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On the conditional infiniteness of primitive weird numbers



Giuseppe Melfi

University of Neuchâtel, Faculty of Economics and Business, Rue A.-L. Breguet, 2,
CH-2000 Neuchâtel, Switzerland

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ABSTRACT

Text. A weird number is a number n for which $\sigma(n) > 2n$ and such that n is not a sum of distinct proper divisors of n . In this paper we prove that $n = 2^k pq$ is weird for a quite large set of primes p and q . In particular this gives an algorithm to generate very large primitive weird numbers, i.e., weird numbers that are not multiple of other weird numbers. Assuming classical conjectures on the gaps between consecutive primes, this also would prove that there are infinitely many primitive weird numbers, a question raised by Benkoski and Erdős in 1974.

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1. Introduction

Let n be a positive integer. Let $\sigma(n)$ denote the sum of divisors of n . If $\sigma(n) > 2n$, then n is called abundant. If n can be expressed as a sum of distinct proper divisors of n , then n is called semiperfect (or sometimes also pseudo-perfect). A weird number is an abundant number that is not semiperfect, i.e., that cannot be expressed as a sum of distinct proper divisors of n .

The term *weird* has been introduced in 1972 by Benkoski [2,3]. In his joint paper with Erdős [4], several results on weird numbers and related questions are proved. In

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particular they proved that there are infinitely many weird numbers, and that indeed, the set of weird numbers has positive asymptotic density.

If n is weird and $p > \sigma(n)$, then np is weird (see for example [6, p. 332]). In this paper we study the properties of *primitive* weird numbers, so those weird numbers that are not a multiple of other weird numbers.

An open problem is to determine whether infinitely many primitive weird numbers exist. Benkoski and Erdős stated this problem as an open question rather than as a conjecture, and in recent literature, as for example in [8, p. 77] and [12, p. 43], this problem still appears as an open question. Computational approaches allowed to provide a list of primitive weird numbers not exceeding $1.8 \cdot 10^9$ [13, Sequence A002975]. Kravitz [10] proved in 1976 that for a prime p , with $p > 2^k$, if $q = [2^k p - (p + 1)] / [(p + 1) - 2^k]$ is prime, then $2^{k-1}pq$ is a primitive weird number, and found eleven weird numbers among which a 53-digit number that has been for many years the largest known primitive weird number, until Klyve [9] announced a 226-digit weird number in 2013.

In this paper we prove the following theorem:

Theorem 1. *Let k be a positive integer and let a and b be positive odd integers such that $p = 2^{k+2} - a$ and $q = 2^{k+2} + b$ are primes. If $b + 3 < a < 2^{(k-1)/2}$ then $n = 2^k pq$ is a primitive weird number.*

Primitive weird numbers of the form $2^k pq$ appear to be quite common: among the first 160 primitive weird numbers, 116 are of this form. The idea of the theorem is to search for such primitive weird numbers with p and q in a close neighborhood of 2^{k+2} . The condition $b + 3 < a$ ensures that n is a primitive abundant number, and the sizes of a and b , both bounded by $2^{(k-1)/2}$, are such that all relevant divisors of n are in a close neighborhood of a multiple of 2^{k+2} . In this way there are quite large intervals of consecutive positive integers that cannot be reached by a sum of distinct proper divisors of n .

Note also that if k is too small, there are no values of a and b that fit the conditions of the theorem. The triple (a, b, k) that fits the conditions of the theorem and yields the least positive integer is $(5, 1, 6)$ for $2^6(2^8 - 5)(2^8 + 1) = 4128448$, that is the 32nd primitive weird number, and there are no other triples with $k \leq 7$.

In 1980 Pajunen [11] characterized primitive weird numbers of the form $2^k pq$ in terms of structure properties, but as far as we know no computational or analytical implications have been investigated since then.

Let $\{p_n\}_{n \in \mathbb{N}}$ denote the increasing sequence of primes. If we assume that for sufficiently large n , $p_{n+1} - p_n < 0.1p_n^{1/2}$, Theorem 1 implies that there are infinitely many primitive weird numbers of the form $2^k pq$. Indeed, infinitely many of such primes should exist: on a hand, Cramér's conjecture [5] claims that $p_{n+1} - p_n = O((\log p_n)^2)$; on the other hand a much weaker conjecture as for example Gonek's conjecture [7, p. 398], claiming that given an arbitrary $\varepsilon > 0$, $p_{n+1} - p_n < p_n^\varepsilon$ for sufficiently large n , is sufficient for the existence of primes postulated in the theorem.

We remind also the celebrated result of Baker, Harman and Pintz [1], who proved that for sufficiently large n , $p_{n+1} - p_n < p_n^{0.525}$. This is very close to what would be sufficient to ensure that there are infinitely many primitive weird numbers.

We will provide some examples of large weird numbers that we computed with a method derived from the conditions of Theorem 1, and that substantially improve the records of Klyve and his collaborators.

Other classical questions on weird numbers, as the existence of odd weird numbers, are still open. Nevertheless, we will discuss a related implication of Theorem 1 concerning odd weird numbers.

2. Preliminary tools

For a given positive integer n , let $\Delta(n) = \sigma(n) - 2n$. For abundant numbers n , we have $\Delta(n) > 0$, and we shall refer to $\Delta(n)$ as the *abundance* of n [8, p. 74].

Lemma 2. *Let n be an abundant positive integer. Then n is weird if and only if $\Delta(n)$ cannot be expressed as a sum of distinct proper divisors of n .*

Proof. Let n be an abundant number and let $\sigma^*(n) = \sigma(n) - n$ be the sum of all proper divisors of n . We have that $\Delta(n) + n = \sigma^*(n)$. So if $\Delta(n)$ is a sum of distinct proper divisors of n , then the sum of proper divisors of n not involved in the sum of $\Delta(n)$ must be n . Analogously, if $\Delta(n)$ cannot be expressed as a sum of distinct proper divisors of n , then also n cannot be expressed as a sum of distinct proper divisors of n . \square

Lemma 3. *If n is weird and $p > \sigma(n)$ is prime, then np is weird.*

Proof. As mentioned above, a proof has been given in [6]. \square

The preceding lemma justifies the definition of *primitive weird number*, a weird number that is not a multiple of another weird number. Since it is well known that an arbitrary multiple of an abundant number is abundant, we shall also consider *primitive abundant numbers*, namely abundant numbers that are not multiple of other abundant numbers.

If n is a primitive abundant number that is weird, then n is a primitive weird number. The converse is also true, i.e., primitive weird numbers are necessarily primitive abundant numbers.

3. Main result

Proof of Theorem 1. As pointed out in the introduction, we may assume $k \geq 8$. Let $p = 2^{k+2} - a$ and $q = 2^{k+2} + b$ be two primes with positive odd integers a and b , and with $b+3 < a < 2^{(k-1)/2}$. Let $n = 2^k pq$. We will prove in three steps that n is a primitive weird number. In the first step we will prove that n is abundant; then we will prove that

is a primitive abundant number and finally by Lemma 2 it will suffice to prove that the abundance $\Delta(n)$ cannot be expressed as a sum of distinct proper divisors of n .

Proof that n is abundant. We have

$$\begin{aligned} \Delta(n) &= \sigma(n) - 2n \\ &= (2^{k+1} - 1)(p + 1)(q + 1) - 2^{k+1}pq \\ &= 2^{k+1}(p + q + 1) - (p + 1)(q + 1) \\ &= 2^{k+1}(2^{k+3} + 1 - a + b) - (2^{k+2} - a + 1)(2^{k+2} + b + 1) \\ &= 2^{k+1}(a - b - 3) + (a - 1)(b + 1) \end{aligned}$$

Since $b + 3 < a$ and $(a - 1)(b + 1) > 0$, this evidently implies that $\Delta(n) > 0$.

Proof that n is a primitive abundant number. We shall prove that

$$\max\{\Delta(n/p), \Delta(n/q), \Delta(n/2)\} < 0.$$

We have

$$\Delta\left(\frac{n}{p}\right) = \sigma(2^k q) - 2^{k+1}q = (2^{k+1} - 1)(q + 1) - 2^{k+1}q = 2^{k+1} - q - 1 < 0.$$

Similarly we have

$$\Delta\left(\frac{n}{q}\right) = 2^{k+1} - p - 1 < 0.$$

Finally we have

$$\begin{aligned} \Delta\left(\frac{n}{2}\right) &= 2^k(2^{k+3} + 1 - a + b) - (2^{k+2} - a + 1)(2^{k+2} + b + 1) \\ &= 2^{2k+3} + 2^k(1 - a + b) - 2^{2k+4} + 2^{k+2}(a - b - 2) + (a - 1)(b + 1) \end{aligned}$$

Note that $1 - a + b < 0$; that $a - b - 2 < 2^{(k-1)/2}$ and that $(a - 1)(b + 1) < 2^k$. Therefore we have

$$\Delta\left(\frac{n}{2}\right) < -2^{2k+3} + 2^{k+2} \cdot 2^{(k-1)/2} + 2^k < 0.$$

Proof that n is weird. The increasing sequence of proper divisors of n starts with $1, 2, 2^2, \dots, 2^k, p, q, 2p, 2q, 4p, 4q, \dots, 2^k p, 2^k q$. Consider $I_0 = \{1, 2, \dots, 2^{k+1} - 1\}$ and for positive integers $h < h_M = 2^{(k-1)/2}$ the sets of consecutive integers $I_h = \{hp, hp + 1, hp + 2, \dots, hq + 2^{k+1} - 1\}$. The only positive integers $\leq 2^{3k/2}$ that are expressible as a sum of distinct proper divisors of n belong necessarily to $I = \bigcup_{i=0}^{h_M} I_i$. Let

$$s = \sum_{j=0}^k \varepsilon_{j,0} 2^j + \sum_{j=1}^{(k-4)/2} \varepsilon_{j,1} 2^j p + \sum_{j=1}^{(k-4)/2} \varepsilon_{j,2} 2^j q < 2^{3k/2},$$

with $\varepsilon_{j,l} \in \{0, 1\}$ be the generic sum of divisors of n not exceeding $2^{3k/2}$. Then $s \in I_{\tilde{h}}$ with

$$\tilde{h} = \sum_{j=1}^{(k-4)/2} (\varepsilon_{j,1} + \varepsilon_{j,2}) 2^j < h_M.$$

Note that the sets I_h have pairwise empty intersection. To prove that, we will prove that for every $h < h_M$, $\max I_h < \min I_{h+1}$. Since $h < h_M$, we have in particular

$$h < \frac{2^{k+1} - a + 1}{a + b}$$

so

$$h(a + b) < 2^{k+1} - a + 1$$

or, equivalently

$$h(q - p) < p - 2^{k+1} + 1$$

and this means that

$$\max I_h = hq + 2^{k+1} - 1 < (h + 1)p = \min I_{h+1}.$$

Note that $\min I_h = 2h \cdot 2^{k+1} - ha$ and $\max I_h = (2h+1) \cdot 2^{k+1} + hb - 1$. Let $h^* = (a-b)/2 - 2$. In order to prove that $\Delta(n)$ is not expressible as a sum of distinct proper divisors of n , we show that $\max I_{h^*} < \Delta(n) < \min I_{h^*+1}$. By subtracting $(2h^* + 1)2^{k+1}$ in each side of these inequalities, the claim is a consequence of the fact that

$$h^*b - 1 < (a - 1)(b + 1) < 2^{k+1} - (h^* + 1)a. \quad \square$$

4. Conclusion

Our attempts to generalize or extend the validity of [Theorem 1](#) unfortunately failed, and we were unable to prove unconditionally that there are infinitely many primitive weird numbers. Results proved here only give a strong evidence that infinitely many primitive weird numbers exist. In particular, under the classical Cramér’s conjecture on the gaps between pairs of consecutive primes [5], or even under the already mentioned Gonek’s conjecture, the following statement is true.

Conjecture 1. *There are infinitely many primitive weird numbers of the form $2^k pq$, with p and q primes.*

An application of [Theorem 1](#) allows to produce quite easily thousands of primitive weird numbers (a few hundreds are known in literature). For example, by using the PARI software, we can easily provide $k = 5898$, $a = 4529$, $b = 4171$ that fit the conditions of the theorem, and so

$$n = 2^{5898} \cdot (2^{5900} - a) \cdot (2^{5900} + b)$$

is a 5328-digits primitive weird number. Other primitive weird numbers we computed correspond to $k = 4998$, $a = 9683$ and $b = 2157$, or $k = 2998$, $a = 5415$ and $b = 3993$.

Another consequence of these discussions is the following conjecture.

Conjecture 2. *Let $\{w_n\}_{n \in \mathbb{N}}$ be the sequence of primitive weird numbers. Then*

$$\liminf_{n \rightarrow \infty} \frac{w_{n+1} - w_n}{w_n} = 0.$$

It is not possible to state the same for the $\limsup_{n \rightarrow \infty} \frac{w_{n+1} - w_n}{w_n}$. A computation shows that there are no primitive weird numbers between $1.74 \cdot 10^8$ and $2.54 \cdot 10^8$, so it is possible that

$$\limsup_{n \rightarrow \infty} \frac{w_{n+1} - w_n}{w_n} > 0.$$

The proof of [Theorem 1](#) can be easily adapted to an *a priori* slightly more general case, by replacing 2^k with an arbitrary *almost perfect number*, i.e., a number m , for which $\sigma(m) = 2m - 1$. Of course, powers of two are almost perfect numbers, but it is unknown whether other integers are almost perfect [[8](#), p. 74]. If m is an almost perfect number, and $p = 4m - a$ and $q = 4m + b$ are primes for odd positive integers a , and b , with $b + 3 < a < \sqrt{m/2}$, then $n = mpq$ is a primitive weird number. In particular, if any odd almost perfect number larger than 1 exists, then a choice of corresponding p and q would provide an odd weird number.

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