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Calculating equivalent permeability: a review

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The purpose of this article is to review the various methods used to calculate the equivalent permeability of a heterogeneous porous medium. It shows how equivalence is defined by using a criterion of flow or of the energy dissipated by viscous forces and explains the two different concepts of effective permeability and block permeability. The intention of this review is to enable the reader to use the various published techniques and to indicate in what circumstances they can be most suitably applied.

Key words: heterogeneous porous medium, monophasic flow, equivalent permeability, upscaling techniques.

NOMENCLATURE

k, f	lightface notations: scalar
$\boldsymbol{x}, \boldsymbol{K}$	boldface notations: tensors
$\boldsymbol{\nabla}$	nabla operator $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$
grad	gradient operator: $\mathbf{grad} h = \boldsymbol{\nabla} h = (\partial h/\partial x, \partial h/\partial y, \partial h/\partial z)$
div	divergence operator: $\mathrm{div}\, \boldsymbol{u} = \boldsymbol{\nabla} \cdot \boldsymbol{u} = \partial u_x/\partial x + \partial u_y/\partial y + \partial u_z/\partial z$
$\langle\rangle$	averaging operator
$E()$	mathematical expectation
$C()$	covariance
σ^2	variance
D	space dimension
$k, \mathbf{K}$	isotropic or anisotropic local permeability
h, H	hydraulic head
$\mathbf{K}_{eq}$	equivalent permeability tensor
$\mathbf{K}_{ef}$	effective permeability tensor
$\mathbf{K}_b$	block permeability tensor
μ_a	arithmetic mean
μ_h	harmonic mean
μ_g	geometric mean
μ_p	mean of order p
$\sigma^2_{\ln k}$	variance of the logarithm of k
T	transmissivity
$\boldsymbol{u}, U$	Darcy filtration velocity

1 INTRODUCTION

The use of numerical models for studying subsurface flow has become common practice in hydrology and petroleum engineering over the last 30 years. However, one of the major questions that still pose a problem is: what parameters to introduce into the models? Although contemporary computers are growing ever more powerful and capable of describing the relevant flows with increasing precision, it is impossible to measure all the parameters at all points.

Flow models can be divided into two groups: those for which the parameters are obtained by calibration on an observed pressure record and those for which there are no such records.[23]

The first type of model obviates the problem of scale change. The parameters are obtained either by manual fitting or inverse methods.[53,96]

A prerequisite for the second type of flow model is a geological model describing the subsurface medium. Among others, Haldorsen,[50] Matheron *et al.*,[82] Terzlaf and Harbaugh,[103] Haldorsen and Damsleth,[51] Kolterman and Gorelick,[68,69] Fayers and Hewett,[38] Journel and Gómez-Hernández,[61] Marsily[22] and Anderson[4] describe methods for generating geological models.

When geostatistical techniques are used, the model should rigorously, be built on the same scale as the supporting measurements. Since laboratory investigations of permeability are made on volumes on the order of $20\,cm^3$ and well log data on volumes of the order of $20\,000\,cm^3$ (from Haldorsen,[50]) the resulting grid might contain from $10^{11}-10^{18}$ meshes depending on the size of the reservoir.[108] In reality, software for flow simulations is, at present, limited to a few million meshes.[1] This makes it necessary to shift to a larger scale. Gómez-Hernández and Journel[46] suggest that the permeabilities be generated geostatistically directly at the scale of the flow model meshes after a phase where the geostatistical parameters are fitted. However, even in this approach, a change of scale is used in the *learning* phase.

A totally different scheme for building geological models is to simulate the sedimentary processes that created the reservoir rocks.[68,103] These techniques produce grids with a very large number of meshes and need a change of scale as well.

The change of scales poses the problem of how to calculate the permeabilities at a larger scale on the basis of the geological model. The matter is complicated by the fact that the permeability is not an additive variable — it is not possible to calculate an equivalent permeability by a simple arithmetic mean. A great number of articles have been published on this subject in the last 20 years (to our knowledge, more than 200). In this review, we have limited our discussion of the equivalent permeability to steady-state, uniform and single-phase flow. Consequently, several important cases such as radial, transient and/or multi-phase flow have been omitted.

The objective of this article is twofold: (i) to describe all the published techniques for calculating the equivalent permeability while providing the reader with the information on how to use them; (ii) to compare these techniques through references to the literature and offer advice on how to choose the most suitable one.

Before we start discussing the techniques, the various equivalence criteria and the distinction between effective and block permeability must be understood.

The two equations that describe flow in porous media are: Darcy's law and the mass balance equation.

$$\boldsymbol{u} = -\mathbf{K}\,\mathbf{grad}(h) \quad \text{and} \quad \operatorname{div}(\boldsymbol{u}) = 0 \tag{1}$$

with $\mathbf{K}$ [m/s] the hydraulic conductivity tensor (later called *permeability tensor*), $\boldsymbol{u}$ the filtration velocity and h the hydraulic head. The mass balance equation is written here for steady-state and incompressible flow. In the following, we will always use the term *permeability*, even if, in two dimensions, the relevant magnitude may rather be the transmissivity.

Generally speaking, by *equivalent permeability* we mean a constant permeability tensor taken to represent a heterogeneous medium. A complete equivalence between the real heterogeneous medium and the fictitious homogeneous one is impossible. It is therefore defined, in a limited sense, according to certain *criteria* that must be equal for both media.

The first criterion ever used is the one of equality of flow.[16,108] The flow at the boundaries of the domain must be identical for the heterogeneous medium and the equivalent homogeneous one, subjected to the same head gradient.

The second criterion is the equality of energy dissipated by the viscous forces in the heterogeneous and the equivalent homogeneous medium.[59,79] This dissipation energy is defined by:

$$e = -\mathbf{grad}(h) \cdot \boldsymbol{u} \tag{2}$$

It is important to note that although these two criteria seem different, they are strictly equivalent in the case of periodic boundary conditions.[13]

Effective permeability is a term used for a medium that is statistically homogeneous on the large scale. In a stochastic context, it is defined by the formula:[79]

$$E(\boldsymbol{u}) = -\mathbf{K}_{\mathrm{ef}} E[\mathbf{grad}(h)] \tag{3}$$

where $E(\boldsymbol{u})$ represents the mathematical expectation of the flow rate in the domain and $E[\mathbf{grad}(h)]$ the head gradient expectation. The effective permeability is an intrinsic physical magnitude, independent of the macroscopic boundary conditions. It has been rigorously studied by two different methods: the stochastic approach[19,43,79] and the homogeneous-equation approach.[36,66,94] In the first case, the permeability is represented by a random function, usually stationary. The heterogeneity is described by correlation lengths. In the second case, the medium is assumed to be spatially periodic. The geometry of the basic cell is assumed to be known. In both cases, an effective permeability emerges, provided that firstly, either the correlation length or the size of the basic cell is much smaller than the whole domain and secondly, that the flow is uniform.[6,8,79,94] The effective permeability tensor has the characteristic of being a second-order, symmetric, positive-definite tensor.

In most engineering situations (flow of water or petroleum), the conditions for the appearance of an effective permeability are not satisfied. *Upscaled permeability* or *block permeability* is the equivalent permeability of a finite-size block. The concept of *statistical homogeneity* disappears because the observed block is too small. Rubin and Gómez-Hernández[98] took the average flow rate over the block as the criterion and defined the effective permeability by the following equation:

$$\frac{1}{V}\int_V \boldsymbol{u}(x)dv = \mathbf{K}_\mathrm{b}\frac{1}{V}\int_V \mathbf{grad}(h)dv \tag{4}$$

Indelman and Dagan[59] suggest that the equality of dissipated energies be used as the criterion:

(i) Necessary condition: The global response must be the same in the real and the numerical solution over homogenized blocks.

(ii) Sufficient condition: Let $\bar{e} = 1/\omega \int_\omega e(x)dx$ be the dissipated energy in the macro-mesh ω and $\tilde{e} = -\mathbf{grad}H \cdot \mathbf{K}_b\mathbf{grad}H$ the energy defined from the macroscopic magnitudes; then, the conditions:

$$E(\tilde{e}) = E(\bar{e}), \qquad C_{\tilde{e}}(r) = C_{\bar{e}}(r) \tag{5}$$

imply the necessary condition. C represents the covariance.

Note that if the block over which the up-scaling is done is large enough, the up-scaled permeability tends toward the effective permeability if it exists.

$$\lim_{\text{block}\to\infty} \mathbf{K}_b = \mathbf{K}_{ef} \tag{6}$$

It is most important to remember that the block permeability is not unique. It depends on the boundary conditions. For example, for the stratified medium in Fig. 1, using the criterion of equal flux, two different results are obtained if the calculation is made with uniform boundary conditions (prescribed hydraulic head varying linearly over all the sides of the block) or with permeameter-type boundary conditions (two sides with a prescribed head and two sides with no flow):

$$\mathbf{K}_b^{\text{uniform}} = \begin{pmatrix} 26{\cdot}61 & 23{\cdot}88 \\ 23{\cdot}88 & 26{\cdot}61 \end{pmatrix},$$

$$\mathbf{K}_b^{\text{permeameter}} = \begin{pmatrix} 9{\cdot}05 & 0 \\ 0 & 9{\cdot}05 \end{pmatrix}$$

Thus, block permeability is not an intrinsic characteristic of the medium as opposed to effective permeability. Moreover, subject to certain boundary conditions for certain media, it has been shown that the block permeability tensor is non-symmetric.[40,85,111,115]

We have divided the methods for calculating the effective or block permeability into three groups: deterministic, stochastic and heuristic. The first ones assume that the geological model is perfectly known as opposed to the stochastic techniques that only assume an approximate knowledge of this model and take a probabilistic view. The heuristic methods propose rules for calculating plausible equivalent permeabilities.

For each group of methods, different calculation techniques can be used: analytical or numerical, exact or approximated. A final criterion of classification is that which distinguishes between local and non-local methods. In the non-local methods, the permeability of a block depends both on the internal and the external permeability values. In the local methods, it only depends on the elementary permeability values inside the block. Most of the methods presented here are local ones.

The article is laid out as follows: Section 2 is devoted to the bounds of the equivalent permeability which are then used in Section 3 in dealing with heuristic methods. Sections 4 and 5 deal respectively with deterministic and stochastic techniques. Section 6 compares the different techniques and offers some recommendations.

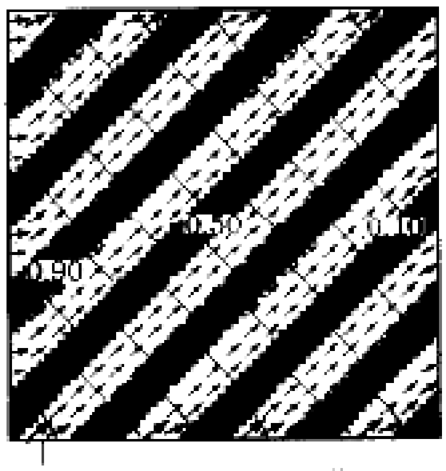


Fig. 1. Results of a simulation on a medium stratified at 45° in relation to the x-axis with uniform boundary conditions (i.e. prescribed heads varying linearly along the sides). The dark bands have a permeability of 1 mD and the light ones a permeability of 100 mD. The lines represent the hydraulic head contours and the arrows the filtration velocity.

2 INEQUALITIES FOR THE EQUIVALENT PERMEABILITY

2.1 Wiener bounds

This inequality is also called the fundamental inequality because it is always valid. It has been demonstrated by a great number of authors, e.g. Wiener,[112] Cardwell and Parsons,[16] Matheron[79] and Dagan,[19] among others.

$$\mu_h \leq K_{ef} \leq \mu_a \tag{7}$$

with μ_h = harmonic mean and μ_a = arithmetic mean.

2.2 Hashin and Shtrikman bounds

They are used for isotropic binary media:

$$\mu_a - \frac{f_1 f_0 (k_1 - k_0)^2}{(D - f_0)k_0 + f_0 k_1} \leq \mathbf{K}_{ef} \leq \mu_a - \frac{f_1 f_0 (k_1 - k_0)^2}{(D - f_1)k_1 + f_1 k_0} \tag{8}$$

where f_0 and f_1 are the fractions of the permeability phases k_0 and k_1. k_1 is higher than k_0 and $\mu_a = f_0 k_0 + f_1 k_1$.

The demonstration of Hashin and Shtrikman[55] is based on a model of the medium built of composite spheres (see Fig. 3). Each composite sphere is defined by an isotropic sphere with a constant permeability K_{in} and an isotropic concentric shell with a K_{out} permeability. By using the method of self-consistent media (see Section 4.3) it is possible to calculate an approximate value for the permeability of the medium. The maximum

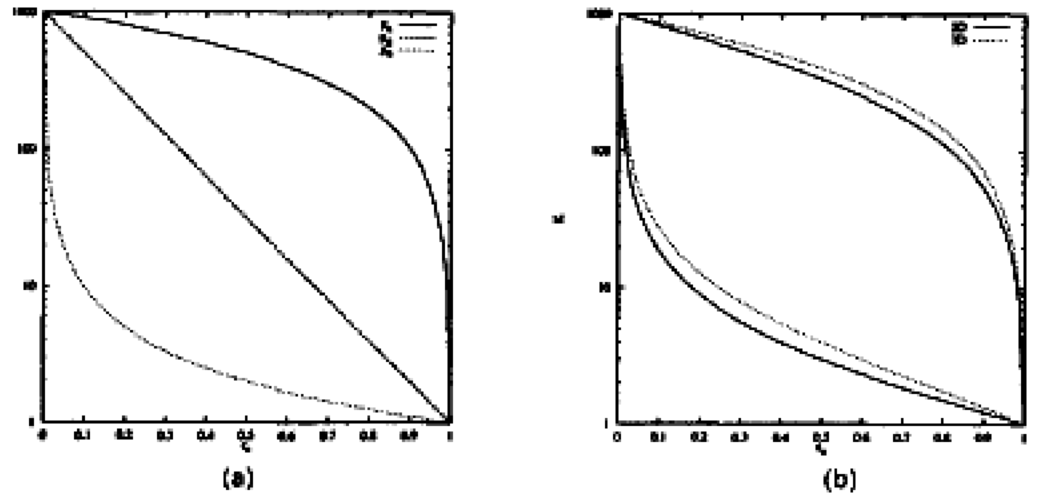


Fig. 2. Bounds for the equivalent permeability in a binary medium ($k_0 = 1$, $k_1 = 1000$). (a) The arithmetic, geometric and harmonic means are plotted versus the proportion of phase zero in the medium, (b) the Hashin and Shtrikman bounds are plotted versus the proportion of phase zero in the medium.

permeability is obtained by assuming that the spheres are the low-permeability medium and the shells are the high-permeability one ($K_{in} = k_0$, $K_{out} = k_1$). The minimum permeability is obtained by inverting these permeabilities: conductive spheres and resistant shells ($K_{in} = k_1$, $K_{out} = k_0$). A more rigorous demonstration, based on a variational principle, gives the same result which prompts the authors to conclude that these bounds are the best possible in terms of volumetric fractions.[54]

But, Fig. 2 shows that there is no large differences between Wiener (Fig. 2a) and Hashin and Shtrikman (Fig. 2b) bounds.

2.3 Cardwell and Parsons bounds

Cardwell and Parsons[16] used an electric analogy and Le Loc'h[74] used a variational method to show that the equivalent permeability in a given direction is bounded by (Fig. 4):

- the arithmetic mean of the harmonic means of the point permeabilities, calculated on each cell line parallel to the given direction (lower bound);
- the harmonic mean of the arithmetic means of the point permeabilities calculated over each slice of a cell perpendicular to the given direction (upper bound).

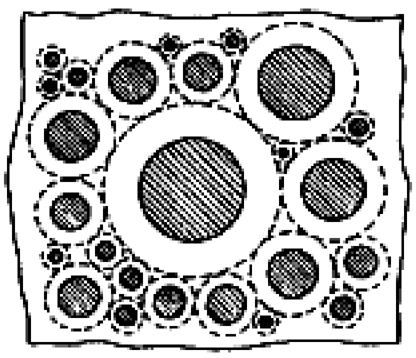


Fig. 3. Assemblage of composite spheres (from Hashin[54]).

$$K_1 = \mu_h^x(\mu_a^z(\mu_a^y)) \le K_{eq}^{xx} \le K_2 = \mu_a^z(\mu_a^y(\mu_h^x)) \tag{9}$$

2.4 Matheron bounds

In the case of an isotropic two-dimensional random mosaic with two phases, Matheron obtains:[80]

$$\begin{aligned} f_0 \ge 0{\cdot}5 &\Rightarrow K_{ef} \ge K_{ac} \\ f_0 \le 0{\cdot}5 &\Rightarrow K_{ef} \le K_{ac} \\ f_1 = 0{\cdot}5 &\Rightarrow K_{ef} = \sqrt{k_0 k_1} \end{aligned} \tag{10}$$

with

$$K_{ac} = \frac{1}{2}\left[(f_1 - f_0)(k_1 - k_0) + \sqrt{(f_1 - f_0)^2(k_1 - k_0)^2 + 4k_0 k_1}\right]$$

Note that this relation remains valid if the mosaic is invariant by a 90° rotation. Furthermore, Matheron[81] demonstrates that:

$$\begin{aligned} f_0 \ge 0{\cdot}5 \Rightarrow K_{ef} &\le K_m = \frac{f_1 k_0 k_1 + f_0 \mu_a \sqrt{k_0(2\mu_a - k_0)}}{f_1 m^* + f_0\sqrt{k_0(2\mu_a - k_0)}} \\ f_0 \le 0{\cdot}5 \Rightarrow K_{ef} &\ge K_m = k_0 k_1 \frac{f_0 \mu_a + f_1\sqrt{k_0(2m^* - k_0)}}{f_0 k_0 k_1 + f_1 m^* \sqrt{k_0(2m^* - k_0)}} \end{aligned} \tag{11}$$

with

$$m^* = f_1 k_0 + f_0 k_1$$
$$(\mu_a = f_1 k_1 + f_0 k_0, \quad \sigma^2 = f_1 f_0 \quad (k_1 - k_0)^2).$$

Figure 5 illustrates that the difference between the maximum and the minimum bounds of Matheron is much smaller than the difference between Wiener or Hashin and Shtrikman maximum and minimum.

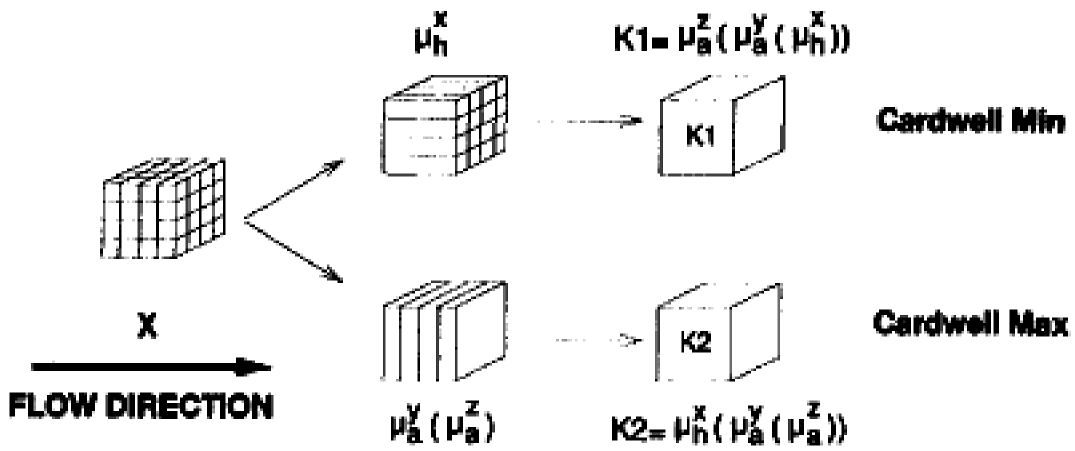


Fig. 4. Calculation of Cardwell and Parsons[16] bounds for flow in the x-direction.

2.5 Ene bounds

Ene[35] uses the method of homogenization (see Section 4.7) and obtains bounds that generalize those of Rubinstein and Torquato[99] in the anisotropic case. These bounds are written as a set of differential equations that require a numerical solution for the general case.

3 HEURISTIC METHODS

3.1 Sampling

The first technique is simply not to change scales. A block is given the permeability measured at its center. This very basic technique is commonly used in the petroleum industry and consists in passing from a measurement at the 10 cm scale to a block on the meter scale.

3.2 Averaging means

The general idea is to take a value between two theoretical bounds.

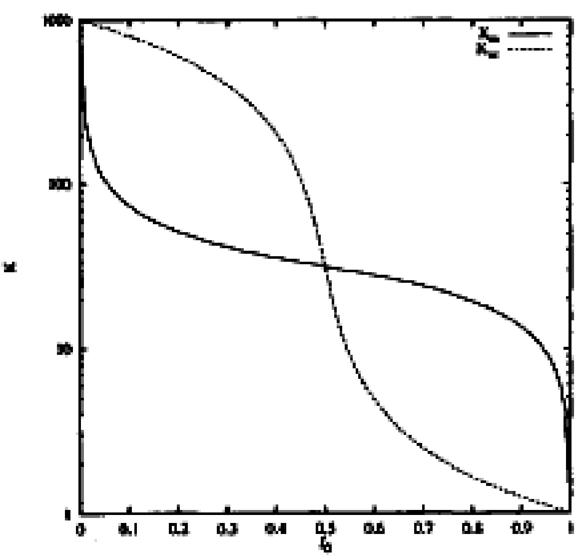


Fig. 5. Matheron bounds for the equivalent permeability in a binary medium. ($k_0 = 1$, $k_1 = 1000$) as a function of the proportion of phase zero in the medium K_m and K_{ac} are the bounds defined in eqns (10) and (11).

3.2.1 Fundamental bounds

Matheron[79] suggests a formula where the effective permeability is a weighted average of Wiener bounds.

$$K_{ef} = \mu_a^{\alpha}\mu_h^{1-\alpha}, \quad \text{with} \quad \alpha \in [0, 1] \tag{12}$$

If the medium is statistically homogeneous and isotropic, the exponent α is given by the equation:

$$\alpha = \frac{D-1}{D} \tag{13}$$

This formula is rigorously exact in the case of a factorized permeability,[80] i.e. of the form:

$$K(x) = k_1(x_1)k_2(x_2)\ldots k_D(x_D)$$

In the case of an anisotropic but statistically homogeneous medium, Ababou[1] takes:

$$\begin{aligned} K_{ef}^{ii} &= (\mu_a^{\alpha_i})(\mu_h^{1-\alpha_i}) \\ \alpha_i &= (D - l_h/l_i)/D \end{aligned} \tag{14}$$

where l_i is the correlation length in the relevant direction and l_h is the harmonic mean of the correlation lengths in the principal directions of anisotropy.

3.2.2 Cardwell and Parsons bounds

According to Kruel-Romeu,[70] Guérillot[47] proposes that we take the geometric mean of the two Cardwell and Parsons bounds, and Lemouzy[75] generalizes this idea by proposing, for three-dimensional media:

$$K_b^{xx} = \sqrt[6]{K_1^2 K_2^2 K_3 K_4} \tag{15}$$

with

$$\begin{aligned} K_1 &= \mu_h^x(\mu_a^y(\mu_a^z)) = \mu_h^x(\mu_a^z(\mu_a^y)) \\ K_2 &= \mu_a^y(\mu_a^z(\mu_h^x)) = \mu_a^z(\mu_a^y(\mu_h^x)) \\ K_3 &= \mu_a^y(\mu_h^x(\mu_a^z)) \\ K_4 &= \mu_a^z(\mu_h^x(\mu_a^y)) \end{aligned}$$

Kruel-Romeu[28,70] introduces exponents that control the influence of anisotropy. In three dimensions, he writes:

$$K_b^{xx} = K_1^{(\theta_{y2}\theta_{z3}+\theta_{z2}\theta_{y3})} K_2^{(1-\theta_{y3}-\theta_{z3})} K_3^{(1-\theta_{y2})\theta_{z3}} K_4^{(1-\theta_{z2})\theta_{y3}} \tag{16}$$

with

$$\theta_{y2} = \frac{\arctan\sqrt{a_y}}{\pi/2} \qquad \theta_{z2} = \frac{\arctan\sqrt{a_z}}{\pi/2}$$

$$\theta_{y3} = \frac{\theta_{y2}(1-\theta_{z2})}{1-\theta_{y2}\theta_{z2}} \qquad \theta_{z3} = \frac{\theta_{z2}(1-\theta_{y2})}{1-\theta_{y2}\theta_{z2}}$$

$$a_y = \frac{k^{yy}}{k^{xx}}\left(\frac{dx}{dy}\right)^2 \qquad a_z = \frac{k^{zz}}{k^{xx}}\left(\frac{dx}{dz}\right)^2$$

a_y and a_z represent the anisotropy factors that are due to the flattening of the grid and to the anisotropy of the local permeabilities (which is assumed constant over all the meshes).

3.3 Power average

Journel *et al.*[60] propose that K_{ef} be equal to a power average (or average of order p) with an exponent p in the interval between -1 and $+1$, depending on the spatial distribution of the permeabilities.

$$\mu_p = \langle k^p \rangle^{1/p} = \left(\frac{1}{V}\int_V k(x)^p dV\right)^{1/p} \tag{17}$$

Note that $p = -1$ corresponds to the harmonic mean, $\lim_{p\to 0}\mu_p$ to the geometric mean and $p = 1$ to the arithmetic mean. For statistically homogeneous and isotropic media, one obtains:[88]

$$p = 1 - \frac{2}{D} \tag{18}$$

This value for the exponent p corresponds to Matheron's conjecture and is discussed in more details further (see Section 5.2).

In the case of a log-normal medium, Ababou *et al.*[2] observe that:

$$\mu_p = \mu_g \exp\left(\frac{p\sigma_{\ln k}^2}{2}\right)$$

For a binary medium, eqn (17) becomes:

$$K_{ef} = [f_0 k_0^p + f_1 k_1^p]^{1/p} \tag{19}$$

Deutsch[27] recommends the use of this type of empirical formula for block permeability; the exponent p is obtained by fitting.

3.4 Flexible grid

One difficulty common to all methods involving block permeability is that the more the permeability variability increases in the volume used for the averaging, the greater is the uncertainty on the up-scaled permeability. One can minimize the error by automatically adjusting the local size of the large-scale grid as a function of, for example, the permeability variance on the small-scale grid. This approach was first used by Garcia *et al.*[41] and seems very popular in, for example, the petroleum industry. Recently, Durlofsky *et al.*,[31] Tran and Journel[105] and Yamada[114] have shown examples where this technique was applied to multi-phase flow. There are several types of flexible grids: grids where the geometry depends only on the permeability field and those that take into account the flow conditions. Some are deformed regular grids, others are more like nested meshes.[37]

4 DETERMINISTIC METHOD

In the deterministic method, the permeability field $\mathbf{K}(x, y, z)$ and the boundary conditions are assumed to be known. For a sufficiently simple permeability field (e.g. a stratified medium), exact analytical solutions can be found. For more general cases, there are theories (percolation, effective medium, streamline, renormalization) that can be used to make approximated calculations with varying precision. An approach which is, in principle, more general consists in solving numerically the diffusion equation. Finally, the homogenization theory, the theory of averaging with closure and the method of moments replace the direct solution by equivalent problems that also require numerical solutions. The last three theories are classified as homogeneous-equation methods.

4.1 Analytical solution of the diffusion equation

The diffusion equation ($\mathrm{div}[\mathbf{K}\,\mathbf{grad}\,h] = 0$) can be solved analytically for uniform flow in stratified media. Its two best known results are that the equivalent permeability for a flow parallel to the strata is the arithmetic mean of the point permeabilities μ_a and, for flow perpendicular to the strata, the harmonic mean μ_h. These results can be used when the point permeabilities are scalar. When the local permeabilities are tensorial, Quintard and Whitaker[94] show that, for a stratified medium composed of two phases with permeabilities $\mathbf{K}_0$ and $\mathbf{K}_1$, the effective permeability tensor is expressed by:

$$\mathbf{K}_{ef} = f_0\mathbf{K}_0 + f_1\mathbf{K}_1 + \frac{f_0 f_1(\mathbf{K}_0 - K_1)\cdot\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\cdot(\mathbf{K}_1 - \mathbf{K}_0)}{\begin{pmatrix}1\\ 0\end{pmatrix}\cdot(f_0\mathbf{K}_1 + f_1\mathbf{K}_0)\cdot\begin{pmatrix}1\\ 0\end{pmatrix}} \tag{20}$$

where $f_0 = V_0/V_{tot}$ is the fraction of the medium with $\mathbf{K}_0$ permeability and $f_1 = V_1/V_{tot}$ is the fraction of the medium with $\mathbf{K}_1$ permeability. This result is one example of the use of the theory of large scale averaging. Note that the same result was obtained by Kasap and Lake[63]

Table 1. Percolation theory, value of the coefficients in eqn (21)

Type of medium	A	n_c	μ	Reference
2D isotropic		0·5	1·1	102
2D	Depends on medium	Depends on medium	1·1	21
3D	Depends on medium	Depends on medium	1·6	21
3D	Depends on medium	Depends on medium	1·8	49

with a grouping of only two blocks with known permeabilities. Kasap and Lake[63] suggest the use of this result and of the rotation formulas (change of reference axis) for analytical calculations of block permeability in media with slanted stratifications.

4.2 Percolation theory

The percolation theory deals statistically with the problem of communication across complex systems constituted by objects that may or may not be connected.

A classical example[21] is that of a sea dotted with islands. If the sea level recedes, the islands become larger and gradually, some of them are strung together by coastal barriers. If the sea continues to shrink, a continent will emerge at a critical point in the sea level. There is then a continuous pathway through the network of coastal barriers by which all the islands are linked. This transition is called *percolation transition* and the critical point is called *percolation threshold*.

This type of transition appears in a great variety of problems: viscosity of polymer materials, conduction in heterogeneous media, diffusion of information or of a virus in a population, etc. More specifically, Guyon *et al.*,[49] and more recently, Berkowitz and Balberg,[12] have studied the applications of this theory to the physics of porous media.

In the field of equivalent permeability, the percolation theory has been applied to materials with two phases, one of which is nonpermeable. Let n be the proportion of permeable medium. Close to the percolation threshold:

$$n < n_c \rightarrow K_{ef} = 0$$

$$n > n_c \rightarrow K_{ef} = A \times (n - n_c)^{\mu} \tag{21}$$

where μ is taken to depend only on the space dimension and A and n_c depend on the geometry of the network (see Table 1).

4.3 Theory of effective media

Apart from effective medium theory (EMT), there are other names for this theory: self-consistent approach, used by Matheron, or embedded matrix method, used by Dagan. The heterogeneous medium constituted by homogeneous blocks placed side by side is replaced by a single inclusion of K permeability embedded in a homogeneous matrix with an unknown permeability K^0 (Fig. 6). The boundary conditions are far enough away from the inclusion for the assumption to be made that the head gradient, and therefore the flow, is constant around the embedded material. If the inclusion has a simple form, there is an analytical solution for the hydraulic head field inside and outside it. The hypotheses are as follows: the perturbations of the head field due to any inclusion do not interfere with the perturbations caused by another inclusion and $K^0 = K_{ef}$.

Dagan[18] made the calculations for spherical inclusions and found the following expressions:

$$K_{ef} = \frac{1}{D}\left[\int_0^{\infty} \frac{f(K)dK}{K(D-1) + K_{ef}}\right]^{-1} \tag{22}$$

where $f(K)$ is the probability density function of the permeability. D is the space dimension. Equation (22) can be integrated numerically by incrementing K_{ef} until equality is obtained. King[65] proposes a similar formula. Dagan[19] and Poley[93] give a more complex formula for ellipsoidal inclusions by which they obtain an anisotropic effective permeability tensor.

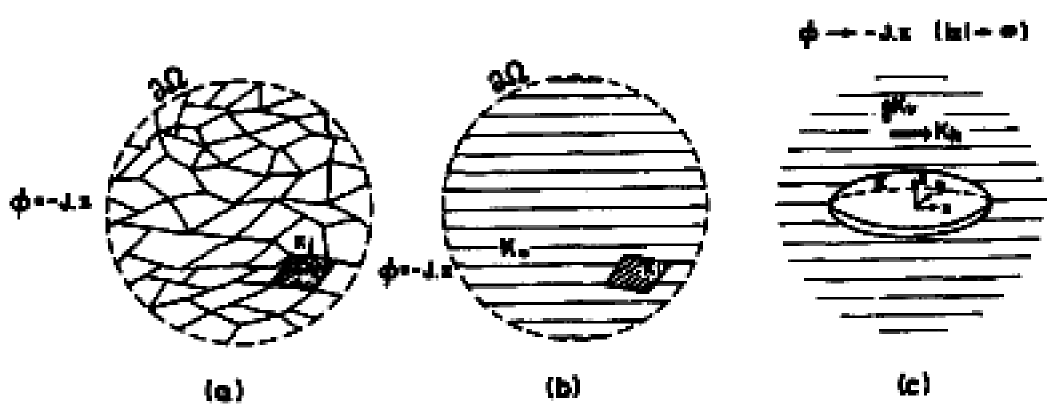


Fig. 6. Schematic representation of the self-consistent model from Dagan.[19] (a) Heterogeneous medium constituted by pieces with constant permeability put side by side, (b) inclusion embedded in a uniform matrix with constant permeability K^0, (c) inclusion represented by a spheroid embedded in an infinite medium.

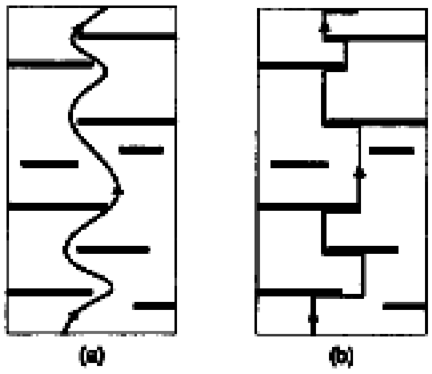


Fig. 7. Approximate method for calculating the length of streamtube: (a) real streamline; (b) approximation by straight-line segments (from Begg and King[10]).

In the case of binary media, eqn (22) becomes:

$$K_{ef} = \frac{1}{D}\left[\frac{f_0}{k_0 + (D-1)K_{ef}} + \frac{f_1}{k_1 + (D-1)K_{ef}}\right]^{-1} \tag{23}$$

According to Desbarats,[25] this formula is satisfactory for $f_0 < 0{\cdot}6$ with phase 0 representing the lowest permeability.

4.4 Streamline

The streamline method is used to calculate the vertical permeability of a binary sand–clay system. The clay formations are described by rectangles (in 2D) or flat parallelepipeds (3D) with zero permeability. The sand is assumed to have k_{hj} and k_{vj} anisotropic permeabilities. The up-scaled permeability is obtained by calculating the head losses along a tortuous tube circulating inside the sand matrix (Fig. 7).[52,78] An improvement on this method is obtained by incorporating statistical parameters on the size and number of clay inclusions and generalizing is to stratified media.[10]

The following formula is taken from the review by Fayers and Hewett:[38]

$$K_v = \frac{(1-F_s)H^2}{N_s \sum_{i=1}^{N_s} \frac{1}{S_i S_{ei}}} \tag{24}$$

where F_s is the fraction of clay inclusions, N_s the number of selected streamlines, H the formation thickness, S_i the length of the ith streamline and S_{ei} the length weighted by the permeability:

$$S_i = \sum_j^{N_j}\left(h_j + \sum_j^{n_j} d_{ijk}\right) \qquad S_{ei} = \sum_k^{N_j}\left(\frac{h_j}{k_{vj}} + \sum_j^{n_j} d_{eijk}\right)$$

where n_j is the number of impervious inclusions found in layer j. The horizontal lengths of the kth step in the ith streamline of the jth layer is given by:

$$d_{ijk} = r_{1ijk}l_{ijk} \quad \text{or} \quad r_{2ijk}w_{ijk}$$

$$d_{eijk} = \frac{r_{1ijk}l_{ijk}}{k_{xj}} \quad \text{or} \quad \frac{r_{2ijk}w_{ijk}}{k_{yj}}$$

where r_{1ijk} and r_{2ijk} are uniform random numbers and l_{ijk} and w_{ijk} are the lengths and thicknesses of the inclusions.

4.5 Renormalization

Historically, renormalization has its origin in statistical physics.[62,113] Its application to the problem of calculating the equivalent permeability is generally associated with King.[65] However, a method of the same type was proposed by Le Loc'h[74] (pp. 107–110).

Renormalization is a recursive algorithm. The equivalent permeability of a fine-mesh grid is determined by a series of successive aggregations. For example, in a space of dimension D, one proceeds from a grid of 2^{nD} meshes to a less finely discretized one with $2^{(n-1)D}$ meshes. This procedure is then repeated until a grid of the desired size is found, ultimately one single mesh (Fig. 8).

The basic operation is the equivalent permeability calculation of a cell with 2D meshes. Since there is no exact formula in two and three dimensions, the calculation has to be approximate. King[65] used the analogy of an electric network to represent a porous medium. The permeability is obtained by successive star–triangle transformations. This expression is equivalent to a solution of the flow problem by a finite difference method with a centered formulation.

Several variations have been proposed, as described below. They usually concern the type of finite difference formulation and the boundary conditions. For example, a *direct* formulation seems more exact (Kruel-Romeu:[70] the permeability is assigned to each *link* between two

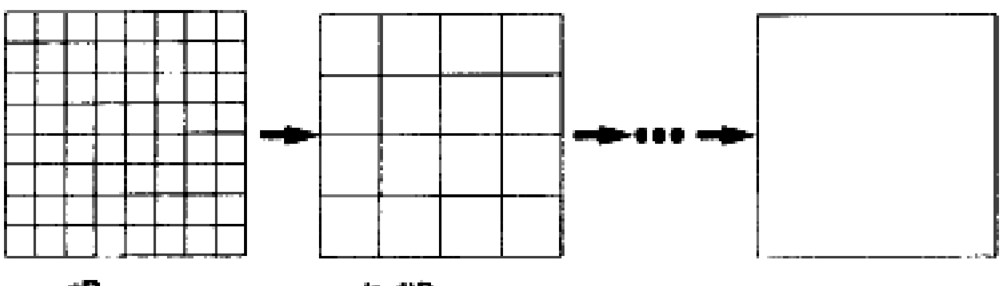


Fig. 8. General principle of renormalization.

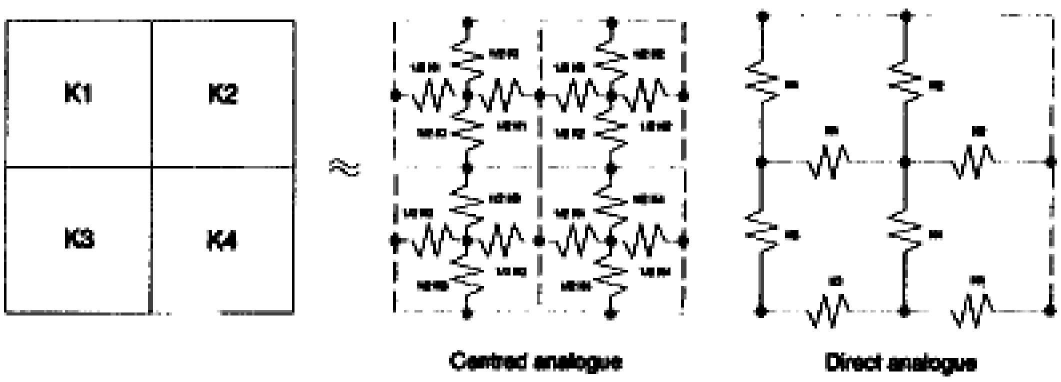


Fig. 9. Electric analogues used to calculate the up-scaled permeability of an elementary cell in two dimensions.

nodes, and not to each *block* surrounding a node). Gautier and Nœtinger[42] calculate a complete tensor by using periodic boundary conditions. Furthermore, they use this method to calculate the velocity field in the medium. Finally, Le Loc'h[74] proposes an original method where the meshes are only grouped two by two.

4.5.1 Directional permeability

In two dimensions, the calculation can be carried out either by successive star–triangle transformations or by writing a linear system of the finite difference type. The two procedures are equivalent and produce the formula:[65]

$$K_b^{xx} = \frac{4(k_1+k_3)(k_2+k_4)[k_2k_4(k_1+k_3)+k_1k_3(k_2+k_4)]}{[k_2k_4(k_1+k_3)+k_1k_3(k_2+k_4)][k_1+k_2+k_3+k_4] \atop +3(k_1+k_2)(k_3+k_4)(k_1+k_3)(k_2+k_4)} \tag{25}$$

In their article, Hinrichsen *et al.*[56] give the Fortran code for calculating this expression, in two and three dimensions.

4.5.2 Tensor

Gautier and Nœtinger[42] propose a method for calculating the complete tensor. The boundary conditions are periodic and the finite difference formulation direct.

Take a square network of four meshes. The meshes are numbered from 1 to 4 (see Fig. 9). To simplify the notation, one assumes that the size of the meshes is 1×1 and the exponent for the diagonal terms of the permeability tensors ($k^{xx} = k^x$) is not repeated. The balance is written at each of the four nodes. Then, the coefficients on the heads are factorized. This results in a system of four equations where the four unknowns are the heads. Note that this system is linked and that it is therefore possible to fix a head value (e.g. $h_4 = 0$) and solve only a 3×3 system.

$$A \cdot \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} k_2^x & k_3^y \\ k_4^{xy} - k_2^x - k_2^{xy} & k_4^y \\ k_4^x & k_4^{xy} - k_3^{xy} - k_3^y \end{pmatrix} \cdot \begin{pmatrix} \Delta H_x \\ \Delta H_y \end{pmatrix}$$

with

$$A = \begin{pmatrix} k_1^x + k_2^x + k_1^y + k_3^y + 2k_1^{xy} & -(k_1^x + k_2^x + k_1^{xy} + k_2^{xy}) & -(k_1^y + k_3^y + k_1^{xy} + k_3^{xy}) \\ -(k_1^x k_2^x + k_1^{xy} + k_2^{xy}) & k_1^x + k_4^y + k_2^x + 2k_2^{xy} + k_2^y & k_1^{xy} + k_4^{xy} \\ -(k_1^y + k_3^y + k_1^{xy} + k_3^{xy}) & k_1^{xy} + k_4^{xy} & k_3^x + k_4^x + k_1^y + 2k_3^{xy} + k_3^y \end{pmatrix}$$

Then, the global flow rates over the domain are calculated by summing the elementary flow rates over each mesh.

$$\begin{cases} U_x = -k_2^x h_1 + (k_2^x + k_2^{xy})h_2 - k_4^x h_3 \\ \qquad + (k_2^x + k_4^x)\Delta H_x + k_4^{xy}\Delta H_y \\ U_y = -k_3^y h_1 - k_4^y h_2 + (k_3^y + k_3^{xy} - k_4^{xy})h_3 \\ \qquad + k_4^{xy}\Delta H_x + (k_3^y + k_4^y)\Delta H_y \end{cases}$$

Using these equations, U_x and U_y are expressed in a complex manner as functions of ΔH_x and ΔH_y. The equivalent tensor is obtained by:

$$\begin{pmatrix} K_{xx} & K_{xy} \\ K_{yx} & k_{yy} \end{pmatrix} \cdot \begin{pmatrix} \Delta H_x \\ \Delta H_y \end{pmatrix} = \begin{pmatrix} U_x(\Delta H_x, \Delta H_y) \\ U_y(\Delta H_x, \Delta H_y) \end{pmatrix}$$

and by taking the special solutions $\Delta H_x = 1$, $\Delta H_y = 0$ and $\Delta H_x = 0$, $\Delta H_y = 1$, one finds:

$$\mathbf{K_b} = \begin{pmatrix} U_x(1,0) & U_y(0,1) \\ U_x(1,0) & U_y(0,1) \end{pmatrix} \tag{26}$$

4.5.3 Simplified renormalization

This method was proposed by Le Loc'h.[74] The calculation of the basic cell is done by successive groupings of two meshes. If the two meshes are in series relative to the flow direction, the point permeabilities are averaged with a harmonic mean. If they are in parallel, an arithmetic mean is used. At each iteration, the direction of the grouping is changed. In three dimensions, for example, a grouping along x is alternated with one along z and finally along y. This order is kept during the whole process (Fig. 10). Depending on the choice of successive directions, the values of the up-scaled permeability will be different.

We shall only discuss the two extreme values that correspond to the cases that start or end by a grouping in series in relation to the flow direction, i.e. by an

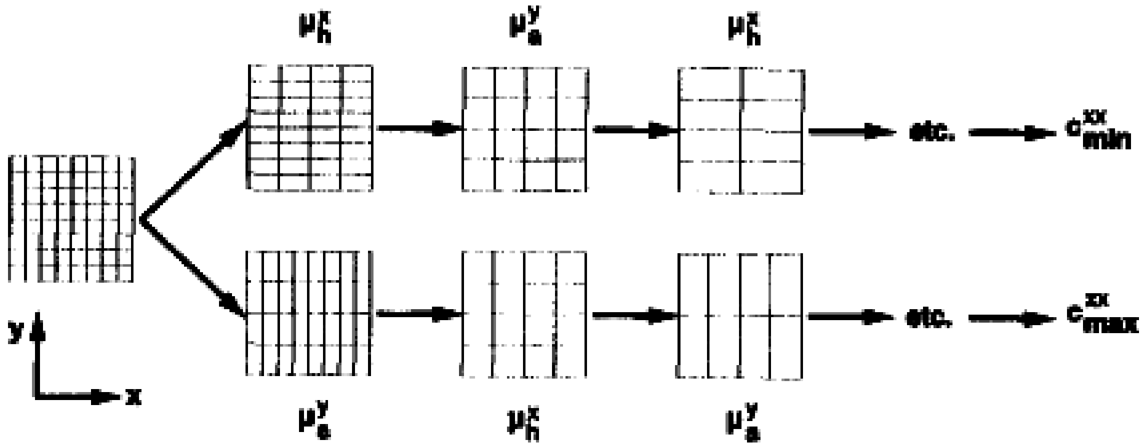


Fig. 10. Example in two dimensions of two possibilities for the successive grouping for calculating the permeability along x. The symbol μ_h^x represents the harmonic mean of two meshes grouped in the x-direction. μ_a^y is the arithmetic mean in y.

harmonic mean. For example, for a flow in x, the maximum and minimum values are:

$$c_{\max}^{xx} = \mu_h^x(\mu_a^y(\mu_a^z)) = \mu_h^x(\mu_a^z(\mu_a^y)) \quad \text{iteratively}$$

$$c_{\min}^{xx} = \mu_a^y(\mu_a^z(\mu_h^x)) = \mu_a^z(\mu_a^y(\mu_h^x))$$

At the end of the run, we take the geometric mean of these two values.

$$K_b^{xx} = \sqrt{c_{\max}^{xx} c_{\min}^{xx}} \qquad (27)$$

4.6 Numerical solution of the diffusion equation

There are several techniques for solving partial differential equations numerically: finite differences, finite elements, mixed finite elements, spectral elements, etc. In the case of the diffusion equation, they make it possible to calculate an approximate solution ($h(x, y, z)$ and $\mathbf{u}(x, y, z)$) for any permeability field and any boundary conditions.

4.6.1 Local methods

Direction permeability. A directional flow, such as in a permeameter, is simulated numerically by prescribing, as boundary conditions, a head gradient of $\Delta H/L$ between two sides of the block and no flow on the perpendicular sides. The numerical simulation gives the flow rate over each mesh. The total flow rate Q crossing the medium is obtained by adding together the elementary flow rates. The block permeability is given by:

$$K_b = -\frac{Q}{S}\frac{L}{\Delta H} \qquad (28)$$

where L is the distance between the two sides with a prescribed head, ΔH is the head difference and S is the cross-section traversed by the flow Q. By rotating the boundary conditions and repeating the flow simulation, one obtains the permeability in the three directions, x, y and z. This type of calculation dates back to the very beginning of the use of numerical methods.[107]

Tensorial permeability. Durlofsky[29] suggests the use of periodic boundary conditions. With this method, it is possible to calculate, for a gradient in a given direction, the fluxes in all directions. The permeability tensor is then given by:

$$K_b^{uv} = -\frac{Q_u^v}{S}\frac{L}{\Delta H^v} \qquad (29)$$

where Q_u^v is the flow rate in the u direction, resulting from a head gradient in the v direction (ΔH^v). When periodic boundary conditions are used, one can obtain a permeability tensor that is always symmetrical and that, in addition to satisfying the equality of fluxes, also satisfies the equality of dissipated energies.[13]

Along the same line of thought, uniform boundary conditions (varying linearly over the side of the block)[40,76] or perturbed conditions[92] have been used to calculate the permeability tensor. Note that uniform conditions do not produce a symmetric permeability as opposed to periodic conditions.

4.6.2 Nonlocal methods

General tensor scaling. This technique was the first nonlocal method to be proposed.[111] It is described in two dimensions but can easily be generalized to three dimensions. With this technique one can calculate transmissivity tensors for up-scaling.

When using it, one must simulate the flow over the entire domain at the scale of the fine-mesh grid with several sets of boundary conditions on the sides of the domain. This may seem excessively computer-intensive but for petroleum engineering where the permeability grid, after upscaling, will be used to simulate multiphase flow, the cost of the single-phase simulation in steady state is low compared to a multi-phase simulation in transient state. In such a case, it becomes worthwhile to use this technique.

With these simulations, it is possible to have the fields of hydraulic head $h(x, y, i)$ and velocity $\mathbf{u}(x, y, i)$ at a small scale for the $i = 1, \ldots, n$ different boundary conditions. The large-scale heads $H(x, y, i)$ are calculated by taking volumetric averages weighted by the storage inside each block of the large-mesh grid and assigned to the center of the large meshes.

$$H(x,y,i) = \frac{\int_V h(x,y,i)S(x,y)dV}{\int_V S(x,y)dV}$$

White and Horne[111] use finite differences for such calculations. The large-scale velocities are calculated by taking the surface average of the velocities obtained on the small scale over the interfaces separating the blocks.

$$U(x,y,i) = \frac{1}{A}\int_A u(x,y,i)dA$$

Having thus obtained the large-scale velocities and heads, one can write a large-scale tensorial Darcy equation for each interface between two blocks.

$$U(x,y,i) = \mathbf{T}_b\mathbf{grad}[K(x,y,i)], \qquad \mathbf{T}_b = \begin{pmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{yy} \end{pmatrix}$$

The unknowns are then the components of tensor T_b. Note that the tensor is not assumed to be, a priori, symmetrical. This problem is over-constrained; it is solved by least squares by minimizing the residuals on the flows. The least squares criterion is written:

$$\zeta = \sum_{i=1}^{n}[U(i) - \mathbf{T}_b I(i)]^2 = \text{minimum}$$

where I is the head gradient between two neighbouring blocks. For an interface between two blocks, White and Horne[110] calculate the gradient as follows:

$$I_x(x+1/2,y,i) = \frac{H(x+1,y,i) - H(x,y,i)}{\Delta x}$$

$$I_y(x+1/2,y,i) = \frac{1}{2}\left[\frac{H(x+1,y+1,i) - H(x+1,y-1,i)}{\Delta y} + \frac{H(x,y+1,i) - H(x,y-1,i)}{\Delta y}\right]$$

The minimization of ζ generates a linear system of equations where the components of tensor T_b are the unknowns. For example, T_{xx} and T_{xy} are solutions of:

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} \cdot \begin{pmatrix} T_{xx} \\ T_{xy} \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \tag{30}$$

with

$$a = \sum_{i=1}^{n} I_x(i)^2 \qquad b = \sum_{i=1}^{n} I_x(i)I_y(i) \qquad d = \sum_{i=1}^{n} I_y(i)^2$$

and

$$S_1 = \sum_{i=1}^{n} = \sum_{i=1}^{n} I_x(i)U_x(i) \qquad S_2 = \sum_{i=1}^{n} I_y(i)U_x(i)$$

A similar system makes it possible to determine T_{yx} and T_{yy}. T_{xx} is replaced by T_{yx} and T_{xy} by T_{yy}, and S_1 and S_2 on the right-hand-side of eqn (30) are replaced by:

$$S_1 = \sum_{i=1}^{n} I_x(i)U_y(i) \qquad S_2 = \sum_{i=1}^{n} I_y(i)U_y(i)$$

Use of the local neighbourhood. Gómez-Hernández and Journel[46] propose that only a *skin* around the block under consideration be used. The flow is simulated over the block surrounded by its closest neighbours and with uniform boundary conditions (i.e. the linearly varying prescribed-head boundary conditions, as described above). The size of the calculations is therefore reduced compared to general tensor scaling, while keeping some of the nonlocal effects and still requiring the calculation of the permeability tensor by least squares. Holden and Lia[57] use the same technique. They note that the calculation of the permeability tensor seems to converge faster than the calculation of the hydraulic-head solution. They therefore propose a change in the iterative numerical solution of the diffusion equation so that the convergence criterion is placed on the equivalent permeability. According to these authors, this would make the numerical method more powerful.

4.7 Homogeneous-equation methods

Wen and Gómez-Hernández[109] include three methods in this group: the theory of homogenization,[11] the method of spatial averaging with closure[17] and the method of moments.[66] These techniques are based on different mathematical concepts but they have a number of points in common, in particular: their objectives, their constituent hypotheses and their results. Bourgeat *et al.*[14] proposes a comparison of the methods of homogenization and spatial averaging with closures.

These three methods pose the problem of scale change in rigorous mathematical terms and determine the equations on the higher scale from constituent equations of a given scale. It is then possible to determine both the parameters and the large-scale equations. In all three cases the techniques are general and can be used to study any physical phenomenon. For example, Ene[36] addressed heat transfer, others study the transfer from the Navier-Stokes to Darcy equations,[5,83,110] multiphase flow[15] or contaminant transport.[95]

The first common hypotheses is that of spatial periodicity in the medium. This allows the study of the domain as a whole to be replaced by that of the basic cell subjected to periodic boundary conditions (Fig. 11). This topic has fuelled heated discussions. Why study periodic media when we know full well that no natural medium is periodic? First of all, because we have the mathematical tools capable of proving the existence and uniqueness of the solution. Furthermore, this hypothesis allows us to obtain new results concerning the macroscopic equations. Finally, it is reasonable to believe that periodic boundary conditions are no more arbitrary

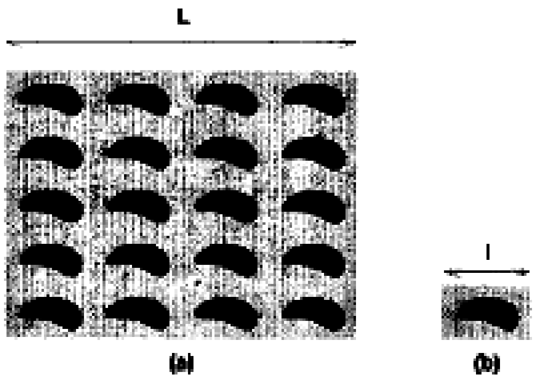


Fig. 11. (a) Periodic medium consisting of two phases; (b) unitary cell of the periodic medium.

than those of a permeameter or uniform type. The latter are part of other, equally constraining, assumptions on the nature of the medium. For example, in the case of conditions of the permeameter type, the no-flow boundary corresponds to an axis of symmetry. Moreover, if the calculation is done for a case where a large-scale permeability emerges, the latter must be independent of the boundary conditions and the use of periodic conditions then does not change the results.

The second hypotheses is that the period is very small compared to the size of the studied domain. This makes it possible to make the necessary approximations and to ensure that the large-scale equation emerges.

The effective permeability is calculated in two steps. First, one solves the boundary problem regarding the ancillary variables $b_1, b_2, \ldots, b_D$. Kitanidis[66] writes:

$$\nabla \cdot (\mathbf{K}\mathbf{grad}b^i) = \nabla \cdot \boldsymbol{K}_i \qquad i = 1, \ldots, D \tag{31}$$

subject to periodic boundary conditions with $\boldsymbol{K}_i$ as the ith column of the K matrix. These differential equations are generally solved numerically. For example, Dykaar and Kitanidis[33,34] use a spectral method and Njifenjou[87] uses mixed hybrid finite elements. The terms of the equivalent permeability tensor are then calculated with the formula:

$$K_{ef}^{ij} = -\frac{1}{2V}\int_V (\boldsymbol{K}_i \cdot \mathbf{grad}b^j + \boldsymbol{K}_j \cdot \mathbf{grad}b^i)dV + \frac{1}{V}\int_V K_{ij}dV \qquad i,j = 1, \ldots, D \tag{32}$$

An analytical solution was calculated by Quintard and Whitaker[94] for a stratified binary medium (eqn (20)).

5 STOCHASTIC METHOD

In order to deal with the uncertainty arising from a partial knowledge of the reservoir properties, the stochastic method considers the studied variables as random functions in space. The definition of the effective permeability is then based on the notion of mathematical expectation (eqn (3)).

The determination of the probability distribution function of the equivalent permeability is expressed in terms of a stochastic differential equation, i.e. as a differential equation linking several random variables. A complete solution would mean that the joint probability distribution for each random function would be determined at all points. This is generally impossible. It is considered sufficient to calculate the first moments (mean, variance, covariance, etc.) or, at least, their approximate value. A great number of methods — which cannot be addressed here without digressing from the main focus of this article — have been used: spectral method, perturbation method, field theory, Monte-Carlo method, etc. We will restrict the discussion to the main results. Major references in this area are Matheron,[79] Dagan[19] and Gelhar;[43] a critical review of the different methods can be found in Kitanidis.[67]

5.1 Rule of geometric averaging

One of the few exact results is the rule of geometric averaging. It was demonstrated by Matheron.[79] This very elegant demonstration is based on the fact that in a two-dimensional space, a 90° rotation transforms a gradient vector into a conservative vector and vice-versa. Subject to the conditions that the permeability k and its inverse $k' = 1/k$ are random functions admitting the same probability distributions and that these functions are invariant by a 90° rotation, the effective permeability is expressed in the following form:

$$K_{ef} = \mu_g = \exp[E(\log k)] \tag{33}$$

The preceding hypotheses are satisfied in the special case of an isotropic log-normal medium and a checker-board binary design. However, this only holds for a uniform (or parallel) flow field and is not satisfied for example for a radial flow. Matheron[79] indicates that in radial flow, the harmonic average (for large distances) should hold.

5.2 Approximated analytical results and conjectures

5.2.1 Matheron's conjecture

Landau and Lifshitz,[73] for the electrodynamic equations, and Matheron,[79] for uniform flow in isotropic and stationary porous media, propose to extend the exact result, in two dimensions, to D dimensional space with a formula which is the first-order approximation of the effective permeability:

$$K_{ef} = \mu_a^{(D-1)/D}\mu_a^{1/D} \tag{34}$$

In three dimensions and for the case of a log-normal permeability distribution, this formula becomes:

$$K_{ef} = \mu_g \exp\left[\sigma_{\ln k}^2\left(\frac{1}{2} - \frac{1}{D}\right)\right] \tag{35}$$

where $\sigma^2_{\ln k}$ represents the variance of the permeability logarithm. A vast number of authors have examined the validity of this formula. In particular, Gutjahr *et al.*,[48] using the method of perturbations and assuming that $\sigma^2_{\ln(k)}$ is small, found:

$$K_{ef} = \mu_g \left[1 + \left(\frac{1}{2} - \frac{1}{D}\right)\sigma^2_{\ln(k)}\right] \tag{36}$$

Equation (36) provides first-order approximations of the classical exact results in one and two dimensions (harmonic mean and geometric mean, respectively). Dagan[20] carries the calculations to the fourth-order and obtains:

$$K_{ef} = \mu_g \left[1 + \left(\frac{1}{2} - \frac{1}{D}\right)\sigma^2_{\ln(k)} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{D}\right)^2 \sigma^4_{\ln(k)}\right] \tag{37}$$

Equations (36) and (37) can then be seen as, respectively, the first- and second-order terms of Taylor's series expansion of the exponential in eqn (35). They have therefore been used to justify conjecture (35). The advantage of eqn (35) over eqns (36) and (37) is that it is not limited to small variances. King[64] and later, Nœtinger,[88] combined the field theory and Green's functions to expand these results. Nœtinger[88] shows that a sub-series of the perturbation expansion can be summed again at all orders. He demonstrates that eqn (35) holds for an uncorrelated isotropic log-normal medium but that the formula remains an approximation. Furthermore, he suggests that eqn (35) can be expressed in the following form:

$$K_{ef} = \langle k^{1-(2/D)} \rangle^{[1-(2/D)]^{-1}} \tag{38}$$

However, the discussion of this topic is still continuing. Abramovich and Indelman,[3] carrying the expansion to the sixth-order and De Wit,[24] using perturbation expansions, show that for correlated isotropic media, the coefficient in $\sigma^2_{\ln(k)}$ depends on the form of the correlation function. This is contrary to Matheron's conjecture.

It is still true, however, that formula (35) has been successfully tested by many authors even for media with correlation lengths greater than the size of the meshes (the correlation length is defined as the distance at which the covariance has been reduced by a factor e from its value at lag zero). Desbarats,[26] for example, studied the power mean and found experimentally that in three dimensions, the exponent is 1/3 for a log-normal-type medium. This corresponds to the formula proposed by Nœtinger (eqn (38)). Dykaar and Kitanidis[34] found a deviation of only 4% between calculations made with the spectral numerical method and Matheron's conjecture (see Fig. 12). These results are confirmed by Newman and Orr[85] up to the values of $\sigma^2_{\ln(k)} = 7$. Sánchez-Vila *et al.*[101] compare the results obtained with three, *a priori*, different methods: a standard numerical method (eqn (28)), the method of Rubin and Gómez-Hernández,[98] eqn (4), and Matheron's conjecture. They show numerically that these three techniques produce comparable results for a log-normal medium where $\sigma^2_{\ln(k)} < 1$. Finally, comparisons with permeability measurements, made at several scales on samples of sandstone and limestone, are also consistent with Matheron's conjecture.[89]

5.2.2 *Taking anisotropy into account*

In the case of an anisotropic medium with an exponential covariance, one can calculate the effective permeabilities in the main directions of anisotropy with

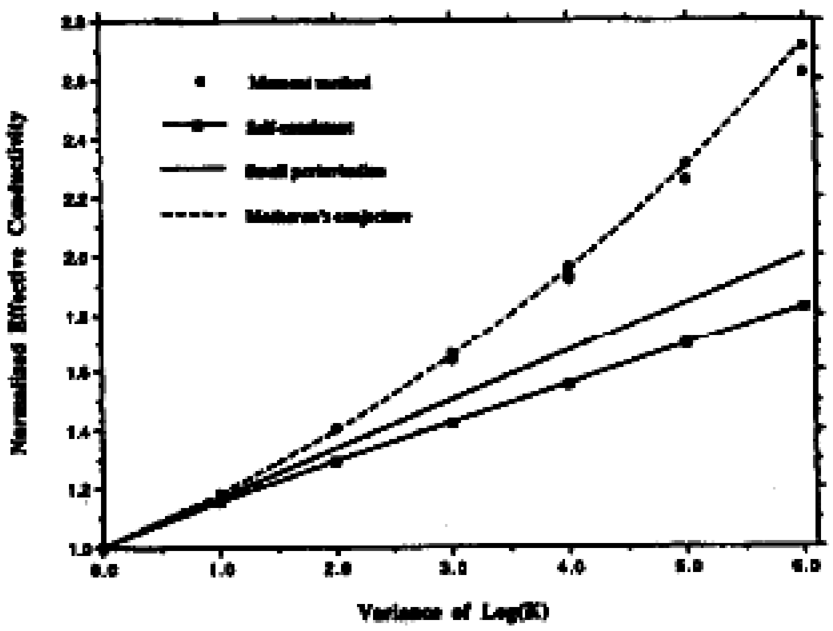


Fig. 12. Comparison between effective permeabilities obtained by the effective-medium theory (labeled self-consistent in the figure), the perturbation method (labeled small perturbation), the moment method and Matheron's conjecture for an isotropic, log-normal, 3D medium. Note the good agreement between the numerical results (moments) and Matheron's conjecture (from Dykaar and Kitanidis[34]).

the formula[43,44] expanded in the first-order with regard to $\sigma^2_{\ln(k)}$:

$$K_{ef}^{ii} = \mu_g\left[1 + \sigma^2_{\ln(k)}\left(\frac{1}{2} - g_{ii}\right)\right] \tag{39}$$

with

$$g_{ii} = \int_{-\infty}^{+\infty} \frac{k_i^2}{k^2}\,\frac{l_1 l_2 l_3 dk_1 dk_2 dk_3}{\pi^2(1 + l_1^2k_1^2 + l_2^2k_2^2 + l_3^2l_3^2)},$$

$$\sum_{i=1,2,3} g_{ii} = 1$$

where l_i is the correlation length in the i direction and k_i is the local permeability in the i direction. This formula is only valid for small $\sigma^2_{\ln(k)}$. As before, if the preceding equation is considered as a Taylor series expansion of the exponential function truncated to the first order in $\sigma^2_{\ln(k)}$, a plausible generalization of eqn (39) for high values of $\sigma^2_{\ln(k)}$ is:

$$K_{ef}^{ii} = \mu_g \exp\left[\sigma^2_{\ln(k)}\left(\frac{1}{2} - g_{ii}\right)\right] = \mu_a \exp\left(-g_{ii}\sigma^2_{\ln(k)}\right) \tag{40}$$

Along the same lines, Ababou[1] takes anisotropy into account with a simplified formula:

$$K_{ef}^{ii} = \mu_g \exp\left(\sigma^2_{\ln(k)}\left[\frac{1}{2} - \frac{1}{D}\,\frac{l_h}{l_i}\right]\right)$$

where l_i is the correlation length in the considered direction and l_h the harmonic mean of the correlation lengths in the principal directions of anisotropy. Neuman[84] proposes the same type of formula:

$$K_{ef} = \mu_g \exp\left(\sigma^2_{\ln(k)}\left[\frac{1}{2} - \frac{\beta}{D}\right]\right)$$

with $0 \le \beta \le D$. More precisely, $\beta = 1$ in the infinite isotropic case, $0 \le \beta \le 1$ in the finite isotropic case and $1 \le \beta \le D$ in the infinite anisotropic case. He does not, however, explain how to calculate β *a priori*.

The development at order two of $\sigma^2_{\ln(k)}$[58] leads to a formula with an order-two coefficient that depends on the variance and on the space dimension but is also dependent on the form of the covariance function $C(r) = \sigma^2_{\ln(k)}\rho(r)$ and not only on the anisotropy ratios. $\rho(r)$ is the autocorrelation function.

$$K_{ef}^{ii} = \mu_g\left\{1 + \left(\frac{1}{2} - \alpha_i\right)\sigma^2_{\ln(k)} + \left[\left(\frac{1}{2} - \alpha_i\right)^2 + \gamma_i\right]\sigma^4_{\ln(k)}\right\} \tag{41}$$

$$\alpha_i = \int \frac{dp}{(2\pi)^2}\,\frac{k_i^2}{k^2}\,\tilde{\rho}(p) \qquad i = 1, \ldots, D$$

$$\gamma_i = \frac{1}{2}\int\int \frac{dq\,dp}{(2\pi)^{2D}}\,\tilde{\rho}(q)\tilde{\rho}(p)\,q\cdot p\,\frac{q^2 + p^2}{(q + p)^2}\left(\frac{q_i}{q^2} + \frac{p_i}{p^2}\right)^2$$

where $\tilde{\rho}$ represents the Fourier transform of the autocorrelation function: $\tilde{\rho} = \int dr\rho(r)\exp(i\mathbf{K}r)$. Note that $g_{ii} = \alpha_i$ in the case where the covariance is exponential. Thus, the conjecture of Gelhar and Axness,[44] eqn (39), is not satisfied at order two.

5.2.3 *Caution*

The results given above may lead one to believe that there are simple rules for calculating the effective permeability. This is only true in an approximate sense. To illustrate this, Matheron[80] used Schwydler's method — permeability is written on the form $k(x) = E(k)[1 + \gamma(x)]$ with $E(\gamma)$, the mathematical expectation of γ, equal to zero — to carry out calculations in the form of a development truncated at order three for three models of different media. The first one is a *random cross-word puzzle*. The second is a medium divided into random convex polygons by Poisson lines. The last one is a medium with an infinitely divisible law. The results for the three models are:

1) $K_{ef} = m\left[1 - \frac{E(\gamma^2)}{2} + 0{\cdot}279E(\gamma^3)\right]$

2) $K_{ef} = m\left[1 - \frac{E(\gamma^2)}{2} + 0{\cdot}307E(\gamma^3)\right]$

3) $K_{ef} = m\left[1 - \frac{E(\gamma^2)}{2} + 0{\cdot}250E(\gamma^3)\right]$

Note that in all three cases, the coefficient of the third-order term is different. This means that for Matheron there is no simple, uniformly valid rule for averaging the permeabilities, not even in the case of isotropic random media. This follows the same line of reasoning as that of Indelman and Abramovich[58] for anisotropic media, described in the preceding section, and that of Abramovich and Indelman[3] and De Wit[24] for log-normal isotropic media.

5.2 Block permeability

Rubin and Gómez-Hernández[98] were the first author to study the problem of block permeability from a stochastic standpoint. The main purpose of their work was to formulate the expectation $E(K_b)$ and the covariance of block permeability C_{K_b} and the cross-variance $C_{K_b,\ln(k)}$ between block permeability and local permeability based on the distribution function of the permeability. These expressions were obtained analytically for block permeability in the case where the local permeability is isotropic and its logarithm has a multi-gaussian distribution.[98] The hypotheses are: infinite medium, uniform flow and small variance of the logarithm of k. For the permeability, Gómez-Hernández[45] writes:

$$E(K_b) = \mu_g\left[1 + \frac{\sigma^2_{\ln(k)}}{2} + \frac{1}{J_a}\left(\sigma^2_{\ln(k),J_x} - \bar{C}_{\ln(k),J_x}\right)\right]$$

$$C_{K_b}(r) = \mu_g^2 \bar{C}_{\ln(k)}(V, V_{+r})$$

$$C_{K_b,\ln(k)}(V, r) = \mu_g \bar{C}_{\ln(k)}(V, r) \tag{42}$$

V represents the size of the block, r is a translation vector, J_a is the modulus of the arithmetic mean of the hydraulic gradient vector J, J_x is the x component of J. $\sigma^2_{\ln(k),J_x}$ is the stationary cross-variance between the permeability logarithm and the hydraulic gradient J_x. The mean covariances $\bar{C}$ are given by:

$$\bar{C}_{\ln(k),J_x} = \frac{1}{V^2}\int_V \int_V C_{\ln(k),J_x}(x - x')dx\,dx'$$

$$\bar{C}_{\ln(k)}(V, V_{+r}) = \frac{1}{V^2}\int_V \int_{V+r} C_{\ln(k)}(x, x')dx\,dx'$$

$$\bar{C}_{\ln(k)}(V, r) = \frac{1}{V} C_{\ln(k)}(x', x + r)dx\,dx'$$

The cross-variance between the permeability logarithm and the hydraulic gradient $C_{\ln(k),J_x}$ can be found in Dagan[19] (sec. 3.7). These analytical results are confirmed by Monte-Carlo simulations.

Note that qualitatively, in two dimensions:

$$\lim_{V \to 0} E(K_b) = \mu_a$$

$$\lim_{V \to \infty} E(K_b) = \mu_g$$

The variance of K_b decreases when V increases but it remains non-negligible for large volumes. The covariance of K_b is different from that of the local permeabilities. The main attraction of this approach is that it makes it possible to condition the block permeabilities by local measurements of h and k. Rubin and Gómez-Hernandez[98] observe that the effect of the conditioning by h is weaker than that of the conditioning by $\ln(k)$. In both cases, the variance of K_b decreases but it decreases more strongly for a conditioning by $\ln(k)$.

More recently, Fenton and Griffiths[39] studied the same problem in two dimensions using Monte-Carlo simulations. They varied the mesh size which gave them the opportunity to investigate the effect of anisotropy on block permeability. They conclude that for blocks that have not been flattened (i.e. where the size of the mesh is identical in the horizontal and vertical directions), the block permeabilities are probably log-normally distributed. For such a case, they propose expressions of the expectation and variance of block permeability:

$$E[\ln(K_b)] = E[\ln(k)]$$

$$\sigma_{\ln(K_b)} = \sigma_{\ln(k)}\sqrt{\gamma R} \tag{43}$$

$\gamma_R = \gamma(R, R)$ is the variance function, R is the anisotropy ratio of the meshes $R = \sqrt{dx/dy}$. The variance function is defined by:[107]

$$\gamma(R_1, R_1) = \frac{4}{(R_1, R_2)^2} \times \int_0^{R_1}\int_0^{R_2}(R_1 - r_1)(R_2 - r_2)\rho(r_1, r_2)dr_1\,dr_2$$

Fenton and Griffiths[39] also note that the flattening factor has a maximum influence when the block size is equal to the correlation length.

In the case of a medium where the permeability is not log-normally distributed and may be anisotropic, Gómez-Hernández and Journel[46] and Gómez-Hernández[45] suggest the use of a numerical method. The objective is to make some fine-mesh simulations of the permeability field. For all simulations, one calculates the inter-block permeability tensor by solving the diffusion equation numerically. The method is the one described above in Section 4.6.2 *use of the local neighbourhood*. These block-scale permeabilities are used to obtain the expectation and the covariance of T_b, and the cross-covariance $C_{T_b,\ln(k)}$. It is then possible to simulate directly the inter-block permeabilities. This method is also used by Tran.[105]

6 DISCUSSION

6.1 Effective or block permeability?

The first possible choice is to describe the heterogeneous medium by a single value (the effective permeability) or by a set of values (block permeability). Durlofsky[30] compares three two-dimensional methods for media with correlated log-normal permeability distributions. These methods are: (1) a technique known as *global* where the heterogeneous medium is replaced by a uniform effective permeability; (2) the technique of sampling; (3) a local technique which consists in calculating the block permeability by taking the geometric mean of the values in the block.

The precision of all three techniques is measured by comparing the solution, after the scale change, to the solution before the change. Thus, Durlofsky[30] calculates mean quadratic errors (L_2) for the head and the flow rate between the results of the numerical simulations (finite differences) on the finely discretized medium and after the change of scale. He then studies the evolution of L_2 as a function of the type of boundary conditions, the correlation length, the size of the blocks and the variance of the permeability logarithm. He observes that the quadratic error can

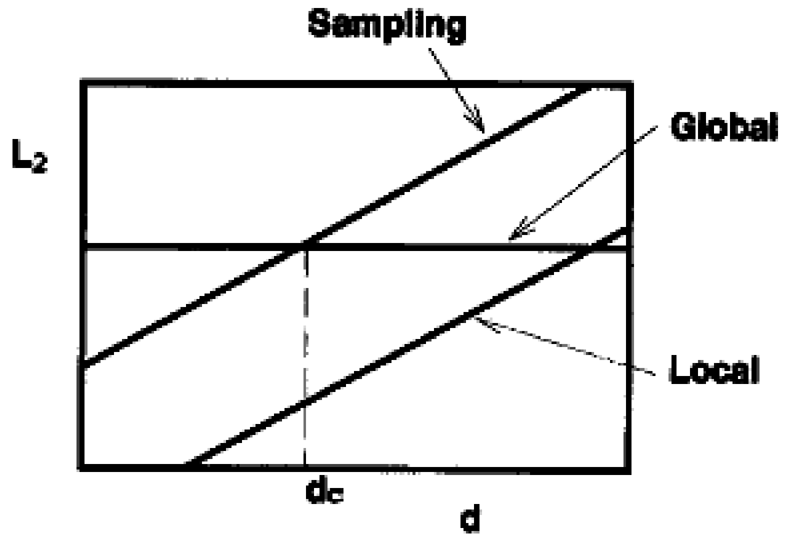


Fig. 13. Evolution of the quadratic error (L_2) with the size of the blocks (d) for three up-scaling techniques. d_c is the critical value for the sampling technique. If $d > d_c$, the global method is more accurate than sampling (from Durlofsky[30]).

be approximated by $L_2 = ad^b$ where d is the size of the blocks for techniques 2 and 3. In the case of the global method, L_2 is independent of d. For each configuration, he determines a critical value d_c, beyond which the technique of sampling or the local method become less accurate than the global one (see Fig. 13). He concludes:

— For flow resulting from permeameter-type boundary conditions, the method of sampling is more accurate than the global one when the size of the blocks is smaller than the correlation length ($d_c \simeq \lambda$). This result is independent of the variance of the permeability logarithm ($\sigma^2_{\ln(k)}$).

— For flow driven by a uniform source term, higher resolution is necessary to make the sampling more accurate than the global method. Moreover, d_c decreases with $\sigma^2_{\ln(k)}$.

— In all cases, the method of local averaging is more accurate than sampling and generally, more precise than the global method.

In other words, if the block permeability is calculated with a reasonably reliable method, it gives a higher accuracy. However, with a much cruder method (sampling), if the size of the blocks is larger than the correlation length, it is better to use a uniform effective permeability.

6.2 Numerical or analytical method?

By analytical method we mean simple algebraic averages as opposed to numerical methods which require a numerical solution of partial differential equations.

All the assumptions on the types of medium and flow pertaining to the techniques described here are listed in Table 2, which clearly shows that analytical techniques have a much more limited range of application than numerical methods. For example, they only give exact results for stratified or isotropic media with log-normal permeability distributions and uniform flow. In all other cases, there is a risk involved in using them. Numerical techniques, on the contrary, are more general: they can be used on any permeability field. Their main drawback is that they are expensive in terms of memory space and computer time.

The choice between analytical and numerical is therefore simple: if one has a case for which analytical methods have been developed, one simply applies the theoretical formula. Otherwise, one must resort to numerical methods. It is, of course, possible that the size of the problem makes it impossible to use numerical techniques. One can then turn to heuristic or approximate methods. Among the latter, the most general one is renormalization.

6.3 What about renormalization?

The results obtained by renormalization are similar to those obtained by numerical techniques, such as finite-differences, provided that the flow lines are not too

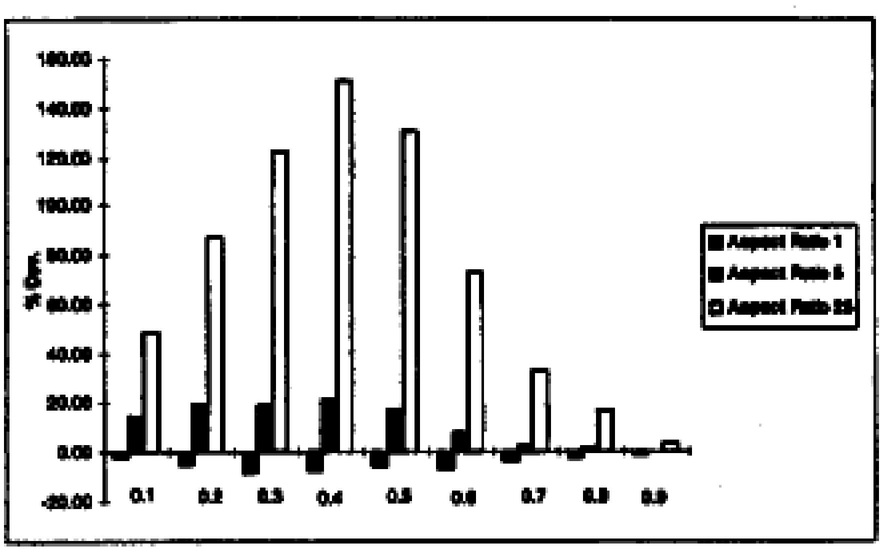
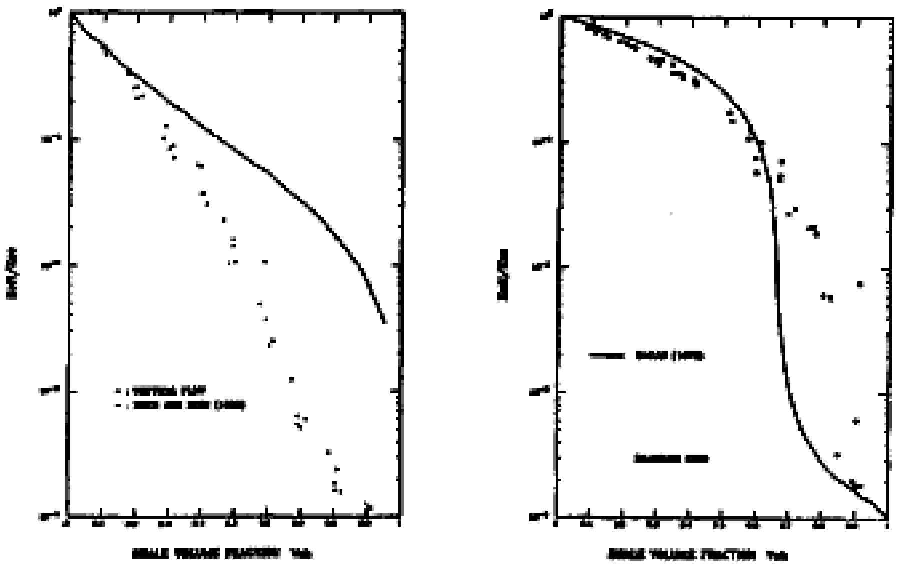


Fig. 14. Deviation between the permeability obtained by solving the diffusion equation and using renormalization, for binary media, versus the density of a low-permeability medium for three anisotropy ratios of the meshes (from Malick and Hewett[77]).

Table 2. Comparison of up-scaling techniques. The column *Speed* gives the rapidity with which the technique can be used. The column *Tensor* indicates whether or not the technique is capable of calculating a permeability tensor and if so, whether it is diagonal (Diag.), symmetric (Sym.) or non-symmetric (N.S.)

Method	Assumptions: Medium	Assumptions: Flow	Speed	Tensor	Remarks
Inequalities for the equivalent permeability					
Wiener bounds	None	Uniform	++	No	Exact analytical results except for Ene bounds
Cardwell and Parson bounds	None	Uniform	++	Diag.	
Hashin and Shtrikman bounds	Isotropic binary	Uniform	++	No	
Matheron bound	Mosaic, random isotropic 2D	Uniform	++	No	
Ene bounds	Periodic	Periodic	–	Sym.	Requires the numerical solution of a system of differential equations
Heuristic method					
Sampling	None	None	+++	No	One of the most commonly used and the least studied method
Power averages	None	None	++	Diag.	Requires the fitting of the exponent
Averaging bounds	None	Uniform	++	Diag.	Provides a range of uncertainties
Analytical solution of the diffusion equation					
Arithmetic mean	Stratified	Parallel to the strata	++	No	
Harmonic mean	Stratified	Perpendicular to the strata	++	No	Exact results within the very narrow framework
Quintard and Whitaker's formula	Binary stratified	Uniform	++	Sym.	of the assumptions, i.e. stratified media
Stochastic method					General analytical results
Geometric mean	2D, K and its inverse have the same distribution and are invariant by 90° rotation	Uniform	++	No	Exact result, special case: isotropic log-normal medium and checker-board
Power mean with exponent 1/3	3D, log-normal isotropic	Uniform	++	No	Approximate result valid even for strong variances
Perturbation	Small variance	Uniform	++	Diag.	Approximate results only for small variances
Effective medium theory	Heterogeneities do not interact	Uniform	++	Diag.	Restricted to weakly heterogeneous media
Streamtube	Binary, one impervious phase	Vertical	++	No	Calculates only the vertical permeability accurate only if the impervious fraction is small
Percolation theory	Close to the percolation threshold	Uniform	++	No	Three parameters to be fitted
Renormalization					Approximate iterative calculations
Simplified	Locally isotropic	None	++	Diag.	Not yet properly tested
Standard	Locally isotropic	None	++	Diag.	Comparable to finite differences if the anisotropy is weak
Tensorial	None	None	++	Sym.	Not yet properly tested
Numerical solution of the diffusion equation					General method, risk of numerical bias
Permeameter boundary conditions	None	Uniform	–	Diag.	
Periodic boundary conditions	Periodic	Uniform	–	Sym.	The periodic boundary conditions prove robust for any real conditions used
Uniform boundary conditions	None		–	N.S.	
Perturbed boundary conditions	None		–	Yes	
General tensor scaling (GTS)	None	None	– – –	N.S.	Takes into account the whole heterogeneous system, sometimes leads to negative diagonal terms
Use of local neighbourhoods	None	Uniform	– –	N.S.	Takes into account the non-local effect while being less cumbersome than GTS
Homogeneous-equation method					
Homogenization	Periodic	Periodic	–	Sym.	These three methods require the numerical solution of partial differential equations
Averaging with closure	Periodic	Periodic	–	Sym.	
Moment method	Periodic	Periodic	–	Sym.	



(a) (b)

Fig. 15. Binary media. Comparison between equivalent permeabilities obtained by numerical solution of the diffusion equation and two approximate methods, versus the proportion of low-permeability medium (V_{sh}). (a) Anisotropic medium, comparison for the vertical permeability in the streamline method (solid line) and the numerical solution (+); (b) Isotropic medium, comparison between numerical results (□) and Dagan's[18] result with the effective-medium theory (solid line). From Desbarats.[25]

tortuous.[65] If this condition is not satisfied, the deviation may be great. Malick and Hewett[77] demonstrate this effect on binary and log-normal anisotropic media by varying the flattening of the grid. They conclude that the error is greater, the flatter the grid. For binary media, the maximum error occurs for average densities around 50% of low-permeability media (see Fig. 14). In log-normal media the error increases with the variance. This appears to be due to boundary conditions of the permeameter type used in the standard method.[65] One can then imagine that the method proposed by Gautier and Nœtinger[42] might be less likely to suffer this limitation as they use periodic boundary conditions and a complete permeability tensor. However, at the moment, no numerical test has been proposed to confirm this assumption.

As to simplified renormalization, we know that it is a non-biased estimator in two dimensions for log-normal isotropic media[74] (p. 197). However, standard renormalization, as well as finite differences, has tendency to under-estimate the permeability. For example, in two dimensions for a log-normal isotropic medium and uniform flow, we know that the equivalent permeability is the geometric mean (μ_g). King's results,[65] in Table 3, show that for an isotropic 2D log-normal medium, finite differences and renormalization give a lower equivalent permeability than the exact theoretical result. For cases where there is no available theoretical value for the equivalent permeability, we can assume that standard renormalization and finite differences are still under-estimating the permeability.

This fact is all the more important as some authors use the method of renormalization to produce a reference equivalent permeability. This value then allows them to validate an algebraic composition rule[100] or to study the influence of the size of the representative elementary volume on synthetic media.[90] The use of renormalization as a reference introduces a systematic bias in the results.

To conclude the discussions on renormalization, one can say that this is a method which has been very popular. However, in its standard form it has two limitations: systematic bias (under-estimation) and arbitrary boundary conditions. One would hope that

Table 3. Example of King's[65] results

$\sigma^2_{\ln k}$	0·5	10
Reference: μ_g	2·0	2·0
Effective medium theory	2·0	2·0
Perturbation (first-order)	1·7	$-3{\cdot}3 \times 10^6$
Finite diff. simulation	1·9	1·3
Renormalization	1·8	1·3

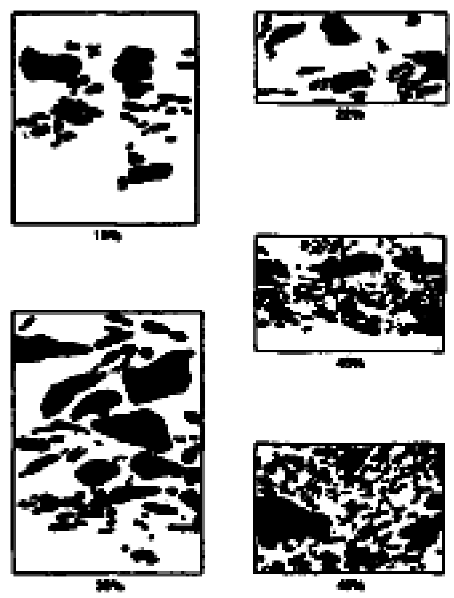


Fig. 16. Binary medium used by Bachu and Cuthiell[7] and Dykaar and Kitanidis[34] to compare up-scaling techniques.

the modifications proposed by Gautier and Nœtinger[42] will lead to a considerable improvement.

6.4 When to use the approximation methods?

The approximation methods are: the streamline method, the effective medium theory, the perturbation method and the percolation theory. The latter only proposes a formula close to the percolation threshold and its usefulness is therefore limited.

The streamline method gives good results according to Begg *et al.*[9] but Desbarats,[25] dealing with binary media, only obtains results comparable to those produced by the numerical solution of the complete flow problem with the small mesh size for very small proportions of impervious medium (Fig. 15a). Desbarats[25] also demonstrates that the formula based on the theory of effective media for binary media (eqn (23)) gives results comparable to those obtained by numerical methods for clay quantities of less than 60% (Fig. 15b). For Dykaar and Kitanidis,[34] the same formula, applied to deterministic binary media (Fig. 16), is in good agreement with the numerical method of moments. Note that the media in question always contain less than 50% clay. Consequently, Dykaar and Kitanidis' experiments[34] do not represent the worst cases described by Desbarats.[25]

For media with a log-normal permeability distribution, the theory of effective media and the formulas based on perturbation calculations produce satisfactory results only if the perturbations are weak.[34] Figure 12 also shows that the results obtained with Dagan's formula (eqn (22)) are not as good as those of perturbation calculations. It seems that in the case of a log-normal distribution, the underlying assumption in this formula of no interaction between the heterogeneous inclusions is not valid.

Finally, it appears that the approximation techniques are mainly valid (i) for small heterogeneities with a low variance, in media with a log-normal distribution of permeability, (ii) when the proportion of low-permeability media is less than 50%, in binary media.

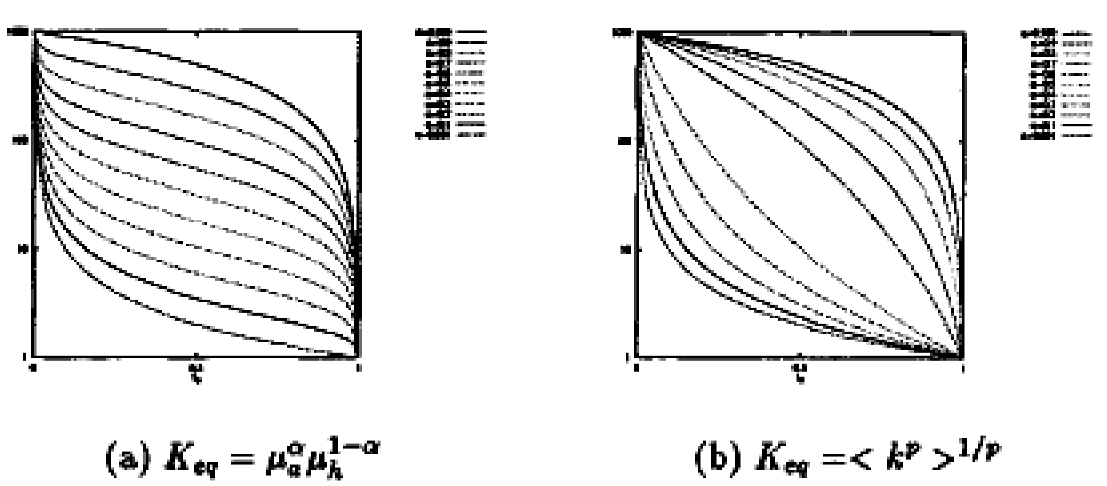
6.5 Are heuristic methods useful?

The two main heuristic methods are: power average and formulas for averaging means.

It must first be emphasized that the behaviours of these averages are very different. For example, for a binary medium with $k_0 = 1$ and $k_1 = 1000$, one can compare the model for averaging arithmetic and harmonic means with the power average p. We have plotted the resulting equivalent permeability as a function of f_0 (fraction of medium 0) for these two averages, for different values of α and p. Figure 17 clearly shows how the behaviours differ.

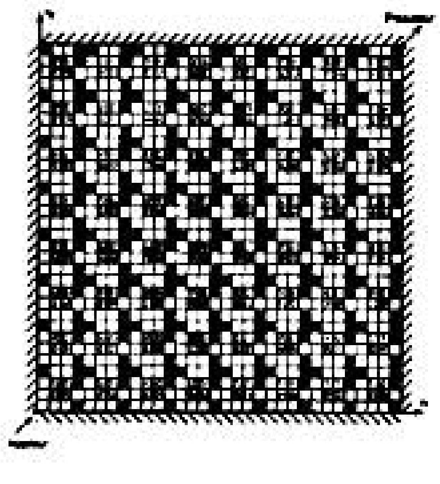
The numerical experiments carried out by Deutsch[27] on binary media demonstrate that when the proportion of low-permeability medium represents less than 40%, the power average expression can be correctly fitted on the numerical results. This author also shows that the percolation theory formula can be successfully fitted on the experimental points but that three parameters have to be fitted, whereas only one parameter, p, has to be calculated for the power average. He ends by recommending the use of power averages in most cases. This method has one limitation: some numerical calculations are required to fit the exponent p. Ababou[1] proposes the use of exponents that depend on the correlation lengths but it remains to be seen whether or not this can be done.

In addition to giving the mean value, the methods of averaging means have the advantage of providing a confidence interval: the two theoretical bounds used in the calculation of the average. The two most general bounds, which also give the most precise interval, are those of Cardwell and Parsons. These bounds are therefore preferable to those of Wiener. Njifenjou[86] compares the method of homogenization to the composition of Cardwell and Parson bounds. He observes

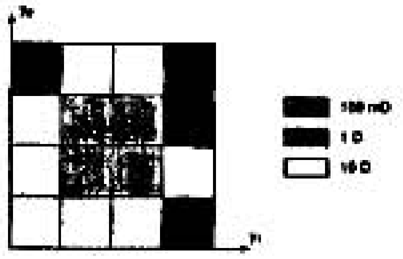


(a) $K_{eq} = \mu_a^\alpha \mu_h^{1-\alpha}$ (b) $K_{eq} = < k^p >^{1/p}$

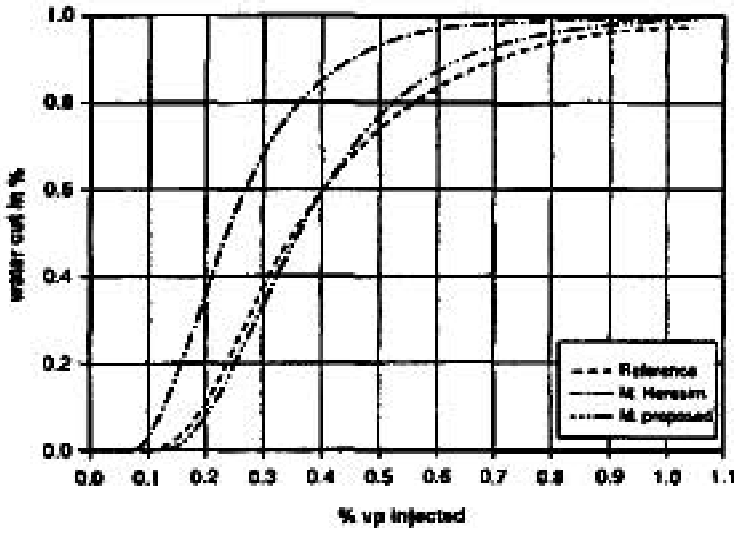
Fig. 17. Binary medium, with $k_0 = 1$ and $k_1 = 1000$. Comparison of two heuristic means, as a function of the proportion of phase zero in the medium. (a) Arithmetic and harmonic mean averages, for different values of the averaging parameter α, see eqn (12); (b) power averages, for different values of $p = 2\alpha - 1$.



(a) permeability field



(b) permeability coding



(c)

Fig. 18. Two-phase flow for a five-spot test on a periodic medium. (a) Permeability field; (b) permeability coding; (c) comparison of three water cut curves in percent, versus injected water volume. The three curves are: reference solution obtained on a fine-mesh grid (– – –) and curves obtained after up-scaling using the homogenization method (· · – · · – ··) and means averaging (– · – · –) (after Njifenjou).[87]

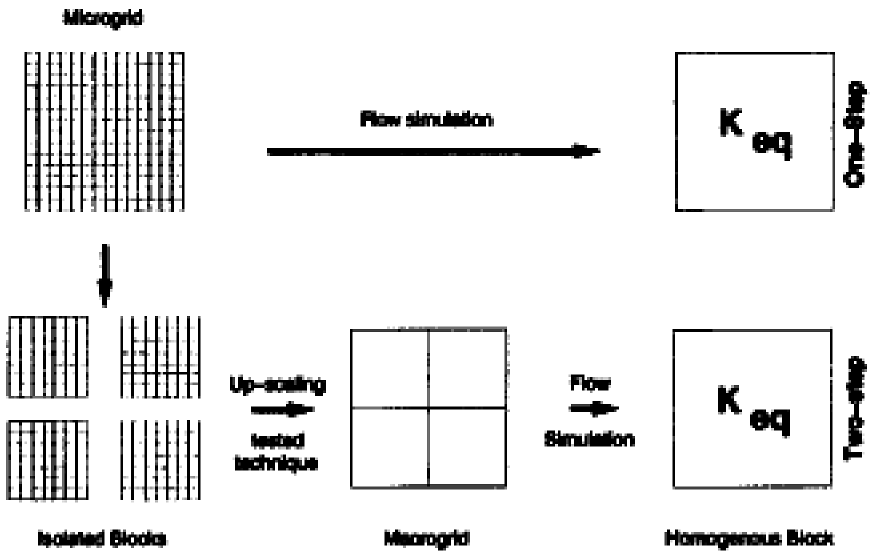
considerable errors on the large-scale fluxes (50%), whereas the maximum errors of the homogenization techniques or the direct numerical methods are on the order of 10% for random or deterministic media. Tests on two-phase flow in periodic media also prove, as expected, the superiority of the numerical method (see Fig. 18). However, here again one would have liked to see a comparison with approximation methods such as the method of renormalization. Finally, the formula proposed by Kruel-Romeu,[70] taking into account anisotropy, seems particularly interesting: the tests[28,70] are positive but the formula deserves more work.

To conclude, it must be emphasized that the heuristic methods are among the fastest and easiest to handle. As opposed to the theoretical analytical results, their fields of application are, *a priori*, wider. However, additional experiments are needed to accurately define their limits.

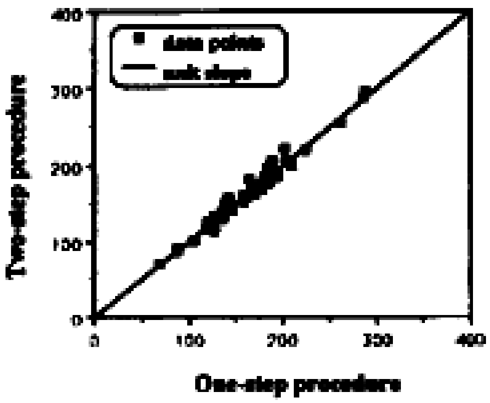
6.6 Which numerical method?

The first task is that of choosing between the numerical methods that solve the diffusion equation directly and those that solve equations on the ancillary variables obtained with homogeneous-equation techniques. The first ones assume that Darcy's law exists at the large scale; the second ones show that, under certain conditions (periodicity, dimension of the module much smaller than that of the domain), Darcy's law emerges at the large scale. The latter is a more rigorous approach than the former, but are the results different in practice?

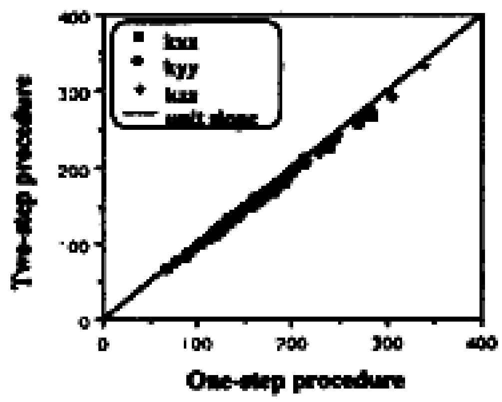
Using the binary medium studied by Bachu,[7] shown in Fig. 16, Dykaar and Kitanidis[33,34] obtained very similar results with the direct numerical method used by Bachu[7] and the method of moments. Njifenjou[86] compares several methods of the homogeneous-equation type with the direct numerical method used by Gallouët



(a) methodology



(b) permeameter-type boundary conditions



(c) periodic boundary conditions

Fig. 19. Comparison of the permeability obtained by the one-step numerical method and by the two-step numerical method while prescribing two types of boundary conditions on the intermediary blocks. The medium are three-dimensional with a correlated log-normal distribution of permeability (from Pickup *et al.*[91]).

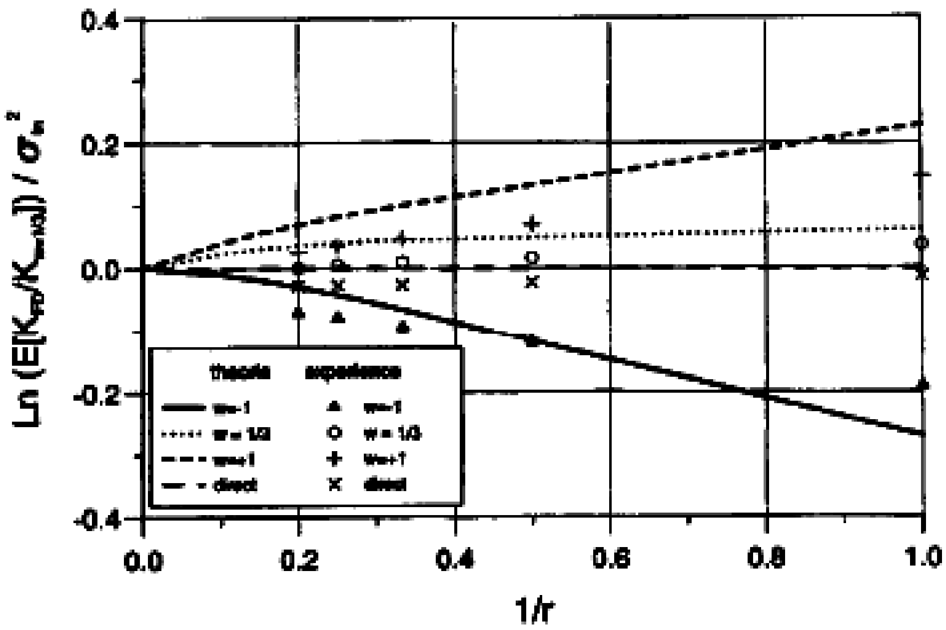
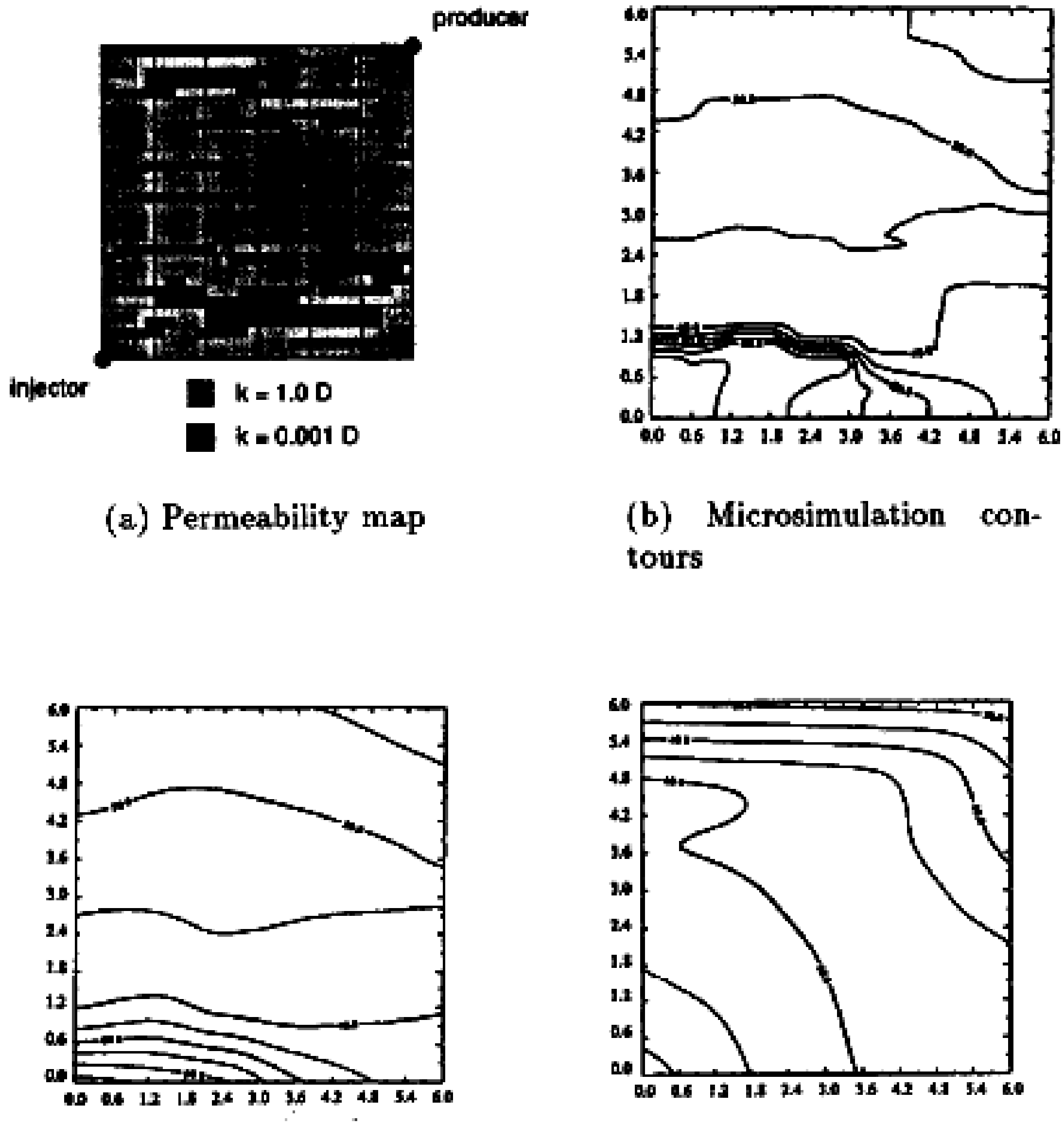


Fig. 20. Three-dimensional isotropic log-normal medium of variance 2. Relative error on the equivalent permeability as a function of the ratio of the mesh size of the grid to the block size for defining the permeability variations. The value of 1 corresponds to the case where the mesh size is equal to the block size for defining K, whereas a value of e.g. 0·2 corresponds to 5 meshes (in each direction) for calculating the head for each block of constant permeability (from Kruel-Romeu and Noetinger[71]).

and Guérillot[40] on random or deterministic media. He observes that these techniques produce similar results for the flux crossing the medium. The comparison is then extended to periodic media where the homogeneous-equation type technique proves superior to the direct numerical one. It must be remembered, however, that the tested numerical method used boundary conditions of a uniform type which are by no means the most suitable conditions for periodic media. What would the results have been of a comparison with Durlofsky's method?[30]

Finally, the techniques of the homogeneous-equation type are without doubt the most robust but to use them, one has to develop a code for solving numerically the boundary-value problem of the ancillary variables. The direct numerical methods can, by contrast, be used with already existing software for simulating single-phase flow. It seems, therefore, that direct numerical methods are preferable except in the special case of periodic media.

The second task is that of choosing the boundary conditions. Some authors work with conditions of the permeameter type, others prefer uniform or periodic conditions. It must be remembered that with permeameter-type conditions it is impossible to calculate a complete permeability tensor, that uniform conditions produce a permeability tensor that may be non-symmetric and that periodic conditions always give a symmetric tensor. Pickup *et al.*[91] have published a comparison of the different types of boundary conditions. Figure 19 shows the technique used in the comparisons and gives the results in two cases: permeameter-type and periodic boundary conditions. The tests carried out by Pickup *et al.*[91] demonstrate that the periodic conditions are the more robust.



(a) Permeability map

(b) Microsimulation contours

(c) GTS macrosimulation

(d) Macrosimulation without cross terms

Fig. 21. Comparison between general tensor scaling and a non-tensorial numerical method. (a) Binary medium, the solid lines define the large-mesh grid, the fine-mesh grid is made of squares with the size of the smallest layers; the flow domain is a section of a five-spot; (b) pressure field calculated on the fine-mesh grid; (c) pressure field calculated on the large-mesh grid with permeabilities given by the general tensor scaling; (d) pressure field calculated on the large-mesh grid with permeabilities given by a numerical method that does not take into account the complete tensor. From Fayers and Hewett.[38]

The main drawback of numerical techniques is that they can be biased.[70,72,97] We have already seen an example of this in King's results (Table 3). These biases are usually weak and can be neglected in the case of homogeneous media. However, they may become large in heterogeneous media and they are difficult to evaluate. Kruel-Romeu and Nœtinger[71] derive the bias of the finite difference method by an analytical calculation, in the case of a log-normal and a checker-board medium in two and three dimensions. The authors show that the bias depends on the method used to calculate the inter-block permeabilities and on the mesh size, when the numerical solution is calculated with a finer mesh than the one defining the permeability variations (Fig. 20). They also show that the only method that almost eliminates the bias is the direct formulation, where the inter-block permeabilities are directly assigned by the geologic model to the bonds between two nodes and not calculated from the values assigned to each node.

Finally, no discussion of numerical methods is complete without mentioning non-local approaches. The advantage of these techniques over the above-mentioned ones is that they take into account the neighbourhood of the block. The typical example is general tensor scaling (GTS).[111] Fayers and Hewett[38] have presented a test that demonstrates the superiority of GTS over a numerical method that neglects the non-diagonal terms of the permeability tensor (see Fig. 21). It would be also interesting to compare GTS to a direct method with, e.g. periodic boundary conditions, in order to evaluate the non-local effect only. Finally, note that with GTS, negative terms may appear on the diagonal of the permeability tensor.

7 CONCLUSION

The purpose of this review of the literature is to give an account of the methods that are currently used to calculate the equivalent permeability for uniform, single-phase, steady-state flow. We have tried to present as complete an inventory as possible of the different techniques, divided into three main categories: deterministic, stochastic and heuristic. These groups have proved to be complementary rather than antagonistic.

It is clear that the stochastic methods are the only ones capable of quantifying the uncertainty linked to a partial knowledge of the exact distribution of the permeability. However, the analytical developments carried out in a stochastic context are based on very restrictive assumptions such as stationarity or log-normal distribution of the permeabilities. Apart from these assumptions, the Monte-Carlo method stands out as the most direct approach with which to evaluate the uncertainty. It is here that the link is established with the deterministic methods: the principle of the Monte-Carlo method is to generate equally probable simulations of the geological model and to use deterministic techniques — upscaling and flow simulations — to evaluate the effect on the flow of the differences in permeability between the simulations of the medium. As to the heuristic methods, they represent an alternative to the deterministic methods. They have the quality of being extremely simple, fast and not limited by the memory space but they deserve to be tested further.

In the introduction, we distinguished between local and non-local methods. In recent studies, the non-local nature of block permeability has been made very clear, whether in a deterministic[57,111] or stochastic[58,85] context. Thus, block permeability is not unique, it depends on the boundary conditions and on the permeabilities inside and outside the block. This leads to a paradoxical situation where one has to know, *a priori*, the boundary conditions in order to determine the block permeability, consistent with the predicted flow conditions, while the model is mostly used to simulate situations where the boundary conditions change with time. This non-local aspect appears as one of the most interesting topics for future research. Another important line of research would be to take into account simultaneously measurements of permeabilities made on different scales.

From a practical point of view, the numerical non-local method called general tensor scaling[111] has the drawback of being very demanding in computer resources and cannot, at the moment, be used on real-size problems. The method that uses a local neighbourhood[46] is easier to handle.

Finally, when faced with a problem of upscaling, one should choose a technique providing block permeabilities rather than a uniform effective permeability. Except in special cases (log-normal or stratified media), a numerical method is preferable. If it is impossible to use a numerical technique (too many meshes), one should use renormalization while making sure that the case in question is not one where the errors are great (flattened meshes, strong heterogeneity). When choosing a numerical method, one must be aware of possible errors linked to the discretization and the numerical formulation. Periodic boundary conditions are a good choice as they seem to be the most robust.

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