\varepsilon > 0$*, there exists a metric of revolution* g_ε *on M such that* $\sigma_1(M, g_\varepsilon) > B_n(L) - \varepsilon$*.*

As a consequence, we get another sharp upper bound that depends only on the dimension n of the hypersurface M:

Corollary (Corollary 3.4)**.** *Let* $n \geq 3$*. Then there exists a bound* $B_n < \infty$ *such that for all hypersurfaces of revolution* (M, g) *in Euclidean space with two boundary components each isometric to* $\mathbb{S}^{n-1}$*, we have*

$$\sigma_1(M, g) < B_n := \frac{(n-2)\left(1+\frac{L_1}{2}\right)^{n-2}}{\left(1+\frac{L_1}{2}\right)^{n-2}-1},$$

where L_1 *is the unique real positive solution of the equation*

$$(1+L/2)^{2n-2} - (n-1)\,(1+L/2)^n - (n-1)^2\,(1+L/2)^{n-2} + n - 1 = 0.$$

Moreover, this bound is sharp: for each $\varepsilon > 0$*, there exists a hypersurface of revolution with two boundary components each isometric to a unit sphere* $(M_\varepsilon, g_\varepsilon)$ *such that* $\sigma_1(M_\varepsilon, g_\varepsilon) > B_n - \varepsilon$*.*

Though these results concern a problem that is slightly different from the one that Colbois and Verma were interested in in [10, Question 8], it allows us to give information about it, see Appendix 3.B.

This corollary justifies the following definition: we say that L_1 is a *finite critical length* associated with $k = 1$. More generally, if we call

$$B_n^k := \sup\{B_n^k(L) : L \in \mathbb{R}_+^*\},$$

we say that $L_k \in \mathbb{R}_+^*$ is a *finite critical length* associated with k if we have $B_n^k = B_n^k(L_k)$. We say that k has a *critical length at infinity* if we have $B_n^k = \lim_{L\to\infty} B_n^k(L)$.

Because we want to give upper bounds for $\sigma_k(M, g)$, it is then natural to ask what qualitative and quantitative information we can provide about the set of finite critical lengths. We obtain the following result:

Theorem (Theorem 3.9)**.** *Let* $n \geq 3$*. Then there exist infinitely many* $k \in \mathbb{N}$ *which have an associated finite critical length. Moreover, if we call* $(k_i)_{i=1}^\infty \subset \mathbb{N}$ *the increasing sequence* *of such k and if we call $(L_i)_{i=1}^{\infty}$ the associated sequence of finite critical lengths, then we have*

$$\lim_{i\to\infty} L_i = 0.$$

The natural question that arises from this statement is open Question 3.22:

Given $n \geq 3$, are there finitely or infinitely many $k \in \mathbb{N}$ which have critical lengths at infinity?

Although this question is still open, we performed numerical experiments whose results can be summed up in the following table:

n	Result of the program investigations
3	From $k = 18$ to $k = 45'601$, only finite critical lengths found.
4	From $k = 408$ to $k = 47'641$, only finite critical lengths found.
5	From $k = 8'400$ to $k = 88'451$, only finite critical lengths found.
6	From $k = 21'112$ to $k = 119'965$, only finite critical lengths found.

From this table, we state the following conjecture that is supported by the program:

Conjecture (Conjecture 4.8)**.** Let $n \geq 3$ be an integer, then there exists a constant $K(n) \in \mathbb{N}$ such that for all $k \geq K(n)$, the kth eigenvalue has an associated finite critical length.

Chapter 1

Upper bounds for Steklov eigenvalues of subgraphs of polynomial growth Cayley graphs

This paper has already been published in *Annals of Global Analysis and Geometry*, see [36]. The reader may encounter a few differences with the original text, due to some minor adjustments.

Abstract

We study the Steklov problem on a subgraph with boundary (Ω, B) of a polynomial growth Cayley graph Γ. We prove that for each $k \in \mathbb{N}$, the kth eigenvalue tends to 0 proportionally to $1/|B|^{\frac{1}{d-1}}$, where d represents the growth rate of Γ. The method consists in associating a manifold M to Γ and a bounded domain $N \subset M$ to a subgraph (Ω, B) of Γ. We find upper bounds for the Steklov spectrum of N and transfer these bounds to (Ω, B) by discretizing N and using comparison theorems.

Keywords: Spectral geometry, Steklov problem, graphs with boundary, polynomial growth Cayley graphs.

1.1 Introduction

HERE, on a smooth compact Riemannian manifold M of dimension $d \geq 2$ with a smooth boundary ∂M, the Steklov problem on M consists in finding all $\sigma \in \mathbb{R}$ such that there exists a non trivial function u satisfying

$$\begin{cases} \Delta u = 0 & \text{on} \quad M \\ \frac{\partial u}{\partial \nu} = \sigma u & \text{on} \quad \partial M, \end{cases}$$

where Δ is the Laplace-Beltrami operator acting on functions on M, and $\frac{\partial}{\partial \nu}$ is the outward normal derivative along ∂M. It is well known that the Steklov spectrum is discrete and forms a sequence such as

$$0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \ldots \nearrow \infty.$$

Among the interesting questions in the study of the eigenvalues of the Steklov problem lies the one consisting in wondering which domain of the Euclidean space maximizes the eigenvalues, and more generally, to get upper bounds, using certain assumptions such as a predefined volume. In [37], Weinstock proves that for simply connected planar domains with an analytic boundary and assigned perimeter, the disk maximizes the first Steklov eigenvalue. In [5], Brock proves that for smooth bounded domains in $\mathbb{R}^d$ with prescribed volume, σ_1 is maximized by the ball. Among other things, in [7] upper bounds for all the eigenvalues of domains of the Euclidean space were given.

As it is already done for the Laplace operator, one can define a discrete Steklov problem, which is a problem defined on graphs with boundary and similar to the Steklov problem defined above. This problem has recently been investigated by various authors, such as [9, 19, 24, 30, 31]. This paper will focus on the discrete Steklov problem; let us begin by describing it briefly.

Definition 1.1. A graph with boundary is a ordered pair (Γ, B), where $\Gamma = (V, E)$ is a simple (that is without loops or multiple edges) connected undirected graph, and $B \subset V$ is a non empty subset of vertices called the boundary. The subset of vertices B^c is called the interior of Γ.

For $i, j \in V$, we write $i \sim j$ when i is adjacent to j, meaning that $\{i, j\} \in E$. For $\Omega \subset V$, we denote $|\Omega|$ the cardinal of Ω, which is the number of vertices contained by Ω. In this paper, all graphs with boundary are finite. The space of real functions u defined on the vertices V is denoted by $\mathbb{R}^V$, it is the Euclidean space of dimension $|V|$. If a real function u is defined only on the boundary B, we will say that $u \in \mathbb{R}^B$, and it corresponds to the Euclidean space of dimension $|B|$. For $u, v \in \mathbb{R}^V$, the scalar product $\langle u, v\rangle$ is the usual scalar product in $\mathbb{R}^{|V|}$. We now introduce the Laplacian operator $\Delta : \mathbb{R}^V \longrightarrow \mathbb{R}^V$, defined by

$$\begin{aligned} \Delta u : V &\longrightarrow \mathbb{R} \\ i &\longmapsto \Delta u(i) = \sum_{j \sim i} (u(i) - u(j)), \end{aligned}$$

as well as the normal derivative $\frac{\partial}{\partial v} : \mathbb{R}^V \longrightarrow \mathbb{R}^B$, defined by

$$\begin{aligned} \frac{\partial u}{\partial v} : B &\longrightarrow \mathbb{R} \\ i &\longmapsto \frac{\partial u}{\partial v}(i) = \sum_{j \sim i} (u(i) - u(j)). \end{aligned}$$

Definition 1.2. The Steklov problem on a graph with boundary (Γ, B) consists in finding all $\sigma \in \mathbb{R}$ such that there exists a non trivial function $u \in \mathbb{R}^V$ satisfying

$$\begin{cases} \Delta u(i) = 0 & \text{if} \quad i \in B^c \\ \frac{\partial u}{\partial v}(i) = \sigma u(i) & \text{if} \quad i \in B. \end{cases}$$

Such a σ is called a Steklov eigenvalue of (Γ, B).

As explained in [31], the Steklov spectrum forms a sequence as follows

$$0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_{|B|-1}.$$

Recent interesting outcomes related to this problem include the following :
In [24], the authors find a Cheeger type inequality for the first non trivial eigenvalue. In [31], the author finds a lower bound for the first non trivial Steklov eigenvalue. Lower bounds for higher Steklov eigenvalues are given in [21].

In [9], the authors described a process, called a discretization, permitting to associate a graph with boundary to a Riemannian manifold and showed some spectral bonds between a manifold and its discretization.

Definition 1.3. Given a graph $\Gamma = (V, E)$ and a finite connected subset $\Omega \subset V$, one can define a graph with boundary $(\bar{\Omega}, E', B)$ included in Γ, by saying

- $B = \{j \in V \backslash \Omega : \exists\, i \in \Omega$ such that $\{i, j\} \in E\}$;
- $\bar{\Omega} = \Omega \cup B$;
- $E' = \{\{i, j\} \in E : i \in \Omega, j \in \bar{\Omega}\}$.

Such a graph is called a subgraph of Γ and is denoted (Ω, B), with Ω the interior and B the boundary.

Following the work of Brock cited above, Han and Hua found in [19] a similar result for subgraphs of the integer lattices:

Theorem 1.4 (Han, Hua, 2019)**.** *Let* $\mathbb{Z}^d$ *be the integer lattice of dimension* d*. Let* (Ω, B) *be a subgraph of* $\mathbb{Z}^d$*. Then we have*

$$\sum_{l=1}^{d} \frac{1}{\sigma_l(\Omega, B)} \geq \bar{C} \cdot |\Omega|^{\frac{1}{d}} - \frac{C'}{|\Omega|},$$

where $\bar{C} = (64 d^3 \omega_d^{\frac{1}{d}})^{-1}, C' = \frac{1}{32d}$ *and* ω_d *is the volume of the unit ball in* $\mathbb{R}^d$*.*

This theorem gives us control over the d first Steklov eigenvalues of a subgraph (Ω, B) of $\mathbb{Z}^d$ and leads to an interesting consequence: for any sequence of subgraphs $(\Omega_l, B_l)_{l=1}^{\infty}$ of Γ such that $|\Omega_l| \underset{l\to\infty}{\longrightarrow} \infty$, we have $\sigma_1(\Omega_l, B_l) \underset{l\to\infty}{\longrightarrow} 0$.

However, unlike Brock, Han and Hua do not get an equality case.

The result of Han and Hua has then been partially extended by Perrin, who found an isoperimetric upper bound for the first eigenvalue of subgraphs of any polynomial growth Cayley graph, see [30]. We will recall in Section 1.2 what a polynomial growth Cayley graph is and the other notions of geometric group theory which are necessary for the understanding of the paper.

Theorem 1.5 (Perrin, 2020)**.** *Let* $\Gamma = (V, E)$ *be a Cayley graph with polynomial growth of order* $d \geq 2$*. There exists* $\tilde{C}(\Gamma) > 0$ *such that for any finite subgraph* (Ω, B) *of* Γ*, we have*

$$\sigma_1(\Omega, B) \leq \tilde{C}(\Gamma) \cdot \frac{1}{|B|^{\frac{1}{d-1}}}.$$

This theorem gives us control over the first Steklov eigenvalue of a subgraph of any polynomial growth Cayley graph and leads to the same consequence as the previous theorem.

It is therefore natural to wonder whether it is possible to extend our control to all of the eigenvalues of a subgraph of any polynomial growth Cayley graph.

The main result of this paper is the following:

Theorem 1.6. *Let* $\Gamma = Cay(G, S)$ *be a polynomial growth Cayley graph of order* $d \geq 2$. *Let* (Ω, B) *be a subgraph of* Γ. *Then there exists a constant* $C(\Gamma) > 0$ *such that for all* $k < |B|$,

$$\sigma_k(\Omega, B) \leq C(\Gamma) \cdot \frac{1}{|B|^{\frac{1}{d-1}}} \cdot k^{\frac{d+2}{d}}.$$

One can observe that the bound depends on the cardinal of the boundary in the same way as Perrin's one in Theorem 1.5.
This leads to a consequence that extends the one of Theorems 1.4 and 1.5:

Corollary 1.7. *Let* Γ *be a polynomial growth Cayley graph of order* $d \geq 2$ *and* $(\Omega_l, B_l)_{l=1}^{\infty}$ *be a sequence of subgraphs of* Γ *such that* $|\Omega_l| \underset{l \to \infty}{\longrightarrow} \infty$. *Fix* $k \in \mathbb{N}$. *Then we have*

$$\sigma_k(\Omega_l, B_l) \underset{l \to \infty}{\longrightarrow} 0.$$

Of course, the number $\sigma_k(\Omega_l, B_l)$ is defined as long as $k < |B_l|$, which is the case for an l big enough due to the assumption that $|\Omega_l| \underset{l \to \infty}{\longrightarrow} \infty$, see Proposition 1.10 for more details.

Our approach to proving this is completely different from [30] and looks more like [19], in the sense that we do not work directly on graphs. However, the detour we make is not the same as the one made by [19]. We use tools described by Colbois, Girouard and Raveendran in [9] to build a manifold associated to Γ and a bounded domain associated to (Ω, B). We then use results from Colbois, El Soufi and Girouard [7] to give upper bounds for the Steklov eigenvalues of the domain, then apply theorems of [9] to transfer these upper bounds to the subgraph (Ω, B) by discretizing the domain into a graph with boundary that corresponds to (Ω, B).

Notation. Throughout the paper, we shall work on graphs and on manifolds. Graphs are denoted $\Gamma = (V, E)$ and manifolds are denoted M. The ordered pair (M, g) means that M is endowed with a Riemannian metric g and we use $|\cdot|_g$ to denote the Riemannian volume of a subset of M, as well as d_M to denote the distance on M. We denote by (N, Σ) a bounded domain of M, with N the interior and Σ the boundary. We shall use the variables e, i, j, v to speak about vertices of graphs and x, y, z for points on manifolds. Several constants will appear, we shall call them $C_1, C_2, \ldots$; each C_l is used exactly once.

Plan of the paper. In Section 1.2 we recall some definitions and results about geometric group theory that are needed for the constructions that will follow. In Section 1.3 we build a manifold M that is modeled on a Cayley graph Γ and prove some Propositions that will allow us to use results that we need on M. In Section 1.4 we prove Theorem 1.6: we first explain how to associate a bounded domain $N \subset M$ to a subgraph (Ω, B) of Γ and we use it to obtain an upper bound for the Steklov eigenvalues of (Ω, B), which will allow us to conclude.

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1.2 Cayley graphs and isoperimetric inequality

Because we will work with graphs with boundary that are subgraphs of a polynomial growth Cayley graph, we recall here some definitions that we will use, as well as some properties satisfied by these graphs. For further details, see [20].

Let G be a finitely generated infinite discrete group and $S \subset G$ be a symmetric ($S = S^{-1}$), finite generating subset such that $e \notin S$, where e denotes the identity of G. The Cayley graph associated to (G, S) is an infinite connected undirected simple graph $Cay(G, S) = (V, E)$ of vertices $V = G$ endowed with the graph structure $E = \{\{i, j\} : i, j \in V \ : \ \exists\, s \in S$ such that $j = is\}$.

Remark 1.8. Hence a Cayley graph is regular, each vertex i has degree $d(i) = |S|$.

A graph is a metric space when endowed with the path distance. A path is a sequence of vertices $i_1 \sim i_2 \sim \ldots \sim i_{n+1}$. If $i = i_1 \sim i_2 \sim \ldots \sim i_{n+1} = j$ is a minimal path joining i to j, then we say that the distance between i and j is n. In particular, for $i, j \in Cay(G, S)$ such that $i \sim j$, we have $d(i, j) = 1$.

We denote by $B(n)$ the ball of radius n centered at $e \in Cay(G, S)$. The growth function of $Cay(G, S)$ is defined by $V(n) = |B(n)|$. If there exists $d \in \mathbb{N}$ and $C_1 \geq 1$ such that for all $n \in \mathbb{N}$, we have

$$C_1^{-1} \cdot n^d \leq V(n) \leq C_1 \cdot n^d,$$

we say that $Cay(G, S)$ is a Cayley graph with a growth rate that is polynomial of order d. It is well known that the growth rate does not depend on the choice of the subset S ([20], chapter VI). Hence we can speak about the growth rate of the group G.

Example 1.9. Let $G = \mathbb{Z}^d$ and $S = \{(\pm 1, 0, \ldots, 0), (0, \pm 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1)\}$. Then $Cay(G, S)$ is a Cayley graph with polynomial growth rate of order d, called the integer lattice of dimension d and simply denoted by $\mathbb{Z}^d$.

Proposition 1.10. *Let G be a group with polynomial growth rate of order d, S a finite symmetric generating set, $e \notin S$ and $Cay(G, S)$ as above. Let (Ω, B) be a subgraph of $Cay(G, S)$. Then there exists a constant C_2 depending only on G and S such that*

$$\frac{|\bar{\Omega}|^{\frac{d-1}{d}}}{|B|} \leq C_2.$$

This isoperimetric control will be very useful to conclude our proof of Theorem 1.6. For a proof of this proposition, one can see Theorem 1 and its illustration in the example below in [13].
From here, we assume $d \geq 2$. This will guarantee that if $|\bar{\Omega}|$ is big, then so is $|B|$.

1.3 Manifold modeled on graphs

In order to use the results presented in [7], we have to work on manifolds. This section is devoted to explaining how we can associate a manifold M to a Cayley graph Γ, in such a way that a subgraph (Ω, B) of Γ corresponds to a discretization of a bounded domain N of M. The idea comes from the work of Colbois, Girouard and Raveendran, see [9], where they construct manifolds with some desired properties that these manifolds share with their discretizations. We shall now explain how to construct a manifold that is modeled on a Cayley graph.

Let $\Gamma = (V, E) = Cay(G, S)$ be a Cayley graph. We build what we call a *fundamental piece* (P, g_0), that is a smooth compact d-dimensional Riemannian manifold with $|S|$ boundary components, homeomorphic to $\mathbb{S}^d$ with $|S|$ holes. Each boundary component possesses a neighborhood which is isometric to the cylinder $[0, 2] \times \mathbb{S}^{d-1}$, with the boundary corresponding to $\{0\} \times \mathbb{S}^{d-1}$, as seen in Figure 1.1. On the cylindrical neighborhood of the boundary, g_0 is expressed as a product metric.

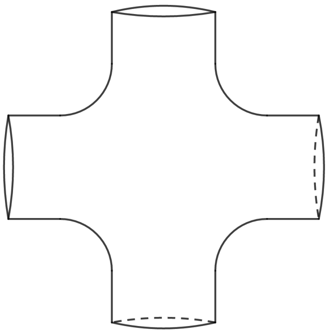

Figure 1.1: A fundamental piece associated with the lattice $\mathbb{Z}^2$.

Remark 1.11. Outside the 2-neighborhood of the boundary, we do not specify the geometry of P. The only thing we impose is that the piece is smooth. In Section 1.4.1, we will particularize the geometry of our piece, but since this specification is not yet relevant, we do not emphasize it now.

From this fundamental piece P, we construct a smooth unbounded complete d-dimensional Riemannian manifold (M, g), gluing infinitely many copies of P. For each vertex $i \in V$ we add one copy of P, denoted P_i. We call P_i a piece of M. It is obvious that these pieces can be glued smoothly along their boundary because of the boundary's cylindrical neighborhood, see Figure 1.2. The metric g comes from the metric g_0 and g is smooth because for each gluing part of M there is a neighborhood isometric to the cylinder $[-2, 2] \times \mathbb{S}^{d-1}$, with the gluing part corresponding to $\{0\} \times \mathbb{S}^{d-1}$ and where g can be expressed as a product metric. As said before, on $[-2, 0] \times \mathbb{S}^{d-1}$ and on $[0, 2] \times \mathbb{S}^{d-1}$, g_0 is a product metric. Hence, g is smooth.

Remark 1.12. We do not specify which diffeomorphism is used to glue the pieces together, hence the manifold M is not entirely well defined. This is not a problem for us, the properties we need about M and that we shall prove are verified by any element of the family of manifolds described by our process. We pick one and call it M for the purpose of this paper.

Because $|V| = \infty$, M is unbounded. Because the number of boundary components of P is exactly $|S|$, this construction leads to a correspondence between Γ and M, in such a way that if

$i \sim j$, then P_i is glued to P_j. Now that we have emphasized the links between M and Γ we can call M a manifold *modeled* on Γ. See [9] for more applications of these manifolds modeled on graphs.

From now we assume that Γ, as above, is a Cayley graph with polynomial growth of order $d \geq 2$ with a growth rate constant C_1.
We now give some properties that are satisfied by such a manifold M modeled on a Cayley graph Γ.
We define

$$\begin{aligned} \varphi : \bigcup_{i\in V} \mathrm{int} P_i &\longrightarrow V \\ x &\longmapsto v_x \end{aligned}$$

where v_x is the vertex of Γ associated with the piece of M to which x belongs.

Remark 1.13. φ is well defined because x does not belong to the boundary of a piece of M. We can then extend φ to the whole manifold: for $x \in M$ that belongs to the boundary of a piece, one of the two possibilities is chosen once and for all. This extended map is called φ again.

Remark 1.14. As explained before, Γ is endowed with the path distance, denoted d_Γ. By construction of M, if x and y do not belong to the same piece, then $d_\Gamma(v_x, v_y)$ represents the number of pieces that must be crossed to go from x to y plus one.

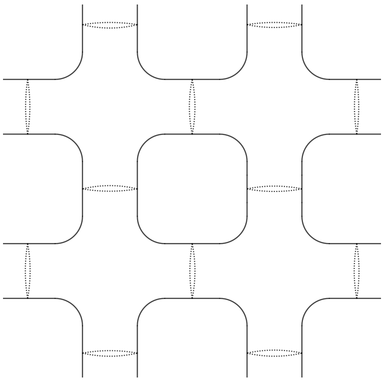

Figure 1.2: Example of a manifold modeled on the lattice $\mathbb{Z}^2$.

We now prove some results about (M, g).

Lemma 1.15. *There exist constants $C_3, C_4 > 0$, depending only on P, such that for all $x, y \in M$ with $d_M(x, y) \geq C_3$, we have*

$$C_4^{-1} \cdot d_\Gamma(v_x, v_y) \leq d_M(x, y) \leq C_4 \cdot d_\Gamma(v_x, v_y).$$

Proof. Let C_3 be the diameter of P plus one, that is $C_3 = \mathrm{diam} P + 1 = \sup\{d_M(x, y) : x, y \in P\} + 1$. Then for $x, y \in M$ such that $d_M(x, y) \geq C_3$, x, y cannot belong to the same piece of M. Let

$C_4 := 2 \cdot \mathrm{diam} P + 1$. Remember that the number $d_\Gamma(v_x, v_y)$ represents the number of pieces that must be crossed to go from x to y plus one. Then for $x, y \in M$ such that $d_M(x, y) > C_3$, we have

$$C_4^{-1} \cdot d_\Gamma(v_x, v_y) \le d_M(x, y) \le C_4 \cdot d_\Gamma(v_x, v_y).$$

♡♠◇♣

Lemma 1.16. *There exist constants* $C_5, C_6 > 0$*, depending only on* Γ *and* P*, such that for all* $x \in M$ *and* $r > C_3$*, we have*

$$C_5 \cdot r^d \le |B(x, r)|_g \le C_6 \cdot r^d.$$

Proof. We use Lemma 1.15 to compare the distance in Γ with the one in M. First we prove the second inequality.

$$\begin{aligned} |B(x,r)|_g &\overset{def}{=} |\{y \in M : d_M(x,y) < r\}|_g \\ &\le |P|_g \cdot |\{v_y : y \in B(x,r)\}| \\ &\le |P|_g \cdot |B(v_x, r)| \cdot C_4 \\ &\le |P|_g \cdot C_1 \cdot r^d \cdot C_4 \\ &=: C_6 \cdot r^d. \end{aligned}$$

Now we prove the first inequality.

$$\begin{aligned} |B(x,r)|_g &\overset{def}{=} |\{y \in M : d_M(x,y) < r\}|_g \\ &\ge |P|_g \cdot |\{v_y : y \in P_{g_y} \subset B(x,r)\}| \\ &\ge |P|_g \cdot |B(v_x, r)| \cdot C_4^{-1} \\ &\ge |P|_g \cdot C_1^{-1} \cdot r^d \cdot C_4^{-1} \\ &=: C_5 \cdot r^d. \end{aligned}$$

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We just showed that for all $x \in M$ and for r big enough, the volume of the ball centered at x with radius r is proportional to r^d. But as M is a d-dimensional manifold, which means that M is locally homeomorphic to $\mathbb{R}^d$, this is also true for r small enough. What we mean is, given $x \in M$, we can choose $r_0 > 0$ sufficiently small to find C_{r_0}, C'_{r_0} that for all $r \le r_0$ we have $C_{r_0} \cdot r^d \le |B(x,r)|_g \le C'_{r_0} \cdot r^d$.

For $x \in P$, call $r_0(x) \in (0, C_3]$ the biggest number such as above and define $r^* = \inf_{x \in P} r_0(x)$. Because P is smooth, the function $x \longmapsto r_0(x)$ is continuous. Moreover, P is compact. Hence the number r^* is strictly positive. Now recall that M is obtained by gluing copies of this unique fundamental piece and we can conclude that r^* is a uniform bound valid for any $x \in M$. We call C_*, C'_* the constant satisfying $C_* \cdot r^d \le |B(x,r)|_g \le C'_* \cdot r^d$ for all $x \in M$ and $r \le r^*$.

So for all $x \in M$, we have that $|B(x,r)|_g \approx r^d$ for r big enough and small enough. This leads to

Proposition 1.17. *There exist constants $C_7, C_8 > 0$, depending only on Γ and P, such that for all $x \in M$ and $r > 0$, we have*

$$C_7 \cdot r^d \leq |B(x,r)|_g \leq C_8 \cdot r^d.$$

Proof. We already know this is true for all $r \leq r^*$ and for all $r > C_3$. Now consider $C_9 := \inf_{x \in P} |B(x,r^*)|_g$ and $C_{10} := \sup_{x \in P} |B(x,C_3)|_g$, which are strictly positive finite numbers following the same arguments as above. Then for all $x \in M$ and all $r^* \leq r \leq C_3$, we have $C_9 \leq |B(x,r)|_g \leq C_{10}$. Define $C_7 := C_* \cdot C_9$ and $C_8 := C_6 \cdot C_{10}$ and we are done.

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The next proposition is a packing property:

Proposition 1.18. *There exists a constant $C_{11} \geq 1$, depending only on Γ and P such that for all $r > 0$ each ball of radius $2r$ in M can be covered by C_{11} balls of radius r.*

Proof. Let $r > 0$, $x \in M$ and $B(x,2r)$ be the ball centered at x with radius $2r$. Choose a maximal set of C_{11} points $x_i \in B(x, \frac{3r}{2})$ such that $d_M(x_i,x_j) \geq \frac{r}{2}$ for $i \neq j$. It is clear by construction that the balls $B(x_i, \frac{r}{4})$ are mutually disjoint. Moreover, the balls $B(x_i,r)$ cover $B(x,2r)$. In order to show this, let $y \in B(x,2r)$. Then there exists $y' \in B(x,\frac{3r}{2})$ such that $d_M(y,y') \leq \frac{r}{2}$. Because the set $\{x_i\}_{i=1}^{C_{11}}$ is maximal, there exists $1 \leq j \leq C_{11}$ such that $d_M(x_j,y') \leq \frac{r}{2}$. Then by the triangle inequality,

$$d_M(x_j,y) \leq d_M(x_i,y') + d_M(y',y) \leq \frac{r}{2} + \frac{r}{2} = r,$$

which means that $y \in B(x_j,r)$.
Then we have

$$\begin{aligned} C_{11} &\leq \frac{|B(x,2r)|_g}{\min_i |B(x_i,\frac{r}{4})|_g} \\ &= \frac{|B(x,2r)|_g}{|B(x_0,\frac{r}{4})|_g} && \text{for a certain } x_0, \\ &\leq \frac{|B(x_0,4r)|_g}{|B(x_0,\frac{r}{4})|_g} && \text{because } B(x,2r) \subset B(x_0,4r), \\ &\leq \frac{C_8 \cdot (4r)^d}{C_7 \cdot (\frac{r}{4})^d} \\ &= 16^d \cdot \frac{C_8}{C_7}. \end{aligned}$$

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1.4 Proof of the main theorem

We split the proof of Theorem 1.6 into several propositions that will lead us to the conclusion.

1.4.1 Domain associated to a subgraph

First we explain how to associate a bounded domain $(N, \Sigma) \subset M$ to a subgraph (Ω, B) of Γ. What we have to keep in mind is that we want to preserve the structure of (Ω, B) in the domain N. For this purpose, as said in Remark 1.11, we have to specify the geometry of our fundamental piece P.

The metric g_0 on P is such that there exists a point $z \in P$ and an annulus $A(z, 1, 3) = \{x \in P : 1 < d_M(z, x) < 3\}$ which is isometric to the cylinder $[0, 2] \times \mathbb{S}^{d-1}$. This annulus does not intersect any cylindrical neighborhood of the boundary, see Figure 1.3.

Remark 1.19. Because P is still a smooth compact d-dimensional Riemannian manifold homeomorphic to $\mathbb{S}^d$ containing $|S|$ holes, with an appropriate neighborhood of the boundary, every proposition we have stated about M are verified.

This annulus is the key point of our construction: if we decided to remove $B(z, 1)$ from P, then we would obtain a manifold which would be homeomorphic to a d-dimensional sphere with $|S| + 1$ holes with the property that each boundary component possesses a neighborhood isometric to the cylinder $[0, 2] \times \mathbb{S}^{d-1}$. We will have to remove this annulus from some particular pieces that shall compose N in order to guarantee the existence of a connected component of Σ on those pieces. We will show in Example 1.22 that this trick is necessary.

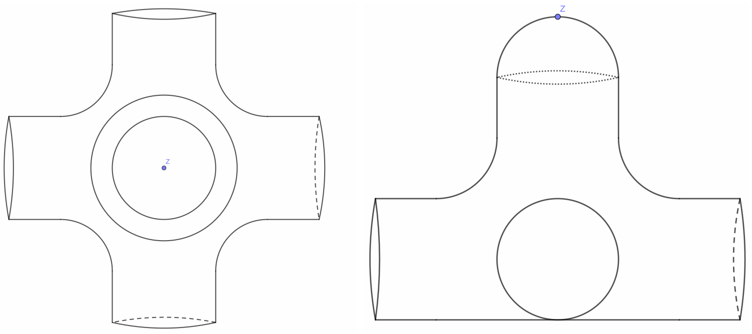

Figure 1.3: Example of a fundamental piece associated with the lattice $\mathbb{Z}^2$. Removing the ball $B(z, 1)$ leads to a fifth hole like the other four ones. The picture on the right is a view from the side.

The copy of the point $z \in P$ on a piece P_i associated to i is denoted z_i.
Given a subgraph (Ω, B) of Γ, we shall associate a bounded domain $(N, \Sigma) \subset M$ to it. Proceed as follows:

- For each $i \in \bar{\Omega}$ we take P_i the piece of M associated to i;
- If $i, j \in \bar{\Omega}$ are such that $i \sim j$ in Γ but $i \nsim j$ in (Ω, B) (which could happen if $i, j \in B$), take the pieces P_i and P_j but disconnect them by removing a cylinder isometric to $[-1, 1] \times \mathbb{S}^{d-1}$, where $\{0\} \times \mathbb{S}^{d-1}$ corresponds to the gluing part of these pieces;
- For all $j \in B$ remove the ball $B(z_j, 1)$ from the piece P_j.

This gives us a natural boundary that we will call Σ, composed of several disjoint copies of $\mathbb{S}^{d-1}$, see Figure 1.4.

Remark 1.20. For $i \in \bar{\Omega}$, we continue to call P_i the piece of N that is associated to i, even if this piece is not a whole one.

Remark 1.21. The purpose of this maneuver is to imitate the structure of (Ω, B) on (N, Σ). Our construction guarantees that for $i, j \in \bar{\Omega}$, we have the equivalence $i \sim j \iff P_i \sim P_j$, where $P_i \sim P_j$ means that P_i is connected to P_j. Moreover, the boundary structure is preserved, that is $j \in B \iff P_j$ contains at least one connected component of the boundary. The trick of the annulus is essential; Example 1.22 shows us that without it, the structure of the subgraph might not be reproduced by the domain.

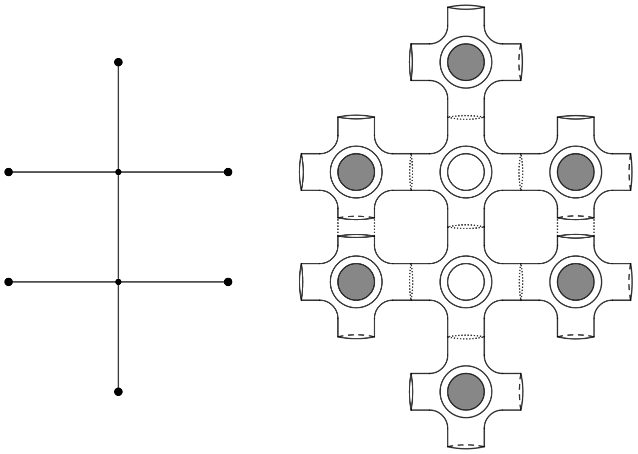

Figure 1.4: Example of a subgraph of $\mathbb{Z}^2$ and an associated domain. On the left, the big dots represent the boundary B while the small ones represent the interior Ω. On the right, the grey balls are removed from the domain.

Example 1.22. Look at the lattice $\mathbb{Z}^2$, and let $\Omega = B(n)\backslash\{e\}$ be the centered ball of radius n, deprived of the origin. Then, for the induced subgraph (Ω, B), we have $e \in B$. Moreover, e is adjacent to each of its four neighbors in $\mathbb{Z}^2$. Then the boundary of the domain would not have any component close to P_e if we did not remove the ball $B(z_e, 1)$ from the piece P_e.

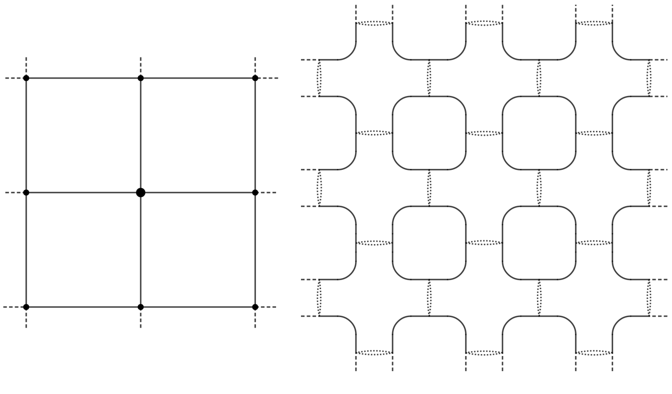

Figure 1.5: Without the trick of the annulus, the boundary structure of the subgraph might not be reproduced on the associated domain: there is no boundary component near the central piece.

Because Ω is chosen connected, so is the domain (N, Σ). We denote by $|\Sigma|$ the $(d-1)$-volume of Σ, that is

$$|\Sigma| := \int_{\Sigma} dv_{\Sigma},$$

where dv_{Σ} is the measure induced by the Riemannian metric of M restricted to Σ.

Remark 1.23. Σ possesses a neighborhood that is isometric to the cylinder $[0,1] \times \mathbb{S}^{d-1}$, for we took care of constructing a fundamental piece P with a boundary that admits a neighborhood isometric to the cylinder $[0,2] \times \mathbb{S}^{d-1}$. Moreover, the annulus $A_i = A(z_i, 1, 3)$ was built in a way that removing $B(z_i, 1)$ leads to a component of the boundary that possesses a neighborhood isometric to $[0,2] \times \mathbb{S}^{d-1}$, see Figure 1.3.

For (N, Σ) a bounded domain in M, we introduce the isoperimetric ratio $I(N)$ defined by

$$I(N) = \frac{|\Sigma|}{|N|_g^{\frac{d-1}{d}}}.$$

The idea is now to compare the eigenvalues of (Ω, B) with N's ones. In order to do it, we state now [7, Theorem 2.2] that gives us an upper bound for the Steklov spectrum of N.

Theorem 1.24. *Let (M, g) be a complete d-dimensional manifold that satisfies properties of Proposition 1.17 and 1.18. Then there exists a constant $C_{12} = C_{12}(g)$ depending only on C_{11}, C_7 and C_8 coming from Propositions 1.17 and 1.18 such that for any bounded domain $(N, \Sigma) \subset (M, g)$, we have for every $k \geq 0$,*

$$\sigma_k(N) \cdot |\Sigma|^{\frac{1}{d-1}} \leq \frac{C_{12}}{I(N)^{\frac{d-2}{d-1}}} \cdot k^{\frac{2}{d}}.$$

For a proof of this, we refer to Theorem 2.2 of [7]. Actually, the result obtained by Colbois, El Soufi and Girouard is more general than that, but this statement is enough for our need.

Remark 1.25. The constant C_{12} does not depend on the subgraph or on the induced domain.

We can rearrange the result of Theorem 1.24 to get

$$\sigma_k(N, \Sigma) \leq C_{12} \cdot \frac{|N|_g^{\frac{d-2}{d}}}{|\Sigma|} \cdot k^{\frac{2}{d}}, \tag{1.4.1}$$

which is more adequate for the purpose of this proof.

1.4.2 Isoperimetric control of a domain associated to a subgraph

This subsection is devoted to state an isoperimetric inequality satisfied by a domain (N, Σ) such as explained in the previous one.

Proposition 1.26. *Let (Ω, B) be a subgraph of Γ, a Cayley graph with polynomial growth rate of order $d \geq 2$. Let M be a modeled manifold and (N, Σ) be the domain of M associated to (Ω, B). Then there exists a constant C_{14} depending only on Γ and P such* *that we have*

$$|N|_g^{\frac{d-2}{d}} \leq C_{14} \cdot |\Sigma|^{\frac{d-2}{d-1}}. \tag{1.4.2}$$

Proof. As stated by Proposition 1.10, we have

$$\frac{|\bar{\Omega}|^{\frac{d-1}{d}}}{|B|} \leq C_2.$$

By construction of the domain N, each vertex $i \in \bar{\Omega}$ adds a piece P_i to N, so we have

$$|N|_g \leq |\bar{\Omega}| \cdot |P|_g,$$

and each vertex $j \in B$ adds at least one copy of $\mathbb{S}^{d-1}$ to the boundary so we have

$$|\Sigma| \geq |B| \cdot |\mathbb{S}^{d-1}|. \tag{1.4.3}$$

Altogether, this gives us

$$\begin{aligned} \frac{|N|_g^{\frac{d-1}{d}}}{|\Sigma|} &\leq \frac{\left(|\bar{\Omega}| \cdot |P|_g\right)^{\frac{d-1}{d}}}{|B| \cdot |\mathbb{S}^{d-1}|} \\ &\leq C_2 \cdot \frac{|P|_g^{\frac{d-1}{d}}}{|\mathbb{S}^{d-1}|} \\ &= C_{13}. \end{aligned}$$

Raising it to the power $\frac{d-2}{d-1}$ gives us

$$\frac{|N|_g^{\frac{d-2}{d}}}{|\Sigma|^{\frac{d-2}{d-1}}} \leq C_{13}^{\frac{d-2}{d-1}},$$

which leads to

$$|N|_g^{\frac{d-2}{d}} \leq C_{14} \cdot |\Sigma|^{\frac{d-2}{d-1}}.$$

♡♠◇♣

1.4.3 Discretization of a Riemannian manifold

Let us begin by explaining the strategy that motivates the next subsection. We have in our possession a subgraph (Ω, B) and a bounded domain (N, Σ) associated to it. Moreover, we know how to estimate the Steklov spectrum of N, see (1.4.1). What we shall do from now is to associate a graph with boundary denoted $(\tilde{V}, \tilde{E}, V_\Sigma)$ to the domain N in such a way that we will be able to estimate the Steklov spectrum of $(\tilde{V}, \tilde{E}, V_\Sigma)$ according to the one of N. This new graph with boundary $(\tilde{V}, \tilde{E}, V_\Sigma)$ is not our starting subgraph (Ω, B) but will be roughly isometric to it (see Definition 1.29), which means that $(\tilde{V}, \tilde{E}, V_\Sigma)$ and (Ω, B) are close enough for us to compare their spectra, which will allow us to conclude.

$$\begin{tikzcd}
\Gamma \arrow[rr, "\text{subgraph}"] \arrow[dd, "\text{modeled}"', "\text{manifold}"] & & (\Omega, B) \arrow[dr, leftrightarrow, "\text{roughly isometric}"] \arrow[dd, leftrightarrow, "\text{structure}"', "\text{preserving}"] & \\
& & & (\tilde{V}, \tilde{E}, V_\Sigma) \\
M \arrow[rr, "\text{domain}"] & & (N, \Sigma) \arrow[ur, "\text{discretization}"'] &
\end{tikzcd}$$

We have to explain how to discretize a manifold. For further investigations on this topic, see [9].

Definition 1.27. We denote by $\mathcal{M} = \mathcal{M}(\kappa, r_0, d)$ the class of all compact Riemannian manifolds N of dimension d with smooth boundary Σ satisfying the following assumptions:
There exist constants $\kappa \geq 0$ and $r_0 \in (0, 1)$ such that

- The boundary Σ admits a neighborhood which is isometric to the cylinder $[0, 1] \times \Sigma$, with the boundary corresponding to $\{0\} \times \Sigma$;
- The Ricci curvature of N is bounded below by $-(d-1)\kappa$;
- The Ricci curvature of Σ is bounded below by $-(d-2)\kappa$;
- For each point $x \in N$ such that $d_M(x, \Sigma) > 1$, $\mathrm{inj}_M(x) > r_0$;
- For each point $x \in \Sigma$, $\mathrm{inj}_\Sigma(x) > r_0$.

Remark 1.28. Because of the regularity of the modeled manifold M and the compactness of the fundamental piece P, it is clear that there exist some κ and r_0 such that each bounded domain $N \subset M$ satisfies the last four assumptions. Moreover, a domain N associated with a subgraph satisfies the first one as well, as stated by Remark 1.23.

Definition 1.29. A rough isometry between two metric spaces (X, d_X) and (Y, d_Y) is a map $\phi : X \longrightarrow Y$ such that there exist constants $C_{15} > 1,\ C_{16}, C_{17} > 0$ satisfying

$$C_{15}^{-1} \cdot d_X(x, y) - C_{16} \leq d_Y(\phi(x), \phi(y)) \leq C_{15} \cdot d_X(x, y) + C_{16}$$

for all $x, y \in X$ and which satisfy

$$\bigcup_{x \in X} B(\phi(x), C_{17}) = Y.$$

Definition 1.30. Given $\varepsilon \in (0, \frac{r_0}{4})$, an ε-discretization of a manifold $N \in \mathcal{M}$ is a procedure which allows us to associate a graph with boundary $(\tilde{V}, \tilde{E}, V_\Sigma)$ to N, such that N is roughly isometric to $(\tilde{V}, \tilde{E}, V_\Sigma)$.

We now explain the procedure of discretization. Given $\varepsilon \in (0, \frac{r_0}{4})$, let V_Σ be a maximal ε-separated set in Σ. Let V'_Σ be a copy of V_Σ located 4ε away from Σ,

$$V'_\Sigma = \{4\varepsilon\} \times V_\Sigma \subset N.$$

Let V_I be a maximal ε-separated set in $N \backslash [0, 4\varepsilon) \times \Sigma$ such that $V'_\Sigma \subset V_I$. The set $\tilde{V} = V_\Sigma \cup V_I$ is endowed with a structure of a graph, declaring

- Any two $v_1, v_2 \in \tilde{V}$ are adjacent if $d_M(v_1, v_2) \leq 3\varepsilon$;
- Any $v \in V_\Sigma$ is adjacent to $v' = (4\varepsilon, v) \in V'_\Sigma$.

The graph $(\tilde{V}, \tilde{E})$ obtained is a graph with boundary $(\tilde{V}, \tilde{E}, V_\Sigma)$, declaring V_Σ as the boundary, V_I as the interior. We shall call it $(\tilde{V}, V_\Sigma)$.

Theorem 1.31. *Given $\varepsilon \in (0, \frac{r_0}{4})$, there exist constants $C_{18}, C_{19} > 0$ depending on κ, r_0, d and ε such that any ε-discretization $(\tilde{V}, V_\Sigma)$ of a manifold $N \in \mathcal{M}(\kappa, r_0, d)$ satisfies*

$$\frac{C_{18}}{k} \leq \frac{\sigma_k(N, \Sigma)}{\sigma_k(\tilde{V}, V_\Sigma)} \leq C_{19},$$

for each $k \leq |V_\Sigma|$.

This theorem is exactly Theorem 3 point 4) of [9], one can look at for a proof.

As an immediate consequence we have

$$\sigma_k(\tilde{V}, V_\Sigma) \leq \frac{\sigma_k(N, \Sigma) \cdot k}{C_{18}}, \tag{1.4.4}$$

which is more useful for us.

Definition 1.32. A rough isometry between two graphs with boundary (Γ_1, B_1) and (Γ_2, B_2) is a rough isometry that sends B_1 to B_2 and such that the restriction of the rough isometry to B_1 is a rough isometry $B_1 \longrightarrow B_2$ when considering extrinsic distances on B_1 and B_2.

Now we shall emphasize the link between a subgraph (Ω, B) and an ε-discretization $(\tilde{V}, V_\Sigma)$ of N, which is the purpose of the following proposition.

Proposition 1.33. *Let $\Gamma = Cay(G, S)$ be a Cayley graph. Let (Ω, B) be a subgraph of Γ, M be a manifold modeled on Γ and $(N, \Sigma) \subset M$ be the bounded domain of M associated to (Ω, B) as before. Let $(\tilde{V}, V_\Sigma)$ be any ε-discretization of N.*
Then there exist constants $C_{15} > 1, C_{16}, C_{17} > 0$ depending only on Γ, M and ε such that there exists a rough isometry $\phi : (\tilde{V}, V_\Sigma) \longrightarrow (\Omega, B)$ with constants C_{15}, C_{16}, C_{17}.

Remark 1.34. The essential point of this proposition is to state that the constants of the rough isometry can be chosen independently of the subgraph (Ω, B).

Proof. Define

$$\phi : (\tilde{V}, V_\Sigma) \longrightarrow (\Omega, B)$$

by:

- If $v \in V_\Sigma$, then v is such that $v \in P_j$ for $j \in B$ and we define $\phi(v) = j$;
- If $v \in V_I$ is such that $v \in P_i$ for $i \in \Omega$, we define $\phi(v) = i$;

- If $v \in V_I$ is such that $v \in P_j$ for $j \in B$, we define $\phi(v) = i$ such that $i \in \Omega$ and $i \sim j$. If there are many such i, one among the at most $|S|$ possibilities is chosen once and for all;
- If v lies on the gluing of two pieces, one of the two possibilities is chosen once and for all.

Define C_{15} as the triple of the cardinal in the biggest set of points ε-separated of P. By compactness of P, C_{15} is finite. From this definition it is straightforward that for $v_1, v_2 \in V$ such that v_1 belongs to the same piece as v_2, we have $d_{\tilde{V}}(v_1, v_2) \leq C_{15}$.
Recall that we chose the domain N such that N gets the same neighbor structure as the subgraph: for $i, j \in \bar{\Omega}$, we have $i \sim j \iff P_i \sim P_j$.
Now we define $C_{16} := C_{15}$. Hence, for all $v_1, v_2 \in \tilde{V}$,

$$d_{\tilde{V}}(v_1, v_2) \leq C_{15} \cdot d_{\bar{\Omega}}(\phi(v_1), \phi(v_2)) + C_{16}.$$

In the same way we also have

$$C_{15}^{-1} \cdot d_{\bar{\Omega}}(\phi(v_1), \phi(v_2)) - C_{16} \leq d_{\tilde{V}}(v_1, v_2).$$

Remark now that ϕ is a surjective map. Hence we have

$$\bigcup_{v \in \tilde{V}} B(\phi(v), C_{17}) = (\Omega, B)$$

for any value of $C_{17} > 0$. We can choose $C_{17} = 1$. ♡♠♢♣

This link between (Ω, B) and $(\tilde{V}, V_\Sigma)$ shall be exploited to give a relationship between the Steklov eigenvalues of these graphs with boundary. We state here Proposition 16 of [9]:

Proposition 1.35. *Given $C_{15} > 1, C_{16}, C_{17} > 0$ there exist constants C_{20}, C_{21} depending only on C_{15}, C_{16}, C_{17} and on the maximal degree of the vertices, such that any two graphs with boundary (Γ_1, B_1) and (Γ_2, B_2) which are roughly isometric with constants C_{15}, C_{16}, C_{17}, satisfy*

$$C_{20} \leq \frac{\sigma_k(\Gamma_1, B_1)}{\sigma_k(\Gamma_2, B_2)} \leq C_{21}$$

for all $k < \min\{|B_1|, |B_2|\}$.

Applied to our graphs, this leads to the existence of constants C_{20}, C_{21} such that

$$C_{20} \leq \frac{\sigma_k(\Omega, B)}{\sigma_k(\tilde{V}, V_\Sigma)} \leq C_{21}$$

for all $k < |B|$, which we can rearrange in

$$\sigma_k(\Omega, B) \leq \sigma_k(\tilde{V}, V_\Sigma) \cdot C_{21}. \tag{1.4.5}$$

Let us conclude our proof of Theorem 1.6 by assembling the different results we obtained before.

$$\begin{aligned}
\sigma_k(\Omega, B) &\overset{(1.4.5)}{\leq} \sigma_k(\tilde{V}, V_\Sigma) \cdot C_{21} \\
&\overset{(1.4.4)}{\leq} \frac{\sigma_k(N, \Sigma) \cdot k}{C_{18}} \cdot C_{21} \\
&\overset{(1.4.1)}{\leq} \frac{C_{12} \cdot \frac{|N|^{\frac{d-2}{d}}}{|\Sigma|} \cdot k^{\frac{2}{d}} \cdot k}{C_{18}} \cdot C_{21} \\
&\overset{(1.4.2)}{\leq} \frac{C_{12} \cdot \frac{C_{14} \cdot |\Sigma|^{\frac{d-2}{d-1}}}{|\Sigma|} \cdot k^{\frac{2}{d}} \cdot k}{C_{18}} \cdot C_{21} \\
&:= C_{22} \cdot \frac{1}{|\Sigma|^{\frac{1}{d-1}}} \cdot k^{\frac{d+2}{d}} \\
&\overset{(1.4.3)}{\leq} C_{22} \cdot \frac{1}{(|B| \cdot |\mathbb{S}^{d-1}|)^{\frac{1}{d-1}}} \cdot k^{\frac{d+2}{d}} \\
&:= C_{23} \cdot \frac{1}{|B|^{\frac{1}{d-1}}} \cdot k^{\frac{d+2}{d}}.
\end{aligned}$$

Throughout the paper, we took care to specify on which parameters the constants depend. It happens that they depend only on Γ, P and ε, not on the subgraph (Ω, B) or the domain (N, Σ) associated. Hence, if we set a fundamental piece P associated to Γ, and if we set a value of ε, the constant C_{23} is now fixed.
Then, given Γ a Cayley graph with polynomial growth rate of order $d \geq 2$, one can find a constant $C_{23} := C(\Gamma)$ such that for any subgraph (Ω, B) of Γ, we have

$$\sigma_k(\Omega, B) \leq C(\Gamma) \cdot \frac{1}{|B|^{\frac{1}{d-1}}} \cdot k^{\frac{d+2}{d}},$$

which proves Theorem 1.6.

From this statement, let us prove Corollary 1.7.

Let $\Gamma = Cay(G, S)$ and $C(\Gamma)$ as above. Let $(\Omega_l, B_l)_{l=1}^{\infty}$ be a family of subgraphs such that $|\Omega_l| \underset{l \to \infty}{\longrightarrow} \infty$.
Because of the isoperimetric control stated by Proposition 1.10, it is clear that $|B_l| \underset{l \to \infty}{\longrightarrow} \infty$ too.

Hence, for all k fixed, we have

$$\sigma_k(\Omega_l, B_l) \leq C(\Gamma) \cdot \frac{1}{|B_l|^{\frac{1}{d-1}}} \cdot k^{\frac{d+2}{d}} \underset{l \to \infty}{\longrightarrow} 0,$$

which proves Corollary 1.7.

Chapter 2

The Steklov problem on triangle-tiling graphs in the hyperbolic plane

This paper has already been published in *The Journal of Geometric Analysis*, see [35]. The reader may encounter a few differences with the original text, due to some minor adjustments. Moreover, Appendix 2.B has been added, providing information about Question 2.36 that we ask in this article.

Abstract

We introduce a graph Γ which is roughly isometric to the hyperbolic plane and we study the Steklov eigenvalues of a subgraph with boundary Ω of Γ. For $(\Omega_l)_{l\geq 1}$ a sequence of subraphs of Γ such that $|\Omega_l| \longrightarrow \infty$, we prove that for each $k \in \mathbb{N}$, the kth eigenvalue tends to 0 proportionally to $1/|B_l|$. The idea of the proof consists in finding a bounded domain N of the hyperbolic plane which is roughly isometric to Ω, giving an upper bound for the Steklov eigenvalues of N and transferring this bound to Ω via a process called discretization.

Keywords: Spectral geometry, Steklov problem, graphs with boundary, discrete Steklov problem.

2.1 Introduction

Enabling (M, g) as a smooth connected compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary ∂M. The Steklov problem on (M, g) consists in finding all $\sigma \in \mathbb{R}$ such that there exists a non-zero harmonic function $f : M \longrightarrow \mathbb{R}$ satisfying $\frac{\partial f}{\partial \nu} = \sigma f$ on ∂M, where $\frac{\partial}{\partial \nu}$ denotes the outward-pointing normal derivative on ∂M.

Such a σ is called a Steklov eigenvalue of M and a corresponding f is called a Steklov eigenfunction. The (ordered) set of eigenvalues is called the Steklov spectrum of (M, g).

It is well known that the Steklov spectrum of M forms a discrete sequence

$$0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \ldots \nearrow \infty,$$

where each eigenvalue is repeated with multiplicity.

There exists a discrete analog to the Steklov problem, which is called the discrete Steklov problem and which is defined on graphs with boundary. Let us begin by defining it.

Definition 2.1. A graph with boundary is a triplet $(\bar{\Omega}, E', B)$, where $(\bar{\Omega}, E')$ is a simple connected undirected graph and $B \subset \bar{\Omega}$ is a non-empty set of vertices, called the boundary. The set B^c is called the interior of the graph.

In this paper, all graphs will always be simple connected and undirected.

For $v, w \in \bar{\Omega}$, we write $v \sim w$ when v is adjacent to w. For $A \subset \bar{\Omega}$, we write $|A|$ the cardinality of A, which is the number of vertices contained in A. For the purpose of this article, all graphs with boundary are finite. We denote by $\mathbb{R}^{\bar{\Omega}}$ the space of all functions $u : \bar{\Omega} \longrightarrow \mathbb{R}$, which is isomorphic to the Euclidean space of dimension $|\bar{\Omega}|$. Similarly, we denote by $\mathbb{R}^B$ the space of functions $u : B \longrightarrow \mathbb{R}$, which is the Euclidean space of dimension $|B|$.

We can now introduce the discrete Laplacian operator $\Delta : \mathbb{R}^{\bar{\Omega}} \longrightarrow \mathbb{R}^{\bar{\Omega}}$, defined by

$$\begin{aligned}\Delta u : \bar{\Omega} &\longrightarrow \mathbb{R}\\ v &\longmapsto \Delta u(v) = \sum_{w \sim v} (u(v) - u(w)).\end{aligned}$$

The normal derivative $\frac{\partial}{\partial v} : \mathbb{R}^{\bar{\Omega}} \longrightarrow \mathbb{R}^B$ is defined by

$$\begin{aligned}\frac{\partial u}{\partial v} : B &\longrightarrow \mathbb{R}\\ v &\longmapsto \frac{\partial u}{\partial v}(v) = \sum_{w \sim v} (u(v) - u(w)).\end{aligned}$$

As one can see, the normal derivative coincides with the restriction of the Laplacian to the boundary. Although this choice may seem strange, it is shown in [9] that it leads to interesting links between the Steklov spectrum of a manifold and the Steklov spectrum of a graph with boundary which *looks like* the manifold, see [9, Theorem 3] for more information about what *looks like* means in this context.

Definition 2.2. The discrete Steklov problem on a finite graph with boundary $(\bar{\Omega}, E', B)$ consists in finding all $\sigma \in \mathbb{R}$ such that there exists a non-zero function $u \in \mathbb{R}^{\bar{\Omega}}$ such that

$$\begin{cases} \Delta u(v) = 0 \text{ if } v \in \Omega \\ \frac{\partial}{\partial v} u(v) = \sigma u(v) \text{ if } v \in B. \end{cases}$$

Such a σ is called a Steklov eigenvalue and a corresponding u is called a Steklov eigenfunction of $(\bar{\Omega}, E', B)$. As said in [31], the Steklov spectrum of a graph with boundary $(\bar{\Omega}, E', B)$ forms a sequence as follows:

$$0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_{|B|-1}.$$

This problem has recently received particular attention, one can cite for instance [19, 24, 30, 31]. An investigation has been made by Colbois, Girouard and Raveendran in [9], allowing us to understand some spectral links between the Steklov problem on a manifold and the discrete Steklov problem of a graph associated to this manifold. These links will be very useful in this paper. The main problem that we will have to face is to place ourselves in the hypotheses of Theorem 3 of [9], in order to use it to our advantage.

Among other things, a question that has been studied by some authors is that of providing an upper bound for the first - and then for the k^{th} - eigenvalue of some particular graphs with boundary. These particular graphs that have been studied are those called *subgraphs* of an (infinite) *host graph*. A subgraph of a host graph can be interpreted as the discrete analog of a bounded domain in a manifold. Let us define what it is exactly.

Definition 2.3. Let $\Gamma = (V, E)$ be a graph and let $\Omega \subset V$ be a finite subset of vertices connected for Γ, i.e for each $v, w \in \Omega$, there exist $l \in \mathbb{N}$ and $v_0 = v, v_1, \ldots, v_l = w \in \Omega$ satisfying $\{v_i, v_{i+1}\} \in E$ for all $i = 0, \ldots, l-1$. The graph with boundary $(\bar{\Omega}, E', B)$ induced by Ω is defined as follows:

- $B = \{w \in V \backslash \Omega : \exists\, v \in \Omega \text{ such that } \{v, w\} \in E\}$;
- $\bar{\Omega} = \Omega \cup B$;
- $E' = \{\{v, w\} \in E : v \in \Omega, w \in \bar{\Omega}\}$.

Such a graph with boundary is simply denoted Ω and is called a subgraph of Γ. The set of vertices B is the boundary of the subgraph. We refer to Γ as the host graph of Ω.

Some interesting results have recently been discovered, providing us with bounds for the eigenvalues, depending on the host graph Γ. A first result, due to Han and Hua, is the following:

Theorem 2.4 (Theorem 1.2 in [19]). *Let $\mathbb{Z}^d$ be the integer lattice of dimension d. Let Ω be a subgraph of $\mathbb{Z}^d$. Then we have*

$$\sum_{l=1}^{d} \frac{1}{\sigma_l(\Omega)} \geq C' \cdot |\Omega|^{\frac{1}{d}} - \frac{C''}{|\Omega|},$$

where $C' = (64 d^3 \omega_d^{\frac{1}{d}})^{-1}, C'' = \frac{1}{32d}$ and ω_d is the volume of the unit ball in $\mathbb{R}^d$.

Another investigation gives some control over the spectrum of a subraph of a *Cayley graph*. We recall that, given a finitely generated group G and a finite generating subset S of G, one can define a graph, called a Cayley graph and denoted $Cay(G, S)$. If G is infinite, then so is $Cay(G, S)$ and we can use it as a host graph. The result provided by Perrin is the following:

Theorem 2.5 (Corollary 1 in [30]). *Let $\Gamma = (V, E)$ be a Cayley graph with polynomial growth of order $d \geq 2$. There exists $\tilde{C}(\Gamma) > 0$ such that for any finite subgraph Ω of Γ, we have*

$$\sigma_1(\Omega) \leq \tilde{C}(\Gamma) \cdot \frac{1}{|B|^{\frac{1}{d-1}}}.$$

This theorem is way more general about the class of host graphs Γ but provides us control over the first non-trivial eigenvalue only, see [30] for details. We gave an extension to this result in a precedent article:

Theorem 2.6 (Theorem 5 in [36]). *Let $\Gamma = Cay(G, S)$ be a polynomial growth Cayley graph of order $d \geq 2$. Let Ω be a subgraph of Γ. Then there exists a constant $\bar{C}(\Gamma) > 0$*

such that for all $k < |B|$,

$$\sigma_k(\Omega) \leq \bar{C}(\Gamma) \cdot \frac{1}{|B|^{\frac{1}{d-1}}} \cdot k^{\frac{d+2}{d}}.$$

As a corollary, we have:

Corollary 2.7 (Corollary 6 in [36]). *Let* Γ *be a polynomial growth Cayley graph of order* $d \geq 2$ *and* $(\Omega_l)_{l=1}^{\infty}$ *be a sequence of subgraphs of* Γ *such that* $|\Omega_l| \xrightarrow[l \to \infty]{} \infty$. *Fix* $k \in \mathbb{N}$. *Then we have*

$$\sigma_k(\Omega_l) \xrightarrow[l \to \infty]{} 0.$$

All these theorems follow from the investigation upon one class of host graphs Γ, which are Cayley graphs of polynomial growth groups. This consideration leads to a natural question:

What can we say about the eigenvalues of subgraphs of a host graph Γ, *whose growth rate is more than polynomial?*

A first class of graphs we can think of is that of trees. In [22], the authors find upper bounds for the eigenvalues of a finite tree. Their investigations lead to the following result:

Theorem 2.8 (Theorem 1.1 and 1.5 in [22]). *Let* $\mathcal{T}$ *be a finite tree with (uniformly) bounded degree D. Let B be the boundary of the tree, i.e the set of vertices of degree one. Then we have*

$$\sigma_1 \leq \frac{4(D-1)}{|B|}.$$

Higher Steklov eigenvalues are bounded as well: for all $k = 2, \ldots, |B| - 1$, *we have*

$$\sigma_k \leq \frac{8(D-1)^2(k-1)}{|B|}.$$

As stated in Remark 1.7 of [22], we can consider as the host graph Γ the Cayley graph of a free group and use this result to estimate the Steklov eigenvalues of a subgraph Ω of Γ. Since the growth rate of such a host graph is exponential, we now have a completely new class of host graphs for which we can estimate their subgraphs eigenvalues.

This paper's objective is to study the subgraphs' eigenvalues of a host graph Γ which is roughly isometric to the hyperbolic plane $\mathbb{H}^2$ (see Definition 2.12). The hyperbolic plane is a Cartan-Hadamard manifold of constant sectional curvature -1. Then Γ can be seen as a discrete analog of such a manifold. Because of its relation with $\mathbb{H}^2$, the growth rate of Γ is exponential, and then Γ does not enter the class of host graphs of Theorems 2.4, 2.5 and 2.6.
Despite a growth rate identical to that of the trees, the structure of Γ is very different from the latter, because of its connection with $\mathbb{H}^2$. Therefore, the method we will use to obtain upper bounds has nothing to do with the one used in [22]. Indeed, He and Hua were able to work directly on the trees and use the great ease of disconnection of the trees as a tool to obtain the

bounds of Theorem 2.8, while on our side we will use the proximity between Γ and $\mathbb{H}^2$ to obtain upper bounds.

There are many graphs which are roughly isometric to the hyperbolic plane. This paper will focus on a particular class of such graphs, coming from a tiling of $\mathbb{H}^2$ associated with a triangle group. We shall refer to such a graph as *triangle-tiling graph*.
Triangle groups are part of the Coxeter groups, which can be seen as groups generated by reflections. These groups have been studied by many authors, see for instance [4, 23, 25]. Triangle groups are Coxeter groups with three generators, that can be regarded as reflections through the sides of a triangle. They lead to many beautiful geometric constructions and tilings, see [2, 12, 29, 40]. We will recall in Section 2.2 hereafter the notions that are required for the understanding of the paper.

Our main result is the following:

Theorem 2.9. *Let Γ be a triangle-tiling graph. Then there exists a constant $C = C(\Gamma) > 0$ such that for all subgraphs Ω of Γ and all $k < |B|$, we have*

$$\sigma_k(\Omega) \leq C(\Gamma) \cdot \frac{1}{|B|} \cdot k^2.$$

As we will see in Section 2.2, the host graph Γ is defined from the choice of three integers. As a consequence, we will see that there are infinitely many triangle-tiling graphs.
As a corollary, we obtain the interesting fact:

Corollary 2.10. *Let $(\Omega_l)_{l\geq 1}$ be a family of subgraphs of Γ such that $|\Omega_l| \underset{l\to\infty}{\longrightarrow} \infty$. Then for all $k \in \mathbb{N}$ fixed,*

$$\sigma_k(\Omega_l) \underset{l\to\infty}{\longrightarrow} 0.$$

The number $\sigma_k(\Omega_l)$ is of course defined if and only if $|B_l| < k$. This condition is satisfied for l big enough thanks to the assumption that $|\Omega_l| \longrightarrow \infty$.

Our approach is sketched this way: we define a triangle-tiling graph Γ that we use as a host graph and show that it is roughly isometric to $\mathbb{H}^2$ (see Definition 2.15). Thanks to the rough isometry, we can naturally associate to a subgraph Ω of Γ a bounded domain N of $\mathbb{H}^2$, whose boundary will be denoted Σ. We can then use results from [7] to give upper bounds for $\sigma_k(N)$. Once this task is completed we use the work of Colbois et al. presented in [9] in order to discretize a Riemannian manifold with boundary (N, g'), obtained as a deformation of the domain N (this deformation is necessary since we have to satisfy the assumptions of [9, Theorem 3]). This discretization will give us a path linking the eigenvalues of N and the ones of Ω, which will allow us to conclude.

Our strategy can be summed up in the diagram below:

$$\begin{tikzcd}
\Gamma \arrow[r, "\text{subgraph}"] \arrow[d, leftrightarrow, "\text{roughly isometric}"] & \Omega \arrow[r, leftrightarrow, "\text{roughly isometric}"] \arrow[d, leftrightarrow, "\text{structure preserved}"] & (\bar{V}, \bar{E}, V_\Sigma) \\
\mathbb{H}^2 \arrow[r, "\text{domain}"] & (N, g) \arrow[r, "\text{change of metric}"] & (N, g') \arrow[u, "\text{discretization}"]
\end{tikzcd}$$

Here, by *structure preserved*, we mean that the structural information of the subgraph Ω can be read in the domain N, see the rest of the paper for more details. Moreover, in the diagram, $P \longleftrightarrow Q$ reflects the idea that P is in some sense analog to Q, and $P \longrightarrow Q$ reflects the idea that Q is obtained from P. More details are given in the rest of the paper.

Our result holds for subgraphs of any triangle-tiling graph. However, there exist many other graphs that are roughly isometric to the hyperbolic plane, and that we could use as host graphs. This remark naturally leads to many interesting interrogations, that we will consider and develop in Section 2.5. In particular, one may ask if the result is still true when using other host graphs roughly isometric to $\mathbb{H}^2$. This leads to the following open question (Question 2.34):

If Γ is any graph roughly isometric to the hyperbolic plane, is there a constant $C = C(\Gamma)$ such that a bound as in Theorem 2.9 *exists?*

Moreover, if $(\Omega_l)_{l \geq 1}$ is a sequence of subgraphs such that $|\Omega_l| \underset{l\to\infty}{\longrightarrow} \infty$, then in many cases (Corollary 2.7, Corollary 2.10, [22, Corollary 1.4]) the behaviour of $\sigma_1(\Omega_l)$ satisfies $\sigma_1(\Omega_l) \underset{l\to\infty}{\longrightarrow} 0$. However, that is not always true, see [19, Example 3.7]. One may ask if the property is preserved under rough isometry (Question 2.36):

Let Γ_1, Γ_2 be two roughly isometric host graphs. Let us assume that in Γ_1, each sequence of subgraphs $(\Omega_l)_{l \geq 1}$ such that $|\Omega_l| \underset{l\to\infty}{\longrightarrow} \infty$ satisfies $\sigma_1(\Omega_l) \underset{l\to\infty}{\longrightarrow} 0$. Does Γ_2 also have this property?

As said before, these interrogations, and others (including some about higher dimensional constructions), will be asked in Section 2.5.

Notation. Throughout this paper, we shall work on graphs, on domains of $\mathbb{H}^2$ and on a manifold obtained from the domains. As stated before, the host graph will be denoted $\Gamma = (V, E)$. A subgraph of Γ is denoted Ω, while N and $\tilde{N}$ are used to speak about domains of $\mathbb{H}^2$. We use g to denote the metric of $\mathbb{H}^2$ and g' the one of the manifold; hence (N, g') is the notation we will use to speak about the manifold. A discretization of the manifold will be called $(\tilde{V}, \tilde{E}, V_\Sigma)$. We shall use the letters v, w to speak about vertices of graphs and x, y, z for elements of the domains or manifold. Several constants will appear, we shall call them $C_1, C_2, \ldots$; each C_l is used exactly once.

Plan of the paper. In Section 2.2, we define precisely what is a triangle-tiling graph. In Section 2.3, we make the constructions. The leading idea is actually simple: we want to associate a domain to a subgraph. However, we encounter some difficulties for different reasons. One of them is the question of the isolated boundary vertices, also called *bad boundary vertices* in [19, Definition 3.1]. We solve this problem in Section 2.3.1. Another difficulty comes from the fact that we want the domain to have a smooth boundary. This is the object of Section 2.3.2.

In Section 2.4 we prove Theorem 2.9. In order to do so, we want to use Theorem 3 of [9]. Therefore we have to make sure that the hypotheses of the theorem are verified, which is the object of Section 2.4.1. Once it is done, we apply the theorem and conclude the proof.

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2.2 Triangle groups and associated triangle-tiling graphs

Let us begin by explaining what triangle groups are and what links they have with tessellations of the model spaces $\mathbb{S}^2, \mathbb{E}^2$ and $\mathbb{H}^2$. When it is done, we can explain how to associate a triangle-tiling graph Γ to a triangle group.

Definition 2.11. Let $p, q, r \geq 2$ be integers. The associated triangle group $T^*(p, q, r)$ is

$$T^*(p,q,r) = \langle P, Q, R : P^2 = Q^2 = R^2 = (PQ)^r = (QR)^p = (RP)^q = 1\rangle.$$

In order to see the links between such an abstract group and a group of reflections, one can think about P, Q, R as reflections through the opposite sides of a triangle with angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ respectively.

It is well known that a triangle with angles α, β, γ satisfies $\alpha + \beta + \gamma > \pi$ in the spherical case, while we have $\alpha + \beta + \gamma = \pi$ in the Euclidean case and that $\alpha + \beta + \gamma < \pi$ in the hyperbolic case. Hence we can regroup the unordered triplets p, q, r according to the value of $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$. If the number obtained is greater than 1 we have to think about a spherical triangle, if it is equal to 1 we have to think about a Euclidean one and if it is less than 1 we have to think about a hyperbolic one.

As said before, we want to work on graphs that have exponential growth rates, therefore we will only consider the third case in this paper. Since one may ask if our result is still true for the two other cases, we remark that in the Euclidean case, the triangle group has polynomial growth rate and then has already been studied in [36]. Regarding the spherical case, the triangle group is finite and hence one can theoretically compute all different possible situations.

Then, from now on, $p, q, r \geq 2$ will be integers satisfying

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

Definition 2.12. We denote by $\mathbb{H}^2$ the hyperbolic plane, represented here by Poincaré's disk model, which is

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\},$$

endowed with the Riemannian metric

$$g(x, y) = 4 \cdot \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}.$$

We denote by $d_g(\cdot,\cdot)$ the distance induced by the metric g.

Remark 2.13. It is a known fact that for any triplet $0 \leq \alpha, \beta, \gamma < \pi$ such that $\alpha+\beta+\gamma < \pi$, there exists a hyperbolic triangle with angles α, β, γ. Moreover, there is a unique one up to isometry [2, Exercise 7.12]. Hence, given p, q, r as before, there exists a unique triangle which has angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$.

We state now Theorem 2.8 of [29]:

Theorem 2.14. *Let P, Q, R be the reflections in the sides of a hyperbolic triangle Δ_0 with angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$. The images of Δ_0 under the action of the distinct elements of the group $T^*(p,q,r)$ generated by P, Q, R fill the hyperbolic plane without gaps and overlapping.*

This means that the choice of the numbers p, q, r gives rise to a tessellation of the hyperbolic plane. Moreover, we know [2, Theorem 7.4.1] that reflections through geodesics are isometries of $\mathbb{H}^2$. Hence, each tile of the tessellation is a triangle which is isometric to the initial one, see Figure 2.1.

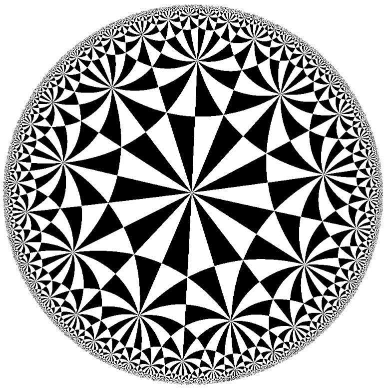

Figure 2.1: Tiling of the hyperbolic plane with congruent triangles of angles $\frac{\pi}{2}$, $\frac{\pi}{3}$ and $\frac{\pi}{9}$, coming from [39].

From such a tiling associated with a triangle group $T^*(p,q,r)$, one can naturally define an infinite simple connected undirected graph $\Gamma = \Gamma(p,q,r)$, called a triangle-tiling graph and that we will use as a host graph. We explain here how to define Γ.

Each triangle contains a point that is the center of its inscribed circle [2, Theorem 7.14.1]. We consider these points. They form the set V of vertices of Γ. The graph structure of Γ is defined as follows: two vertices $v_1, v_2 \in V$ are joined by an edge $\{v_1, v_2\}$ if and only if they belong to two adjacent triangles.

It is then obvious that $\Gamma = (V, E)$ is an infinite, 3-regular graph.

We can see Γ as a metric space when endowed with the path metric: each edge is of length 1, the distance between two vertices $v_1, v_2 \in V$ is the minimal number of edges we have to cross to go from v_1 to v_2.

Because of its links with $\mathbb{H}^2$, it is clear that Γ has an exponential growth rate. Hence, as said in Section 2.1, Γ does not enter the class of graphs concerned by Theorems 2.4, 2.5 and 2.6. Moreover, Γ has cycles, therefore it is not a tree. Hence, it does not enter the class of graphs of Theorem 2.8 either.

We recall that, given a connected locally finite graph X and any vertex v of X, the number of ends of X is $\lim_{n\to\infty} \|X\backslash B(v,n)\|$, where $B(v,n)$ is the ball centered at v with radius n and $\|X\backslash B(v,n)\|$ is the number of infinite connected components of $X\backslash B(v,n)$. It is well known that two roughly isometric graphs have the same number of ends, see [28, Proposition 8.2.8]. It is obvious that Γ has 1 end while a Cayley graph of a free group has infinitely many. Therefore, as said before, the structure of Γ is completely different from the graphs concerned by Theorem 2.8 and this difference will be felt in the way we solve the problem.

Definition 2.15. A rough isometry between two metric spaces (X, d_X) and (Y, d_Y) is a map $\phi : X \longrightarrow Y$ such that there exist constants $C_1 > 1, C_2,\ C_3 > 0$ satisfying

$$C_1^{-1} \cdot d_X(x_1, x_2) - C_2 \le d_Y(\phi(x_1), \phi(x_2)) \le C_1 \cdot d_X(x_1, x_2) + C_2$$

for all $x_1, x_2 \in X$ and satisfying

$$\bigcup_{x\in X} B(\phi(x), C_3) = Y.$$

If there is such a map, we say that X is roughly isometric to Y.

Proposition 2.16. *The host graph Γ constructed above is roughly isometric to $(\mathbb{H}^2, g)$, with constants that depend on the value of p, q, r.*

Proof. Take $\phi : \Gamma \longrightarrow \mathbb{H}^2$ as the canonical inclusion and take the constants as the triangle's diameter.

♡♠◇♣

2.3 Construction of the domain N

We consider a finite subset of vertices $\Omega \subset V$, connected for Γ, giving rise to a subgraph with boundary Ω as in Definition 2.3. We recall that each vertex is the center of a triangle of the tiling and that all triangles are isometric.

This section aims to detail a method allowing us to associate a smooth bounded domain N to the subgraph Ω. The relevance of the domain N lies within its structural links with the subgraph Ω: we will transcribe the structure of Ω onto N.

Before starting, we want to give an overview of the problems that could happen and that we will avoid.

The structural information of Ω is of two types: the neighborhood structure and the interior/boundary structure. Hence, we have to make sure that the domain of $\mathbb{H}^2$ we will associate to Ω is able to reflect these two pieces of information.
In other words, for two vertices $v_1, v_2 \in \bar{\Omega}$, we want v_1 to be near v_2 in Ω if and only if v_1 is near v_2 in the domain. Moreover, for $v \in B$, we want to guarantee the existence of a part of Σ near v. Reciprocally, for each $x \in \Sigma$, we want to guarantee the existence of a vertex $v \in B$ near x. The sense of the word *near* is the following: the proximity between x and v does not depend on the subgraph Ω. This proximity shall be quantified by Proposition 2.32.

As already spotted by Han and Hua in [19], one of the difficulties comes from the isolated boundary vertices. If $v \in B$ is isolated, we have to be clever to make sure there is $x \in \Sigma$ which is near v, see Example 2.20.

A second difficulty is the following: we want the domain N to be smooth. This will give us the possibility to make a change of metric on N, in order to use Theorem 3 of [9].

Hence the process contains two steps: first we find a domain $\tilde{N}$ which is structurally related to Ω but whose boundary $\tilde{\Sigma}$ is not smooth, and secondly we change this domain slightly by smoothing the angles in order to get the required domain N.

2.3.1 Construction of the domain $\tilde{N}$

Let us begin by considering a vertex $v \in \bar{\Omega}$ and the associated triangle T_v. In this section, v will always refer to this particular triangle. We call A_1, A_2, A_3 the vertices of T_v, respectively at angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$. We define a map $H : \{A_1, A_2, A_3\} \longrightarrow \mathbb{H}^2$ as follows:
$H(A_1)$ is the unique point of the geodesic segment $[v, A_1]$ such that $d_g(v, H(A_1)) = \frac{9}{10} \cdot d_g(v, A_1)$. The points $H(A_2)$ and $H(A_3)$ are defined similarly.

We then connect $H(A_1), H(A_2)$ and $H(A_3)$ with geodesic segments. This gives rise to a new triangle, denoted T'_v. By convexity, T'_v is strictly contained inside the initial triangle T_v. It is also easy to see that v is contained inside T'_v.

If $w \in \bar{\Omega}$ is another vertex of the subgraph, then by construction there is a triangle T_w of the tiling associated to w and there is an isometry $\psi_{v,w} : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ such that $\psi_{v,w}(T_v) = T_w$. This isometry is not necessary unique. If there are several, we just pick one and call it $\psi_{v,w}$. We consider this isometry and call $T'_w := \psi_{v,w}(T'_v)$.
We apply this process to each vertex of $\bar{\Omega}$. Hence we have now at our disposal $|\bar{\Omega}|$ new triangles, disjoint from each other and isometric to each other.

If $v_1, v_2 \in \bar{\Omega}$ are such that $v_1 \sim v_2$ in the subgraph, then by definition of Γ, v_1 and v_2 represent the centers of two triangles, let us say T_1 and T_2, having one side in common. Thus T_1 has two vertices x, y which are also vertices of the triangle T_2. As we said before, there is an isometry ψ_{v,v_1} of $\mathbb{H}^2$ such that $\psi_{v,v_1}(T_v) = T_1$. Without loss of generality, say that $\psi_{v,v_1}(A_1) = x$ and $\psi_{v,v_1}(A_2) = y$.

We denote $x_1 := \psi_{v,v_1}(H(A_1))$ and $y_1 := \psi_{v,v_1}(H(A_2))$, which are vertices of the triangle $T'_1 = \psi_{v,v_1}(T'_v)$. Similarly, we denote $x_2 := \psi_{v,v_2}(H(A_1))$ and $y_2 := \psi_{v,v_2}(H(A_2))$ which are vertices of the triangle $T'_2 = \psi_{v,v_2}(T'_v)$.

We then connect x_1 to x_2 by a geodesic segment, and we do the same with y_1 and y_2, see Figure 2.2.

We write $T'_1 \sim T'_2$ in order to say that we have connected the triangles T'_1 and T'_2.

This process connecting the triangles according to the structure of Ω allows us to notice the following relation: for two vertices $v_1, v_2 \in \bar{\Omega}$ which are the centers of two triangles T'_1, T'_2, we have

$$v_1 \sim v_2 \iff T'_1 \sim T'_2.$$

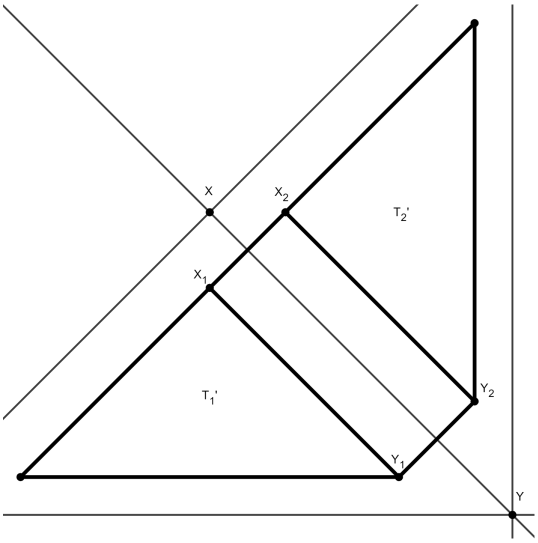

Figure 2.2: The vertices x_1, y_1 of T'_1 are connected respectively to the vertices x_2, y_2 of T'_2 because of the assumption that $v_1 \sim v_2$ in the subraph Ω.

Let us suppose that z is the common vertex of $2p$ triangles such that their centers $v_1, \ldots, v_{2p}$ satisfy $v_1 \sim v_2 \sim v_3 \sim \ldots \sim v_{2p} \sim v_1$ in Ω. Without loss of generality, let us say that $\psi_{v,v_1}(A_1) = z$. We denote $z_1 = \psi_{v,v_1}(H(A_1)), \ldots, z_{2p} = \psi_{v,v_{2p}}(H(A_1))$ as before. By applying the process described above, we connect z_1 to z_2, z_2 to $z_3, \ldots, z_{2p}$ to z_1 by geodesic segments, see Figure 2.3.
Of course, there is nothing specific about p and the same holds for q and r.

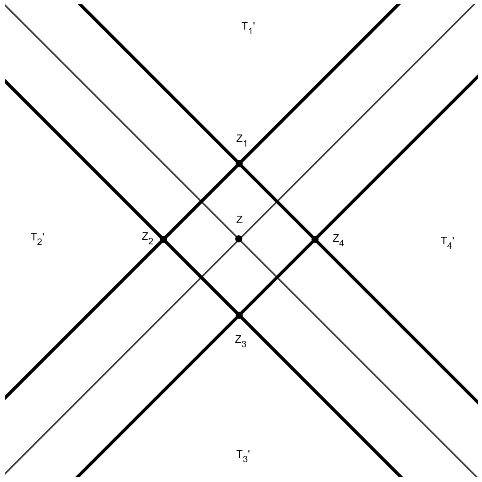

Figure 2.3: We connected z_1 to z_2, z_2 to z_3, z_3 to z_4 and z_4 to z_1 because of the assumption that $v_1 \sim v_2 \sim v_3 \sim v_4 \sim v_1$ in Ω.

Remark 2.17. The previous construction naturally generates different *simple polygons* contained inside the hyperbolic plane $\mathbb{H}^2$, of which the exhaustive list is the following:

- Each vertex $w \in \bar{\Omega}$ adds one triangle T'_w;
- Each pair of vertices $v_1, v_2 \in \bar{\Omega}$ such that $v_1 \sim v_2$ adds one quadrilateral;

- Each vertex z which is the common vertex of $2p$ triangles such that their centers $v_1, \dots, v_{2p}$ satisfy $v_1 \sim v_2 \sim v_3 \sim \dots \sim v_{2p} \sim v_1$ in Ω adds one $2p$-gon;
- Each vertex z which is the common vertex of $2q$ triangles such that their centers $v_1, \dots, v_{2q}$ satisfy $v_1 \sim v_2 \sim v_3 \sim \dots \sim v_{2q} \sim v_1$ in Ω adds one $2q$-gon;
- Each vertex z which is the common vertex of $2r$ triangles such that their centers $v_1, \dots, v_{2r}$ satisfy $v_1 \sim v_2 \sim v_3 \sim \dots \sim v_{2r} \sim v_1$ in Ω adds one $2r$-gon.

Definition 2.18. We call K the compact subset of $\mathbb{H}^2$ obtained by considering the closure of the union of all the simple polygons generated by the previous construction. We also call $\hat{N}$ the bounded domain of $\mathbb{H}^2$ defined by $\hat{N} = \overset{\circ}{K}$ and we call $\hat{\Sigma}$ the boundary of $\hat{N}$.

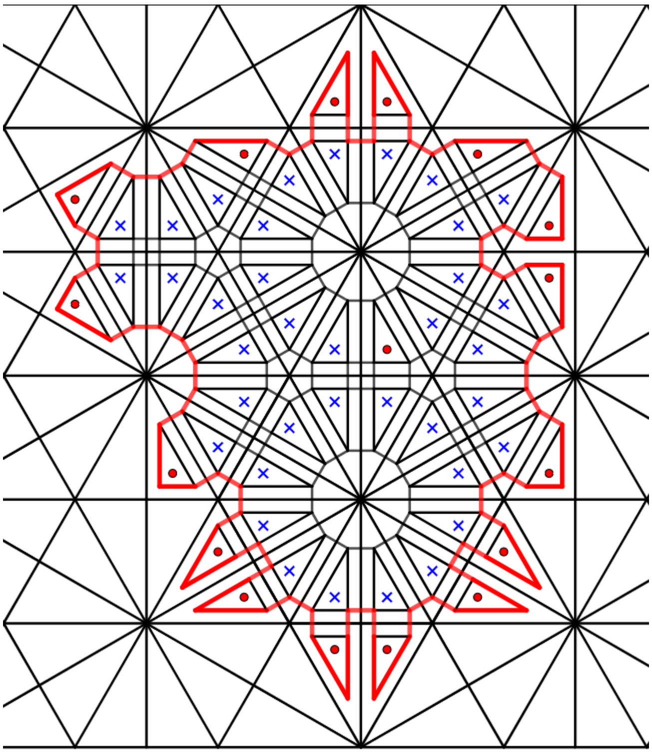

Figure 2.4: The crosses represent the interior Ω of the subgraph, the dots represent the boundary B of the subgraph. The polygonal curve represents the boundary $\hat{\Sigma}$ while the polygon (of which $\hat{\Sigma}$ is the boundary) is the interior $\hat{N}$.

At this point of the paper, one may ask why we do not simply define the domain as the thickening of the union of all T_w, for $w \in \bar{\Omega}$. The reason is that by doing so, the domain would not be able to reflect the neighborhood structure of the subgraph.
Indeed, we recall that by definition of a subgraph, two boundary vertices are never connected by an edge. Let us consider two vertices $w_1, w_2 \in B$ such that T_{w_1} is adjacent to T_{w_2} (meaning that $\{w_1, w_2\} \in E$). Gluing the two triangles T_{w_1}, T_{w_2} would give the information that w_1 is adjacent to w_2 in the subgraph, which is not the case because they are two boundary vertices.
This mismatch between the structure of the domain and the structure of the subgraph would then jeopardize one of our next constructions, namely the rough isometry of Proposition 2.32.

This proposition claims the existence of a rough isometry whose constants do not depend on the subgraph Ω chosen. In order to prove the existence of such a rough isometry, it is crucial that the domain N we are building reflects the neighborhood structure of the subgraph Ω. We give more details about this problem in Appendix 2.A.

Remark 2.19. We recall that, by construction, the domain $\hat{N}$ has the same neighborhood structure as the subgraph Ω. Indeed, we already saw that for $v_1, v_2 \in \bar{\Omega}$,

$$v_1 \sim v_2 \iff T'_1 \sim T'_2.$$

However, the boundary structure of $\hat{N}$ is not analog to the one of Ω. We already have one implication: for all $x \in \hat{\Sigma}$, there exists $w \in B$ such that w is near x.
The reciprocal is not verified. If $w \in B$, there is no guarantee that there exists $x \in \hat{\Sigma}$ such that x is near w. To see that, one can look at Example 2.20.

Example 2.20. Choose a vertex v^* of the host graph and define Ω as the ball of radius n deprived of v^*. This will give rise to a subgraph Ω, for which $v^* \in B$. However, there is no $x \in \hat{\Sigma}$ near v^*. Indeed, the bigger n is, the bigger the distance between $\hat{\Sigma}$ and v^* is. Hence the proximity between $\hat{\Sigma}$ and v^* depends on the subgraph, which we want to avoid. This kind of situation also appears on Figure 2.4, where we can see an isolated boundary vertex.

To remedy this problem, we proceed to do a surgery of this domain $\hat{N}$: for each $w \in B$, we remove the ball centered at w of radius $\frac{\rho}{2}$, where ρ denotes the radius of the circle inscribed in T'_w, see Figure 2.5

Definition 2.21. We call $\tilde{N}$ the domain obtain after the removal of the balls, and we call $\tilde{\Sigma}$ its boundary.

Remark 2.22. This last surgery obviously gives us the reciprocal we lacked until now: for each $w \in B$, there exists $x \in \tilde{\Sigma}$ such that x is near w.

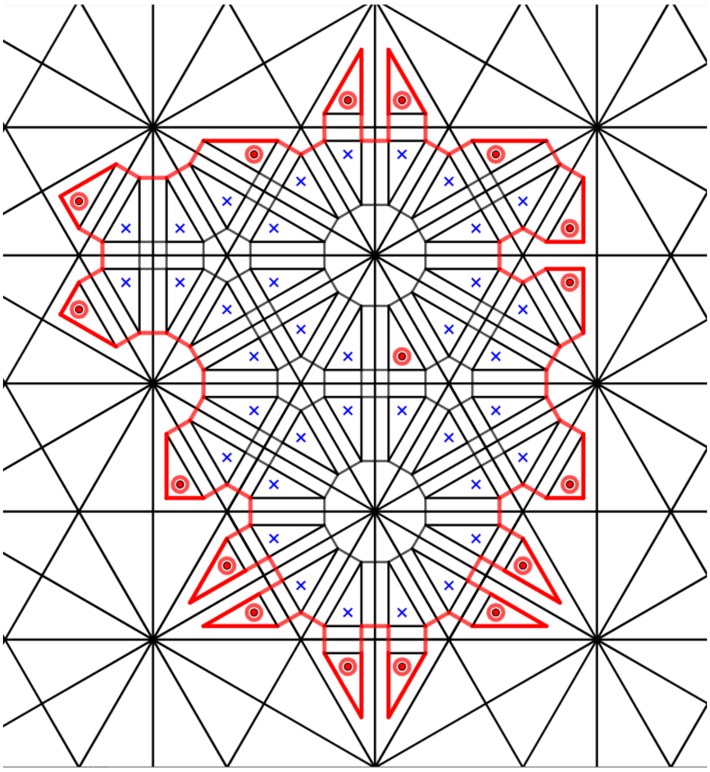

Figure 2.5: The crosses represent the vertices of Ω, the dots represent the boundary B. The balls surrounding the boundary vertices are removed from the domain and the structure of the subgraph is readable on the domain.

This bounded domain $\tilde{N}$ is not our final domain because we want one with a smooth boundary.

2.3.2 Smoothing of the domain $\tilde{N}$

As we said in the introduction, we want to discretize the domain in order to find an upper bound for the Steklov spectrum of Ω. One way to do this consists in using Theorem 3 of [9]. Of course, we have to make sure the assumptions of this theorem are verified before using it. However, the domain $\tilde{N}$ does not satisfy all these assumptions, see Remark 2.26. This section is devoted to modifying the domain $\tilde{N}$ and getting a new domain N which has the advantage to have a smooth boundary.

Note that, as always in this paper, we have to make sure that the operations we make do not depend on the subgraph Ω, but only on the host graph Γ.

We recall that $\tilde{\Sigma}$ is composed of the union of $\hat{\Sigma}$ and many circles. Each circle is already a smooth connected component of $\tilde{\Sigma}$, hence we only have to smooth $\hat{\Sigma}$ out. Each connected component of $\hat{\Sigma}$ is a simple closed C^∞ piecewise curve, composed of geodesic segments. Note that by construction, there exist at most $4 \times 3 - 3 = 9$ different segments (two isometric segments are identified). We shall designate by *corner* the intersection of two geodesic segments forming $\hat{\Sigma}$. A corner is therefore a point of the curve whose neighborhoods are of class C^0, but not of class C^1. By construction, a corner is always located on a vertex of a triangle T'.

The regularity of our construction allows us to state that the domain $\tilde{N}$ has at most $\begin{pmatrix} 4 \\ 2 \end{pmatrix} \times 3 = 18$ different internal angles (two congruent angles are identified).

The interest of these comments is to simplify considerably the smoothing of the domain $\tilde{N}$. Indeed, there are at most 18 different types of angles to smooth out.

Let us call $\lambda_1, \dots, \lambda_9$ the length of the geodesic segments and let us denote

$$\lambda := \min\{\lambda_1, \dots, \lambda_9\}.$$

If $\tilde{\Sigma}$ has n corners, let us call them $z_1, \dots z_n$. For each corner z_i, there exist exactly two points $x_i, x_i' \in \tilde{\Sigma}$ such that

$$d_g(x_i, z_i) = d_g(x_i', z_i) = \frac{\lambda}{10}.$$

Let us consider a corner z_i as well as the two associated points x_i, x_i'.

We then create a smooth curve

$$\alpha_1 : [0,1] \longrightarrow \mathbb{H}^2$$

such that

- $\alpha_1(0) = x_i$, $\alpha_1(1) = x_i'$;
- For all $t \in (0,1)$ we have $\alpha_1(t) \in \tilde{N}$;
- For all $t \in [0,1]$ we have $d_g(\alpha_1(t), z_i) \leq \frac{\lambda}{10}$;
- A curve whose image is

  $$[z_{i-1}, x_i] \cup \alpha_1([0,1]) \cup [x_i', z_{i+1}]$$

  is smooth, see Figure 2.6.

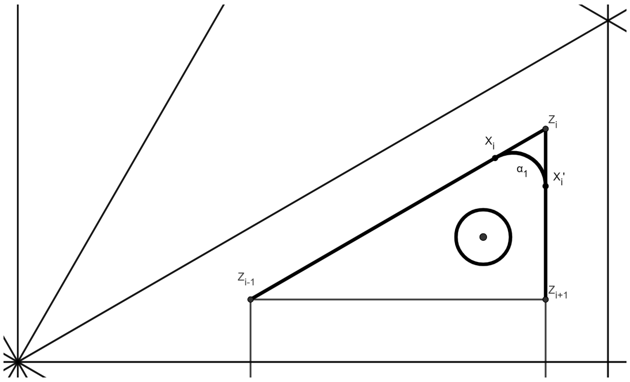

Figure 2.6: The curve α_1 can be seen as a smoothing of the angle at the corner z_i.

Then suppose that z_i is a corner associated with an angle which is not congruent to the previous one. We then create a smooth curve

$$\alpha_2 : [0,1] \longrightarrow \mathbb{H}^2$$

with the same four properties as the previous curve, see Figure 2.7.

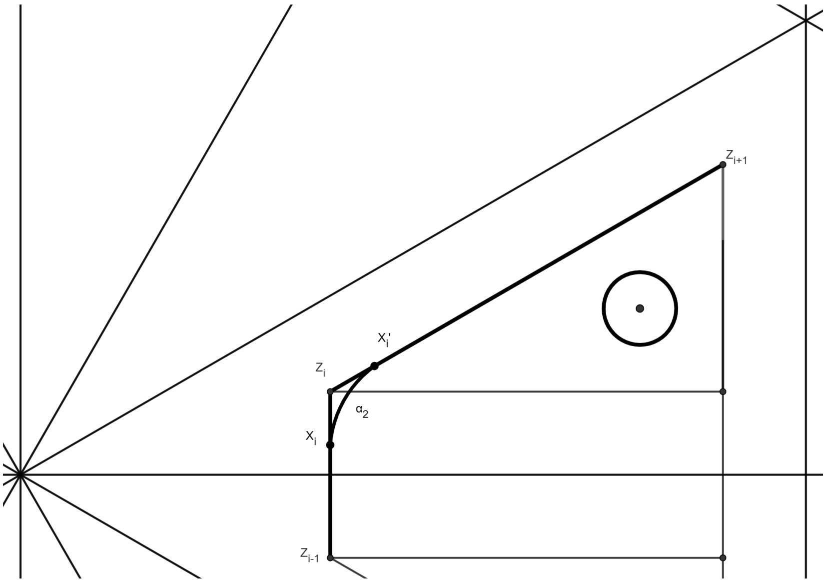

Figure 2.7: The curve α_2 is a smoothing of the angle at z_i.

We continue the process and create a smooth curve for each type of angle, at most 18 times as said before.

Remark 2.23. If z_j is another corner of the same type as z_i, meaning that the angle at z_j is congruent to the angle at z_i, there is then an isometry $\Psi : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ which sends the angle at z_i onto the angle at z_j. The smoothing curve at angle z_j is then given by $\Psi \circ \alpha_\mu$, where $\mu \in \{1, \dots, 18\}$ depends on the nature of the angle.

Thus, we smooth out the domain $\tilde{N}$ with these 18 curves and obtain a new connected domain with smooth boundary.
We obtain the domain N that we wanted, whose boundary is denoted Σ. By construction, the domain N has the following characteristics:

- N is connected;
- The boundary Σ is smooth;
- Σ is composed of at most 28 types of curve:
  - The 9 geodesic segments (coming from triangles and quadrilaterals);
  - The 18 smoothing curves $\alpha_1, \dots, \alpha_{18}$;
  - The circles resulting from the removal of the balls.

Moreover, the domain N is constructed in a way that the structure of the subgraph Ω is readable in it. Indeed, if we call *smoothed triangle* a region of N of the form $N \cap T'_w$, for $w \in \bar{\Omega}$, then

- A smoothed triangle $N \cap T'_{v_1}$ is connected to a neighbor $N \cap T'_{v_2}$ if and only if $v_1 \sim v_2$ in Ω;

- A vertex w is part of B if and only if there exists $x \in \Sigma$ such that x is near w. As said before, Proposition 2.32 will clarify the sense of the word *near*.

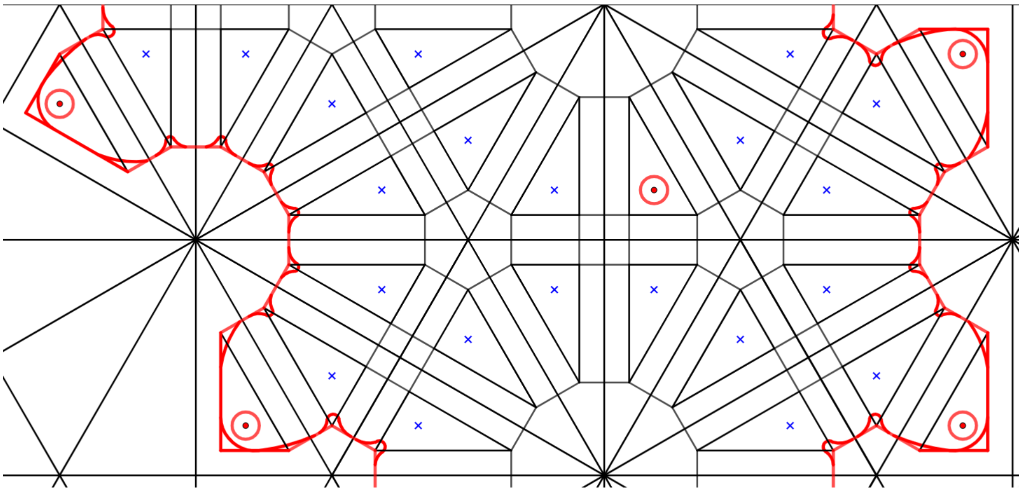

Figure 2.8: The crosses represent the vertices of Ω, the dots represent the boundary B. The balls surrounding the boundary vertices are removed from the domain, the structure of the subgraph is readable in the domain and Σ is smooth.

Remark 2.24. Since each $w \in B$ adds one connected component of Σ as a circle, we have the inequality

$$|\Sigma| \geq C_4 \cdot |B|, \tag{2.3.1}$$

where C_4 corresponds to the perimeter of a circle of radius $\frac{\rho}{2}$.

2.4 Proof of the main theorem

Let us begin by recalling Theorem 1.2 of [7].

Theorem 2.25. *There exists a constant C_5 such that for all bounded domains N of the hyperbolic space $\mathbb{H}^2$ and for all $k \geq 0$,*

$$\sigma_k(N, g) \leq C_5 \cdot \frac{k}{|\Sigma|}. \tag{2.4.2}$$

Actually, the result of Colbois et al. is more general than that, but this statement is enough for our needs.

The domain N being structurally similar to the subgraph Ω, we will show that a bound of the same type exists for the subgraph's spectrum. The goal of this section is to transfer this result to the subgraph.

To do this, we want to discretize the domain N. Let us recall the conditions that the domain must satisfy to be discretized:
We have to assume the existence of constants $\kappa > 0$ and $r_0 \in (0, 1)$ such that

- The boundary Σ admits a neighborhood which is isometric to the cylinder $[0,1]\times\Sigma$, whose boundary corresponds to $\{0\}\times\Sigma$;
- The Ricci curvature of N is bounded below by $-\kappa$;
- The Ricci curvature of Σ is bounded below by 0;
- For all $x \in N$ such that $d_g(x,\Sigma) > 1$, we have $\mathrm{inj}_M(x) > r_0$;
- For all $x \in \Sigma$, we have $\mathrm{inj}_\Sigma(x) > r_0$.

For further investigation on this topic and to understand why these assumptions are made, one can see [9].

Remark 2.26. The last four conditions are trivially satisfied by N. Moreover, the constants κ, r_0 do not depend on the subgraph Ω. Indeed, the regularity of the construction of the domain N allows to give constants κ, r_0 valid for any domain N obtained by the process described above. In other words, if we call $\mathcal{M} = \mathcal{M}(\kappa, r_0)$ the class of 2-dimensional manifolds which satisfy the last four properties, then $N \in \mathcal{M}$ whatever the chosen subgraph Ω.

On the other hand, the first assumption is not satisfied by the domain. Indeed, Σ does not have a neighborhood isometric to a cylinder. To remedy this, we will proceed to do a change of metric on N in order to obtain a new Riemannian manifold which satisfies the five properties.

2.4.1 Changing the metric on the domain

The main difficulty of this subsection is proceeding to do a change of metric which is uniform for all domains N obtained by the procedure described in Section 2.3.
Here, the word *uniform* reflects the existence of a constant C_6 as in Proposition 2.28 which is valid for all domains.

Let us denote

$$N(\delta) = \{x \in N \ :\ d_g(x,\Sigma) \le \delta\}$$

the δ-neighborhood of the boundary.

Proposition 2.27 (Lemma 34 of [9]). *There exist on N a $\delta > 0$ (depending only on the 28 types of curves) and a Riemannian metric g' such that*

- *$(N(\delta), g')$ is isometric to $[0,1]\times\Sigma$;*
- *The metrics g and g' are homothetic on $N\backslash N(3\delta)$.*

Proof. We will use the Fermi parallel coordinates: we parametrize each connected component of Σ by arc-length and call s the parameter. We then use the distance t to Σ as a second parameter to describe the points of N lying in a close neighborhood of Σ. In these coordinates, the hyperbolic metric is expressed by

$$g(s,t) = \varphi(s,t)\cdot ds^2 + dt^2,$$

where φ is a smooth positive function satisfying $\varphi(s,0) = 1$.

Let $\delta > 0$ be small enough to have $\frac{1}{2} \leq \varphi(s,t) \leq 2$ on $N(3\delta)$ (such a δ exists because φ is smooth).
We call g_0 the product metric which, in the Fermi coordinates (s,t), is expressed by

$$g_0(s,t) = ds^2 + dt^2.$$

We then take a smooth function

$$\chi : [0, 3\delta] \longrightarrow [0, 1]$$

such that $\chi \equiv 0$ on $[0, \delta]$, $\chi \equiv 1$ on $[2\delta, 3\delta]$ and such that χ is strictly increasing on $[\delta, 2\delta]$.
Then we define the metric

$$g_\delta(s,t) = \chi(t)g(s,t) + (1 - \chi(t))g_0(s,t).$$

This metric coincides with the hyperbolic metric on $N(3\delta)\backslash N(2\delta)$, then it can be extended all over the domain N into a metric that we continue to call g_δ.
Moreover, endowed with this metric, $N(\delta)$ is isometric to $[0, \delta] \times \Sigma$. We then define the metric

$$g' := \frac{1}{\delta^2} g_\delta,$$

for the cylindrical neighborhood to have length 1.

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The value of δ depends only on the 28 types of curves composing Σ. That is the reason we built the domain N with such regularity. Thanks to the process, we can choose δ independently of the subgraph Ω chosen.

Proposition 2.28 (Lemma 34 of [9]). *There exists a constant $C_6 > 1$, that does not depend on the subgraph Ω, such that for all $x \in N$ and all $v \in T_xN$, $v \neq 0$, we have*

$$\frac{1}{C_6} \leq \frac{g'(x)(v,v)}{g(x)(v,v)} \leq C_6.$$

Proof. We distinguish three cases:

- $x \in N\backslash N(2\delta)$;
- $x \in N(\delta)$;
- $x \in N(2\delta)\backslash N(\delta)$.

Let us start with the first one. Let $x \in N\backslash N(2\delta)$ and $0 \neq v \in T_xN$. We have

$$\frac{g'(x)(v,v)}{g(x)(v,v)} = \frac{\frac{1}{\delta^2} g_\delta(x)(v,v)}{g(x)(v,v)} = \frac{\frac{1}{\delta^2} g(x)(v,v)}{g(x)(v,v)} = \frac{1}{\delta^2}$$

because on $N\backslash N(2\delta)$, the metric g_δ coincides with the hyperbolic metric g.

For the second case, let $x \in N(\delta)$ and $0 \neq v \in T_xN.$ we have

$$\begin{aligned}
\frac{g'(x)(v,v)}{g(x)(v,v)} &= \frac{g'(x)(v,v)}{(\varphi(s,t)ds^2 + dt^2)(v,v)} \leq \frac{g'(x)(v,v)}{(\frac{1}{2}ds^2 + \frac{1}{2}dt^2)(v,v)} \\
&= \frac{\frac{1}{\delta^2}g_\delta(x)(v,v)}{\frac{1}{2}(ds^2 + dt^2)(v,v)} = \frac{\frac{1}{\delta^2}g_0(x)(v,v)}{\frac{1}{2}g_0(x)(v,v)} \\
&= \frac{2}{\delta^2}
\end{aligned}$$

because g_δ coincides with the product metric g_0 on $N(\delta)$.
In a similar way, we have

$$\begin{aligned}
\frac{g'(x)(v,v)}{g(x)(v,v)} &= \frac{g'(x)(v,v)}{(\varphi(s,t)ds^2 + dt^2)(v,v)} \geq \frac{g'(x)(v,v)}{(2ds^2 + 2dt^2)(v,v)} \\
&= \frac{\frac{1}{\delta^2}g_\delta(x)(v,v)}{2(ds^2 + dt^2)(v,v)} = \frac{\frac{1}{\delta^2}g_0(x)(v,v)}{2g_0(x)(v,v)} \\
&= \frac{1}{2\delta^2}.
\end{aligned}$$

Let us now look at the third case. Let $x \in N(2\delta)\backslash N(\delta)$ and $0 \neq v \in T_xN.$
We recall that on $N(2\delta)\backslash N(\delta)$, the metric g_δ interpolates the product metric g_0 and the hyperbolic metric g with the help of a smooth increasing function χ.
Then we have

$$\begin{aligned}
\frac{g'(x)(v,v)}{g(x)(v,v)} &= \frac{\frac{1}{\delta^2}g_\delta(x)(v,v)}{g(x)(v,v)} = \frac{\frac{1}{\delta^2}(\chi(t)g(s,t) + (1-\chi(t))g_0(s,t))(v,v)}{g(x)(v,v)} \\
&= \frac{1}{\delta^2}\left(\chi(t) + (1-\chi(t))\frac{g_0(s,t)(v,v)}{g(x)(v,v)}\right) \geq \frac{1}{\delta^2}\left(\chi(t) + (1-\chi(t))\frac{g_0(s,t)(v,v)}{2g_0(x)(v,v)}\right) \\
&= \frac{\chi(t)}{\delta^2} + \frac{1-\chi(t)}{2\delta^2} \geq \frac{1}{2\delta^2}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\frac{g'(x)(v,v)}{g(x)(v,v)} &= \frac{\frac{1}{\delta^2}g_\delta(x)(v,v)}{g(x)(v,v)} = \frac{\frac{1}{\delta^2}(\chi(t)g(s,t) + (1-\chi(t))g_0(s,t))(v,v)}{g(x)(v,v)} \\
&= \frac{1}{\delta^2}\left(\chi(t) + (1-\chi(t))\frac{g_0(s,t)(v,v)}{g(x)(v,v)}\right) \leq \frac{1}{\delta^2}\left(\chi(t) + (1-\chi(t))\frac{g_0(s,t)(v,v)}{\frac{1}{2}g_0(x)(v,v)}\right) \\
&= \frac{\chi(t)}{\delta^2} + \frac{1-\chi(t)}{\frac{1}{2}\delta^2} \\
&\leq \frac{1}{\frac{1}{2}\delta^2} = \frac{2}{\delta^2}.
\end{aligned}$$

Then the ratio is bounded for all $x \in N$ and for all $v \in T_xN,\ v \neq 0$, and we can choose

$$C_6 := \frac{2}{\delta^2}.$$

Moreover, this constant C_6 does not depend on the chosen subgraph Ω. Indeed, the function φ depends only on the, at most, 28 types of curves forming Σ (which we have fixed once and for all), and δ depends only on φ. Thus, as said before, the constant $\delta > 0$ can be chosen independently of the subgraph, which allows us to fix a universal value of $C_6 > 1$ for all the domains N obtained thanks to the procedure described in Section 2.3.

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We now have at our disposal a new Riemannian manifold with boundary, denoted (N, g'), which is related to (N, g) in the sense of Proposition 2.28. We recall now Proposition 32 of [9]:

Proposition 2.29. *Let N be a Riemannian manifold of dimension m, compact with smooth boundary and let g, g' be two Riemannian metrics on N. Let us assume that there exists a constant $C_6 > 1$ such that for all $x \in N$ and for all $v \in T_xN,\ v \neq 0$, we have*

$$\frac{1}{C_6} \leq \frac{g'(x)(v,v)}{g(x)(v,v)} \leq C_6.$$

Then we have

$$\frac{1}{C_6^{2m+1}} \leq \frac{\sigma_k(N,g')}{\sigma_k(N,g)} \leq C_6^{2m+1}.$$

The assumption is exactly what we prove in Proposition 2.28. Hence we can apply this result to (N, g) and (N, g') in order to get:

$$\sigma_k(N,g') \leq C_6^5 \cdot \sigma_k(N,g). \tag{2.4.3}$$

2.4.2 Discretization of the manifold (N, g')

Let us recall that we proceeded to a change of metric on N in order to give it the ability to be discretized, according to constants r_0 and κ, as said in Remark 2.26. There exist several ways to discretize a manifold. In this paper, we apply the process described in [9], for we want the discretization to have a spectral link with the manifold.

This process is the following:
We choose $\varepsilon \in (0, r_0/4)$ and we choose V_Σ a maximal ε-separated subset of Σ. Then we call V'_Σ the copy of V_Σ lying 4ε away from the boundary:

$$V'_\Sigma = \{4\varepsilon\} \times V_\Sigma.$$

Then we choose V_I a maximal ε-separated subset of $N \backslash [0, 4\varepsilon] \times \Sigma$ such that $V'_\Sigma \subset V_I$.
Then we consider the subset $\tilde{V} = V_\Sigma \cup V_I$ and grant it the structure of a graph by imposing

- Two vertices $v, w \in \tilde{V}$ are adjacent as soon as $d_{g'}(v, w) \leq 3\varepsilon$;
- A vertex $v \in V_\Sigma$ is adjacent to its counterpart $v' \in V'_\Sigma$.

This process gives a graph with boundary $(\tilde{V}, \tilde{E}, V_\Sigma)$, simply denoted $(\tilde{V}, V_\Sigma)$ hereafter, whose boundary is V_Σ and that we call an ε-discretization of N.

Theorem 3 point 4) of [9] allows us to state:

Theorem 2.30. *There exists a constant $C_7 > 0$ depending only on κ, r_0 and ε such that for all $k \leq |V_\Sigma|$, we have*

$$\sigma_k(\tilde{V}, V_\Sigma) \leq C_7 \cdot \sigma_k(N, g') \cdot k. \tag{2.4.4}$$

2.4.3 Rough isometry between $(\tilde{V}, V_\Sigma)$ and Ω

We now want to exploit the graph $(\tilde{V}, V_\Sigma)$ for which we have an upper bound relative to its spectrum to control the spectrum of our initial subgraph Ω. In order to do it, we will have to deal with the concept of rough isometry once again. This will allow us to use Proposition 16 of [9] to compare the Steklov spectra of the graphs. The main difficulty here is that we have to make sure the constants of the rough isometry are independant of the subgraph Ω. Let us begin by defining what a rough isometry is in the context of graphs with boundary.

Definition 2.31. A rough isometry ϕ between two graphs with boundary, say $(\bar{\Omega}_1, E_1', B_1)$ and $(\bar{\Omega}_2, E_2', B_2)$, is a rough isometry which sends B_1 onto B_2 and such that the restriction of ϕ to B_1 is a rough isometry $B_1 \longrightarrow B_2$ when considering extrinsic distances on B_1 and B_2.

Proposition 2.32. *There exists a rough isometry $\bar{\phi} : (\tilde{V}, V_\Sigma) \longrightarrow \bar{\Omega}$ whose constants C_1, C_2, C_3 are independent of the subgraph Ω.*

Proof. We have to define a map $\bar{\phi} : (\tilde{V}, V_\Sigma) \longrightarrow \bar{\Omega}$ and show that it is a rough isometry.
Remark that the vertices v of $\tilde{V}$ can be of different types. There are boundary vertices coming from the 28 different kinds of curves forming Σ, and there are interior vertices coming from N. As a consequence, the definition of $\bar{\phi}$ is a little bit heavy, but the idea to define the rough isometry is very natural: each vertex $v \in \tilde{V}$ is sent onto the vertex w of $\bar{\Omega}$ which is of the same nature (interior or boundary) and which is the nearest to it.

Let us define

$$\bar{\phi} : (\tilde{V}, V_\Sigma) \longrightarrow \bar{\Omega}.$$

For the vertices of the boundary:

- For $v \in V_\Sigma$ such that v is part of a side of a triangle T', we choose $\bar{\phi}(v) \in B$ the vertex at the center of T';
- For $v \in V_\Sigma$ such that v is part of the boundary of a ball that had been removed, we choose $\bar{\phi}(v) \in B$ the vertex at the center of the removed ball;
- For $v \in V_\Sigma$ such that v is part of a side of a quadrilateral, we find the side of a triangle closest to v and we choose $\bar{\phi}(v) \in B$ as if v were on this triangle's side;
- For $v \in V_\Sigma$ such that v is part of a smoothing curve, we find the side of a triangle closest to v and choose $\bar{\phi}(v) \in B$ as if v were on this triangle's side.

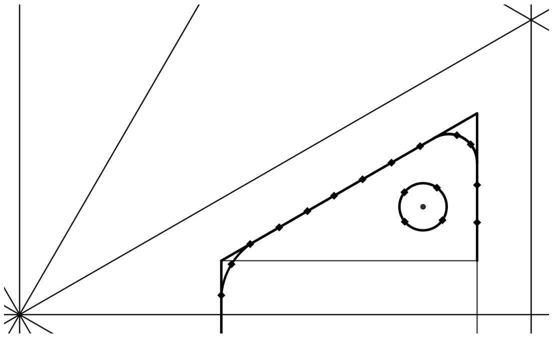

Figure 2.9: The vertices of V_Σ are represented by diamonds, the dot vertex belongs to B. All of the diamonds are sent to the dot by $\bar{\phi}$.

And for the interior vertices:

- For $v \in V_I$ such that v is part of a triangle whose center is $w \in \Omega$, we choose $\bar{\phi}(v) = w$;
- For $v \in V_I$ such that v is part of a triangle whose center is $w \in B$, then there exists at least one $w' \in \Omega$ such that $w \sim w'$. We then choose $\bar{\phi}(v) = w'$. If there are several possibilities, we choose one once and for all;
- For $v \in V_I$ such that v is part of a quadrilateral, then two opposite sides of this quadrilateral are the sides of two triangles T'_1, T'_2. At least one of them has a center $w \in \Omega$. We then choose $\bar{\phi}(v) = w$. If there are two possibilities, we choose one once and for all;
- For $v \in V_I$ such that v is part of a $2p$-gon (respectively $2q$-gon, $2r$-gon), then this $2p$-gon (resp. $2q$-gon, $2r$-gon) is surrounded by $2p$ (resp. $2q, 2r$) triangles $T'_1, \dots, T'_{2p}$ (resp. T'_{2q}, T'_{2r}) of which at least p (resp. q, r) have a center $w \in \Omega$. We then choose $\bar{\phi}(v) = w$ once and for all.

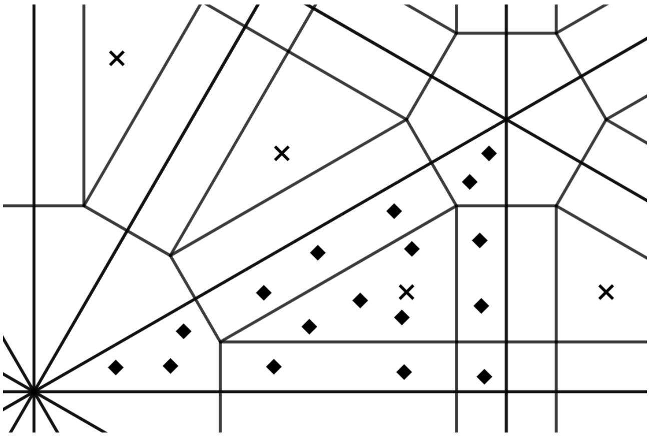

Figure 2.10: The diamond vertices are part of V_I, the cross vertices belongs to Ω. All of the diamonds vertices are send to the bottom left cross vertex by $\bar{\phi}$.

In order to show that $\bar{\phi}$ is a rough isometry, let us partition the domain N into cobblestones: a cobblestone C is defined as the intersection of a triangle T of the initial tiling with N. If $w \in \bar{\Omega}$ is the center of a triangle T_w, we denote by C_w the associated cobblestone. We also write $C_w \sim C_{w'}$ to say that two cobblestones are adjacent.

Then we choose C_1 as the cardinality of the biggest possible ε-separated set contained inside a cobblestone multiplied by $\max\{p, q, r\}$. Then we choose $C_2 = C_1$. Thus, if two vertices $v_1, v_2 \in \tilde{V}$ belong to the same cobblestone, we have $d_{\tilde{V}}(v_1, v_2) \leq C_1$.

We recall that by our construction of the domain N, for $w, w' \in \bar{\Omega}$ we have

$$w \sim w' \iff C_w \sim C_{w'},$$

i.e the neighborhood structure of the subgraph is readable onto the domain. Therefore, for $w_1, w_2 \in \bar{\Omega}$, $w_1 \neq w_2$, the distance $d_{\bar{\Omega}}(w_1, w_2)$ represents the number of cobblestones that separate w_1 from w_2 plus one. Thus, if $v_1, v_2 \in \tilde{V}$ are such that $\bar{\phi}(v_1) = w_1$ and $\bar{\phi}(v_2) = w_2$, then we have

$$C_1^{-1} d_{\tilde{V}}(v_1, v_2) - C_2 \leq d_{\bar{\Omega}}(w_1, w_2) \leq C_1 d_{\tilde{V}}(v_1, v_2) + C_2.$$

Moreover, $\bar{\phi}$ is a surjective map so we can choose $C_3 = 1$ and we get

$$\bigcup_{v \in \tilde{V}} B(\bar{\phi}(v), C_3) = \bar{\Omega}.$$

♡♠◇♣

We can now recall Proposition 16 of [9]:

Proposition 2.33. *Given* $C_1 \geq 1, C_2, C_3 \geq 0$, *there exist some constants* C_8, C_9 *depending only on* C_1, C_2, C_3 *and of the maximal degree of the vertices such that for all graphs with boundary* $(\Gamma_1, B_1), (\Gamma_2, B_2)$ *roughly isometric with constants* C_1, C_2, C_3, *we have*

$$C_8 \leq \frac{\sigma_k(\Gamma_1, B_1)}{\sigma_k(\Gamma_2, B_2)} \leq C_9.$$

Applied to this situation, we obtain

$$\sigma_k(\Omega) \leq \frac{1}{C_8} \sigma_k(\tilde{V}, V_\Sigma). \tag{2.4.5}$$

2.4.4 Conclusion

In this section, we prove Theorem 2.9 and Corollary 2.10.

Proof. Throughout the paper, we got different results, that we can now assemble to finally obtain Theorem 2.9:

$$\begin{aligned}
\sigma_k(\Omega) &\overset{(2.4.5)}{\leq} \frac{1}{C_8} \cdot \sigma_k(\tilde{V}, V_\Sigma) \\
&\overset{(2.4.4)}{\leq} \frac{1}{C_8} \cdot C_7 \cdot \sigma_k(N, g') \cdot k \\
&\overset{(2.4.3)}{\leq} \frac{1}{C_8} \cdot C_7 \cdot C_6^5 \cdot \sigma_k(N, g) \cdot k \\
&\overset{(2.4.2)}{\leq} \frac{1}{C_8} \cdot C_7 \cdot C_6^5 \cdot C_6 \cdot \frac{k}{|\Sigma|} \cdot k \\
&\overset{(2.3.1)}{\leq} \frac{1}{C_8} \cdot C_7 \cdot C_6^5 \cdot C_6 \cdot \frac{k}{C_4 \cdot |B|} \cdot k \\
&=: C \cdot \frac{1}{|B|} \cdot k^2.
\end{aligned}$$

All along the paper, we took care of specifying on which parameters the constants depend. It happens that they do not depend on the subgraph Ω chosen. They only depend on the host graph Γ and on ε. Therefore, if we set a value for ε, we can take the same constant C for all subgraphs Ω of Γ; it is now fixed once and for all.

As a consequence, for a choice of three integers $p, q, r \geq 2$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, giving rise to a tessellation of the hyperbolic plane and to a host graph Γ as defined in Section 2.2, there exists a constant $C = C(\Gamma)$ such that for any subgraph Ω of Γ, we have

$$\sigma_k(\Omega) \leq C(\Gamma) \cdot \frac{1}{|B|} \cdot k^2.$$

♡♠◇♣

From this statement, let us prove Corollary 2.10.

Proof. It is enough to notice the following fact: for $(\Omega_l, B_l)_{l \geq 1}$ a family of subgraphs of Γ such that $|\Omega_l| \longrightarrow \infty$, then we also have $|B_l| \longrightarrow \infty$.

Therefore, for all $k \in \mathbb{N}$ fixed, we have

$$\sigma_k(\Omega_l, B_l) \leq C(\Gamma) \cdot \frac{1}{|B_l|} \cdot k^2 \underset{l \to \infty}{\longrightarrow} 0.$$

♡♠◇♣

2.5 Consideration and interrogation

All the constructions above were about a host graph Γ, which was a triangle-tiling graph. However, one may have noticed that we could have used other polygons rather than triangles and still obtained the result. The information we used is the finite number of possible situations, like the 28 different kinds of curves composing Σ or the 18 types of angles to smooth out.
All these constructions could have emerged from any exact tessellation of the hyperbolic plane, as long as the tiles are compact and the number of different polygons in the tessellation is finite (the tessellation is exact if and only if each edge of a tile is an edge of exactly two polygons

of the tessellation). If we used other polygons rather than triangles, the number of different possible situations would have been larger, and the constants would have been different. Nevertheless, the result would have been the same.

This comment shows that the result we get in this paper is more general than it primarily seems. Unfortunately, it has its limits. If we get interested in a tiling of the hyperbolic plane which has infinitely many kinds of tiles, then our construction is not relevant anymore. In the same way, if a tile of the tessellation is not compact, we cannot use our method either.

This consideration leads to an open question:

Question 2.34. If Γ is any graph roughly isometric to the hyperbolic plane, is there a constant $C = C(\Gamma)$ such that a bound as in Theorem 2.9 exists?

This question naturally leads to a more general interrogation. In order to properly define the problem, let us give a definition.

Definition 2.35. We say that a host graph Γ has the property (S) if for each $k \in \mathbb{N}$ and each family $(\Omega_l)_{l\geq 1}$ of subgraphs of Γ, we have

$$|\Omega_l| \underset{l\to\infty}{\longrightarrow} \infty \implies \sigma_k(\Omega_l) \underset{l\to\infty}{\longrightarrow} 0.$$

Now we can ask the following open question:

Question 2.36. Let Γ_1, Γ_2 be two roughly isometric graphs. Let us assume that Γ_1 has the property (S). Does Γ_2 also have the property (S)?

Reformulated in the language of geometric group theory, the question becomes

Is Property (S) a large scale invariant?

This question, apparently not so hard, appears to be more thorny than expected.
If positively answered, it would automatically generalise our result to any graph roughly isometric to the hyperbolic plane, and it would certainly have many other applications.

Another interesting interrogation one may have consists in wondering if some similar constructions could be done in the hyperbolic space $\mathbb{H}^n$, with $n \geq 3$. In particular, a first question is the following:

Is there a natural class of graphs, analogous to triangle-tiling graphs, that would be roughly isometric to $\mathbb{H}^n$?

The answer to this question is *yes.* Using [32, Section 6.8], we can generate tessellations of $\mathbb{H}^n$ with polyhedra, for any $n \geq 2$. From such a tessellation, we can define a host graph Γ in the same manner as we did in this paper. It could be interesting to study such a host graph and see if some results analogous to Theorem 2.9 hold in higher dimension. This consideration leads to the following open question:

Question 2.37. Let Γ be a graph coming from a polyhedral tessellation of $\mathbb{H}^n, n \geq 3$. Does a constant $C = C(\Gamma)$ exist, such that a bound as in Theorem 2.9 holds?

2.A About the importance of the small triangles in our construction

We provide here an example which shows that, given a subgraph Ω of Γ, we cannot simply consider the domain that we get when thickening the union of T_w for all $w \in \bar{\Omega}$.

Let us consider the subgraph given by the following figure:

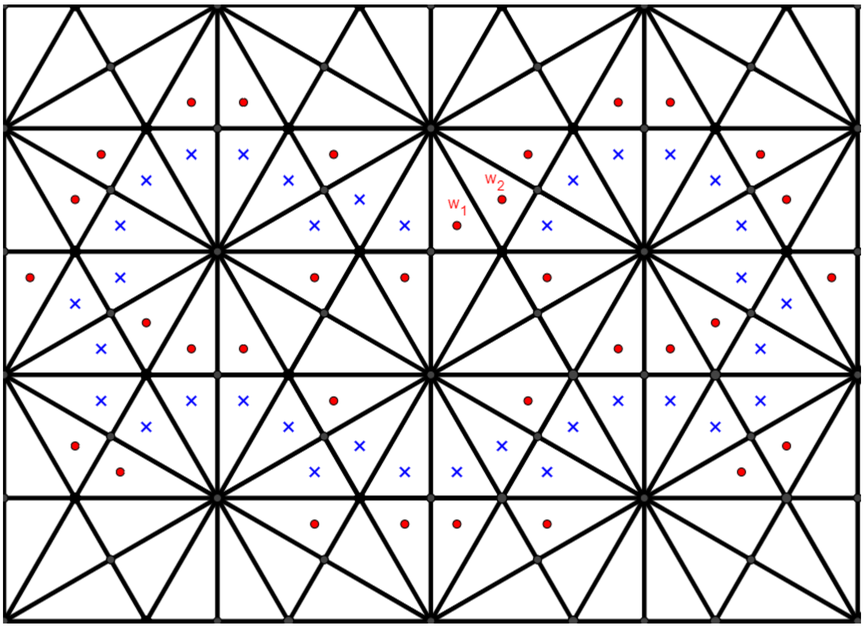

Figure 2.11: The cross vertices form the interior of the subgraph, the dot vertices form the boundary.

We are particularly interested in the boundary vertices named w_1 and w_2 in Figure 2.11. Here are two properties that w_1 and w_2 have:

- w_1 is close to w_2 in the host graph. Indeed, they belong to two adjacent triangles of the tessellation. Therefore, $d_\Gamma(w_1, w_2) = 1$ (where we used the notation d_Γ for the distance in the host graph).
- w_1 is far from w_2 in the subgraph. Indeed, by definition there is no edge between w_1 and w_2 in the subgraph. In fact, we have $d_\Omega(w_1, w_2) = 33$, which is the diameter of the subgraph (we used the notation d_Ω for the distance in the subgraph).

Because we are building a domain which is a sort of analog of the subgraph, we have to make sure that the distance between w_1 and w_2 is large in the domain.
The domain $\hat{N}$ that we get from this subgraph, using the strategy presented in this paper (using the small triangles), is the following, see Figure 2.12.

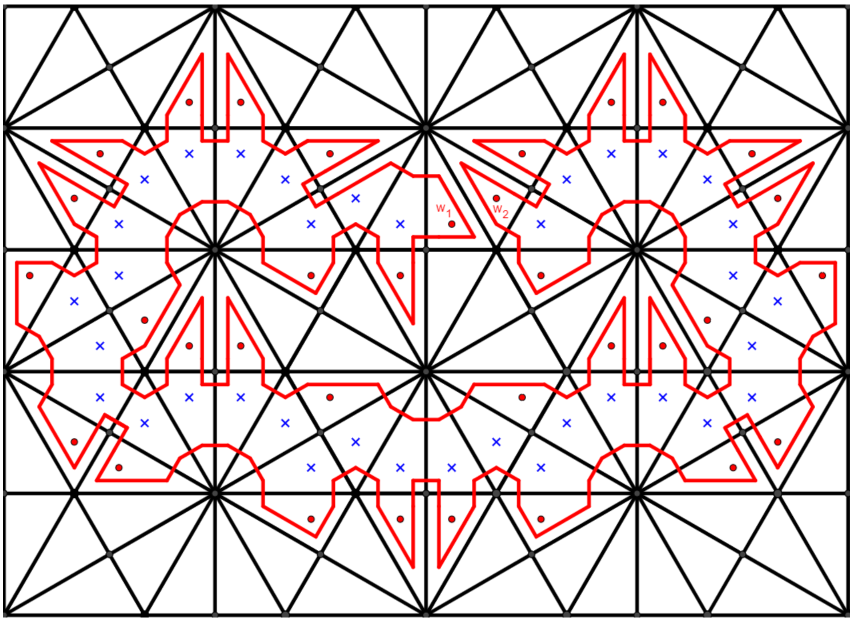

Figure 2.12: Using now $d_{\hat{N}}$ as a notation for the distance in $\hat{N}$, we can easily see that $d_{\hat{N}}(w_1, w_2)$ is large, roughly as the diameter of $\hat{N}$.

Here is now the domain that we get while considering the union of triangle T_w for all $w \in \bar{\Omega}$:

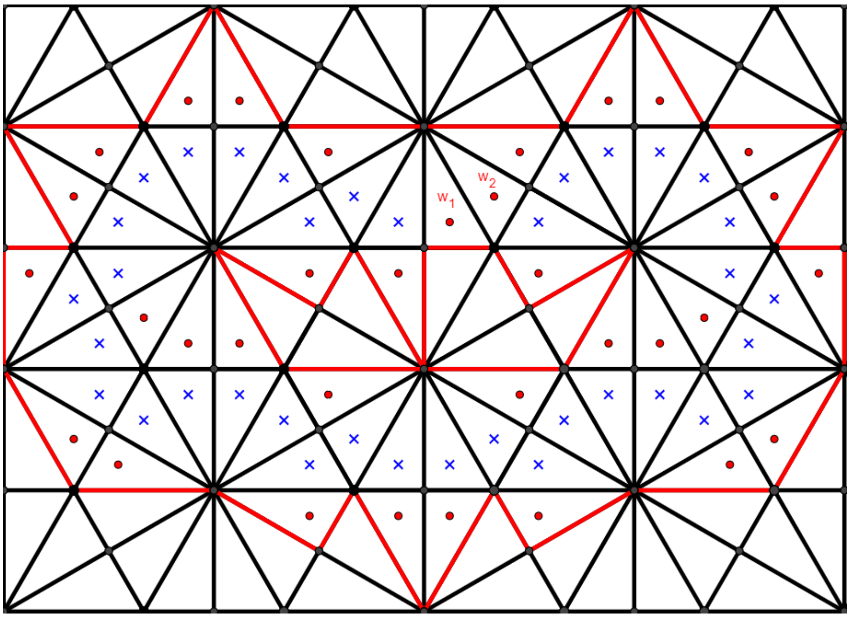

Figure 2.13: We can see that the distance in the domain between w_1 and w_2 is small.

If we were to pursue our construction with the domain given by Figure 2.13, we would have a real problem when building the rough isometry of Proposition 2.32.

Indeed, let us now consider a family of subgraphs $(\Omega_l)_{l \geq 1}$, such that $|\Omega_l| \underset{l \to \infty}{\longrightarrow} \infty$ and such that each subgraph of the family has the same particular property as the subgraph of Figure 2.11 (the property concerning w_1 and w_2 we discussed above). In that case, the constants in the rough isometry would then have to be chosen according to each subgraph (the diameter of each subgraph would do). This would obviously destroy our result.

2.B Is Property (S) a large scale invariant?

We recall that a host graph Γ has the property (S) if for all $k \in \mathbb{N}$ and for all sequences $(\Omega_l)_{l\in\mathbb{N}}$ of subgraphs of Γ, we have

$$|\Omega_l| \underset{l\to\infty}{\longrightarrow} \infty \implies \sigma_k(\Omega_l) \underset{l\to\infty}{\longrightarrow} 0.$$

Many host graphs have the property (S). Indeed, any Cayley graph with polynomial growth of order $d \geq 2$ has (S), see Corollary 1.7. As we saw in this section, this is also the case for host graphs that are coming from a tessellation of $\mathbb{H}^2$, if the tessellation is regular enough.

However, there exist some host graphs that do not have the property (S). For example, a host graph coming from a discretization of a sequence of truncated n-spheres ($n \geq 3$), see [33, Theorem 4.8], or some particular host graphs, such as the one given by Figure 2.14, see [19, Example 3.7].

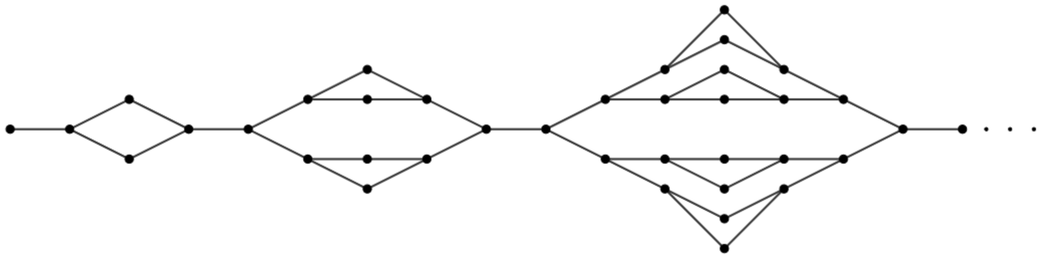

Figure 2.14: This infinite host graph does not have the property (S).

As far as we know, graphs that have the property (S) are very different from those which do not have it. Here, the meaning of the word *different* is that they are not roughly isometric.

This consideration leads to Question 2.36:

If Γ_1 *is roughly isometric to* Γ_2*, and if* Γ_2 *has (S), does* Γ_1 *have (S) too?*

A first attempt to answer positively this question consists in the following reasoning:

1. Given a family of subgraphs $(\Omega_l^1)_{l\in\mathbb{N}}$ of Γ_1, use the rough isometry to construct a family of subgraphs $(\Omega_l^2)_{l\in\mathbb{N}}$ of Γ_2, in a way that there exists $C \geq 1$ such that for all l, the subgraph Ω_l^1 is roughly isometric to Ω_l^2 with constant C.
2. Use [9, Proposition 16] to conclude that

$$\sigma_k(\Omega_l^1) \leq C \cdot \sigma_k(\Omega_l^2) \underset{l\to\infty}{\longrightarrow} 0,$$

meaning that Γ_1 also has the property (S).

This approach, although simple and elegant, is doomed to failure in a general setting. Indeed, the first step is impossible to do in general. We provide here an example in which the first step cannot be achieved.

Example 2.38 (Bottinelli-Kaiser)**.** Let Γ_1 and Γ_2 be the host graphs[1] given by Figure 2.15:

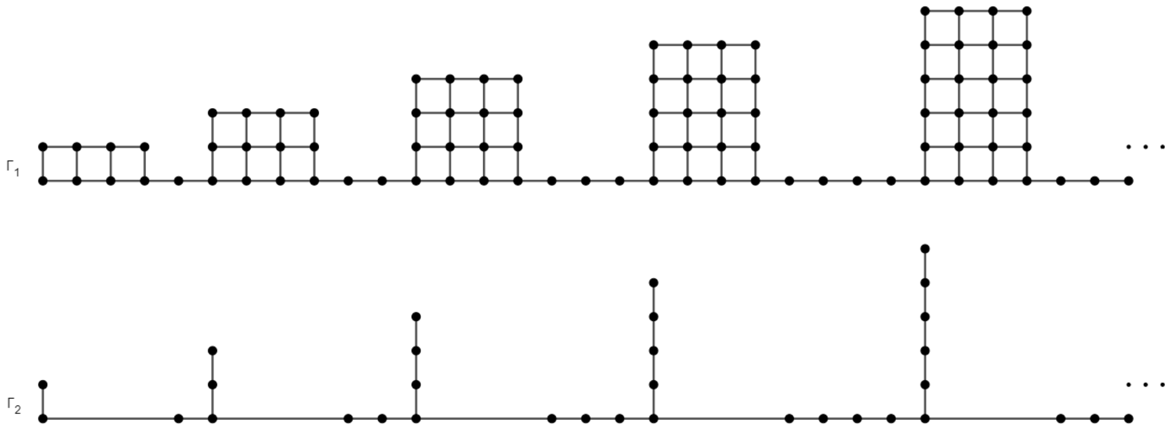

Figure 2.15: Here are two graphs, Γ_1 and Γ_2, which are roughly isometric.

The canonical embedding $\Gamma_2 \hookrightarrow \Gamma_1$ is clearly a rough isometry, where the constant can be chosen as 4.

To show that the first step is impossible to achieve here, let us find a family $(\Omega_l^1)_l$ of subgraphs of Γ_1 such that there is no family of subgraphs of Γ_2 that is uniformly roughly isometric to (Ω_l^1). For each $l \geq 2$, we choose as Ω_l^1 the connected set of vertices that forms the largest cycle in the l^{th} vertical parts of Γ_1, see Figure 2.16.

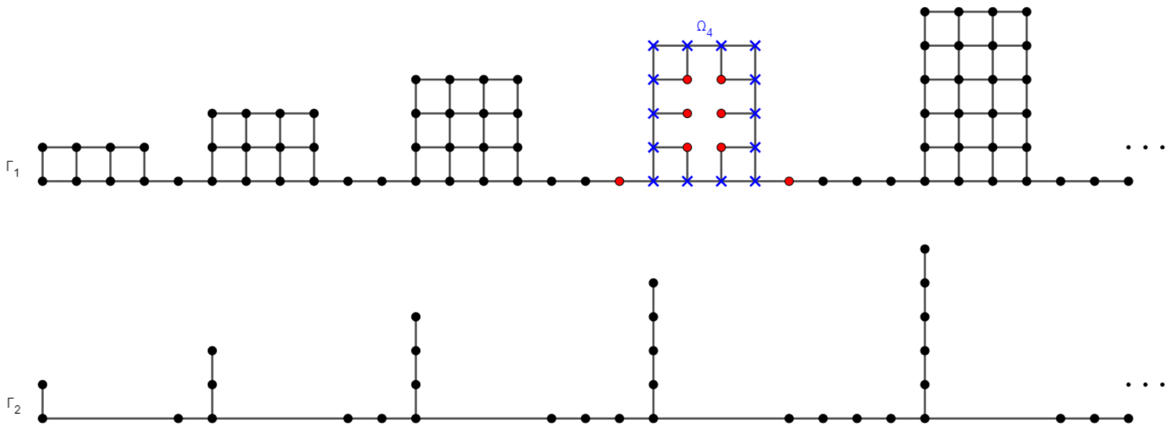

Figure 2.16: The choice of Ω_4^1, (the crosses), induces the boundary B_4^1, according to the rules presented in Definition 2.3.

With this setting, for each l the interior Ω_l^1 of the subgraph is a cycle, and the larger l is, the bigger the cycle is also. However, there is no family of subgraphs $(\Omega_l^2)_l$ in Γ_2 that is uniformly roughly isometric to $(\Omega_l^1)_l$. Let us show this statement.

Let us call $(\Omega_l^2)_l$ a family of subgraphs of Γ_2. Because of the construction of the host graph Γ_2, for each l the interior of a subgraph Ω_l^2 can be either:

---

[1] Remark that neither of them have the property (S) since in both of them, we can choose the interior Ω as the vertical parts and apply [31, Theorem 1].

1. Uniformly roughly isometric to a linear-shape graph, as in Figure 2.17:

Figure 2.17: The interior Ω_l^2 of the subgraph have a linear-shape.

2. Uniformly roughly isometric to a T-shape graph, as in Figure 2.18:

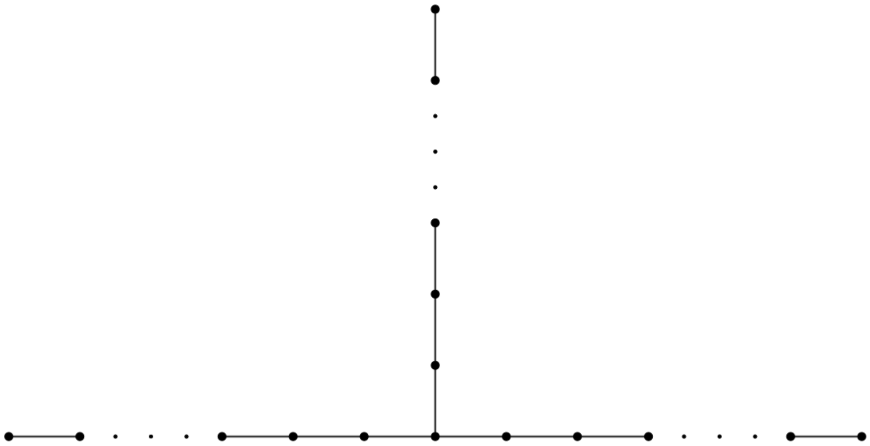

Figure 2.18: The interior of Ω_l^2 is a T-shape graph.

3. Uniformly roughly isometric to a multi T-shape graph, that is the concatenation of several T-shape graph.

None of these families of graphs can be uniformly roughly isometric to a family of cycles, that is the shape of Ω_l^1 for each l. Let us show that for a family of linear-shape graphs.

For all $l \geq 2$, let us call

$$\Psi_l : \Omega_l^2 \longrightarrow \Omega_l^1$$

a rough isometry between the linear-shape graph Ω_l^2 and the cycle Ω_l^1. For a contradiction, we suppose that there exists a constant $C \geq 1$ such that for all l, the map Ψ_l is a C-rough isometry.

Let us call i_0 and i_1 the two vertices of the linear-shape graph Ω_l^2 that have only one neighbour. Let us fix $\Psi_l(i_0)$ as any vertex of Ω_l^1 (in a cycle, there is no distinguished vertex). Then $\Psi_l(i_1) \in \Omega_l^1$ satisfy

$$d_1(\Psi_l(i_0), \Psi_l(i_1)) \geq \frac{d_2(i_0, i_1) - C}{C},$$

where we write d_1 for the distance in the cycle Ω_l^1 and d_2 the distance in Ω_l^2. We remark that this expression goes to infinity with $l \to \infty$.

Let us now consider γ and γ' the two different direct paths that connect $\Psi_l(i_0)$ to $\Psi_l(i_1)$. The smallest of them gives the distance between $\Psi_l(i_0)$ and $\Psi_l(i_1)$, Let us now consider j and j' two vertices of γ and γ' respectively, such that

$$|d_1(\Psi_l(i_0), j) - d_1(j, \Psi_l(i_1))| \leq 1, \quad |d_1(\Psi_l(i_0), j') - d_1(j', \Psi_l(i_1))| \leq 1.$$

Because Ψ_l is a rough isometry, Ψ_l is roughly surjective and therefore there exist $i, i' \in \Omega_l^2$ such that

$$\max\{d_2(\Psi_l(i), j), d_2(\Psi_l(i'), j')\} \leq C,$$

see Figure 2.19.

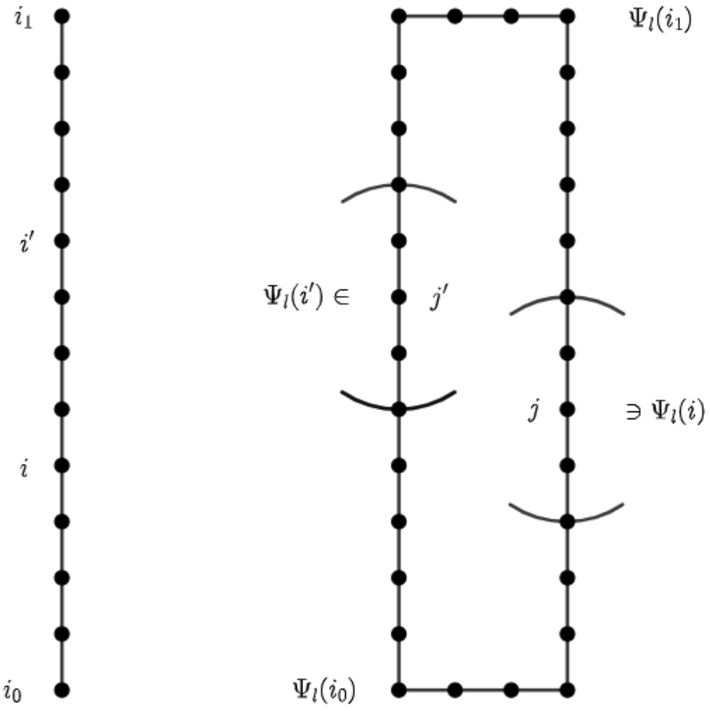

Figure 2.19: The image $\Psi_l(i)$ of i belongs to the ball $B(j, C)$, and in the same manner we have $\Psi_l(i') \in B(j', C)$.

Moreover, since

$$d_1(\Psi_l(i_0), j) \geq \frac{d_1(\Psi_l(i_0), \Psi_l(i_1)) - 1}{2} \xrightarrow[l\to\infty]{} \infty,$$

we have also

$$d_2(i_0, i) \xrightarrow[l\to\infty]{} \infty.$$

The same is true for i' and i_1. We write $\gamma_{i,i'}$ the direct path from i to i' in Ω_l^2. We remark that for every vertex $\bar{i}$ of $\gamma_{i,i'}$, we have

$$\min\{d_2(i_0, \bar{i}), d_2(\bar{i}, i_1)\} \xrightarrow[l\to\infty]{} \infty.$$

However, working from neighbour to neighbour in the path $\gamma_{i,i'}$, we find a vertex i^* of $\gamma_{i,i'}$ such that

$$\min\{d_1(\Psi_l(i_0), \Psi_l(i^*)), d_1(\Psi_l(i^*), \Psi_l(i_1))\} \le C. \qquad \lightning$$

We can use similar arguments to show that families of T-shape and multi T-shape graphs cannot be uniformly roughly isometric to a growing family of cycles.

Therefore, there is no family of subgraphs $(\Omega_l^2)_l$ of Γ_2 that is uniformly roughly isometric to the family of subgraphs $(\Omega_l^1)_l$ of Γ_1.

This example shows that one of the natural attempts to solve Question 2.36 cannot work; however it does not imply that the answer is negative. In particular, it is still to be explored if this approach can be achieved, *should we add a few assumptions on the host graphs.* This leads to the following open question:

Question 2.39. Let Γ_1 and Γ_2 be two host graphs roughly isometric to $\mathbb{H}^2$. Let $(\Omega_l^1)_l$ be a family of subgraphs of Γ_1. Is there a family of subgraphs $(\Omega_l^2)_l$ of Γ_2 which is uniformly roughly isometric to $(\Omega_l^1)_l$?

Chapter 3

Sharp upper bounds for Steklov eigenvalues of a hypersurface of revolution with two boundary components in Euclidean space

This paper has already been accepted for publication in the *Annales Mathématiques du Québec*, see [34]. The reader may encounter a few differences with the original text, due to some minor adjustments. Moreover, details had been added in some proofs, and Appendix 3.B has been added, providing information about an open problem which is a variant of the one studied in this paper.

Abstract

We investigate the question of sharp upper bounds for the Steklov eigenvalues of a hypersurface of revolution in Euclidean space with two boundary components, each isometric to $\mathbb{S}^{n-1}$. For the case of the first non zero Steklov eigenvalue, we give a sharp upper bound $B_n(L)$ (that depends only on the dimension $n \geq 3$ and the meridian length $L > 0$) which is reached by a degenerated metric g^* that we compute explicitly. We also give a sharp upper bound B_n which depends only on n. Our method also permits us to prove some stability properties of these upper bounds.

Keywords: Spectral geometry, Steklov problem, hypersurfaces of revolution, sharp upper bounds.

3.1 Introduction

BEGIN with (M, g) a smooth compact connected Riemannian manifold of dimension $n \geq 2$ with smooth boundary Σ. The Steklov problem on (M, g) consists in finding the real numbers σ and the harmonic functions $f : M \longrightarrow \mathbb{R}$ such that $\partial_\nu f = \sigma f$ on Σ, where ν denotes the outward normal on Σ. Such a σ is called a Steklov eigenvalue of (M, g). It is well known that the Steklov spectrum forms a discrete sequence $0 = \sigma_0(M, g) < \sigma_1(M, g) \leq \sigma_2(M, g) \leq \ldots \nearrow \infty$. Each eigenvalue is repeated with its multiplicity, which is finite. If the context is clear, then we simply write $\sigma_k(M)$ for $\sigma_k(M, g)$.

It is known [6, Theorem 1.1] that for any connected compact manifold (M, g) of dimension $n \geq 3$, there exists a family (g_ε) of Riemannian metrics conformal to g which coincide with g on the boundary of M, such that

$$\sigma_1(M, g_\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \infty.$$

Therefore, to obtain upper bounds for the Steklov eigenvalues, it is necessary to study manifolds that satisfy certain additional constraints. We refer to [11] for an overview of the current state-of-the-art on geometric upper bounds for the Steklov eigenvalues.

Recently, authors investigated the Steklov problem on manifolds of revolution [16, 17, 41, 42]. A natural constraint for the manifolds is that they are (hyper)surfaces of revolution in Euclidean space. Some work has already been done on these kinds of manifolds, see for example [8, 10]. We refer to [8, Section 3.1] for a review about what these manifolds are, and consider a particular case in this paper that we define below (see Definition 3.1).

This work led to the discovery of lower and upper bounds for the Steklov eigenvalues of a hypersurface of revolution. We begin by recalling some recent results.

We first consider results for hypersurfaces of revolution with one boundary component that is isometric to $\mathbb{S}^{n-1}$. In dimension $n = 2$, it is proved in [8, Proposition 1.10] that each surface of revolution $M \subset \mathbb{R}^3$ with boundary $\mathbb{S}^1 \subset \mathbb{R}^2 \times \{0\}$ is Steklov isospectral to the unit disk. In dimension $n \geq 3$, many bounds were given. It is proved that each hypersurface of revolution $M \subset \mathbb{R}^{n+1}$ with one boundary component isometric to $\mathbb{S}^{n-1}$ satisfies $\sigma_k(M) \geq \sigma_k(\mathbb{B}^n)$, where $\mathbb{B}^n$ is the Euclidean ball and equality holds if and only if $M = \mathbb{B}^n \times \{0\}$, see [8, Theorem 1.8]. In [10, Theorem 1], the authors show the following upper bound: if $M \subset \mathbb{R}^{n+1}$ is a hypersurface of revolution with one boundary component isometric to $\mathbb{S}^{n-1}$, then for each $k \geq 1$, we have

$$\sigma_{(k)}(M) < k + n - 2,$$

where $\sigma_{(k)}(M)$ is the kth distinct Steklov eigenvalue of M. Although there exists no equality case within the collection of hypersurfaces of revolution, this upper bound is sharp. Indeed, for each $\varepsilon > 0$ and each $k \geq 1$, there exists a hypersurface of revolution M_ε such that $\sigma_{(k)}(M_\varepsilon) > k + n - 2 - \varepsilon$.

These results concern hypersurfaces of revolution that have one boundary component isometric to $\mathbb{S}^{n-1}$. Therefore, the goal of this paper is to investigate the Steklov problem on a hypersurface of revolution with two boundary components. As it was already done in [8] and in [10], we will consider hypersurfaces with boundary components isometric to $\mathbb{S}^{n-1}$. We begin by defining the context.

Definition 3.1. An n-dimensional compact hypersurface of revolution (M, g) in Euclidean space with two boundary components each isometric to $\mathbb{S}^{n-1}$ is the warped product $M = [0, L] \times \mathbb{S}^{n-1}$ endowed with the Riemannian metric

$$g(r, p) = dr^2 + h^2(r)g_0(p),$$

where $(r, p) \in [0, L] \times \mathbb{S}^{n-1}$, g_0 is the canonical metric of the $(n-1)$-sphere of radius one and $h : [0, L] \longrightarrow \mathbb{R}_+^*$ is a smooth function which satisfies:

(1) $|h'(r)| \leq 1$ for all $r \in [0, L]$;

(2) $h(0) = h(L) = 1$.

Assumption (1) comes from the fact that (M, g) is a hypersurface in Euclidean space $\mathbb{R}^{n+1}$, see [8, Section 3.1] for more details. Assumption (2) implies that each component of the boundary is isometric to $\mathbb{S}^{n-1}$, as commented in Figure 3.1.

We now make some remarks on the terminology used throughout this paper. If $M = [0, L] \times \mathbb{S}^{n-1}$ and $h : [0, L] \longrightarrow \mathbb{R}_+^*$ satisfies the properties above, we say that M is a *hypersurface of revolution*, we say that $g(r, p) = dr^2 + h^2(r)g_0(p)$ is a *metric of revolution on* M *induced by* h and we call the number L the *meridian length* of M.

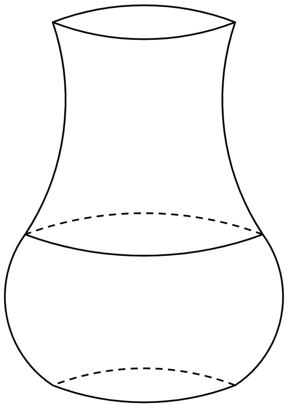

Figure 3.1: Since $h(0) = h(L) = 1$, the boundary of M consists of two copies of $\mathbb{S}^{n-1}$.

Some lower bounds have already been obtained is this case. Indeed, [8, Theorem 1.11] states that if $M \subset \mathbb{R}^{n+1}$, $n \geq 3$, is a hypersurface of revolution (in the sense of Definition 3.1), and $L > 2$ is the meridian length of M, then for each $k \geq 1$,

$$\sigma_k(M) \geq \sigma_k(\mathbb{B}^n \sqcup \mathbb{B}^n).$$

Moreover, this inequality is sharp. In the case $0 < L \leq 2$, a lower bound is also obtained:

$$\sigma_k(M) \geq \left(1 - \frac{L}{2}\right)^{n-1} \sigma_k(C_L, dr^2 + g_0),$$

However, this inequality does not appear to be sharp.

In this paper, we will look for *upper bounds* for the Steklov eigenvalues of hypersurfaces of revolution. First, we recall that there exists a bound $B_n^k(L)$ such that for all metrics of revolution g on M, we have $\sigma_k(M, g) < B_n^k(L)$. Indeed, Proposition 3.3 of [8] states that if $M = [0, L] \times \mathbb{S}^{n-1}$ is a hypersurface of revolution, then we have

$$\sigma_k(M) \leq \left(1 + \frac{L}{2}\right)^{n-1} \sigma_k(C_L, dr^2 + g_0).$$

As such, a natural question is the following:

Given the dimension $n \geq 3$ and the meridian length L of M, does a metric of revolution g^* on M exist, such that $\sigma_k(M, g) \leq \sigma_k(M, g^*)$ for all metrics of revolution g on M?

Our investigations show that the answer is negative. Indeed, a sharp upper bound $B_n^k(L)$ exists, but no metric of revolution on $M = [0, L] \times \mathbb{S}^{n-1}$ achieves the equality case. However, there exists a non-smooth metric g^*, that we will call a *degenerated maximizing metric*, which maximizes the kth Steklov eigenvalue, for each $k \in \mathbb{N}$. This metric is non-smooth, therefore g^* is not a metric of revolution on M in the sense of Definition 3.1. Endowed with this metric, (M, g^*) can be seen as two annuli glued together; we provide more information about this degenerated maximizing metric g^* and the geometric representation of (M, g^*) in Section 3.3.

We state our first result:

Theorem 3.2. *Let $(M = [0, L] \times \mathbb{S}^{n-1}, g_1)$ be a hypersurface of revolution in Euclidean space with two boundary components each isometric to $\mathbb{S}^{n-1}$ and meridian length L. We suppose $n \geq 3$. Then there exists a metric of revolution g_2 on M such that for each $k \geq 1$,*

$$\sigma_k(M, g_1) < \sigma_k(M, g_2).$$

This result implies that among all metrics of revolution on M, none maximizes the kth non zero Steklov eigenvalue. Nevertheless, given any metric of revolution g_1 on M, we can iterate Theorem 3.2 to generate a sequence of metrics $(g_i)_{i=1}^{\infty}$ on M. This sequence converges to a unique non-smooth metric g^* on M, which is quite simple (see Section 3.3) and which maximizes the kth Steklov eigenvalue. That is why we call g^* the degenerated maximizing metric. Hence, as we search for the optimal bounds $B_n^k(L)$, we must use the piece of information contained in g^*.

We start by studying the case $k = 1$. We fix $n \geq 3$ and $L > 0$ and search for a sharp upper bound $B_n(L)$ for $\sigma_1(M, g)$. In this case, we are able to calculate an expression for $B_n(L)$:

Theorem 3.3. *Let $(M = [0, L] \times \mathbb{S}^{n-1}, g)$ be a hypersurface of revolution in Euclidean space with two boundary components each isometric to $\mathbb{S}^{n-1}$ and dimension $n \geq 3$. Then the first non trivial Steklov eigenvalue $\sigma_1(M, g)$ is bounded above, by a bound that depends only on the dimension n and the meridian length L of M:*

$$\sigma_1(M, g) < B_n(L) := \min\left\{ \frac{(n-2)(1+L/2)^{n-2}}{(1+L/2)^{n-2}-1}, \frac{(n-1)((1+L/2)^n - 1)}{(1+L/2)^n + n - 1} \right\}.$$

Moreover, this bound is sharp: for each $\varepsilon > 0$, there exists a metric of revolution g_ε on M such that $\sigma_1(M, g_\varepsilon) > B_n(L) - \varepsilon$.

We have the following asymptotic behaviour:

$$B_n(L) \underset{L\to\infty}{\longrightarrow} n-2$$
$$B_n(L) \underset{L\to 0}{\longrightarrow} 0,$$

see Figure 3.4.

We also study the function $L \longmapsto B_n(L)$. This allows us to find a sharp upper bound B_n such that for all meridian lengths $L > 0$ and metrics of revolution g on M, we have $\sigma_1(M, g) < B_n$:

Corollary 3.4. *Let $n \geq 3$. Then there exists a bound $B_n < \infty$ such that for all hypersurfaces of revolution (M, g) in Euclidean space with two boundary components each* *isometric to* $\mathbb{S}^{n-1}$*, we have*

$$\sigma_1(M,g) < B_n := \frac{(n-2)\left(1+\frac{L_1}{2}\right)^{n-2}}{\left(1+\frac{L_1}{2}\right)^{n-2}-1},$$

where L_1 *is the unique real positive solution of the equation*

$$(1+L/2)^{2n-2} - (n-1)\,(1+L/2)^{n} - (n-1)^2\,(1+L/2)^{n-2} + n - 1 = 0.$$

Moreover, this bound is sharp: for each $\varepsilon > 0$*, there exists a hypersurface of revolution with two boundary components each isometric to a unit sphere* $(M_\varepsilon, g_\varepsilon)$ *such that* $\sigma_1(M_\varepsilon, g_\varepsilon) > B_n - \varepsilon$*.*

We say that L_1 is a *critical length* associated with $k = 1$, see Definition 3.8.

Proposition 3.5. *Let* $n \geq 3$*, and let* $L_1 = L_1(n)$ *be the critical length associated with* $k = 1$*. Then we have:*

$$\lim_{n\to\infty} L_1(n) = 0 \ \textit{ and } \ \lim_{n\to\infty} B_n = \infty.$$

Note that the behaviour of L_1 is surprising since we know that when n is fixed, then $L \ll 1$ implies $\sigma_1(M,g) \ll 1$. Indeed, by [8, Proposition 3.3], we have

$$\sigma_1(M) \leq \left(1+\frac{L}{2}\right)^{n-1} \sigma_1(C_L) \underset{L\to 0}{\longrightarrow} 0.$$

Now that we have provided information about sharp upper bounds for $\sigma_1(M,g)$, it is natural to wonder what kind of stability properties the hypersurfaces of revolution possess. A first interesting question is the following:

Given the information that $\sigma_1(M = [0,L]\times\mathbb{S}^{n-1}, g)$ *is close to the sharp upper bound* B_n*, can we conclude that the meridian length* L *of* M *is close to the critical length* L_1*?*

The answer to this question is positive. Indeed we will prove that if L is not close to L_1, then $\sigma_1(M,g)$ is not close to B_n. Additionally, given the information that $\sigma_1(M,g)$ is δ-close to B_n, we will show that the distance between L and L_1 is less than δ, up to a constant of proportionality which depends only on the dimension n.

Theorem 3.6. *Let* $M = [0,L]\times\mathbb{S}^{n-1}$*, with* $L > 0$ *and* $n \geq 3$*. We suppose* $L \neq L_1$*. Then there exists a constant* $C(n,L) > 0$ *such that for all metrics of revolution* g *on* M*, we have*

$$B_n - \sigma_1(M,g) \geq C(n,L).$$

Moreover, there exists a constant $C(n) > 0$ *such that for all* $0 < \delta < \frac{B_n-(n-2)}{2}$*, we have*

$$|B_n - \sigma_1(M,g)| < \delta \implies |L_1 - L| < C(n)\cdot\delta.$$

We also consider the following question about stability properties:

Given the information that $\sigma_1(M, g)$ is close to the sharp upper bound $B_n(L)$, can we conclude that the metric of revolution g is close (in a sense that is defined below) to the degenerated maximizing metric g^*?

We prove that if g is not close to g^*, then $\sigma_1(M, g)$ is not close to $B_n(L)$.

For this purpose, given $m \in [1, 1 + L/2)$, we define

$$\begin{aligned}\mathcal{M}_m := &\{\text{metrics of revolution } g \text{ on } M \\ &\quad \text{induced by a function } h \text{ such that } \max_{r\in[0,L]}\{h(r)\} \leq m\}.\end{aligned}$$

The collection $\mathcal{M}_m$ can be thought of as the set of all metrics of revolution that are not close to the degenerated maximizing metric g^*, where the qualitative appreciation of the word "close" is given by the parameter m. The larger m is, the closer to g^* the metrics in $\mathcal{M}_m$ can be.

We get the following result:

Theorem 3.7. *Let $(M = [0, L] \times \mathbb{S}^{n-1}, g)$ be a hypersurface of revolution in Euclidean space with two boundary components each isometric to $\mathbb{S}^{n-1}$ and dimension $n \geq 3$. Let $m \in [1, 1 + L/2)$ and $\mathcal{M}_m$ as above. Then there exists a constant $C(n, L, m) > 0$ such that for all $g \in \mathcal{M}_m$, we have*

$$B_n(L) - \sigma_1(M, g) \geq C(n, L, m).$$

These results solve the case $k = 1$. Therefore, it would be interesting to find the same kind of results for any $k \geq 1$. After having calculated sharp upper bounds for some higher values of k in Section 3.6.1 and 3.6.2, we will see that in order to get an expression for $B_n^k(L)$, we need to distinguish between many cases. As such, giving a general formula for $B_n^k(L)$ or $B_n^k := \sup_{L\in\mathbb{R}_+^*}\{B_n^k(L)\}$ via this method seems difficult. We discuss this in Remark 3.20.

Definition 3.8. We say that $L_k \in \mathbb{R}_+^*$ is a finite critical length associated with k if we have $B_n^k = B_n^k(L_k)$. We say that k has a critical length at infinity if it satisfies $B_n^k = \lim_{L\to\infty} B_n^k(L)$.

These lengths are critical in the following sense: if $L_k \in \mathbb{R}_+^*$ is a finite critical length for a certain $k \in \mathbb{N}$ and if we write g^* the degenerated maximizing metric on $M_k = [0, L_k] \times \mathbb{S}^{n-1}$, then

$$B_n^k = \sigma_k(M_k, g^*).$$

Given $n \geq 3$, there exist some k which have a finite critical length associated with them. Indeed, thanks to Corollary 3.4, we know that $k = 1$ has this property. Moreover, we know that there exist some k which have a critical length at infinity, see Section 3.6.1.

Since we want to study upper bounds for the Steklov eigenvalues, it is then natural to ask what qualitative and quantitative information we can provide about these critical lengths.

We get the following result:

Theorem 3.9. *Let $n \geq 3$. Then there exist infinitely many $k \in \mathbb{N}$ which have a finite critical length associated with them. Moreover, if we call $(k_i)_{i=1}^{\infty} \subset \mathbb{N}$ the increasing sequence of such k and if we call $(L_i)_{i=1}^{\infty}$ the associated sequence of finite critical lengths, then we have*

$$\lim_{i\to\infty} L_i = 0.$$

The existence of finite critical lengths is something surprising when we compare with what happens in the case of hypersurfaces of revolution with one boundary component. Indeed, using our vocabulary, we can state that in the case of hypersurfaces of revolution with one boundary component, each $k \in \mathbb{N}$ has a critical length at infinity, see [10, Proposition 7]. Nevertheless, in our case, Theorem 3.9 guarantees that there exist infinitely many $k \in \mathbb{N}$ which have a finite critical length associated with them. Moreover, we will show in Section 3.6.1 that there exist some k which have a critical length at infinity. However, we do not know if there are *infinitely many* of them. This consideration leads to the following open question (Question 3.22):

Given $n \geq 3$, are there finitely or infinitely many $k \in \mathbb{N}$ such that k has a critical length at infinity?

Plan of the paper. In Section 3.2, we recall the variational characterizations of the Steklov eigenvalues before giving the expression of eigenfunctions on hypersurfaces of revolution, and we introduce the notion of mixed Steklov-Dirichlet and Steklov-Neumann problems and state some propositions about them. We will then have enough information to prove Theorem 3.2 in Section 3.3. This will allow us to prove Theorem 3.3, Corollary 3.4 and Proposition 3.5 in Section 3.4. Then we prove the stability properties of hypersurfaces of revolution, i.e Theorem 3.6 and Theorem 3.7 in Section 3.5. We continue by performing some calculations for sharp upper bounds for higher eigenvalues in Section 3.6. We conclude by proving Theorem 3.9 in Section 3.7.

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3.2 Variational characterization of the Steklov eigenvalues and mixed problems

We state some general facts about Steklov eigenfunctions and we define the mixed Steklov-Dirichlet and Steklov-Neumann problems.

3.2.1 Variational characterization of the Steklov eigenvalues

Let (M, g) be a Riemannian manifold with smooth boundary Σ. Then we can characterize the kth Steklov eigenvalue of M by the following formula:

$$\sigma_k(M,g) = \min\left\{R_g(f) : f \in H^1(M),\ f \perp_\Sigma f_0, f_1, \dots, f_{k-1}\right\}, \tag{3.2.1}$$

where

$$R_g(f) = \frac{\int_M |\nabla f|^2 dV_g}{\int_\Sigma |f|^2 dV_\Sigma}$$

is called the Rayleigh quotient and

$$f \perp_\Sigma f_i \iff \int_\Sigma f f_i dV_\Sigma = 0.$$

Another way to characterize the kth eigenvalue of M is given by the Min-Max principle:

$$\sigma_k(M,g) = \min_{E \in \mathcal{H}_{k+1}(M)} \max_{f \in E} \left\{ R_g(f) : \int_\Sigma f^2 dV_\Sigma \neq 0 \right\}, \tag{3.2.2}$$

where $\mathcal{H}_{k+1}$ is the set of all $(k+1)$-dimensional subspaces in the Sobolev space $H^1(M)$.

We state now a proposition that provides us with information about the expression of the Steklov eigenfunctions of a hypersurface of revolution.

We denote by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \nearrow \infty$ the spectrum of the Laplacian on $(\mathbb{S}^{n-1}, g_0)$ and we consider $(S_j)_{j=0}^\infty$ an orthonormal basis of eigenfunctions associated to $(\lambda_j)_{j=0}^\infty$.

Proposition 3.10. *Let (M,g) be a hypersurface of revolution as in Definition 3.1. Then each eigenfunction on M can be written as $f_k(r,p) = u_l(r)S_j(p)$, where u_l is a smooth function on* $[0,L]$.

This property is well known for warped product manifolds (and thus for our case of hypersurfaces of revolution) and it is used often, see for example [14, Remark 1.1], [15, Lemma 3], [33, Proposition 3.16] or [41, Proposition 9].

3.2.2 Mixed problems and their variational characterizations

Let $(N, \partial N)$ be a smooth compact connected Riemannian manifold and $A \subset N$ be a domain which satisfies $\partial N \subset \partial A$. We suppose that ∂A is smooth and we call $\partial_{int} A$ the intersection of ∂A with the interior of N.

Definition 3.11. The Steklov-Dirichlet problem on A is the eigenvalue problem

$$\begin{cases} \Delta f = 0 & \text{in } A \\ \partial_\nu f = \sigma f & \text{on } \partial N \\ f = 0 & \text{on } \partial_{int} A. \end{cases}$$

It is well known that this mixed problem possesses solutions that form a discrete sequence

$$0 < \sigma_0^D(A) \leq \sigma_1^D(A) \leq \ldots \nearrow \infty.$$

The variational characterization of the kth Steklov-Dirichlet eigenvalue is the following:

$$\sigma_k^D(A) = \min_{E \in \mathcal{H}_{k+1,0}(A)} \max_{0 \neq f \in E} \frac{\int_A |\nabla f|^2 dV_A}{\int_\Sigma |f|^2 dV_\Sigma},$$

where $\mathcal{H}_{k+1,0}$ is the set of all $(k+1)$-dimensional subspaces in the Sobolev space

$$H_0^1(A) = \{f \in H^1(A) : f = 0 \text{ on } \partial_{int}A\}.$$

Definition 3.12. The Steklov-Neumann problem on A is the eigenvalue problem

$$\begin{cases} \Delta f = 0 & \text{in } A \\ \partial_\nu f = \sigma f & \text{on } \partial N \\ \partial_\nu f = 0 & \text{on } \partial_{int}A. \end{cases}$$

It is well known that this mixed problem possesses solutions that form a discrete sequence

$$0 = \sigma_0^N(A) \leq \sigma_1^N(A) \leq \ldots \nearrow \infty.$$

The variational characterization of the kth Steklov-Neumann eigenvalue is the following:

$$\sigma_k^N(A) = \min_{E\in\mathcal{H}_{k+1}(A)} \max_{0\neq f\in E} \frac{\int_A |\nabla f|^2 dV_A}{\int_\Sigma |f|^2 dV_\Sigma},$$

where $\mathcal{H}_{k+1}$ is the set of all $(k+1)$-dimensional subspaces in the Sobolev space $H^1(A)$.

3.2.3 Mixed problems on annular domains

Let $\mathbb{B}_1$ and $\mathbb{B}_R$ be the balls in $\mathbb{R}^n$, $n \geq 3$, with radius 1 and $R > 1$ respectively centered at the origin. The annulus A_R is defined as follows: $A_R = \mathbb{B}_R \backslash \overline{\mathbb{B}}_1$. We say that this annulus is of inner radius 1 and outer radius R. This particular kind of domain shall be useful in this paper.

For such domains, it is possible to compute $\sigma_{(k)}^D(A_R)$ explicitly, which is the (k)th eigenvalue of the Steklov-Dirichlet problem on A_R, counted without multiplicity.

We state here Proposition 4 of [10]:

Proposition 3.13. *For A_R as above, consider the Steklov-Dirichlet problem*

$$\begin{cases} \Delta f = 0 & \textit{in } A_R \\ \partial_\nu f = \sigma f & \textit{on } \partial\mathbb{B}_1 \\ f = 0 & \textit{on } \partial\mathbb{B}_R. \end{cases}$$

Then, for $k \geq 0$, the (k)th eigenvalue (counted without multiplicity) of this problem is

$$\sigma_{(k)}^D(A_R) = \frac{(k+n-2)R^{2k+n-2} + k}{R^{2k+n-2} - 1}.$$

By [10, Proposition 4], it is possible to get the expression of the eigenfunctions of the Steklov-Dirichlet problem on an annular domain.

Lemma 3.14. *Each eigenfunction φ_l of the Steklov-Dirichlet problem on the annulus A_R can be expressed as $\varphi_l(r,p) = \alpha_l(r)S_l(p)$, where S_l is an eigenfunction for the l^{th} harmonic of the sphere $\mathbb{S}^{n-1}$.*

It is possible to compute $\sigma^N_{(k)}(A_R)$ explicitly, which is the (k)th eigenvalue of the Steklov-Neumann problem on A_R, counted without multiplicity.

We state now Proposition 5 of [10]:

Proposition 3.15. *For A_R as above, consider the Steklov-Neumann problem*

$$\begin{cases} \Delta f = 0 & \text{in } A_R \\ \partial_\nu f = \sigma f & \text{on } \partial\mathbb{B}_1 \\ \partial_\nu f = 0 & \text{on } \partial\mathbb{B}_R. \end{cases}$$

Then, for $k \geq 0$, the (k)th eigenvalue (counted without multiplicity) of this problem is

$$\sigma^N_{(k)}(A_R) = k\frac{(k+n-2)(R^{2k+n-2}-1)}{kR^{2k+n-2}+k+n-2}.$$

In the same manner as before, we have the following expression for the Steklov-Neumann eigenvalues, see [10, Proposition 5].

Lemma 3.16. *Each eigenfunction ϕ_l of the Steklov-Neumann problem on the annulus A_R can be expressed as $\phi_l(r,p) = \beta_l(r)S_l(p)$, where S_l is an eigenfunction for the l^{th} harmonic of the sphere $\mathbb{S}^{n-1}$.*

3.3 The degenerated maximizing metric

A particular case of hypersurfaces of revolution is the following: let $M = [0,L] \times \mathbb{S}^{n-1}$ be endowed with a metric of revolution $g(r,p) = dr^2 + h^2(r)g_0(p)$. Let us suppose that there exists $\varepsilon > 0$ such that $h(r) = 1 + r$ on $[0,\varepsilon]$. Let us consider the connected component of the boundary $\mathbb{S}_0$ associated with $h(0)$. Then the ε-neighborhood of $\mathbb{S}_0$ is an annulus with inner radius 1 and outer radius $1+\varepsilon$.

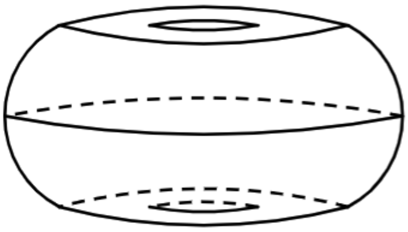

Figure 3.2: On $[0,\varepsilon]$, we have $h(r) = 1 + r$ and on $[L-\varepsilon, L]$, we have $h(r) = -r + L + 1$. This implies that the ε-neighborhood of the boundary consists of two disjoint copies of an annulus with inner radius 1 and outer radius $1+\varepsilon$.

This particular case is the key idea that we use to prove Theorem 3.2. We prove it now.

Proof. We write $g_1(r,p) = dr^2 + h_1^2(r)g_0(p)$. Because h_1 is smooth and $|h_1'| \leq 1$, we have $h_1(r) < 1 + \frac{L}{2}$ for all $r \in [0,L]$. Since h_1 is continuous and $[0,L]$ is compact, h_1 reaches its maximum on $[0, L]$. We call

$$m := \max_{r\in[0,L]}\{h_1(r)\}.$$

Notice that $1 \leq m < 1 + \frac{L}{2}$.

We define a smooth function $h_2 : [0, L] \longrightarrow \mathbb{R}$ by

$$h_2(r) = \begin{cases} 1 + r & \text{if } \ 0 \leq r \leq m - 1 \\ 1 + L - r & \text{if } \ L - m + 1 \leq r \leq L. \end{cases}$$

For $r \in (m - 1, L - m + 1)$, we only require that $h_2(r) > m$, that $h_2(L/2) = \frac{1+L/2+m}{2}$ and that

$$g_2(r, p) := dr^2 + h_2^2(r)g_0(p)$$

is a symmetric metric of revolution on M, i.e for all $r \in [0, L]$, we have $h_2(r) = h_2(L - r)$. Note that we have $h_2 \geq h_1$ and that for $r \in (m - 1, L - m + 1)$ we have $h_2(r) > h_1(r)$.

Besides, for f a smooth function on M, we have

$$R_{g_1}(f) = \frac{\int_M |\nabla f|^2_{g_1} dV_{g_1}}{\int_\Sigma |f|^2 dV_\Sigma} = \frac{\int_M \left((\partial_r f)^2 + \frac{1}{h_1^2}|\tilde{\nabla} f|^2_{g_0}\right) h_1^{n-1} dV_{g_0} dr}{\int_\Sigma |f|^2 dV_\Sigma}$$

and

$$R_{g_2}(f) = \frac{\int_M |\nabla f|^2_{g_2} dV_{g_2}}{\int_\Sigma |f|^2 dV_\Sigma} = \frac{\int_M \left((\partial_r f)^2 + \frac{1}{h_2^2}|\tilde{\nabla} f|^2_{g_0}\right) h_2^{n-1} dV_{g_0} dr}{\int_\Sigma |f|^2 dV_\Sigma},$$

where $\tilde{\nabla} f$ is the gradient of f seen as a function of p.

Since $n \geq 3$, for all functions $f \in H^1(M)$, we have $R_{g_1}(f) \leq R_{g_2}(f)$. Using the Min-Max principle, we can conclude that for all $k \geq 1$, we have $\sigma_k(M, g_1) \leq \sigma_k(M, g_2)$. However, here we want to show a strict inequality.

Because of the existence of a continuum of points r for which $h_1(r) < h_2(r)$, if $\partial_r f$ does not vanish on any interval, then the inequality is strict.

Let $k \geq 1$ be an integer. Let $E_{k+1} := Span(f_{0,2}, \dots, f_{k,2})$, where $f_{i,2}$ is a Steklov eigenfunction associated with $\sigma_i(M, g_2)$. We can choose these functions such that for all $i = 0, \dots, k$, we have

$$\int_\Sigma (f_{i,2})^2 dV_\Sigma = 1,$$

and hence

$$\int_M |\nabla f_{i,2}|^2_{g_2} dV_{g_2} = \sigma_i(M, g_2).$$

Let $f^* = \sum_{i=0}^k a_i f_{i,2} \in E_{k+1}$ be such that $\max_{f\in E_{k+1}} R_{g_1}(f) = R_{g_1}(f^*)$.

We now consider two cases:

1. Let us suppose $f^* = a_k f_{k,2}$ with $a_k \neq 0$, i.e f^* is an eigenfunction associated with $\sigma_k(M, g_2)$. Then by Proposition 3.10, we have $f^*(r, p) = u_j(r)S_j(p)$. Moreover, using [10, Proposition 2], we know that u_j is a non trivial solution of the ODE

$$\frac{1}{h_2^{n-1}}\frac{d}{dr}\left(h_2^{n-1}\frac{d}{dr}u_j\right) - \frac{1}{h_2^2}\lambda_j u_j = 0.$$

   a) If $\lambda_j = 0$, which means $S_j = S_0 = const$, then u_j cannot be locally constant. Indeed, otherwise f^* would be locally constant, but since f^* is harmonic, this implies that f^* is constant, see [1]. That is not the case because $k \geq 1$.

   b) If $\lambda_j \neq 0$, then u_j cannot be locally constant, otherwise the ODE is not satisfied.

   Hence u_j is not locally constant and then $\partial_r f^*$ does not vanish on any interval. Therefore, using the Min-Max principle (3.2.2), we have

$$\sigma_k(M, g_1) \leq \max_{f \in E_{k+1}} R_{g_1}(f) = R_{g_1}(f^*) < R_{g_2}(f^*) = \sigma_k(M, g_2).$$

2. Let us suppose $f^* = \sum_{i=0}^k a_i f_{i,2}$ such that there exists $0 \leq i < k$ such that $a_i \neq 0$.

   Then by the Min-Max principle (3.2.2), we have

$$\begin{aligned}\sigma_k(M, g_1) &\leq \max_{f \in E_{k+1}} R_{g_1}(f) = R_{g_1}(f^*) \leq R_{g_2}(f^*)\\ &= \frac{\int_M \sum_{i=0}^k a_i^2 |\nabla f_{i,2}|^2 dV_{g_2}}{\int_\Sigma (\sum_{i=0}^k a_i f_{i,2})^2 dV_\Sigma}\\ &= \frac{\sum_{i=0}^k a_i^2 \sigma_i(M, g_2)}{\sum_{i=0}^k a_i^2} \quad \text{since } \int_\Sigma f_{i,2} f_{j,2} dV_\Sigma = \delta_{ij}\\ &< \sigma_k(M, g_2).\end{aligned}$$

In both cases, we have

$$\sigma_k(M, g_1) < \sigma_k(M, g_2).$$

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Remark 3.17. We never used the assumption that g_2 is a *symmetric* metric of revolution on M in the previous proof. However, it will be useful in the proofs of the theorems that follow.

The process that constructs the metric g_2 from g_1 can then be repeated to create a third metric g_3, and so on. This generates a sequence of metrics (g_i), obtained from a sequence of functions (h_i).

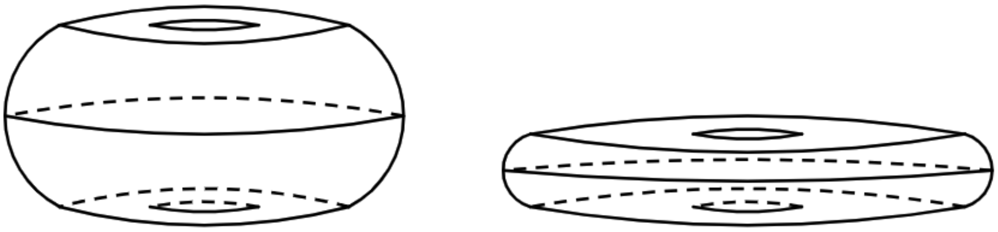

Figure 3.3: On the left, $M = [0, L] \times \mathbb{S}^{n-1}$ is endowed with a metric g_i of the sequence. On the right, $M = [0, L] \times \mathbb{S}^{n-1}$ is endowed with another metric g_j of the sequence, $j > i$.

The sequence (h_i) uniformly converges to the function

$$\begin{aligned} h^* : [0, L] &\longrightarrow \mathbb{R} \\ r &\longmapsto \begin{cases} 1 + r & \text{if } \; 0 \le r \le L/2 \\ 1 + L - r & \text{if } \; L/2 \le r \le L. \end{cases} \end{aligned}$$

This function is not smooth. Hence (M, g^*), where $g^* = dr^2 + h^{*2}(r)g_0$, is not a hypersurface of revolution in the sense of Definition 3.1. In the limit, (M, g^*) can be seen as the gluing of two copies of an annulus of inner radius 1 and outer radius $1 + L/2$. The metric g^* is therefore a maximizing metric, but is degenerated since it is induced by the function h^* which is non-smooth. That is why, as already mentioned, we call g^* the *degenerated maximizing metric on* M.

3.4 The first non trivial eigenvalue

In this section, we prove Theorem 3.3. The idea consists in comparing $\sigma_1(M, g)$ with the Rayleigh quotient of a test function that is obtained from an eigenfunction for a mixed problem (Steklov-Dirichlet or Steklov-Neumann) introduced in Section 3.2.2. Then, to show that the upper bound $B_n(L)$ given is sharp, we take a metric of revolution g_ε on M that is close to the degenerated maximizing metric g^* and show that $\sigma_1(M, g_\varepsilon)$ is close to $B_n(L)$.

Proof. Let $(M = [0, L] \times \mathbb{S}^{n-1}, g)$ be a hypersurface of revolution, where $L > 0$ is the meridian length of M. We recall that the boundary Σ of M consists of two disjoint copies of $\mathbb{S}^{n-1}$. We want to find a sharp upper bound $B_n(L)$ for $\sigma_1(M, g)$.

We consider $A_{1+L/2}$ the annulus of inner radius 1 and outer radius $1 + L/2$. Let φ_0 be an eigenfunction for the first eigenvalue of the Steklov-Dirichlet problem on $A_{1+L/2}$, i.e.

$$\sigma_0^D(A_{1+L/2}) = \frac{\int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r \varphi_0)^2 + \frac{1}{(1+r)^2} |\tilde{\nabla} \varphi_0|^2 \right) (1 + r)^{n-1} dV_{g_0} dr}{\int_{\mathbb{S}^{n-1}} \varphi_0^2(0, p) dV_{g_0}}.$$

We define a new function

$$\begin{aligned} \tilde{\varphi}_0 : [0, L] \times \mathbb{S}^{n-1} &\longrightarrow \mathbb{R} \\ (r, p) &\longmapsto \begin{cases} \varphi_0(r, p) & \text{if } 0 \le r \le L/2 \\ -\varphi_0(L - r, p) & \text{if } L/2 \le r \le L. \end{cases} \end{aligned} \tag{3.4.3}$$

The function $\tilde{\varphi_0}$ is continuous and we can check that

$$\begin{aligned}\int_\Sigma \tilde{\varphi_0}(r,p)dV_\Sigma &= \int_{\mathbb{S}^{n-1}} \tilde{\varphi_0}(0,p)dV_{g_0} + \int_{\mathbb{S}^{n-1}} \tilde{\varphi_0}(L,p)dV_{g_0}\\ &= \int_{\mathbb{S}^{n-1}} \varphi_0(0,p)dV_{g_0} - \int_{\mathbb{S}^{n-1}} \varphi_0(0,p)dV_{g_0}\\ &= 0.\end{aligned}$$

Hence, thanks to Formula (3.2.1), the function $\tilde{\varphi_0}$ can be used as a test function for $\sigma_1(M,g)$. We have

$$\begin{aligned}\sigma_1(M,g) &\le R_g(\tilde{\varphi_0})\\ &< R_{\tilde{g}}(\tilde{\varphi_0}) \quad \text{where } \tilde{g} = dr^2 + \tilde{h}^2 g_0 \text{ comes from Theorem 3.2}\\ &= \frac{\int_0^L \int_{\mathbb{S}^{n-1}} \left((\partial_r\tilde{\varphi_0})^2 + \frac{1}{\tilde{h}(r)^2}|\tilde{\nabla}\tilde{\varphi_0}|^2\right)\tilde{h}(r)^{n-1}dV_{g_0}dr}{\int_\Sigma \tilde{\varphi_0}^2(0,p)dV_\Sigma}\\ &= \frac{2\times\int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r\tilde{\varphi_0})^2 + \frac{1}{\tilde{h}(r)^2}|\tilde{\nabla}\tilde{\varphi_0}|^2\right)\tilde{h}(r)^{n-1}dV_{g_0}dr}{2\times\int_{\mathbb{S}^{n-1}} \tilde{\varphi_0}^2(0,p)dV_{g_0}} \quad \text{since } \tilde{g} \text{ is symmetric}\\ &= \frac{\int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r\varphi_0)^2 + \frac{1}{\tilde{h}(r)^2}|\tilde{\nabla}\varphi_0|^2\right)\tilde{h}(r)^{n-1}dV_{g_0}dr}{\int_{\mathbb{S}^{n-1}} \varphi_0^2(0,p)dV_{g_0}}\\ &< \frac{\int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r\varphi_0)^2 + \frac{1}{(1+r)^2}|\tilde{\nabla}\varphi_0|^2\right)(1+r)^{n-1}dV_{g_0}dr}{\int_{\mathbb{S}^{n-1}} \varphi_0^2(0,p)dV_{g_0}}\\ &= \sigma_0^D(A_{1+L/2}),\end{aligned} \tag{3.4.4}$$

where the second strict inequality comes from the existence of a continuum of points $r \in [0, L/2]$ such that $\tilde{h}(r) < 1 + r$.

If ϕ_1 is an eigenfunction for the first non trivial eigenvalue of the Steklov-Neumann problem on $A_{1+L/2}$, i.e

$$\sigma_1^N(A_{1+L/2}) = \frac{\int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r\phi_1)^2 + \frac{1}{(1+r)^2}|\tilde{\nabla}\phi_1|^2\right)(1+r)^{n-1}dV_{g_0}dr}{\int_{\mathbb{S}^{n-1}} \phi_1^2(0,p)dV_{g_0}},$$

then we define a new function

$$\begin{aligned}\tilde{\phi_1} : [0,L]\times\mathbb{S}^{n-1} &\longrightarrow \mathbb{R}\\ (r,p) &\longmapsto \begin{cases} \phi_1(r,p) & \text{if } 0 \le r \le L/2\\ \phi_1(L-r,p) & \text{if } L/2 \le r \le L.\end{cases}\end{aligned}$$

The function $\tilde{\phi}_1$ is continuous and we can check that

$$\begin{aligned}\int_\Sigma \tilde{\phi}_1(r,p)dV_\Sigma &= \int_{\mathbb{S}^{n-1}} \tilde{\phi}_1(0,p) + \int_{\mathbb{S}^{n-1}} \tilde{\phi}_1(L,p)\\ &= 0+0\\ &= 0,\end{aligned}$$

hence we can use it as a test function for $\sigma_1(M,g)$. The same calculations as in (3.4.4) show that

$$\sigma_1(M,g) < \sigma_1^N(A_{1+L/2}). \tag{3.4.5}$$

Putting Inequality (3.4.4) and Inequality (3.4.5) together, we get

$$\begin{aligned}\sigma_1(M,g) < B_n(L) :&= \min\left\{\sigma_0^D(A_{1+L/2}),\ \sigma_1^N(A_{1+L/2})\right\}\\ &= \min\left\{\frac{(n-2)\,(1+L/2)^{n-2}}{(1+L/2)^{n-2}-1}, \frac{(n-1)\,((1+L/2)^n-1)}{(1+L/2)^n+n-1}\right\}.\end{aligned} \tag{3.4.6}$$

We will now prove that the bound $B_n(L)$ is sharp. This means that for each $\varepsilon > 0$, there exists a metric of revolution g_ε on M such that $\sigma_1(M,g_\varepsilon) > B_n(L) - \varepsilon$.

Let $\varepsilon > 0$. Let $M = [0,L]\times\mathbb{S}^{n-1}$ and let $g_\varepsilon(r,p) = dr^2 + h_\varepsilon^2(r)g_0(p)$ be a metric of revolution on M such that:

1. The function h_ε is symmetric: for all $r\in[0,L]$, we have $h_\varepsilon(r) = h_\varepsilon(L-r)$;
2. For all $r\in[0,L/2-\delta]$, we have $h_\varepsilon(r) = (1+r)$, with δ small enough to guarantee that for all $r\in[0,L/2]$, we have

$$\max\{(1+r)^{n-3} - h_\varepsilon(r)^{n-3}, (1+r)^{n-1} - h_\varepsilon(r)^{n-1}\} < \frac{\varepsilon}{B_n(L)} =: \varepsilon^*.$$

   Geometrically, this means that (M,g_ε) looks like two copies of an annulus joined by a smooth curve, see Figure 3.3.

Let f_1 be an eigenfunction for $\sigma_1(M,g_\varepsilon)$. Because (M,g_ε) is symmetric, then we can choose f_1 symmetric or anti-symmetric [27, Section 3.2.2], which means that for all $r\in[0,L]$ and $p\in\mathbb{S}^{n-1}$, we have $|f_1(r,p)| = |f_1(L-r,p)|$.
Moreover, it results from the calculations in (3.4.4) that for any symmetric or anti-symmetric function f, we have

$$R_{g_\varepsilon}(f) = \frac{\int_0^{L/2}\int_{\mathbb{S}^{n-1}}\left((\partial_r f)^2 + \frac{1}{h_\varepsilon(r)^2}|\tilde{\nabla} f|^2\right)h_\varepsilon(r)^{n-1}dV_{g_0}dr}{\int_{\mathbb{S}^{n-1}} f^2(0,p)dV_{g_0}}.$$

We will compare

$$R_{g_\varepsilon}(f_1) = \frac{\int_0^{L/2}\int_{\mathbb{S}^{n-1}}\left((\partial_r f_1)^2 + \frac{1}{h_\varepsilon(r)^2}|\tilde{\nabla} f_1|^2\right)h_\varepsilon(r)^{n-1}dV_{g_0}dr}{\int_{\mathbb{S}^{n-1}}(f_1)^2(0,p)dV_{g_0}}$$

with

$$R_{A_{1+L/2}}(f_1) = \frac{\int_0^{L/2}\int_{\mathbb{S}^{n-1}}\left((\partial_r f_1)^2 + \frac{1}{(1+r)^2}|\tilde{\nabla} f_1|^2\right)(1+r)^{n-1}dV_{g_0}dr}{\int_{\mathbb{S}^{n-1}}(f_1)^2(0,p)dV_{g_0}}.$$

If we call $S := R_{A_{1+L/2}}(f_1) - R_{g_\varepsilon}(f_1)$, we have

$$\begin{aligned}
S &= \frac{\int_0^{L/2}\int_{\mathbb{S}^{n-1}}(\partial_r f_1)^2\left((1+r)^{n-1} - h_\varepsilon(r)^{n-1}\right) + |\tilde{\nabla} f_1|^2\left((1+r)^{n-3} - h_\varepsilon(r)^{n-3}\right)dV_{g_0}dr}{\int_{\mathbb{S}^{n-1}}(f_1)^2(0,p)dV_{g_0}}\\
&< \frac{\int_0^{L/2}\int_{\mathbb{S}^{n-1}}((\partial_r f_1)^2\cdot\varepsilon^* + |\tilde{\nabla} f_1|^2\cdot\varepsilon^*)dV_{g_0}dr}{\int_{\mathbb{S}^{n-1}}(f_1)^2(0,p)dV_{g_0}}\\
&= \varepsilon^*\cdot\frac{\int_0^{L/2}\int_{\mathbb{S}^{n-1}}((\partial_r f_1)^2 + |\tilde{\nabla} f_1|^2)dV_{g_0}dr}{\int_{\mathbb{S}^{n-1}}(f_1)^2(0,p)dV_{g_0}}\\
&< \varepsilon^*\cdot\frac{\int_0^{L/2}\int_{\mathbb{S}^{n-1}}((\partial_r f_1)^2 h_\varepsilon(r)^{n-1} + |\tilde{\nabla} f_1|^2 h_\varepsilon(r)^{n-3})dV_{g_0}dr}{\int_{\mathbb{S}^{n-1}}(f_1)^2(0,p)dV_{g_0}} \qquad \text{since } h_\varepsilon \geq 1\\
&= \varepsilon^*\cdot\sigma_1(M,g_\varepsilon) \qquad \text{since } f_1 \text{ is an eigenfunction}\\
&< \varepsilon^*\cdot B_n(L)\\
&= \varepsilon.
\end{aligned}$$

Hence, we have

$$R_{A_{1+L/2}}(f_1) < \sigma_1(M,g_\varepsilon) + \varepsilon. \tag{3.4.7}$$

We now have two cases:

1. f_1 can be written as $f_1(r,p) = u_0(r)S_0(p)$, where S_0 is a trivial harmonic function of the sphere, i.e S_0 is constant (we can choose $S_0 \equiv 1/\mathrm{Vol}(\mathbb{S}^{n-1})$), and u_0 is smooth. Hence f_1 is constant on $\{0\}\times\mathbb{S}^{n-1}$,

$$\int_{\{0\}\times\mathbb{S}^{n-1}} f_1(r,p)dV_{g_0} = u_0(0) \neq 0.$$

Moreover, since $|f_1(r,p)| = |f_1(L-r,p)|$ for all $r \in [0,L]$ and since

$$\int_\Sigma f_1(r,p)dV_\Sigma = 0,$$

we have

$$f_1\left(\frac{L}{2},p\right) = 0.$$

Therefore, we can use $f_1|_{[0,L/2]\times\mathbb{S}^{n-1}}$ as a test function for $\sigma_0^D(A_{1+L/2})$, and we can state

$$\sigma_0^D(A_{1+L/2}) \leq R_{A_{1+L/2}}(f_1).$$

2. f_1 can be written as $f_1(r,p) = u_1(r)S_1(p)$, where S_1 is a non constant harmonic function of the sphere associated with the first non zero eigenvalue and u_1 is smooth. Hence

$$\int_{\{0\}\times\mathbb{S}^{n-1}} f_1(r,p)dV_{g_0} = 0.$$

   Moreover, we have $u_1(L/2) > 0$.

   Added with the fact that $|f_1(r,p)| = |f_1(L-r,p)|$ for all $r \in [0,L]$ and because f_1 is smooth, we can conclude

$$\partial_r f_1\left(\frac{L}{2}, p\right) = 0.$$

   Therefore, we can use $f_1|_{[0,L/2]\times\mathbb{S}^{n-1}}$ as a test function for $\sigma_1^N(A_{1+L/2})$ and we can state

$$\sigma_1^N(A_{1+L/2}) \le R_{A_{1+L/2}}(f_1).$$

But we defined $B_n(L)$ as

$$B_n(L) = \min\{\sigma_0^D(A_{1+L/2}), \sigma_1^N(A_{1+L/2})\}.$$

Hence we have

$$B_n(L) \le R_{A_{1+L/2}}(f_1) \overset{(3.4.7)}{<} \sigma_1(M, g_\varepsilon) + \varepsilon$$

and then

$$\sigma_1(M, g_\varepsilon) > B_n(L) - \varepsilon.$$

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From this result we can prove Corollary 3.4.

Proof. By Theorem 3.3, Inequality (3.4.6) holds which is

$$\sigma_1(M,g) < \min\left\{\frac{(n-2)(1+L/2)^{n-2}}{(1+L/2)^{n-2}-1}, \frac{(n-1)((1+L/2)^n-1)}{(1+L/2)^n+n-1}\right\}.$$

We consider the two functions

$$L \longmapsto \frac{(n-2)(1+L/2)^{n-2}}{(1+L/2)^{n-2}-1} = \sigma_0^D(A_{1+L/2})$$
$$L \longmapsto \frac{(n-1)((1+L/2)^n-1)}{(1+L/2)^n+n-1} = \sigma_1^N(A_{1+L/2}).$$

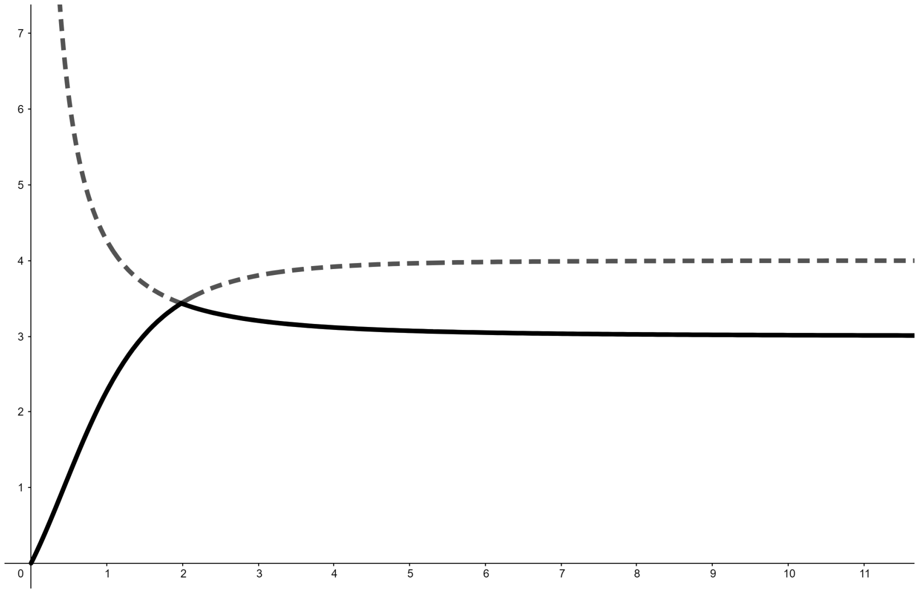

Figure 3.4: Representation of the case $n = 5$. The decreasing smooth curve is $L \longmapsto \sigma_0^D(A_{1+L/2})$ while the increasing smooth curve is $L \longmapsto \sigma_1^N(A_{1+L/2})$. The solid curve is the bound $B_5(L)$ given by Theorem 3.3.

We can show that $L \longmapsto \sigma_0^D(A_{1+L/2})$ is strictly decreasing with L. Indeed, let $L' > L$ and let φ_0 be an eigenfunction for $\sigma_0^D(A_{1+L/2})$. We consider

$$\bar{\varphi}_0 : [0, \frac{L'}{2}] \times \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$$

the extension by 0 of φ_0 to the annulus $A_{1+L'/2}$. We get

$$\sigma_0^D(A_{1+L'/2}) < R_{A_{1+L'/2}}(\bar{\varphi}_0) = R_{A_{1+L/2}}(\varphi_0) = \sigma_0^D(A_{1+L/2}),$$

where the strict inequality comes from the fact that $\bar{\varphi}_0$ is not an eigenfunction associated with $\sigma_0^D(A_{1+L'/2})$. Indeed, if we suppose that $\bar{\varphi}_0$ is an eigenfunction for $\sigma_0^D(A_{1+L'/2})$, then it is harmonic in $A_{1+L'/2}$ (since it satisfies the Steklov-Dirichlet problem), and since $\bar{\varphi}_0$ vanishes on the open set $A_{1+L'/2} \backslash A_{1+L/2}$, then by [1] $\bar{\varphi}_0$ is constant, which is a contradiction.

In the same way, we can show that $L \longmapsto \sigma_1^N(A_{1+L/2})$ is strictly increasing with L. Indeed, let $L' > L$ and let ϕ_1 be an eigenfunction for $\sigma_1^N(A_{1+L'/2})$. We consider

$$\bar{\phi}_1 : [0, \frac{L}{2}] \times \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$$

the restriction of ϕ_1 to the annulus $A_{1+L/2}$. We get

$$\sigma_1^N(A_{1+L/2}) \leq R_{A_{1+L/2}}(\bar{\phi}_1) < R_{A_{1+L'/2}}(\phi_1) = \sigma_1^N(A_{1+L'/2}).$$

Hence the bound we gave possesses a maximum depending only on the dimension n, given by

$$\sigma_1(M, g) < B_n := \frac{(n-2)\,(1 + L_1/2)^{n-2}}{(1 + L_1/2)^{n-2} - 1}$$

where L_1 is the unique positive solution of the equation

$$(1 + L/2)^{2n-2} - (n-1)\,(1 + L/2)^n - (n-1)^2\,(1 + L/2))^{n-2} + n - 1 = 0.$$

In order to prove that this bound is sharp, let $\varepsilon > 0$. We define $M_\varepsilon := [0, L_1] \times \mathbb{S}^{n-1}$. Theorem 3.3 guarantees that there exists a metric of revolution g_ε on M_ε such that $\sigma_1(M_\varepsilon, g_\varepsilon) > B_n(L_1) - \varepsilon = B_n - \varepsilon$, which ends the proof.

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We continue by proving Proposition 3.5.

Proof. We know that there exists a unique positive value of L, that we call $L_1 = L_1(n)$, such that the equality

$$(1 + L/2)^{2n-2} - (n-1)\,(1 + L/2)^n - (n-1)^2\,(1 + L/2))^{n-2} + n - 1 = 0$$

holds. To ease notation, we substitute $(1 + L/2)$ by R and we can state that there is a unique value of $R \in (1, \infty)$ such that the equality

$$R^{2n-2} - (n-1)R^n - (n-1)^2 R^{n-2} + n - 1 = 0$$

holds. This equation is equivalent to

$$R^{n-2}\left(R^n - (n-1)R^2 - (n-1)^2\right) + n - 1 = 0,$$

and we call $R_1 = R_1(n)$ its unique solution in $(1, \infty)$. We prove that $R_1(n) \underset{n\to\infty}{\longrightarrow} 1$.

We call

$$\psi_n(R) := R^n - (n-1)R^2 - (n-1)^2$$

and

$$\Psi_n(R) := R^{n-2}\left(R^n - (n-1)R^2 - (n-1)^2\right) + n - 1.$$

Then, for R_1 to be such that $\Psi_n(R_1) = 0$, it is necessary that $\psi_n(R_1) < 0$.

Thus,

$$\begin{aligned} R_1^n &< (n-1)R_1^2 + (n-1)^2 \\ &< (n-1)^2(R_1^2 + 1) \\ &< (n-1)^2 \cdot 2R_1^2 && \text{since } R_1 > 1 \\ &< (n-1)^3 R_1^2 && \text{since } n - 1 \geq 2. \end{aligned}$$

Therefore,

$$n \ln(R_1) < 3 \ln(n-1) + 2 \ln(R_1)$$

so

$$\ln(R_1) < \frac{3\ln(n-1)}{n-2}$$

and

$$R_1 < e^{\frac{3\ln(n-1)}{n-2}}.$$

As we substituted $(1 + L/2)$ by R, and we can state that

$$L_1(n) < 2\left(e^{\frac{3\ln(n-1)}{n-2}} - 1\right).$$

Therefore, since $\frac{3\ln(n-1)}{n-2} \underset{n\to\infty}{\longrightarrow} 0$, we have

$$L_1(n) \underset{n\to\infty}{\longrightarrow} 0.$$

Moreover, we have

$$n - 1 > B_n > n - 2 \underset{n\to\infty}{\longrightarrow} \infty.$$

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3.5 Stability properties of hypersurfaces of revolution

The goal of this section is to prove Theorem 3.6 and Theorem 3.7, which show some stability properties of the hypersurfaces we are studying in this paper. For Theorem 3.6, the key idea is to choose $L \neq L_1$ and compare $\sigma_1(M = [0, L] \times \mathbb{S}^{n-1}, g)$ with the first non trivial eigenvalue of M when endowed with the degenerated maximizing metric, namely $B_n(L)$. For the case of Theorem 3.7, the strategy consists in showing that among all metrics of revolution that are not close (in a sense properly defined) to the degenerated maximizing metric, none of them induces a first non trivial eigenvalue that is close to $B_n(L)$. We prove these theorems now.

3.5.1 Proof of Theorem 3.6

Recall that here we suppose $L \neq L_1$.

Proof. Let g be any metric of revolution on $M = [0, L] \times \mathbb{S}^{n-1}$. Then we have

$$\sigma_1(M, g) < B_n(L),$$

where $B_n(L)$ is given by Theorem 3.3.

We define $C(n, L) := B_n - B_n(L)$, which is strictly positive since we assumed $L \neq L_1$. Then we have

$$B_n - \sigma_1(M, g) \geq B_n - B_n(L) = C(n, L).$$

Let $0 < \delta < \frac{B_n-(n-2)}{2}$, and let us suppose $|B_n - \sigma_1(M,g)| < \delta$. Therefore, we have $|B_n - \sigma_1(M,g^*)| < \delta$, where we wrote g^* the degenerated maximizing metric on M. We consider two cases:

1. We suppose $L_1 < L$. In this case, we have $B_n(L) = \sigma_0^D(A_{1+L/2}) = \frac{(n-2)(1+L/2)^{n-2}}{(1+L/2)^{n-2}-1}$. We write

   $$R := 1 + L/2 \text{ and } \sigma_1(R) := \frac{(n-2)R^{n-2}}{R^{n-2}-1}.$$

   Hence we have $|B_n - \sigma_1(R)| < \delta \implies R \in [R_1, R_\delta]$, where $R_1 = 1 + L_1/2$ and R_δ is defined by $\sigma_1(R_\delta) = B_n - \delta$. Note that R_δ exists since we assumed $\delta < B_n - (n-2)$. We can calculate that

   $$R_\delta = \left(\frac{B_n - \delta}{B_n - (n-2) - \delta}\right)^{\frac{1}{n-2}} \text{ and } R_1 = \left(\frac{B_n}{B_n - (n-2)}\right)^{\frac{1}{n-2}}.$$

   Thus, we have

   $$|R_1 - R| \leq R_\delta - R_1 = \left(\frac{B_n - \delta}{B_n - (n-2) - \delta}\right)^{\frac{1}{n-2}} - \left(\frac{B_n}{B_n - (n-2)}\right)^{\frac{1}{n-2}}.$$

   To estimate this expression, we use the identity $x^{n-2} - y^{n-2} = (x-y)(x^{n-3} + x^{n-4}y + \ldots + xy^{n-4} + y^{n-3})$, with $x = R_\delta$ and $y = R_1$. On the one hand, we can compute that

   $$R_\delta^{n-2} - R_1^{n-2} = \frac{(n-2)\delta}{(B_n - (n-2) - \delta)(B_n - (n-2))} \leq \frac{2(n-2)\delta}{(B_n - (n-2))^2},$$

   where the inequality comes from the assumption $\delta < \frac{B_n-(n-2)}{2}$. On the other hand, we can compute that

   $$R_\delta^{n-3} + R_\delta^{n-4}R_1 + \ldots + R_\delta R_1^{n-4} + R_1^{n-3} \geq (n-2) \cdot \left(\frac{B_n}{B_n - (n-2)}\right)^{\frac{n-3}{n-2}}.$$

   Therefore,

   $$R_\delta - R_1 \leq \frac{2/(B_n - (n-2))^2}{(B_n/(B_n - (n-2)))^{\frac{n-3}{n-2}}} \cdot \delta := C_1(n) \cdot \delta.$$

   Since we wrote $R = 1 + L/2$, we can conclude that, for $L_1 < L$ and $0 < \delta < \frac{B_n-(n-2)}{2}$, we have

   $$B_n - \sigma_1(M,g) < \delta \implies L - L_1 < 2C_1(n) \cdot \delta.$$

2. Now we suppose $L < L_1$ and we do a similar calculation, this time with $B_n(L) = \sigma_1^N(A_{1+L/2}) = \frac{(n-1)((1+L/2)^n-1)}{(1+L/2)^n+n-1}$. For the convenience of the reader, we provide the details in this thesis.

   This time we write

   $$R := 1 + L/2 \text{ and } \sigma_1(R) := \frac{(n-1)(R^n - 1)}{R^n + n - 1}.$$

Therefore, we have $|B_n - \sigma_1(R)| < \delta \implies R \in [R_\delta, R_1]$, where $R_1 = 1 + L_1/2$ and R_δ is defined by $\sigma_1(R_\delta) = B_n - \delta$. We can compute that

$$R_1 = \left(\frac{(n-1)(B_n+1)}{n-1-B_n}\right)^{\frac{1}{n}} \text{ and } R_\delta = \left(\frac{(n-1)(B_n+1-\delta)}{n-1-B_n+\delta}\right)^{\frac{1}{n}}.$$

Therefore, we have

$$|R_1 - R| \leq R_1 - R_\delta = \left(\frac{(n-1)(B_n+1)}{n-1-B_n}\right)^{\frac{1}{n}} - \left(\frac{(n-1)(B_n+1-\delta)}{n-1-B_n+\delta}\right)^{\frac{1}{n}}.$$

Now we use the identity $x^n - y^n = (x-y)(x^n + x^{n-1}y + \ldots + xy^{n-1} + y^n)$, with $x = R_1$ and $y = R_\delta$. We have

$$R_1^n - R_\delta^n = \frac{(n-1)(2B_n - (n-2))\delta}{(n-1-B_n+\delta)(n-1-B_n)} \leq \frac{(n-1)(2B_n-(n-2))\delta}{(n-1-B_n)^2}.$$

Now we also compute that

$$\begin{aligned} R_1^n + R_1^{n-1}R_\delta + \ldots + R_1R_\delta^{n-1} + R_\delta^n &\geq (n+1)\cdot R_\delta^n \\ &\geq (n+1)\cdot R^n_{\frac{B_n-(n-2)}{2}} \\ &= (n+1)\cdot\frac{(n-1)(B_n+n)}{n-B_n}. \end{aligned}$$

Hence, we have

$$R_1 - R_\delta \leq \frac{(n-1)(2B_n-(n-2))/(n-1-B_n)^2}{(n+1)(n-1)(B_n+n)/(n-B_n)}\cdot\delta := C_2(n)\cdot\delta.$$

Now we have a new constant $C_2(n)$ such that

$$B_n - \sigma_1(M,g) < \delta \implies |L_1 - L| \leq 2C_2(n)\cdot\delta.$$

Defining $C(n) := 2\cdot\max\{C_1(n), C_2(n)\}$ concludes the proof.

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3.5.2 Proof of Theorem 3.7

Recall that we fixed $m \in [1, 1 + L/2)$ and that we defined $\mathcal{M}_m :=$ {metrics of revolution g induced by a function h such that $\max_{r\in[0,L]}\{h(r)\} \leq m$}.

Proof. Let $g \in \mathcal{M}_m$, and let $h : [0,L] \longrightarrow \mathbb{R}_+^*$ be the function which induces g. We define a new function $h_m : [0,L] \longrightarrow \mathbb{R}_+^*$ as follows:

$$h_m(r) = \begin{cases} 1+r & \text{if } 0 \leq r \leq m-1 \\ m & \text{if } m-1 \leq r \leq L-m+1 \\ 1+L-r & \text{if } L-m+1 \leq r \leq L. \end{cases}$$

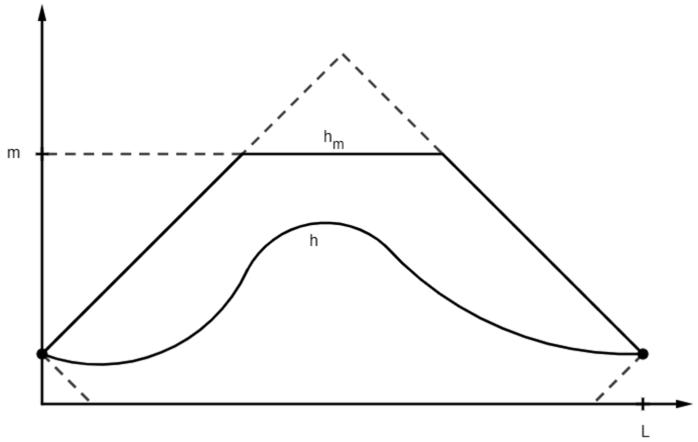

Figure 3.5: Since $g \in \mathcal{M}_m$, the function h which induces g satisfies $h \leq h_m$.

We call g_m the metric induced by h_m. Notice that g_m is not a metric of revolution in the sense of Definition 3.1 since h_m is not smooth.

In the same spirit as in Section 3.3, for any smooth function f on M, we have

$$R_g(f) = \frac{\int_M \left((\partial_r f)^2 + \frac{1}{h^2}|\tilde{\nabla} f|^2_{g_0}\right) h^{n-1} dV_{g_0} dr}{\int_\Sigma |f|^2 dV_\Sigma}$$

and

$$R_{g_m}(f) = \frac{\int_M \left((\partial_r f)^2 + \frac{1}{h_m^2}|\tilde{\nabla} f|^2_{g_0}\right) h_m^{n-1} dV_{g_0} dr}{\int_\Sigma |f|^2 dV_\Sigma}.$$

Therefore, since $n \geq 3$ and $h \leq h_m$, we have

$$\sigma_1(M, g) \leq \sigma_1(M, g_m).$$

We can now consider a new function $\tilde{h}_m$, obtained from h_m by smoothing out the two non-smooth points, with $\tilde{h}_m$ satisfying:

1. For all $r \in [0, L]$, we have $h_m(r) \leq \tilde{h}_m(r)$;
2. The metric $\tilde{g}_m$ induced by $\tilde{h}_m$ is a metric of revolution in the sense of Definition 3.1.

Remark that since $h_m \leq \tilde{h}_m$, we have $\sigma_1(M, g_m) \leq \sigma_1(M, \tilde{g}_m)$.

We define $C(n, L, m) := B_n(L) - \sigma_1(M, \tilde{g}_m)$, which is strictly positive by Theorem 3.3. Then we have

$$B_n(L) - \sigma_1(M, g) \geq B_n(L) - \sigma_1(M, \tilde{g}_m) = C(n, L, m).$$

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3.6 Upper bounds for higher Steklov eigenvalues

In this section, we want to compute some sharp upper bound for higher Steklov eigenvalues of hypersurfaces of revolution. Therefore, we will have to deal with the multiplicity of the eigenvalues. We write $\lambda_{(k)}, \sigma_{(k)}, \sigma^D_{(k)}, \sigma^N_{(k)}$ for the (k)th eigenvalue counted without multiplicity.

Before we can state and prove our results, we first recall some known properties of the multiplicities of the eigenvalues under consideration.

Given a hypersurface of revolution $(M = [0, L] \times \mathbb{S}^{n-1}, g)$, we want to provide information about the multiplicity of the Steklov eigenvalues of (M, g).

For the classical Laplacian problem $\Delta S = \lambda S$ on $(\mathbb{S}^{n-1}, g_0)$, we know [3, Page 160-162] that the set of eigenvalues is $\{\lambda_{(k)} = k(n + k - 2) : k \geq 0\}$, where the multiplicity m_0 of $\lambda_{(0)} = 0$ is 1 and the multiplicity of $\lambda_{(k)}$ is

$$m_k := \frac{(n + k - 3)(n + k - 4) \ldots n(n - 1)}{k!}(n + 2k - 2). \tag{3.6.8}$$

As such, given $k \geq 0$, there exist m_k independent functions $S^1_k, \ldots, S^{m_k}_k$ such that $\Delta S^i_k = \lambda_{(k)} S^i_k$, $i = 1, \ldots, m_k$.

Given $k \geq 0$, there are m_k independent Steklov-Dirichlet eigenfunctions associated with the eigenvalue $\sigma^D_{(k)}(A_{1+L/2})$, that can be written $\varphi^i_k(r, p) = \alpha_k(r) S^i_k(p)$, $i = 1, \ldots, m_k$. For the Steklov-Neumann case, the eigenfunctions associated with $\sigma^N_{(k)}(A_{1+L/2})$ can be written $\phi^i_k(r, p) = \beta_k(r) S^i_k(p)$, $i = 1, \ldots, m_k$. Indeed, for each of these problems, the multiplicity of the (k)th eigenvalue is exactly m_k, see, for example, [10, Proposition 3].

3.6.1 Upper bound for $\sigma_2(M, g), \ldots, \sigma_{m_1}(M, g)$

In this section, we prove the following theorem:

Theorem 3.18. *Let $(M = [0, L] \times \mathbb{S}^{n-1}, g)$ be a hypersurface of revolution in Euclidean space with two boundary components each isometric to $\mathbb{S}^{n-1}$ and dimension $n \geq 3$. Let m_1 be the multiplicity of the first non trivial eigenvalue of the classical Laplacian problem on $(\mathbb{S}^{n-1}, g_0)$. Then we have*

$$\sigma_2(M, g) = \ldots = \sigma_{m_1}(M, g) < B^2_n(L) = \ldots = B^{m_1}_n(L) =: \sigma^N_{(1)}(A_{1+L/2}).$$

Moreover, this bound is sharp: for all $\varepsilon > 0$ there exists a metric of revolution g_ε on M such that

$$\sigma_2(M, g_\varepsilon) = \ldots = \sigma_{m_1}(M, g_\varepsilon) > \sigma^N_{(1)}(A_{1+L/2}) - \varepsilon.$$

Proof. We consider two cases.

1. Let $M = [0, L] \times \mathbb{S}^{n-1}$, with $L \leq L_1$. We write f^1_1 an eigenfunction associated with $\sigma_1(M, g)$. Since $L \leq L_1$, we have $B_n(L) = \sigma^N_1(A_{1+L/2})$ and therefore $f^1_1(r, p) = u_1(r) S^1_1(p)$.

   We consider now a new function denoted f^2_1 given by $f^2_1(r, p) = u_1(r) S^2_1(p)$. We can check that

$$\int_\Sigma f_1^2(r,p)dV\Sigma = 0 \quad \text{and} \quad \int_\Sigma f_1^1(r,p)f_1^2(r,p)dV\Sigma = 0.$$

Moreover, we have

$$\sigma_1(M,g) = R_g(f_1^1) = R_g(f_1^2).$$

In the same way, we write

$$f_1^i(r,p) = u_1(r)S_1^i(p), \quad i = 1,\dots,m_1$$

and we can conclude

$$\sigma_1(M,g) = \sigma_2(M,g) = \ldots = \sigma_{m_1}(M,g).$$

Therefore, we already have a sharp upper bound for these eigenvalues, which is given by $\sigma_1^N(A_{1+L/2})$.

2. Let $M = [0,L]\times\mathbb{S}^{n-1}$, with $L > L_1$. We call f_1 an eigenfunction associated with $\sigma_1(M,g)$. Since $L > L_1$, we have $B_n(L) = \sigma_0^D(A_{1+L/2})$. Therefore $f_1(r,p) = u_0(r)S_0(p)$.

   We write now $f_2^1(r,p) = u_2(r)S_1^1(p)$ an eigenfunction associated with $\sigma_2(M,g)$. As before, we then consider m_1 functions denoted $f_2^i(r,p) = u_2(r)S_1^i(p), i = 1,\dots,m_1$ and we get

$$\sigma_2(m,g) = \ldots = \sigma_{m_1+1}(M,g).$$

   We consider a function $\phi_1(r,p) = \beta_1(r)S_1(p)$ associated with $\sigma_{(1)}^N(A_{1+L/2})$. In the same spirit as before, we define a function

$$\begin{aligned}\tilde{\phi}_1 : [0,L]\times\mathbb{S}^{n-1} &\longrightarrow \mathbb{R}\\ (r,p) &\longmapsto \begin{cases} \phi_1(r,p) & \text{if } 0\le r\le L/2\\ \phi_1(L-r,p) & \text{if } L/2\le r\le L.\end{cases}\end{aligned}$$

   We can check that the function $\tilde{\phi}_1$ is continuous and that $\int_\Sigma \tilde{\phi}_1 dV_\Sigma = 0$. Moreover, it is immediate that $\int_\Sigma \tilde{\phi}_1 f_1 dV_\Sigma = 0$. Hence we can use $\tilde{\phi}_1$ as a test function for $\sigma_2(M,g)$ and as we did before, we can see that

$$\begin{aligned}
\sigma_2(M,g) &\leq R_g(\tilde{\phi_1}) \\
&< R_{\tilde{g}}(\tilde{\phi_1}) \text{ where } \tilde{g} \text{ comes from Theorem 3.2} \\
&= \frac{\int_0^L \int_{\mathbb{S}^{n-1}} \left((\partial_r \tilde{\phi_1})^2 + \frac{1}{\tilde{h}(r)^2} |\tilde{\nabla} \tilde{\phi_1}|^2 \right) \tilde{h}(r)^{n-1} dV_{g_0} dr}{\int_\Sigma \tilde{\phi_1}^2(0,p) dV_{g_0}} \\
&= \frac{2 \times \int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r \tilde{\phi_1})^2 + \frac{1}{\tilde{h}(r)^2} |\tilde{\nabla} \tilde{\phi_1}|^2 \right) \tilde{h}(r)^{n-1} dV_{g_0} dr}{2 \times \int_{\mathbb{S}^{n-1}} \tilde{\phi_1}^2(0,p) dV_{g_0}} \\
&= \frac{\int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r \phi_1)^2 + \frac{1}{\tilde{h}(r)^2} |\tilde{\nabla} \phi_1|^2 \right) \tilde{h}(r)^{n-1} dV_{g_0} dr}{\int_{\mathbb{S}^{n-1}} \phi_1^2(0,p) dV_{g_0}} \\
&< \frac{\int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r \phi_1)^2 + \frac{1}{(1+r)^2} |\tilde{\nabla} \phi_1|^2 \right) (1+r)^{n-1} dV_{g_0} dr}{\int_{\mathbb{S}^{n-1}} \phi_1^2(0,p) dV_{g_0}} \\
&= \sigma_{(1)}^N(A_{1+L/2}).
\end{aligned}$$

Therefore, regardless of the value of $L > 0$, we have

$$\sigma_2(M,g) = \ldots = \sigma_{m_1}(M,g) < B_n^2(L) = \ldots = B_n^{m_1}(L) := \sigma_{(1)}^N(A_{1+L/2}).$$

Moreover, this bound is sharp : for all $\varepsilon > 0$, there exists a metric g_ε on $M = [0,L] \times \mathbb{S}^{n-1}$ such that $\sigma_2(M,g_\varepsilon) = \ldots = \sigma_{m_1}(M,g_\varepsilon) > \sigma_{(1)}^N(A_{1+L/2}) - \varepsilon$. Indeed, as before it is sufficient to choose the metric $g_\varepsilon = dr^2 + h_\varepsilon^2 g_0$, with the function h_ε such that

1. h_ε is symmetric;
2. For all $r \in [0, L/2 - \delta]$, we have $h_\varepsilon(r) = 1 + r$, with δ small enough.

The proof of sharpness goes as in the proof of Theorem 3.3.

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The upper bound we gave, namely $\sigma_{(1)}^N(A_{1+L/2})$, depends on the dimension of M and the meridian length L of M. It is easy to see that $\sigma_{(1)}^N(A_{1+L/2})$, which is strictly increasing, satisfies

$$\sigma_{(1)}^N(A_{1+L/2}) = \frac{(n-1)\left((1+L/2)^n - 1\right)}{(1+L/2)^n + n - 1} \xrightarrow[L\to\infty]{} n-1.$$

Therefore, we have got a bound that depends only on the dimension n of M. Given a hypersurface of revolution (M,g) with two boundary components, we have

$$\sigma_2(M,g) = \ldots = \sigma_{m_1}(M,g) < B_n^2 = \ldots = B_n^{m_1} := n-1.$$

Moreover, this bound is sharp, in the sense that for all $\varepsilon > 0$, there exists a hypersurface of revolution $(M_\varepsilon, g_\varepsilon)$ such that $\sigma_2(M_\varepsilon, g_\varepsilon) = \ldots = \sigma_{m_1}(M_\varepsilon, g_\varepsilon) > n - 1 - \varepsilon$. Indeed, we can choose L_ε large enough for $\sigma_1^N(A_{1+L_\varepsilon/2})$ to be $\frac{\varepsilon}{2}$-close to $n-1$, and then define $M_\varepsilon := [0, L_\varepsilon] \times \mathbb{S}^{n-1}$. Now we can put a metric g_ε on M_ε such that $\sigma_2(M_\varepsilon, g_\varepsilon) = \ldots = \sigma_{m_1}(M_\varepsilon, g_\varepsilon) > \sigma_1^N(A_{1+L_\varepsilon/2}) - \frac{\varepsilon}{2}$, and we are done.

Our calculations showed that the eigenvalues $k = 2, \ldots, m_1$ have a critical length at infinity.

3.6.2 Upper bound for $\sigma_{m_1+1}(M, g)$

Now we are interested in the next eigenvalue, namely $\sigma_{m_1+1}(M, g)$. For that reason, we define a new special meridian length L_2: it is the unique solution of the equation $\sigma_0^D(A_{1+L/2}) = \sigma_{(2)}^N(A_{1+L/2})$. We remark that we have $L_2 < L_1$. Indeed, for all $L > 0$, we have $\sigma_{(2)}^N(A_{1+L/2}) > \sigma_{(1)}^N(A_{1+L/2})$, and the function $L \longmapsto \sigma_0^D(A_{1+L/2})$ is strictly decreasing. Hence, comparing the intersection of the curves gives $L_2 < L_1$. We prove the following theorem:

Theorem 3.19. *Let $(M = [0, L] \times \mathbb{S}^{n-1}, g)$ be a hypersurface of revolution in Euclidean space with two boundary components each isometric to $\mathbb{S}^{n-1}$ and dimension $n \geq 3$. Let m_1 be the multiplicity of the first non trivial eigenvalue of the classical Laplacian problem on $(\mathbb{S}^{n-1}, g_0)$. Then we have*

$$\sigma_{m_1+1}(M, g) < B_n^{m_1+1}(L) := \begin{cases} \sigma_{(2)}^N(A_{1+L/2}) & \textit{if } L \leq L_2 \\ \sigma_{(0)}^D(A_{1+L/2}) & \textit{if } L_2 < L \leq L_1 \\ \sigma_{(1)}^N(A_{1+L/2}) & \textit{if } L_1 < L. \end{cases}$$

Moreover, this bound is sharp: for all $\varepsilon > 0$, there exists a metric of revolution g_ε on M such that

$$\sigma_{m_1+1}(M, g_\varepsilon) > B_n^{m_1+1}(L) - \varepsilon.$$

A plot of the function $L \longmapsto B_n^{m_1+1}(L)$ can be useful to visualize the sharp upper bound, see Figure 3.6.

Proof. Now we have to distinguish three cases.

1. Let $M = [0, L] \times \mathbb{S}^{n-1}$, with $L \leq L_2$. We call $f_1^1(r, p) = u_1(r)S_1^1(p), \ldots, f_1^{m_1}(r, p) = u_1(r)S_1^{m_1}(p)$ the Steklov eigenfunctions associated with $\sigma_{(1)}(M, g) = \sigma_1(M, g) = \ldots = \sigma_{m_1}(M, g)$.

   There exists an eigenfunction $\phi_2(r, p) = \beta_2(r)S_2(p)$ associated with the Steklov-Neumann eigenvalue $\sigma_{(2)}^N(M, g) = \sigma_{m_1+1}^N(M, g)$. We define a new function

$$\begin{aligned} \tilde{\phi}_2 : [0, L] \times \mathbb{S}^{n-1} &\longrightarrow \mathbb{R} \\ (r, p) &\longmapsto \begin{cases} \phi_2(r, p) & \text{if } 0 \leq r \leq L/2 \\ \phi_2(L - r, p) & \text{if } L/2 \leq r \leq L. \end{cases} \end{aligned}$$

   This function is continuous, satisfies $\int_\Sigma \tilde{\phi}_2 dV_\Sigma = 0$ and we can check that for all $i = 1, \ldots, m_1$,

$$\int_\Sigma \tilde{\phi}_2 f_1^i dV_\Sigma = 0.$$

   Hence we can use $\tilde{\phi}_2$ as a test function for $\sigma_{m_1+1}(M, g)$. The same kind of calculations as in Inequality (3.4.5) show that we have

$$\sigma_{m_1+1}(M, g) < \sigma_{(2)}^N(A_{1+L/2}),$$

which is a sharp upper bound.

2. Let $M = [0, L] \times \mathbb{S}^{n-1}$, with $L_2 < L \leq L_1$. We call $f_1^1(r, p) = u_1(r)S_1^1(p), \dots, f_1^{m_1}(r, p) = u_1(r)S_1^{m_1}(p)$ the Steklov eigenfunctions associated with $\sigma_{(1)}(M, g) = \sigma_1(M, g) = \dots = \sigma_{m_1}(M, g)$.

   There exists an eigenfunction $\varphi_0(r, p) = \alpha_0(r)S_0(p)$ associated with $\sigma_0^D(M, g)$. We use the function $\tilde{\varphi_0}$ we defined before, namely

   $$\begin{aligned} \tilde{\varphi_0} : [0, L] \times \mathbb{S}^{n-1} &\longrightarrow \mathbb{R} \\ (r, p) &\longmapsto \begin{cases} \varphi_0(r, p) & \text{if } 0 \leq r \leq L/2 \\ -\varphi_0(L - r, p) & \text{if } L/2 \leq r \leq L. \end{cases} \end{aligned}$$

   We already saw that $\tilde{\varphi_0}$ is continuous, that $\int_\Sigma \tilde{\varphi_0} dV_\Sigma = 0$ and we can check that for all $i = 1, \dots, m_1$,

   $$\int_\Sigma \tilde{\varphi_0} f_1^i dV_\Sigma = 0.$$

   Using $\tilde{\varphi_0}$ as a test function, we get

   $$\sigma_{m_1+1}(M, g) < \sigma_{(0)}^D(A_{1+L/2}),$$

   which is a sharp upper bound.

3. Let $M = [0, L] \times \mathbb{S}^{n-1}$, with $L_1 \leq L$. Then $\sigma_1(M, g) < \sigma_2(M, g) = \dots = \sigma_{m_1+1}(M, g)$. We already dealt with this case in the proof of Theorem 3.18 and we saw that

   $$\sigma_{m_1+1}(M, g) < \sigma_{(1)}^N(A_{1+L/2}),$$

   which is a sharp upper bound.

Therefore, given a hypersurface of revolution $(M = [0, L] \times \mathbb{S}^{n-1}, g)$, we have a sharp upper bound for $\sigma_{m_1+1}(M, g)$, depending on n and L, given by

$$\sigma_{m_1+1}(M, g) < B_n^{m_1+1}(L) := \begin{cases} \sigma_{(2)}^N(A_{1+L/2}) & \text{if } L \leq L_2 \\ \sigma_{(0)}^D(A_{1+L/2}) & \text{if } L_2 < L \leq L_1 \\ \sigma_{(1)}^N(A_{1+L/2}) & \text{if } L_1 < L. \end{cases}$$

The proof of sharpness goes as in the proof of Theorem 3.3.

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From this, one can once again look for a sharp upper bound for $\sigma_{m_1+1}(M, g)$ that depends only on the dimension n of M. This bound is given by

$$\begin{aligned} \sigma_{m_1+1}(M, g) < B_n^{m_1+1} &:= \max\left\{\sigma_0^D(A_{1+L_2/2}), n - 1\right\} \\ &= \begin{cases} \sigma_0^D(A_{1+L_2/2}) & \text{if } 3 \leq n \leq 6 \\ n - 1 & \text{if } 7 \leq n. \end{cases} \end{aligned} \tag{3.6.9}$$

A proof of (3.6.9) is given in Appendix 3.A.

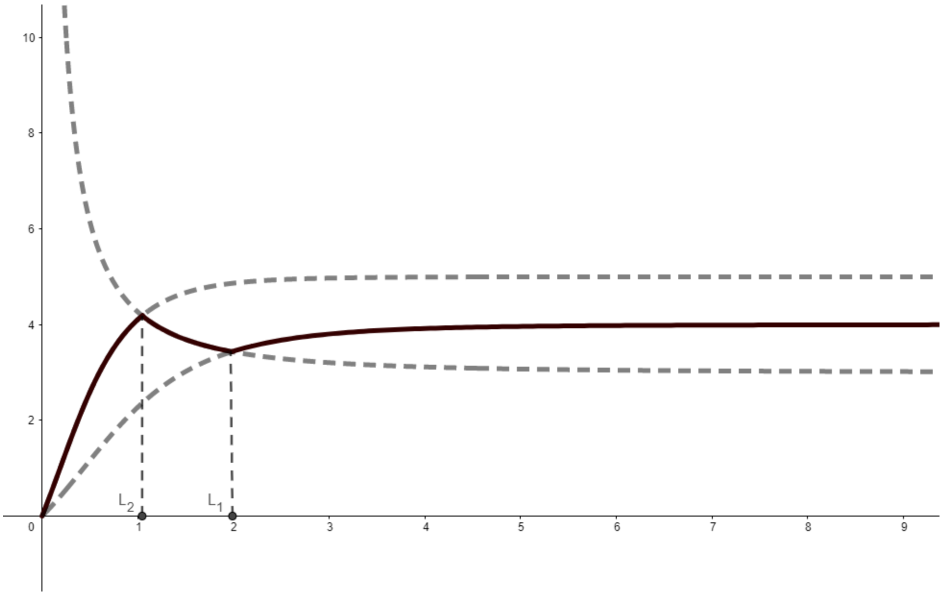

Figure 3.6: Representation of the case $n = 5$. The solid curve is the bound given in Theorem 3.19.

Therefore, the eigenvalue $k = m_1 + 1$ possesses a finite critical length if $3 \leq n \leq 6$, and it has a critical length at infinity if $7 \leq n$.

Remark 3.20. It is then tempting to search for an expression for $B_n^k := \sup_{L\in\mathbb{R}_+^*}\{B_n^k(L)\}$ for any n and k; but it seems to be hard to give an explicit formula for it. Indeed, as Section 3.6.1 and 3.6.2 suggest, the function $L \longmapsto B_n^k(L)$ is hard to determine and can be either smooth (as in Section 3.6.1 for instance) or piecewise smooth (as Section 3.6.2 for instance). In the second case, there are possibly many irregular points that we have to consider. Moreover, depending on the value of n and k:

1. Either k has a finite critical length, i.e $B_n^k = B_n^k(L_k)$ for a certain $L_k \in \mathbb{R}_+^*$. That is for instance the case of $\sigma_1(M, g)$ or σ_{m_1+1} if $n = 3, 4, 5$ or 6;

2. Or k has a critical length at infinity, i.e $B_n^k = \lim_{L\to\infty} B_n^k(L)$. That is for instance the case of $\sigma_2(M, g), \ldots, \sigma_{m_1}(M, g)$.

Furthermore, we will prove in Section 3.7 that for all $n \geq 3$, there are infinitely many k that have a finite critical length associated to them. In all these cases, the function $L \longmapsto B_n^k(L)$ is piecewise smooth.

3.7 Critical lengths of hypersurfaces of revolution

We recall that given $n \geq 3$, we are interested in giving information about the set of finite critical lengths. We want to prove Theorem 3.9, i.e that there are infinitely many k such that $B_n^k = B_n^k(L_k)$ for a certain finite $L_k \in \mathbb{R}_+^*$, and that the sequence of critical lengths converges to 0.

Proof. As before, for $j \geq 0$, we denote by m_j the number given by Formula (3.6.8), which is the multiplicity of $\sigma_{(j)}^D(A_R)$ as well as the multiplicity of $\sigma_{(j)}^N(A_R)$. Let $i \geq 2$ be an integer. We claim that for all k such that

$$m_0 + \sum_{j=1}^{i-1} 2m_j + m_i < k \leq m_0 + \sum_{j=1}^{i} 2m_j, \tag{3.7.10}$$

we have

$$B_n^k = B_n^k(L_k)$$

for a certain $L_k \in \mathbb{R}_+^*$.

Indeed, let k satisfy (3.7.10). Then, because of the asymptotic behaviour of the functions $L \longmapsto \sigma_{(j)}^D(A_{1+L/2})$ and $L \longmapsto \sigma_{(j)}^N(A_{1+L/2})$ as $L \longrightarrow \infty$, there exists $C > 0$ such that for all $L > C$, we have $B_n^k(L) = \sigma_{(i)}^D(A_{1+L/2})$. But we can compute that

$$\frac{\partial}{\partial L}\sigma_{(i)}^D(A_{1+L/2}) = -\frac{4(L+2)(2i+n-2)^2(1+L/2)^{2i+n}}{\left(4(1+L/2)^{2i+n} - L^2 - 4L - 4\right)^2} < 0,$$

which means that the function $L \longmapsto \sigma_{(i)}^D(A_{1+L/2})$ is strictly decreasing. Hence, for $L > L' > C$, we have $B_n^k(L) < B_n^k(L')$. Therefore, for such a k, we have

$$B_n^k = B_n^k(L_k)$$

with L_k finite, that is L_k is a finite critical length associated to k.

Then, defining $k_1 := 1$ and for each $i \geq 2$, defining $k_i := m_0 + \sum_{j=1}^{i} 2m_j$, we get a sequence $(k_i)_{i=1}^{\infty}$ such that

$$B_n^{k_i} = B_n^{k_i}(L_i)$$

for a certain $L_i \in \mathbb{R}_+^*$ finite.

Now we want to prove that the sequence of finite critical lengths $(L_i)_{i=1}^{\infty}$ converges to 0.
Let $i \geq 1$. We know that L_i has to be a solution of the equation $\sigma_{(j_1)}^D(A_{1+L/2}) = \sigma_{(j_2)}^N(A_{1+L/2})$ for a certain ordered pair $(j_1, j_2) \in \mathbb{N}^2$. As said before, for $L > C$, we have $B_n^{k_i}(L) = \sigma_{(i)}^D(A_{1+L/2})$, hence $L_i \leq L_i^*$, where L_i^* is the unique solution of the equation

$$\sigma_{(i)}^D(A_{1+L/2}) = \sigma_{(i+1)}^N(A_{1+L/2}).$$

Therefore, in order to prove that $L_i \underset{i\to\infty}{\longrightarrow} 0$, we prove that $L_i^* \underset{i\to\infty}{\longrightarrow} 0$.
Using Propositions 3.13 and 3.15, making some calculations and substituting $(1 + L/2)$ by R, we can see that solving

$$\sigma_{(i)}^D(A_{1+L/2}) = \sigma_{(i+1)}^N(A_{1+L/2})$$

is equivalent to finding the unique value $R_i \in (1, \infty)$ which solves the equation

$$(i+1)R^{2i+n-2}\left(R^{2i+n} - R^2(2i+n-1) - \frac{(i+n-1)(2i+n-1)}{i+1}\right) + (i+n-1) = 0.$$

We call

$$\underbrace{(i+1)R^{2i+n-2}\left(\underbrace{R^{2i+n}-R^2(2i+n-1)-\frac{(i+n-1)(2i+n-1)}{i+1}}_{=:\psi_i(R)}\right)+(i+n-1)}_{=:\Psi_i(R)}.$$

Because $(i+1)R^{2i+n-2} > 0$ and $(i+n-1) > 0$, then for R_i to be the solution of the equation $\Psi_i(R) = 0$, it is necessary that $\psi_i(R_i) < 0$.

Then we have

$$\begin{aligned} R^{2i+n} &< R^2(2i+n-1) + \frac{(i+n-1)(2i+n-1)}{i+1} \\ &< R^2\left((2i+n-1) + \frac{(i+n-1)(2i+n-1)}{i+1}\right) \\ &= R^2\left(\frac{(2i+n-1)(2i+n)}{i+1}\right) \\ &< R^2(2i+n)^2. \end{aligned}$$

Therefore we have

$$\ln(R) < \frac{2\ln(2i+n)}{2i+n-2}$$

and thus

$$R < e^{\frac{2\ln(2i+n)}{2i+n-2}}$$

Remember that we substituted $(1+L/2)$ by R, and then the unique solution of the equation

$$\sigma^D_{(i)}(A_{1+L/2}) = \sigma^N_{(i+1)}(A_{1+L/2})$$

is a value L_i^* wich satisfies

$$0 < L_i^* < 2\left(e^{\frac{2\ln(2i+n)}{2i+n-2}} - 1\right).$$

Therefore, since $\frac{2\ln(2i+n)}{2i+n-2} \underset{i\to\infty}{\longrightarrow} 0$, we have

$$L_i < L_i^* \underset{i\to\infty}{\longrightarrow} 0.$$

In particular, for each $\delta > 0$ there exists $k_0 \in \mathbb{N}$ such that for each $k > k_0$ which has a finite critical length L_k, then $L_k < \delta$.

♡♠◇♣

Remark 3.21. The condition given by (3.7.10) is sufficient, but is not a necessary one. Indeed, $k = 1$ does not meet Condition (3.7.10) but we have $B_n^1 = B_n^1(L_1)$, where L_1 is given by Corollary 3.4. This consideration naturally leads to the following open question:

Question 3.22. Given $n \geq 3$, are there finitely or infinitely many $k \in \mathbb{N}$ such that k has a critical length at infinity?

3.A Proof of Equality (3.6.9)

We know that there exists a unique $L_2 > 0$ such that $\sigma^D_{(0)}(A_{1+L_2/2}) = \sigma^N_{(2)}(A_{1+L_2/2})$. We want to choose, depending on the value of n, if $\sigma^D_{(0)}(A_{1+L_2/2})$ is bigger or smaller than $n-1$. For this purpose, we call L_D the unique positive value such that $\sigma^D_{(0)}(A_{1+L_D/2}) = n-1$, and we call L_N the unique positive value such that $\sigma^N_{(2)}(A_{1+L_N/2}) = n-1$.

Then we have the following fact: if $L_N < L_D$, we have $\sigma_{(0)}(A_{1+L_2/2}) > n-1$. On the contrary, if $L_D < L_N$, we have $\sigma_{(0)}(A_{1+L_2/2}) < n-1$.

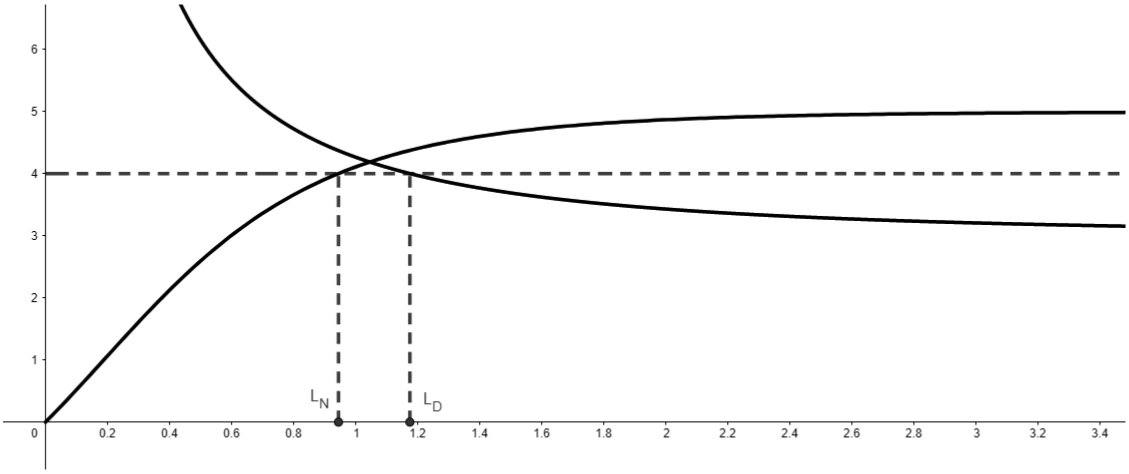

Figure 3.7: Representation of the case $n = 5$. Since $L_N < L_D$, then $\sigma_{(0)}(A_{1+L_2/2}) > n-1$.

Hence, we solve the equation $\sigma^D_{(0)}(A_{1+L_D/2}) = n-1$, i.e we find the unique $L_D > 0$ such that

$$\frac{(n-2)(1+L_D/2)^{n-2}}{(1+L_D/2)^{n-2}-1} = n-1.$$

We find

$$L_D = 2(n-1)^{\frac{1}{n-2}} - 2.$$

Similarly, solving the equation

$$\frac{2n((1+L_N/2)^{n+2}-1)}{2(1+L_N/2)^{n+2}+n} = n-1$$

leads to

$$L_N = 2\left(\frac{n(n+1)}{2}\right)^{\frac{1}{n+2}} - 2.$$

We have to find for which values of n we have $L_D < L_N$ and vice versa. This leads to the inequality

$$\left(\frac{n(n+1)}{2}\right)^{\frac{1}{n+2}} > (n-1)^{\frac{1}{n-2}},$$

which is equivalent to

$$\left(\frac{n(n+1)}{2}\right)^{\frac{n-2}{n+2}} > n-1.$$

We suppose $n \geq 9$.

$$\left(\frac{n(n+1)}{2}\right)^{\frac{n-2}{n+2}} > \left(\frac{n^2}{2}\right)^{\frac{n-2}{n+2}} = \frac{1}{2^{\frac{n-2}{n+2}}} n^{\frac{2n-4}{n+2}} > \frac{1}{2} n^{\frac{2n-4}{n+2}} \geq \frac{1}{2} n^{\frac{14}{11}}.$$

We analyze the function $f : [9, \infty) \longrightarrow \mathbb{R}, x \longmapsto \frac{1}{2} x^{\frac{14}{11}}$. We have $f(9) > 8$, and $f'(x) = \frac{14}{22} x^{\frac{3}{11}}$. We can compute that $f'(9) > 1$ and since $f''(x) = \frac{42}{242} x^{\frac{-9}{11}} > 0$ for all $x \in [9, \infty)$, we can conclude $f'(x) > 1$ for all $x \in [9, \infty)$. Hence, $f(x) > x - 1$ for all $x \in [9, \infty)$.

Therefore, for all integers $n \geq 9$, we have

$$\left(\frac{n(n+1)}{2}\right)^{\frac{n-2}{n+2}} > n-1$$

and then $L_D < L_N$. We can compute the cases $n = 3, \ldots, 8$ and we can conclude that

$$\begin{cases} L_N < L_D & \text{if } 3 \leq n \leq 6 \\ L_D < L_N & \text{if } 7 \leq n. \end{cases}$$

Therefore, we have

$$\max\left\{\sigma_0^D(A_{1+L_2/2}), n-1\right\} = \begin{cases} \sigma_0^D(A_{1+L_2/2}) & \text{if } 3 \leq n \leq 6 \\ n-1 & \text{if } 7 \leq n. \end{cases}$$

3.B A variation: fix the distance between the boundary components

A variant of the problem studied in this paper is the one proposed by Colbois and Verma in [10, Question 8]. Let us recall what it is.

We consider the Euclidean space $\mathbb{R}^{n+1}$, with $n \geq 3$, and we fix $\delta > 0$. Then we define

$$S_1 := \left\{ x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} x_i = 1, x_{n+1} = 0 \right\} = \mathbb{S}^{n-1} \times \{0\},$$

and S_2 which is a translation of $\delta \cdot e_{n+1}$ of S_1, that is

$$S_2 = \mathbb{S}^{n-1} \times \{\delta\}.$$

Hence, S_1 and S_2 both are isometric to $\mathbb{S}^{n-1}$. Then we consider the set $\mathcal{H}_\delta$ of all hypersurfaces of revolution whose boundary is $S_1 \sqcup S_2$. Therefore two elements of $\mathcal{H}_\delta$ have the same boundary, the extrinsic distance between the two boundary components being δ. However, they may have different meridian lengths $L \geq \delta$, as shown in Figure 3.8.

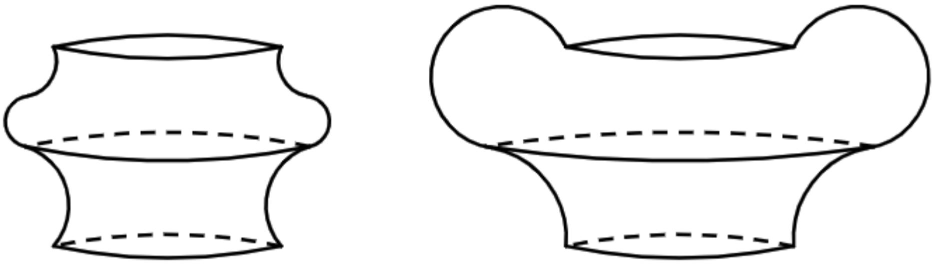

Figure 3.8: The extrinsic distance between the boundary components is fixed and equal to δ. However, the meridian length is not fixed.

The questions are the following:

1. For $k \in \mathbb{N}^*$ fixed, is there a hypersurface of revolution in $\mathcal{H}_\delta$ which maximizes the kth Steklov eigenvalue?
2. If such a manifold exists, is it unique? Can we determine it?

After reading Colbois and Verma's article [10], one can be tempted to think that in order to maximize the kth Steklov eigenvalue, we have to consider a family $(H_i)_{i\in\mathbb{N}} \subset \mathcal{H}_\delta$ with meridian length $L_i \to \infty$ as $i \to \infty$, as is the case of one boundary component hypersurfaces of revolution. However, we can show as a consequence of our previous constructions that this is not the case in a general setting, as stated by the following proposition.

Proposition 3.23. *Let $n \geq 3$. Then there exist $\delta > 0$, a hypersurface $(M_{L_1}, g_{L_1}) \in \mathcal{H}_\delta$ (with meridian length L_1) and $L_\delta > L_1$ such that for all $(M, g) \in \mathcal{H}_\delta$ with meridian length $L > L_\delta$, we have*

$$\sigma_1(M, g) < \sigma_1(M_{L_1}, g_{L_1}).$$

Proof. Let $0 < \varepsilon < \frac{B_n-(n-2)}{2}$. By Corollary 3.4, there exists a hypersurface of revolution $(M_{L_1} = [0, L_1] \times \mathbb{S}^{n-1}, g_{L_1})$ such that

$$\sigma_1(M_{L_1}, g_{L_1}) > B_n - \varepsilon.$$

Its boundary is then given by

$$\mathbb{S}^{n-1} \times \{0\} \sqcup \mathbb{S}^{n-1} \times \{\delta\}$$

for a certain $\delta > 0$, therefore we have $(M_{L_1}, g_{L_1}) \in \mathcal{H}_\delta$.

Let $L_\delta \in (L_1, \infty)$ be the unique meridian length such that $B_n(L_\delta) = B_n - \varepsilon$. We remark that L_δ exists because $B_n - \varepsilon > n - 2$, and it is unique since we chose it only in the interval (L_1, ∞), see Figure 3.9

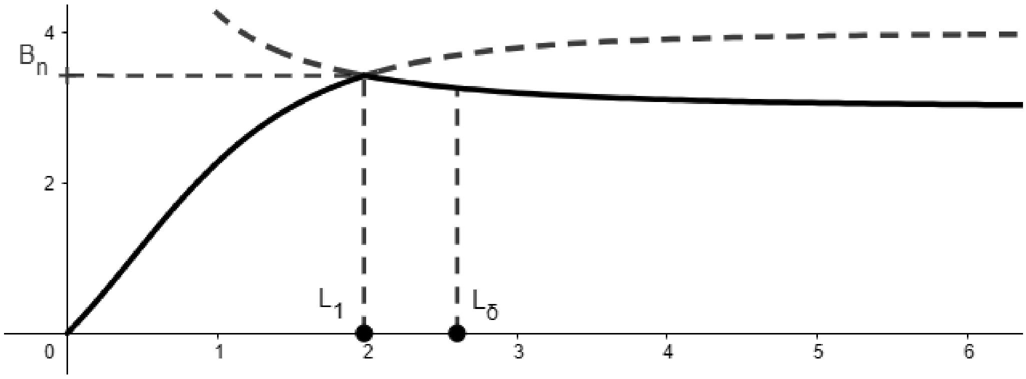

Figure 3.9: Illustration of the case $n = 5$. We chose L_δ the unique value of L such that $L_\delta > L_1$ and such that $B_n(L_\delta) = B_n - \varepsilon$.

Let $L > L_\delta$ and $(M = [0, L] \times \mathbb{S}^{n-1}, g)$ be a hypersurface of revolution. Then

$$\sigma_1(M_{L_1}, g_{L_1}) > B_n - \varepsilon = B_n(L_\delta) > B_n(L) > \sigma_1(M, g).$$

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This proposition shows that if the extrinsic distance δ between the boundary components is smaller than $B_n - (n-2)$, then a hypersurface of revolution in $\mathcal{H}_\delta$ with a big meridian length cannot maximize the first non zero Steklov eigenvalue.

We remark that a similar construction can be done for every k that does not have a critical length at infinity.

However, the same construction cannot be done if k does have a critical length at infinity. For all those k (for instance $k = 2$), it is worth trying to show that for δ fixed, a maximizing family $(H_i)_{i\in\mathbb{N}} \subset \mathcal{H}_\delta$ has an associated meridian length family (L_i) satisfying $L_i \underset{i\to\infty}{\longrightarrow} \infty$.

Chapter 4

Critical lengths of Steklov eigenvalues of hypersurfaces of revolution in Euclidean space: numerical experiments

Abstract

We study the intriguing phenomenon of critical length which was discovered while investigating the Steklov problem on hypersurfaces of revolution in the Euclidean space. We propose an algorithm that can be used in order to perform numerical experiments on this topic. These experiments support a conjecture which answer an open question about the number of eigenvalues that have a critical length at infinity.

Keywords: Spectral geometry, Steklov problem, hypersurfaces of revolution, numerical experiments.

4.1 Introduction

GIVEN (M, g) a smooth compact connected Riemannian manifold of dimension $n \geq 2$ with smooth boundary Σ, the Steklov problem on (M, Σ) consists in finding the real numbers σ and the functions $f : M \longrightarrow \mathbb{R}$ such that

$$\begin{cases} \Delta f = 0 \text{ in } M \\ \partial_\nu f = \sigma f \text{ on } \Sigma, \end{cases}$$

where ν denotes the outward normal on Σ. Such a σ is called a Steklov eigenvalue of (M, g). It is well known that the Steklov spectrum forms a discrete sequence $0 = \sigma_0(M, g) < \sigma_1(M, g) \leq \sigma_2(M, g) \leq \ldots \nearrow \infty$. Each eigenvalue is repeated with its multiplicity, which is finite.

In this chapter, we are interested in hypersurfaces of revolution in Euclidean space. Let us start by recalling what they are.

Definition 4.1. A n-dimensional compact hypersurface of revolution (M, g) in Euclidean space with two boundary components is the warped product $M = [0, L] \times \mathbb{S}^{n-1}$ endowed with the Riemannian metric

$$g(r, p) = dr^2 + h^2(r) g_0(p),$$

where $(r, p) \in [0, L] \times \mathbb{S}^{n-1}$, g_0 is the canonical metric of the $(n-1)$-sphere of radius one and $h : [0, L] \longrightarrow \mathbb{R}_+^*$ is a smooth function which satisfies: $|h'(r)| \leq 1$ for all $r \in [0, L]$. This assumption on h comes from the fact that we are in the Euclidean space, see [8, Section 3] for more details.

As it was done in [8] and [10], in this chapter we will only consider hypersurfaces of revolution whose boundary components are isometric to a unit $(n-1)$-sphere. Therefore, all chapter long we will also assume that $h(0) = h(L) = 1$.

If $M = [0, L] \times \mathbb{S}^{n-1}$ and $h : [0, L] \longrightarrow \mathbb{R}_+^*$ satisfies the properties above, we say in this chapter that M is a *hypersurface of revolution*, we say that $g(r, p) = dr^2 + h^2(r)g_0(p)$ is a *metric of revolution on M induced by h* and we call the number L the *meridian length* of M.

We refer to Chapter 3 for an overview of what is already known concerning upper and lower bounds for the Steklov spectrum of hypersurfaces of revolution. Here, because we want to investigate Question 3.22, we will focus on the upper bounds. As such, the meridian length L is precisely the parameter which interests us in this chapter. Indeed, when trying to maximize the eigenvalues of a hypersurface of revolution, we understand that choosing the correct meridian length L is crucial, see Chapter 3. More precisely, for a given eigenvalue, let us say the kth, we have to choose the meridian length L corresponding to k in order to maximize this kth eigenvalue. We understand that for some k, the best meridian length for the kth eigenvalues is a finite number $L_k \in \mathbb{R}_+^*$. However, for some other k, the best meridian length is at infinity, meaning that the larger L is, the better. This motivates the following definitions.

Definition 4.2. Let $n \geq 3$ and $k \geq 1$ be integers and $L \in \mathbb{R}_+^*$.

1. We write $B_n^k(L) < \infty$ the sharp upper bound for the kth eigenvalue of a hypersurface of revolution of dimension n and meridian length L.
2. We write $B_n^k := \sup_{L \in \mathbb{R}_+^*}\{B_n^k(L)\}$.
3. We say that $L_k \in \mathbb{R}_+^*$ is a finite critical length associated with k if we have $B_n^k = B_n^k(L_k)$.
4. We say that k has a critical length at infinity if it satisfies $B_n^k = \lim_{L \to \infty} B_n^k(L)$.

As explained in Chapter 3, given a fixed dimension $n \geq 3$, the set of eigenvalues which have a finite critical length is non empty. Indeed, Corollary 3.4 guarantees that $k = 1$ has a finite critical length. Moreover, the set of eigenvalues which have a critical length at infinity is also non empty. Indeed, in Section 3.6.1 it was shown that $k = 2$ has an infinite critical length.
We state here Theorem 3.9:

Theorem 4.3. *Let $n \geq 3$. Then there exist infinitely many $k \in \mathbb{N}$ which have a finite critical length associated with them. Moreover, if we call $(k_i)_{i=1}^{\infty} \subset \mathbb{N}$ the increasing sequence of such k and if we call $(L_i)_{i=1}^{\infty}$ the associated sequence of finite critical lengths, then we have*

$$\lim_{i \to \infty} L_i = 0.$$

This result immediately raises the following open question:

Given $n \geq 3$, are there finitely or infinitely many $k \in \mathbb{N}$ such that k has a critical length at infinity?

It seems hard to answer this question. Here is a tool that we will use to explore the phenomenon of finite / infinite critical length.

Theorem 4.4. *Let $(M = [0, L] \times \mathbb{S}^{n-1}, g)$ be a hypersurface of revolution in Euclidean space with two boundary components isometric to two copies of $\mathbb{S}^{n-1}$ of dimension $n \geq 3$ and let $k \geq 1$. Then there exists an algorithm, called* extension process, *that computes a finite number $B_n^k(L)$, depending only on n, k and L, such that*

$$\sigma_k(M, g) < B_n^k(L).$$

Moreover, the bound computed is sharp: for all $\varepsilon > 0$, there exists a metric of revolution g_ε on M such that $\sigma_k(M, g_\varepsilon) > B_n^k(L) - \varepsilon$.

This result has two corollaries.

Corollary 4.5. *Let $n \geq 3$ and $k \geq 1$. Then there exists a bound $B_n^k < \infty$ such that for all hypersurfaces of revolution (M, g) in Euclidean space with two boundary components each isometric to $\mathbb{S}^{n-1}$, we have*

$$\sigma_k(M, g) < B_n^k,$$

given by

$$B_n^k := \sup\{B_n^k(L) : L \in \mathbb{R}_+^*\}.$$

Moreover, this bound is sharp: for each $\varepsilon > 0$, there exists a hypersurface of revolution $(M_\varepsilon, g_\varepsilon)$ such that $\sigma_k(M_\varepsilon, g_\varepsilon) > B_n^k - \varepsilon$.

Corollary 4.6. *Let $(M_i, g_i)_{i=1}^\infty$ be a family of hypersurfaces of revolution in Euclidean space with two boundary components each isometric to $\mathbb{S}^{n-1}$, where $M_i = [0, L_i] \times \mathbb{S}^{n-1}$, $n \geq 3$. Let $k \in \mathbb{N}$. Let us suppose that $L_i \underset{i\to\infty}{\longrightarrow} 0$. Then we have*

$$\sigma_k(M_i, g_i) \underset{i\to\infty}{\longrightarrow} 0.$$

Remark 4.7. Corollary 4.6 is already contained in Proposition 3.3 of [8].

As already explained, in this chapter we do not answer the open question but we perform some numerical experiments in order to see clearer into the phenomenon of finite / infinite critical lengths. As we will see, these experiments support the following conjecture:

Conjecture 4.8. Let $n \geq 3$ be an integer. Then there exists a constant $K = K(n) \in \mathbb{N}$ such that for every $k \geq K$, the kth eigenvalue has an associated finite critical length.

Plan of the chapter. In Section 4.2, we recall the context and fix some notation we will use throughout the chapter. In Section 4.3, we describe the extension process and prove Corollary 4.5 and Corollary 4.6. In Section 4.4, we plot the sharp upper bound as a function of L. In Section 4.5, we add on these plots the mixed Steklov-Dirichlet and Steklov-Neumann eigenvalues. Finally in Section 4.6, we give code for a program which checks if the eigenvalues have a finite critical length or a critical length at infinity.

Acknowledgment. I would like to warmly thank my thesis supervisor Bruno Colbois for letting me work on this topic. Moreover, I would like to thank Maxime Welcklen for his help on the use of Python when coding the functions used in the extension process. I would also like to thank Prof. Pascal Felber who greatly improved the computing time of the codes, allowing me to search way further in a decent time.

4.2 Hypersurfaces of revolution and mixed problems

As explained in Chapter 3, maximizing the Steklov eigenvalues of hypersurfaces of revolution is deeply linked to the comprehension of the mixed Steklov-Dirichlet and Steklov-Neumann problem on annular domains. We recall what these mixed problems are in this section, as well as recalling what these links are.

4.2.1 Characterization of the Steklov eigenvalues and eigenfunctions

For (M, g) a Riemannian manifold with smooth boundary Σ, we can characterize its kth Steklov eigenvalue by:

$$\sigma_k(M,g) = \min\left\{R_g(f) : f \in H^1(M),\ f \perp_\Sigma f_0, f_1, \dots, f_{k-1}\right\}, \tag{4.2.1}$$

where

$$R_g(f) = \frac{\int_M |\nabla f|^2 dV_g}{\int_\Sigma |f|^2 dV_\Sigma}$$

is the Rayleigh quotient and

$$f \perp_\Sigma f_i \iff \int_\Sigma f f_i dV_\Sigma = 0.$$

In the special setting of this chapter, where (M, g) is a hypersurface of revolution, the corresponding eigenfunctions have a special expression. Here is the context:
We denote by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \nearrow \infty$ the spectrum of the Laplacian on $(\mathbb{S}^{n-1}, g_0)$ and we consider $(S_j)_{j=0}^{\infty}$ an orthonormal basis of eigenfunctions associated.

Proposition 4.9. *Let (M, g) be a hypersurface of revolution as above. Then each eigenfunction on M can be written as $f_l(r, p) = u_l(r)S_j(p)$, where u_j is a smooth function on $[0, L]$.*

This expression for the eigenfunctions is true in the more general case of warped product manifolds (and therefore for hypersurfaces of revolution) and it is often used, see for example [14, Remark 1.3], [15, Lemma 3], [33, Proposition 3.16] or [41, Proposition 9].

4.2.2 Mixed problems on annular domains

Let $\mathbb{B}_1$ and $\mathbb{B}_R$ be the balls in $\mathbb{R}^n$, with $R > 1$ and $n \geq 3$, centered at the origin. The annulus A_R is defined as follows: $A_R = \mathbb{B}_R \backslash \overline{\mathbb{B}}_1$. We say that this annulus is of inner radius 1 and outer radius R. This particular kind of domains shall be useful in this chapter.

For such domains, it is possible to compute explicitly $\sigma^D_{(k)}(A_R)$, which is the (k)th eigenvalue of the Steklov-Dirichlet problem on A_R, counted without multiplicity.

We state here Proposition 4 of [10]:

Proposition 4.10. *For A_R as above, consider the Steklov-Dirichlet problem*

$$\begin{cases} \Delta f = 0 & \text{in } A_R \\ \partial_\nu f = \sigma f & \text{on } \partial\mathbb{B}_1 \\ f = 0 & \text{on } \partial\mathbb{B}_R. \end{cases}$$

Then, for $k \geq 0$, the (k)th eigenvalue (counted without multiplicity) of this problem is

$$\sigma^D_{(k)}(A_R) = \frac{(k+n-2)R^{2k+n-2}+k}{R^{2k+n-2}-1}.$$

It is also possible to get the expression of the eigenfunctions of the Steklov-Dirichlet problem on an annular domain.

Lemma 4.11. *Each eigenfunction φ_l of the Steklov-Dirichlet problem on the annulus A_R can be expressed as $\varphi_l(r,p) = \alpha_l(r)S_l(p)$, where S_l is an eigenfunction for the l^{th} harmonic of the sphere $\mathbb{S}^{n-1}$.*

Similarly we can compute explicitly $\sigma^N_{(k)}(A_R)$, which is the (k)th eigenvalue of the Steklov-Neumann problem on A_R, counted without multiplicity.

We state now Proposition 5 of [10]:

Proposition 4.12. *For A_R as above, consider the Steklov-Neumann problem*

$$\begin{cases} \Delta f = 0 & \text{in } A_R \\ \partial_\nu f = \sigma f & \text{on } \partial\mathbb{B}_1 \\ \partial_\nu f = 0 & \text{on } \partial\mathbb{B}_R. \end{cases}$$

Then, for $k \geq 0$, the (k)th eigenvalue (counted without multiplicity) of this problem is

$$\sigma^N_{(k)}(A_R) = k\frac{(k+n-2)(R^{2k+n-2}-1)}{kR^{2k+n-2}+k+n-2}.$$

In the same manner as before, we have the following:

Lemma 4.13. *Each eigenfunction ϕ_l of the Steklov-Neumann problem on the annulus A_R can be expressed as $\phi_l(r,p) = \beta_l(r)S_l(p)$, where S_l is an eigenfunction for the l^{th} harmonic of the sphere $\mathbb{S}^{n-1}$.*

4.2.3 Multiplicity of the eigenvalues

Because we will have to deal with several problems and the multiplicity of the eigenvalues, we start by giving some notation, that are summarized in the table below:

Problem	kth eigenvalue	kth eigenfunction
Laplace	λ_k	S_k
Steklov	σ_k	$f_k = u_k S_j$
Steklov-Dirichlet	σ_k^D	$\varphi_k = \alpha_k S_k$
Steklov-Neumann	σ_k^N	$\phi_k = \beta_k S_k$

We also write $\lambda_{(k)}, \sigma_{(k)}, \sigma_{(k)}^D, \sigma_{(k)}^N$ for the (k)th eigenvalue counted without multiplicity.

In the case of the classical Laplacian problem $\Delta S = \lambda S$ on $(\mathbb{S}^{n-1}, g_0)$, it is known that the set of eigenvalues is $\{\lambda_{(k)} = k(n+k-2) : k \geq 0\}$, see [3, Page 160-162]. Besides, the multiplicity m_0 of $\lambda_{(0)} = 0$ is 1 and the multiplicity of $\lambda_{(k)}$ is

$$m_k := \frac{(n+k-3)(n+k-4)\ldots n(n-1)}{k!}(n+2k-2). \tag{4.2.2}$$

Thus, given $k \geq 0$, there are m_k independent functions $S_k^1, \ldots, S_k^{m_k}$ which satisfy

$$\Delta S_k^i = \lambda_{(k)} S_k^i, \text{ for all } i = 1, \ldots, m_k.$$

Moreover, for both the Steklov-Dirichlet and Steklov-Neumann problems on annular domains, the multiplicity of the (k)th eigenvalue is exactly m_k, as stated by [10, Proposition 3].
Hence, using the notation above, given $k \geq 0$,

1. There are exactly m_k linearly independent Steklov-Dirichlet eigenfunctions associated with $\sigma_{(k)}^D(A_{1+L/2})$, that can be written $\varphi_k^i(r,p) = \alpha_k(r) S_k^i(p),\ i = 1, \ldots, m_k$.
2. There are exactly m_k linearly independent Steklov-Neumann eigenfunctions associated with $\sigma_{(k)}^N(A_{1+L/2})$, that can be written $\phi_k^i(r,p) = \beta_k(r) S_k^i(p),\ i = 1, \ldots, m_k$.

4.3 Extension process

It is natural to wonder if we can get a general formula giving a sharp upper bound for the kth Steklov eigenvalue of (M, g), depending on L and n or even only on n. The expression for such a general bound is difficult to catch. However, one can introduce a process, that we call *extension process*, leading to a sharp bound $B_n^k(L)$ such that $\sigma_k(M, g) < B_n^k(L)$. This process is the following.

Let $(M = [0, L] \times \mathbb{S}^{n-1}, g)$ be a hypersurface of revolution and let $k \in \mathbb{N}^*$. We define

$$l_0 := \max\left\{ l \in \mathbb{N} : \sum_{i=0}^{l} m_i \leq k \right\}.$$

Let us now consider the finite set

$$E := \left\{ \sigma_{(0)}^D(A_{1+L/2}), \ldots, \sigma_{(l_0)}^D(A_{1+L/2}), \sigma_{(1)}^N(A_{1+L/2}), \ldots, \sigma_{(l_0+1)}^N(A_{1+L/2}) \right\}.$$

One can rearrange this set in ascending order, i.e we choose $\pi \in Sym(E)$ such that the finite sequence

$$\left(\pi(\sigma^D_{(0)}(A_{1+L/2})), \ldots, \pi(\sigma^D_{(l_0)}(A_{1+L/2})), \pi(\sigma^N_{(1)}(A_{1+L/2})), \ldots, \pi(\sigma^N_{(l_0+1)}(A_{1+L/2}))\right)$$

is increasing. For practical reasons, we rename this sequence

$$\left(\nu_0, \ldots, \nu_{l_0}, \nu_{l_0+1}, \ldots, \nu_{2l_0+1}\right).$$

Let us now consider the corresponding multiplicities. For $i \in \{0, \ldots, 2l_0 + 1\}$, we write μ_i for the multiplicity of ν_i. This gives us a finite sequence

$$(\mu_0, \ldots, \mu_{l_0}, \mu_{l_0+1}, \ldots, \mu_{2l_0+1}).$$

We define

$$l_1 := \min\left\{ l \in \{0, \ldots, 2l_0 + 1\} : \sum_{i=0}^{l} \mu_i \geq k \right\},$$

and we will prove that

$$B^k_n(L) := \nu_{l_1}$$

is a sharp upper bound for $\sigma_k(M, g)$.

Example 4.14. We take the case $n = 5$ and $L = 1$, and we choose $k = 127$. Using the formula

$$m_i = \frac{(n+i-3)(n+i-4)\ldots n(n-1)}{i!}(n+2i-2),$$

we have $m_0 = 1,\ m_1 = 5,\ m_2 = 14,\ m_3 = 30,\ m_4 = 55,\ m_5 = 91$. Therefore, $\sum_{l=0}^{4} m_l = 105 \leq k = 127$, but $\sum_{l=0}^{5} = 196 > k$, which means that we have $l_0 = 4$.

We consider the set

$$E = \{\sigma^D_{(0)}(A_{3/2}), \sigma^D_{(1)}(A_{3/2}), \sigma^D_{(2)}(A_{3/2}), \sigma^D_{(3)}(A_{3/2}), \sigma^D_{(4)}(A_{3/2}),\\ \sigma^N_{(1)}(A_{3/2}), \sigma^N_{(2)}(A_{3/2}), \sigma^N_{(3)}(A_{3/2}), \sigma^N_{(4)}(A_{3/2}), \sigma^N_{(5)}(A_{3/2})\}.$$

Using Propositions 4.10 and 4.12, we can determine the value of the 10 numbers belonging to the set E, and arrange them in an ascending order. We get

$$(\sigma^N_{(1)}(A_{3/2}) =: \nu_0 \approx 2.27, \sigma^N_{(2)}(A_{3/2}) =: \nu_1 \approx 4.11, \sigma^D_{(0)}(A_{3/2}) =: \nu_2 \approx 4.26,\\ \sigma^D_{(1)}(A_{3/2}) =: \nu_3 \approx 4.76, \sigma^D_{(2)}(A_{3/2}) =: \nu_4 \approx 5.44, \sigma^N_{(3)}(A_{3/2}) =: \nu_5 \approx 5.55,\\ \sigma^D_{(3)}(A_{3/2}) =: \nu_6 \approx 6.24, \sigma^N_{(4)}(A_{3/2}) =: \nu_7 \approx 6.78, \sigma^D_{(4)}(A_{3/2}) =: \nu_8 \approx 7.13,\\ \sigma^N_{(5)}(A_{3/2}) =: \nu_9 \approx 7.89).$$

Writing μ_i for the multiplicity of ν_i, we can associate to $(\nu_i)_{i=0}^{9}$ the sequence $(\mu_i)_{i=0}^{9}$. We get

$$(\mu_0 = 5, \mu_1 = 14, \mu_2 = 1, \mu_3 = 5, \mu_4 = 14, \mu_5 = 30, \mu_6 = 30, \mu_7 = 55, \mu_8 = 55, \mu_9 = 91).$$

We determine the value of l_1: since $\sum_{i=0}^{6}\mu_i = 99 < k$ and $\sum_{i=0}^{7}\mu_i = 154 \geq k$, we have $l_1 = 7$. We have a sharp upper bound for $\sigma_{127}([0,1]\times\mathbb{S}^4, g)$, given by

$$\sigma_{127}([0,1]\times\mathbb{S}^4, g) < B_5^{127}(1) = \nu_7 = \sigma_{(4)}^N(A_{3/2}) \approx 6.78.$$

One can then vary the value of L in order to find a sharp upper bound $B_5^{127}(L)$, that we can represent in an axis system, see Figure 4.1.

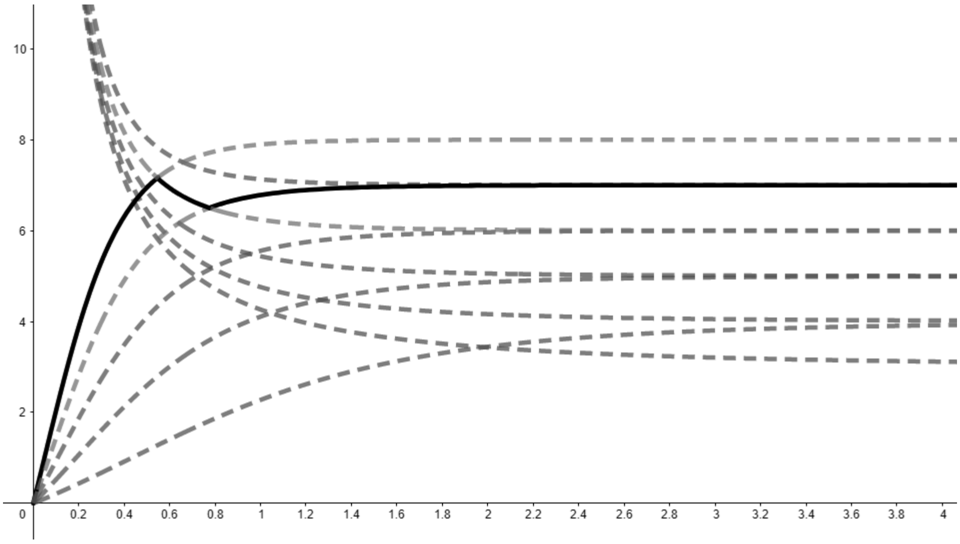


Figure 4.1: Representation of the case $n = 5$. The solid curve is the sharp upper bound $B_5^{127}(L)$ that we obtain thanks to the extension process.

Let us prove that $B_n^k(L)$ is indeed the bound that we were looking for.

Proof. We will use the same strategy that we already used in the proof of Theorem 3.3 at Section 3.4. Let us start by showing that $\sigma_k(M,g) < B_n^k(L)$.

Since ν_{l_1} has multiplicity μ_{l_1}, there exist μ_{l_1} independent eigenfunctions $\xi_1, \dots, \xi_{\mu_{l_1}}$ associated with ν_{l_1}.
Now we consider the function

$$\xi_{\mu_{l_1}} : [0, \frac{L}{2}] \times \mathbb{S}^{n-1} \longrightarrow \mathbb{R},$$

and look at two cases:

1. Let us suppose that $\xi_{\mu_{l_1}}$ is a Steklov-Dirichlet eigenfunction. Then $\xi_{\mu_{l_1}}(L/2) = 0$ and we can extend $\xi_{\mu_{l_1}}$ to a new function

$$\begin{aligned}\tilde{\xi}_{\mu_{l_1}} &: [0,L]\times\mathbb{S}^{n-1} \longrightarrow \mathbb{R}\\ (r,p) &\longmapsto \begin{cases} \xi_{\mu_{l_1}}(r,p) & \text{if } 0 \leq r \leq L/2 \\ -\xi_{\mu_{l_1}}(L-r,p) & \text{if } L/2 \leq r \leq L. \end{cases}\end{aligned}$$

   The function $\tilde{\xi}_{\mu_{l_1}}$ is continuous. Let us call $f_0 = u_0 S_0,\ f_1 = u_1 S_1, \dots,\ f_{k-1} = u_{k-1} S_{k-1}$ some Steklov eigenfunctions associated with the k first Steklov eigenvalues $\sigma_0(M,g) =$

0, $\sigma_1(M,g),\dots,$ $\sigma_{k-1}(M,g)$ respectively. Then, since $\xi_{\mu_{l_1}}$ is a Steklov-Dirichlet eigenfunction and

$$\sum_{i=0}^{l_1}\mu_i \geq k,$$

we have $\tilde{\xi}_{\mu_{l_1}} = \alpha_l S_l$, with $l \geq k-1$. Then

$$\int_\Sigma f_i \tilde{\xi}_{\mu_{l_1}} dV_\Sigma = 0 \ \text{ for all } \ i = 0,\dots,k-1.$$

2. Let us suppose that $\xi_{\mu_{l_1}}$ is a Steklov-Neumann eigenfunction. Then we can extend $\xi_{\mu_{l_1}}$ to a new function

$$\begin{aligned} \tilde{\xi}_{\mu_{l_1}} : [0,L]\times\mathbb{S}^{n-1} &\longrightarrow \mathbb{R} \\ (r,p) &\longmapsto \begin{cases} \xi_{\mu_{l_1}}(r,p) & \text{if } 0\leq r\leq L/2 \\ \xi_{\mu_{l_1}}(L-r,p) & \text{if } L/2\leq r\leq L. \end{cases} \end{aligned}$$

The function $\tilde{\xi}_{\mu_{l_1}}$ is continuous. Let us call $f_0 = u_0S_0$, $f_1 = u_1S_1,\dots,$ $f_{k-1} = u_{k-1}S_{k-1}$ some Steklov eigenfunctions associated with the k first Steklov eigenvalues $\sigma_0(M,g) = 0$, $\sigma_1(M,g),\dots,$ $\sigma_{k-1}(M,g)$ respectively. Then, since $\xi_{\mu_{l_1}}$ is a Steklov-Neumann eigenfunction and

$$\sum_{i=0}^{l_1}\mu_i \geq k,$$

we have $\tilde{\xi}_{\mu_{l_1}} = \beta_l S_l$, with $l \geq k$. Then

$$\int_\Sigma f_i \tilde{\xi}_{\mu_{l_1}} dV_\Sigma = 0 \ \text{ for all } \ i = 0,\dots,k-1.$$

Therefore, in both cases we have $\int_\Sigma f_i \tilde{\xi}_{\mu_{l_1}} dV_\Sigma = 0$ for all $i = 0,\dots,k-1$. Hence we can use $\tilde{\xi}_{\mu_{l_1}}$ as a test function for $\sigma_k(M,g)$, thanks to Equation (4.2.1). We get

$$\begin{aligned}
\sigma_k(M,g) &\leq R_g(\tilde{\xi}_{\mu_{l_1}}) \\
&< R_{\tilde{g}}(\tilde{\xi}_{\mu_{l_1}}) \quad \text{where } \tilde{g} = dr^2 + \tilde{h}^2 g_0 \text{ comes from Theorem 3.2} \\
&= \frac{\int_0^L \int_{\mathbb{S}^{n-1}} \left((\partial_r \tilde{\xi}_{\mu_{l_1}})^2 + \frac{1}{\tilde{h}(r)^2} |\tilde{\nabla} \tilde{\xi}_{\mu_{l_1}}|^2 \right) \tilde{h}(r)^{n-1} dV_{g_0} dr}{\int_\Sigma \tilde{\xi}^2_{\mu_{l_1}}(0,p) dV_\Sigma} \\
&= \frac{2 \times \int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r \tilde{\xi}_{\mu_{l_1}})^2 + \frac{1}{\tilde{h}(r)^2} |\tilde{\nabla} \tilde{\xi}_{\mu_{l_1}}|^2 \right) \tilde{h}(r)^{n-1} dV_{g_0} dr}{2 \times \int_{\mathbb{S}^{n-1}} \tilde{\xi}^2_{\mu_{l_1}}(0,p) dV_{g_0}} \qquad (\tilde{g} \text{ is symmetric}) \\
&= \frac{\int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r \xi_{\mu_{l_1}})^2 + \frac{1}{\tilde{h}(r)^2} |\tilde{\nabla} \xi_{\mu_{l_1}}|^2 \right) \tilde{h}(r)^{n-1} dV_{g_0} dr}{\int_{\mathbb{S}^{n-1}} \xi^2_{\mu_{l_1}}(0,p) dV_{g_0}} \\
&< \frac{\int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r \xi_{\mu_{l_1}})^2 + \frac{1}{(1+r)^2} |\tilde{\nabla} \xi_{\mu_{l_1}}|^2 \right) (1+r)^{n-1} dV_{g_0} dr}{\int_{\mathbb{S}^{n-1}} \xi^2_{\mu_{l_1}}(0,p) dV_{g_0}} \\
&= \nu_{l_1} \\
&= B_n^k(L),
\end{aligned}$$

where the second strict inequality comes from the existence of a continuum of points $r \in [0, L/2]$ such that $\tilde{h}(r) < 1 + r$.

We still have to show that the upper bound $B_n^k(L)$ is sharp, i.e for all $\varepsilon > 0$, there exists a metric of revolution g_ε on M such that

$$\sigma_k(M, g_\varepsilon) > B_n^k(L) - \varepsilon.$$

Let $\varepsilon > 0$. Let $M = [0, L] \times \mathbb{S}^{n-1}$ and let $g_\varepsilon(r,p) = dr^2 + h_\varepsilon^2(r) g_0(p)$ be a metric of revolution on M such that

1. For all $r \in [0, L]$, we have $h_\varepsilon(r) = h_\varepsilon(L - r)$ (i.e h_ε is symmetric).
2. For all $r \in [0, L/2 - \delta]$, we have $h_\varepsilon(r) = (1 + r)$, with δ small enough to guarantee that for all $r \in [0, L/2]$, we have $\max\{(1+r)^{n-3} - h_\varepsilon(r)^{n-3}, (1+r)^{n-1} - h_\varepsilon(r)^{n-1}\} < \frac{\varepsilon}{B_n^k(L)} =: \varepsilon^*$.

We write f_k a Steklov eigenfunction associated with $\sigma_k(M, g_\varepsilon)$. Because g_ε is symmetric, we can choose f_k symmetric or anti-symmetric. Writing

$$R_{A_{1+L/2}}(f_k) := \frac{\int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r f_k)^2 + \frac{1}{(1+r)^2} |\tilde{\nabla} f_k|^2 \right) (1+r)^{n-1} dV_{g_0} dr}{\int_{\mathbb{S}^{n-1}} (f_k)^2(0,p) dV_{g_0}},$$

it is an easy computation to check that

$$R_{A_{1+L/2}}(f_k) < \sigma_k(M, g_\varepsilon) + \varepsilon.$$

We once again split the proof into two cases:

1. Let us suppose that f_k is anti-symmetric, i.e we have $f_k(r,p) = u_k(r)S_l(p)$ with u_k anti-symmetric and $B_n^k(L) = \sigma_j^D(A_{1+L/2})$ for a certain $j \leq l$. Then we can check that
$$f_k(L/2,p) = 0 \text{ and } \int_{\{0\}\times\mathbb{S}^{n-1}} \varphi_i f_k dV_{g_0} = 0 \text{ for all } i = 0,\dots,l-1.$$
2. Let us suppose that f_k is symmetric, i.e we have $f_k(r,p) = u_k(r)S_l(p)$ with u_k symmetric and $B_n^k(L) = \sigma_j^N(A_{1+L/2})$ for a certain $j \leq l$. Then we can check that
$$\partial_r f_k(L/2,p) = 0 \text{ and } \int_{\{0\}\times\mathbb{S}^{n-1}} \phi_i f_k dV_{g_0} = 0 \text{ for all } i = 0,\dots,l-1.$$

In both cases, we can use $f_k|_{[0,L/2]\times\mathbb{S}^{n-1}}$ as a test function for $B_n^k(L)$, and we get
$$B_n^k(L) < R_{A_{1+L/2}}(f_k) < \sigma_k(M,g_\varepsilon) + \varepsilon.$$

♡♠◇♣

From this statement, let us prove Corollary 4.5.

Proof. Let $k \geq 1$. We want to show the existence of a bound $B_n^k < \infty$ such that for all hypersurfaces of revolution (M,g) of dimension $n \geq 3$, we have $\sigma_k(M,g) < B_n^k$. A way to do it is the analyse to function $L \longmapsto B_n^k(L)$, and defining B_n^k as follows:
$$B_n^k := \sup\{B_n^k(L) : L \in \mathbb{R}_+^*\}.$$

By construction, we have $\sigma_k(M,g) < B_n^k$. We only have to show that B_n^k is finite. Indeed, for all $L \in \mathbb{R}_+^*$, we have
$$B_n^k(L) < \sigma_{(k)}^N(L).$$

Therefore, we have
$$\begin{aligned} B_n^k &= \sup\{B_n^k(L) : L \in \mathbb{R}_+^*\} \\ &\leq \sup\{\sigma_{(k)}^N(A_{1+L/2}) : L \in \mathbb{R}_+^*\} \\ &= \lim_{L\to\infty} \sigma_{(k)}^N(A_{1+L/2}) \\ &= n+k-2. \end{aligned}$$

Now we can prove that B_n^k is a sharp upper bound, that is for all $\varepsilon > 0$, there exists a hypersurface of revolution $(M_\varepsilon, g_\varepsilon)$ such that $\sigma_k(M_\varepsilon,g_\varepsilon) > B_n^k - \varepsilon$.

Let $\varepsilon > 0$. There exists $L^* \in \mathbb{R}_+^*$ such that $B_n^k - B_n^k(L^*) < \frac{\varepsilon}{2}$. We define $M_\varepsilon := [0,L^*]\times\mathbb{S}^{n-1}$. Thanks to Theorem 4.4, there exists a metric of revolution g_ε on M_ε such that
$$\sigma_k(M_\varepsilon,g_\varepsilon) > B_n^k(L^*) - \frac{\varepsilon}{2} > B_n^k - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = B_n^k - \varepsilon.$$

♡♠◇♣

From Theorem 4.4, let us prove Corollary 4.6.

Proof. Let us notice that given $j \in \mathbb{N}$ fixed, we have

$$\lim_{L \to 0} \sigma^D_{(j)}(A_{1+L/2}) = \infty \quad \text{and} \quad \lim_{L \to 0} \sigma^N_{(j)}(A_{1+L/2}) = 0.$$

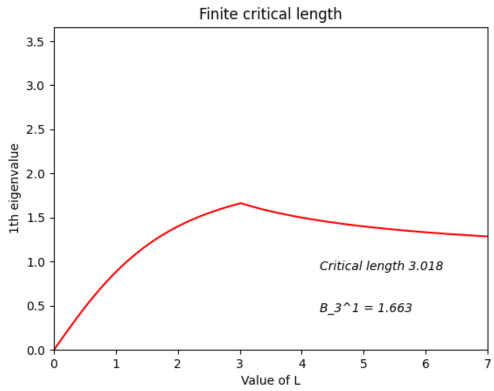
Let $k \in \mathbb{N}$ and let $\varepsilon > 0$. Let $L_0 > 0$ be small enough so that $\sigma^N_{(1)}(A_{1+L_0/2}) < \ldots < \sigma^N_{(k)}(A_{1+L_0/2}) < \varepsilon$. Let $0 < L < L_0$. Then the finite set $E = E(n, k, L)$ can be ordered and $\nu_0, \ldots, \nu_{l_0+1}$ are equal respectively to $\sigma^N_{(1)}(A_{1+L/2}), \ldots, \sigma^N_{(l_0+1)}(A_{1+L/2})$, where $l_0 + 1 < k$.
Hence $\nu_{l_1} \in \{\sigma^N_{(1)}(A_{1+L/2}), \ldots, \sigma^N_{(l_0+1)}(A_{1+L/2})\}$ and therefore, writing $M := [0, L] \times \mathbb{S}^{n-1}$,

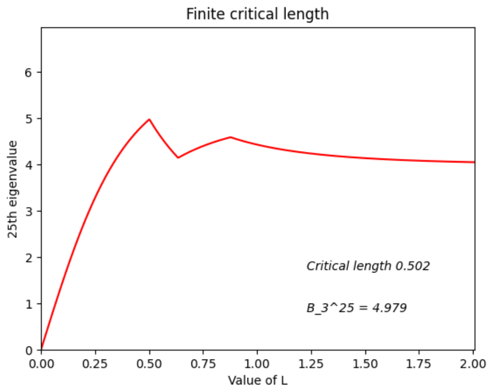
$$\sigma_k(M, g) < \nu_{l_1} \leq \max\{\sigma^N_{(1)}(A_{1+L/2}), \ldots, \sigma^N_{(l_0+1)}(A_{1+L/2})\} < \varepsilon.$$

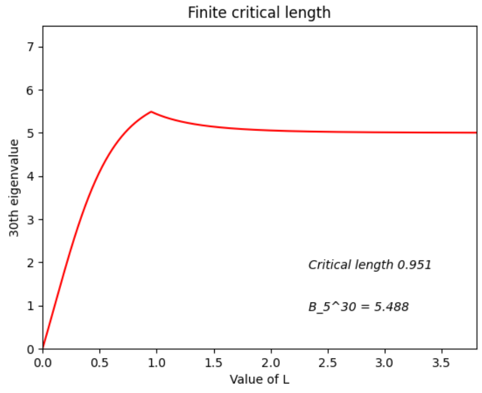
♡♠♢♣

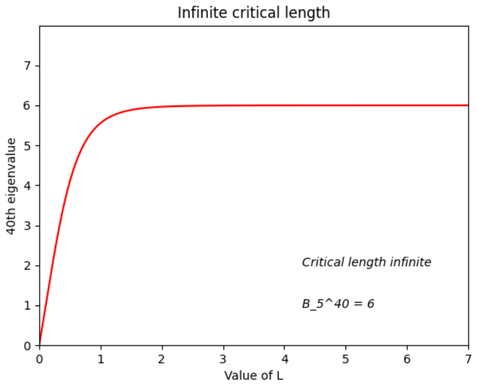
4.4 Plotting sharp upper bounds

We can implement the extension process in a computer and let the value of L vary in order to plot the sharp upper bound as a function of L, where we see n and k as parameters.
Here are some figures obtained. On the bottom-right part of each graphic, one can see written "`B_n^k`", meaning that we studied the nth dimension and the kth eigenvalue. Moreover, for each values of n and k, the graphic indicates if we found a finite or infinite critical length, and provides an estimation of the critical length as well as an estimation of the sharp upper bound B^k_n.









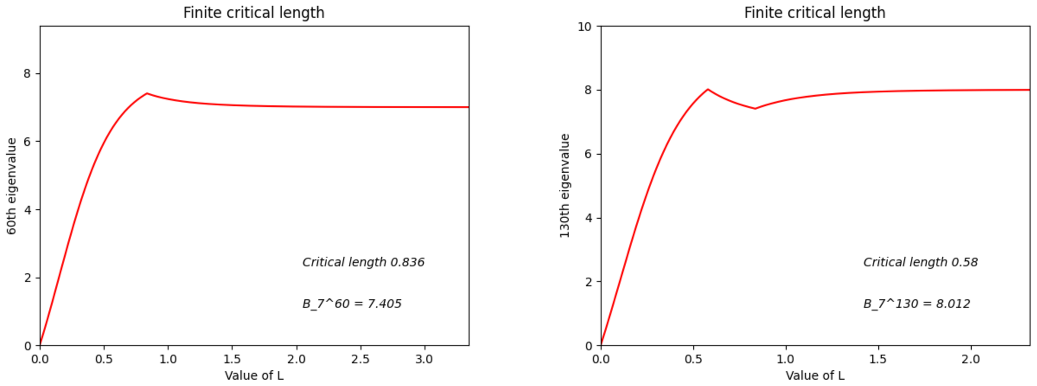
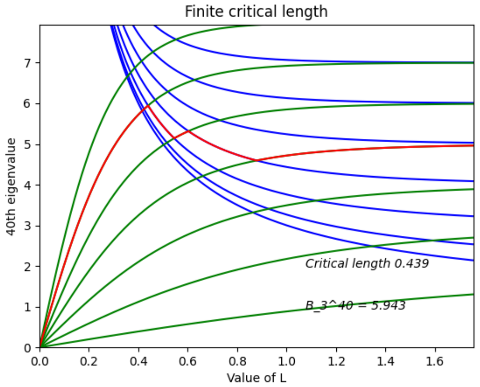
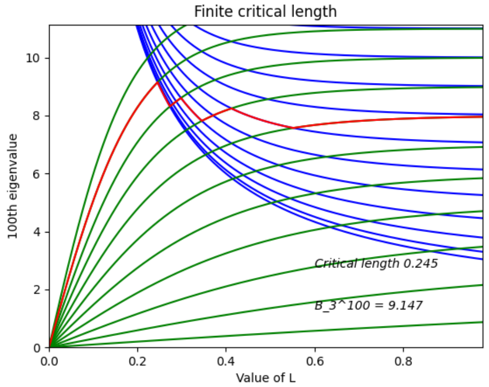


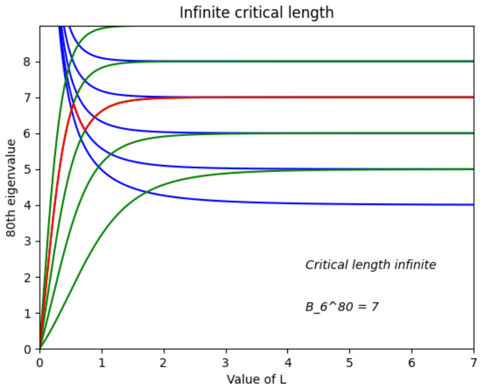
Figure 4.2: As one can see, the behaviour of the function $L \longmapsto B_n^k(L)$ is hard to predict, and its properties (smoothness, infinite critical length, finite critical length) depend on the values of n and k.

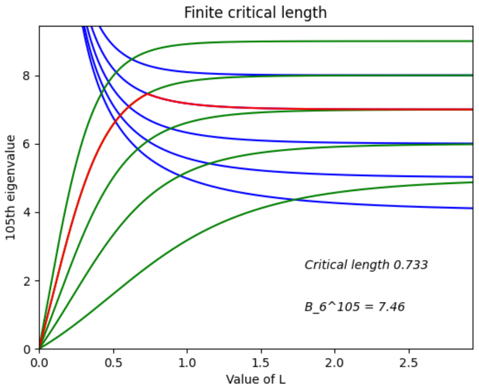
4.5 Adding the mixed eigenvalues

Since the function $L \longmapsto B_n^k(L)$ comes from the mixed Steklov-Dirichlet and Steklov-Neumann eigenvalues, we can add them in the graphics to understand the function better.
We got the following figures.









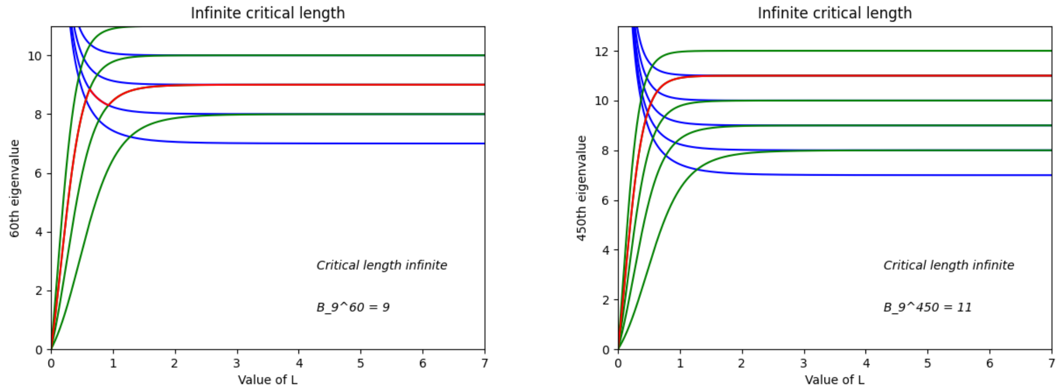


Figure 4.3: The blue curves are the Steklov-Dirichlet eigenvalues, the green ones are the Steklov-Neumann eigenvalues.

4.6 Question and conjecture

We also give code for a program that, given a dimension n and an integer k, checks if the eigenvalues with index smaller or equal to k have a finite critical length or have a critical length at infinity.

Such a program requires a lot of time of computation, at least if we want to check many eigenvalues. Here are some considerations that we can use in order to optimize the program that we can implement in a programming language.

Definition 4.15. Given a dimension $n \geq 3$, we say that $k \in \mathbb{N}$ is a *diagnosis eigenvalue* if $k = \sum_{j=0}^{i-1} 2m_j$ for a certain integer i.

This definition is motivated by the following lemma:

Lemma 4.16. *Let $n \geq 3$ and $i \in \mathbb{N}$. Let $k := \sum_{j=0}^{i-1} 2m_j$ be the ith diagnosis eigenvalue. Let us suppose that k has an associated finite critical. Then, for each $k' \in \{k, k+2m_i-1\}$, the eigenvalue k' has an associated finite critical length.*

Proof. Since $k = \sum_{j=0}^{i-1} 2m_j$, there exists $C > 0$ such that $B_n^k(L) = \sigma_{(i)}^N(A_{1+L/2})$ for all $L > C$. Moreover, since $L \longmapsto \sigma_{(i)}^N(A_{1+L/2})$ is strictly increasing, then the critical length L_k (which is finite by assumption) satisfies $L_k \leq C$.

Let us fix $L^* > C$. Then for each k' such that $k \leq k' < k + m_i$, we have $B_n^{k'}(L^*) = \sigma_{(i)}^N(A_{1+L^*/2})$. Moreover, we have $B_n^{k'}(L_k) \geq B_n^k(L_k)$. Thus, we have

$$\sigma_{(i)}^N(A_{1+L^*/2}) < B_n^k(L_k) \leq B_n^{k'}(L_k) \leq \sup_{L \in (0,C]} B_n^{k'}(L),$$

and k' has a finite critical length.

Then, for k' such that

$$k + m_i \leq k' < k + 2m_i,$$

we have $B_n^{k'}(L) = \sigma_{(i)}^D(A_{1+L/2})$ for all $L > C$. Moreover, since $L \longmapsto \sigma_{(i)}^D(A_{1+L/2})$ is decreasing, we know that k' has a finite critical length.

Therefore, for all $k' \in \{k, k + 2m_i - 1\}$, the eigenvalue k' has an associated finite critical length.

♡♠◇♣

Example 4.17. Let $n = 5$. The sequence of diagnosis eigenvalues is

$$(2, 12, 40, 100, 210, 392, 672, 1'080, 1'650, 2'420, 3'432, 4'732, 6'370, 8'400, 10'880, \ldots).$$

The lemma allows us to state that in dimension 5, *if* the 14th diagnosis eigenvalue $k = 8'400$ has an associated finite critical length, *then* for each $k' \in \{8'400, 10'879\}$, the k'th eigenvalue has an associated finite critical length.

Therefore, it is relevant to consider only these diagnosis eigenvalues to save time while checking what could be the answer to our open question.

Here are some values given by this program:

n	i	Result of the program investigations
3	150	From $k = 18$ to $k = 45'601$, only finite critical lengths found.
4	40	From $k = 408$ to $k = 47'641$, only finite critical lengths found.
5	25	From $k = 8'400$ to $k = 88'451$, only finite critical lengths found.
6	18	From $k = 21'112$ to $k = 119'965$, only finite critical lengths found.

We draw attention to the fact that the program *does not say* that in dimension 3, the eigenvalue number 17 has a critical length at infinity, it only says that it has not found any infinite critical length above 17.
Actually, in dimension $n = 3$, we have not found any infinite critical length associated with the eigenvalue k, if $k > 8$.

These results given by our program motivate Conjecture 4.8, which is:

Let $n \geq 3$ be an integer. Then there exists $K = K(n) \in \mathbb{N}$ such that for every $k \geq K$, the kth eigenvalue has an associated finite critical length.

If this conjecture were to be proven, it would then be interesting to study the function

$$n \longmapsto K_{min}(n),$$

where $K_{min}(n)$ is the smallest integer in the conjecture. From our numerical experiments, it seems that this function, *if it exists*, grows quickly with n.

4.A Python codes

The codes that are provided here can also be found on Github, with some other documents (such as results, time of computation, etc.), at the address: https://github.com/Tchatchu5/python

4.A.1 Functions

We code for some functions that are used in the extension process.

```
import functools

from math import factorial

# function multiplicity of the Laplace eigenvalues on a sphere
@functools.cache
def mu(n, k):
  return int((n + 2 * k - 2) * factorial(n + k - 3) /
         (factorial(k) * factorial(n - 2)))


# Steklov - Dirichlet eigenvalue on an annulus
def sigma_dirichlet(n, D, L):
  if L == 0:
    return float("inf")
  return ((n + D - 2) * (1 + L / 2)**(2 * D + n - 2) + D) / \
    ((1 + L / 2)**(2 * D + n - 2) - 1)


# Steklov - Neumann eigenvalue on an annulus
def sigma_neumann(n, N, L):
  return N * ((N + n - 2) * ((1 + L / 2)**(2 * N + n - 2) - 1)) / \
    (N * (1 + L / 2)**(2 * N + n - 2) + N + n - 2)


# function which return the integer l_0 in the extension process
@functools.cache
def l_0(n, k):
  list = []
  for i in range(0, k + 1):
    list.append((mu(n, i)))
  sum = 0
  j = 0
  for i in range(0, k + 1):
    if sum + list[i] > k:
      return j
    sum = sum + list[i]
    j = j + 1
  return None


# function which return the integer l_1 in the extension process
def l_1(m2_list: list, k: int) -> int:
  """Compute the L_1 number representing the number of incremental addition to perform from M2's list values in order to
  go above or equal to K
  Args
  ----
    * m2_list: list of M sorted according to its respective value in E
    * K: Knt eigenvalue
  Returns
  ----
    The number of additions performed
```

```python
  """
  i, accumulator = 0, 0
  while accumulator < k:
    accumulator += m2_list[i]
    i += 1
  # Reverse latest addition that brought us above K
  return i - 1


# function which gives the sharp upper bound in the extension process
def sharp_upper_bound(n, k, L):
  E = []
  for i in range(0, l_0(n, k) + 1):
    E.append(sigma_dirichlet(n, i, L))
  for i in range(1, l_0(n, k) + 2):
    E.append(sigma_neumann(n, i, L))

  M = []
  for i in range(0, l_0(n, k) + 1):
    M.append(mu(n, i))
  for i in range(1, l_0(n, k) + 2):
    M.append(mu(n, i))

  #print(f"Before, e:{E}\nm:{M}")
  link_dict = {}
  for i in range(len(E)):
    key = E[i]
    value = M[i]
    try:
      link_dict[key].append(value)
    except KeyError as e:
      link_dict[key] = [value]

  #print(f"Link dict: {link_dict}")
  E.sort()
  M2 = [link_dict[e_value].pop(0) for e_value in E]

  l1 = l_1(M2, k)

  #print(f"sorted E: {E}\nSorted M2:{M2}")
  #print(f"Computing addition number to obtain a value >= K")
  return E[l1]
  #print(f"From list: {M2}, There need to be {l1} addition to obtain accumulator >= K = {k}")
  #print(f"From list {E}, at index {l1} = {E[l1]}")
```

4.A.2 Sharp upper bound

Here is a first program which takes three inputs (the dimension n, the kth eigenvalue and the meridian length L) and gives as output the sharp upper bound $B_n^k(L)$.

```python
import functions


n=int(input("Which dimension do you want to study?"))
k=int(input("Which eigenvalue do you want to study?"))
```

```python
L=float(input("Which meridian length o you want to study?"))

print(f"The sharp upper bound for the {k}th Steklov eigenavlue of a
    hypersurface of revolution in the Euclidean space, with meridian length
    {L} and dimension {n} is {functions.sharp_upper_bound(n, k, L)}")
```

4.A.3 Plot sharp upper bound

Now we code for a program that takes the dimension n and the kth eigenvalue k as inputs, and produces a plot of the function

$$\begin{aligned} \mathbb{R}_+^* &\longrightarrow \mathbb{R} \\ L &\longmapsto B_n^k(L). \end{aligned}$$

```python
import matplotlib.pyplot as plt
import numpy as np

import functions

n = int(input("Which dimension do you want to study?"))
k = int(input("Which eigenvalue do you want to study?"))

# first interval that we want to study
X = np.linspace(0, 7, max(400, k**2))
# for each value of L, find the sharp upper bound depending on n, k, L
Y = []
for l in X:
    Y.append(functions.sharp_upper_bound(n, k, l))
# find the critical length
y_max_index = Y.index(max(Y))
if max(Y) > Y[-1]:
    critical_length = round(X[y_max_index], 3)
else:
    critical_length = "infinite"

# refine the interval according to the critical length
if critical_length == "infinite":
    pass
else:
    X = np.linspace(0, min(critical_length * 4, 7), max(400, k**2))
    Y = []
    for l in X:
        Y.append(functions.sharp_upper_bound(n, k, l))
    # find the critical length
    y_max_index = Y.index(max(Y))
    if max(Y) > Y[-1]:
        critical_length = round(X[y_max_index], 3)
    else:
        critical_length = "infinite"


if max(Y) > Y[-1]:
    critical_length2 = "Finite critical length"
else:
```

```
    critical_length2 = "Infinite critical length"

# find the sharp upper bound depending on n, k
if max(Y) > Y[-1]:
    upper_bound = round(max(Y), 3)
else:
    upper_bound = round(max(Y))

# centering the figure
if critical_length == round(X[y_max_index], 3):
    center_x = min(critical_length * 4, 7)
else:
    center_x = 7
# plotting the sharp upper bound
plt.plot(X, Y, "r")
# lengending according to the indicators found
plt.title(critical_length2)
plt.xlabel("Value of L")
plt.ylabel(f"{k}th eigenvalue")
plt.figtext(0.6, 0.3, f"Critical length {critical_length}", style="italic")
plt.figtext(0.6, 0.2, f"B_{n}^{k} = {upper_bound}", style="italic")
plt.axis([0, center_x, 0, max(Y) + 2])
plt.show()
```

4.A.4 Adding mixed eigenvalues

Because we know that the function $L \longmapsto B_n^k(L)$ comes from the mixed Steklov-Dirichlet and Steklov-Neumann eigenvalues, we code for a program that add them in the previous plots.

```
import matplotlib.pyplot as plt
import numpy as np

import functions

n = int(input("Which dimension do you want to study?"))
k = int(input("Which eigenvalue do you want to study?"))

# first interval that we want to study
X = np.linspace(0, 7, max(400, k ** 2))
# for each value of L, find the sharp upper bound depending on n, k, L
Y = []
for l in X:
    Y.append(functions.sharp_upper_bound(n, k, l))
# find the critical length
y_max_index = Y.index(max(Y))
if max(Y) > Y[-1]:
    critical_length = round(X[y_max_index], 3)
else:
    critical_length = "infinite"

# refine the interval according to the critical length
if critical_length == "infinite":
    pass
else:
    X = np.linspace(0, min(critical_length * 4, 7), max(400, k ** 2))
    Y = []
```

```
    for l in X:
        Y.append(functions.sharp_upper_bound(n, k, l))
    # find the critical length
    y_max_index = Y.index(max(Y))
    if max(Y) > Y[-1]:
        critical_length = round(X[y_max_index], 3)
    else:
        critical_length = "infinite"
# endtry


if max(Y) > Y[-1]:
    critical_length2 = "Finite critical length"
else:
    critical_length2 = "Infinite critical length"

# find the sharp upper bound depending on n, k
if max(Y) > Y[-1]:
    upper_bound = round(max(Y), 3)
else:
    upper_bound = round(max(Y))

# plotting Steklov-Dirichlet eigenvalues
for i in range(0, functions.l_0(n, k) + 1):
    dirichlet_i = []
    for l in X:
        dirichlet_i.append(functions.sigma_dirichlet(n, i, l))
    plt.plot(X, dirichlet_i, "b")
# plotting Steklov-Neumann eigenvalues
for i in range(1, functions.l_0(n, k) + 2):
    neumann_i = []
    for l in X:
        neumann_i.append(functions.sigma_neumann(n, i, l))
    plt.plot(X, neumann_i, "g")
# centering the figure
if critical_length == round(X[y_max_index], 3):
    center_x = min(critical_length * 4, 7)
else:
    center_x = 7
# plotting the sharp upper bound
plt.plot(X, Y, "r")
# lengending according to the indicators found
plt.title(critical_length2)
plt.xlabel("Value of L")
plt.ylabel(f"{k}th eigenvalue")
plt.figtext(0.6, 0.3, f"Critical length {critical_length}", style="italic")
plt.figtext(0.6, 0.2, f"B_{n}^{k} = {upper_bound}", style="italic")
plt.axis([0, center_x, 0, max(Y) + 2])
plt.show()
```

4.A.5 Critical lengths

We code for a program that takes as inputs the dimension n and an integer k, and gives as output the highest number $j \leq k$ such that the jth eigenvalue has an infinite critical length.

```
import numpy as np
```

```

import functions


n=int(input("Which dimension do you want to study?"))
k=int(input("To which eigenvalue do you want to check?"))

#interval that we want to study
X = np.linspace(0, 7, max(400, k**2))
Y=[]
for m in range(1, k+1):
    Y_m=[]
    # for each value of L, find the sharp upper bound depending on n, m, L
    for l in X:
        Y_m.append(functions.sharp_upper_bound(n,m,l))


    #find the critical length
    ym_max_index = Y_m.index(max(Y_m))
    if max(Y_m) > Y_m[-1]:
        critical_length_m = X[ym_max_index]
    else:
        critical_length_m = -1

    Y.append(critical_length_m)
#print(Y)

#find the largest index which is equal to -1
i = 0
if Y[len(Y) - i - 1] == -1:
    print(f"In dimension {n}, the {k}th eigenvalue has a critical length at infinity.")
while Y[len(Y)-i-1] != -1:
    i=i+1
    if Y[len(Y)-i-1] ==-1:
        print(f"In dimension {n}, from the eigenvalue number {len(Y)-i+1} and to the {k}th, I only found finite critical lengths.")
```

4.A.6 Diagnosis eigenvalues

We code for a program that takes two arguments in inputs: the dimension n and and integer i, and checks for infinite critical length until the ith diagnosis eigenvalue.

```
import numpy as np
import sys
import timeit
import functions


# python -m cProfile -s time valeurs_propres_diagnostiques.py

def main():
  args = sys.argv[1:]

  n = int(args[0] if len(args) > 0 else input("Which dimension do you want to study? "))
```

```
  i = int(args[1] if len(args) > 1 else input("To which diagnosis eigenvalue do you want to study? "))

  k = 2
  K = [k]
  Diag_critic_length = []

  t_0 = timeit.default_timer()
  for j in range(1, i):
    k = k + 2 * functions.mu(n, j)
    K.append(k)
    t_1 = timeit.default_timer()
    elapsed_time = round((t_1 - t_0) * 10 ** 3, 3)
    print(f"{j:02d}:", K, f"-- Elapsed time: {elapsed_time} ms")
    t_0 = timeit.default_timer()
    # first interval that we want to study
    X_j = np.linspace(0, 7, max(400, k))
    # for each value of L, find the sharp upper bound depending on n, k, L
    Y_j = []
    for l in X_j:
      Y_j.append(functions.sharp_upper_bound(n, k, l))
    if max(Y_j) > Y_j[-1]:
      diagnosis_eigenvalue_j = 1
    else:
      diagnosis_eigenvalue_j = -1
    Diag_critic_length.append(diagnosis_eigenvalue_j)

  print(Diag_critic_length)

  c = 0
  if Diag_critic_length[len(Diag_critic_length) - c - 1] == -1:
    print(
      f"In dimension {n}, the {K[len(Diag_critic_length) - c - 1]}th eigenvalue has a critical length at infinity.")

  while Diag_critic_length[len(Diag_critic_length) - c - 1] != -1:
    c = c + 1
    if Diag_critic_length[len(Diag_critic_length) - c - 1] == -1:
      print(
        f"In dimension {n}, from the eigenvalue number {K[len(Diag_critic_length)-c+1]} and to the {K[-1]+2*functions.mu(n, i)-1}th, I only found finite critical lengths.")


if __name__ == "__main__":
  main()
```

List of Figures

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"Odi panem quid meliora.
Ça veut rien dire, mais je trouve que
ça boucle bien."

- Loth d'Orcanie.