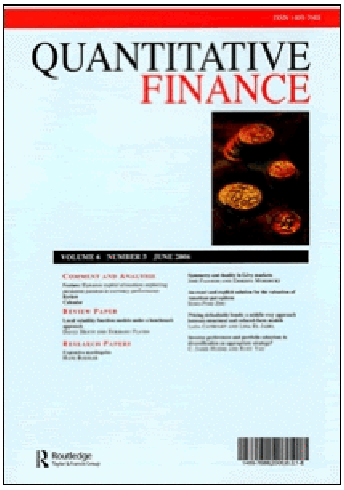

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An empirical analysis of multivariate copula models

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An empirical analysis of multivariate copula models

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Since the pioneering work of Embrechts and co-authors in 1999, copula models have enjoyed steadily increasing popularity in finance. Whereas copulas are well studied in the bivariate case, the higher-dimensional case still offers several open issues and it is far from clear how to construct copulas which sufficiently capture the characteristics of financial returns. For this reason, elliptical copulas (i.e. Gaussian and Student-*t* copula) still dominate both empirical and practical applications. On the other hand, several attractive construction schemes have appeared in the recent literature promising flexible but still manageable dependence models. The aim of this work is to empirically investigate whether these models are really capable of outperforming its benchmark, i.e. the Student-*t* copula and, in addition, to compare the fit of these different copula classes among themselves.

Keywords: KS-copula; Hierarchical Archimedian; Product copulas; Pair-copula decomposition

1. Introduction

The increasing linkages between countries, markets and companies require an accurate and realistic modelling of the underlying dependence structure. This applies to financial markets and, in particular, to the financial assets traded there-on. For a long time both practitioners and theorists have relied on the multivariate normal (Gaussian) distribution as statistical foundation, seemingly ignoring that this model assigns too little probability mass to extremal events. In order to overcome this drawback but still maintain many of the attractive properties, elliptical distributions (e.g. multivariate Student-*t* or multivariate generalized hyperbolic distribution) occasionally have found their way into the financial literature. Though being able to model heavy tails, elliptical distributions fail to capture asymmetric dependence structures. The copula concept, in contrast, which originally dates back to Sklar (1959) but was made popular in finance through the pioneering work of Embrechts and co-authors (1999), provides a flexible tool to capture different patterns of dependence. Within this work we assume that the reader is already familiar with the notion of copulas. Otherwise, we refer the reader to to Nelsen (2006) or Joe (1997). Whereas copulas are well studied in the bivariate case, construction schemes for higher dimensional copulas are not. Recently, several publications on high-dimensional copulas have appeared (e.g. Morillas 2005, Palmitesta and Provasi 2005, Liebscher 2006, Savu and Trede 2006, Aas *et al.* 2009). Each of them claims to provide a flexible dependence model, but there is no comprehensive comparison among these approaches, as far as we know. In particular, no comparisons are found to the Student-*t* copula (i.e. the copula associated to the multivariate Student-*t* distribution) which is known for its excellent fit to multivariate financial return data.

The outline of this work is as follows: section 2 overviews and connects several recent construction schemes of multivariate copulas. A short digression on goodness-of-fit measures can be found in section 3. Section 4 is dedicated to the description of the underlying data sets, whereas the empirical results are summarized and discussed in section 5. As in the bivariate case, both Student-*t* copula and pair-copulas based on bivariate Student-*t* copulas, as proposed in Aas *et al.* (2006), outperform their competitors also in the higher dimensional case.

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2. Constructing multivariate non-elliptical copulas

Among the classes of non-elliptical copulas, Archimedean copulas and its generalizations (section 2.1) enjoy great popularity. Beyond that, so-called pair-copula constructions are reviewed in section 2.2, where the joint distribution is decomposed into simple building blocks, so-called pair-copulas. Thirdly, we pick up the copulas associated with Koehler–Symanowski distributions in section 2.3 which have been successfully applied by Palmistesta and Provasi (2005) as models for financial returns. Finally, Liebscher's (2006) recent proposal to generalize given d-copulas is reviewed in section 2.4.

2.1. Multivariate Archimedean copulas

2.1.1. Classical multivariate Archimedean copulas. Let φ: $[0,\ 1] \to [0,\ \infty]$ be a continuous, strictly decreasing and convex function with $\varphi(1)=0$, $\varphi(0)\leq\infty$ and let $\varphi^{[-1]}$ be the so-called pseudo-inverse of φ defined by

$$\varphi^{[-1]}(t) \equiv \begin{cases} \varphi^{-1}(t) & 0 \leq t \leq \varphi(0), \\ 0 & \varphi(0) \leq t \leq \infty. \end{cases}$$

It can be shown (see, e.g. Nelsen, 2006) that

$$C(u_1, u_2) = \varphi^{[-1]}\Big(\varphi(u_1) + \varphi(u_2)\Big)$$

defines a class of bivariate copulas, the so-called Archimedean copulas. The function φ is called the (additive) generator of the copula. Furthermore, if $\varphi(0)=\infty$ the pseudo-inverse describes an ordinary inverse function (i.e. $\varphi^{[-1]}=\varphi^{-1}$) and in this case φ is known as a strict generator.

Given a strict generator $\varphi : [0,\ 1] \to [0,\ \infty]$, bivariate Archimedean copulas can be extended to the d-dimensional case. For every $d \geq 2$ the function $C : [0, 1]^d \to [0, 1]$ defined as

$$C(\mathbf{u}) = \varphi^{-1}\Big(\varphi(u_1) + \varphi(u_2) + \cdots + \varphi(u_d)\Big) \tag{1}$$

is a d-dimensional Archimedean copula if and only if φ^{-1} is completely monotonic on $\mathbb{R}_+$, i.e. if $\varphi^{-1} \in \mathcal{L}_\infty$ with

$$\mathcal{L}_m \equiv \Big\{\phi : \mathbb{R}_+ \to [0, 1] \Big| \phi(0) = 1, \phi(\infty) = 0, \\ (-1)^k \phi^{(k)}(t) \geq 0, \quad k = 1, \ldots, m, \Big\}.$$

The Gumbel copula is derived from the generator $\varphi(t)=(-\ln t)^\theta$, $\theta \geq 1$ and the Clayton copula is generated by $\varphi(t)=\theta^{-1}(t^{-\theta}-1)$, $\theta>0$. For an overview of further Archimedean copulas and the properties of the aforementioned ones, we refer the reader to the monographs of Nelson (2006) and Joe (1997).

2.1.2. Non-exchangeable Archimedean copulas. In order to increase flexibility and to allow for non-exchangeable dependence structures, several generalizations have emerged in the recent literature: a simple one—the so-called fully nested Archimedean (FNA) copulas—can be found in Joe (1997, p. 89), Whelan (2004) and Savu and Trede (2006), and requires $d-1$ generator functions $\varphi_1, \ldots, \varphi_{d-1}$ with $\varphi_1^{-1}, \ldots, \varphi_{d-1}^{-1} \in \mathcal{L}_\infty$ and $\varphi_{i+1} \circ \varphi_i^{-1}(t) = \varphi_{i+1}(\varphi_i^{-1}(t)) \in \mathcal{L}_\infty^*$ for

$$\mathcal{L}_d^* = \Big\{\phi : \mathbb{R}_+ \to \mathbb{R}_+ \Big| \phi(0) = 0, \phi(\infty) = \infty, \\ (-1)^{k-1} \phi^{(k)}(t) \geq 0, \quad k = 1, \ldots, d, \Big\}.$$

The structure of FNA d-copulas is rather simple: one first couples u_1 and u_2, then the copula of u_1 and u_2 with u_3 to a new copula which is coupled afterwards with u_4 and so on. Hence the FNA 4-copula is of the form

$$C(\mathbf{u}) = \varphi_3^{-1}\big[\varphi_3\big(\varphi_2^{-1}\big[\varphi_2\big(\varphi_1^{-1}[\varphi_1(u_1) + \varphi_1(u_2)]\big) + \varphi_2(u_3)\big]\big) \\ + \varphi_3(u_4)\big]. \tag{2}$$

Figure 1 illustrates one possible FNA copula for dimension $d=5$.

Alternatively, mixing ordinary Archimedean and FNA copulas, partially nested Archimedean (PNA) copulas may be used. Again, for ease of notation, we focus on the 4-variate case

$$C(\mathbf{u}) = \varphi^{-1}\big[\varphi\big(\varphi_{12}^{-1}[\varphi_{12}(u_1) + \varphi_{12}(u_2)]\big) \\ + \varphi\big(\varphi_{34}^{-1}[\varphi_{34}(u_3) + \varphi_{34}(u_4)]\big)\big]. \tag{3}$$

Note that φ, φ_{12}, φ_{34} are generators with $\varphi^{-1}, \varphi_{12}^{-1}, \varphi_{34}^{-1} \in \mathcal{L}_\infty$ and $\varphi \circ \varphi_{12}^{-1}$, $\varphi \circ \varphi_{34}^{-1} \in \mathcal{L}_\infty^*$. Obviously, one first couples the pairs u_1, u_2 and u_3, u_4 with distinct generators. The resulting copula pair is then coupled using a third generator φ (which in turn might be coupled with an additional variable u_5 using a fourth generator ψ for an extension to the 5-dimensional case). Another possible structure of a PNA copula is illustrated in figure 1.

Thirdly, copula C from (3) is also an example of a so-called hierarchical Archimedean (HA) copula. Borrowing the notation of Savu and Trede (2006), the basic idea of this approach is to build a hierarchy of Archimedean copulas with L different levels, indexed by $l=1, \ldots, L$. At each level l there are n_l distinct objects, indexed by $j=1, \ldots, n_l$. In a first step (i.e. in level 1), the

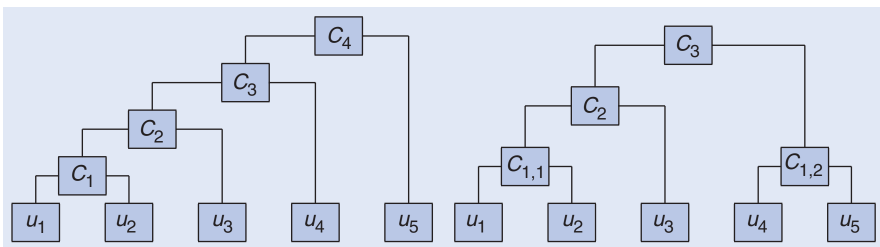

Figure 1. FNA copula (left) and PNA copula (right) for $d=5$.

variables $u_1, \ldots, u_d$ are grouped into n_1 ordinary multivariate Archimedean copulas

$$C_{1,j}(\mathbf{u}_{1,j}) = \varphi_{1,j}^{-1}\left(\sum_{\mathbf{u}_{1,j}} \varphi_{1,j}(\mathbf{u}_{1,j})\right), \quad j = 1, \ldots, n_1.$$

with (possibly different) generators $\varphi_{1,j}$ and where $\mathbf{u}_{1,j}$ denotes the set of elements of $u_1, \ldots, u_d$ belonging to $C_{1,j}$. All copulas of the first level are again grouped into copulas at level $l=2$. These copulas $C_{2,j}$ with generator function $\varphi_{2,j}$, $j=1, \ldots, n_2$ are generalized Archimedean copulas, whose dependence structure is only of partial exchangeability and consists of copulas from the previous level (as elements), denoted by

$$C_{2,j}(\mathbf{C}_{2,j}) = \varphi_{2,j}^{-1}\left(\sum_{\mathbf{C}_{2,j}} \varphi_{2,j}(\mathbf{C}_{2,j})\right),$$

where $\mathbf{C}_{2,j}$ represents the set of all copulas from level $l=1$ entering copula $C_{2,j}$. This procedure continues until only a single hierarchical Archimedean copula $C_{L,1}$ is achieved at level L. In order to ensure that $C_{L,1}$ is a proper copula, we have to proclaim that $\varphi_{l,j}^{-1} \in \mathcal{L}_\infty$ for $l=1, \ldots, L$ and $j=1, \ldots, n_l$, and that $\varphi_{l+1,i} \circ \varphi_{l,j}^{-1} \in \mathcal{L}_\infty^*$ for all $l=1, \ldots, L$ and $j=1, \ldots, n_l$, $i=1, \ldots, n_{l+1}$ such that $C_{l,j} \in \mathbf{C}_{l+1,i}$. Moreover, a hierarchy is established if the number of copulas decreases at each level, if the top level contains only a single object and if at each level the dimensions of the copulas add up to d. Figure 2 displays the possible construction of a 5-dimensional HA-copula.

Savu and Trede (2006) also derive the HA-copula density

$$\frac{\partial^d C_{L,1}}{\partial u_1 \ldots \partial u_d} = \sum \frac{\partial^{d-i} C_{L,1}}{\partial C_{L-1,1}^{k_1} \ldots \partial C_{L-1,n_{L-1}}^{k_n,L-1}} \times \prod_{r=1}^{n_{L-1}} \sum_{u=\upsilon_1,\ldots,\upsilon_r} \frac{\partial^{|\upsilon_1|} C_{L-1,r}}{\partial \upsilon_1}, \ldots, \frac{\partial^{|\upsilon_r|} C_{L-1,r}}{\partial \upsilon_r},$$

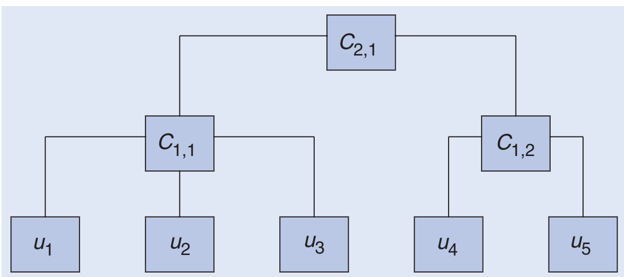

Figure 2. HA copula for $d=5$.

where the outer sum extends over all sets of integers $k_1, \ldots, k_{n,L-1} \in \mathbb{N}_0$ such that $\max_j k_j \le d_{L-1,j}$ and $\sum_{j=1}^{n_{L-1}} k_j = d-i$ for all $i=0, \ldots, d-n_{L-1}$. These terms are the outer derivatives of the copula with respect to the elements of $\mathbf{C}_{L,1}$, i.e. the n_{L-1} copulas from level $L-1$. The second part of the formula comprises the inner derivatives, corresponding to the derivatives of the copulas at level $L-1$ with respect to their arguments $\mathbf{u}_{L-1,j}$.

2.1.3. Generalized multiplicative Archimedean copulas. In this section we focus on methods recently proposed by Morillas (2005) and Liebscher (2006). Both approaches are based on a second functional representation of Archimedean copulas via so-called multiplicative generators (see Nelsen 2006). Setting $\vartheta(t) \equiv \exp(-\varphi(t))$ and $\vartheta^{[-1]}(t) \equiv \varphi^{[-1]}(-\ln t)$, equation (1) can be rewritten as

$$C(u_1, \ldots, u_d) = \vartheta^{[-1]}\Big(\vartheta(u_1) \cdot \vartheta(u_2) \cdot \ldots \cdot \vartheta(u_d)\Big). \quad (4)$$

The function ϑ is called the multiplicative generator of C. Due to the relationship between φ and ϑ, the function ϑ : $[0, 1] \to [0, 1]$ is continuous, strictly increasing and concave with $\vartheta(1)=1$ and $\vartheta^{[-1]}(t)=0$ if $0 \le t \le \vartheta(0)$ and $\vartheta^{[-1]}(t)=\vartheta^{-1}(t)$ if $\vartheta(0) \le t \le 1$.

Equation (4) can also be expressed using the independence copula $C^\perp(\mathbf{u}) = \prod_{i=1}^d u_i$:

$$C(u_1, \ldots, u_d) = \vartheta^{[-1]}\Big(C^\perp(\vartheta(u_1), \ldots, \vartheta(u_d))\Big).$$

Morillas (2005) substitutes $C^\perp$ by an arbitrary d-copula C in order to obtain

$$C_\vartheta(u_1, \ldots, u_d) = \vartheta^{[-1]}\Big(C(\vartheta(u_1), \vartheta(u_2), \ldots, \vartheta(u_d))\Big) \quad (5)$$

and proves that C_ϑ is a d-copula if $\vartheta^{[-1]}$ is *absolutely monotonic of order* d on $[0, 1]$, i.e. if $\vartheta^{[-1]}(t)$ satisfies $(\vartheta^{[-1]})^{(k)}(t) = \mathrm{d}^k \vartheta^{[-1]}(t)/\mathrm{d}t^k \ge 0$ for $k=1,2,\ldots,d$ and $t \in (0, 1)$.

Examples of generator functions are stated in Morillas (2005). Notice that not every generator given there is absolutely monotonic for arbitrary $d>1$: as one can easily verify, the generator $\vartheta(t)=t^r/(2-t^r)$, $r \in (0, 1/3]$ (see table 1, no. 9 in Morillas, 2005) has no absolutely monotonic pseudo-inverse of order $d \ge 3$, because the third derivative of $\vartheta^{[-1]}$ becomes negative. Hence this generator is suitable only for a construction of generalized bivariate copulas. Concerning the basic properties of such Morillas copulas we refer to Morillas (2005).

Table 1. German stock returns.

Start	End	N	Stocks	μ	s^2	$\mathbb{S}$	$\mathbb{K}$	$\mathcal{LB}(10)$	$\mathcal{LM}(10)$
02-01-90	12-11-03	3486	HVB	0.004	5.61	−0.033	8.16	24.45	621.08
02-01-90	12-11-03	3486	BMW	0.046	4.33	−0.132	7.19	28.96	366.49
02-01-90	12-11-03	3486	Allianz	−0.002	4.87	−0.07	8.37	29.74	517.14
02-01-90	12-11-03	3486	MunichRe	0.02	5.06	−0.027	8.75	50.53	508.59

Another way of generalizing Archimedean copulas is the method proposed by Liebscher (2006) who introduces the following copula representation

$$C(u_1, \ldots, u_d) = \Psi\left(\frac{1}{m}\sum_{j=1}^{m} \psi_{j1}(u_1) \cdot \psi_{j2}(u_2) \cdot \ldots \cdot \psi_{jd}(u_d)\right), \quad (6)$$

where Ψ and $\psi_{jk} : [0, 1] \to [0, 1]$ are functions satisfying the following conditions: firstly, it is assumed that $\Psi^{(d)}$ exists with $\Psi^{(k)}(u) \geq 0$ for $k = 1, 2, \ldots, d$ and $u \in [0, 1]$, and that $\Psi(0) = 0$. Secondly, ψ_{jk} is assumed to be differentiable and monotone increasing with $\psi_{jk}(0) = 0$ and $\psi_{jk}(1) = 1$ for all k, j. Thirdly, Liebscher's construction requires that

$$\Psi\left(\frac{1}{m}\sum_{j=1}^{m} \psi_{jk}(v)\right) = v \quad \text{for } k = 1, 2, \ldots, d \text{ and } v \in [0, 1].$$

The three conditions guarantee that C defined in (6) is actually a copula.

It is easily seen that the approaches of Morillas (2005) and Liebscher (2006) coincide for $m = 1$, $\vartheta^{[-1]} = \Psi$ in (6) and $C_\vartheta = C^\perp$ in (5).

Liebscher (2006) also states a general method for deriving appropriate functions ψ_{jk}. Let $h_{jk} : [0, 1] \to [0, 1]$, $j = 1, \ldots, m$, $k = 1, \ldots, d$ be a differentiable and bijective function such that $h'_{jk}(u) > 0$ for $u \in (0, 1)$, $h_{jk}(0) = 0$, $h_{jk}(1) = 1$ and $m \cdot u = \sum_{j=1}^{m} h_{jk}(u)$, $u \in [0, 1]$, $k = 1, \ldots, d$. Let $\psi = \Psi^{-1}$ be the differentiable inverse function of Ψ. An appropriate choice is setting $\psi_{jk}(u) = h_{jk}(\psi(u))$, since $\psi'_{jk}(u) = h'_{jk}(\psi(u)) \cdot \psi'(u) > 0$ for $j = 1, \ldots, m$ and $u \in [0, 1]$. Considering $m = 2$, define

$$h_{1k}(u) \equiv u^{\delta_k}, \quad h_{2k}(u) \equiv 2u - u^{\delta_k} \text{ with } \delta_k \in [1, 2]. \quad (7)$$

Choosing further

$$\Psi(t) = -\frac{1}{\theta}\ln(1 - (1 - e^{-\theta})t), \text{ and } \psi(u) = \frac{1 - e^{-\theta u}}{1 - e^{-\theta}}, \theta > 0$$

and defining $\psi_{jk}(u) \equiv h_{jk}(\psi(u))$ a generalized Frank copula (GMLF) is obtained. Setting $\delta_k = 1$ for all but one k, $k = 1, \ldots, d$, it is easily verified that the GMLF copula reduces to the common Frank copula. Setting $m = 2$ and h_{jk} as in (7) but now choosing (see table 2, no. 2, p. 8 in Liebscher (2006))

$$\Psi(t) = \frac{(\delta - t)^{-\theta} - \delta^{-\theta}}{(\delta - 1)^{-\theta} - \delta^{-\theta}}, \text{ and}$$

$$\psi(u) = \delta - \left(\delta^{-\theta}(1 - u) + u(\delta - 1)^{-\theta}\right)^{-1/\theta}, \quad \theta > 0, \delta > 1$$

a copula is obtained, which will be termed as the GML2 copula henceforth.

In the field of insurance pricing the function ψ_{jk} is known as a distortion function (for a definition see Freez and Valdez 1998) and the methods proposed by e.g. Freez and Valdez (1998) or Wang (1998) appear as special cases in (6). The same holds for the approach given by Morillas (2005) where the function ϑ also satisfies the requirements of a distortion function.

2.2. *Pair-copula decompositions*

2.2.1. Pair-copula decomposition: the general case. One way of calculating a multivariate density is by decomposing it into a product of marginal densities and conditional densities. The latter can be replaced stepwise by so-called pair-copulas.

Again, let $\mathbf{X} = (X_1, \ldots, X_d)'$ have the joint density function

$$f(x_1, \ldots, x_d) = f(x_d) \cdot f(x_{d-1}|x_d) \cdot f(x_{d-2}|x_{d-1}, x_d) \cdot \ldots \cdot f(x_1|x_2, \ldots, x_d), \quad (8)$$

which is unique up to a relabelling of the variables. Because of

$$f(x_1, \ldots, x_d) = c_{12\ldots d}(F_1(x_1), \ldots, F_d(x_d)) \cdot f_1(x_1) \cdots f_d(x_d),$$

with $c_{12\ldots d}(\cdot)$ being the d-variate copula density, $f(x_d|x_{d-1})$, for example, may also be expressed by $c_{12}(F_1(x_1), F_2(x_2)) \cdot f_1(x_1)$, where $c_{12}(\cdot, \cdot)$ is called *pair-copula* density for the respective transformed variables. Similarly, $f(x_{d-2}|x_{d-1}, x_d)$, can be decomposed into

$$c_{(d-2)d|d-1}\left(F_{d-2|d-1}(x_{d-2}|x_{d-1}), F_{d|d-1}(x_d, x_{d-1})\right) \cdot f(x_{d-2}|x_{d-1}).$$

Using $f(x_{d-2}|x_{d-1}) = c_{(d-2)(d-1)}(F_{d-2}(x_{d-2}), F_{d-1}(x_{d-1})) \cdot f_{d-2}(x_{d-2})$ results in

$$f(x_{d-2}|x_{d-1}, x_d) = c_{(d-2)d|d-1}(F_{d-2|d-1}(x_{d-2}|x_{d-1}), F_{d|d-1}(x_d, x_{d-1})) \cdot c_{(d-2)(d-1)}(F_{d-2}(x_{d-2}), F_{d-1}(x_{d-1})) \cdot f_{d-2}(x_{d-2}),$$

which is not unique anymore, because one may also condition on x_d instead of x_{d-1}. This leads to a different decomposition. The *general formula* reads as

$$f(x|\mathbf{v}) = c_{xv_j|\mathbf{v}_{-j}}(F(x|\mathbf{v}_{-j}, F(v_j|\mathbf{v}_{-j})) \cdot f(x|\mathbf{v}_{-j}) \quad (9)$$

Table 2. Exchange rates.

Start	End	N	FX Rate	μ	s^2	$\mathbb{S}$	$\mathbb{K}$	$\mathcal{LB}(10)$	$\mathcal{LM}(10)$
02-01-90	31-12-04	8054	CAD	0.002	0.09	−0.004	6.75	12.65	912.18
02-01-90	31-12-04	8054	YEN	−0.015	0.56	−0.002	6.11	12.5	429.24
02-01-90	31-12-04	8054	SFR	0.003	0.36	0.132	6.84	55.79	485.26
02-01-90	31-12-04	8054	BRP	−0.013	0.44	−0.723	13.33	34.48	176.20

for a d-dimensional vector v with components v_j. The vector $\mathbf{v}_{-j}$ denotes v excluding the component v_j. For methods and formulas to calculate $F(x|\mathbf{v})$ we refer to Joe (1996).

As seen above, every (conditional) d-dimensional density can be split up into a pair-copula and a $(d-1)$-dimensional (conditional) density. For $d>2$ you can iteratively repeat this splitting for the $(d-1)$-dimensional conditional density. Eventually, a product of univariate densities and pair-copulas is obtained. As shown in the trivariate case, this decomposition is not unique but there are various ways of doing this.

In order to sort the different decomposition constructs, so-called *regular vines* (see Bedford and Cooke 2001, 2002) are defined. Vines are graphical models that present complete decomposition schemes. Following Aas *et al.* (2007) we choose the structure of the D-vine. The joint density $f(x_1, \ldots, x_d)$ can be expressed as

$$\prod_{k=1}^{d} f(x_k) \prod_{j=1}^{d-1} \prod_{i=1}^{d-j} c_{i,i+j|i+1,\ldots,i+j-1}(F(x_i|x_{i+1}, \ldots, x_{i+j-1}), F(x_{i+j}|x_{i+1}, \ldots, x_{i+j-1})).$$

The decomposition of a four-dimensional density according to the D-vine scheme is

$$\begin{aligned} f(x_1, x_2, x_3, x_4) = {} & f(x_1) \cdot f(x_2) \cdot f(x_3) \cdot f(x_4) \\ & \cdot c_{12}(F(x_1), F(x_2)) \cdot c_{23}(F(x_2), F(x_3)) \\ & \cdot c_{34}(F(x_3), F(x_4)) \cdot c_{13|2}(F(x_1|x_2), F(x_3|x_2)) \\ & \cdot c_{24|3}(F(x_2|x_3), F(x_4|x_3)) \\ & \cdot c_{14|23}(F(x_1|x_2, x_3), F(x_4|x_2, x_3)). \end{aligned} \quad (10)$$

2.2.2. Pair-copula decomposition of a copula. Originally, the pair-copula decomposition (PCD) decomposes the common density f of d random variables. Of course, one may also apply the pair-copula decomposition to the underlying copula density c, as we will show in this subsection. To simplify notation, we restrict ourselves to $d=4$ and the D-vine decomposition. As an immediate consequence of Sklar's (1959) theorem,

$$c(F(x_1), F(x_2), F(x_3), F(x_4)) = \frac{f(x_1, x_2, x_3, x_4)}{f(x_1) \cdot f(x_2) \cdot f(x_3) \cdot f(x_4)}.$$

Substituting the common density by its PCD given in (10),

$$\begin{aligned} & c(F(x_1), F(x_2), F(x_3), F(x_4)) \\ & = c_{12}(F(x_1), F(x_2)) \cdot c_{23}(F(x_2), F(x_3)) \cdot c_{34}(F(x_3), F(x_4)) \\ & \quad \cdot \ c_{13|2}(F(x_1|x_2), F(x_3|x_2)) \cdot c_{24|3}(F(x_2|x_3), F(x_4|x_3)) \\ & \quad \cdot \ c_{14|23}(F(x_1|x_2, x_3), F(x_4|x_2, x_3)) \end{aligned}$$

with $c_{i|j}(\cdot, \cdot)$ being a pair-copula density and its indices i, j refer to x_i and x_j. According to Joe (1996),

$$F(x|\mathbf{v}) = \frac{\partial\, C_{x,v_j|\mathbf{v}_{-j}}(F(x|\mathbf{v}_{-j}), F(v_j|\mathbf{v}_{-j}))}{\partial\, F(v_j|\mathbf{v}_{-j})}$$

with $\mathbf{v}_{-j}$ being the vector $\mathbf{v}$ except the element v_j. In the univariate case (i.e. $\mathbf{v}=v$),

$$F(x|v) = \frac{\partial C_{x|v}(F_X(x), F_V(v))}{\partial F_V(v)} \equiv h(x, v, \theta),$$

where θ is the parameter vector of the copula $C_{x|v}$. The copula density decomposition can be written as follows: it is obvious that $F(x_1|x_2)=h(x_1, x_2, \theta_{12})$ with θ_{12} being the parameter (vector) of the copula C_{12}. Analogously, $F(x_3|x_2)=h(x_3, x_2, \theta_{23})$, $F(x_2|x_3)=h(x_2, x_3, \theta_{23})$ and $F(x_4|x_3)=h(x_4, x_3, \theta_{34})$. $F(x_1|x_2, x_3)$, and again, can be iteratively simplified to

$$\begin{aligned} & \frac{\partial C_{13|2}(F(x_1|x_2), F(x_3|x_2))}{\partial F(x_3|x_2)} \\ & \quad = h(h(x_1, x_2, \theta_{12}), h(x_3, x_2, \theta_{32}), \theta_{13|2}). \end{aligned}$$

Analogously, $F(x_4|x_2, x_3)$ can be written as

$$\begin{aligned} & \frac{\partial C_{24|3}(F(x_4|x_3), F(x_2|x_3))}{\partial F(x_2|x_3)} \\ & \quad = h(h(x_4, x_3, \theta_{43}), h(x_2, x_3, \theta_{23}), \theta_{24|3}). \end{aligned}$$

Finally, define $u_1=F(x_1)$, $u_2=F(x_2)$, $u_3=F(x_3)$, $u_4=F(x_4)$. The formula for the four-dimensional PCD copula density now reads as

$$\begin{aligned} c(\mathbf{u}) = {} & c_{12}(u_1, u_2) \cdot c_{23}(u_2, u_3) \cdot c_{34}(u_3, u_4) \\ & \cdot c_{13|2}(h(u_1, u_2, \theta_{12}), h(u_3, u_2, \theta_{23})) \\ & \cdot c_{24|3}(h(u_2, u_3, \theta_{23}), h(u_4, u_3, \theta_{34})) \\ & \cdot c_{14|23}(h(h(u_1, u_3, \theta_{13}), h(u_2, u_3, \theta_{23}), \theta_{13|2}), \\ & \qquad h(h(u_4, u_3, \theta_{43}), h(u_2, u_3, \theta_{23}), \theta_{24|3})). \end{aligned}$$

To summarize, in order to specify a d-dimensional (copula) density, two main steps have to be taken (see Aas *et al.* 2007): firstly, an appropriate decomposition scheme has to be selected. Secondly, the pair-copulas have to be specified: e.g. Gaussian, Student's t, Archimedean or Gumbel copula. It is possible to use one copula model for all pair-copulas or decide individually.

2.3. Koehler–Symanowski (KS) copulas

Koehler and Symanowski (1995) introduce a multivariate distribution as follows: for the index set $V=\{1, 2, \ldots, d\}$, let $\mathcal{V}$ denote the power set of V and $\mathcal{I} \equiv \{I \in \mathcal{V}$ with $|I| \geq 2\}$. Further let $\mathbf{X}$ denote a d-dimensional random vector with univariate marginal distributions $F_i(x_i)$, $i \in V$. For all subsets $I \in \mathcal{I}$ let $\alpha_I \in \mathbb{R}_0^+$ and $\alpha_i \in \mathbb{R}_0^+$ for all $i \in V$ such that $\alpha_{i+}=\alpha_i+\sum_{I \in \mathcal{I}} \alpha_I>0$ for $i \in I$. Then the common cdf F is defined by

$$F(x_1, \ldots, x_d) = \frac{\prod_{i \in V} F_i(x_i)}{\prod_{I \in \mathcal{I}} \left[\sum_{i \in I} \prod_{j \in I, j \neq i} F_j(x_j)^{\alpha_{j+}} - (|I|-1) \prod_{i \in I} F_i(x_i)^{\alpha_{i+}} \right]^{\alpha_I}}. \quad (11)$$

The terms $K_I=\sum_{i \in I} \Pi_{j \in I, j \neq i} F_j(x_j)^{\alpha j+} - (|I|-1) \Pi_{i \in I} \times F_i(x_i)^{\alpha_{i+}}$ are called association terms. Moreover, Koehler and Symanowski (1995) showed that the joint density function exists if the marginal density functions f_i exist for all $i \in V$. Due to the design of the Koehler–Symanowski distribution the corresponding copula has a similar functional form: setting $u_i = F_i(x_i)$ for all $i \in V$, the KS copula is

$$C(u_1, \ldots, u_d) = \frac{\prod_{i \in V} u_i}{\prod_{I \in \mathcal{I}} \left[\sum_{i \in I} \prod_{j \in I, j \neq i} u_j^{\alpha_{j+}} - (|I| - 1) \prod_{i \in I} u_i^{\alpha_{i+}} \right]^{\alpha_I}}. \quad (12)$$

In contrast to the cumulative distribution function the functional representation of the density is quite complicated due to complex factors with additive components. Koehler and Symanowski (1995) gave an explicit formula for the special case of a so-called KS(2)-distribution (Caputo 1998), where all parameters α_I are set equal to zero for $|I| > 2$. The corresponding copula will be called the KS(2) copula henceforth. Assuming that $\alpha_{ij} \equiv \alpha_{ji} \geq 0$ for all $(i, j) \in V \times V$ and $\alpha_{i+} = \alpha_{i1} + \alpha_{i2} + \cdots + \alpha_{id} > 0$ for all $i \in V$, the KS(2)-copula simplifies to

$$C(u_1, u_2, \ldots, u_d) = \prod_{i=1}^{d} u_i \prod_{i \leq j} \prod K_{ij}^{-\alpha_{ij}} \quad (13)$$

with $K_{ij} \equiv u_i^{1/\alpha_{i+}} + u_j^{1/\alpha_{j+}} - u_i^{1/\alpha_{i+}} u_j^{1/\alpha_{j+}} = K_{ji}$.

Palmitesta and Provasi (2005) apply this particular KS copula to weekly log-returns. They also argue that this copula has the ability to model complex dependence structures among subsets of marginal distribution but they do not present any goodness-of-fit measure or any comparison with other copulas.

2.4. Multiplicative Liebscher copulas

By now, different methods of how to construct d-variate copulas have been reviewed. Liebscher (2006) discusses how to combine or connect a given set of k possibly different d-copulas $C_1, \ldots, C_k$ to a new d-copula C in order to increase flexibility and/or introduce asymmetry. His proposal focuses on multiplicative connections of d-copulas of the form

$$C(u_1, \ldots, u_d) = \prod_{j=1}^{k} C_j(g_{j1}(u_1), \ldots, g_{jd}(u_d)) \quad (14)$$

with a set of $k \cdot d$ admissible functions $g_{11}, \ldots, g_{1d}, \ldots, g_{k1}, \ldots, g_{kd}$, each of which being bijective, monotonously increasing or identically equal 1 satisfying

$$\prod_{j=1}^{k} g_{ji}(v) = v, \quad i = 1, \ldots, d. \quad (15)$$

Note that (15) reduces to $g_{1i}(v) = v$ for $k = 1$ and $i = 1, \ldots, d$, and C is recovered. In accordance to Liebscher (2006), possible choices are

$$g_{ji}(v) \equiv v^{\theta_{ji}} \text{ with } \theta_{ji} > 0 \text{ and } \sum_{j=1}^{k} \theta_{ji} = 1 \text{ for } i = 1, \ldots, d \quad (16)$$

or $g_{1i}(v) \equiv f(v), \quad g_{2i}(v) \equiv v \cdot \frac{1}{f(v)}, \quad f(v) = \left(\frac{1 - e^{-\theta_i v}}{1 - e^{-\theta_i}} \right)^{\alpha},$
$\theta > 0, \alpha \in (0, 1).$ (17)

We consider four different generalized Clayton copulas based on (14). The 'Generalized Clayton of Liebscher type I' (L_1) is obtained by setting $k = 2$, choosing the Clayton copula for C_1, the independence copula for C_2 and $g_{ji}(v)$ as in (16). Applying (17) rather than (16), the 'Generalized Clayton of Liebscher type II' (L_2) copula with $d + 2$ dependence parameters is constructed. Similarly, combining two d-variate Clayton copulas and using g from (16) we obtain the d-dimensional copula family with $d + 2$ parameters, termed as the 'Generalized Clayton of Liebscher type III' (L_3) in the sequel. Finally, applying again (17) rather than (16), the 'Generalized Clayton of Liebscher type IV' (L_4) is obtained.

3. Goodness-of-fit measures

We now tackle the problem of comparing the goodness-of-fit (GOF) of the different copula models from section 2, noting that most of them are not nested. As we apply maximum likelihood (ML) methods to obtain estimators for the unknown parameter vector, the first choice is the log-likelihood value ℓ or—in order to take the different numbers of parameters into account—the Bayesian information criterion $BIC = -2\ell + K \ln(N)$, where K and N denote the number of parameters to be fitted and the number of observations, respectively. However, comparing log-likelihood values for non-nested models may produce misleading conclusions. Therefore, other GOF tests may come to application. Following Breymann *et al.* (2003), Chen *et al.* (2004) or recently Berg and Bakken (2006), the main idea is to project the multivariate problem onto a set of independent and uniform $U(0, 1)$ variables, given the multivariate distribution and to calculate the distance (e.g. Anderson–Darling, Kolmogorov–Smirnov, Cramér–von Mises) between the transformed variables and the uniform distribution. In contrast to the authors above, we are not primarily interested whether the data stem from the specified copula model. Instead, we use these distances as a criterion itself. The proceeding is roughly as follows.

By means of the Rosenblatt (1952) transformation the random vector $\mathbf{X} = (X_1, \ldots, X_d)'$ is mapped onto a random vector $\boldsymbol{Z}^* = (Z_1^*, \ldots, Z_d^*)'$ via

$$Z_1^* \equiv F_1(X_1) \text{ and } Z_i^* \equiv F_{X_i|X_1,\ldots,X_{i-1}}(X_i|X_1, \ldots, X_{i-1}), \quad i = 2, \ldots, d. \quad (18)$$

It can be shown that $\boldsymbol{Z}^*$ is uniformly distributed on $[0, 1]^d$ with independent components $Z_1^*, \ldots, Z_d^*$. Assume that the cumulative distribution function of $\boldsymbol{X}$ admits the decomposition

$$F_{\boldsymbol{X}}(x_1, \ldots, x_d) = C(F_{X_1}(x_1), \ldots, F_{X_d}(x_d)),$$

where $C(\cdot)$ denotes a parametric copula which is the common distribution function of $\boldsymbol{U} = (U_1, \ldots, U_d)'$ with

$U_i \equiv F_{X_i}(X_i)$. Define $C(u_1, \ldots, u_j) \equiv C(u_1, \ldots, u_j, 1, \ldots, 1)$ for $j \le d$. Furthermore, the conditional distribution of $U_i | U_1, \ldots, U_{i-1}$ is given by

$$C_i(u_i) \equiv \frac{\partial^{i-1} C(u_1, \ldots, u_i)}{\partial u_1 \ldots \partial u_{i-1}} \bigg/ \frac{\partial^{i-1} C(u_1, \ldots, u_{i-1})}{\partial u_1 \ldots \partial u_{i-1}}.$$

According to (18), the variables

$$Z_1 \equiv C(U_1) = U_1 \text{ and } Z_i \equiv C_i(U_i), \quad i = 2, \ldots, d \quad (19)$$

are independent and uniform on [0, 1]. Consequently, the sample $X_1, \ldots, X_N$ from a parametric copula and with marginals given by $F_1, \ldots, F_d$ can be mapped onto an *iid* sample $Z_1, \ldots, Z_N$ from a uniform distribution on $[0, 1]^d$.

Breymann *et al.* (2003) suggest transforming each random vector $Z_i = (Z_{i1}, \ldots, Z_{id})'$ into a (univariate) chi-square variable χ_j with d degrees of freedom through $\chi_j = \sum_{i=1}^{d} \Phi^{-1}(Z_{ji})^2$, $j = 1, \ldots, N$, where $\Phi^{-1}(u)$ denotes the standard normal quantile function. If the margins are unknown, they may be replaced by the corresponding empirical counterparts. Breymann *et al.* state that 'we do assume that the χ^2-distribution will not be significantly affected by the use of the empirical distribution functions used to transform the marginal data'.

4. The data set

The data sets we used to compare the different copula models come from three different markets (German stock market, foreign exchange (FX) market and commodity markets). From each market, four typical representatives were selected, provided that the corresponding sample period is sufficiently large. Instead of analysing the prices themselves, we calculated and considered (percentual) continuously compounded returns ('log-returns') $R_t = 100(\log P_t - \log P_{t-1})$, $t = 2, \ldots, N$. In order to account for possible time-dependencies (which are common to most financial return series), we also fitted univariate GARCH models of the form $R_t = \mu + \gamma_1 R_{t-1} + \cdots + \gamma_k R_{t-k} + h_t \varepsilon_t$ with variance equations $h_t^2 = \alpha_0 + \alpha_1 R_{t-1}^2 + \cdots + \alpha_1 R_{t-p}^2 + \beta_1 h_{t-1}^2 + \cdots + \beta_q h_{t-q}^2$ to each of the series and considered standardized residuals ε_t rather than the original returns R_t. Secondly, as we are not primarily interested in parametric models for the marginal distributions, all observations (i.e. returns or standardized residuals) were transformed into uniform ones by means of the (empirical) probability integral transform, i.e.

$$U_t = F_N(R_t) \text{ with } F_N(x) = \frac{\{\# R_t | R_t \le x\}}{\# R_t} \text{ and } U_t^* = F_N(\epsilon_t).$$

4.1. German stock returns

From the German stock market, we selected prices of HVB AG, BMW AG, Allianz AG and Munich Re AG, all of them being part of the German stock market index DAX which measures the performance of the Prime Standard's 30 largest German companies in terms of order book volume and market capitalization. Figure 3 contains the series of prices and returns. Table 1 summarizes descriptive and inductive statistics. All series feature negative skewness and high kurtosis (measured by the third and fourth standardized moment $\mathbb{S}$ and $\mathbb{K}$). Moreover, there is empirical evidence for (slight) serial correlation and GARCH effects as the Ljung-Box statistic $\mathcal{LB}$ and Engle's Lagrange multiplier statistic $\mathcal{LM}$ indicate (the critical χ^2-value is given by 18.307 in both cases for $\alpha = 0.05$).

4.2. Exchange rate returns

Data from foreign exchange markets (FX-markets) are available from the PACIFIC Exchange Rate Service†. This service offered by Prof. Werner Antweiler at UBC's Sauder School of Business provides access to current and historic daily exchange rates through an on-line database retrieval and plotting system. In contrast to the volume notation, where values are expressed in units of the target currency per unit of the base currency, the price notation is used within this work which corresponds to the numerical inverse of the volume notation. All values are expressed in units of the base currency (here US-Dollar) per unit of the target currency. Table 2 summarizes the statistics of the four exchanges rates (Canadian Dollar, Japanese Yen, Swiss Franc, British Pound) which are used later on. Again, prices and log-returns are shown in figure 4 below.

4.3. Metal returns

The London Metal Exchange‡ (LME) is the world's premier non-ferrous metals market with a turnover value of some US$2000 billion per annum. For a detailed introduction on metal markets with emphasis on the London metal exchange see Crowson and Sampson (2001). Among the different metals, emphasis is placed on aluminium, copper, lead and nickel. All prices are quoted in US-Dollar per tonne. Table 3 again contains the basic summary statistics. Prices and log-returns are displayed in figure 5.

5. Empirical results

Firstly, we selected the Clayton copula (**CLA**), the Gumbel copula (**GUM**) and its rotated version (**roGUM**) from the Archimedean class. From the generalized Archimedean copula family, two hierarchical copula models (i.e. **HA-CLA** and **HA-GUM**) are included, based on the Clayton and the Gumbel copula, respectively. Moreover, six representatives of Morrillas' construction scheme (i.e. **MO-CLA1**, **MO-CLA2**, **MO-CLA3**, **MO-GUM1**, **MO-GUM2**, **MO-GUM3**) involving the Clayton,

† Download under the URL-link http://pacific.commerce.ubc.ca.
‡Download under http://www.lme.co.uk/.

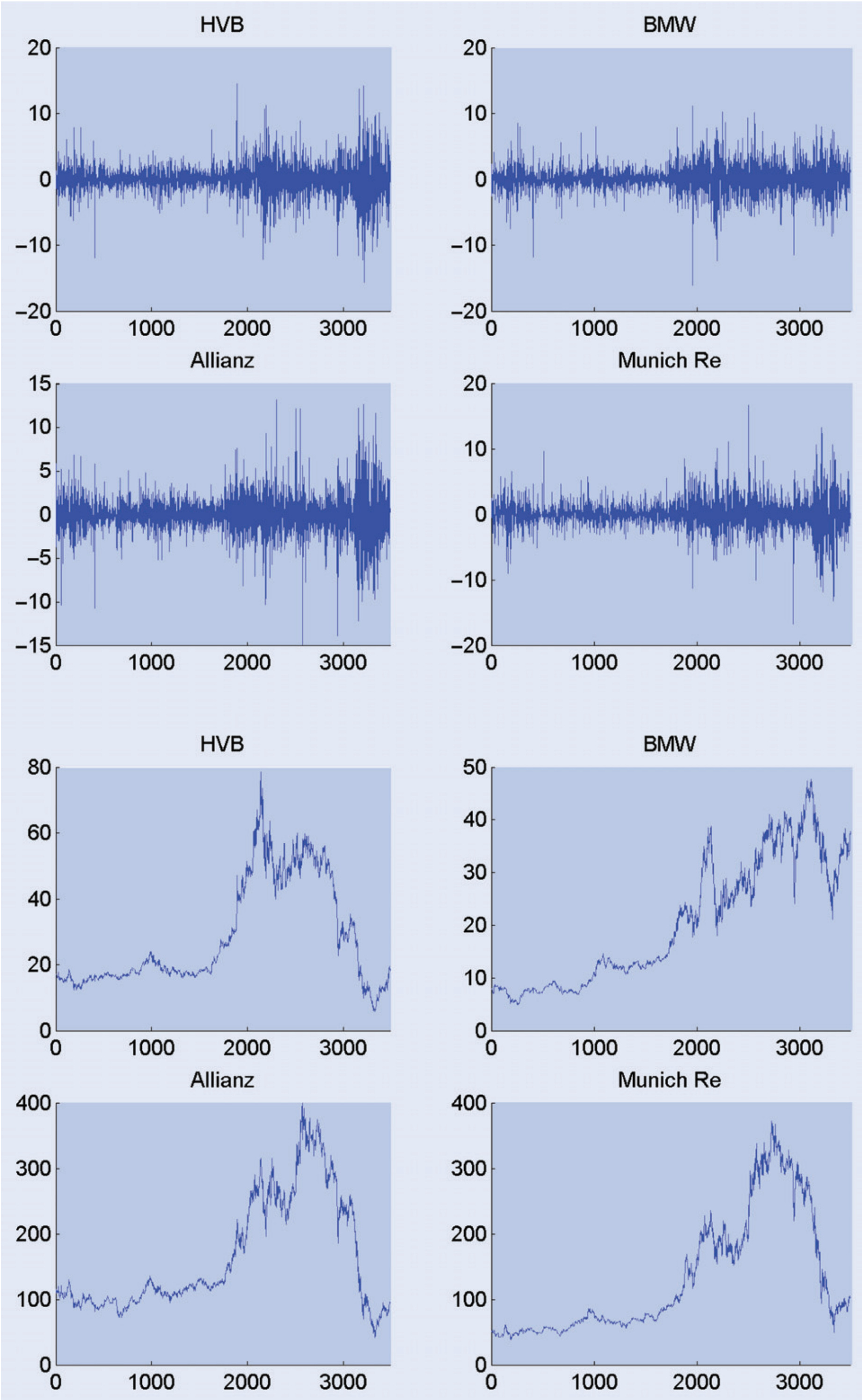

Figure 3. German stock prices and stock returns.

the Gumbel and different generator functions (no. 3, 2, 4 in Morillas 2005) are included as well. In addition, two versions of Liebscher's proposal (**GMLF**, **GML2**) are used. Beyond that, representing the 'elliptical copula world', the Gaussian copula (**NORM**) and—as ultimate benchmark—the Student-*t* (**T**) copula are also included. From the pair-copula decomposition we chose five representatives (i.e. **PC-NORM**, **PC-T**, **PC-CLA**, **PC-GUM**, **PC-roGUM**) each of them derived from one single copula model (i.e. we used no decompositions based on different copulas). Additionally, we fit the **KS(2)-**copula of Palmitesta and Provasi (2005) as well as two generalized versions (i.e. the augmented KS(2)-copula, denoted by **aKS(2)**, where the four-dimensional association parameter is added and the fully specified model, briefly **KSC**) of Koehler and Symanowski (1995). Finally, four different types of multiplicative Liebscher copulas (**L1**, **L2**, **L3**, **L4**) are considered.

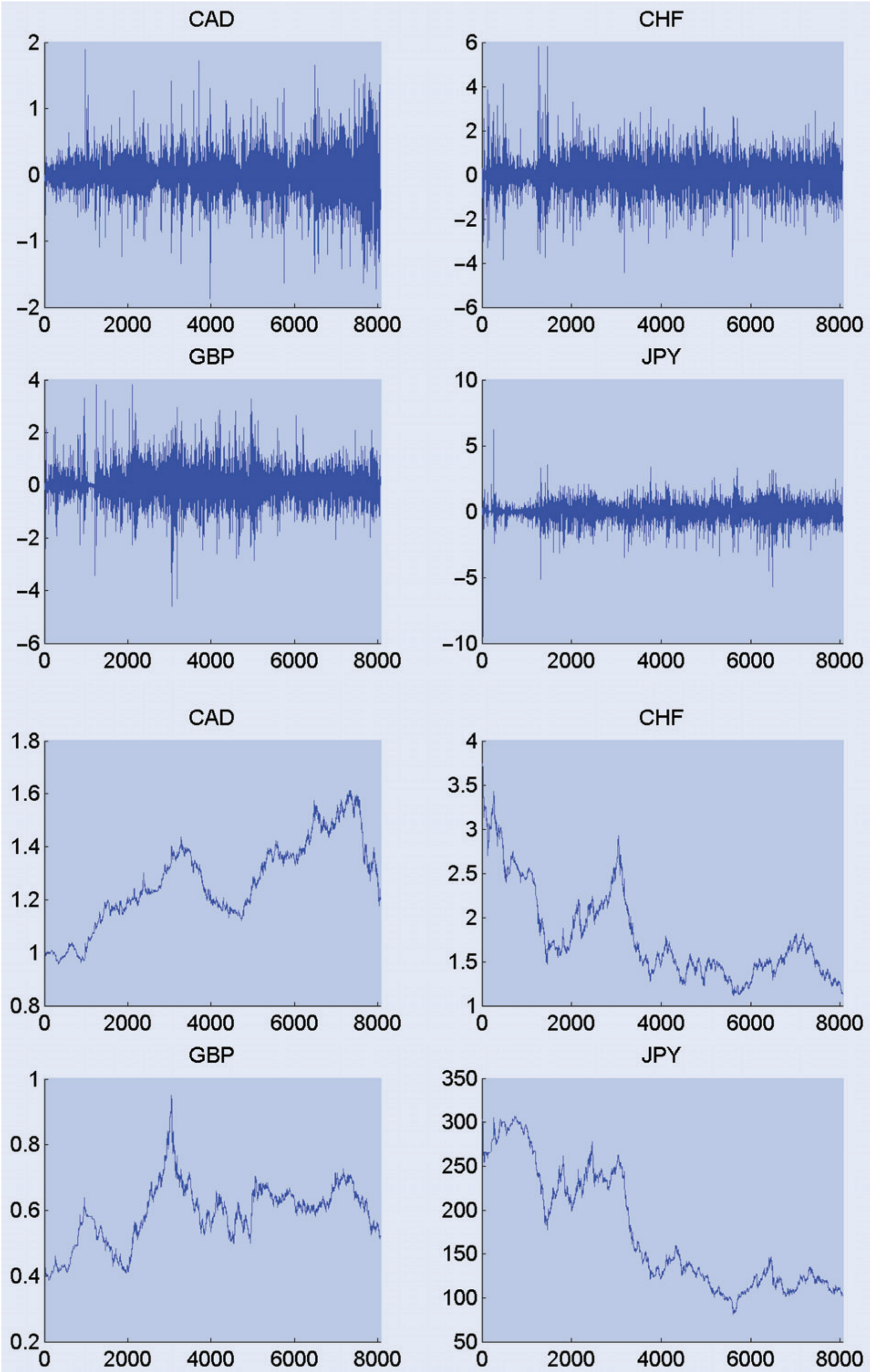

Figure 4. Exchange rates: prices versus returns.

Table 3. Metals: prices versus returns.

Start	End	N	Metal	μ	s^2	$\mathbb{S}$	$\mathbb{K}$	$\mathcal{LB}(10)$	$\mathcal{LM}(10)$
26-03-99	07-08-06	1093	Lead	0.034	1.22	−0.555	8.72	29.74	161.56
26-03-99	07-08-06	1093	Tin	0.084	4.21	−0.368	5.59	30.11	100.37
26-03-99	07-08-06	1093	Nickel	0.142	2.38	−0.139	5.13	12.95	127.66
26-03-99	07-08-06	1093	Zinc	0.13	4.97	−0.618	7.9	10.72	21.45

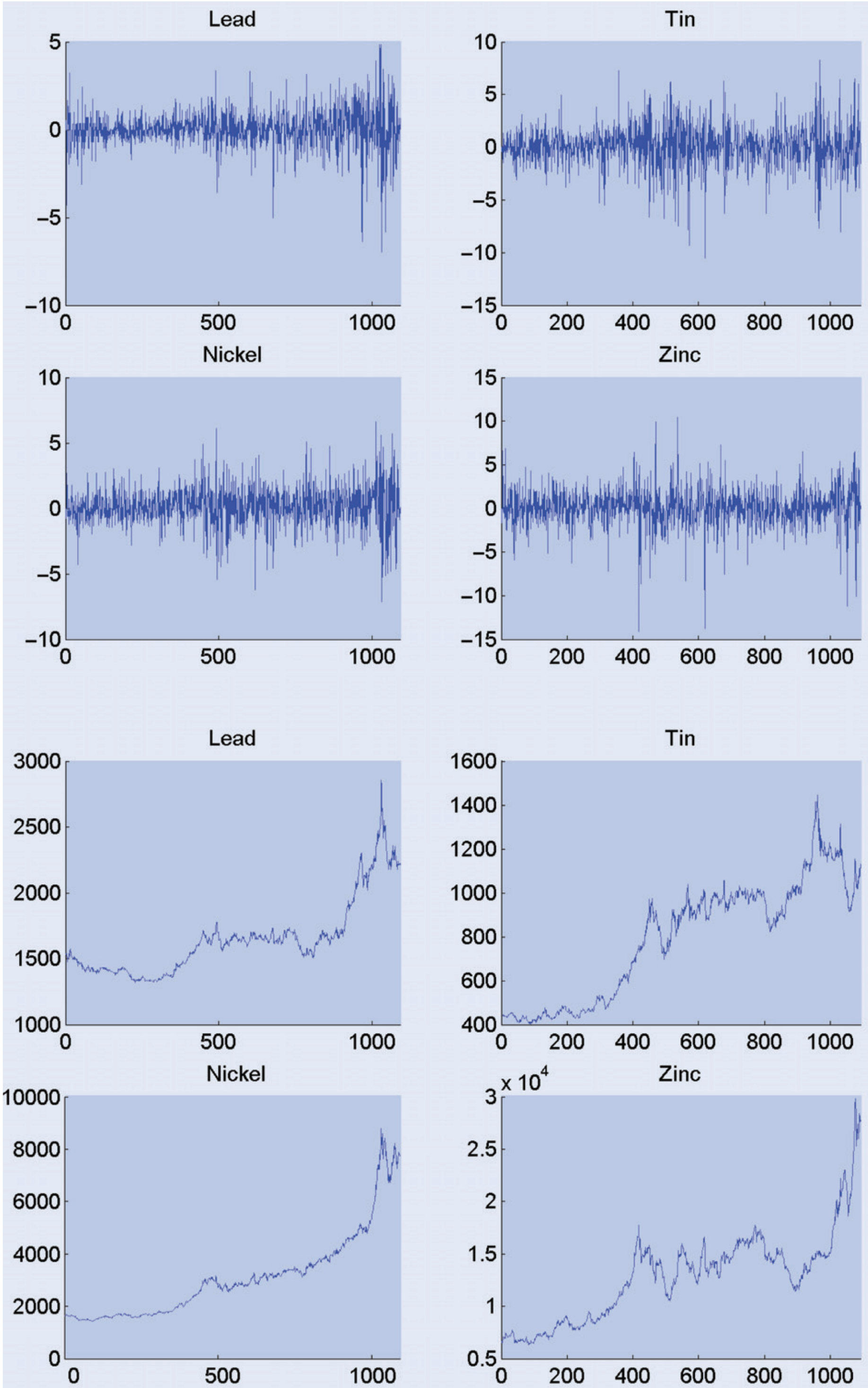

Figure 5. Metals: prices versus returns.

The computer code for the ML-estimation was implemented in Matlab 7.1. For maximization purposes we used the line-search algorithm of Matlab. We calculated parameter estimates and their standard errors† as well as the different goodness-of-fit measures for the GARCH-residuals and all copula models mentioned above. As stated above, goodness-of-fit is measured by the Log-likelihood value and the BIC criterion. Above that,

†It should be pointed out that the standard errors in the subsequent tables (which we extracted from the estimated Fisher information) should be interpreted as 'rough-and-ready' estimators for the true (unknown) ones. As we make use of the semi-parametric estimation procedure (where the unknown marginals are replaced by the empirical distribution function), the correct choice would be to adopt the proposal of Genest *et al.* (1995). Because we are primarily interested in comparing different goodness-of-fit measures and not in checking significances, this procedure (which requires derivatives of the log-density of the underlying copula models) is omitted.

Table 4. German stock returns (GARCH residuals): parameter estimates and corresponding standard errors (in parenthesis).

Copula	θ	Copula	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	Copula	θ_{11}	θ_{12}	θ_{21}			
CLA	0.6145	PC-CLA	0.5509	0.6899	0.9086	0.5524	0.1453	0.0724	HA-CLA	0.5984	0.8856	0.5780			
	(0.014)		(0.0275)	(0.0284)	(0.0324)	(0.0312)	(0.0201)	(0.0181)		(0.0261)	(0.0314)	(0.0156)			
GUM	1.3820	PC-GUM	1.3158	1.4146	1.5678	1.3537	1.0658	1.0468	HA-GUM	1.3818	1.5794	1.3573			
	(0.0099)		(0.0162)	(0.0186)	(0.0201)	(0.0185)	(0.0104)	(0.0099)		(0.0175)	(0.0197)	(0.0118)			
roGUM	1.3866	PC-roGUM	1.3437	1.4444	1.5898	1.3741	1.0742	1.0334							
	(0.0099)		(0.0174)	(0.0192)	(0.021)	(0.0194)	(0.0115)	(0.0103)							
Copula	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{23}	ρ_{24}	ρ_{34}	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6			
NORM	0.4220	0.5558	0.3985	0.4888	0.3729	0.5770									
	(0.0129)	(0.0102)	(0.0129)	(0.0116)	(0.0133)	(0.0098)									
T	0.4327	0.5715	0.4097	0.5046	0.3831	0.5785	10.0271								
	(0.0129)	(0.0102)	(0.0131)	(0.0115)	(0.0138)	(0.0101)	(0.835)								
PC-NORM	0.4219	0.4887	0.5768	0.4420	0.1276	0.0906									
	(0.0129)	(0.0117)	(0.0104)	(0.013)	(0.017)	(0.0174)									
PC-T	0.4330	0.5047	0.5816	0.4565	0.1307	0.0945	8.3406	6.9444	8.8916	8.2465	27.2068	42.3716			
	(0.0043)	(0.0107)	(0.0104)	(0.0127)	(0.0163)	(0.0158)	(1.2034)	(0.8046)	(1.4645)	(1.1718)	(12.688)	(26.0791)			
Copula	a_{11}	a_{12}	a_{13}	a_{14}	a_{22}	a_{23}	a_{24}	a_{33}	a_{34}	a_{44}	a_{123}	a_{124}	a_{134}	a_{234}	a_{1234}
KS(2)	0.0231	0.077	0.2173	0.0631	0.1912	0.0000	0.0000	0.171	0.0000	0.6181					
	(0.0076)	(0.0213)	(0.0186)	(0.0177)	(0.0234)	(0.0021)	(0.0087)	(0.098)	(0.0067)	(0.0316)					
aKS(2)	0.0731	0.0465	0.1016	0.0206	0.1465	0.0545	0.0159	0.002	0.113	0.1098					0.2347
	(0.0126)	(0.0103)	(0.0095)	(0.0089)	(0.0181)	(0.009)	(0.0102)	(0.002)	(0.0106)	(0.0164)					(0.0193)
KSC	0.0586	0.0317	0.0393	0.0046	0.1021	0.0074	0.0059	0.0051	0.0526	0.0892	0.0874	0.0155	0.0887	0.0848	0.1746
	(0.0142)	(0.0113)	(0.0088)	(0.009)	(0.0162)	(0.0064)	(0.0091)	(0.0041)	(0.0104)	(0.0158)	(0.011)	(0.007)	(0.0124)	(0.0124)	(0.0174)
Copula	θ	r	Copula	γ	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6					
MO-CLA1	0.1991	−4.3968	L1	1.5158	0.739	0.6373	0.8806	0.7143							
	(0.036)	(0.5436)		(0.1166)	(0.025)	(0.0235)	(0.0249)	(0.0267)							
MO-GUM1	1.1024	−3.6399	L2	6.2663	0.4761	0.3957	0.3203	0.3946	0.358						
	(0.0097)	(0.1979)		(1.1326)	(0.0254)	(0.039)	(0.0341)	(0.0434)	(0.0413)						
MO-CLA2	5.2056	8.4711	L3	1.9788	0.3798	0.8363	0.2498	1.0000	0.4922						
	(0.2048)	(0.4735)		(0.2814)	(0.1074)	(0.6536)	(0.6238)	(0.9957)	(0.643)						
MO-GUM2	1.3820	2.0029	L4	0.4292	8.6085	0.848	0.9492	0.6666	1.0000	0.6007					
	(0.0099)	(1.0003)		(0.0691)	(7.2814)	(0.1103)	(1.2434)	(0.8584)	(1.7084)	(0.8595)					
MO-CLA3	0.1991	0.8147													
	(0.0356)	(0.0187)													
MO-GUM3	1.1024	0.7845													
	(0.0112)	(0.0169)													
Copula	δ_1	δ_2	δ_3	δ_4	θ_1	θ_2									
GMLF	1.6408	1.4282	2.0000	1.709	2.3507										
	(0.0739)	(0.1931)}	(1.1696)	(0.1543)	(0.2486)										
GML2	1.6376	1.425	2.0000	1.7065	1.1039	0.0000									
	(76.1131)	(301.5028)}	(3.9515)	(94.3773)	(0.1136)	(2.1235)									

Table 5. Exchange rate returns (GARCH residuals): parameter estimates and corresponding standard errors (in parenthesis).

Copula	θ	Copula	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	Copula	θ_{11}	θ_{12}	θ_{21}		
CLA	0.3935	PC-CLA	0.1690	1.1182	0.5315	0.0607	0.3906	0.0230	HA-CLA	0.3772	0.5047	0.3771		
	(0.0082)		(0.0149)	(0.0227)	(0.0175)	(0.0118)	(0.0171)	(0.0110)		(0.6497)	(0.0381)	(0.1278)		
GUM	1.2397	PC-GUM	1.1115	1.7143	1.3244	1.0209	1.2780	1.0052	HA-GUM	1.2256	1.3516	1.2255		
	(0.0056)		(0.0081)	(0.0149)	(0.0105)	(0.0060)	(0.0115)	(0.0049)		(1.7784)	(0.1365)	(0.3434)		
roGUM	1.2388	PC-roGUM	1.1082	1.7407	1.3454	1.0305	1.2662	1.0121						
	(0.0056)		(0.0089)	(0.0156)	(0.0109)	(0.0067)	(0.0112)	(0.0051)						
Copula	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{23}	ρ_{24}	ρ_{34}	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6		
NORM	0.1640	0.1550	0.1079	0.6339	0.5274	0.4082								
	(0.0104)	(0.0108)	(0.0110)	(0.0056)	(0.0070)	(0.0080)								
T	0.1744	0.1668	0.1182	0.6500	0.5353	0.4239	8.7118							
	(0.0109)	(0.0106)	(0.0120)	(0.0057)	(0.0078)	(0.0088)	(0.3964)							
PC-NORM	0.1639	0.6337	0.4080	0.0671	0.3806	0.0182								
	(0.0107)	(0.0056)	(0.0084)	(0.0113)	(0.0092)	(0.0111)								
PC-T	0.1769	0.6498	0.4232	0.0688	0.3756	0.0196	9.3166	4.8162	6.5620	89.1416	9.0486	24.2253		
	(0.0106)	(0.0059)	(0.0082)	(0.0091)	(0.0093)	(0.0111)	(1.0428)	(0.2658)	(0.4799)	(7.0893)	(0.9279)	(5.8091)		
Copula	a_{11}	a_{12}	a_{13}	a_{14}	a_{22}	a_{23}	a_{24}	a_{33}	a_{34}	a_{44}	a_{123}	a_{124}	a_{134}	a_{234}	a_{1234}
KS(2)	0.2129	0.0667	0.0382	0.0000	0.3101	0.0000	0.0000	0.2398	0.0000	2.4371					
	(0.1238)	(0.0459)	(0.0366)	(0.1818)	(0.1509)	(0.0998)	(0.0996)	(0.1188)	(0.0995)	(1.1580)					
aKS(2)	1.9981	0.0201	0.0640	0.0113	0.0000	0.1356	0.0610	0.005	0.0218	0.0781					0.1785
	(0.3941)	(0.0959)	(0.0167)	(0.1475)	(0.1003)	(0.0478)	(0.0209)	(0.0150)	(0.0074)	(0.0136)					(0.0921)
KSC	0.2423	0.0101	0.0197	0.0105	0.0087	0.0994	0.0445	0.0479	0.0184	0.0779	0.0366	0.0167	0.0026	0.1487	0.0361
	(0.0229)	(0.0039)	(0.0054)	(0.0054)	(0.0047)	(0.0074)	(0.0052)	(0.0064)	(0.0048)	(0.0079)	(0.006)	(0.0045)	(0.0039)	(0.0102)	(0.0056)
Copula	θ	r	Copula	γ	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6					
MO-CLA1	0.1700	−1.3573	L1	2.4106	0.0997	0.8246	0.7746	0.5517							
	(0.0144)	(0.0967)		(0.1307)	(0.0093)	(0.0154)	(0.0148)	(0.0145)							
MO-GUM1	1.0683	−1.4955	L2	8.5186	0.8328	0.0000	1.0000	1.0000	1.0000						
	(0.0072)	(0.1114)		(11.1918)	(0.3091)	(0.9853)	(1.3167)	(1.2173)	(1.3685)						
MO-CLA2	0.9146	2.3242	L3	5.2755	0.3182	0.032	0.5553	0.5293	0.4905						
	(0.9167)	(2.3015)		(0.4312)	(0.0226)	(0.0059)	(0.0217)	(0.0208)	(0.0158)						
MO-GUM2	1.2397	2.0047	L4	8.3605	0.2161	0.3337	1.0000	0.0000	0.0000	0.0000					
	(0.0056)	(1.0000)		(13.4778)	(0.1655)	(0.2601)	(1.0186)	(1.1042)	(1.0702)	(0.966)					
MO-CLA3	0.1700	0.5758													
	(0.0152)	(0.0197)													
MO-GUM3	1.0683	0.5993													
	(0.0070)	(0.0180)													
Copula	δ_1	δ_2	δ_3	δ_4	θ_1	θ_2									
GMLF	1.0000	2.0000	1.9966	1.7436	1.3955										
	(0.9136)	(1.212)	(0.0351)	(0.2155)	(0.8127)										
GML2	1.0000	2.0000	1.9966	1.7512	1.3472	0.0000									
	(1.0057)	(1.0472)	(0.1189)	(0.331)	(0.0005)	(0.001)									

Table 6. Metal returns (GARCH residuals): parameter estimates and corresponding standard errors (in parenthesis).

Copula	θ	Copula	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	Copula	θ_{11}	θ_{12}	θ_{21}			
CLA	0.5773	PC-CLA	0.3967	0.8440	0.7863	0.4752	0.2084	0.0732	HA-CLA	0.5528	0.7627	0.5527			
GUM	1.3286	PC-GUM	1.2022	1.4919	1.4242	1.2639	1.1250	1.0226	HA-GUM	1.3126	1.4395	1.3125			
	(0.0168)		(0.0271)	(0.0359)	(0.0341)	(0.0326)	(0.0237)	(0.0176)		(0.7023)	(0.0623)	(0.1566)			
roGUM	1.3369	PC-roGUM	1.2323	1.5398	1.4799	1.2952	1.1139	1.0410							
	(0.0170)		(0.0270)	(0.0366)	(0.0353)	(0.0312)	(0.0233)	(0.0180)							
Copula	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{23}	ρ_{24}	ρ_{34}	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6			
NORM	0.3222	0.4783	0.3094	0.5536	0.4333	0.5104									
	(0.0272)	(0.0214)	(0.0275)	(0.0200)	(0.0227)	(0.0204)									
T	0.3271	0.4818	0.3159	0.5613	0.4367	0.5174	18.8384								
	(0.0254)	(0.0198)	(0.0262)	(0.0173)	(0.0227)	(0.0191)	(4.4703)								
PC-NORM	0.3215	0.5528	0.5098	0.3814	0.2106	0.0722									
	(0.0262)	(0.0189)	(0.0209)	(0.0266)	(0.0301)	(0.0312)									
PC-T	0.3301	0.5634	0.5209	0.3819	0.2110	0.0744	27.4447	11.6176	13.9478	12.2527	45.5236	21.8855			
	(0.0247)	(0.0205)	(0.0152)	(0.0270)	(0.0295)	(0.0311)	(19.3849)	(1.3362)	(4.5763)	(4.6006)	(22.5881)	(12.6221)			
Copula	a_{11}	a_{12}	a_{13}	a_{14}	a_{22}	a_{23}	a_{24}	a_{33}	a_{34}	a_{44}	a_{123}	a_{124}	a_{134}	a_{234}	a_{1234}
KS(2)	0.3199	0.0000	0.0714	0.0010	0.0940	0.1018	0.0604	0.0000	0.0897	0.1900	0.4068				
	(0.0944)	(0.0588)	(0.3934)	(0.1641)	(0.0429)	(0.0335)	(0.1991)	(0.1077)	(0.3132)	(0.0464)	(0.4325)				
aKS(2)	0.2190	0.0053	0.0660	0.0011	0.0971	0.1251	0.0570	0.0000	0.0817	0.1346					0.2334
	(0.0776)	(0.1216)	(0.5615)	(0.4344)	(0.0977)	(0.0586)	(0.2890)	(0.0937)	(0.3054)	(0.0667)					(0.3417)
KSC	0.1789	0.0003	0.0289	0.0047	0.0746	0.0336	0.0376	0.0000	0.0273	0.1089	0.1103	0.0110	0.0944	0.1396	0.2069
	(0.0564)	(0.2338)	(0.2767)	(0.2158)	(0.0317)	(0.0450)	(0.2659)	(0.0975)	(0.1692)	(0.0460)	(0.0295)	(0.0656)	(0.0884)	(0.2596)	(0.0629)
Copula	θ	r	Copula	γ	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6					
MO-CLA1	0.2649	−2.7984	L1	1.2887	0.5963	0.8355	0.9115	0.6852							
	(0.0603)	(0.6879)		(0.1555)	(0.0436)	(0.0457)	(0.0384)	(0.0478)							
MO-GUM1	1.0554	−4.1123	L2	3.3950	0.7079	0.1801	0.598	0.4441	0.3642						
	(0.0355)	(2.1040)		(3.8834)	(0.1842)	(0.1814)	(0.3011)	(0.2651)	(0.2380)						
MO-CLA2	10.6418	18.4340	L3	1.6396	0.3698	0.0000	1.0000	1.0000	0.7786						
	(0.5169)	(1.3396)		(0.4167)	(0.1049)	(0.5822)	(1.0301)	(0.9952)	(0.6362)						
MO-GUM2	1.3286	2.0028	L4	3.1137	0.4960	0.4821	1.0000	0.0000	0.0000	0.0000					
	(0.0169)	(1.0000)		(8.6242)	(0.1652)	(0.7503)	(1.2217)	(1.1204)	(1.1360)	(1.3661)					
MO-CLA3	0.2649	0.7367													
	(0.0590)	(0.0474)													
MO-GUM3	1.0554	0.8044													
	(0.0173)	(0.0282)													
Copula	δ_1	δ_2	δ_3	δ_4	θ_1	θ_2									
GMLF	1.2633	1.9066	2.0000	1.6572	2.0387										
	(0.3132)	(0.2643)	(1.0524)	(0.3010)	(0.3144)										
GML2	1.2470	1.9057	2.0000	1.6499	1.1666	0.0978									
	(0.8259)	(0.3968)	(2.2361)	(0.4812)	(0.1372)	(0.7335)									

Table 7. Goodness-of-fit measures: German stock returns (left), Exchange rates (middle) and Metal returns (right).

	German stocks (GARCH-Residuals)					Exchange rates (GARCH-Residuals)					Metal data (GARCH-Residuals)				
Copula	*l*	*BIC*	*KS*	*AKS*	$\mathcal{L}_2$	*l*	*BIC*	*KS*	*AKS*	$\mathcal{L}_2$	*l*	*BIC*	*KS*	*AKS*	$\mathcal{L}_2$
CLA	1489.3	−2970.5	3.98	0.56	0.67	1779.7	−3550.5	5.49	0.89	0.68	410.2	−813.4	1.80	0.23	0.51
GUM	1486.9	−2965.7	2.51	0.51	0.52	1786.0	−3563.1	4.64	0.93	0.64	363.0	−719.0	1.24	0.21	0.40
roGUM	1609.7	−3211.2	2.43	0.45	0.47	1818.6	−3628.1	4.73	0.93	0.64	409.1	−811.2	1.06	0.20	0.37
NORM	1931.9	−3815.0	3.10	0.44	0.51	3564.5	−7075.1	4.44	0.72	0.55	523.6	−1005.5	1.36	0.17	0.36
T	2041.9	−4026.8	0.68	0.08	0.10	3872.6	−7682.2	1.69	0.16	0.13	536.3	−1023.8	0.69	0.07	0.15
PC-NORM	1931.8	−3814.7	3.11	0.81	0.72	3564.4	−7074.8	4.45	1.36	0.77	523.0	−1004.3	1.40	0.30	0.51
PC-T	2063.7	−4029.6	0.77	0.16	0.15	3958.3	−7808.6	1.74	0.29	0.19	542.7	−1001.9	0.61	0.09	0.16
PC-CLA	1622.1	−3195.3	3.60	1.01	0.91	2950.9	−5847.9	5.20	1.54	0.89	456.6	−871.5	1.48	0.39	0.66
PC-GUM	1745.3	−3441.8	1.99	0.53	0.48	3407.1	−6760.2	3.56	1.03	0.57	431.8	−821.9	1.28	0.28	0.47
PC-roGUM	1912.4	−3776.0	1.76	0.49	0.42	3561.5	−7068.9	3.14	0.90	0.51	515.3	−988.8	0.80	0.14	0.26
GML2	1723.7	−3398.5	2.62	0.40	0.46	2711.8	−5369.6	5.64	0.84	0.71	459.5	−877.3	1.47	0.17	0.40
GMLF	1723.7	−3406.7	2.66	0.40	0.46	2712.2	−5379.5	5.52	0.83	0.69	459.4	−884.0	1.49	0.18	0.40
KS(2)	713.9	−1346.2	4.47	0.63	0.83	164.7	−239.4	6.84	1.13	0.87	135.7	−201.8	2.39	0.36	0.79
aKS(2)	1779.1	−3468.7	2.20	0.35	0.45	3440.9	−6783.0	2.76	0.41	0.35	464.5	−852.5	1.25	0.15	0.41
KSC	1865.7	−3609.1	1.60	0.28	0.35	3540.0	−6945.1	2.15	0.34	0.29	485.3	−866.3	1.27	0.15	0.39
MO-CLA1	1616.7	−3217.2	3.65	0.44	0.59	1967.3	−3916.6	4.97	0.79	0.63	438.0	−862.0	1.59	0.18	0.44
MO-GUM1	1695.5	−3374.7	2.38	0.36	0.40	1991.6	−3965.2	4.37	0.78	0.58	436.5	−859.2	1.06	0.14	0.29
MO-CLA2	1489.3	−2962.3	3.98	0.56	0.67	1779.7	−3541.5	5.49	0.89	0.68	410.2	−806.4	1.80	0.23	0.51
MO-GUM2	1486.9	−2957.5	2.51	0.51	0.52	1786.0	−3554.1	4.64	0.93	0.64	363.0	−712.1	1.24	0.21	0.40
MO-CLA3	1616.7	−3217.2	3.65	0.44	0.59	1967.3	−3916.6	4.97	0.79	0.63	438.0	−862.0	1.59	0.18	0.44
MO-GUM3	1695.5	−3374.7	2.38	0.36	0.40	1991.6	−3965.2	4.37	0.78	0.58	436.5	−859.2	1.06	0.14	0.29
L1	1618.1	−3195.5	2.87	0.33	0.47	3044.9	−6044.8	4.58	0.56	0.50	445.1	−855.5	1.30	0.16	0.40
L3	1648.0	−3247.1	2.27	0.29	0.37	3104.9	−6155.8	3.42	0.47	0.40	447.5	−853.3	1.55	0.19	0.45
L2	1622.3	−3195.7	2.66	0.31	0.43	2399.0	−4744.0	5.14	0.75	0.64	437.8	−833.9	1.35	0.15	0.39
L4	1724.7	−3392.4	2.30	0.31	0.39	2278.2	−4493.5	2.99	0.51	0.39	433.7	−818.6	1.13	0.13	0.32
HA-CLA	1543.1	−3061.7	3.59	0.54	0.64	1805.3	−3583.5	5.36	0.88	0.68	418.9	−817.0	1.65	0.23	0.50
HA-GUM	1614.8	−3205.1	2.02	0.42	0.44	1913.6	−3800.3	4.38	0.84	0.59	375.0	−729.1	1.27	0.19	0.37

three distance measures,

$$\mathrm{KS} = \sqrt{N} \max_{j=1,\ldots,N} \left| F_{\chi^2(d)}(\chi_j) - F_{N,\chi}(\chi_j) \right|,$$

$$\mathrm{AKS} = \frac{1}{\sqrt{N}} \sum_{j=1,\ldots,N} \left| F_{\chi^2(d)}(\chi_j) - F_{N,\chi}(\chi_j) \right| \quad \text{and}$$

$$\mathcal{L}_2 = \left\| F_{\chi^2(d)} - F_{N,\chi} \right\|_2$$

are calculated to quantify the distance after application of the Rosenblatt transformation (based on the different parametric copula models).

The subsequent tables summarize the estimation results for the different copula models under consideration. In contrast to tables 4, 5 and 6 which are dedicated to the parameter estimates and their approximate standard error, table 7 displays five goodness-of-fit measures for every copula and every data set. As already mentioned above, we only presented the results for the GARCH residuals, emphasizing that using the original data instead doesn't change the estimation results substantially. In particular, the ordering of the goodness-of-fit measures is essentially preserved. Above that, parameter estimates of the dependence parameters are roughly stable for most of the copulas under consideration. In general, the results in table 7 seem to be rather stable across all data sets and distance measures. The most important conclusions are the following:

First of all, both Student-t copula (T) and the pair copula built from bivariate Student-t copulas (PC-T) provide the best fit over all measures. Taking the comparatively large number of parameters of the PC-T into account, the Student-t copula should be preferred from a practical point of view.

Secondly, within the class of pair-copulas itself, the pair-copula approach based on different bivariate rotated Gumbel copulas dominates the approaches based on bivariate Clayton and bivariate Gumbel copulas, though being outperformed by the above-mentioned PC-t approach. Among the different construction schemes of multivariate copulas, the pair-copula approach has to be pointed out.

Thirdly, the Gaussian copula seems to attain more attraction if the number of dimensions increases. Whereas Archimedean copulas frequently outperform this dependence model in the bivariate case, the situation seems to reverse for the higher-dimensional case. There is empirical evidence that the overall goodness-of-fit of the Gaussian copula (measured by the log-likelihood) seems to be very good, whereas distance measures (with more emphasis on the tail area) noticeably worsen compared to other copulas.

Fourthly, the KS(2)-copula which was advocated by Palmitesta and Provasi (2005) provides only a poor fit to the return series. In contrast to the original specification of Koehler and Symanowski (1995), it neglects three- and four-dimensional association parameters. Taking a global (i.e. four-dimensional) association parameter into account, the corresponding augmented KS copula (aKS(2)) clearly improves all goodness-of-fit measures (note that most of distance measures cut in half) which can be further improved if the fully specified KS copula (KSC) is used. The latter proves to be a serious alternative to the elliptical competitors, at least in the four-dimensional case.

Fifth, the fit of 'plain' Archimedean copulas significantly improves if a generalized model based on the proposals of Liebscher and Morillas is taken into consideration. Within both classes, there seems to be further discussion as to how to choose the underlying generating functions. Basically, the multiplicative Liebscher copulas $L1$–$L4$ as well as GML2 and GLMF tend to outperform the representatives of Morillas' class.

Sixth, focusing on hierarchical Archimedean copula families, we found only slight improvement regarding the goodness-of-fit, at least for our data sets. However, we admit that one might further improve the results with another hierarchy which might be found on the basis of cluster algorithms.

To sum up, the 4-variate Student-t distribution still plays a predominant role. Some of the recently proposed construction schemes are partially competitive while others are more likely to be overestimated in the relevant literature. Finally, our findings are derived from four-dimensional data sets. For the higher dimensional case, some results are expected to become still more evident, while some of the models under consideration (e.g. KSC) will be no longer estimable.

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References

Aas, K., Czado, C., Frigessi, A. and Bakken, H., Pair copula constructions of multiple dependence. *Insur. Math. Econ.*, 2009, **44**(2), 182–198.

Berg, D. and Bakken, H., A copula goodness-of-fit test based on the probability integral transform. Working Paper, 2006 (Norwegian Computing Center).

Breymann, W., Dias, A. and Embrechts, P., Dependence structures for multivariate high-frequency data in finance. *Quant. Finan.*, 2003, **1**, 1–14.

Bedford, T. and Cooke, R.M., Probability density decomposition for conditionally dependent random variables modeled by vines. *Ann. Math. Artific. Intelligence*, 2001, **32**, 245–268.

Bedford, T. and Cooke, R.M., Vines — a new graphical model for dependent random variables. *Ann. Stat.*, 2002, **30**(4), 1031–1068.

Caputo, A., Some properties of the family of Koehler Symanowski distributions. The Collaborative Research Center (SBF) 386, Working Paper No. 103, 1998 (LMU: München).

Chen, X., Fan, Y. and Patton, A., Simple tests for models of dependence between multiple financial time series, with applications to U.S. equity returns and exchange rates. Working Paper, 2004. Available online at: http://papers.ssrn.com/sol3/papers.cfm?abstract_id = 513024

Crowson, P. and Sampson, R., *Managing Metal Price Risk with the London Metal Exchange*, 2001 (London Metal Exchange: London).

Embrechts, P., McNeil, A. and Straumann, D., Correlation: pitfalls and alternatives. *Risk*, 1999, **5**, 69–71.

Frees, E.W. and Valdez, E.A., Understanding relationship using copulas. *North Am. Actuar. J.*, 1998, **2**(1), 1–25.

Genest, C., Ghoudi, K. and Rivest, L.-P., A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika*, 1995, **82**(3), 543–552.

Joe, H., Families of m-variate distributions with given margins and $m(m-1)/2$ bivariate dependence parameters, In *Distributions with Fixed Marginals and Related Topics*, edited by L. Rüschendorf, B. Schweizer, and M.D. Taylor, pp. 120–141, 1996 (Institute of Mathematical Statistics: Hayward, CA).

Joe, H., Multivariate models and dependence concepts, *Monographs on Statistics and Applied Probability*, Vol. 37, 1997 (Chapman and Hall: London).

Koehler, K.J. and Symanowski, J.T., Constructing multivariate distributions with specific marginal distributions. *J. Multivar. Distrib.*, 1995, **55**, 261–282.

Liebscher, E., Modelling and estimation of multivariate copulas. Working Paper, 2006 (University of Applied Sciences Merseburg).

Morillas, P.M., A method to obtain new copulas from a given one. *Metrika*, 2005, **61**, 169–184.

Nelsen, R.B., *An Introduction to Copulas*. *Springer Series in Statistics*, 2006 (Springer: Berlin).

Palmitesta, P. and Provasi, C., Aggregation of dependent risks using the Koehler–Symanowski copula function. *Comput. Econ.*, 2005, **25**, 189–205.

Rosenblatt, M., Remarks on multivariate transformation. *Ann. Stat.*, 1952, **23**, 470–472.

Savu, C. and Trede, M., Hierarchical Archimedean copulas. Working Paper, 2006 (University of Münster).

Sklar, A., Fonctions de répartition á n dimensions et leurs marges. *Publ. Inst. Stat.*, 1959, **8**, 229–231.

Wang, S., Aggregation of correlated risk portfolios: models and algorithms. Working Paper, 1998. Available online at: www.casact.org.

Whelan, N., Sampling from Archimedean copulas. *Quant. Finan.*, 2004, **4**, 339–352.