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https://doi.org/10.1007/s11749-019-00681-3

INVITED PAPER

Deville and Särndal's calibration: revisiting a 25-years-old successful optimization problem

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Abstract
In 1992, in a famous paper, Deville and Särndal proposed the calibration method in order to adjust samples on known population totals. This paper had a very important impact in the theory and practice of survey statistics. In this paper, we propose a rigorous formalization of the calibration problem viewed as an optimization problem. We examine the main calibration functions and we discuss the question of the existence of solutions. We also propose an alternate way of solving the optimization problem given by the calibration principle. We finally present a set of simulations in order to compare the different methods.

Keywords Calibration · Estimation · Regression · Sampling · Survey · Weight

Mathematics Subject Classification 62-03 · 62D05

1 Introduction

In 1992, Jean-Claude Deville and Carl-Eric Särndal published an article entitled "Calibration Estimators in Survey Sampling" in the *Journal of the American Statistical Association* (Deville and Särndal 1992). This article was soon followed by a supplement one entitled "Generalized Raking Procedures in Survey Sampling" by Jean-Claude Deville, Carl-Erik Särndal and Olivier Sautory (Deville et al. 1993). This second article deals more specifically with calibration on marginal totals and variance estimation.

This invited paper is discussed in comments available at: http://dx.doi.org/10.1007/s11749-019-00682-2; http://dx.doi.org/10.1007/s11749-019-00683-1; http://dx.doi.org/10.1007/s11749-019-00684-0; http://dx.doi.org/10.1007/s11749-019-00686-y; http://dx.doi.org/10.1007/s11749-019-00687-x.

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Published online: 22 November 2019

Adjustment of statistical tables on marginal totals is a very old topic. Yule (1912, p. 589) already adjusted a table on marginal totals by the raking ratio method. The method was used to adjust sample data on census totals by Deming and Stephan (1940), Stephan (1942), Deming (1948, 1950). Hartley and Rao (1968) also adjusted sample data by a constrained likelihood method that was precursor to the empirical likelihood method. The development of the generalized regression estimator was done by Cassel et al. (1976); Särndal (1980); Fuller and Isaki (1981); Isaki and Fuller (1982); Bethlehem and Keller (1987). The publication of Deville and Särndal must also be put in the context of a French school in official statistics (see Thionet 1976; Froment and Lenclud 1976; Lemel 1976; Thionet 1959; Cholakian 1980; Madre 1980; Deville 1988; Deville and Särndal 1990) in the *Institut National de la Statistique et des Études Économiques* (INSEE) in which Jean-Claude Deville was the head of the Unit of Statistical Methodology.

The paper by Deville and Särndal had a considerable impact on survey statistics. For their contributions to the survey methodology, Carl-Erik Särndal and Jean-Claude Deville have received the Waksberg Award in 2007 and 2018, respectively. This prize is given in honour of Joseph Waksberg to recognize the contributions of a prominent survey statistician for her/his contributions to survey methodology.

The SAS® macros CALMAR and CALMAR2 implement several calibration methods. They were developed by Sautory (1993) and Le Guennec and Sautory (2002) and have been widely distributed. SPSS or R versions were also developed (Vanderhoeft 2001; Lumley 2010; Matei and Tillé 2016). Calibration methods are now applied on almost all surveys in official statistics. Why did this article have such an impact? We are convinced that the most important contribution is the focus on the calibration weights.

Cornfield (1944) had proposed an innovation that went a little unnoticed but greatly simplified the way the problem was viewed by separating the value taken by the variable of interest y on unit k from the random indicator variable a_k indicating membership in the sample. This new notation is now adopted in all the publications about survey sampling.

Deville and Särndal also clearly separated two aspects by making the distinction between the interest variable y_k and the weights w_k. This distinction allows us the calculation of a unique weighting system to estimate the parameters for all the variables of interest and allows a greatly simplified implementation. Indeed, it is enough to create a weighting system to be able to realize all the estimations. The practical aspect of this formulation was probably decisive in the dissemination of the method.

The reformulation in terms of weights directly opens the door to new questions: Are the calibrated weights negative, null or too large? Are the calibrated weights very different from the original ones? The estimates are the sum of the products $w_k y_k$, which clarifies the notion of extreme value. Indeed, an observation is extreme if the product $w_k y_k$ is an outlier. Calibration can also be used in an abusive way, for example by overcalibrating on too many variables for aesthetic reasons. It should be kept in mind that calibration is an estimation method and its interest should be evaluated in terms of the efficiency and robustness of the estimators it produces.

In what follows, we propose a formulation that we think is original to the problem by focusing on issues related to optimization. We list all the distances that have been proposed, and we discuss the existence of solutions including when limits are imposed

on weights. Section 2 is devoted to the general formulation of calibration estimators. A widely used method, namely the distance minimization approach, is then discussed in Sect. 3. The theory of generalized calibration is explained in Sect. 4.1. In Sect. 5, variance estimation is discussed and we conclude this paper with some numerical experiments.

2 Calibration estimators

Let us consider a finite population $\mathcal{U} = \{1, \ldots, k, \ldots, N\}$ together with a sampling design $p(\cdot)$, i.e. a probability measure on the set of all possible samples $s \subseteq \mathcal{U}$. We also define for $k, l \in s$ the inclusion probabilities $\pi_k = \sum_{s:k\in s} p(s)$ and $\pi_{kl} = \sum_{s:k,l\in s} p(s)$ of first and second orders, respectively. Throughout this paper, we assume that $\pi_k > 0$ for all $k, l \in s$. For a variable of interest y taking value y_k on unit $k \in \mathcal{U}$, our goal is to estimate the total

$$t_y = \sum_{k\in\mathcal{U}} y_k. \tag{1}$$

In order to simplify the notations, we use $\sum_A$ to denote $\sum_{k\in A}$ for any subset $A \subseteq \mathcal{U}$. We also use this notation for a set of variables indexed over A, i.e. we will write $(u_k)_A$ for $(u_k)_{k\in A}$, where u_k is a (known or unknown) variable associated with unit $k \in A$. Furthermore, vectors are denoted using boldface and matrices with capital letters and boldface. The letter t is used for totals and hat symbols to denote estimated values.

The classic estimator of (1) is the so-called Horvitz–Thompson (HT) or Horvitz–Thompson–Narain estimator defined as

$$\widehat{t}_{y\pi} = \sum_s \frac{y_k}{\pi_k} = \sum_s d_k y_k, \tag{2}$$

where $d_k = \pi_k^{-1}$ for $k \in s$. It is well known that $\widehat{t}_{y\pi}$ is design unbiased under the assumption that $\pi_k > 0$, i.e. $\mathbb{E}_p\left(\widehat{t}_{y\pi}\right) = t_y$ (Narain 1951; Horvitz and Thompson 1952), where $\mathbb{E}_p(\cdot)$ denotes the expectation with respect to the sampling design $p(\cdot)$.

The idea of calibration is to build a new set of (sample dependent) weights using auxiliary information in order to reduce the variance of the estimator. This method was formally introduced by Deville and Särndal (1992) and Deville et al. (1993) but several authors addressed similar issues before but without focussing on the weights (Lemel 1976; Huang and Fuller 1978; Bethlehem and Keller 1987; Deville 1988).

In order to perform calibration, let us assume that a vector of auxiliary information $\boldsymbol{x}_k = (x_{k1}, \ldots, x_{kj}, \ldots, x_{kp})^\top$ is observed for every unit $k \in s$ and furthermore that the vector of totals $\boldsymbol{t_x} = \sum_{\mathcal{U}} \boldsymbol{x}_k$ is accurately known. We would like to modify the HT weights d_k and build a set of weights $(w_k)_s$ satisfying the *calibration equations*

$$\sum_s w_k \boldsymbol{x}_k = \boldsymbol{t_x}. \tag{3}$$

The idea is that the new weights reflect a property which is known at the population level and can help improve the estimation of (1). In particular, one goal is to decrease the variance of the estimator. Based on this result, we then define the new calibrated estimator

$$\widehat{t}_{yw} = \sum_s w_k y_k. \tag{4}$$

We point out that the weights are sample dependent, meaning that for every new sample it is necessary to compute new $w_k = w_k(s)$. This is not the case of the original weights d_k which are defined beforehand. From (4), calibration can be seen as a linearly weighting method. The information $\boldsymbol{t_x}$ may come from different sources such as another survey or census data. A simple example is the case where $\boldsymbol{x}_k = (1, x_k)^\top$ and x is a binary categorical variable. In this case, the weights are adjusted to the population size and the number of persons in each category. Under this assumption, there exist infinitely many systems of weights $(w_k)_s$ satisfying (3) if $n > p$. Using the above notation, (3) can be written as $\boldsymbol{X}_s^\top \boldsymbol{w} = \boldsymbol{t_x}$, where $\boldsymbol{w} = (w_1, \ldots, w_n)^\top$. Note that the fact that the full matrix $\boldsymbol{X}_\mathcal{U}$ has rank p does not necessarily imply that $\boldsymbol{X}_s$ does. We point out that it is still possible to solve (3) if $\boldsymbol{X}_s$ does not have rank p as long as $\boldsymbol{t_x} \in \mathrm{Im}(\boldsymbol{X}_s^\top)$. The question is then how to choose "good" weights, and one possible method is presented in the next section.

3 The distance minimization approach

As discussed in the previous section, we use the calibrated estimator defined in (4) in place of the HT estimator defined in (2). Using this new set of weights, it is possible to obtain an estimator with a smaller variance than that of (2). However, it is not possible to guarantee that the design unbiasedness property still holds. The idea is then to choose weights close enough to the d_k in order to introduce only little bias in the new estimator. In order to do this, let us consider for $k \in \mathcal{U}$ a function $G_k(\cdot, d) : \mathbb{R} \to \mathbb{R}$ such that for every $d > 0$ there exists an open interval $\mathcal{D}_k(d)$ containing d and such that:

(A1) $G_k(\cdot, d) : \mathcal{D}_k(d) \to \mathbb{R}$ is nonnegative, differentiable, strictly convex and $G_k(d, d) = 0$;

(A2) the function

$$g_k(w, d) = \frac{\partial G_k}{\partial w}(w, d)$$

is continuous on $\mathcal{D}_k(d)$ and $g_k(\cdot, d) : \mathcal{D}_k(d) \to \mathcal{R}_k(d)$ is injective, where $\mathcal{R}_k(d) = \mathrm{Im}\,(g_k(\cdot, d))$.

Remark A function f from U to V is said to be injective if for all a and b in U, whenever $f(a) = f(b)$, then $a = b$. A function f from U to V is said to be bijective if each element of U is paired with exactly one element of V, and each element of the V is paired with exactly one element of U. The image of a function f on U is the set $\mathrm{Im}\ f = \{f(x), x \in U\}$.

These conditions imply that $g_k(d, d) = 0$ and that $g_k(\cdot, d)$ is strictly increasing. Let $F_k : \mathcal{R}_k \to \mathcal{D}_k$ be such that $d_k F_k : \mathcal{R}_k(d_k) \to \mathcal{D}_k(d_k)$ is the inverse of $g_k(\cdot, d_k)$,

where $\mathcal{R}_k(d_k) = \{d_k u \mid u \in \mathcal{R}_k\}$ and $\mathcal{D}_k(d_k) = \{d_k v \mid v \in \mathcal{D}_k\}$. Since $g_k(d_k, d_k) = 0$, we have $F_k(0) = 1$. Furthermore, let us assume that $F_k'(0) > 0$. We point out that F_k does not depend on d_k. In the sequel, we will use $\mathcal{D}_k(d_k)$ and $\mathcal{R}_k(d_k)$ to denote the intervals on which $G_k(\cdot, d_k)$ and $g_k(\cdot, d_k)$ satisfy the above assumptions.

The function G_k is a pseudo-distance since it is not symmetric and is used as a measure between the original and the new weights. We minimize the average distance $\sum_s G_k(w_k, d_k)$ under constraints (3), i.e. we want to solve

$$\min_{\boldsymbol{w} \in \mathcal{A}} \sum_s G_k(w_k, d_k) \tag{5}$$

where $\mathcal{A} = \mathcal{A}(\boldsymbol{t_x}, \boldsymbol{X}_s) = \left\{\boldsymbol{v} \in \mathbb{R}^n \mid \boldsymbol{X}_s^\top \boldsymbol{v} = \boldsymbol{t_x}\right\}$ denotes the set of admissible solutions of (3). In order to do this, let us define the Lagrangian

$$\mathcal{L}(\boldsymbol{w}, \boldsymbol{d}, \boldsymbol{\lambda}) = \sum_s G_k(w_k, d_k) - \boldsymbol{\lambda}^\top \left(\sum_s w_k \boldsymbol{x}_k - \boldsymbol{t_x} \right). \tag{6}$$

Deriving (6) with respect to w_k for some $k \in s$ yields

$$0 = \frac{\partial \mathcal{L}(\boldsymbol{w}, \boldsymbol{d}, \boldsymbol{\lambda})}{\partial w_k} = g_k(w_k, d_k) - \boldsymbol{\lambda}^\top \boldsymbol{x}_k.$$

Using the above assumptions, if $\boldsymbol{\lambda}^\top \boldsymbol{x}_k \in \mathcal{R}_k(d_k)$, this equation admits a unique solution defined through

$$w_k = d_k F_k \left(\boldsymbol{\lambda}^\top \boldsymbol{x}_k \right). \tag{7}$$

Replacing w_k in (3) yields a nonlinear system of p equations and p unknowns, namely

$$\boldsymbol{t_x} = \sum_s d_k F_k \left(\boldsymbol{\lambda}^\top \boldsymbol{x}_k \right) \boldsymbol{x}_k. \tag{8}$$

The goal is then to find $\boldsymbol{\lambda}$ satisfying this equation. We point out that the existence of such $\boldsymbol{\lambda}$ is not guaranteed. It depends on $\boldsymbol{t_x}$, $\boldsymbol{X}_s$ and $\mathcal{D}_k(d_k)$. This question together with that of solving Eq. (8) is addressed in Sect. 3.2. Once $\boldsymbol{\lambda}$ is determined, the weights are computed using (7). Deville and Särndal proved that estimator (4) is asymptotically design unbiased and design consistent in the sense of Fuller and Isaki (1981) and Isaki and Fuller (1982).

Remark 1 Under assumptions (A1)–(A2), G_k defines a unique function F_k. We see from (7) that the properties of w_k are a consequence of those of F_k. Hence, it might be desirable to choose F_k instead of G_k. We show here that under some assumptions on F_k, we can define equivalently a pseudo-distance function G_k for a prescribed F_k. Let $F_k : \mathcal{R}_k \to \mathrm{Im}(F_k)$ be a continuous and strictly increasing function such that $F_k(0) = 1 \in \mathrm{Im}(F_k)$, where $0 \in \mathrm{int}(\mathcal{R}_k)$. It follows that F_k is bijective and we can define $g_k(\cdot, d_k) : \mathcal{D}_k(d_k) \to \mathcal{R}_k(d_k)$ as the inverse of $d_k F_k$, where $\mathcal{D}_k(d_k) = d_k \mathrm{Im}(F_k)$. This function is also continuous and strictly increasing. For $w \in \mathcal{D}_k(d_k)$, let

$G_k(w, d_k) = \text{sign}\,(w - d_k) \int_{d_k}^{w} g_k(r, d_k)dr$. Then, $G_k(\cdot, d_k) \in \mathcal{C}^1(\mathcal{D}_k(d_k))$ is strictly convex since its derivative is strictly increasing and $G_k(d_k, d_k) = 0$. Finally, since $G_k(\cdot, d_k)$ is strictly convex and $g_k(d_k, d_k) = 0$, we have that it attains its minimum at d_k and so it is nonnegative. Hence, $G_k(\cdot, d_k)$ satisfies assumptions (A1) and (A2). It follows that we can define interchangeably G_k and F_k. Choosing G_k, we can gain interpretability of the distance function. On the other hand, since the new weights satisfying the calibration equations (3) are defined through (7), choosing F_k can allow us to obtain weights w_k satisfying some desirable properties.

Weights $q_k > 0$ are tuning parameters. In most of the cases, $q_k = 1, k \in s$. However, if the population is assumed to be generated from a linear model with heteroscedastic error terms, the q_k can be used to take the heteroscedasticity factor into account and to choose F_k such that $F_k'(0) = q_k$. For instance, heteroscedasticity often appears in business surveys or in situations where the sizes of the units are very contrasted. An alternative consists in using $q_k = d_k - 1$, which avoids the modification of the units selected by a very large inclusion probability that is desirable.

Remark 2 As we will see later, it is desirable to obtain positive weights which are not too large. Hence, an "ideal" function should take positive values and not increase too fast. We then look for functions $d_k F_k : \mathcal{R}_k(d_k) \to \mathcal{D}_k(d_k)$ satisfying the following properties:

1. $\mathcal{R}_k(d_k) = \mathbb{R}$;
2. $\mathcal{D}_k(d_k) \subset \mathbb{R}_+$ for all $d_k > 0$;
3. $\left|F_k'(z)\right| \leq C_k$ for all $z \in \mathcal{R}_k$ and some constant $C_k > 0$.

Assumption 1 is used for computational reasons as we will see in Sect. 3.2. By assumptions (A1)–(A2), we have that F_k is strictly increasing and bijective. We point out that the linear (Sect. 3.1.1) and raking ratio (Sect. 3.1.2) methods do not satisfy 2 and 3, respectively. Hence, the most widely used methods in practice are not based on "ideal" functions.

In the following section, we present some of the distance functions that can be found in the literature and discuss some of their features. Davies et al. (2015) discussed the effect of the different calibration functions on the weights.

3.1 Different distances

As already mentioned before, the distance minimization approach relies on the choice of the distance function G_k (or equivalently the function F_k). The idea is that intuitively, if the weights w_k are close to the original weights d_k, then we only introduce a little bias in the estimator $\widehat{t}_{yw}$. Over the years, different functions were considered. However, the most widely used in practice were already introduced and discussed in Deville and Särndal (1992).

In many cases, the function is of the form $G_k(w, d) = (d/q_k)G\,(w/d)$ for some given factor q_k. In that setting, we define $g = G'$ and $F = g^{-1}$. Then, $g_k(w, d) = (1/q_k)g\,(w/d)$. The constant q_k is defined beforehand and is used to consider unequal weighting on certain units. In most applications, $q_k = 1$ is considered for all $k \in s$ but in some cases it is useful to choose unequal factors (see Remark 5).

3.1.1 Linear method

The most widely used method in practice is the so-called *linear method* or *Chi-square method*. Its name comes from the fact that both the functions g and F are linear. In that case, we have $G_{\text{lin}}(t) = (1/2)(t-1)^2$ for all $t \in \mathbb{R}$. It yields for $u, w \in \mathbb{R}$

$$G_{k,\text{lin}}(w,d) = \frac{(w-d)^2}{2dq_k}, \qquad g_{k,\text{lin}}(w,d) = \frac{w-d}{dq_k}, \qquad F_{k,\text{lin}}(u) = 1 + q_k u.$$

This distance is a weighted ℓ^2-norm which measures how much the ratio w_k/d_k deviates from 1. For this method, it holds $\mathcal{R}_k(d_k) = \mathcal{D}_k(d_k) = \mathbb{R}$ for all $d_k > 0$. This implies that it always admits a solution. Furthermore, it is possible to derive an explicit formula for this solution. Indeed, from (7), we have

$$w_k = d_k F_{k,\text{lin}}\left(\boldsymbol{\lambda}^\top \boldsymbol{x}_k\right) = d_k\left(1 + q_k \boldsymbol{\lambda}^\top \boldsymbol{x}_k\right).$$

Inserting this in (3) yields

$$\boldsymbol{t}_{\boldsymbol{x}} = \sum_s w_k \boldsymbol{x}_k = \sum_s d_k \boldsymbol{x}_k + \left(\sum_s d_k q_k \boldsymbol{x}_k \boldsymbol{x}_k^\top\right)\boldsymbol{\lambda} = \widehat{\boldsymbol{t}}_{\boldsymbol{x}\pi} + \boldsymbol{T}_s \boldsymbol{\lambda}, \tag{9}$$

so that $\boldsymbol{\lambda} = \boldsymbol{T}_s^{-1}\left(\boldsymbol{t}_{\boldsymbol{x}} - \widehat{\boldsymbol{t}}_{\boldsymbol{x}\pi}\right)$ and

$$w_k = d_k\left(1 + q_k \boldsymbol{\lambda}^\top \boldsymbol{x}_k\right) = d_k\left[1 + \left(\boldsymbol{t}_{\boldsymbol{x}} - \widehat{\boldsymbol{t}}_{\boldsymbol{x}\pi}\right)^\top \boldsymbol{T}_s^{-1} q_k \boldsymbol{x}_k\right],$$

where $\boldsymbol{T}_s = \left(\sum_s d_k q_k \boldsymbol{x}_k \boldsymbol{x}_k^\top\right)$ and $\widehat{\boldsymbol{t}}_{\boldsymbol{x}\pi} = \sum_s d_k \boldsymbol{x}_k$. Finally, we obtain

$$t_{yw,\text{lin}} = \widehat{t}_{y\pi} + \left(\boldsymbol{t}_{\boldsymbol{x}} - \widehat{\boldsymbol{t}}_{\boldsymbol{x}\pi}\right)^\top \left(\sum_s d_k q_k \boldsymbol{x}_k \boldsymbol{x}_k^\top\right)^{-1} \sum_s d_k q_k y_k \boldsymbol{x}_k. \tag{10}$$

The matrix $\boldsymbol{T}_s$ can be written as $\boldsymbol{T}_s = \boldsymbol{X}_s^\top \boldsymbol{D}\boldsymbol{Q}\boldsymbol{X}_s$, where $\boldsymbol{D}$ and $\boldsymbol{Q}$ are diagonal matrices containing the elements d_k and q_k on the diagonal, respectively. Since $d_k, q_k > 0$ and $\boldsymbol{X}_s$ is assumed to have rank p, $\boldsymbol{T}_s$ is a symmetric positive definite matrix and so is invertible. This formula makes the linear method very attractive because the minimization is already solved theoretically. Furthermore, $\boldsymbol{T}_s \in \mathbb{R}^{p\times p}$ and since $p < n$ is in general small, it is possible to solve system (9) at very low computational costs. If the full rank assumption on $\boldsymbol{X}_s$ is not fulfilled, the method can still be applied but a generalized inverse needs to be used, as, for instance, in the SAS® macro CALMAR2 (Le Guennec and Sautory 2002) or the `calib` function of the R `sampling` package (Tillé and Matei 2016).

This estimator is in direct relation to the generalized regression estimation (GREG) and more precisely the linear GREG (Särndal et al 1992, Section 6.4). In that frame-

work, we consider the estimator

$$\widehat{t}_{y,\mathrm{GREG}} = \widehat{t}_{y\pi} + \left(\sum_{\mathcal{U}} \widehat{y}_k - \sum_{s} d_k \widehat{y}_k \right),$$

where $\widehat{y}_k$ is a predicted value of y_k based on the known auxiliary information $\boldsymbol{x}_k$ for $k \in \mathcal{U}$. More precisely, let us assume that there is a linear model ξ such that $\mathbb{E}_\xi(y_k) = \boldsymbol{\beta}^\top \boldsymbol{x}_k$ and $\mathrm{Var}_\xi(y_k) = \sigma_k^2$. Using least squares fit with $q_k = 1/\sigma_k^2$, we obtain predicted values of y_k given by $\widehat{y}_k = \widehat{\boldsymbol{b}}_s^\top \boldsymbol{x}_k$, where

$$\widehat{\boldsymbol{b}}_s = \left(\sum_{s} d_k q_k \boldsymbol{x}_k \boldsymbol{x}_k^\top \right)^{-1} \sum_{s} d_k q_k y_k \boldsymbol{x}_k.$$

This yields

$$\widehat{t}_{y,\mathrm{GREG}} = \widehat{t}_{y\pi} + \left(\sum_{\mathcal{U}} \boldsymbol{x}_k - \sum_{s} d_k \boldsymbol{x}_k \right)^\top \widehat{\boldsymbol{b}}_s = \widehat{t}_{yw,\mathrm{lin}}.$$

This result also helps us interpret the estimator obtained by the linear method due to the underlying model. A more complete discussion on the links and differences between the GREG and the calibration approaches can be found in Särndal (2007, Sections 3 and 4).

Remark 3 In Result 5 p.379, Deville and Särndal (1992) show that if the population fulfils some superpopulation assumptions as in Fuller and Isaki (1981); Isaki and Fuller (1982), then all estimators obtained from other distances [satisfying assumptions (A1) and (A2)] are asymptotically equivalent to (10). This is a consequence of the fact that linearizing F_k around zero yields $F_{k,\mathrm{lin}}$. All these results make the linear method very attractive and popular. However, one drawback which is often pointed out is that since $\mathrm{Im}(F_k) = \mathbb{R}$, it is possible to obtain negative weights which is commonly seen as undesirable.

Remark 4 The pseudo-distance associated with the linear method allows us to give an upper bound for the bias if $y_k \geq 0$ and $q_k = 1$ for all $k \in s$. In that particular case, we have

$$\begin{aligned}
\left| \mathbb{E}_p\left(\widehat{t}_{yw,\mathrm{lin}}\right) - t_y \right| = \left| \mathbb{E}_p\left(\widehat{t}_{yw,\mathrm{lin}} - \widehat{t}_{y\pi}\right) \right| &\leq \mathbb{E}_p \left[\sum_{s} |d_k - w_k| \, y_k \right] \\
&\leq \mathbb{E}_p \left\{ \left[\sum_{s} 2 d_k y_k \right]^{\frac{1}{2}} \left[\sum_{s} \frac{(d_k - w_k)^2}{2 d_k} y_k \right]^{\frac{1}{2}} \right\} \\
&\leq \left\{ \mathbb{E}_p \left[\sum_{s} 2 d_k y_k \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E}_p \left[\sum_{s} \frac{(d_k - w_k)^2}{2 d_k} y_k \right] \right\}^{\frac{1}{2}}
\end{aligned}$$

$$\leq \sqrt{2t_y \max(y_k)} \left\{ \mathbb{E}_p \left[\sum_s G_k(w_k, d_k) \right] \right\}^{\frac{1}{2}},$$

where we have used Cauchy–Schwarz inequality on the expectation and $0 \leq y_k \leq \max(y_k)$ for all $k \in s$. Hence, the relative bias can be bounded as

$$\left| \frac{\mathbb{E}_p\left(\widehat{t}_{yw,\mathrm{lin}}\right) - t_y}{t_y} \right| \leq \sqrt{\frac{2\max(y_k)}{t_y}} \mathbb{E}_p \left[\sum_s G_k(w_k, d_k) \right]^{\frac{1}{2}}.$$

This bound is not computationally useful but gives an intuition to the use of the distance minimization approach. Indeed, we see that if the new weights are close to the original ones with respect to the pseudo-distance considered, then the resulting bias is negligible.

Remark 5 In practice, the case $q_k = 1$ for all $k \in s$ is in general considered. However, choosing unequal q_k can be justified in some situations. A classic example (Deville and Särndal 1992, Example 1) is the case $p = 1$ and $q_k = 1/x_k$ for all $k \in s$. In that setting, we have $\boldsymbol{T}_s = \sum_s d_k q_k x_k x_k = \sum_s d_k x_k = \widehat{t}_{x\pi}$, so that

$$\lambda = \boldsymbol{T}_s^{-1}(\boldsymbol{t}_{\boldsymbol{x}} - \widehat{\boldsymbol{t}}_{\boldsymbol{x}\pi}) = \frac{1}{\sum_s d_k x_k} \left(\sum_{\mathcal{U}} x_k - \sum_s d_k x_k \right) = \frac{\sum_{\mathcal{U}} x_k}{\sum_s d_k x_k} - 1 = \frac{t_x}{\widehat{t}_{x\pi}} - 1.$$

Then,

$$w_k = d_k(1 + q_k x_k \lambda) = d_k(1 + \lambda) = d_k \frac{t_x}{\widehat{t}_{x\pi}},$$

which yields the so-called ratio estimator $\widehat{t}_{yw} = t_x \widehat{t}_{y\pi} / \widehat{t}_{x\pi}$.

3.1.2 Raking ratio

Based on the Kullback–Leibler information, the *raking ratio method* is also widely used. It is obtained by taking $G_{\mathrm{rak}}(t) = t\log(t) - t + 1$ and the associated pseudo-distance for $w \in \mathbb{R}_+^*$

$$G_{k,\mathrm{rak}}(w, d) = \frac{1}{q_k} \left[w \log\left(\frac{w}{d}\right) + d - w \right].$$

This yields for $w \in \mathbb{R}_+^*$ and $u \in \mathbb{R}$

$$g_{k,\mathrm{rak}}(w, d) = \frac{1}{q_k} \log\left(\frac{w}{d}\right), \qquad F_{k,\mathrm{rak}}(u) = \exp(q_k u).$$

This circumvents the problem of negative weights since $\mathcal{D}(d_k) = \mathbb{R}_+^*$. In this case, we have $\mathcal{R}_k(d_k) = \mathbb{R}$. The problem is that the exponential functional becomes very steep so that the new weights are likely to be very large, whence creating potential

outliers. Due to the multiplicative property of the exponential function, this method is also known as the *multiplicative method*.

Remark 6 The multiplicative property of the exponential function is especially useful when calibration is performed over known marginal counts in a frequency table as explained in Deville et al. (1993). In this particular case, the vector $\boldsymbol{x}_k$ is composed of 0 everywhere except for a 1 entry that indicates to which cell unit $k \in s$ belongs.

3.1.3 Minimum entropy

This method is obtained by taking the inverse Kullback–Leibler information. The pseudo-distance is defined for $w \in \mathbb{R}_+^*$ as

$$G_{k,\text{ent}}(w, d) = \frac{1}{q_k}\left[d \log\left(\frac{d}{w}\right) + w - d\right].$$

This gives for $w \in \mathbb{R}_+^*$ and $u \in (-\infty, 1/q_k)$

$$g_{k,\text{ent}}(w, d) = \frac{1}{q_k}\left(1 - \frac{d}{w}\right), \qquad F_{k,\text{ent}}(u) = \frac{1}{1 - q_k u}.$$

In that case, we have $\mathcal{R}_k(d_k) = \mathcal{R}_k = (-\infty, 1/q_k)$ and $\mathcal{D}_k(d_k) = \mathbb{R}_+^*$. Furthermore, the weights are given in this case by

$$w_{k,\text{ent}} = d_k F_{k,\text{ent}}\left(\boldsymbol{\lambda}^\top \boldsymbol{x}_k\right) = \frac{d_k}{1 - q_k \boldsymbol{\lambda}^\top \boldsymbol{x}_k}. \tag{11}$$

We point out that this method tends to deliver even more extreme weights than the raking ratio method. This can be seen in Fig. 1. Furthermore, the term $1 - q_k \boldsymbol{\lambda}^\top \boldsymbol{x}_k$ in the denominator can lead to numerical problems as explained in Sect. 3.2.2.

3.1.4 Generalized pseudo-distance

Deville and Särndal (1992) mention a generalized pseudo-distance. The linear method, the minimum entropy method and the raking ratio methods are covered by this parametrized function. Define for $\alpha \in \mathbb{R}$, $w \in \mathbb{R}$ and $d > 0$

$$G_k^\alpha(w, d) = \begin{cases} \dfrac{1}{\alpha(\alpha - 1)q_k}\left[\dfrac{|w|^\alpha}{d^{\alpha-1}} + (\alpha - 1)\, d - \alpha w\right] & \alpha \in \mathbb{R}\backslash(0, 1), \\ \dfrac{1}{q_k}\left[|w| \log\left(\dfrac{|w|}{d}\right) + d - w\right] & \alpha = 1, \\ \dfrac{1}{q_k}\left[d \log\left(\dfrac{d}{|w|}\right) + w - d\right] & \alpha = 0. \end{cases}$$

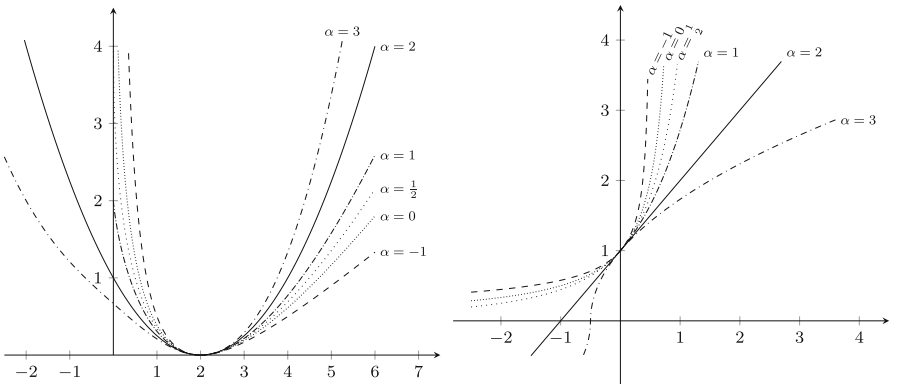

Fig. 1 Function associated with the generalized pseudo-distance for several values of α. The parameters are $d_k = 2$ and $q_k = 1$. The function $G_k^\alpha(\cdot, d_k)$ is presented in **a** while F_k^α is depicted in **b**

Table 1 Domain and codomain of the function $g_k^\alpha(\cdot, d_k)$ for different values of α

α	$\mathcal{D}_k(d_k)$	$\mathcal{R}_k(d_k)$
$]1, \infty[$	$\mathbb{R}$	$\mathbb{R}$
1	$\mathbb{R}_+^*$	$\mathbb{R}$
$]-\infty, 1[$	$\mathbb{R}_+^*$	$\left]-\infty, \frac{1}{(1-\alpha)q_k}\right[$

Considering these restricted intervals, the functions $G_k^\alpha(\cdot, d_k)$ and $g_k^\alpha(\cdot, d_k)$ fulfil (A1)–(A2)

This yields for $u, w \in \mathbb{R}$

$$g_k^\alpha(w, d) = \begin{cases} \dfrac{1}{(\alpha-1)q_k}\left[\dfrac{\text{sign}(w)\,|w|^{\alpha-1}}{d^{\alpha-1}} - 1\right] & \alpha \in \mathbb{R}\backslash 1, \\ \dfrac{\text{sign}(w)}{q_k}\log\left(\dfrac{|w|}{d}\right) & \alpha = 1, \end{cases}$$

$$F_k^\alpha(u) = \begin{cases} \text{sign}\left[1 + q_k u\,(\alpha-1)\right]\left|1 + q_k u\,(\alpha-1)\right|^{\frac{1}{\alpha-1}} & \alpha \in \mathbb{R}\backslash 1, \\ \exp\left(q_k u\right) & \alpha = 1. \end{cases}$$

The linear method is obtained for $\alpha = 2$ while the multiplicative method corresponds to $\alpha = 1$. We point out that the function is defined so that $\mathcal{D}_k(d_k) = \mathbb{R}$ for all $d_k > 0$. However, $G_k^\alpha(\cdot, d)$ does not satisfy assumption (A1) in that setting. In order to obtain the desired properties, we consider the domains and codomains presented in Table 1 for the function $g_k^\alpha(\cdot, d_k)$.

As already pointed out, several methods fit in this generalized pseudo-distance framework for some specific choices of α. We present in Table 2 some of these methods and the associated parameter. In Fig. 1, the functions $G_k^\alpha(\cdot, d_k)$ and F_k are depicted for several values of α. The domains $\mathcal{D}_k(d_k)$ and $\mathcal{R}_k(d_k)$ can be seen from both plots.

Table 2 Some well-known methods fitting in the framework of the generalized pseudo-distance approach for different values of the parameter α

α	$G_k^\alpha(w,d)$	$g_k^\alpha(w,d)$	$F_k^\alpha(u)$	Method
2	$\frac{(w-d)^2}{2dq_k}$	$\frac{w-d}{dq_k}$	$1+q_ku$	Linear
1	$\frac{1}{q_k}[w\log(\frac{w}{d})+d-w]$	$\frac{1}{q_k}\log(\frac{w}{d})$	$\exp(q_ku)$	Raking ratio
$\frac{1}{2}$	$\frac{2}{q_k}(\sqrt{w}-\sqrt{d})^2$	$\frac{2}{q_k}(1-\sqrt{\frac{d}{w}})$	$(1-\frac{q_ku}{2})^{-2}$	Hellinger distance
0	$\frac{1}{q_k}[d\log(\frac{d}{w})+w-d]$	$\frac{1}{q_k}(1-\frac{d}{w})$	$\frac{1}{1-q_ku}$	Minimum entropy
-1	$\frac{(w-d)^2}{2dq_k}$	$\frac{1}{2q_k}[1-(\frac{d}{w})^2]$	$(1-2q_ku)^{-\frac{1}{2}}$	Inverse chi-square

3.1.5 Deville's calibration function

The linear and raking ratio methods are the basic calibration techniques. However, one drawback is that they might deliver either negative or very large weights, respectively. Many efforts have been made over the years to develop methods yielding reasonable weights, in the sense that they are neither negative nor very large. Due to (7), the properties of w_k are a direct consequence of those of F_k. Hence, the challenge is to find a function F_k satisfying the assumptions of Remark 2. During a private meeting, Jean-Claude Deville proposed to use the function

$$F_{k,\mathrm{Dev}}(u) = q_ku + \sqrt{1+q_k^2u^2}, \qquad u \in \mathbb{R}.$$

The associated functions are defined for $w \in \mathbb{R}_+^*$ as

$$g_{k,\mathrm{Dev}}(w,d) = \frac{d}{q_k}\left[\frac{(w/d)^2-1}{2w}\right], \qquad G_{k,\mathrm{Dev}}(w,d) = \frac{d}{4q_k}\left[\left(\frac{w}{d}\right)^2 - 1 - 2\ln\left(\frac{w}{d}\right)\right].$$

We see that these functions fit the framework presented at the beginning of Sect. 3.1, namely $g_{k,\mathrm{Dev}}(w,d) = (1/q_k)g_{\mathrm{Dev}}(w/d)$ and $G_{k,\mathrm{Dev}}(w,d) = (w/q_k)G_{\mathrm{Dev}}(w/d)$ with

$$g_{\mathrm{Dev}}(z) = \frac{z^2-1}{2z}, \qquad G_{\mathrm{Dev}}(z) = \frac{1}{2}\left[\frac{z}{2} - \frac{1}{2} - \ln(z)\right].$$

Furthermore, in this case we have $\mathcal{R}_k(d_k) = \mathbb{R}$ and $\mathcal{D}_k(d_k) = \mathbb{R}_+^*$ for all $d_k > 0$.

Since $\lim_{u\to\infty}[F_k(u) - 2q_ku] = 0$, we have that F_k behaves asymptotically as $2q_ku$, meaning that it increases linearly away from zero. As a consequence, this method does not tend to give extremely large weights. Furthermore, since $F_k(u) > q_ku + \sqrt{q_k^2u^2} = q_ku + |q_ku| \geq 0$, this method always yields strictly positive weights. In particular, we see that this method satisfies all three assumptions of Remark 2 in order to be an "ideal" calibration function.

3.1.6 Roy and Vanheuverzwyn's method

Roy and Vanheuverzwyn (2001) present a parametrized pseudo-distance which yields an "ideal" calibration function in the sense of Remark 2. Given $\alpha \in \mathbb{R}^*$, let us define $G^{\alpha}_{\mathrm{RV}}(z) = (1/2\alpha)\int_1^z \sinh\left[\alpha\left(t - (1/t)\right)\right] dt$, where $\sinh(t) = (1/2)(e^t - e^{-t})$. This gives

$$G^{\alpha}_{k,\mathrm{RV}}(w,d) = G^{\alpha}_{\mathrm{RV}}\left(\frac{w}{d}\right)\frac{d}{q_k} = \frac{d}{q_k}\int_1^{\frac{w}{d}} \frac{1}{2\alpha}\sinh\left[\alpha\left(t - \frac{1}{t}\right)\right] dt.$$

The associated functions are

$$g^{\alpha}_{k,\mathrm{RV}}(w,d) = \frac{1}{2\alpha}\sinh\left[\alpha\left(\frac{w}{d} - \frac{d}{w}\right)\right],$$

$$F^{\alpha}_{k,\mathrm{RV}}(u) = \frac{\operatorname{arsinh}(2\alpha q_k u)}{2\alpha} + \sqrt{1 + \left(\frac{\operatorname{arsinh}(2\alpha q_k u)}{2\alpha}\right)^2} = F_{k,\mathrm{Dev}}\left[\frac{\operatorname{arsinh}(2\alpha q_k u)}{2\alpha q_k}\right],$$

where $\operatorname{arsinh}(t) = \ln\left(t + \sqrt{1+t^2}\right)$ for $t \in \mathbb{R}$. The functions $G^{\alpha}_{k,\mathrm{RV}}(\cdot, d)$, $g^{\alpha}_{k,\mathrm{RV}}(\cdot, d)$ and $F^{\alpha}_{k,\mathrm{RV}}$ are even with respect to α, i.e. $F^{\alpha}_{k,\mathrm{RV}} = F^{-\alpha}_{k,\mathrm{RV}}$ for all $\alpha \in \mathbb{R}^*$. It holds

$$\begin{aligned}\frac{dF^{\alpha}_{k,\mathrm{RV}}}{du}(u) &= \frac{q_k}{\sqrt{1+(2\alpha q_k u)^2}}\left(1 + \frac{\frac{\operatorname{arsinh}(2\alpha q_k u)}{2\alpha}}{\sqrt{1+\left(\frac{\operatorname{arsinh}(2\alpha q_k u)}{2\alpha}\right)^2}}\right)\\ &> \frac{q_k}{\sqrt{1+(2\alpha q_k u)^2}}\left[1 + \frac{\frac{\operatorname{arsinh}(2\alpha q_k u)}{2\alpha}}{\left|\frac{\operatorname{arsinh}(2\alpha q_k u)}{2\alpha}\right|}\right] \geq 0.\end{aligned}$$

Hence, $F^{\alpha}_{k,\mathrm{RV}}$ is strictly increasing, continuous and since

$$\lim_{u\to\pm\infty}\operatorname{arsinh}(u) = \pm\infty, \qquad \lim_{u\to-\infty} F_{k,\mathrm{Dev}}(u) = 0, \qquad \lim_{u\to\infty} F_{k,\mathrm{Dev}}(u) = \infty,$$

we obtain by continuity

$$\lim_{u\to-\infty} F^{\alpha}_{k,\mathrm{RV}}(u) = 0, \qquad \lim_{u\to\infty} F^{\alpha}_{k,\mathrm{RV}}(u) = \infty.$$

The domain and codomain of $F^{\alpha}_{k,\mathrm{RV}}$ are $\mathcal{R}_k(d_k) = \mathbb{R}$ and $\mathcal{D}_k(d_k) = \mathbb{R}^*_+$ for all $d_k > 0$, respectively. Whence $F_{k,\mathrm{RV}}$ satisfies all the assumption of Remark 1. Furthermore,

$$\left|\frac{dF^{\alpha}_{k,\mathrm{RV}}}{du}(u)\right| \leq \frac{2q_k}{\sqrt{1+(2\alpha q_k u)^2}} \leq 2q_k,$$

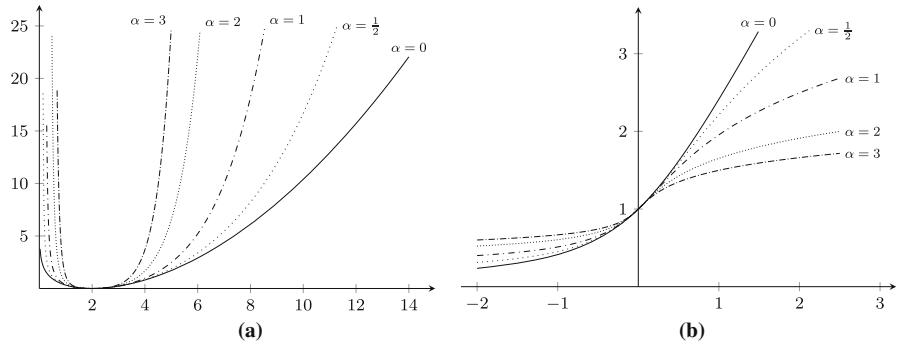

Fig. 2 Function associated with the Roy and Vanheuverzwyn pseudo-distance for several values of $\alpha > 0$. The parameters are $d_k = 2$ and $q_k = 1$. The function $G^{\alpha}_{k,\mathrm{RV}}(\cdot, d_k)$ is presented in **a** while $F^{\alpha}_{k,\mathrm{RV}}$ is depicted in **b**

so that this method satisfies all three properties presented in Remark 2. More precisely, it can be shown that

$$\lim_{u\to\infty}\left[F^{\alpha}_{k,\mathrm{RV}}(u) - \frac{\ln(4\alpha q_k u)}{\alpha q_k}\right] = 0,$$

so that $F^{\alpha}_{k,\mathrm{RV}}$ behaves asymptotically as ln. We also point out that for all $u \in \mathbb{R}$, it holds

$$\lim_{\alpha\to 0} F^{\alpha}_{k,\mathrm{RV}}(u) = q_k u + \sqrt{1 + q_k^2 u^2} = F_{k,\mathrm{Dev}}(u),$$

so that this method is a generalization of Deville's calibration function (see Sect. 3.1.5). In Fig. 2, we present the graph of $G^{\alpha}_{k,\mathrm{RV}}(\cdot, d)$ and $F^{\alpha}_{k,\mathrm{RV}}$ for different values of $\alpha > 0$.

3.1.7 Logistic method

Since the range of the weights w_k is only defined by that of F_k and d_k, Deville and Särndal proposed to circumvent this problem by considering a function which has a restricted range. More precisely, given $0 \le L \le 1 \le U$ such that $L < U$, we want to ensure that $\mathrm{g}_k = w_k/d_k \in [L, U]$, i.e. $d_k L \le w_k \le d_k U$. The weights g_k are often called g-weights in the literature. We then consider a function F_k such that $\mathrm{Im}(F_k) = [L, U]$ (or $\mathrm{Im}(F_k) =]L, U[$). Indeed, due to (7), we have $\mathrm{g}_k = F_k(\boldsymbol{\lambda}^{\top}\boldsymbol{x}_k) \in [L, U]$. In order to achieve this, we consider a logistic function. More precisely, let

$$F_{k,\log}(u) = \frac{L(U-1) + U(1-L)\exp(Aq_k u)}{U - 1 + (1-L)\exp(Aq_k u)},$$

where $A = (U-L)/((1-L)(U-1)) > 0$. It follows then

$$G_{k,\log}(w, d) = \frac{d}{Aq_k}\left[\left(\frac{w}{d} - L\right)\log\left(\frac{(w/d) - L}{1-L}\right) + \left(U - \frac{w}{d}\right)\log\left(\frac{U - (w/d)}{U-1}\right)\right],$$

$$g_{k,\log}(w,d) = \frac{1}{Aq_k}\left[\log\left(\frac{(w/d)-L}{1-L}\right) - \log\left(\frac{U-(w/d)}{U-1}\right)\right].$$

We can show that

$$\lim_{u\to-\infty} F_{k,\log}(u) = L, \qquad \lim_{u\to\infty} F_{k,\log}(u) = U,$$

and $F'_{k,\log}(u) > 0$ for all $u \in \mathbb{R}$. Hence, $F_{k,\log}$ satisfies the assumptions presented in Remark 1 so that the associated functions $G_{k,\log}(\cdot, d_k)$ and $g_{k,\log}(\cdot, d_k)$ fulfil (A1)–(A2). In this case, we have $\mathcal{R}_k(d_k) = \mathbb{R}$ and $\mathcal{D}_k(d_k) =]Ld_k, Ud_k[$ for all $d_k > 0$.

3.1.8 Truncated method

Deville and Särndal proposed another way to obtain g-weights that satisfy some prescribed bounds $g_k \in [L, U]$. They consider one of the above functions F_k and truncate it. Let F_k satisfy the assumptions of Remark 1 and be such that $[L, U] \subset \mathrm{Im}\,(F_k)$. Hence, there exist $\widetilde{L}_k = F_k^{-1}(L)$ and $\widetilde{U}_k = F_k^{-1}(U)$. Furthermore, they satisfy $\widetilde{L}_k < \widetilde{U}_k$ since F_k is strictly increasing and $L < U$. Then, define

$$F_{k,\mathrm{trun}} = F_k|_{[\widetilde{L}_k, \widetilde{U}_k]},$$

i.e. the restriction of F_k to the range $[\widetilde{L}_k, \widetilde{U}_k]$. A similar way to define $F_{k,\mathrm{trun}}$ which is usually considered in the literature is

$$\widetilde{F}_{k,\mathrm{trun}}(u) = \begin{cases} F_k(u) & \text{if } \widetilde{L}_k \le u \le \widetilde{U}_k, \\ L & \text{if } u < \widetilde{L}_k, \\ U & \text{if } u > \widetilde{U}_k. \end{cases}$$

It might be useful to consider this alternative definition for computational reasons (see Sect. 3.2). However, it does not fulfil the required conditions on F_k while $F_{k,\mathrm{trun}}$ does. Hence we use $F_{k,\mathrm{trun}}$ in the following. The function $g_{k,\mathrm{trun}}(\cdot, d_k)$ is defined as the inverse of $d_k F_{k,\mathrm{trun}}$ and $G_{k,\mathrm{trun}}(\cdot, d_k)$ as in Remark 1. We point out that in this case it is possible to have weights on the boundary, i.e. such that $w_k = Ld_k$ or $w_k = Ud_k$. This is not the case of the logistic method presented in Sect. 3.1.7 since this can only be obtained asymptotically.

The question then becomes how to choose L and U. First, we point out that the conditions $0 \le L \le 1 \le U$ are not necessary. They are just commonly used in practice. Indeed, $L \ge 0$ ensures that the weights are positive and the condition $L \le 1 \le U$ is used to control how much w_k can deviate from d_k since the condition is on the g-weights g_k. In practice, people are interested in finding L and U such that $U - L$ is minimal. The idea behind that principle is the same as the one for the distance minimization, namely that we do not want the new weights to deviate too much from the original ones. The problem is that taking bounds so that $U - L$ is very small might lead to a problem which does not admit any solution as discussed in Sect. 3.2. The bounds need to be chosen so that the calibration equations (3) admit a solution in

$\otimes_{k\in s}[L_k, U_k]$, where $L_k = d_k L$ and $U_k = d_k U$. Equivalently, we can consider the calibration equations for the g-weights

$$\sum_s d_k \boldsymbol{x}_k \mathrm{g}_k = \boldsymbol{t}_{\boldsymbol{x}}. \tag{12}$$

Then, we need to choose L and U such that (12) admits a solution satisfying $\mathrm{g}_k \in [L, U]$ for all $k \in s$.

Example 1 The most well-known method fitting that framework is the *linear truncated method*. In that particular case, we have

$$\widetilde{F}_{k,\mathrm{trun}}(u) = \begin{cases} 1 + q_k u & \text{if } \frac{L-1}{q_k} \le u \le \frac{U-1}{q_k}, \\ L & \text{if } u < \frac{L-1}{q_k}, \\ U & \text{if } u > \frac{U-1}{q_k}. \end{cases}$$

This yields

$$\widetilde{g}_{k,\mathrm{trun}}(w, d) = \begin{cases} \dfrac{w-d}{dq_k} & \text{if } Ld \le w \le Ud, \\ \infty & \text{if } w > U, \\ -\infty & \text{if } w < L, \end{cases}$$

$$\widetilde{G}_{k,\mathrm{trun}}(w, d) = \begin{cases} \dfrac{(w-d)^2}{2dq_k} & \text{if } Ld \le w \le Ud, \\ \infty & \text{else.} \end{cases}$$

3.1.9 Pseudo-empirical likelihood

Empirical likelihood function was originally proposed by Owen (1988) and has been adapted to survey sampling by Chen and Qin (1993), Chen and Sitter (1999), Chen and Wu (1999), Wu and Rao (2006), Chaudhuri et al. (2008), Berger and De La Riva Torres (2016), Berger (2018). Berger and De La Riva Torres (2016) and Berger (2018) empirical likelihood approaches will not be presented. In this section, we describe the pseudo-empirical likelihood approach proposed by Chen and Sitter (1999), because it is related to the calibration approaches described in the previous sections. Furthermore, it is linked to the minimum entropy method from Sect. 3.1.3. Consider the pseudo-empirical log-likelihood function

$$\widehat{\ell}(\boldsymbol{p}) = \sum_s d_k \log p_k,$$

where p_k is the probability associated with unit k. We want to minimize $\widehat{\ell}(p)$ under the constraint that $0 \le p_k \le 1$ for all $k \in s$ and

$$\sum_s p_k = 1, \qquad \sum_s p_k \left(\boldsymbol{x}_k - \frac{\boldsymbol{t}_{\boldsymbol{x}}}{N} \right) = 0. \tag{13}$$

In order to compute the p_k, let us consider the Lagrangian

$$\mathcal{L}(\boldsymbol{p}, \boldsymbol{\gamma}, \beta) = \sum_s d_k \log(p_k) - \boldsymbol{\gamma}^\top \sum_s p_k \left(\boldsymbol{x}_k - \frac{\boldsymbol{t}_{\boldsymbol{x}}}{N} \right) - \beta \left(\sum_s p_k - 1 \right).$$

By cancelling the derivatives of $\mathcal{L}$ with respect to p_k, we obtain

$$p_k = \frac{1}{1 + \boldsymbol{\lambda}^\top \left(\boldsymbol{x}_k - \frac{\boldsymbol{t}_{\boldsymbol{x}}}{N} \right)} \frac{d_k}{\sum_s d_k},$$

where we have set $\boldsymbol{\lambda} = \left(\sum_s d_k \right)^{-1} \boldsymbol{\gamma}$. The value of $\boldsymbol{\lambda}$ can then be computed by solving the nonlinear system

$$\mathbf{0} = \sum_s p_k \left(\boldsymbol{x}_k - \frac{\boldsymbol{t}_{\boldsymbol{x}}}{N} \right) = \frac{1}{\sum_s d_k} \sum_s \frac{d_k \left(\boldsymbol{x}_k - \frac{\boldsymbol{t}_{\boldsymbol{x}}}{N} \right)}{1 + \boldsymbol{\lambda}^\top \left(\boldsymbol{x}_k - \frac{\boldsymbol{t}_{\boldsymbol{x}}}{N} \right)}.$$

Total (1) can then be estimated by choosing $w_{k,\text{lik}} = N p_k$, which yields

$$\widehat{t}_{yw,\text{lik}} = N \sum_s p_k y_k.$$

For this specific choice of $w_{k,\text{lik}}$, we have $\sum_s w_{k,\text{lik}} = N$. We point out that the computational issues discussed in the case of the minimum entropy method also apply here.

The pseudo-empirical likelihood is related to the calibration with the minimum entropy method presented in Sect. 3.1.3. Indeed, in Expression (11), if $q_k = 1$ and $w_{k,\text{ent}}$ is a solution to the minimum entropy method such that $\sum_s w_{k,\text{ent}} = N$ and p_k is defined according to

$$p_k = \frac{w_{k,\text{ent}}}{\sum_s w_{k,\text{ent}}},$$

then $\boldsymbol{p}$ is a solution associated with the pseudo-empirical likelihood method (see Kim and Park 2010).

Chaudhuri et al. (2008) use an empirical likelihood function and incorporate population level information via constraints on functions of the model parameters that can be nonlinear. Berger and De La Riva Torres (2016) proposed to use directly an empirical likelihood function (and not the pseudo-empirical likelihood) and incorporate constraints which gives consistent design-based confidence intervals. With this method, simple confidence intervals can be computed which can be calculated without the need of variance estimates (see also Berger 2018).

3.2 Solving calibration equations

In this section, we discuss methods to solve the minimization problem (5). Before describing procedures to solve this system, we first briefly discuss the existence of such solution.

3.2.1 Existence of solutions

This question was already addressed by Vanderhoeft (2001) for some specific distances. We review and extend it here. Consider $\mathcal{D} = \otimes_s \mathcal{D}_k(d_k)$, which is the product space associated with the domain of the function $\sum_s G_k(\cdot, d_k)$ that we aim at minimizing. The set of feasible solutions is then $\mathcal{F} = \mathcal{A} \cap \mathcal{D}$, where $\mathcal{A} = \{\boldsymbol{v} \in \mathbb{R}^n | \, \boldsymbol{X}_s^\top \boldsymbol{v} = \boldsymbol{t}_{\boldsymbol{x}}\}$ is the set of admissible solution to the calibration equations.

Lemma 1 *Assume that $\mathcal{F} \neq \emptyset$ and that one of the following holds:*

1. *$\mathcal{D}$ is closed*
2. *every $\mathcal{D}_k(d_k)$ is open and*

$$\lim_{\substack{w \to a_k \\ > }} |G_k(w, d_k)| = \lim_{\substack{w \to b_k \\ < }} |G_k(w, d_k)| = \infty,$$

where $\mathcal{D}_k(d_k) =]a_k, b_k[$.

Then, under (A1)*, the minimization problem* (5) *admits a unique solution $\boldsymbol{z} \in \mathcal{F}$ such that*

$$\sum_s G_k(z_k, d_k) = \min_{\boldsymbol{w} \in \mathcal{A}} \sum_s G_k(w_k, d_k).$$

Proof We use the convention that $s = \{1, \ldots, n\}$ and iteration over elements $k \in s$ is seen as an enumeration over $k = 1, \ldots, n$ so that $\mathcal{A}$ and $\mathcal{D}$ are viewed as subsets of $\mathbb{R}^n$. Furthermore, to simplify the notations, we denote $h(\boldsymbol{v}) = \sum_s G_k(v_k, d_k)$. It is clear that $\mathcal{A}$ is convex and since $\mathcal{D}$ is the product of intervals, it is convex as well. Hence, $\mathcal{F}$ is a convex set. If we can prove that the solution exists, the uniqueness follows from the assumption that $G_k(\cdot, d_k)$ is strictly convex (Güler 2010, Theorem 4.32). First, consider 1. We distinguish two cases.

Case 1 ($\mathcal{D}$ *is bounded*) Since $\mathcal{A}$ is closed, then $\mathcal{F}$ is closed and bounded, so that it is compact. It follows that the continuous function h admits a minimum.

Case 2 ($\mathcal{D}$ *is unbounded*) Let $(\boldsymbol{v}_k)_{\mathbb{N}} \subset \mathcal{F}$ be such that $\lim_{k \to \infty} h(\boldsymbol{v}_k) = \inf_{\boldsymbol{x} \in \mathcal{F}} h(\boldsymbol{x})$. Using the fact that h is strictly convex and satisfies $h(\boldsymbol{d}) = 0$, it can be shown that $\lim_{\|\boldsymbol{v}\| \to \infty} h(\boldsymbol{v}) = \infty$. Hence, the sequence $(\boldsymbol{v}_k)_{\mathbb{N}}$ is bounded. It follows from Bolzano–Weierstrass theorem that there exist subsequences $\left(\boldsymbol{v}_{k_j}\right)_{\mathbb{N}}$ and $\boldsymbol{z}$ satisfying $\lim_{j \to \infty} \boldsymbol{v}_{k_j} = \boldsymbol{z}$, so that $h(\boldsymbol{z}) = \inf_{\boldsymbol{x} \in \mathcal{F}} h(\boldsymbol{x})$. Since $\mathcal{F}$ is closed, $\boldsymbol{z} \in \mathcal{F}$.
The proof of 2 follows a similar argument to Case 2 above. □

Remark 7 All the methods presented in Sect. 3 except the raking ratio satisfy one of the two conditions of Lemma 1. Indeed, for the raking ratio,

$$\lim_{\substack{w \to 0 \\ > }} G_{k,\mathrm{rak}}(w, d) = \frac{d}{q_k}.$$

We also point out that to fill in the assumptions of the above lemma, the truncated method needs to be considered on open intervals. Furthermore, since $\mathcal{D}_k(d)$ is open, the solution lies inside $\mathcal{D}$.

3.2.2 An algorithm to solve the distance minimization problem

As discussed at the beginning of Sect. 3, if we can find $\boldsymbol{\lambda} \in \mathbb{R}^p$ such that

$$\boldsymbol{t}_{\boldsymbol{x}} = \sum_s d_k F_k \left(\boldsymbol{\lambda}^\top \boldsymbol{x}_k \right) \boldsymbol{x}_k,$$

then the new weights can be computed through (7). This is a (possibly nonlinear) system of p equations and p unknowns. We distinguish two cases, namely $\mathcal{R}_k(d) = \mathbb{R}$ and $\mathcal{R}_k(d) \subsetneq \mathbb{R}$, and present two algorithms accordingly.

First, assume that $\mathcal{R}_k(d) = \mathbb{R}$ for all $d > 0$ and consider the functional $\boldsymbol{\Phi} : \mathbb{R}^p \to \mathbb{R}^p$ defined as

$$\boldsymbol{\Phi}(\boldsymbol{\lambda}) = \sum_s d_k \left[F_k \left(\boldsymbol{\lambda}^\top \boldsymbol{x}_k \right) - 1 \right] \boldsymbol{x}_k.$$

We then need to find $\boldsymbol{\lambda}$ such that $\boldsymbol{\Phi}(\boldsymbol{\lambda}) = \boldsymbol{t}_{\boldsymbol{x}} - \widehat{\boldsymbol{t}}_{\boldsymbol{x}\pi}$. One way to do this is to apply Newton method which was proposed by Deville and Särndal (1992). The algorithm is the one presented in Algorithm 1.

Algorithm 1 Newton method for the distance minimization problem in the case $\mathcal{R}_k(d) = \mathbb{R}$. tol is a prescribed parameter to control the error

$\boldsymbol{\lambda}^{(0)} = \mathbf{0},\ \boldsymbol{w}^{(0)} = \mathbf{0},\ \boldsymbol{w}^{(1)} = \boldsymbol{d},\ m = 0$
while $\left\| w^{(m+1)} - w^{(m)} \right\|_\infty > tol$ **do**
  $\boldsymbol{\lambda}^{(m+1)} = \boldsymbol{\lambda}^{(m)} - \left[\nabla \boldsymbol{\Phi} \left(\boldsymbol{\lambda}^{(m)} \right) \right]^{-1} \left[\boldsymbol{\Phi} \left(\boldsymbol{\lambda}^{(m)} \right) - \boldsymbol{t}_{\boldsymbol{x}} + \widehat{\boldsymbol{t}}_{\boldsymbol{x}\pi} \right]$
  $w_k^{(m+1)} = d_k F_k \left[\left(\boldsymbol{\lambda}^{(m+1)} \right)^\top \boldsymbol{x}_k \right]$
  $m \leftarrow m + 1$
end while

If $\boldsymbol{X}_s$ is full rank, matrix $\nabla \boldsymbol{\Phi}(\boldsymbol{\lambda}) \in \mathbb{R}^{p \times p}$ defined for $\boldsymbol{\lambda} \in \mathbb{R}^p$ as

$$\nabla \boldsymbol{\Phi}(\boldsymbol{\lambda}) = \sum_s d_k F_k' \left(\boldsymbol{\lambda}^\top \boldsymbol{x}_k \right) \boldsymbol{x}_k \boldsymbol{x}_k^\top$$

is always invertible. Indeed, by assumption, for any $w \in \mathbb{R}$ it holds $F_k'(w) > 0$. The above matrix can then be written as $\nabla \boldsymbol{\Phi}(\boldsymbol{\lambda}) = \boldsymbol{X}_s^\top \boldsymbol{F}'(\boldsymbol{\lambda}) \boldsymbol{D} \boldsymbol{X}_s$, where $\boldsymbol{D}$ and $\boldsymbol{F}'(\boldsymbol{\lambda})$ are

diagonal matrices with strictly positive entries, namely d_k and $F_k'(\boldsymbol{\lambda}^\top \boldsymbol{x}_k)$, respectively. Hence, $\nabla \boldsymbol{\Phi}(\boldsymbol{\lambda})$ is symmetric positive definite due to the assumption that $\boldsymbol{X}_s$ has rank p. The Newton algorithm can also be applied with a Moore–Penrose generalized inverse in the case where $\boldsymbol{X}_s$ does not have full rank. This is implemented in the SAS® macro CALMAR2 (Le Guennec and Sautory 2002) or the `calib` function of the R `sampling` package (Tillé and Matei 2016).

Let us then turn to the case where $\mathcal{R}_k(d_k) \subsetneq \mathbb{R}$. It means that we would need to apply the Newton method for $\boldsymbol{\lambda}$ in the polyhedron defined by the conditions $\boldsymbol{\lambda}^\top \boldsymbol{x}_k \in \mathcal{R}_k(d_k)$. Let us write $\mathcal{R}_k(d_k) =]a_k, b_k[$. Then, different situations can happen, such as

(i) $\lim_{u \to a_k} F_k(u), \lim_{u \to b_k} F_k(u) < \infty$, which is the case, for instance, of the truncated method. In this case, it is in principle possible to consider an extension of the function F_k as in Sect. 3.1.8 and use Algorithm 1. We have that $F_k'(u) = 0$ for every $u \in \mathbb{R} \backslash \mathcal{R}_k(d_k)$. It follows that if for some iteration of the Newton process the element exits the domain of F_k, i.e. $\boldsymbol{\lambda}^\top \boldsymbol{x}_k \in \mathbb{R} \backslash \mathcal{R}_k(d_k)$, then $F_k'(\boldsymbol{\lambda}^\top \boldsymbol{x}_k) = 0$ and $F_k(\boldsymbol{\lambda}^\top \boldsymbol{x}_k) = L$ or $F_k(\boldsymbol{\lambda}^\top \boldsymbol{x}_k) = U$. It is then not possible anymore to guarantee that the matrix $\nabla \boldsymbol{\Phi}(\boldsymbol{\lambda})$ is invertible. This happens in particular when the solution lies close to the boundary of $\mathcal{R}(\boldsymbol{d})$, which is the case when one wants, for example, to consider the truncated method with the tightest bounds possible.

(ii) $\lim_{u \to a_k} F_k(u) = \lim_{u \to b_k} F_k(u) = \infty$, which is, for instance, the case of the minimum entropy method. In this situation, we have seen that the solution to the minimization problem lies inside the domain. As in the previous case, it can happen that some iteration gets out of $\mathcal{R}(\boldsymbol{d})$, leading to a point where the function is not defined. We then use a damping factor, i.e. we select $\delta \in]0, 1[$ such that $\delta \boldsymbol{\lambda}^\top \boldsymbol{x}_k \in]a_k, b_k[$ for all $k \in s$. In other words, we choose a $\boldsymbol{\lambda}$ smaller in magnitude to ensure that the current iteration lies inside the domain of F_k.

In order to circumvent this possible drawback, we propose to consider the original problem (5) and apply a primal-dual interior-point method as described, for instance, in Boyd and Vandenberghe (2004, Section 11.7). In order to do this, the problem needs to be written in the form

$$\begin{aligned} \underset{z}{\text{minimize}} \quad & h(z) \\ \text{subject to} \quad & \boldsymbol{A}z = \boldsymbol{b},\ c_i(z) \leq 0,\ i = 1, \ldots, m, \end{aligned}$$

for some convex functions $h, c_1, \ldots, c_m : \mathbb{R}^n \to \mathbb{R}$. In our case, $h(z) = \sum_s G_k(z_k, d_k)$, $\boldsymbol{A} = \boldsymbol{X}_s^\top$, $\boldsymbol{b} = \boldsymbol{t}_{\boldsymbol{x}}$ and the functions $c_1, \ldots, c_m$ are linear functions describing the conditions $y_k \in \mathcal{D}_k(d_k)$, i.e. these functions take the form $a_k - x_k$ or $x_k - b_k$ if $\mathcal{D}_k(d_k) =]a_k, b_k[$. We point out that this method requires h to be twice differentiable and the functions $c_1, \ldots, c_m$ to be differentiable. This is the case of all the functions that we have introduced so far when considered on the domain $\mathcal{D}$. Since this is beyond the scope of this paper, we do not explain how the interior-point method works but only present a simple version applied to the distance minimization problem. For a sake of simplicity in Algorithm 2, we write $\boldsymbol{c}(\boldsymbol{w}) = [c_1(\boldsymbol{w}), \ldots, c_m(\boldsymbol{w})]^\top$. Furthermore, the matrix $\boldsymbol{\Theta}(\boldsymbol{w}, \boldsymbol{\lambda})$ and vectors $\boldsymbol{r}(\boldsymbol{w}, \boldsymbol{\lambda}, \boldsymbol{v}, \gamma)$ are defined for $\boldsymbol{w} \in \mathbb{R}^n$,

$\boldsymbol{\lambda} \in \mathbb{R}^m$, $\boldsymbol{\nu} \in \mathbb{R}^p$ and $\gamma \in \mathbb{R}$ as

$$\boldsymbol{\Theta}(\boldsymbol{w}, \boldsymbol{\lambda}) = \begin{pmatrix} \nabla^2 h(\boldsymbol{w}) & \nabla \boldsymbol{c}(\boldsymbol{w})^\top & \boldsymbol{X}_s \\ -\text{diag}\,(\boldsymbol{\lambda})\,\nabla \boldsymbol{c}(\boldsymbol{w}) & -\text{diag}\,(\boldsymbol{c}(\boldsymbol{w})) & \boldsymbol{0} \\ \boldsymbol{X}_s^\top & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix},$$

$$\boldsymbol{r}(\boldsymbol{w}, \boldsymbol{\lambda}, \boldsymbol{\nu}, \gamma) = \begin{pmatrix} -\nabla h(\boldsymbol{w}) - \nabla \boldsymbol{c}(\boldsymbol{w})^\top \boldsymbol{\lambda} - \boldsymbol{X}_s \boldsymbol{\nu} \\ \text{diag}\,(\boldsymbol{\lambda})\,\boldsymbol{c}(\boldsymbol{w}) + \frac{1}{\gamma}\boldsymbol{1} \\ \boldsymbol{t}_{\boldsymbol{x}} - \boldsymbol{X}_s^\top \boldsymbol{w} \end{pmatrix} = \begin{pmatrix} \boldsymbol{r}_{\text{dual}} \\ \boldsymbol{r}_{\text{cent}} \\ \boldsymbol{r}_{\text{primal}} \end{pmatrix},$$

where $\nabla^2 h$ is the hessian of h and $\text{diag}(\boldsymbol{\lambda})$ is the diagonal matrix with elements λ_i on the diagonal. Furthermore to simplify the notations in the algorithm, we write $\boldsymbol{y} = (\boldsymbol{w}, \boldsymbol{\lambda}, \boldsymbol{\nu})^\top$. The primal-dual interior-point method applied to the minimization problem (5) is presented in Algorithm 2. We point out that this method can also be used if $\mathcal{R}_k(d) = \mathbb{R}$.

Algorithm 2 Primal-dual interior-point method for the distance minimization problem in the case $\mathcal{R}_k(d) \subsetneq \mathbb{R}$. In the algorithm, $\mu > 1$ is a prescribed parameter. We have used the notation $\boldsymbol{y} = (\boldsymbol{w}, \boldsymbol{\lambda}, \boldsymbol{\nu})^\top$

$\boldsymbol{\lambda} = \boldsymbol{1}$, $\boldsymbol{w} = \boldsymbol{d}$, $\boldsymbol{\nu} = \boldsymbol{0}$
while $iter <$ maxiter **do**
  $\eta = -\boldsymbol{c}(\boldsymbol{w})^\top \boldsymbol{\lambda}$
  $\gamma = \mu m / \eta$
  solve $\boldsymbol{\Theta}(\boldsymbol{y}) \Delta \boldsymbol{y} = \boldsymbol{r}(\boldsymbol{y}, \gamma)$
  find $s > 0$ such that $\boldsymbol{c}(\boldsymbol{w} + s\Delta \boldsymbol{w}) \leq 0$
  $\boldsymbol{y} = \boldsymbol{y} + s\Delta \boldsymbol{y}$
  if $\|\boldsymbol{r}_{\text{dual}}\| < tol$, $\|\boldsymbol{r}_{\text{primal}}\| < tol$ and $\eta < tol$ **then**
    break
  end if
  $iter \leftarrow iter + 1$
end while

In most cases, the Newton method converges and delivers satisfactory results. However, due to the problem that we pointed out above, it is not possible to guarantee that the method will converge. On the other hand, the method described in Algorithm 2 converges if the unique solution lies inside $\mathcal{D}$ (see Boyd and Vandenberghe (2004, Section 11.7)). However, if the Newton algorithm converges, it is much faster. Indeed, at each step a system of size $p \times p$ needs to be solved while every iteration of Algorithm 2 requires the solution of a $(n+m+p) \times (n+m+p)$ system, where in general $m = 2n$. Since in most cases $n \gg p$, the system size is much larger in the second method and in both cases the matrices are full. A summary of the different methods presented in this section is given in Table 3.

Table 3 Summary of the different methods presented in Sect. 3

Method	$G_k(w,d)$	$g_k(w,d)$	$F_k(w)$	$\mathcal{D}_k(d)$	$\mathcal{R}_k(d)$	α
Linear	$\frac{(w-d)^2}{2dq_k}$	$\frac{w-d}{dq_k}$	$1+q_kw$	$\mathbb{R}$	$\mathbb{R}$	–
Raking ratio	$\frac{1}{q_k}[w\log(\frac{w}{d})+d-w]$	$\frac{1}{q_k}\log(\frac{w}{d})$	$\exp(q_kw)$	$\mathbb{R}_+^*$	$\mathbb{R}$	–
Minimum entropy	$\frac{1}{q_k}[d\log(\frac{d}{w})+w-d]$	$\frac{1}{q_k}(1-\frac{d}{w})$	$\frac{1}{1-q_kw}$	$\mathbb{R}_+^*$	$(-\infty,\frac{1}{q_k})$	–
Generalized	$\frac{1}{\alpha(\alpha-1)q_k}[\frac{\lvert w\rvert^\alpha}{d^{\alpha-1}}+(\alpha-1)d-\alpha w]$	$\frac{1}{(\alpha-1)q_k}[\frac{\text{sign}(w)\lvert w\rvert^{\alpha-1}}{d^{\alpha-1}}-1]$	$\text{sign}[1+q_kw(\alpha-1)]\,\lvert 1+q_kw(\alpha-1)\rvert^{\frac{1}{\alpha-1}}$	$\mathbb{R}_+^*$	$\left]-\infty,\frac{1}{(1-\alpha)q_k}\right[$	$]-\infty,0[\cup]0,1[$
	$\frac{1}{q_k}[d\log(\frac{d}{\lvert w\rvert})+w-d]$	$\frac{1}{(\alpha-1)q_k}[\frac{\text{sign}(w)\lvert w\rvert^{\alpha-1}}{d^{\alpha-1}}-1]$	$\text{sign}[1+q_kw(\alpha-1)]\,\lvert 1+q_kw(\alpha-1)\rvert^{\frac{1}{\alpha-1}}$	$\mathbb{R}_+^*$	$\left]-\infty,\frac{1}{q_k}\right[$	0
	$\frac{1}{q_k}[\lvert w\rvert\log(\frac{\lvert w\rvert}{d})+d-w]$	$\frac{\text{sign}(w)}{q_k}\log(\frac{\lvert w\rvert}{d})$	$\exp(q_kw)$	$\mathbb{R}_+^*$	$\mathbb{R}$	1
	$\frac{1}{\alpha(\alpha-1)q_k}[\frac{\lvert w\rvert^\alpha}{d^{\alpha-1}}+(\alpha-1)d-\alpha w]$	$\frac{1}{(\alpha-1)q_k}[\frac{\text{sign}(w)\lvert w\rvert^{\alpha-1}}{d^{\alpha-1}}-1]$	$\text{sign}[1+q_kw(\alpha-1)]\,\lvert 1+q_kw(\alpha-1)\rvert^{\frac{1}{\alpha-1}}$	$\mathbb{R}$	$\mathbb{R}$	$]1,\infty[$
Deville	$\frac{d}{4q_k}[(\frac{w}{d})^2-1-2\ln(\frac{w}{d})]$	$\frac{d}{q_k}[\frac{(\frac{w}{d})^2-1}{2w}]$	$q_kw+\sqrt{1+q_k^2w^2}$	$\mathbb{R}_+^*$	$\mathbb{R}$	-
Roy and Vanheuverzwyn	$\frac{d}{q_k}\int_1^{\frac{w}{d}}\frac{1}{2\alpha}\sinh[\alpha(t-\frac{1}{t})]dt$	$\frac{1}{2\alpha}\sinh[\alpha(\frac{w}{d}-\frac{d}{w})]$	$\frac{\text{arsinh}(2\alpha q_kw)}{2\alpha}+\sqrt{1+[\frac{\text{arsinh}(2\alpha q_kw)}{2\alpha}]^2}$	$\mathbb{R}_+^*$	$\mathbb{R}$	$\mathbb{R}\backslash\{0\}$
Logistic	$\frac{d}{Aq_k}[\xi\log(\frac{\xi}{1-L})+\zeta\log(\frac{\zeta}{U-1})]$	$\frac{1}{Aq_k}[\log(\frac{\xi}{1-L})-\log(\frac{\zeta}{U-1})]$	$\frac{L(U-1)+U(1-L)\exp(Aq_kw)}{U-1+(1-L)\exp(Aq_kw)}$	$]Ld,Ud[$	$\mathbb{R}$	-
Truncated	$G_k(w,d)$	$g_k(w,d)$	$F_k(w)$	$[Ld,Ud]$	$\left[F_k^{-1}(L),F_k^{-1}(U)\right]$	-

The ranges $\mathcal{R}_k(d)$ and $\mathcal{D}_k(d)$ are those such that the assumptions (A1)–(A2) on the functions $G_k(\cdot,d):\mathcal{D}_k(d)\to\mathcal{R}_k(d)$ and $g_k(\cdot,d):\mathcal{D}_k(d)\to\mathcal{R}_k(d)$ hold. In the case of the logistic method, we have $A=\frac{U-L}{(1-L)(U-1)}$, $\xi=\frac{w}{d}-L$ and $\zeta=U-\frac{w}{d}$

4 Alternative approaches

The first goal of calibration is to decrease the sampling variance by using known auxiliary information. However, it could be used for other purposes such as treating nonresponse (see Sect. 4.2).

4.1 Generalized calibration

In recent years, a series of authors (Deville 1998; Estevao and Särndal 2000, 2006; Deville 2000, 2002, 2004; Kott 2006; Chang and Kott 2008; Kott 2009; Kott and Chang 2010; Särndal 2007) have discussed a generalization of the method presented in the previous section. There are several motivations to this generalization, such as introducing more complex information. The idea, introduced by Estevao and Särndal (2000), is to remove the requirement that the w_k minimize a certain distance but only follow a certain functional form based on (7). This can then be used, for instance, to deal with nonresponse (see Sect. 4.2).

We present in the sequel a framework for *generalized calibration*, also known as *instrument vector approach*. The following theory was introduced by Deville (2002) and Kott (2006). As already discussed, the properties of the calibrated weights are a direct consequence of those of the function F_k. Let us consider an instrumental vector $(\boldsymbol{z}_k)_s \subset \mathbb{R}^q$ for some $q \geq p$ and denote $\boldsymbol{Z}_s$ the associated matrix. We then look for weights w_k of the form

$$w_k = d_k F_k \left(\boldsymbol{\lambda}^\top \boldsymbol{z}_k\right),$$

for $k \in s$ and some $\boldsymbol{\lambda} \in \mathbb{R}^q$ such that the weights satisfy the calibration equations

$$\boldsymbol{t}_{\boldsymbol{x}} = \sum_s d_k F_k \left(\boldsymbol{\lambda}^\top \boldsymbol{z}_k\right) \boldsymbol{x}_k.$$

The function F_k is assumed to fulfil the assumptions of Remark 1. Let $\mathcal{P}(\boldsymbol{Z}_s) = \mathrm{Im}(\boldsymbol{Z}_s^\top)$ and $\mathcal{B} = \mathcal{B}(\boldsymbol{Z}_s, \boldsymbol{d}) = \boldsymbol{dF}(\mathcal{P}(\boldsymbol{Z}_s) \cap \mathcal{R}(\boldsymbol{d}))$, where $\mathcal{R}(\boldsymbol{d}) = \otimes_s \mathcal{R}_k(d_k)$ and $\boldsymbol{dF} = [d_1 F_1, \ldots, d_n F_n]^\top$. It follows that if $\mathcal{B}(\boldsymbol{Z}_s, \boldsymbol{d}) \cap \mathcal{A}(\boldsymbol{t}_{\boldsymbol{x}}, \boldsymbol{X}_s) \neq \emptyset$, then this problem admits a solution.

4.2 Calibration to account for nonresponse

The papers of Dupont (1994) abd Fuller et al. (1994) were the first to present a method based on calibration to deal with nonresponse. Since then, several authors have worked on the same topic (Lundström and Särndal 1999; Deville 2000; Särndal and Lundström 2005; Kott 2006; Brick 2013). There are essentially two approaches to tackle this issue, which are well explained in Haziza and Lesage (2016).

The first one is the so-called *two-step approach*. The first step addresses the issue of nonresponse while the second one uses calibration to adjust the weights. Let us consider a *response set* $r \subset s$ defined as the set of units for which the study variable y is observed. We then assume that the response distribution $q(r|s)$ is known with

corresponding known probabilities $\theta_k = \Pr(k \in r|s)$ independent of the sample s. Furthermore, we assume that $\theta_k > 0$ for every unit, which is in some cases not true due to hard-core nonresponse as Kott (1994) points out. The modified HT estimator taking into account nonresponse is then given by

$$\widehat{t}_{y\pi\theta} = \sum_r \frac{d_k}{\theta_k} y_k. \tag{14}$$

The main issue is that in practice the value of θ_k is unknown and so needs to be estimated. There are different approaches to tackle this problem but we follow the more general case discussed by Haziza and Lesage (2016). Let us consider the model

$$\theta_k = m(\boldsymbol{z}_k, \boldsymbol{\gamma}),$$

where m is a prescribed function, $\boldsymbol{z}_k$ is a vector of auxiliary information and $\boldsymbol{\gamma}$ is a vector of unknown parameters. The choice of the function m and how $\boldsymbol{\gamma}$ can be estimated is a question that will not be treated here since it is beyond the scope of this paper. We then consider an estimator $\widehat{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$ and define $\widehat{\theta}_k = m(\boldsymbol{z}_k, \widehat{\boldsymbol{\gamma}})$. The propensity-score-adjusted (PSA) estimator of t_y is then obtained by replacing θ_k by $\widehat{\theta}_k$ in (14). It is defined as

$$\widehat{t}_{y,\mathrm{PSA}} = \sum_r \frac{d_k}{\widehat{\theta}_k} y_k = \sum_r \widetilde{w}_k y_k.$$

In the second step, we apply calibration with $\widetilde{w}_k$ as original weights, i.e. we look for weights of the form

$$w_{k,\mathrm{NRTS}} = \widetilde{w}_k F_k\left(\boldsymbol{\lambda}^\top \boldsymbol{x}_k\right),$$

where F_k is one of the functions presented in Sect. 3.1.

The acronym NRTS stands for nonresponse two steps. The goal of the first step is to decrease the bias introduced by nonresponse, while the second one is performed in order to reduce the sampling variance of the estimator. This is discussed in Särndal and Lundström (2005, Section 6.1) for the case of the linear method, i.e. $F_k(w) = 1 + q_k w$. y_k and $\boldsymbol{x}_k$. However, in multipurpose surveys it is very unlikely that this relation is satisfied for every variable of interest.

In the second approach, everything is performed in one step, whence the name of *one-step approach* (Särndal and Lundström 2005, Section 6.1). In this setting, we consider two different levels of information:

(i) $\mathcal{U}$-level: a vector of auxiliary information $\boldsymbol{x}_k^\star$ is known for $k \in r$ together with $\boldsymbol{t}_{\boldsymbol{x}^\star} = \sum_{\mathcal{U}} \boldsymbol{x}_k^\star$;
(ii) s-level: a vector of auxiliary information $\boldsymbol{x}_k^\circ$ is known for $k \in s$ together with $\widehat{\boldsymbol{t}}_{\boldsymbol{x}^\circ} = \sum_s d_k \boldsymbol{x}_k^\circ$.

The vector of auxiliary variables and the associated totals are then defined as

$$\boldsymbol{x}_k = \begin{pmatrix} \boldsymbol{x}_k^\star \\ \boldsymbol{x}_k^\circ \end{pmatrix}, \qquad \boldsymbol{t}_{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{t}_{\boldsymbol{x}^\star} \\ \widehat{\boldsymbol{t}}_{\boldsymbol{x}^\circ} \end{pmatrix}.$$

The calibration is then performed based on this information. In order to obtain an asymptotically zero bias due to nonresponse, Haziza and Lesage (2016) discuss the idea that in this second approach one needs to have $F_k(\boldsymbol{\lambda}^\top \boldsymbol{x}_k) = \theta_k^{-1}$, where $\boldsymbol{\lambda}$ is such that the calibration equations are satisfied. Hence, selecting a calibration function F_k is in some sense equivalent to selecting a nonresponse model. In some cases, this situation is not even possible. Haziza and Lesage (2016) compare these approaches and in particular discuss bias reduction for both cases. The authors advocate for the two-step approach. Lesage et al (2018) also determine the possible drawbacks of the one-step approach when generalized calibration is used to deal with nonignorable nonresponse. The choice of the calibration function can have an important impact on the variance.

5 Variance estimation

Since one of the main goals of calibration is to reduce the variance of the HT estimator (2) by introducing auxiliary information, it is crucial to be able to determine whether the variance of the calibrated estimator is indeed smaller. In their original paper, Deville and Särndal (1992) already address this question. The variance estimator that they derive is based on the result that all linearly weighted estimators obtained through the distance minimization approach [with a pseudo-distance function satisfying assumptions (A1)–(A2)] are asymptotically equivalent to the linear GREG estimator. It follows (Särndal et al 1992, Result 6.6.1) that their asymptotic variance is given by

$$\mathrm{AV}\left(\widehat{t}_{yw}\right) = \sum_{k\in\mathcal{U}}\sum_{l\in\mathcal{U}} \Delta_{kl} d_k d_l E_k E_l,$$

where $\Delta_{kl} = \pi_{kl} - \pi_k\pi_l$, $E_k = y_k - \boldsymbol{x}_k^\top \boldsymbol{b}_\mathcal{U}$ and $\boldsymbol{b}_\mathcal{U} = (\sum_\mathcal{U} q_k \boldsymbol{x}_k \boldsymbol{x}_k^\top)^{-1} \sum_\mathcal{U} q_k y_k \boldsymbol{x}_k$. The $\boldsymbol{b}_\mathcal{U}$ is the minimizer of the least square residuals $S_\mathcal{U} = \sum_\mathcal{U} q_k E_k^2$. Since $\boldsymbol{b}_\mathcal{U}$ depends on the whole population, it cannot be computed and so we replace it by $\widehat{\boldsymbol{b}}_{sw} = (\sum_s w_k q_k \boldsymbol{x}_k \boldsymbol{x}_k^\top)^{-1} \sum_s w_k q_k y_k \boldsymbol{x}_k$. For $e_{kw} = y_k - \boldsymbol{x}_k^\top \widehat{\boldsymbol{b}}_{sw}$, a variance estimator of $\widehat{t}_{yw}$ is then given by

$$\widehat{\mathrm{Var}}\left(\widehat{t}_{yw}\right) = \sum_{k\in s}\sum_{l\in s} \frac{\Delta_{kl} w_k w_l e_{kw} e_{lw}}{\pi_{kl}}. \tag{15}$$

As Deville and Särndal (1992) point out, the weights d_k can be equivalently used in (15) instead of w_k. However, w_k tend to give reduced bias for smaller samples.

There exist other variance estimators for calibrated estimators. However, they are all based on the use of residuals of the variable of interest by the calibration variable. According to the way of linearizing the calibrated estimator, slightly different formulas have been proposed by Estevao and Särndal (2000, 2006), Demnati and Rao (2004), Kim and Park (2010) and Graf (2011). For estimating the vector of regression coefficients, these authors have proposed to use

$$\widehat{\boldsymbol{b}}_{sh} = \left(\sum_s d_k F_k'(\boldsymbol{\lambda}_s^\top \boldsymbol{x}_k) \boldsymbol{x}_k \boldsymbol{x}_k^\top \right)^{-1} \sum_s d_k F_k'(\boldsymbol{\lambda}_s^\top \boldsymbol{x}_k) \boldsymbol{x}_k y_k.$$

The regression coefficient thus depends on the calibration function F_k, which is not the case in the estimator proposed by Deville and Särndal. In the case of the linear method, $F_k'(\boldsymbol{\lambda}_s^\top \boldsymbol{x}_k) = q_k$ and $\widehat{\boldsymbol{b}}_{sw} = \widehat{\boldsymbol{b}}_{sh}$.

6 Calibration in practice

Calibration is widely used in practice and has been shown over the years to be a very powerful tool to decrease the sampling variance. However, there are some safeguards that must be considered and that are discussed in the following.

First of all, it is important to point out that calibration cannot circumvent poorly selected sample. For instance, let us assume that the population is partitioned into strata and that some of the auxiliary variables are used for post-stratification with respect to this partition. If in the selected sample s, there exists a stratum U_j such that $U_j \cap s = \emptyset$, i.e. no unit from U_j was selected in the sample, then calibration will not be able to correct this problem. From a strictly mathematical point out view, this leads to a zero column in $\boldsymbol{X}_s$, meaning that $\boldsymbol{X}_s$ cannot have rank p.

In general, the gain due to calibration relies essentially on the correlation between the observed value and the auxiliary variables. Indeed, from (15), we see that in the linear method the estimated variance decreases if y is indeed correlated with $\boldsymbol{X}_s$. This principle can be observed as well for other methods. As in the case of regression, the principle of parsimony applies. New auxiliary variables should be used only if they are correlated with y. Otherwise, we might in fact increase the variance instead of decreasing it. Hence, even if adding external information might be tempting, for instance, for cosmetic reasons, one should always ask themselves whether such link to y exists. Nascimento Silva and Skinner (1997) have proposed a method to select the calibration variables. Several publications are dedicated to the implementation of ridge regression in calibration (Chen et al 2002; Park and Yang 2008; Beaumont and Bocci 2008; Goga and Shehzad 2010) or to penalized calibration (Guggemos and Tillé 2010) in order to relax the constraints when many auxiliary variables are available. Wu and Sitter (2001) avoid the problem of high-dimensional minimization problem by using fitted values to realize the calibration.

The use of calibration can be generalized for harmonizing a set of samples. Guandalini and Tillé (2017) give a general framework to handle situations where several surveys are calibrated on each other's and discussed the cases of embedded, overlapping or non-overlapping samples.

Another interesting question is the choice of the bounds L and U when one wants to either use the logistic or the truncated method. In general, intervals of the form $[1/m, m]$ for some $m \in \mathbb{N}$ are considered. As the size of s increases, it is possible to consider tightest bounds since the g-weights draw closer to 1. In practice, people are interested in choosing the tightest bounds possible. However, choosing too tight bounds might lead to a problem with no solution as discussed in Sect. 3.1.8. To avoid

such problem, one could consider the method from Roy and Vanheuverzwyn (2001) discussed in Sect. 3.1.6.

In order to compute calibrated weights, both methods presented in Sect. 3.2.2 can be applied in a straightforward manner. We point out that several softwares with built-in macros already exist. CALMAR (Sautory 1993) package is a SAS® macro which was developed at INSEE. In the second version of this macro (Sautory and Le Guennec 2003), it is possible to perform generalized calibration. Other SAS® macros were also implemented such as generalized estimation system (GES) at Statistics Canada (Estevao et al. 1995), CLAN at Statistics Sweden (Andersson and Lennart 1998) and SUDAAN at the Research Triangle Institute (RTI) (Shah et al. 1977; Shah 1981; Shah et al. 1984, 1993, 1997). Bascula is a Windows-based software designed at Statistics Netherlands (Nieuwenbroek and Boonstra 2002). The SPSS macro `gCALIB` (Vanderhoeft 2001; Vanderhoeft et al. 2001) was developed at *Statistics Belgium* (STATBEL). Finally, we mention the R package `sampling` (Matei and Tillé 2007) which can be used for calibration as well as for other survey techniques and the R package icarus that enables to compute the bonds with the minimum interval (Rebecq 2017).

7 Numerical experiments

We present here two experiments to illustrate some of the aspects that have been discussed throughout the paper. In the first one, simulations are run to show that calibration can indeed decrease the variance of the estimator relatively differently according to the used distance. In the second example, we want to investigate the impact of the choice of the bounds while considering the truncated method.

In the following examples, we consider the MU284 data set used in Sarndal et al. (1992, Appendix B). The data are taken from 284 Swedish municipalities in the 1980s and are composed of 11 variables. We are interested in the variable RMT85 which represents the revenue of each municipality from taxation in 1985. Poisson sampling with equal inclusion probabilities is considered, i.e. it holds $\pi_k = n/N$ for all $k \in \mathcal{U}$. Calibration is performed on the following variables

- $\mathbb{1}$: a vector of ones and we impose that $\sum_s w_k = N$,
- REV84: real estate values according to 1984 assessment,
- S82: number of total seats in municipal council.

We point out that since $\sum_s w_k = N$ and $\pi_k = n/N$ for all $k \in \mathcal{U}$, we have $\sum_s w_k \pi_k = n$. For all sample sizes n in the sequel, we have selected $M = 10{,}000$ samples of expected sample size n and then computed the per cent relative bias ($\mathrm{RB}_{\mathrm{sim}}$) and the mean squared error ($\mathrm{MSE}_{\mathrm{sim}}$) defined as

$$\mathrm{RB}_{\mathrm{sim}} = \frac{100}{M t_y} \sum_{j=1}^{M} \left(\widehat{t}_{y s_j} - t_y \right),$$

$$\mathrm{MSE}_{\mathrm{sim}} = \frac{1}{M-1} \sum_{j=1}^{M} \left(\widehat{t}_{y s_j} - t_y \right)^2,$$

Table 4 Relative bias RB_{sim} for different sample sizes n and different distances. The Roy and Vanheuverzwyn method was used with $\alpha = 1$

Method	60	100	140	180
Horvitz Thomson estimator	−0.2766	−0.1008	−0.0517	−0.0377
Linear (Sect. 3.1.1)	−4.5246	−2.1625	−0.9289	−0.2688
Raking ratio (Sect. 3.1.2)	−4.5545	−2.1192	−0.8000	−0.1132
Minimum entropy (Sect. 3.1.3)	−4.4303	−1.9983	−0.6151	0.1016
Deville (Sect. 3.1.5)	−4.4977	−2.1358	−0.8532	−0.1730
Roy and Vanheuverzwyn (Sect. 3.1.6)	−4.2545	−2.1469	−0.9845	−0.3275

where $\widehat{t}_{ys_j}$ denotes the estimator obtained from the j-th sample s_j. In our example, $t_y = 8339$. All computations were performed using the R programming language with the help of the `sampling` package (Matei and Tillé 2007). For the methods associated with a distance function satisfying $\mathcal{R}_k(d_k) = \mathbb{R}$, we have used Algorithm 1 while computations have been performed with Algorithm 2 for functions such that $\mathcal{R}_k(d_k) \subsetneq \mathbb{R}$.

7.1 The impact of calibration

The goal of calibration is to decrease the variance of the HT estimator. There exists no theoretical result that allows to choose the distance in function of the mean squared error. In this first study, we want to investigate the impact of the choice of the calibration function on the bias and the mean squared error of the estimator. Furthermore, we expect that calibrated estimators have larger bias than the HT estimator. We performed the computation for several distances, so that we can also investigate the differences between the different methods. The results for RB_{sim} are presented in Table 4 for each sample size while Table 5 corresponds to MSE_{sim}.

As expected, we see that the bias is smaller in the case of the HT estimator while the MSE is larger compared to all calibrated estimators. The different distances yield very similar results in terms of bias and MSE, which agrees with Deville and Särndal (1992) statement that all methods deliver asymptotically the same estimator. We also point out that both the bias and the MSE decrease as n increases, which is the expected behaviour. In Fig. 3, we present the density of the g-weights for different methods and a sample of size $n = 191$. We see that for all distances, the calibrated weights are very close.

7.2 The effect of bounds on the variance

The goal of this experiment is to determine whether the choice of the bounds used in the truncated method (Sect. 3.1.8) has an impact on the bias and the variance of the associated estimator. In order to do this, we compute in the first step, given a sample s, the tightest bounds $0 < L \leq 1 \leq U < \infty$ such that the calibration equations (3) admit a solution in $\otimes_s [d_k L, d_k U]$. Then, we consider the truncated linear method using the

Table 5 Mean squared error MSE_{sim} for different sample sizes n and different distances. The Roy and Vanheuverzwyn method was used with $\alpha = 1$

Method	60	100	140	180
Horvitz Thomson estimator	436946828	216999444	122388041	67282420
Linear (Sect. 3.1.1)	76116122	43739730	24473129	12863236
Raking ratio (Sect. 3.1.2)	78511333	46374164	26147373	13539850
Minimum entropy (Sect. 3.1.3)	80860746	49478717	28571191	14769366
Deville (Sect. 3.1.5)	77251292	45325506	25472503	13284227
Roy and Vanheuverzwyn (Sect. 3.1.6)	74047975	42513623	23995349	12797530

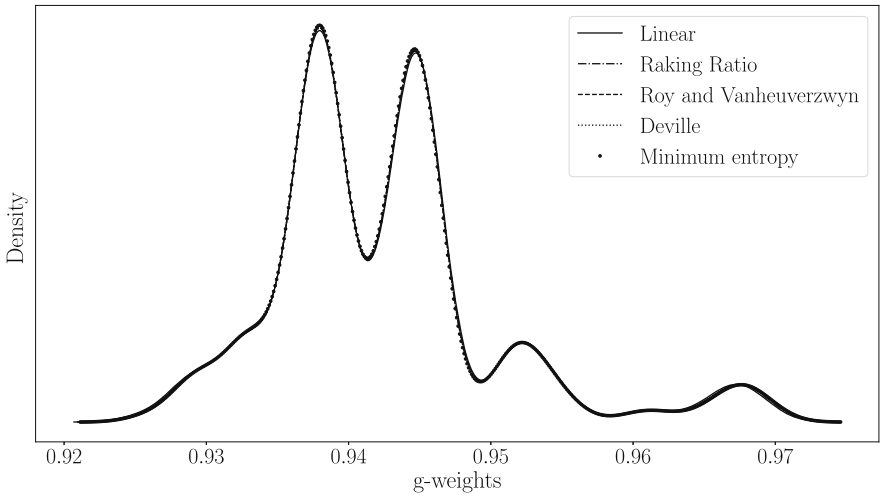

Fig. 3 Density of g-weights for different calibration methods in the framework of Sect. 7.1. The results are presented for a random sample of size $n = 191$

Table 6 Relative bias RB_{sim} for different expected sample sizes n using the truncated linear method for different perturbation parameters $\delta > 0$ as well as the standard linear method

δ	60	100	140	180
0.1	−4.2563	−2.4054	−1.4316	−0.9481
0.2	−4.4944	−2.5784	−1.5165	−0.9194
0.5	−4.5324	−2.3033	−1.1284	−0.5177
0.8	−4.5133	−2.2050	−1.0036	−0.3829
Linear	−4.5210	−2.1538	−0.9453	−0.3263

The values in the first column correspond to the value of δ

bounds $L_\delta = \max(L - \delta, 0)$ and $U_\delta = U + \delta$, where $\delta > 0$ is a prescribed perturbation parameter. We also compare the results to the unconstrained linear method. The relative bias and the MSE_{sim} are presented in Tables 6 and 7, respectively.

We see that as the bounds get larger, the bias decreases while the mean squared error increases. This means that the more we constrain the weights, the smaller the variance is. The effect on the bias is more visible for larger sample sizes. It is also

Table 7 Mean squared error $\mathrm{MSE}_{\mathrm{sim}}$ for different expected sample sizes n using the truncated linear method for different perturbation parameters $\delta > 0$ as well as the standard linear method

δ	60	100	140	180
0.1	63576144	35687345	20702892	11579681
0.2	67429678	37855993	21769450	12210672
0.5	72935408	41067769	23333886	12910721
0.8	74496347	42445119	23901620	13158124
Linear	75360750	43470046	24385658	13298633

The values in the first column correspond to the value of δ

interesting to notice that the estimator obtained through the truncated method tends to the one obtained with the classic linear method when either $\delta > 0$ increases or the average sample size increase. This coincides with the results obtained in the previous example where we have seen that the different methods tend to yield the same errors when n increases. The fact that the results tend to coincide when δ increases can be easily explained by noticing that $\lim_{\delta\to\infty} F^{\delta}_{k,\mathrm{trun}} = F_{k,\mathrm{lin}}$, where $F^{\delta}_{k,\mathrm{trun}}$ is the function associated with the truncated linear method for the bounds $[d_k L_\delta, d_k U_\delta]$.

8 Conclusion

Throughout this paper, we have presented an overview of the calibration approach in the framework of survey sampling. Formally introduced by Deville and Särndal (1992), this technique aims at decreasing the variance of a total estimator. The idea is to build a new set of (sample dependent) weights such that totals known at the population level can be verified at the sample level. Since there exists infinitely many systems of weights satisfying this, the question is then how to select a good one. One commonly used method is the so-called distance minimization approach in which the new weights minimize a prescribed pseudo-distance to the original weights associated with the HT estimator. We have then presented several candidates from the literature for such pseudo-distance and discuss some of their properties. In particular, we have defined what in our point of view an "ideal" function should be. To the best of our knowledge, the existence and uniqueness of a solution to the calibration problem had not been entirely covered so far. We then gave some sufficient conditions for this problem to have a solution.

Once the pseudo-distance and the calibration variables have been chosen, one needs to actually compute a solution to this problem. As we pointed out, the classic approach based on Newton method might not converge in some unlikely cases. An alternative algorithm has then been presented. In the last section, different numerical experiments have been conducted. No theoretical result proves that calibration actually increases the quality of the total estimator. However, 25 years of computations and application of this method have shown how powerful it is.

That is why we believe that this a very important tool for the estimation of totals in survey sampling. Finally, we acknowledge the work performed by Deville and Särndal (1992) and point out that this paper contained fundamental results that built a solid framework for further research in this topic.

Acknowledgements The authors thank the Swiss Federal Statistical Office (FSO) which partially supported this work as well as the three reviewers for their comments and efforts towards improving our manuscript.

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