

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/309008726>

On strict convergence of stochastic gradients

Article · October 2016

CITATION

1

READS

130

1 author:



[Michel Benaim](#)

Université de Neuchâtel

133 PUBLICATIONS 4,302 CITATIONS

SEE PROFILE

On strict convergence of stochastic gradients

Michel Benaim
 Institut de Mathématiques
 Université de Neuchâtel, Switzerland

October 12, 2016

Abstract

We discuss conditions ensuring the (strict) convergence of stochastic gradient algorithms.

Keywords Asymptotic pseudo trajectories, Gradient like systems, Lojasiewicz inequalities, Shadowing, Stochastic approximation, Stochastic gradients

1 Introduction

A stochastic gradient algorithm is a process (x_n) , $x_n \in \mathbb{R}^m$, verifying a recursion of the form

$$x_{n+1} - x_n = -\gamma_{n+1}(\nabla V(x_n) + U_{n+1}) \quad (1)$$

where $V : \mathbb{R}^m \mapsto \mathbb{R}$ is a smooth potential, (γ_n) a decreasing sequence of nonnegative weights (e.g $\gamma_n = \frac{A}{n}$) and (U_n) a \mathbb{R}^m valued sequence of perturbations (e.g a martingale difference sequence).

Under classical assumptions on (γ_n) and (U_n) , the limit set of (x_n) is a compact connected subset of the *critical set* of V :

$$\text{crit}(V) = \{x \in \mathbb{R}^m : \nabla V(x) = 0\}.$$

For generic V , critical points are isolated, so that the process converges to one of them. The purpose of this note is to discuss conditions ensuring convergence for general (possibly degenerate) potentials. The main result follows from Lojasiewicz type convergence results for gradient like systems established in [5], combined with a shadowing theorem proved in [11] and classical estimates for stochastic approximation processes.

As an illustration we get assertion (ii) of Theorem 1.1 below. Assertion (i) is classical (see e.g Proposition 2.3 and Remark 2.4 below) and recalled here for completeness.

For $p \in \text{crit}(V)$ let $D^2V(p)$ be the Hessian of V at p and let $\Lambda(p)$ denote the spectrum (i.e the collection of eigenvalues) of $-D^2V(p)$. For any set $C \subset \text{crit}(V)$ let $\Lambda(C) = \bigcup_{p \in C} \Lambda(p)$ and $\mathcal{R}(C) = \mathbb{R} \setminus \Lambda(C)$.

Theorem 1.1 *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_n\}$, (U_n) a sequence of \mathbb{R}^m valued random variables adapted to $\{\mathcal{F}_n\}$ and $x_0 \in \mathbb{R}^m$ an \mathcal{F}_0 measurable random variable. Assume*

- (a) $\gamma_n = \frac{A}{n}$ for some $A > 0$;
- (b) $\mathbf{E}(U_{n+1} | \mathcal{F}_n) = 0$;
- (c) (U_n) is bounded in L^2 (i.e $\sup_n \|U_n\|^2 < \infty$);
- (d) The process (x_n) solution to (1) is almost surely bounded.

Let $\mathcal{L}(x_n)$ denote the limit set of (x_n) .

- (i) Suppose V is C^r for some $r \geq m$. Then $\mathcal{L}(x_n)$ is a compact connected subset of $\text{crit}(V)$.
- (ii) Suppose V is real analytic. Let C be a compact connected subset of $\text{crit}(V)$. Assume that

$$\left] -\frac{1}{2A}, 0 \right[\cap \mathcal{R}(C) \neq \emptyset. \quad (2)$$

Then there exists a random variable $x_\infty \in C$ and $c > 0$ such that

$$\|x_n - x_\infty\| = O\left(\frac{1}{\log(n)^c}\right)$$

almost surely on the even

$$\{\mathcal{L}(x_n) \subset C\}.$$

Remark 1.2 If V is proper, there exists $A^* > 0$ such that for all $A \leq A^*$ and every component $C \subset \text{crit}(V)$ condition (2) is satisfied.

A point $p \in \text{crit}(V)$ is called *linearly unstable* if $\Lambda(p) \cap \mathbb{R}_*^+ \neq \emptyset$. By analogy we say that $C \subset \text{crit}(V)$ is *linearly unstable* if every $p \in C$ is linearly unstable. Observe that whenever $C \subset \text{crit}(V)$ is a connected linearly unstable set, then either C reduces to a singleton, or $0 \in \Lambda(p)$ for all $p \in C$.

A byproduct of the previous theorem is the following non convergence result.

Theorem 1.3 *With the notation of Theorem 1.1, let V and C be like in Theorem 1.1 (ii). Assume furthermore that C is linearly unstable and that*

(i) $\sup_n E(\|U_n\|^q | \mathcal{F}_n) < \infty$ for some $q > 2$;

(ii)

$$\liminf_{n \rightarrow \infty} \lambda_{\min}[\mathbf{E}(U_{n+1}U_{n+1}^T | \mathcal{F}_n)] > 0$$

where $\lambda_{\min}[\cdot]$ denotes the smallest eigenvalue.

Then, the event $\{\mathcal{L}(x_n) \subset C\}$ has zero probability.

Outline Section 2 introduces some notation and background. The main result (Theorem 3.3) is given in section 3 and is applied in section 4 to provide condition ensuring that the process cannot converge toward an degenerate set of unstable equilibria (Theorem 4.2).

2 Notation and background

Let $F : \mathbb{R}^m \mapsto \mathbb{R}^m$ be a C^1 globally integrable vector field, $\Phi = \{\Phi_t\}$ the induced flow and $\text{Eq}(F) = \{x \in \mathbb{R}^m : F(x) = 0\}$ the *equilibrium set*.

A *strict Lyapounov* function for Φ (or F) is a continuous map $V : \mathbb{R}^m \mapsto \mathbb{R}$ such that

$$V(\Phi_t(x)) < V(x)$$

for all $x \in \mathbb{R}^m \setminus \text{Eq}(F)$ and $t > 0$. When such a V exists, F is called a *gradient-like* vector field.

Using the terminology coined in [3], a continuous function $X : \mathbb{R}^+ \mapsto \mathbb{R}^m$ is called a *asymptotic pseudo trajectory* (in short APT) of Φ , if for all $T > 0$

$$\lim_{t \rightarrow \infty} \sup_{0 \leq h \leq T} \|X(t+h) - \Phi_h(X(t))\| = 0.$$

Its *limit set* is the (possibly empty) set defined as

$$\mathcal{L}(X) = \bigcap_{t \geq 0} \overline{\{X(t+s), s \geq 0\}}.$$

Example 2.1 Let (x_n) be solution to the recursion

$$x_{n+1} - x_n = \gamma_{n+1}(F(x_n) + U_{n+1})$$

where $\gamma_n \geq 0, \gamma_n \rightarrow 0$ and $\sum_n \gamma_n = \infty$.

Let $\tau_0 = 0, \tau_n = \sum_{i=1}^n \gamma_i$ and let X be the continuous interpolated process defined by

- (a) $X(\tau_n) = x_n$,
- (b) X is affine on each interval $[\tau_n, \tau_{n+1}]$.

Assume that

- (i) F is Lipschitz and bounded on a neighborhood of (x_n) (this holds for instance whenever (x_n) is bounded),

- (ii)

$$\lim_{n \rightarrow \infty} \sup \left\{ \left\| \sum_{i=n}^{k-1} \gamma_{i+1} U_{i+1} \right\| : k \geq 1, \tau_k \leq \tau_n + T \right\} = 0.$$

Then X is an asymptotic trajectory of the flow induced by F (see Proposition 4.1 in [1]).

Example 2.2 [*Robbins Monro Algorithm*] With the notation of example 2.1, assume that

- (i) The sequence (U_n) is a stochastic process adapted to some filtration $\{\mathcal{F}_n\}$, x_0 is \mathcal{F}_0 measurable and

$$\mathbf{E}(U_{n+1}|\mathcal{F}_n) = 0;$$

- (ii) $\gamma_n \geq 0, \sum_n \gamma_n = \infty$;

- (iii) There exists $q \geq 2$ such that $\sum_n \gamma_n^{1+q/2} < \infty$ and

$$\sup_n \mathbf{E}(\|U_n\|^q) < \infty;$$

- (iv) F is Lipschitz and bounded on a neighborhood of (x_n) .

Then X is an asymptotic trajectory of the flow induced by F (see Proposition 4.3 in [1]).

By Theorem 0.1 in [3] (see also Theorem 5.7 in [1]), limit sets of bounded APTs are *internally chain transitive* for Φ . By this we mean that, if \mathcal{L} is such a set, then \mathcal{L} is compact, connected Φ -invariant and the restricted flow $\Phi|_{\mathcal{L}}$ is chain recurrent in the sense of Conley [6].

When F is gradient like this implies the following result (see Proposition 6.4 in [1]).

Proposition 2.3 *Suppose F is gradient like with strict Lyapounov function V and induced flow Φ . Let X be a bounded APT for Φ and let $\mathcal{L} = \mathcal{L}(X)$ be its limit set. If $V(\mathcal{L} \cap \text{Eq}(F))$ has empty interior, then \mathcal{L} is a (compact connected) subset of $\text{Eq}(F)$ and $V|_{\mathcal{L}}$ is constant.*

Remark 2.4 If $\text{Eq}(F) \cap \mathcal{L}$ is countable (for instance if equilibria of F are isolated) Proposition 2.3 implies convergence of $(X(t))$ to an equilibrium point.

If $\text{Eq}(F) \subset \text{Crit}(V)$ and V is C^r with $r \geq m$. Then Sard's theorem implies that $V(\text{Eq}(F))$ has empty interior and the conclusion of Proposition 2.3 holds.

3 A strict convergence result

We assume here that F is a C^1 vector field which is gradient like with a C^1 strict Lyapounov function V .

We let X denote a bounded APT. The *error rate* of X is the number $e(X) \in [-\infty, 0]$ defined as

$$e(X) = \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left(\sup_{0 \leq h \leq T} \|X(t+h) - \Phi_h(X(t))\| \right) \leq 0$$

By Lemma 8.2 in [1], $e(X)$ doesn't depend of the choice of $T > 0$.

If $e(X) \leq \lambda < 0$. Then, following [8] and [3], X is called a λ *pseudotrajectory*.

Example 3.1 [*Robbins Monro, continued*] With the notation of example 2.2 and $\gamma_n = \frac{A}{n \log(n)^\beta}$, $A > 0, 0 \leq \beta \leq 1$. Then

$$e(X) = -\frac{1}{2A}$$

if $\beta = 0$; and

$$e(X) = -\infty$$

if $\beta > 0$. This follows from Proposition 8.3 and Remark 8.4 in [1].

The map V is said to satisfy a *Lojasiewick inequality* at point $p \in \mathbb{R}^m$ if there exists a neighborhood U of p and constants $0 < \theta \leq 1/2$ and $c_0 \geq 0$ such that for all $x \in U$

$$\|V(x) - V(p)\|^{1-\theta} \leq c_0 \|\nabla V(x)\|.$$

It was proved by Lojasiewicz ([9]) that V satisfies such an inequality whenever it is real analytic in a neighborhood of p and this inequality was used to prove that bounded trajectories of real analytic gradient vector fields have finite length, hence converge. This result was later extended in [5] to gradient-like systems verifying a certain *angle condition*. We say that (F, V) satisfies an angle condition at p if there exists a neighborhood U of p and a constant $c_1 > 0$ such that

$$|\langle \nabla V(x), F(x) \rangle| \geq c_1 \|\nabla V(x)\| \|F(x)\|$$

for all $x \in U$.

Let $\mathcal{L} = \mathcal{L}(X)$ denote the limit set of X and let $\mathcal{R}(\mathcal{L}) \subset \mathbb{R}$ denote its *Sacker-Sell resolvent* (also called in the literature the *dynamical resolvent*). Because \mathcal{L} is internally chain transitive, it follows from Lemma 3 in [14], that $\mathcal{R}(\mathcal{L})$ can be defined as the set of $\lambda \in \mathbb{R}$ such that for all $x \in \mathcal{L}$ and $v \in \mathbb{R}^m \setminus \{0\}$

$$\sup_t \|e^{-\lambda t} D\Phi_t(x)v\| = \infty.$$

In other words, $\lambda \in \mathcal{R}(\mathcal{L})$ means that there is no bounded solution to the differential system

$$\begin{cases} \frac{dx}{dt} = F(x) \\ \frac{dv}{dt} = DF(x)v - \lambda v \end{cases}$$

with initial condition $(x, v) \in \mathcal{L} \times \mathbb{R}^m \setminus \{0\}$.

For each $p \in \text{Eq}(F)$ let $\Lambda(p)$ denote the set of real parts of eigenvalues of $DF(p)$ (the jacobian matrix of F at p). The next proposition can be used to compute or estimate $\mathcal{R}(\mathcal{L})$. Its proof follows directly from Proposition 2.3, Remark 2.4 and the above definition of $\mathcal{R}(\mathcal{L})$.

Proposition 3.2 *Consider the following assertions:*

- (i) V is C^r for $r \geq m$ and $\text{Eq}(F) \subset \text{crit}(V)$,
- (ii) $V(\mathcal{L} \cap \text{Eq}(F))$ has empty interior,
- (iii) $\mathcal{L} \subset \text{Eq}(F)$,
- (iv) $\mathcal{R}(\mathcal{L}) = \mathbb{R} \setminus \bigcup_{p \in \mathcal{L} \cap \text{Eq}(F)} \Lambda(p)$.

Then (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv).

Theorem 3.3 *Assume that :*

- (i) V satisfies a Lojasiewick inequality and an angle condition at some point $p \in \mathcal{L} \cap \text{Eq}(F)$;
- (ii) $]e(X), 0[\cap \mathcal{R}(\mathcal{L}) \neq \emptyset$.

Then \mathcal{L} reduces to $\{p\}$. If furthermore $|\langle \nabla V(x), F(x) \rangle| \geq \beta \|\nabla V(x)\|^2$ in a neighborhood of p , then

$$\|X(t) - p\| = \begin{cases} O(e^{ct}) & \text{if } \theta = 1/2 \\ O(t^{-\theta/(1-2\theta)}) & \text{if } 0 < \theta < 1/2 \end{cases}$$

where $c = \max(-\frac{\beta}{2c_0^2}, \inf]e(X), 0[\cap \mathcal{R}(\mathcal{L}))$.

Proof. For every \mathbb{R}^m -valued sequence $\xi = (\xi_k)_{k \geq 0}$ and every $r \geq 1$ set

$$\|\xi\|_r = \sum_{k \geq 0} r^k \|y_k\|.$$

Given such a sequence and $x \in \mathbb{R}^m$, define the sequences

$$g(\xi) = (g_k(\xi))$$

and

$$h(x, \xi) = (h_k(x, \xi))$$

by

$$g_k(\xi) = \xi_{k+1} - \Phi_1(\xi_k)$$

and

$$h_k(x, \xi) = \Phi_k(x) - \xi_k$$

for all $k \geq 0$.

Let C be a compact connected invariant set (say internally chain transitive), $\mu \in \mathcal{R}(C) \cap \mathbb{R}^-$ and $r = e^{-\mu}$. The following result follows from the more general Theorem 1.4.5 in [11]

Proposition 3.4 (Theorem 1.4.5 in [11]) *There exist positive numbers d_0, L and a neighborhood W of C such that if the sequence ξ is contained in W and verifies $\|g(\xi)\|_r \leq d \leq d_0$ then there exists $x \in \mathbb{R}^m$ such that $\|h(x, \xi)\|_r \leq Ld$.*

In order to use this proposition, let

$$\xi_k = X(T + k)$$

where $T > 0$ will be chosen later and let $C = \mathcal{L}$ be the limit set of X .

Choose (thank to condition (ii)) $\mu \in]e(X), 0[\cap \mathcal{R}(C)$ and $e(X) < \alpha < \mu$. Then, for t large enough

$$\sup_{0 \leq h \leq 1} \|X(t+h) - \Phi_h(X(t))\| \leq e^{\alpha t}.$$

Therefore, with $r = e^{-\mu}$ and T sufficiently large, $\|g(\xi)\|_r \leq \frac{e^{\alpha T}}{1-e^{\alpha-\mu}} \leq d$. Thus, by the latter proposition, $\|h(x, \xi)\|_r \leq Ld$ for some x . It then follows that for all $0 \leq h \leq 1$ and $k \geq 0$

$$\begin{aligned} \|X(T+k+h) - \Phi_{k+h}(x)\| &\leq \|X(T+k+h) - \Phi_h(X(T+k))\| + \|\Phi_h(X(T+k)) - \Phi_h(\Phi_k(x))\| \\ &\leq e^{\alpha(T+k)} + e^L \|g_k(\xi)\| = o(e^{\mu k}) \end{aligned}$$

where L is a Lipschitz constant of F on a neighborhood of $X(\mathbb{R}^+)$. Then

$$\|X(T+t) - \Phi_t(x)\| = o(e^{\mu t}).$$

In particular $X(\cdot)$ and $\{\Phi_t(x), t \geq 0\}$ have the same limit set. Now, by assumption (i) and Theorem 1 in [5]¹, $\Phi_t(x) \rightarrow p$ as $t \rightarrow \infty$. Under the supplementary assumption that $|\langle \nabla V(x), F(x) \rangle| \geq \beta \|\nabla V(x)\|^2$ then, by Theorem 2 in [5]

$$\|\Phi_t(x) - p\| = \begin{cases} O(e^{-ct}) & \text{if } \theta = 1/2 \\ O(t^{-\theta/(1-2\theta)}) & \text{if } 0 < \theta < 1/2 \end{cases}$$

■

Proof of Theorem 1.1 Assertion (i) follows from Remark 2.4 and Proposition 2.3. Assertion (ii) follows from Theorem 3.3, Example 3.1 and the fact that real analytic maps satisfies Lojasiewick inequality. ■

¹This theorem is stated under the assumption that V is C^2 and satisfies a global angle condition, but the proof works for C^1 with an angle condition at point p

Discussion of condition (ii) in Theorem 3.3 If we no longer assume that F is gradient like (but continue to assume that X is a bounded APT with limit set $\mathcal{L} = \mathcal{L}(X)$) the condition

$$]e(X), 0[\cap \mathcal{R}(\mathcal{L}) \neq \emptyset \quad (3)$$

implies (by Theorem 1.4.5 in [11] as in the proof of Theorem 3.3) that there exists $x \in \mathbb{R}^m$ such that

$$\limsup_{t \rightarrow \infty} \frac{\log(\|X(t) - \Phi_t(x)\|)}{t} \leq \lambda$$

with $\lambda = \inf]e(X), 0[\cap \mathcal{R}(\mathcal{L}) < 0$.

This shadowing property is reminiscent of the shadowing Theorem 9.3 in [3] (see also Theorem 8.9 in [1]). This latter result which easily follows from Hirsch's shadowing theorem ([8], Theorem 3.2) asserts the following. Let

$$\mathcal{E}(\Phi, \mathcal{L}) = \lim_{t \rightarrow \infty} \frac{1}{t} \log(\inf_{x \in \mathcal{L}} \|D\Phi_{-t}(\Phi_t(x))\|^{-1})$$

be the *expansion rate* of Φ at \mathcal{L} . If

$$e(X) < \min(0, \mathcal{E}(\Phi, \mathcal{L})) \quad (4)$$

then there exists $x \in \mathbb{R}^m$ such that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(\|X(t) - \Phi_t(x)\|) \leq e(X).$$

By a theorem of Schreiber ([15])

$$\mathcal{E}(\Phi, \mathcal{L}) = \inf_{\mu \in \mathcal{P}_{erg}(\mathcal{L})} \lambda_1(\mu)$$

where $\mathcal{P}_{erg}(\mathcal{L})$ is the set of Φ -invariant ergodic measures supported by \mathcal{L} and for each $\mu \in \mathcal{P}_{erg}(\mathcal{L})$ $\lambda_1(\mu)$ is the smallest Lyapounov exponent ² of μ .

Now, by Theorem 2 in [14] the *dynamical spectrum* $\Sigma(\mathcal{L}) = \mathbb{R} \setminus \mathcal{R}(\mathcal{L})$ is the union of $k \leq m$ compact intervals

$$\Sigma(\mathcal{L}) = [a_1, b_1] \cup \dots \cup [a_k, b_k]$$

²By a lyapounov exponent of μ we mean a lyapounov exponent of the skew product flow $(x, v) \in \mathcal{L} \times \mathbb{R}^m \mapsto (\Phi_t(x), D\Phi_t(x)v)$

with $a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_k \leq b_k$; and by Theorem 2.3 in [12] for every $\mu \in \mathcal{P}_{erg}(\mathcal{L})$ all the lyapounov exponents of μ are contained in $\Sigma(\mathcal{L})$ and every point in $\partial\Sigma(\mathcal{L}) = \{a_1, b_1, \dots, a_k, b_k\}$ is a lyapounov exponent for some $\mu \in \mathcal{P}_{erg}(\mathcal{L})$. It then follows that

$$a_1 = \mathcal{E}(\Phi, \mathcal{L})$$

so that condition (4) implies condition (3).

4 Non convergence toward (degenerate) unstable set of equilibria

We discuss here some implication of Theorem 3.3 to the problem of non convergence toward degenerate set of equilibria, for stochastic approximation processes.

Let $X = (X(t))_{t \geq 0}$ be a \mathbb{R}^m valued continuous stochastic process. For every open set $U \subset \mathbb{R}^m$ and $s \geq 0$, let T_s^U be the stopping time (with respect to the canonical filtration of X) defined by

$$T_s^U = \inf\{t \geq s : X(t) \in \mathbb{R}^m \setminus U\}.$$

Let $p \in \mathbb{R}^m$. We say that p is *repulsive* for X provided there exists a neighborhood U of p such that for all $n \in \mathbb{N}$

$$\mathbf{P}(T_n^U < \infty) = 1.$$

This clearly implies that

$$\mathbf{P}(\lim_{t \rightarrow \infty} X(t) = p) = 0$$

but the converse is false as shown by the next example.

Example 4.1 Consider a two colors Polya urn process. Initially, there are two balls, say one black and one white, in the urn. At each time a ball is randomly chosen, and replaced in the urn with a new ball of the same color. Let W_n denote the number of white balls at time n and $x_n = \frac{W_n}{n+2} \in [0, 1]$ its proportion. Then (x_n) satisfies (2.1) with $\gamma_n = \frac{1}{n+2}$ and $F(x) = 0$. It is well known that (x_n) (hence the interpolated process X) converges almost surely toward a random variable X_∞ having a uniform distribution on $[0, 1]$. Thus, for all $p \in [0, 1]$ $\mathbf{P}(\lim_{t \rightarrow \infty} X(t) = p) = 0$ but p is not repulsive for X .

The following result is a straightforward consequence of Theorem 3.3.

Theorem 4.2 *Let F and V be like in section 3 and let $X : \mathbb{R}^+ \mapsto \mathbb{R}^m$ be a continuous stochastic process, almost surely bounded and such that $e(X) \leq e < 0$. Let $C \subset \mathbf{Eq}(F)$ be a compact connected set of equilibria. Assume that*

- (i) *V satisfies a Lojasiewick inequality and an angle condition at every point $p \in C$;*
- (ii) *$]e, 0[\cap \mathcal{R}(C) \neq \emptyset$;*
- (iii) *Every point $p \in C$ is repulsive for X .*

Then

$$\mathbf{P}(\mathcal{L}(X) \subset C) = 0.$$

Proof. By assumptions and compactness, there exist points $p_1, \dots, p_k \in C$ and neighborhoods U_1, \dots, U_k such that $C \subset \bigcup U_i$, $p_i \in U_i$, and $\mathbf{P}(\mathcal{L}(X) \subset U_i) = 0$. Now, on the event $\mathcal{L}(X) \subset C$ there exists $i \in \{1, \dots, k\}$ such that $\mathcal{L}(X) \cap U_i \neq \emptyset$, hence by Theorem 3.3, $\mathcal{L}(X) \subset U_i$. Thus $\mathbf{P}(\mathcal{L}(X) \subset C) \leq \sum_i \mathbf{P}(\mathcal{L}(X) \subset U_i) = 0$. ■

Theorem 4.2 requires the verification that every point in C is repulsive. Beginning with a seminal paper by Pemantle [10], the literature on stochastic approximation and urn processes has produced several results showing that, under reasonable assumptions, a process given by (2.1) cannot converge toward an "unstable" equilibrium of the associated vector field. It turns out that, while these results are usually formulated as "non convergence results", a careful reading of the proofs shows that they actually prove repulsiveness of the unstable point.

The following result, due to Brandière and Duflo [4] (see also Chapter 3 in [7]) was first proved by Pemantle [10] when F is C^2 and p hyperbolic, for bounded noise and $\gamma_n = 1/n$. Pemantle's theorem was later extended to more general invariant sets (including non hyperbolic equilibria) in [1], (Section 9, Theorem 9.1). Under stronger assumptions on the noise sequence the smoothness assumption on F can be weakened to C^1 (see Theorem 3.12 in [2]). Tarrès'

PhD thesis ([16]) contains interesting generalizations that allow to deal with unstable (but not linearly unstable) points, especially in dimension one.

Let p be an equilibrium of a C^1 vector field $F : \mathbb{R}^m \mapsto \mathbb{R}^m$. Then \mathbb{R}^m can be written as the direct sum of E_p^s, E_p^u, E_p^c the generalized eigenspaces corresponding to the eigenvalues of the jacobian matrix $DF(p)$ having, respectively, negative real parts, null real parts and positive real parts. Equilibrium p is said to be *linearly unstable* if $E_p^u \neq \{0\}$ and *hyperbolic* if $E_p^c = \{0\}$.

Theorem 4.3 ([4], **Theorem 1**) *Let F be a C^1 vector field whose Jacobian is locally Lipschitz. Let p be a linearly unstable (non necessarily hyperbolic) equilibrium of F . Consider the process given in example 2.1 where (U_n) and γ_n are like in example 2.2 with $q = 2$. Assume furthermore that*

$$\sup_n \mathbf{E}(\|U_{n+1}\|^2 | \mathcal{F}_n) < \infty$$

and

$$\liminf_{n \rightarrow \infty} \mathbf{E}(\|U_{n+1}^u\| | \mathcal{F}_n) > 0,$$

where U_n^u stands for the projection of U_n on E_p^u along $E_p^s \oplus E_p^c$. Then p is repulsive for X .

Proof. Theorem 1 in [4] states that $\mathbf{P}(\lim_{n \rightarrow \infty} x_n = p) = 0$ so we need to explain how the proof implies that p is repulsive. We can always assume $p = 0$. By the center-stable manifold theorem (see e.g [13], Section 5.10.2), there exists a neighborhood $U = U_1 \times U_2$ of the origin with $U_1 \subset E_p^s \oplus E_p^c, U_2 \subset E_p^u$, and a C^1 map $G : U_1 \mapsto U_2$ with $G(0) = 0, DG(0) = 0$ whose graph W_{loc}^{cs} is locally invariant under $\{\Phi_t\}$. Furthermore, if $\Phi_t(x) \in U$ for all $t \in \mathbb{R}$, then $x \in W_{loc}^{cs}$. In particular, if X is an asymptotic pseudo trajectory of Φ , then

$$\mathcal{L}(X) \subset U \Rightarrow \mathcal{L}(X) \subset W_{loc}^{cs}.$$

Let X be the interpolated process associated to (x_n) (as defined in example 2.1). Set $X(t) = (X^1(t), X^2(t)) \in (E_p^s \oplus E_p^c) \times E_p^u$. The proof of Theorem 1 in [4] shows that

$$\mathbf{P}(\lim_{t \rightarrow \infty} X^2(t) - G(X^1(t)) = 0) = 0.$$

Thus,

$$\mathbf{P}(\mathcal{L}(X) \subset U) = 0.$$

■

Proof of Theorem 1.3 follows from Theorems 4.2 and 4.3 and the discussion in ([4], section I.2, 1) which shows that the assumptions on the noise given in Theorem 1.3 imply the assumptions given in Theorem 4.3.

References

- [1] M Benaïm, *Dynamics of stochastic approximation algorithms*, Séminaire de Probabilités, XXXIII, Lecture Notes in Math., vol. 1709, Springer, Berlin, 1999, pp. 1–68.
- [2] M. Benaïm and M. Faure, *Stochastic approximations, cooperative dynamics and supermodular games*, The Annals of Applied Probability **22** (2012), no. 5, 2133–2164.
- [3] M Benaïm and M W. Hirsch, *Asymptotic pseudotrajectories and chain recurrent flows, with applications*, J. Dynam. Differential Equations **8** (1996), no. 1, 141–176. MR 1388167 (97d:58165)
- [4] O. Brandiere and M. Duflo, *Les algorithmes stochastiques contournent-ils les pièges?*, Annales de l’I. H. P. Probabilités et statistiques **32** (1996), no. 3, 395–427.
- [5] R. Chill, A. Haraux, and M. Ali-Jendoubi, *Applications of the lojasiewicz simon, gradient inequality to gradient-like evolution equations*, Analysis and Applications **7** (2009), no. 4, 351–372.
- [6] C. Conley, *Isolated invariant sets and the morse index*, AMS, Providence, 1978, Series in Math.
- [7] M. Duflo, *Algorithmes stochastiques*, Springer Paris, 1996.

- [8] M. W Hirsch, *Asymptotic phase, shadowing and reaction diffusion systems*, Differential equations, dynamical systems and control science, Lecture Notes in Pure and Applied Math., vol. 152, Marcel Dekker, New York, 1994, pp. 87–99.
- [9] S. Lojasiewicz, *Une propriété topologique des sous-ensembles analytiques réels*, Les Équations aux Dérivées Partielles (1963), 87–89, Éditions du C.N.R.S, Paris.
- [10] R. Pemantle, *Nonconvergence to unstable points in urn models and stochastic approximations.*, ANN. PROB. **18** (1990), no. 2, 698–712.
- [11] S. Y. Pilyugin, *Shadowing in dynamical systems*, Springer, 1999, Lecture Notes in Mathematics, 1706.
- [12] K. J. Palmer R. A. Johnson and G. R. Sell, *Ergodic properties of linear dynamical systems*, SIAM J. MATH. ANAL. **18** (1987), no. 1, 1–33.
- [13] Clark. Robinson, *Dynamical Systems, Stability, Symbolic Dynamics, and Chaos*, 1999, second edition.
- [14] R. J. Sacker and G. R. Sell, *A spectral theory for linear differential systems*, Journal of Differential Equations **27** (1978), 320–358.
- [15] S. Schreiber, *Expansion rates and lyapounov exponents*, Discrete and Continuous Dynamical Systems **3** (1997), no. 4, 433–438.
- [16] P. Tarrès, *Pièges des algorithmes stochastiques et marches aléatoires renforcées par sommets*, Ph.D. thesis, ENS Cachan, 2001.

Acknowledgments

This work was supported by the SNF grant 2000020 – 149871/1