An explicit approach to the Baum-Connes conjecture for some semi-direct products

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Abstract

We investigate the Baum-Connes assembly map through concrete examples. It is known that the Baum-Connes conjecture holds for large classes of groups including a-T-menable groups, thanks to work of Higson and Kasparov. However neither of these works describes the K-groups. In this thesis we describe explicitly the Baum-Connes assembly map for some a-T-menable groups namely $\mathrm{BS}(1, n)$, where $n > 1$ and $F \wr \mathbb{F}_n$, where $n \geq 1$ and F is a finite group. Our explicit approach by nature involves computations of K-theory and equivariant K-homology with the latter being tightly related to Bredon homology. Our main tools to compute the K-theory is the Pimsner-Voiculescu 6-term exact sequence and to compute the equivariant K-homology we employ spectral sequences, namely suitable versions of Atiyah-Hirzebruch and Lyndon-Hochschild-Serre. In order to provide a complete and clear picture of the conjecture for these groups, we include concrete models for their classifying spaces. Moreover, we present natural sets of generators for the K-groups and identify them via the assembly map. In doing so, we reprove the Baum-Connes conjecture for these groups. Finally, in the context of the (modified) trace conjecture, we directly calculate the image of the induced trace and verify that this is the desired subring of $\mathbb{R}$.

Keywords: The Baum-Connes conjecture, wreath product, K-theory, equivariant K-homology, Bredon homology, the modified trace conjecture.

Résumé

On investigue la conjecture de Baum-Connes pour quelque exemples concrets. Cette conjecture est vraie pour plusieurs classes des groupes, en particulier les groupes a-T-moyennables, grâce au travail de Higson-Kasparov. En faisant référence à ce résultat on n'obtient qu'un isomorphisme abstrait. Dans cette thèse on décrira explicitement le morphisme d'assemblage des groupes a-T-moyennables; BS$(1, n)$ où $n > 1$ et $F \wr \mathbb{F}_n$ où F est fini et $n \geq 1$. Tous ces groupes peuvent être écrits comme un produit semidirect. Cette approche explicite, naturellement, contient calculs de K-theórie et K-homologie équivariante, celle-ci étans reliée à l'homologie de Bredon. Notre outil pour les calculs en K-théorie est la suite exacte à 6-termes de Pimsner-Voiculescu et pour calculer la K-homologie équivariante, on exploite les suites spectrales d'Atiyah-Hirzebruch et de Lyndon-Hochschild-Serre. Pour avoir une image claire et complète de la conjecture dans ces cas, on ajoute la description des modèles concrets d'espaces classifiants. En plus, on présente des bases naturelles de K-groupes et on les identifie par le morphisme d'assemblage. En faisant cela, on redémontre la conjecture de Baum-Connes pour ces groupes. Finalement, en considérant la conjecture de la trace modifiée, on calcule directement l'image de la trace induite.

Mots clefs: La conjecture de Baum-Connes, K-théorie, K-homologie équivariante, l'homologie de Bredon, la conjecture de la trace modifiée

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Contents

Chapter 1

Introduction

The Baum-Connes conjecture was introduced by Paul Baum and Alain Connes in an IHES preprint in 1982. This paper got published only 18 years later [BC00] with no changes except for the list of references! During those 18 years, not only a lot of beautiful works were motivated and done, but also, for instance, the trace conjecture got a *counterexample.* See Section 4.2 of this thesis. In 1994, Paul Baum, Alain Connes and Nigel Higson announced a reformulation (and a generalisation) of the Baum-Connes conjecture [BCH94]. While, the Baum-Connes conjecture of 1982 was set for discrete groups, the one of 1994 was conjectured to hold true for all (second countable) locally compact groups and all coefficients rather than only $\mathbb{C}$. The latter is known as the Baum-Connes conjecture with coefficients. Higson and Kasparov in [HK01] proved that this stronger conjecture is satisfied for all a-T-menable groups. We make a pause to provide the statement of the Baum-Connes conjecture. This reads as follows.

Conjecture 1.0.1. *Let G be a group. The assembly map*

$$\mu_i^G\colon \mathrm{K}_i^G(\underline{\mathrm{E}}G) \to \mathrm{K}_{\mathrm{i}}(\mathrm{C}^*_{\mathrm{r}}(G)), \quad i = 0, 1$$

is an isomorphism.

The object on the left-hand side of the assembly map is the G-equivariant K-homology of the *classifying space* for proper actions of G and the one on the right-hand side is the K-theory of the reduced C^*-algebra of G. These notions will be explained in Section 3.5.1 and Section 2.2. Alternatively, we refer to the left-hand side as the topological side and the right-hand side as the analytical side. Philosophically speaking, the validity of conjecture would provide a way to describe the K-theory of the difficult object $\mathrm{C}^*_{\mathrm{r}}(G)$ via the more accessible object namely the equivariant K-homology of $\underline{\mathrm{E}}G$.

Let us explain how (the left-hand side of) this assembly map was developed. Originally, Baum and Connes stated their conjecture for *discrete* groups and in order to describe the left-hand side they defined a *geometric* K-theory denoted by $\mathrm{K}^*(X,G)$ as a collection of certain K-cycles for (X,G), where X is a G-manifold. Using *elliptic* operators and index theory, they constructed a "natural" map from their geometrical K-theory to the analytical K-theory. They claimed that

$$\mu \colon \mathrm{K}^i(X,G) \to \mathrm{K}_i(\mathrm{C}_0(X) \rtimes_{\mathrm{r}} G), \quad i = 0,1$$

is an isomorphism. In this formulation $\mathrm{C}_0(X)$ denotes the algebra of continuous functions on X vanishing at infinity, and $\rtimes_{\mathrm{r}}$ denotes the reduced crossed product. See Section 2.1.2. In their article in 1994, they presented their conjecture as in Conjecture 1.0.1 and predicted that it holds true for *all* locally compact, second countable groups. The key point for them, as they said, was the use of $\underline{\mathrm{E}}G$ which was introduced only in 1988 in [BC88]. Employing KK-theory enabled them to state their conjecture in full generality. They defined then the *equivariant* K-*homology group* $\mathrm{K}_i^G(\underline{\mathrm{E}}G)$ for $i = 0,1$. A class in this group is represented by an abstract G-equivariant elliptic operator on $\underline{\mathrm{E}}G$ which is supported on a G-invariant subset $X \subset \underline{\mathrm{E}}G$ with $G\backslash X$ compact. Finally, they associated to such operator its index which is lying in $\mathrm{K}_i(\mathrm{C}^*_{\mathrm{r}}(G))$.

Comparing the two descriptions of the left-hand side for *discrete* groups, one wants to see formally that they are equivalent however folklorely they were! Only in 2007, Baum, Higson and Schick [BHS07] showed that the two are indeed equivalent.

There is yet another description of the assembly map which was obtained in a totally different setting. James F. Davis and Wolfgang Lück [DL98] in 1998, provided a categorical/topological definition of the assembly map by means of spectra over the orbit category of a discrete group G. The key point for them was to define their K-homology functor $\mathrm{K}_*^G(\cdot)$ satisfying

$$\mathrm{K}_*^G(\{pt\}) \cong \mathrm{K}_*(\mathrm{C}^*_{\mathrm{r}}(G)).$$

Once this was constructed, applying it to the G-equivariant map $\underline{\mathrm{E}}G \to \{\mathrm{pt}\}$ yields an assembly map $\mathrm{K}_*^G(\underline{\mathrm{E}}G) \to \mathrm{K}_*(\mathrm{C}^*_{\mathrm{r}}(G))$. In Section 3.5.1 we discuss in detail this approach. Hambleton and Pedersen [HP04] in 2004 proved that this assembly map is identified with the one of [BCH94] for discrete groups. Remark that the assembly map of Davis and Lück is defined only for discrete groups however the one of Baum, Connes and Higson is defined for all (second countable) locally compact groups. Nevertheless, the one of Baum, Connes and Higson requires G to be second countable while such an assumption is not required in the case of Davis and Lück. One motivation of Davis and Lück for their construction is, as they say, to set the stage for explicit computations. And indeed, they developed tools from algebraic topology to compute the left-hand side, namely a suitable version of the Atiyah-Hirzebruch spectral sequence.

After all, the formulation of the Baum-Connes conjecture in either of its descriptions as well as its confirming results are technical and abstract. As a consequence only specialists can access the literature and utilise its powerful applications. In this thesis, we explicitly describe the Baum-Connes conjecture for some concrete groups, namely the solvable Baumslag-Solitar groups and wreath products of a finite group with free groups. All these groups can be viewed as a semidirects product with some free group. Hence natural tools are available in order to compute the K-groups. Our aim is to shed light on the assembly map of these groups from a different perspective instead of Higson and Kasparov's. In doing so we will be able to explain the K-groups and their natural bases and moreover to identify them via the assembly map. Similarity and duality between topological and the analytical parts will be visible through our computations.

This thesis is organised as follows.

Chapter 2 recalls some well-known vocabulary and tools that will be used throughout our work to understand the right-hand side of the Baum-Connes conjecture. In particular we describe K-theory and explain the Pimsner-Voiculescu 6-term exact sequence. This exact sequence will be our main tool for computations on the right-hand side.

Chapter 3 provides all background required to understand the topological side of the Baum-Connes conjecture. As our reference for the left-hand side will be Davis-Lück's approach, we will spend a good amount of time to provide all requirements. Among all, we will introduce Bredon homology which is employed in the Atiyah-Hirzebruch spectral sequence to describe equivariant K-homology of $\underline{E}G$. Furthermore, we will introduce an equivariant Lyndon-Hochschild-Serre spectral sequence in Bredon homology. It will be our main tool for computations on the left-hand side.

Chapter 4 is a survey on the Baum-Connes conjecture and the (modified) trace conjecture.

Chapter 5 provides a short summary of the main results of this thesis.

Chapter 6, Chapter 7 and Chapter 8 contain the main results, respectively, on $\mathrm{BS}(1,n)$, $F \wr \mathbb{Z}$ and $F \wr \mathbb{F}_n$. These include computation of

K-theory, equivariant K-homology, presenting bases for these K-groups and identification of theses basis via the Baum-Connes assembly map. Furthermore we calculate directly the image of the induced trace.

Chapter 9 is the conclusion and contains some questions related to our work.

Chapter 2

On the right-hand side of the assembly map

This chapter comprises general material related to the C*-algebraic part of the Baum-Connes conjecture. We intend to highlight the basic definitions, constructions, and results that will be required later. The main references for this chapter are [LLR00], [Mu90], [Bla86] and [Weg93].

2.1 C*-algebra

Let A be an algebra over $\mathbb{C}$. An *involution* is a conjugate-linear map $a \mapsto a^*$ on A such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. A *Banach $*$-algebra* is a $*$-algebra together with a complete submultiplicative norm such that $\|a\| = \|a^*\|$ for all $a \in A$. If in addition A has a unit, such that $\|1\| = 1$ then A is called a *unital Banach $*$-algebra.* A C**-algebra* is a Banach $*$-algebra such that

$$\|aa^*\| = \|a\|^2, \qquad \forall a \in A.$$

One very first example of C*-algebra next to the trivial C*-algebra $\mathbb{C}$, is the algebra of bounded operators on a Hilbert space H

$$\mathcal{B}(H) = \{T\colon H \to H \,|\, T \text{ is linear and bounded}\}.$$

The algebra of *compact operators* on a Hilbert space H denoted by $\mathcal{K}(H)$ is a norm and $*$-closed two sided ideal in $\mathcal{B}(H)$, and hence it is a C*-algebra.

In a C*-algebra two types of elements play an important role.

Definition 2.1.1. (*projection, isometry, unitary*)
Let A be a unital C*-algebra.

- An element $p \in A$ is a *projection* if it is a self-adjoint idempotent i.e. it satisfies $p = p^*$ and $p^2 = p$.
- An element $v \in A$ is an *isometry* if $v^*v = 1$.
- An element $u \in A$ is a *unitary* if it is an isometry which satisfies $uu^* = 1$.

We may consider elements of an abstract C*-algebras as operators in $\mathcal{B}(H)$ for some Hilbert space H. This result is due to Gelfand, Naimark and Segal. Here is the statement.

Theorem 2.1.2.
For each C**-algebra* A *there exists a Hilbert space* H *with an isometric* $*$-*homomorphism* $\varphi\colon A \to \mathcal{B}(H)$. *In other words, every* C**-algebra is isometric to a sub*-C**-algebra of* $\mathcal{B}(H)$. *If* A *is separable, then* H *can be chosen to be a separable Hilbert space.*

Let X be a locally compact topological space. The algebra of continuous functions on X vanishing at infinity, denoted $\mathrm{C}_0(X)$, is a commutative C*-algebra. It turns out that all commutative C*-algebras can be viewed as such. This is known as the *Gelfand* Theorem. (see [Mu90, Theorem 2.1.10]).

Example 2.1.3. *(matrix algebra)*
Let $A \subset \mathcal{B}(H)$ *be a* C**-algebra. For* $n \in \mathbb{N}$, *the matrix algebra* $\mathrm{M}_n(A) \subset \mathcal{B}(H^n)$ *is a* C**-algebra.*

A C*-algebra is called a *finite dimensional* C*-algebra, if it is finite dimensional as a complex vector space. Here is a very useful result regarding the structure of finite dimensional C*-algebras.

Proposition 2.1.4.
Let A *be a finite dimensional* C**-algebra. Then there exist positive integers* r *and* $n_1, \ldots, n_r$ *such that*

$$A \cong \mathrm{M}_{n_1}(\mathbb{C}) \oplus \mathrm{M}_{n_2}(\mathbb{C}) \oplus \ldots \oplus \mathrm{M}_{n_r}(\mathbb{C}).$$

Moreover, r *is unique and* $n_1, \cdots, n_r$ *are unique up to a permutation.*

2.1.1 Inductive limit

In this part we introduce the notion of inductive limit, which is a colimit in the categorical sense, in an arbitrary category **C**. Later, we focus only on the category of C*-algebras, C*-**Alg**, and category of abelian groups, **Ab**.

This construction provides us with more examples of C*-algebras and abelian groups, respectively.
Let $(A_n)_{n=1}^{\infty}$ be a sequence of objects in **C** and consider a sequence $\varphi_n \colon A_n \to A_{n+1}$ of morphisms in **C**. For $m > n$ we consider the composed morphisms

$$\varphi_{m,n} = \varphi_{m-1} \circ \varphi_{m-2} \circ \cdots \circ \varphi_n \ : A_n \to A_m.$$

We refer to φ_n and $\varphi_{m,n}$ as the connecting morphisms. An *inductive limit or a direct limit* of the inductive sequence

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

in **C** is a system $(A, (\eta_n)_{n=1}^{\infty})$, where A is an object in **C**, with $\eta_n \colon A_n \to A$ a morphism in **C** for $n \in \mathbb{N}$, and where the following two conditions hold.

1. The diagram

$$\begin{tikzcd}
A_n \arrow[rr, "\varphi_n"] \arrow[dr, "\eta_n"'] & & A_{n+1} \arrow[dl, "\eta_{n+1}"] \\
& A &
\end{tikzcd}$$

   commutes for each $n \in \mathbb{N}$.

2. If $(B, (\lambda_n)_{n=1}^{\infty})$ is a system, where B is an object in **C**, $\lambda_n \colon A_n \to B$ is a morphism in $\mathscr{C}$ for each $n \in \mathbb{N}$, then there is unique morphism $\lambda \colon A \to B$ making the diagram

$$\begin{tikzcd}
& A_n \arrow[dl, "\lambda_n"'] \arrow[dr, "\eta_n"] & \\
A \arrow[rr, "\lambda"] & & B
\end{tikzcd}$$

   commutative for each $n \in \mathbb{N}$.

Inductive limits, when they exists, are unique in the sense that if $(A, (\eta_n)_{n=1}^{\infty})$ and $(B, (\lambda_n)_{n=1}^{\infty})$ are both inductive limits of the sequence above, then there is exactly one isomorphism $\lambda \colon A \to B$ making the diagram above commutative. We denote the inductive limit of the sequence above by

$$\varinjlim(A_n, \varphi_n) \quad \text{or} \quad \varinjlim A_n.$$

Remark 2.1.5. In C*-**Alg** and **Ab**, inductive limits exist. More precisely, in the category C*-**Alg** and in the above notations, if $(A, (\eta_n)_{n=1}^{\infty})$ is the inductive limit of the system $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$, then

$$A = \overline{\bigcup_n \mu_n(A_n)}, \quad \text{and} \quad \|\mu_n(a)\| = \lim_{m\to\infty} ||\varphi_{m,n}(a)||, \forall n \in \mathbb{N}, \forall a \in A_n.$$

Similarly, in the category **Ab**, if $(G, (\eta_n)_{n=1}^{\infty})$ is the direct limit of a direct system $G_1 \xrightarrow{\varphi_1} G_2 \xrightarrow{\varphi_2} G_3 \xrightarrow{\varphi_3} \cdots$ we have that

$$G = \bigcup_n \mu_n(G_n).$$

Note that $\mu_n(A_n)$ (and respectively $\mu_n(G_n)$) is an increasing sequence of C*-algebras (and respectively of abelian groups). For more details on direct limits, we refer to Chapter 6 of [LLR00].

One of the very well-known class of C*-algebras are the so-called *AF-algebras*. An AF-algebra is a C*-algebra which is isomorphic to the inductive limit of a sequence of finite dimensional C*-algebras. The term "AF" is an abbreviation of Approximately Finite Dimensional.
Uniformly Hyper-Finite C*-algebras, *UHF-algebras*, form a class of particularly accessible AF-algebras.
A UHF-algebra is a C*-algebra which is isomorphic to the inductive limit of a sequence

$$\mathrm{M}_{k_1}(\mathbb{C}) \xrightarrow{\phi_1} \mathrm{M}_{k_2}(\mathbb{C}) \xrightarrow{\phi_2} \mathrm{M}_{k_3}(\mathbb{C}) \xrightarrow{\phi_3} \cdots ,$$

for some natural numbers $k_1, k_2, k_3, \ldots$ and some unit preserving connecting $*$-homomorphisms $\varphi_1, \varphi_2, \varphi_3, \ldots$. Remark that simplicity of $\mathrm{M}_n(\mathbb{C})$ ensures that ϕ_n is injective.
These algebras were studied and classified by Glimm in [G60] before general AF-algebras were considered.

Remark 2.1.6. By reversing all arrows appearing in the construction of inductive limit, we come up with the notion of *projective limit* or *inverse limit.*

2.1.2 Crossed product

Crossed product construction is a source of producing new C*-algebras out of dynamical systems involving group actions. In this section we introduce these algebras with a particular attention towards *group* C**-algebras*, an important subclass of crossed products.
We can obtain C*-algebras from groups. These are known as *group* C**-algebras.* Here we describe how to construct them.
Through this thesis, G is always a discrete group unless different is stated. For a group G we denote by λ its *left regular representation*

$$\lambda \colon G \to \mathcal{B}(\ell^2(G)) \qquad \lambda_s(\delta_r) = \delta_{sr}, \quad r, s \in G,$$

where $\{\delta_r \colon r \in G\} \subset \ell^2(G)$ is the canonical orthonormal basis. The *group ring* of G denoted by $\mathbb{C}G$ is the set of finite formal sums $\sum_{r \in G} a_r r$, where

$a_r \in \mathbb{C}$. Multiplication and involution are defined by

$$(\sum_{s\in G} a_s s)(\sum_{t\in G} a_t t) = \sum_{s,t} a_s a_t st, \qquad (\sum_{s\in G} a_s s)^* = \sum_{s\in G} \bar{a_s} s^{-1}.$$

Remark that the left regular representation can be extended to an injective $*$-homomorphism

$$\lambda\colon \mathbb{C}G \to \mathcal{B}(\ell^2(G)),$$

which is still denoted by λ. The *reduced group* C*-*algebra* of G, denoted by $\mathrm{C}^*_{\mathrm{r}}(G)$ is the completion of $\mathbb{C}G$ with respect to the norm

$$\|x\|_{\mathrm{r}} = \|\lambda(x)\|_{\mathcal{B}(\ell^2(G))},$$

in other words

$$\mathrm{C}^*_{\mathrm{r}}(G) = \overline{\lambda(\mathbb{C}G)}^{\|\cdot\|_{\mathrm{r}}}.$$

The *universal group* C*-*algebra* of G, denoted $\mathrm{C}^*(G)$, is the completion of $\mathbb{C}G$ with respect to the norm

$$\|x\|_u = \sup_\pi \|\pi(x)\|,$$

where the supremum is taken over all $*$-representations $\pi\colon \mathbb{C}G \to \mathcal{B}(H)$. Note that the supremum is finite as unitaries have norm one.

We might refer to the image of an element $g \in G$ in the unitary group $\mathcal{U}(\ell^2(G))$ by u_g. Here is a relation between these two group C*-algebras. Consider

$$\lambda\colon G \to \mathrm{C}^*_{\mathrm{r}}(G) \subset \mathcal{B}(\ell^2(G)), \qquad g \mapsto u_g.$$

This homomorphism induces the surjection

$$\lambda\colon \mathrm{C}^*(G) \to \mathrm{C}^*_{\mathrm{r}}(G), \qquad u_g \mapsto u_g.$$

Therefore we have that

$$\mathrm{C}^*_{\mathrm{r}}G \cong \mathrm{C}^*(G)/\ker\lambda.$$

Definition 2.1.7. (*Pontryagin dual*)
For an abelian topological group G, the *dual group* $\hat{G}$ is the group of all *characters* of G that means all continuous homomorphism from G with values in the *circle group* $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$.

Remark 2.1.8. For a discrete group G, its dual group $\hat{G}$ is compact.

Remark 2.1.9. For a discrete abelian group G, the Fourier transform yields the identification

$$\mathrm{C}^*(G) \cong \mathrm{C}(\hat{G}),$$

where $\mathrm{C}(\hat{G})$ indicates the C*-algebra of continuous functions on $\hat{G}$.

There is a useful relation between amenability of a group and its group C*-algebras. Let us first define amenability of a group, which means that we give one out of approximately $10^{10^{10}}$ different definitions/characterisations of amenability as Brown and Ozawa remarked in their book [BrOz07].

Definition 2.1.10. (*amenable group*)
A group G is *amenable* if there is a state μ that is a positive linear functional of norm one on $\ell^\infty(G)$ which is invariant under the left translation action: for all $g \in G$ and $f \in \ell^\infty(G)$, $\mu(gf) = \mu(f)$. Such a state μ is called an *invariant mean.*

Here is the relation between the two group C*-algebras and amenability, as we promised earlier.

Theorem 2.1.11. *[BrOz07, Theorem 2.6.8]*
Let G be a discrete group. The group G is amenable if and only if $\mathrm{C}^*_\mathrm{r}(G) \cong \mathrm{C}^*(G)$.

There is a huge class of groups containing all amenable groups.

Definition 2.1.12. (*Haagerup property or a-T-menability*)
An isometric action of a discrete group G on an affine Hilbert space H is called *metrically proper* if for every bounded subsets $B \subset H$ the set

$$\{g \in G \colon gB \cap B \neq \emptyset\}$$

is finite. We say a group G has *Haagerup property* or is *a-T-menable* if it admits a metrically proper isometric action on some (affine) Hilbert space.

Remark 2.1.13. Haagerup property à priory is defined in a different setting than a-T-menability. For discrete groups they are equivalent.

The most classical examples of groups with Haagerup property are free groups $\mathbb{F}_n$ [Ha78] and amenable groups [BCV95]. Remark that the class of such groups is closed under passing to subgroups and passing to finite extensions. However, a semidirect product of groups having Haagerup property does not need to have the Haagerup property, as can be seen by the example of $\mathbb{Z}^2 \rtimes \mathrm{S}L_2(\mathbb{Z})$ [BHV08].
We may define crossed products from (noncommutative) dynamical systems.

Definition 2.1.14. ($G \overset{\alpha}{\curvearrowright} A$)
Let G be a (discrete) group and A be a C*-algebra. An *action* of G on A, denoted $G \overset{\alpha}{\curvearrowright} A$, is a group homomorphism α from G into the group of $*$-automorphisms of A. A C*-algebra equipped with a G-action is called a G-C*-algebra.

Suppose A is a unital C*-algebra. We construct a C*-algebra $A \rtimes_\alpha G$ encoding the action of G on A, with the property that there is a copy of G inside the unitary group of $A \rtimes_\alpha G$, and there is a natural inclusion $A \subset A \rtimes_\alpha G$ such that

- $A \rtimes_\alpha G$ is generated by A and G, and
- $\alpha_g(a) = gag^*$ for all $a \in A$ and $g \in G$.

For a G-C*-algebra A, we denote by $C_c(G, A)$ the linear space of finitely supported functions on G with values in A. An element $S \in C_c(G, A)$ can be written as a finite sum

$$S = \sum_{s \in G} a_s s.$$

For two such elements $S, T \in C_c(G, A)$, the multiplication which is *α-twisted convolution product* and $*$-operation are defined as follows.

$$S *_\alpha T = \sum_{s,t \in G} a_s \alpha_s(b_t) st \quad \text{and} \quad S^* = \sum_{s \in G} \alpha_{s^{-1}}(a_s^*) s^{-1}.$$

Therefore $C_c(G, A)$ is a $*$-algebra. In order to make it into a C*-algebra we need to introduce certain representations.

Definition 2.1.15. (*covariant representation*)
A *covariant representation* (u, π, H) of a G-C*-algebra A is constructed from a unitary representation $u \colon G \to \mathcal{U}(H)$ and a $*$-representation $\pi \colon A \to \mathcal{B}(H)$ such that

$$u_s \pi(a) u_s^* = \pi(\alpha_s(a)) \qquad s \in G, a \in A.$$

The *maximal crossed product* or *universal crossed product* of a C*-dynamical system (A, α, G) denoted $A \rtimes_\alpha G$ is the completion of $C_c(G, A)$ with respect to the norm

$$\|x\|_u = \sup \|\pi(x)\|,$$

where the supremum is taken over all $*$-homomorphisms $\pi \colon C_c(G, A) \to \mathcal{B}(H)$.

Proposition 2.1.16. *[BrOz07, Proposition 4.1.3] (universal property)*
For every covariant representation (u, π, H) of a G-C-algebra A, there is a $*$-homomorphism $\varphi \colon A \rtimes_\alpha G \to \mathcal{B}(H)$ such that*

$$\varphi(\sum_{s \in G} a_s s) = \sum_{s \in G} \pi(a_s) u_s,$$

for all $\sum_{s \in G} a_s s \in C_c(G, A)$.

We can complete $C_c(G, A)$ in the norm coming from the left regular representation. Let A be represented on a Hilbert space H. The *reduced crossed product* of a C*-dynamical system (A, G, α), denoted $A \rtimes_{\alpha,r} G$, is the norm closure of the image of a left regular representation $C_c(G, A) \to \mathcal{B}(H \otimes \ell^2(G))$. We close this section with two remarks.

Remark 2.1.17. If G is an amenable group, then

$$A \rtimes_{\alpha,\mathrm{r}} G \cong A \rtimes_\alpha G.$$

Remark 2.1.18. If we let $A = \mathbb{C}$ in the crossed product construction above, then we obtain the group C*-algebras associated to a group as defined earlier.

2.2 K-theory of C*-algebras

In this section we introduce one of the fundamental objects of this thesis namely K-theory of C*-algebras.

One of the most considerable recent achievements in C*-algebra theory has been the introduction of homological algebraic methods. Specifically, the K-theory of C*-algebras has had some magnificent success in solving long-open problems. Furthermore, regarding Elliott's classification program for C*-algebras, K-theoretic information plays an important role. One of the early achievements of this program was that AF-algebras are classifiable using purely K-theoretic data.

The basic idea of K-theory is to associate with each C*-algebra A, two abelian groups $\mathrm{K}_0(A)$ and $\mathrm{K}_1(A)$ which reflect some structure of A.

2.2.1 K_0-group

Let A be a unital C*-algebra. For $n \in \mathbb{N}$, we consider the matrix algebra $\mathrm{M}_n(A)$ with the inclusions

$$\mathrm{M}_n(A) \subset \mathrm{M}_{n+1}(A), \qquad M \mapsto \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}.$$

We consider

$$\mathrm{M}_\infty(A) = \bigcup_n \mathrm{M}_n(A), \quad \mathrm{P}_\infty(A) = \{p \in \mathrm{M}_\infty(A) \colon p \,\text{is a projection}\},$$

and define a binary operation

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}, \qquad p, q \in \mathrm{P}_\infty(A).$$

Now consider the Murray-von Neumann equivalence relation on $\mathrm{P}_\infty(A)$

$$p \sim q \iff \exists v \in \mathrm{M}_\infty(A) \quad \text{such that} \quad p = vv^*, q = v^*v, \qquad p, q \in P_\infty(A).$$

Set $V(A) = \mathrm{P}_\infty(A)/\!\sim$ and define

$$[p] + [q] = [p \oplus q].$$

One sees that $(V(A), +)$ is an *abelian semigroup.* We may obtain an abelian group from every abelian semigroup in an analogous way to obtaining $\mathbb{Z}$ from $\mathbb{N}$. This construction is known as the *Grothendieck group.*

Definition 2.2.1. (K_0*-group*)
Let A be a unital C*-algebra. Then $\mathrm{K}_0(A)$ is the Grothendieck group of the abelian semigroup $(V(A), +)$. The *standard picture* of K_0 is

$$\mathrm{K}_0(A) = \{[p] - [q] \colon p, q \in \mathrm{P}_\infty(A)\}.$$

Moreover, $[0_A] = 0$, where 0_A is the *zero projection* in A.

So far we have defined K_0 for unital C*-algebras, it is also possible to define it for *non-unital* ones. Let A be a C*-algebra. Consider its *unitisation*, denoted $\tilde{A}$, which is a suitable completion of the vector space $A \oplus \mathbb{C}$. Now let $\tau \colon \tilde{A} \to \mathbb{C}$ be the canonical $*$-homomorphism. We consider the induced homomorphism at the level of K_0 which is denoted by $\tau_* \colon \mathrm{K}_0(\tilde{A}) \to \mathrm{K}_0(\mathbb{C})$. We define

$$\mathrm{K}_0(A) = \mathrm{Ker}(\tau_*),$$

hence $\mathrm{K}_0(A)$ is a subgroup of $\mathrm{K}_0(\tilde{A})$. Here are a few examples.

Example 2.2.2.

- $\mathrm{K}_0(\mathbb{C}) = \mathrm{K}_0(\mathrm{M}_n(\mathbb{C})) = \mathrm{K}_0(\mathcal{K}(H)) = \mathbb{Z}$.
- $\mathrm{K}_0(\mathcal{B}(H)) = 0$. *(for an infinite dimensional Hilbert space)*
- $\mathrm{K}_0(A \otimes \mathcal{K}) = \mathrm{K}_0(A)$. *(stability of* K_0*)*
- $\mathrm{K}_0(\mathrm{C}(\mathbb{T})) = \mathbb{Z}$.

2.2.2 Higher K-groups

The other K-group, K_1, captures information related to the unitaries of a C*-algebra and its matrix algebras.

Definition 2.2.3. (K_1-group)
Let A be a C*-algebra, not necessarily unital. Denote by $\mathrm{U}_n(\tilde{A})$ the unitary matrix over $\tilde{A}$ and consider the inclusion

$$\mathrm{U_n}(\tilde{A}) \subset \mathrm{U_{n+1}}(\tilde{A}), \quad u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

Let

$$\mathrm{U}_\infty(\tilde{A}) = \bigcup_{n\in\mathbb{N}} \mathrm{U}_n(\tilde{A}).$$

Then define the equivalence relation

$$u \sim v \iff \exists m, \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}_{m\times m} \sim_h \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}_{m\times m}, \quad u, v \in \mathrm{U}_\infty(\tilde{A}),$$

where "h" indicates *homotopy* (i.e. there is a path in $\mathrm{U}_m(\tilde{A})$ connecting continuously $u \oplus 1$ to $v \oplus 1$).
We define

$$\mathrm{K}_1(A) = \{[u] \colon u \in \mathrm{U}_\infty(\tilde{A})\}.$$

This is an abelian group with a multiplication given by

$$[u][v] = [uv] = [u \oplus v]$$

Note that then $[1] = 0$.

Example 2.2.4.

- $\mathrm{K}_1(\mathbb{C}) = \mathrm{K}_1(\mathrm{M}_n(\mathbb{C})) = \mathrm{K}_1(\mathcal{K}(H)) = 0$.
- $\mathrm{K}_1(\mathcal{B}(H)) = 0$.
- $\mathrm{K}_1(A \otimes \mathcal{K}) = \mathrm{K}_1(A)$, *(stability of* K_1*)*
- $\mathrm{K}_1(\mathrm{C}(\mathbb{T})) = \mathbb{Z}$

We may define the K_1-group (and the higher K-groups) from the K_0-group. Let A be a C*-algebra, we define its *suspension* to be the C*-algebra

$$\mathrm{S}(A) = \{f \in \mathrm{C}_0([0,1], A) \colon f(0) = f(1) = 0\}.$$

Now we define

$$\mathrm{K}_n(A) = \mathrm{K}_0(S^n(A)), \qquad n \in \mathbb{N},$$

and in particular we have

$$\mathrm{K}_1(A) = \mathrm{K}_0(\mathrm{S}(A)),$$

(see Theorem 2.2.8).

2.2.3 Functoriality of K-theory

Let C***-Alg** be the category of C*-algebras, whose objects are C*-algebra and its morphisms are the $*$-homomorphism between C*-algebras. We may view the K-groups as functors from the category C***-Alg** to the category **Ab** of abelian groups. Let $\varphi\colon A \to B$ be a $*$-homomorphism between C*-algebras. It induces a unique homomorphism

$$\begin{aligned} \varphi_*\colon \mathrm{K}_i(A) &\to \mathrm{K}_i(B), \qquad i = 0,1 \\ \varphi_*([p]) &= [\varphi(p)]. \end{aligned}$$

Now if $\psi\colon B \to C$ is another $*$-homomorphism of C*-algebras we have that $(\psi\varphi)_* = \psi_*\varphi_*$ and $(\mathrm{id}_A)_* = \mathrm{id}_{\mathrm{K}_i(A)}$. Thus we have a *covariant functor*

$$\begin{aligned} \mathrm{K}_i\colon \mathrm{C}^*\textbf{-Alg} &\to \textbf{Ab} \\ A &\mapsto \mathrm{K_i}(A), \\ \varphi &\mapsto \varphi_* \qquad i = 0,1. \end{aligned}$$

In the sequel we recall some of the fundamental results in K-theory.

Theorem 2.2.5. *(Continuity of K-theory)*
If a C**-algebra* A *appears as the inductive limit of a system* (A_n, φ_{nm}) *of* C**-algebras, then* $(\mathrm{K}_*(A_n), \varphi_{nm*})$ *is a directed systems of abelian groups and*

$$\mathrm{K}_i(A) = \mathrm{K}_i(\varinjlim A_n) \cong \varinjlim \mathrm{K}_i(A_n), \qquad i = 0,1.$$

For $1 \leq i,j \leq n$, let the element $e_{ij} \in \mathrm{M}_n(\mathbb{C})$ denote the matrix with all its entries 0 except for the (i,j) entry, which is 1. The matrices e_{ij}'s form a linear basis for $\mathrm{M}_n(\mathbb{C})$, called the *canonical basis*. Clearly e_{ii}'s are (rank 1) projections. For an infinite separable Hilbert space H, we have $H \simeq \ell^2(\mathbb{N})$ with the canonical basis $\{e_i\colon i \in \mathbb{N}\}$. We define for $i,j \in \mathbb{N}$ the operator e_{ij} such that

$$e_{ij}(e_k) = \delta_{jk}e_i, \qquad k \in \mathbb{N}.$$

Theorem 2.2.6. *(stability of K-theory)*
Consider the morphism $A \to A \otimes \mathcal{K}$ *sending* $a \mapsto a \otimes e_{11}$*, where* e_{11} *is a rank* 1 *projection in* $\mathcal{K}$*. This induces an isomorphism*

$$\mathrm{K}_i(A) \simeq \mathrm{K}_i(A \otimes \mathcal{K}) \qquad i = 0,1.$$

In particular, if A *and* B *are stably isomorphism, i.e.* $A \otimes \mathcal{K} \simeq B \otimes \mathcal{K}$*. Then there is an isomorphism* $\mathrm{K}_i(A) \simeq \mathrm{K}_i(B)$*.*

Theorem 2.2.7. *(half exactness of K-theory)*
An exact sequence $0 \to J \xrightarrow{\iota} A \xrightarrow{\pi} A/J \to 0$, *where* J *is an ideal in* A, *induces a short exact sequence of* K*-groups*

$$\mathrm{K}_i(J) \xrightarrow{\iota_*} \mathrm{K}_i(A) \xrightarrow{\pi_*} \mathrm{K}_i(A/J) \qquad i = 0, 1.$$

Remark that if $0 \to A \xrightarrow{j} B \xrightarrow{\varphi} C \to 0$ is a short exact of sequence of C*-algebras that *splits*, i.e. there exists a $*$-homomorphism $\psi\colon B \to A$ such that $\varphi\psi = \mathrm{id}_B$, then $0 \to \mathrm{K_i}(A) \xrightarrow{j_*} \mathrm{K_i}(B) \xrightarrow{\varphi_*} \mathrm{K_i}(C) \to 0$ for $i = 0, 1$ is a split short exact sequence in K-theory as well. In particular, we have that the K-theory functor is *additive* in the sense that

$$\mathrm{K_i}(A \oplus B) \simeq \mathrm{K_i}(A) \oplus \mathrm{K_i}(B) \qquad i = 0, 1.$$

Theorem 2.2.8. *For every* C**-algebra* A *there is a natural* $*$*-isomorphism*

$$\alpha_A\colon \mathrm{K}_1(A) \to \mathrm{K}_0(\mathrm{S}(A))$$

Theorem 2.2.9. *(Bott periodicity)*
For every C**-algebra* A *there is a natural isomorphism*

$$\beta_A\colon \mathrm{K}_0(A) \to \mathrm{K}_1(\mathrm{S}(A)).$$

Now combining Theorem 2.2.8 and Theorem 2.2.9 we obtain

$$\mathrm{K}_0(A) \simeq \mathrm{K}_0(\mathrm{S}^2(A)) \quad \text{and} \quad \mathrm{K}_1(A) \simeq \mathrm{K}_1(\mathrm{S}^2(A)).$$

Therefore K-theory of C*-algebras is 2-periodic.

Consider a short exact sequence $0 \to J \to A \to A/J \to 0$ of C*-algebras. According to half exactness of the K_i-functors we have two 3-term exact sequences. We may define a connecting map called *exponential map*

$$\partial_1\colon \mathrm{K}_1(A/J) \to \mathrm{K}_0(J),$$

to make these two pieces into a long exact sequence

$$\mathrm{K}_1(J) \to \mathrm{K}_1(A) \to \mathrm{K}_1(A/J) \xrightarrow{\partial_1} \mathrm{K}_0(J) \to \mathrm{K}_0(A) \to \mathrm{K}_0(A/J).$$

In order to connect the two ends of this exact sequence we use *Bott periodicity.* Here we explain how. In Section 2.2.2 we introduced the suspension C*-algebra S. Applying this C*-algebra which is also an exact functor, [Mu90, Theorem 7.5.8], to the exact sequence above, we obtain the short exact sequence

$$0 \to S(J) \to S(A) \to S(A/J) \to 0.$$

Hence we have again an exact sequence of C*-algebras. Applying the K-theory functor, we arrive at

$$\mathrm{K}_1(S(J)) \to \mathrm{K}_1(S(A)) \to \mathrm{K}_1(S(A/J)) \xrightarrow{\tilde{\partial}_1} \mathrm{K}_0(S(J)) \to \mathrm{K}_0(S(A)) \to \mathrm{K}_0(S(A/J)),$$

where $\tilde{\partial}_1$ is the exponential map mentioned above. Finally, in order to introduce our missing map, we employ $\beta_{A/J}$ in Theorem 2.2.9, $\tilde{\partial}_1$ in the above long exact sequence and α_J in Theorem 2.2.8, and we define

$$\partial_0 = \alpha_J^{-1} \circ \tilde{\partial}_1 \circ \beta_{A/J}.$$

We call the boundary map ∂_0 an *index map.*

Therefore we get a 6-term exact cyclic diagram.

Theorem 2.2.10. *(6-term exact sequence)*
For a short exact sequence $0 \to J \xrightarrow{\iota} A \xrightarrow{\pi} A/J \to 0$ *of* C*-*algebras, the cyclic diagram*

$$\begin{tikzcd}
\mathrm{K}_0(J) \arrow[r, "\iota_*"] & \mathrm{K}_0(A) \arrow[r, "\pi_*"] & \mathrm{K}_0(A/J) \arrow[d, "\partial_0"] \\
\mathrm{K}_1(A/J) \arrow[u, "\partial_1"] & \mathrm{K}_1(A) \arrow[l, "\pi_*"'] & \mathrm{K}_1(J) \arrow[l, "\iota_*"']
\end{tikzcd}$$

is exact everywhere.

This exact sequence is a very efficient tool to describe the K-theory for C*-algebras related to an extension. There are also other useful tools to compute the K-theory of certain C*-algebras involving crossed products and tensor products, namely *Pimsner-Voiculescu 6-term exact sequence*, *Connes' Thom isomorphism*, and the *Künneth theorem.* In the next section we describe the Pimsner-Voiculescu exact sequence which will be relevant to our computations.

We close this section by a characterisation of the K-theory functor. The two functors K_0 and K_1 are covariant, homotopy invariant, half exact, split exact, continuous, additive, stable functors from the category C*-**Alg** to the category **Ab**. A functor from the category C*-**Alg** to the category **Ab** is called a *Bott functor* if it is half exact, homotopy invariance, and stable.

Theorem 2.2.11. *[Weg93, Theorem 11.2.1]*
Every Bott functor K *has Bott periodicity that means there is a natural isomorphism* $\mathrm{K}(A) \simeq \mathrm{K}(\mathrm{S}^2(A))$ *for every* C*-*algebra* A. *In particular, associated to a short exact sequence* $0 \to A \to B \to C \to 0$ *in* C*-*algebras we get a 6-term cyclic exact sequence in abelian groups*

$$\begin{tikzcd}
\mathrm{K}A \arrow[r] & \mathrm{K}B \arrow[r] & \mathrm{K}C \arrow[d] \\
\mathrm{KS}(C) \arrow[u] & \mathrm{KS}(B) \arrow[l] & \mathrm{KS}(A) \arrow[l]
\end{tikzcd}$$

The following result by Cuntz [Cu82] shows that K-theory is completely determined by its functorial properties.

Theorem 2.2.12. *(functorial characterisation of K-theory) [Weg93, Theorem 11.2.3]*
Let K *be a continuous Bott functor from the category of* C^**-algebras* C^***-Alg** *to the category of abelian groups* **Ab**. *If* $\mathrm{K}(\mathbb{C}) = \mathbb{Z}$ *and* $\mathrm{K}(\mathrm{S}(\mathbb{C})) = 0$*, then* $\mathrm{K}(A) = \mathrm{K}_0(A)$ *for* A *in a large subcategory* $\mathbf{C}$ *of* C^**-algebras (the Bootstrap class). If* $\mathrm{K}(\mathbb{C}) = 0$ *and* $\mathrm{K}(\mathrm{S}(\mathbb{C})) = \mathbb{Z}$*, then* $\mathrm{K} = \mathrm{K}_1(A)$*.*

This functorial characterisation of K-theory is interesting in particular in analogy with functorial constructions in Section 3.5.1.

2.3 Pimsner-Voiculescu 6-term exact sequence

This section is devoted to two 6-term cyclic exact sequences due to M. Pimsner and D. Voiculescu introduced in [PV80] and [PV82]. These exact sequences will be our main tool to compute the K-theory of C*-algebras of Chapter 6, Chapter 7 and Chapter 8. Let A be C*-algebra on which the integer group $\mathbb{Z}$ acts by α via the unitary $u \in A \rtimes \mathbb{Z}$ such that $\alpha(a) = uau^*$ for $a \in A$. Pimsner and Voiculescu in 1980 constructed the 6-term exact sequence

$$\begin{tikzcd}
\mathrm{K}_0(A) \arrow[r, "\mathrm{Id} - \alpha_*"] & \mathrm{K}_0(A) \arrow[r, "\iota_*"] & \mathrm{K}_0(A \rtimes_\alpha \mathbb{Z}) \arrow[d, "\partial_0"] \\
\mathrm{K}_1(A \rtimes_\alpha \mathbb{Z}) \arrow[u, "\partial_1"] & \mathrm{K}_1(A) \arrow[l, "\iota_*"'] & \mathrm{K}_1(A), \arrow[l, "\mathrm{Id} - \alpha_*"']
\end{tikzcd} \tag{2.3.1}$$

where ι_* is the induced map at the level of K-theory of the inclusion

$$\iota \colon A \hookrightarrow A \rtimes_\alpha \mathbb{Z}.$$

This diagram was obtained from the short exact sequence

$$0 \longrightarrow \mathcal{K} \xrightarrow{\varphi} \mathrm{C}^*(S) \xrightarrow{\psi} \mathrm{C}(\mathbb{T}) \longrightarrow 0.$$

Let us introduce the terms of this exact sequence. The operator S is a non-unitary isometry, for instance the right shift on $\ell^2(\mathbb{N})$, and $\mathrm{C}^*(S)$ is the C*-algebra generated by it. This is a unital C*-algebra. We set $T = I - SS^*$. The *Toeplitz algebra* associated to A and α is the C*-algebra generated by $A \otimes I$ and $u \otimes S$ inside the C*-algebra $(A \rtimes_\alpha \mathbb{Z}) \otimes \mathrm{C}^*(S)$. It is usually denoted

by $\mathcal{T}_{A,\alpha}$ or simply $\mathcal{T}$.
The $*$-homomorphisms φ and ψ are given as follows:

$$\varphi(e_{ij}) = S^i T S^{*^j},$$

for e_{ij}'s the operators defined in Section 4.2.3, and

$$\psi(S) = \mathrm{id}_{\mathbb{T}},$$

where $\mathbb{T} \subset \mathbb{C}$. Now tensoring by $A \rtimes_\alpha \mathbb{Z}$ all terms of this short exact sequence (which are nuclear) we arrive at an exact sequence

$$0 \longrightarrow (\mathcal{K} \otimes A) \otimes (A \rtimes_\alpha \mathbb{Z}) \longrightarrow \mathrm{C}^*(S) \otimes (A \rtimes_\alpha \mathbb{Z}) \longrightarrow \mathrm{C}(\mathbb{T}) \otimes (A \rtimes_\alpha \mathbb{Z}) \longrightarrow 0.$$

Pimsner and Voiculescu defined

$$\tilde{\varphi} \colon \mathcal{K} \otimes A \to \mathcal{T}, \quad \tilde{\varphi}(e_{ij} \otimes a) = \varphi(e_{ij}) \otimes u^i a u^{*j},$$

and showed that

$$\mathrm{Im}(\tilde{\varphi}) = (\varphi(\mathcal{K}) \otimes (A \rtimes_\alpha \mathbb{Z})) \cap \mathcal{T}.$$

They deduced the exact sequence

$$0 \longrightarrow \mathcal{K} \otimes A \longrightarrow \mathcal{T} \longrightarrow A \rtimes_\alpha \mathbb{Z} \longrightarrow 0.$$

The last step was to apply the K-theory functor together with the identifications

$$\mathrm{K}_i(\mathcal{K} \otimes A) \cong \mathrm{K}_i(A) \cong \mathrm{K}_i(\mathcal{T}).$$

The invention of this tool was motivated by understanding all equivalence classes of projections in the so-called *irrational rotation algebra* A_θ, a non-commutative version of the C*-algebra of the torus $\mathbb{T}^2$. This was one of the first non trivial applications of K-theory to the structure of C*-algebras. A bit later Cuntz calculated the K-theory of the so-called *Cuntz algebras* $\mathcal{O}_n$, certain C*-algebras generated by n isometries.
Pimsner and Voiculescu [PV82] in 1982 generalised their work and obtained a generalised version of the digram 2.3.1. Here is the setting.
Let A be a C*-algebra. Let $\mathbb{F}_n$ be the free group on n generators $\{\gamma_1, \cdots, \gamma_n\}$. Suppose for $i = 1, \cdots, n$ each generator γ_i acts on A via α_i in such a way that they form an action $\alpha \colon \mathbb{F}_n \curvearrowright A$. Then we have the following cyclic diagram.

$$\begin{tikzcd}
\bigoplus_{i=1}^n \mathrm{K}_0(A) \arrow[r, "\sigma_*"] & \mathrm{K}_0(A) \arrow[r, "\iota_*"] & \mathrm{K}_0(A \rtimes_{\mathrm{r}} \mathbb{F}_n) \arrow[d, "\partial_0"] \\
\mathrm{K}_1(A \rtimes_{\mathrm{r}} \mathbb{F}_n) \arrow[u, "\partial_1"] & \mathrm{K}_1(A) \arrow[l, "\iota_*"'] & \bigoplus_{i=1}^n \mathrm{K}_1(A), \arrow[l, "\sigma_*"']
\end{tikzcd} \tag{2.3.2}$$

where

$$\sigma_* = \sum_{i=1}^{n} \mathrm{id} - \alpha_{i*},$$

and ι_* is the induced map at the level of K-theory of the inclusion

$$\iota\colon A \hookrightarrow A \rtimes_\alpha \mathbb{F}_n.$$

With this tool, Pimsner and Voiculescu computed the K-theory of $\mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_n)$ presented in [PV82]. More precisely, they showed that $\mathrm{K}_0(\mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_n)) = \mathbb{Z}$ and $\mathrm{K}_1(\mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_n)) = \mathbb{Z}^n$. The fact that $\mathrm{K}_1(\mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_n)) = \mathbb{Z}^n$ allowed to prove that the $\mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_n)$'s are pairwise non-isomorphic.
The reduced C*-algebra of $\mathbb{F}_2$, $\mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_2)$, was known to be simple due to work of Powers [Pow75]. Together with the fact that $\mathrm{K}_0(\mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_2)) = \mathbb{Z}$, this C*-algebra provided an example of a projectionless simple C*-algebra, hence a negative answer to a question posed by J. Dixmier on the non-existence of such C*-algebras. Moreover, later it turned out that using this data, the Baum-Connes conjecture was proved for the free groups.

When a discrete group G decomposes as a semidirect product $G = H \rtimes_\alpha \mathbb{F}_n$, its reduced C*-algebra can be viewed as the reduced crossed product

$$\mathrm{C}^*_{\mathrm{r}}(G) = \mathrm{C}^*_{\mathrm{r}}(H) \rtimes_{\hat{\alpha},\mathrm{r}} \mathbb{F}_n,$$

where $\hat{\alpha}$ is the induced action of $\mathbb{F}_n$ on $\mathrm{C}^*_{\mathrm{r}}(H)$ defined by $\gamma_i \cdot u_g = u_{\gamma_i \cdot g}$, for $i \in \{1, \ldots, n\}$ and $g \in G$. In this case, for instance, we may apply the Pimsner-Voiculescu 6-term exact sequence to study its K-theory.

It is interesting to note that for the group $\mathrm{SL}_3(\mathbb{Z})$, up to date, there is no decomposition that allows to understand the K-theory of its reduced group C*-algebra.

2.4 Kasparov's KK-theory

In this final section of this chapter, we want to point out one very important and powerful notion in operator algebra theory, namely KK-theory. We do not intend to state a formal definition since we will not really need it. Our emphasise will be on its functoriality. The reason to include this short section here is to provide a reference for later when we want to discuss different approaches to the (left-hand side of the) Baum-Connes assembly map (see Section 3.5.1 and Section 4.1). A comprehensive reference on KK-theory can be found in [Bla86]. This part is partially taken from [C87].

A contravariant counterpart of K-theory of topological spaces is the so-called K-*homology*. Roughly speaking, it deals with a kind of *Fredholm* operators on certain manifolds and it uses methods from differential geometry.

Similar to K-theory, K-homology produces exact sequence and satisfies Bott periodicity. However, it is a *contravariant* functor.

Kasparov's KK-theory introduced in [K80] somewhat merges these two (dual) theories. In fact, KK-theory is a bifunctor i.e. it depends on two C*-algebras whereas K-theory is a one variable functor. Kasparov's basic idea was to view K-theory not as a functor of one variable but as a functor $\mathrm{KK}(A, B)$ of two variables, where A and B are C*-algebras. The first variable of KK represents K-homology while the second one represents K-theory. The two variables do *not* appear exactly symmetric. The first one has more algebraic nature while the second one more topological. KK-theory associates to two C*-algebras A and B two abelian groups $\mathrm{KK}_0(A, B)$ and $\mathrm{KK}_1(A, B)$. The definitions of KK-groups are made in such a way that if we take for one of the variables the trivial C*-algebra $\mathbb{C}$ then we have

$$\begin{aligned} &\mathrm{KK}_i(\mathbb{C}, A) \simeq \mathrm{K}_i(A), \qquad i = 0, 1, \\ &\mathrm{KK}_i(A, \mathbb{C}) \simeq \mathrm{K}^i(A), \qquad i = 0, 1, \end{aligned} \tag{2.4.1}$$

where the latter one is the K-homology group.

It is worth mentioning that the basic properties characterising KK-theory are *split exactness* and *stability* in both variables.

An equivariant version of this theory was also introduced in the same article [K80]. This includes an action of a (locally compact) group G and is denoted by $\mathrm{KK}^G(A, B)$ where A and B are G-C*-algebras in the sense of Definition 2.1.14.

In analogy with 2.4.1, and under the condition that G is respectively a discrete group and a compact group and A and B are G-C*-algebras we have that

$$\begin{aligned} &\mathrm{KK}_i^G(A, \mathbb{C}) \simeq \mathrm{K}^i(A \rtimes G), \qquad i = 0, 1, \\ &\mathrm{KK}_i^G(\mathbb{C}, A) \simeq \mathrm{K}_i(A \rtimes G), \qquad i = 0, 1. \end{aligned} \tag{2.4.2}$$

The latter isomorphism in 2.4.2 is known as Green-Julg theorem.

Chapter 3

On the left-hand side of the assembly map

This chapter is concerned with the necessary background on the left-hand side of the Baum-Connes conjecture. In view of Davis-Lück's approach [DL98] toward the assembly map, which is the one that we are interested in, the equivariant K-homology can be computed by means of Bredon homology. We will provide all ingredients required to describe these notions. Moreover, we will discuss spectral sequences as main tools to compute the equivariant K-homology.

3.1 Group actions and group constructions

3.1.1 Semidirect product

Given two groups, there is a useful construction which produces new groups.

Definition 3.1.1. (*semidirect product*)
Let G be a group and let H and N be subgroups with N normal. The group G is said to be the semidirect product of H and N if $G = HN$ and $H \cap N = \{1\}$. Conversely, let H and N be two groups, let $\varphi \colon H \to \mathrm{Aut}(N)$ be a given group homomorphism. We construct a semidirect product as follows. We let G be the set of all pairs $(n, h) \in N \times H$. We define the product by

$$(n, h) \cdot (m, k) = (n\varphi_h(m), hk), \qquad n, m \in N, h, k \in H.$$

Therefore $G = N \rtimes_\varphi H$.

There is yet another construction related to semidirect products which is called *wreath product*, see Theorem 7.1.1. Our main results will be mainly concerned with this type of groups.

Definition 3.1.2. (*wreath product*)
Let N and H be two groups. The *wreath product* $N \wr H$ is the semidirect product

$$\bigoplus_{h\in H} N \rtimes H,$$

where H acts on the direct sum $\bigoplus_{h\in H} N$ by shifting the indices on the left that means

$$k(n_h)_{h\in H} = (n_{k^{-1}h})_{h\in H}, \qquad k \in H,\ (n_h)_{h\in H} \in \bigoplus_{h\in H} N.$$

See Section 7.1 for more discussion on wreath products.

3.1.2 Group actions

A *G-space* is a topological space X equipped with a (continuous) left G-action

$$G \times X \to X, \qquad (g, x) \mapsto gx,$$

satisfying $ex = x$ and $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$, where e is the neutral element of the group G.
Sometimes we might use the notation $G \curvearrowright X$ for denoting the action of G on X.

Definition 3.1.3. (*stabiliser or isotropy*)
Let $x \in X$. The subgroup $G_x := \{g \in G \colon gx = x\}$ is called the stabiliser of x or the isotropy group of x in G.

Let H be a subgroup of G. We denote by X^H the subspace of *H-fixed points* of X and by X/H the space of H-orbits $\{Hx \colon x \in X\}$ (with quotient topology). A G-space X is called *free* if for every $\{e\} \neq H < G$ we have that X^H is empty, equivalently $G_x = \{e\}$ for every $x \in X$.

Definition 3.1.4. (*G-equivariant homomorphism*)
Let $\varphi \colon H \to K$ be a homomorphism of groups on which the group G acts by automorphisms. The homomorphism φ is called *G-equivariant* if

$$g\varphi(h) = \varphi(gh), \qquad \forall g \in G, h \in H.$$

Let X be an H-space and $H \leq G$. The *induced G-space* is

$$\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} X = G \times_H X := (G \times X)/H \quad \text{via the H-action} \quad (h, (g, x)) \mapsto (gh^{-1}, hx).$$

The left G-action on $G \times X$ then passes to a G-action on $G \times_H X$ via $(g, \overline{(k, x)}) \mapsto \overline{(gk, x)}$, where the overline shows that these elements belong to the orbit space.

3.1.3 Proper actions

We start with the definition of proper actions.

Definition 3.1.5. (*proper action*)
A G-space is called *proper* if there are finite subgroups $H_i < G$ and open H_i-invariant subspaces $X_i \subset X$ such that the G-maps $G \times_{H_i} X_i \to X$ are G-homeomorphisms onto their images, and $X = \bigcup_i G \times_{H_i} X_i$.

Alternatively, we may define a proper action on a G-space as follows.

Definition 3.1.6. (*proper action*)
The action of G on a Hausdorff space X is *proper* if for every pair of compact subsets K, L of X, the set

$$\{g \in G \colon g \cdot K \cap L \neq \emptyset\} \text{ is finite.}$$

Definition 3.1.7. (*pushout*) [F03, p.42]
A commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ C \arrow[r, "g"] & P \end{tikzcd}$$

is a *pushout* diagram if for every pair of maps $\varphi \colon B \to X$ and $\psi \colon C \to X$ such that

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "\varphi"] \\ C \arrow[r, "\psi"] & X \end{tikzcd}$$

commutes, and there is a unique $\eta \colon P \to X$ such that

$$\eta \circ f = \varphi, \quad \eta \circ g = \psi.$$

Example 3.1.8. *The categorical pushout construction specialises to a number of well-known constructions. Notably, in the*

- *category of topological spaces: adjunction spaces via gluing,*
- *category of abelian groups: direct sum or certain quotients of direct sums,*
- *category of groups: free product with amalgamation,*

are pushouts.

Let us fix some notations. We denote the standard unit sphere and unit ball in $\mathbb{R}^n$ respectively by S^{n-1} and B^n. Here is a definition.

Definition 3.1.9. (G-CW-*complex*)
A G-CW-complex X is a G-space together with a G-*invariant* filtration

$$X^0 \subset X^1 \subset X^2 \subset \cdots \subset \cup_{n\geq 0} X^n = X,$$

such that X carries the colomit topology with respect to this filtration and X^n is obtained from X^{n-1} for each $n \geq 0$ by attaching *equivariant* n-dimensional cells, i.e. there exists a G-pushout

$$\begin{tikzcd}
\Delta_n \times S^{n-1} \arrow[r] \arrow[d] & X^{n-1} \arrow[d] \\
\Delta_n \times \mathrm{B}^n \arrow[r] & X^n,
\end{tikzcd}$$

with Δ_n a discrete G-space for every $n \in \mathbb{N}$.

Let us make a few remarks.

- If $G = \{e\}$, then the construction above defines a CW-complex.
- Given a G-CW-complex X, a G-*subcomplex* of X is a G-invariant subspace $Y \subset X$ containing some of the cells of X. Remark that every G-subcomplex is a G-CW-complex.
- The space X^n is called the n-*skeleton* of X.
- A G-CW-complex is called n-*dimensional* if $X^n = X$ and $X \neq X^{n-1}$.
- For a G-CW-complex X, X/G is an ordinary CW-complex and X is called G-*finite* or *cocompact* if X/G is a finite CW-complex.

Lemma 3.1.10. *[Lu89, Theorem 1.23]*
Let G be a discrete group. A G-CW-complex X is proper if and only if all its isotropy groups are finite.

In particular a free G-CW-complex is always proper.

3.2 Classifying space for proper actions

In this section, we describe the notion of the classifying space and in particular the one for *proper actions*. This topological space is constructed from a group and it appears on the left-hand side of the Baum-Connes assembly map. In the sequel we will present some examples and will state some results helping us to understand the behaviour of this object.

Definition 3.2.1. (*classifying space*) [W94, Definition 6.10.4]
A CW-complex with fundamental group G and contractible universal covering space is called *classifying space* for G, or a model for $\mathrm{B}G$, by abuse of notation, we call such a space $\mathrm{B}G$, and we write $\mathrm{E}G$ for its universal covering space.

Recall that two G-maps φ_0 and $\varphi_1\colon X \to Y$ are *G-homotopic* if they are homotopic through G-maps, that is, if there exists a homotopy $\{\varphi_t\}$ where $0 \leq t \leq 1$ with each φ_t a G-map.
In [BCH94] Baum, Connes and Higson defined the classifying space for proper actions of G, denoted $\underline{\mathrm{E}}G$, as follows.

Definition 3.2.2. (*universal example* or *classifying space for proper actions*)
A *universal example* for proper actions of G is a proper G-space with the following property: If X is any proper G-space then there exists a G-map $f\colon X \to \underline{\mathrm{E}}G$, and two G-maps from X to $\underline{\mathrm{E}}G$ are G-homotopic.

A model for this object can be constructed by functoriality using cosets G/H where $H < G$ is finite. For a recipe of this construction see page 6 of [MV03].

Theorem 3.2.3. *(characterisation for classifying space for proper actions) A proper G-CW-complex is a classifying space for proper actions of G, or a model for $\underline{\mathrm{E}}G$ if for any finite subgroup $H \leq G$ the stabiliser X^H is contractible and for other subgroups it is empty.*

Here is a list of examples.

3.2.1 Examples of classifying space

- If G is a finite group, a one-point space $\{pt\}$ is a model for $\underline{\mathrm{E}}G$.
- On the other extreme, if G is torsion free, we have that $\underline{\mathrm{E}}G = \mathrm{E}G$.
- If $H \leq G$, then $\underline{\mathrm{E}}G$ is also a model for $\underline{\mathrm{E}}H$.
- For a free group $\mathbb{F}_n$, its Cayley graph is a model for $\underline{\mathrm{E}}\mathbb{F}_n$ and in general, any tree T with G-action which makes T a G-CW-complex with finite vertex stabilisers is a model for $\underline{\mathrm{E}}G$. (cf. [S80])
- If G is a discrete group, then the set of finitely supported probability measures

$$\Big\{\varphi\colon G \to [0,1] \,|\, \varphi \text{ has finite support and } \sum_{g\in G} \varphi(g) = 1\Big\},$$

equipped with the action of G by translation and the topology determined by the metric

$$d(\varphi, \psi) = \sup_{g \in G} |\varphi(g) - \psi(g)|$$

is a model for $\underline{E}G$. [BCH94, cf. p. 248]

More examples are provided in Section 2 of [BCH94] and in Section 9.6 of [L15].

It is also possible to construct models for $\underline{E}G$ out of the ones from certain subgroups of the group G. For instance, when the group G decomposes as semidirect product $H \rtimes \mathbb{F}_n$, we can show that a suitable quotient of the product of models for $\underline{E}H$ and $\underline{E}\mathbb{F}_n$ could provide us with a model for $\underline{E}G$. For more detail see [Flu11], Section 8.6.1 and Section 9.2.1 of this note.

It is known that dimension is not a G-homotopy invariant. Since a good amount of our explicit description of the Baum-Connes assembly map is concerned with homological computations of $\underline{E}G$, knowing the minimum dimension of this CW-complex is naturally of importance simplifying the computations.
More generally, understanding *finiteness conditions* for classifying spaces proved useful and as a subject of research attracts both topologists and group-theorists. See for example [Lu05] or [Lu15].
We close this section with a result on the dimension of a models for extensions due to Lück. This will be relevant to our main results in the coming chapters.

Theorem 3.2.4. *[Lu00, Theorem 3.1]*
Let $1 \to N \to G \to Q \to 1$ be an exact sequence of groups. Suppose that there exists $d \in \mathbb{Z}_{\geq 0}$ which bounds the orders of finite subgroups of Q. Suppose that $\underline{E}N$ has a k-dimensional N-CW-model and $\underline{E}Q$ has a m-dimensional Q-CW-model. Then $\underline{E}G$ has a $(dk + m)$-dimensional G-CW-model.

3.3 Homology

This section is devoted to some core concepts and results in homology. As we will define a more generalised version of ordinary homology, this section serves as a reminder of the topic in the classical sense. This section is mainly taken from [W94] and [Bro82].

Let R be an associative unital ring. A graded R-module is a sequence $C := (C_n)_{n \in \mathbb{Z}}$ of R-modules. If C and D are graded R-modules, a graded homomorphism of *degree* d from C to D is a sequence $f := \{f_n \colon C_n \to$

$D_{n+d}\}_{n\in\mathbb{Z}}$ of R-module homomorphisms. A *chain complex* over R is a pair (C,δ) of a graded R-module C and a homomorphism $\delta\colon C\to C$ of degree -1 such that $\delta^2=0$, that means we have

$$\cdots \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} C_{n-2} \cdots \to C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0.$$

We may refer to δ_n as *differential* or *boundary maps.* Moreover, elements of $B_n(X)=\mathrm{Im}(\delta_{n+1})$ are called *boundaries* and the elements of $Z_n(C)=\ker(\delta_n)$ are called *cycles.* We define then the n-th homology group

$$\mathrm{H}_n(C)=\ker\delta_n/\mathrm{Im}\,\delta_{n-1}=Z_n(C)/B_n(C).$$

A natural homology theory for CW-complexes is *cellular homology.*

Definition 3.3.1. (*cellular homology*)
For a CW-complex X, with n-skeleton X^n, the cellular homology is defined as the homology groups of the *cellular complex*

$$\cdots \to \mathrm{C}^{\mathrm{cell}}_{n+1}(X) \xrightarrow{\delta_{n+1}} \mathrm{C}^{\mathrm{cell}}_{n}(X) \xrightarrow{\delta_n} \mathrm{C}^{\mathrm{cell}}_{n-1}(X) \xrightarrow{\delta_{n-1}} \cdots .$$

The group $\mathrm{C}^{\mathrm{cell}}_{n}(X)$ is a *free abelian* group generated by the n-cells of X, namely e_n^{α}'s for $\alpha\in I$. The boundary maps δ_n are defined by

$$\delta_n(e^n_\alpha)=\sum_\beta d_{\alpha\beta}e^{n-1}_\beta,$$

where $d_{\alpha\beta}$ is the degree of the map $S^{n-1}_\alpha\to X^{n-1}\to S^{n-1}_\beta$ that is the composition of the attaching map of e^n_α with the quotient map collapsing $X^{n-1}\backslash e^{n-1}_\beta$ to a point.

Definition 3.3.2. (**Ab**-*category*)
A category $\mathbf{A}$ is an $\mathbf{Ab}$-category if every hom-set $\mathrm{Hom}_{\mathbf{A}}(A,B)$ in $\mathbf{A}$ is given the structure of an abelian group such that composition distributes over addition. In particular given a diagram in $\mathbf{A}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g'"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] & C \arrow[r] & D, \end{tikzcd}$$

we have $h(g+g')f=hgf+hg'f$ in $\mathrm{Hom}(A,D)$.

An $\mathbf{Ab}$-category with one object is the same thing as a ring.

Definition 3.3.3. (*additive category*)
A category $\mathbf{A}$ is an $\mathbf{Ab}$-category with a zero object 0 and a product $A\times B$ for every pair A, B of objects of $\mathbf{A}$.

Definition 3.3.4. (*abelian category*)
An abelian category is an additive category **A** such that

1. every map in **A** has a kernel and cokernel.
2. every monic in **A** is the kernel of its cokernel.
3. every epi in **A** is the cokernel of its kernel.

A first example of an abelian category is the category **mod-R** of R-modules.

Definition 3.3.5. (*projective*)
An object P in an abelian category **A** is called *projective* if it satisfies the following lifting property:
Given an epimorphism $g\colon B \to C$ and a map $\gamma\colon P \to C$, there is a map $\beta\colon P \to B$ such that $\gamma = g \circ \beta$, i.e. the following diagram commutes

$$\begin{tikzcd}
 & P \arrow[dl, "\exists\beta"'] \arrow[d, "\gamma"] & \\
B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

Proposition 3.3.6. *An R-module is projective if and only if it is a direct summand of a free R-module.*

We say a category **A** has *enough projectives* if for every object A of **A** there is a surjection $P \to A$ with P projective.

Definition 3.3.7. (*resolution*)
Let M be an object of **A**. A left *resolution* of M is a complex P with $P_i = 0$ for $i < 0$, together with a map $\epsilon\colon P_0 \to M$ so that the augmented complex

$$\cdots \xrightarrow{d} P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\varepsilon} M \to 0$$

is exact. It is a *projective (resp. free)* resolution if each P_i is projective (resp. free).

Lemma 3.3.8. *[W94, Lemma 2.2.5]*
Every R-module M has a projective resolution. More generally, if an abelian category **A** *has enough projectives, then every object M in* **A** *has a projective resolution.*

We say that a functor F is right *exact* if it maps a right exact sequence $0 \to A \to B \to C \to 0$ to the exact sequence $F(A) \to F(B) \to F(C) \to 0$. We may define a left exact functor similarly.

Definition 3.3.9. (*left derived functor*)
Let $F\colon \mathbf{A} \to \mathbf{B}$ be a right exact functor between two abelian categories. If $\mathbf{A}$ has enough projectives, we can construct the *left derived functor* as follows. Take $A \in \mathbf{A}$ and a projective resolution $P \to A$. We now define

$$\mathrm{L}_n F(A) = \mathrm{H}_n(F(P)). \qquad n \geq 0.$$

As a result of right exactness we always have $\mathrm{L}_0 F(A) \cong F(A)$.

The rest of this section involves group actions. Let G be a group. Let $\mathbb{Z}G$ be the free $\mathbb{Z}$-module generated by the elements of G. Thus elements of $\mathbb{Z}G$ are uniquely expressed with finitely many summands in the form $\sum_{g \in G} a_g g$, where $a_g \in \mathbb{Z}$. The multiplication on G extends uniquely to a product which makes $\mathbb{Z}G$ a ring, called the *integral group ring.*
A (left) *G-module* is an abelian group A on which G acts (on the left). Let G-**mod** be the category of left-G-modules. We can identify this category with the category $\mathbb{Z}G$-**mod** and also with the functor category $\mathbf{Ab}^G$ of functors from the category "G"(one object, G being its endomorphisms) to the category **Ab** of abelian groups. We recall that a *functor category* between an arbitrary category **C** and an abelian category A is the abelian category $\mathbf{A}^{\mathbf{C}}$ whose objects are functors $F\colon \mathrm{C} \to A$ and whose morphisms are natural transformations.

Remark 3.3.10. The tensor product $M \otimes_R N$ is defined whenever M is a right R-module and N is a left R-module. In the special case where $R = \mathbb{Z}G$ we may take any right R-module as a left one and vice versa by defining $mg = g^{-1}m$ for $m \in M, g \in G$.

Note that $M \otimes_G N$ is obtained from $M \otimes N$ subjects to the relations

$$g^{-1}m \otimes n = m \otimes gn \qquad g \in G, m \in M, n \in N.$$

In particular, we have that $M \otimes_G N \cong N \otimes_G M$.

Definition 3.3.11. *(homology of a group (with coefficient))*
Let P be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$-modules. The *homology groups* of G are defined by

$$\mathrm{H}_*(G) = \mathrm{H}_*(P \otimes_G \mathbb{Z}).$$

Now take in addition a $\mathbb{Z}G$-module M. The *homology groups* of G with *coefficients* in M are defined by

$$\mathrm{H}_*(G; M) = \mathrm{H}_*(P \otimes_G M),$$

where $P \otimes_G M$ is the complex obtained from P by applying the functor $- \otimes_G M$.

A natural generalisation of $\mathrm{H}_*(G, \cdot)$ is called $\mathrm{Tor}_*^G(\cdot, \cdot)$. It is obtained by taking projective resolutions $F \to M$ and $P \to N$ of two arbitrary G-modules M and N and set

$$\mathrm{Tor}_*^G(M, N) = \mathrm{H}_*(F \otimes_G N) = \mathrm{H}_*(F \otimes_G P) = \mathrm{H}_*(M \otimes_G P).$$

Homology group $\mathrm{H}_*(G, \cdot)$ can be recovered by considering $\mathrm{Tor}_*^G(\mathbb{Z}, \cdot)$.

3.4 Representation theory

In this section we state relevant definitions and results on the representation theory of discrete groups with a focus on finite groups.

Definition 3.4.1. (*unitary representation*)
Let $\mathcal{U}(H)$ be the unitary group on a (complex) Hilbert space H. A *unitary representation* of a group G is a group homomorphism $\pi\colon G \to \mathcal{U}(H_\pi)$.
When there is no confusion we might ignore the subscript π and only write H.

A well-known example is the *(left) regular representation* which is defined by

$$\lambda\colon G \to \mathcal{U}(\ell^2(G)), \qquad \lambda(g)f(t) = f(g^{-1}t) \;\; g, t \in G, f \in \ell^2(G).$$

Definition 3.4.2. (*unitary equivalence*)
Two representations π and ρ of G are said to be *unitary equivalent* if there is a unitary operator $U\colon H_\pi \to H_\rho$ such that $\pi(g) = U^*\rho(g)U$ for $g \in G$. In this case we write $\pi \cong \rho$ and let $[\pi]$ denote the (unitary) equivalence class of the representation π.

Let (π, H_π) and (ρ, H_ρ) be two representations of a group G. We define the *(direct) sum*

$$\pi \oplus \rho\colon G \to \mathcal{U}(H_\pi \oplus H_\rho), \quad g \mapsto \pi(g) \oplus \rho(g).$$

A representation π is called *irreducible* if H_π has no non trivial closed invariant subspace.
For a group G we denote by $\hat{G}$ its *dual* which is the collection of equivalence classes of irreducible representations of G. If G is abelian, then this is its *Pontryagin dual* which is the set of all one dimensional homomorphisms $\varphi\colon G \to \mathbb{T}$ i.e. *characters*. See Definition 2.1.7.

Now let G be a finite group, and let V be a finite dimensional complex vector space. Let $\dim(V) = n$. Consider the group GL(V) of automorphisms of V, equivalently (after choice of a basis) $\mathrm{GL}_n(\mathbb{C})$ the group of complex invertible square matrices of size n. A (complex) *representation of dimension* n is a group homomorphism

$$\pi\colon G \to \mathrm{GL(V)} \cong \mathrm{GL}_n(\mathbb{C}).$$

This is equivalent to giving a linear G-action on V or alternatively a $\mathbb{C}G$-module structure on V. We remark that in fact any complex representation of G, say (π, V), is *unitarisable* i.e. the space V can be equipped with a scalar product such that π is unitary; If $\langle\cdot,\cdot\rangle$ is any scalar product, then

$$\langle\xi,\eta\rangle_G = \frac{1}{|G|}\sum_{g\in G}\langle\pi(g)\xi,\pi(g)\eta\rangle$$

is a G-invariant scalar product.
We may associate to a representation π a *character*

$$\pi\colon G \to \mathbb{C}, \qquad \chi_\pi(g) = \mathrm{Tr}(\pi(g)),$$

where Tr is the trace of $\pi(g)$ as a matrix in $\mathrm{GL}_n(\mathbb{C})$.
Given a subgroup $H \leq G$, we may speak of *induced* representation $\mathrm{Ind}_H^G(\rho)$ and *restricted* representation $\mathrm{Res}_H^G(\rho)$. Let us briefly describe them.
Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G. The *restricted* representation, denoted $\mathrm{Res}_H^G(\rho)$, or ρ_H, is defined by letting $\rho_H(h) = \rho(h)$ for $h \in H$.
Let $\pi\colon H \to \mathrm{GL}(W)$ be a representation of a group H. We may view W as a $\mathbb{C}H$-module. The *induced representation* is

$$\mathrm{Ind}_H^G(\pi) = \mathbb{C}G \otimes_{\mathbb{C}H} W.$$

Proposition 3.4.3. *(Frobenius reciprocity)*
Let G be a finite group, H a subgroup of G, ρ a representation of G and π a representation of H. Then we have

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G(\pi),\rho) \cong \mathrm{Hom}_H(\pi,\mathrm{Res}_H^G(\rho))$$

We close this section by introducing the notion of the representation ring of a finite group.

Definition 3.4.4. (*representation ring*)
The complex *representation ring* of a finite group F, denoted $\mathrm{R}_{\mathbb{C}}(F)$, is the free abelian group generated by the unitary equivalence classes of irreducible representations of F and the ring structure is induced by tensor product of representations.

3.5 Bredon homology

In this section we introduce *Bredon homology.* Bredon homology was introduced by Bredon in [Bre67] for G-CW-complexes in the case of finite groups. Later, this was defined for all groups. This homology theory is applied to describe the left-hand side of the Baum-Connes conjecture. This section is mainly based on part I of [MV03].

Let us first fix some notations. Let $\mathfrak{F}$ be a family of subgroups of the group G. We denote by $\mathbf{O}_{\mathfrak{F}}$ the *orbit category* whose objects are left coset spaces G/H with $H \in \mathfrak{F}$ and morphism sets, i.e. $\mathrm{Mor}(G/H, G/K)$ are G-maps $G/H \to G/K$.

$$\text{orbit category}\, \mathbf{O}_{\mathfrak{F}}\colon \begin{cases} \text{objects:} \quad \text{left cosets} \quad G/H \quad \text{where} \quad H \in \mathfrak{F}, \\ \text{morphisms:} \quad G\text{-maps} \quad G/H \to G/K. \end{cases}$$

Now we define two new categories using these the orbit category. As before **Ab** denotes the category of abelian groups.

$$G\text{-}\mathbf{mod}_{\mathfrak{F}} : \begin{cases} \text{objects:} \qquad N\colon \mathbf{O}_{\mathfrak{F}} \to \mathbf{Ab} \quad \text{where}\, N \,\text{is a}\ \textit{covariant}\ \text{functor,} \\ \text{morphisms:} \quad \text{natural transformations of functors.} \end{cases}$$

$$\mathbf{mod}_{\mathfrak{F}}\text{-}G : \begin{cases} \text{objects:} \qquad N\colon \mathbf{O}_{\mathfrak{F}} \to \mathbf{Ab} \quad \text{where}\, N \,\text{is a}\ \textit{contravariant}\ \text{functor,} \\ \text{morphisms:} \quad \text{natural transformations of functors.} \end{cases}$$

We refer to the objects of the categories G-$\mathbf{mod}_{\mathfrak{F}}$ and $\mathbf{mod}_{\mathfrak{F}}$-$G$ as $\mathbf{O}_{\mathfrak{F}}$-modules. An $\mathbf{O}_{\mathfrak{F}}$-module P is *projective* if the functor $\mathrm{Mor}(P, \cdot)\colon \mathbf{mod}_{\mathfrak{F}}\text{\_}G \to$ **Ab** is *exact.* To be exact in $\mathbf{mod}_{\mathfrak{F}}$-$G$ means that for every $K \in \mathfrak{F}$ the sequence $M(G/K) \to N(G/K) \to L(G/K)$ of abelian groups is exact.
Note that the category of Bredon modules is a *functor category*, it follows that in order to define different concepts, we only need to define them objectwise, similar to exactness.

In analogy with the topological approach to group homology via free G-CW complexes, we construct a projective resolution of the trivial G-module $\mathbb{Z}$ in the category $\mathbf{O}_{\mathfrak{F}}$ from the proper G-CW-complex X. Note that since $\mathbf{O}_{\mathfrak{F}}$ is an abelian category, projective resolutions can be defined.
We describe then how a G-CW complex gives rise to projectives. Let X be a G-CW complex, and let $\mathfrak{F}$ be a family of subgroups of G that contains the isotropy groups of the action. Let Δ_i, for $i \geq 0$, be the discrete G-set of i-cells of X appearing in the definition of the i skeleton of X in Definition 3.1.9. For every $H \in \mathfrak{F}$, we consider

$$\mathrm{C}_i(X^H) = \mathbb{Z}[\Delta_i^H] \quad \text{free abelian group generated by the fixed-point set } \Delta_i^H.$$

Therefore $\mathrm{C}_i(X^H)$ is the cellular chain complex of X^H at degree i where the boundary maps

$$\delta_i^H \colon \mathbb{Z}[\Delta_i^H] \to \mathbb{Z}[\Delta_{i-1}^H],$$

are given by the boundary maps of the cellular complex described in Definition 3.3.1. Using this we define a contravariant $\mathbf{O}_{\mathfrak{F}}G$-module, that is

$$\underline{\mathrm{C}_i(X)} \colon \mathbf{O}_{\mathfrak{F}}G \to \mathbf{Ab}, \qquad \underline{\mathrm{C}_i(X)}(G/H) = C_i(X^H),$$

and introduce $\underline{\mathbb{Z}}$ to be the constant Bredon module with value $\mathbb{Z}$

$$\underline{\mathbb{Z}} \colon \mathbf{O}_{\mathfrak{F}}G \to \mathbf{Ab}, \qquad \underline{\mathbb{Z}}(G/H) = \mathbb{Z}.$$

Now we can define a chain complex of contravariant (or right) Bredon modules as follows.

$$\cdots \longrightarrow \underline{\mathrm{C}_i(X)} \xrightarrow{\underline{\delta_i}} \underline{\mathrm{C}_{i-1}(X)} \longrightarrow \cdots \longrightarrow \underline{\mathrm{C}_1(X)} \xrightarrow{\underline{\delta_1}} \underline{\mathrm{C}_0(X)} \xrightarrow{\underline{\varepsilon}} \underline{\mathbb{Z}} \longrightarrow 0,$$

where the boundary maps

$$\underline{\delta_i} \colon \underline{\mathrm{C}_i(X)} \to \underline{\mathrm{C}_{i-1}(X)}, \qquad \underline{\delta}(G/H) = \delta_i^H \colon \mathrm{C}_i(X^H) \to \mathrm{C}_{i-1}(X^H),$$

where δ^H is the boundary map of the cellular chain complex associated to X^H. Moreover, the augmentation

$$\underline{\varepsilon} \colon \underline{\mathrm{C}_0(X)} \to \underline{\mathbb{Z}}, \qquad \underline{\varepsilon}(G/H) = \varepsilon \colon \mathrm{C}_0(X^H) \to \mathbb{Z},$$

defined by mapping basis elements of $\mathbb{Z}[\Delta_0^H]$ to $1 \in \mathbb{Z}$.
Clearly if $i > \dim(X)$ then $\underline{\mathrm{C}_i(X)}$ is trivial.

Definition 3.5.1. (*categorical tensor product)*
Let M and N be $\mathbf{O}_{\mathfrak{F}}G$-modules. The categorical tensor product $M \otimes_{\mathfrak{F}} N$ is the abelian group

$$\bigoplus_{\mathfrak{F}} M(G/K) \otimes N(G/K)/\sim.$$

where the equivalence relation is given by $M(\phi)(m) \otimes n \sim m \otimes N(\phi)(n)$ for morphism $\phi \colon G/K \to G/L$, and elements $m \in M(G/K)$ and $n \in N(G/L)$.

Using this we define a functor $\cdot \otimes_{\mathfrak{F}} N \colon \mathbf{mod}_{\mathfrak{F}}\text{-}G \to \mathbf{Ab}$.
Now we have all ingredients to define Bredon homology groups with coefficients, for G-CW-complexes.

Definition 3.5.2. (*Bredon homology of a G-*CW*-complex*)
Let X be a G-CW-complex and $N \in G\text{-}\mathbf{mod}_{\mathfrak{F}}$ with $\mathfrak{F}$ containing all isotropy groups of the G-action in X. The Bredon homology groups of X with coefficients in N are

$$\mathrm{H}_i^{\mathfrak{F}}(X; N) = \mathrm{H}_i(\underline{\mathrm{C}_*(X)} \otimes_{\mathfrak{F}} N) \qquad i \geq 0.$$

Some remarks about this homology theory:

1. homology groups just defined are *independent* of $\mathfrak{F}$ as long as it contains all isotropy groups of its action.
2. We can define the Bredon homology groups of a group G with coefficients in $N \in G\text{-}\mathbf{mod}_{\mathfrak{F}}$ via Tor as the left derived functor of the categorical tensor product functor $\cdot \otimes_{\mathfrak{F}} N$, (cf. Chapter 3), that is

$$\mathrm{H}_i^{\mathfrak{F}}(G; N) = \mathrm{Tor}_i^{\mathfrak{F}}(\underline{\mathbb{Z}}; N) \qquad i \geq 0.$$

3. In the case of $\underline{\mathrm{E}}G$ the two functors $\mathrm{H}_*^{\mathfrak{F}}(G, \cdot)$ and $\mathrm{H}_*^{\mathfrak{F}}(\underline{\mathrm{E}}G, \cdot)$ are equivalent, since

$$\underline{\mathrm{C}_*(\underline{\mathrm{E}}G)} \to \underline{\mathbb{Z}}$$

is a projective resolution.
4. The functor $\mathrm{H}_*^{\mathfrak{F}}(X, \cdot)$ depends *only* on the G-homotopy type of X.

From now on we consider only the family $\mathfrak{F}$ of *finite* subgroups of G and we introduce a specific $\mathbf{O}_{\mathfrak{F}}G$-module which is the suitable coefficient of Bredon homology in the context of the Baum-Connes conjecture. The suitable coefficient module is the functor with values in the *representation ring* of finite subgroups of G,

$$\mathrm{R}_{\mathbb{C}} \colon \mathbf{O}_{\mathfrak{F}}G \to \mathbf{Ab}, \quad G/H \mapsto \mathrm{R}_{\mathbb{C}}(H),$$

and for a G-map $G/H \to G/K$ between objects of $\mathbf{O}_{\mathfrak{F}}G$, which always implies that $gHg^{-1} \subset K$ for some $g \in G$, we associate (uniquely up to conjugation by elements in K) the group homomorphism

$$\mathrm{Ind}_{gHg^{-1}}^{K} \mathrm{R}_{\mathbb{C}}(H) = \mathrm{R}_{\mathbb{C}}(gHg^{-1}) \to \mathrm{R}_{\mathbb{C}}(K),$$

where Ind denotes the *induced* representation defined in Section 3.4.

Remark 3.5.3. In the context of the Baum-Connes conjecture we are particularly interested in the groups $\mathrm{H}_i^{\mathfrak{F}}(\underline{\mathrm{E}}G; \mathrm{R}_{\mathbb{C}})$ for $i \geq 0$.

Note that the zero degree Bredon homology group is

$$\mathrm{H}_0^{\mathfrak{F}}(\underline{\mathrm{E}}G;\mathrm{R}_{\mathbb{C}}) = \underline{\mathbb{Z}} \otimes_{\mathfrak{F}} \mathrm{R}_{\mathbb{C}} = \mathrm{colim}_{G/H \in \mathbf{O}_{\mathfrak{F}}} \mathrm{R}_{\mathbb{C}}(H).$$

If a group G admits a one-dimensional model for $\underline{\mathrm{E}}G$, then there is a concrete tool to compute the Bredon homology groups.

Theorem 3.5.4. *([MV03], Theorem I.3.17)*
Suppose $\underline{\mathrm{E}}G$ has a model of dimension 1 hence a tree. Then $\mathrm{H}_i^{\mathfrak{F}}(G,\mathrm{R}_{\mathbb{C}}) = 0$ for $i > 1$, and there is an exact sequence

$$0 \longrightarrow \mathrm{H}_1^{\mathfrak{F}}(G;\mathrm{R}_{\mathbb{C}}) \longrightarrow \bigoplus_{[e]} \mathrm{R}_{\mathbb{C}}(G_e) \longrightarrow \bigoplus_{[v]} \mathrm{R}_{\mathbb{C}}(G_v) \longrightarrow \mathrm{H}_0^{\mathfrak{F}}(G;\mathrm{R}_{\mathbb{C}}) \longrightarrow 0,$$

where the direct sums are respectively taken over the orbits of edges and vertices of the tree $\underline{\mathrm{E}}G$, and G_e and G_v denote the stabilisers of the edges and the vertices, respectively.

After discussing the Bredon homology groups, our next step is to understand its relation with the left-hand side of the Baum-Connes conjecture.

3.5.1 Equivariant K-homology à la Davis-Lück

The left-hand side of the Baum-Connes assembly map can be described via two different approaches. The original approach is due to Baum, Connes and Higson [BCH94] and is set in the context of Kasparov's KK-theory. The other one is due to Davis and Lück [DL98] and is defined functorially/topologically using spaces and spectra over the orbit category of a discrete group. The latter approach is going to be our reference. This approach comes equipped with topological tools which are suitable for the purpose of explicit calculations.
We provide some basic material on the theory of spectra.

Definition 3.5.5. (*wedge sum*)
Let (X, x_0) and (Y, y_0) be two pointed spaces. The *wedge sum* $X \vee Y$ is the quotient of the disjoint union $X \coprod Y$ obtained by identifying x_0 and y_0 to a single point.

Definition 3.5.6. (*smash product*)
The *smash product* $X \wedge Y$ of pointed spaces X and Y is the quotient $X \times Y/X \vee Y$.

Definition 3.5.7. (*spectrum*)
A *spectrum* **S** is a collection of pointed CW-complexes $\{\mathrm{S}_i\}_{i\in\mathbb{N}}$ together with pointed maps

$$\sigma_i\colon \mathrm{S}_i \wedge \mathrm{S}^1 \to \mathrm{S}_{i+1},$$

called *structure* maps. A morphism between spectra **S** and **T** is a sequence of pointed maps $f_i\colon \mathrm{S}_i \to \mathrm{T}_i$ which are compatible with the structure maps σ_i, σ_i' that means

$$f_{i+1}\circ\sigma_i = \sigma_{i+1}'\circ(f_i\wedge \mathrm{id}_{\mathrm{S}^1}), \qquad i\in\mathbb{N}.$$

Let X be a pointed topological space. We denote by ΩX the *loop space* of X, which is the space of loops, i.e. continuous maps $S^1 \to X$, in X with the compact-open topology.

Definition 3.5.8. (Ω-*spectrum)*
Given a structure map $S_i \to \Omega S_{i+1}$. *In the homotopoy category:*

$$\sigma_i \in \mathrm{Hom}(S_i\wedge S^1, S_{i+1}) \overset{\textit{adjoint formula}}{\cong} \mathrm{Hom}(S_i, \Omega S_{i+1}).$$

Therefore we obtain a unique up to homotopy adjoint of σ_i *in the space* $\mathrm{Hom}(S_i, \Omega S_{i+1})$. *Now an* Ω*-spectrum is a spectrum such that the adjoint of the structure* σ_i *is a weak homotopy equivalence.*

Denote the category of CW-complexes by **CW** and the category of spectra by **Sp**. We define two functors in the sequel with which we define G-homology.

An $\mathbf{O}_{\mathfrak{F}}$-space is a *contravariant* functor $\mathbf{O}_{\mathfrak{F}}(G) \to \mathbf{CW}$. A typical example of such a functor is given by a G-CW-complex X and defining $X(G/H) := X^H$. An $\mathbf{O}_{\mathfrak{F}}$-spectrum **S** is a *covariant* functor

$$\mathbf{S}\colon \mathbf{O}_{\mathfrak{F}}(G) \to \mathbf{Sp}.$$

Spectra are convenient objects to describe homology theories in the category of CW-complexes. A spectrum **S** can define a G-homology theory by means of *homotopy* groups. We explain it, in more detail, in one particular case suitable for the Baum-Connes conjecture.

K-theory spectrum

The Baum-Connes conjecture relates the K-theory of $\mathrm{C}^*_{\mathrm{r}}(G)$ to the representation rings of finite subgroups of G via equivariant K-homology of $\underline{\mathrm{E}}G$. This equivariant K-homology is defined by means of the *non-connective* K-*theory*

spectrum $\mathbf{K}^G$ constructed in [DL98]. Let us recall its construction and its properties.
Davis and Lück considered the spectrum over the orbit category

$$\mathbf{K}^G\colon \mathbf{O}_{\mathfrak{F}}(G) \to \Omega\text{-}\mathbf{Sp},$$

which is related to the classical functor

$$\mathbf{K}\colon \mathrm{C}^*\text{-}\mathbf{Alg} \to \Omega\text{-}\mathbf{Sp},$$

in such a way that these two functors (spectra) satisfy the property that

$$\mathbf{K}^G(G/H) \quad \text{weak homotopy type of} \quad \mathbf{K}(\mathrm{C}^*_{\mathrm{r}}(H)),$$

meaning that at the level of i-th homotopy groups, we have

$$\pi_i(\mathbf{K}^G(G/H)) \simeq \pi_i(\mathbf{K}(\mathrm{C}^*_{\mathrm{r}}(H))), \quad i \in \mathbb{Z}.$$

In view of Lemma 2.4 of [DL98] we have

$$\pi_n(\mathbf{K}^G(G/H)) \simeq \mathrm{K}_n(\mathrm{C}^*_{\mathrm{r}}(H)),$$

where on the right-hand side we have the C*-algebraic K-theory, of Section 2.2, which in particular enjoys Bott periodicity.

Definition 3.5.9. (*equivariant* K-*homology*)
Let X be a G-CW-complex. Set $X_+ = X \coprod \{pt\}$. We define the *equivariant* K-*homology of* X denoted $\mathrm{K}^G_*(X)$ as

$$\mathrm{K}^G_i(X) = \pi_i(X_+ \otimes_G \mathbf{K}^G), \quad i \in \mathbb{Z}.$$

Therefore K^G_* is the generalised G-equivariant homology theory associated to the $\mathbf{O}$-spectrum $\mathbf{K}^G$. It follows that

$$\mathrm{K}^G_i(\{pt\}) \cong \mathrm{K}_i(\mathrm{C}^*_{\mathrm{r}}(G)).$$

This *equivariant* K-*homology* satisfies

- if $G = \operatorname{colim} G_\alpha$, then $\mathrm{K}^G_i(\underline{\mathrm{E}}G) \cong \operatorname{colim} \mathrm{K}^{G_\alpha}(\underline{\mathrm{E}}G_\alpha)$.

- if X is free G-CW- complex, then $\mathrm{K}^G_i(X) \cong \mathrm{K}_i(X/G)$.

In particular, when G is torsion free we have that

$$\mathrm{K}^G_i(\underline{\mathrm{E}}G) = \mathrm{K}^G_i(\mathrm{E}G) = \mathrm{K}_i(\mathrm{B}G).$$

3.6 Spectral sequence in homology

In the previous section we discussed equivariant K-homology, our next aim is to compute it. In general *spectral sequences* are very useful tools for computing homology. In this section we describe these tools. The material of this section is mainly based on [W94].

Definition 3.6.1. (*double complex or bicomplex*)
A double complex or bicomplex in **A** is a family $\{\mathrm{C}_{p,q}\}$ of objects of **A**, together with maps

$$d^h \colon \mathrm{C}_{p,q} \to \mathrm{C}_{p-1,q} \quad \text{and} \quad d^v \colon \mathrm{C}_{p,q} \to \mathrm{C}_{p,q-1},$$

where h and v stand for horizontal and vertical differentials, and such that $d^h \circ d^h = d^v \circ d^v = d^v d^h + d^h d^v = 0$.

We may imagine this as a lattice in which the maps d^h go horizontally the maps d^v go vertically, and each square anticommutes. Each row C_{*q} and each column C_{p*} is a chain complex.
We say that a double complex C is *bounded* if C has only finitely many nonzero terms along each diagonal line $p + q = n$. For example if C is concentrated in the first quadrant of the plane.

Definition 3.6.2. (*spectral sequence*)
A homological spectral sequence in an abelian category **A** consists of

1. A family $\{\mathrm{E}^r_{p,q}\}$ of objects in **A** for every $r \geq 0$ and $p, q \in \mathbb{Z}$,
2. Maps $d^r_{p,q} \colon \mathrm{E}^r_{p,q} \to \mathrm{E}^r_{p-r,q+r-1}$ that are differentials in a sense that $d^r \circ d^r = 0$, so that the lines of slope $-(r+1)/r$ in the lattice $\mathrm{E}^r_{*,*}$ form chain complexes.
3. Isomorphisms between $\mathrm{E}^{r+1}_{p,q}$ and the homology of $\mathrm{E}^r_{*,*}$ at the spot $\mathrm{E}^r_{p,q}$

$$\mathrm{E}^{r+1}_{p,q} \cong \ker(d^r_{p,q})/\mathrm{Im}(d^r_{p+r,q-r+1}).$$

Let us make a few remarks.

- $\mathrm{E}^{r+1}_{p,q}$ is a subquotient of $\mathrm{E}^r_{p,q}$.
- Related to the second part of the definition, we say the *bidegree* of the differentials d^r on the r-th page is $(-r, r-1)$.

Definition 3.6.3. *(stabilise (for object))*
We say that the objects on the spot (p, q) *stabilises* if for r large enough we have

$$\mathrm{E}^r_{p,q} \cong \mathrm{E}^{r+1}_{p,q} \cong \mathrm{E}^{r+2}_{p,q} \cong \cdots .$$

In this situation we write $\mathrm{E}^\infty_{p,q}$ for this object.

Another useful notion that we will need later is the following definition.

Definition 3.6.4. (*collapse or degenerate (for spectral sequence)*)
A spectral sequence *collapses* at E^s if there is at most one nonzero row or column in E^s, and all differentials are zero (which is automatic if $s \geq 2$). Moreover, a spectral sequence *degenerates* at page E^r if for $s \geq r$ all differentials d^s are zero.

Clearly if a spectral sequence collapses at some page, then it degenerates and also the objects of that page are stabilised.

Definition 3.6.5. (*bounded-convergence*)
A (first and forth quadrant) homological spectral sequence is said to be *bounded* if for any n there are finitely many nonzero terms of total degree n in $\mathrm{E}^r_{*,*}$.
A bounded spectral sequence *converges* to H_*, denoted

$$\mathrm{E}^r_{p,q} \Rightarrow \mathrm{H}_{p+q},$$

if we are given a family of objects H_n, each having a finite filtration for each $n \geq 0$

$$0 = F_{-1}\mathrm{H}_n \subset F_0\mathrm{H}_n \subset F_1\mathrm{H}_n \subset \ldots \subset F_{n-1}\mathrm{H}_n \subset F_n\mathrm{H}_n = \mathrm{H}_n,$$

and we are given isomorphisms

$$\mathrm{E}^\infty_{p,q} \cong F_p\mathrm{H}_{p+q}/F_{p-1}\mathrm{H}_{p+q}.$$

In the next two sections we discuss two well-known spectral sequences which are useful for computing Bredon homology.

3.6.1 Equivariant Atiyah-Hirzebruch spectral sequence

The *Atiyah-Hirzebruch* spectral sequence provides a tool to compute a *generalised* homology theory h_*, satisfying certain necessary axioms, from the *cellular* homology with coefficients in $\mathrm{h}_*(\{pt\})$. Given such h_* and a CW-complex X, there is a spectral sequence $\{\mathrm{E}^r_{p,q}, d^r\}$ such that

$$\mathrm{E}^2_{p,q} = \mathrm{H}_p(X; \mathrm{h}_q(\{pt\})) \Rightarrow \mathrm{h}_{p+q}(X).$$

An equivariant version of the Atiyah-Hirzebruch spectral sequence for equivariant homology theories, involving Bredon homology, has been defined as well.

In Section 3.5.1 we defined $\mathrm{K}_*^G(X)$ for a G-CW-complex X. We recall that a fundamental property of K^G-homology is that for any subgroup H of G, we have

$$\mathrm{K}_i^G(G/F) \cong \mathrm{K}_i(\mathrm{C}_\mathrm{r}^*(F)).$$

Let F be a finite group and let $\mathrm{M}_{\dim \pi}(\mathbb{C})$ denote a complex matrix of size $\dim \pi$. Bredon modules $\mathrm{K}_i^G(\cdot)$ and $\mathrm{R}_\mathbb{C}$ satisfy the identification

$$\begin{aligned}\mathrm{K}_i^G(G/F) &= \mathrm{K}_i(\mathbb{C}F)\\ &= \{[p] : p \,\text{is a minimal projection in } \bigoplus_{\pi\in\hat{F}} \mathrm{M}_{\dim\pi}(\mathbb{C})\}\\ &= \mathrm{R}_\mathbb{C}(F) = \mathrm{R}_\mathbb{C}(G/F),\end{aligned}$$

where the last equality is by the definition of such a module over the orbit category of G.

Therefore for finite subgroup F of G, we obtain

$$\mathrm{K}_i^G(G/F) = \mathrm{K}_i(\mathbb{C}F) = \begin{cases}\mathrm{R}_\mathbb{C}(F) & i = 0,\\ 0 & \text{otherwise.}\end{cases} \tag{3.6.1}$$

Since equivariant K-homology is a generalised (equivariant) homology theory, its value on an arbitrary G-CW-complex can be computed via Atiyah-Hirzebruch spectral sequence in Bredon homology whose coefficient is the $\mathbf{O}_\mathfrak{F}$-module given by $\mathrm{R}_\mathbb{C}(F)$ for $F \in \mathfrak{F}$. Let us explain it briefly here.

Consider again $\mathrm{K}_i^G(\cdot)$ as an object in G-$\mathbf{mod}_\mathfrak{F}$, for a proper G-CW-complex X we have the associated Bredon homology group $\mathrm{H}_p^\mathfrak{F}(X; \mathrm{K}_q^G(\cdot))$ obtained as homology groups of the chain complex $\underline{C_*(X)} \otimes_\mathfrak{F} \mathrm{K}_*^G(\cdot)$. Using the skeleton filtration $\{X^p\}_{p\in\mathbb{N}}$ of X we obtain a spectral sequence whose second page is

$$\mathrm{E}_{p,q}^2 = \mathrm{H}_p^\mathfrak{F}(X; \mathrm{K}_q^G(\cdot)) \Rightarrow \mathrm{K}_{p+q}^G(X). \tag{3.6.2}$$

This spectral sequence lives in *first* and *forth* quadrant i.e. it is defined for $p, q \in \mathbb{Z}$ with $p \geq 0$.

We specialise to the case of proper G-CW-complex with dimension 2.

Lemma 3.6.6. *If* $\dim \underline{\mathrm{E}}G \leq 2$*, the Atiyah-Hirzebruch spectral sequence for* $\mathrm{K}_{p+q}^G(\underline{\mathrm{E}}G)$ *degenerates on the second page.*

Proof. In view of 3.6.2 the E^2-page of this spectral sequence is defined by $\mathrm{E}_{p,q}^2 = \mathrm{H}_p^\mathfrak{F}(G; \mathrm{K}_q^G(\cdot))$. We want to show that for $r \geq 2$ all differentials are zero. Let us first consider the case where $r \geq 3$. In this case, since the dimension of $\underline{\mathrm{E}}G$ is at most 2 and the bidegree is $(-r, r-1)$, all differentials are zero since either source or target lie outside of the stripe $0 \leq p \leq 2$. This means that the spectral sequence degenerates the latest on the third page. We investigate the differentials on the second page. Note that due to 3.6.1 for an *odd* q and a finite subgroup F of G we have

$$\mathrm{K}_q^G(G/F) = \mathrm{K}_q(\mathbb{C}F) \overset{\text{Bott priodicity}}{=} \mathrm{K}_1(\mathbb{C}F) = 0.$$

For arbitrary (p, q) consider the associated differential

$$d^2_{p,q} \colon \mathrm{E}^2_{p,q} \to \mathrm{E}^2_{p-2,q+1}.$$

Since the bidegree is $(-2, 1)$, either q or $q-1$ is odd. This implies that either source or target is zero, and hence $d^2_{p,q}$ is zero. Having all $d^2_{*,*}$ zero means that the spectral sequence degenerates at $r = 2$. $\square$

Theorem 3.6.7. *([MV03], Theorem I.5.27)*
If G is a group with dim$(\underline{\mathrm{E}}G) \leq 2$, then there is a short exact sequence

$$0 \longrightarrow \mathrm{H}_0^{\mathfrak{F}}(G; \mathrm{R}_{\mathbb{C}}) \longrightarrow \mathrm{K}_0^G(\underline{E}G) \longrightarrow \mathrm{H}_2^{\mathfrak{F}}(G; R_{\mathbb{C}}) \longrightarrow 0,$$

and an isomorphism $\mathrm{H}_1^{\mathfrak{F}}(G; \mathrm{R}_{\mathbb{C}}) \simeq \mathrm{K}_1^G(\underline{E}G)$.

The proof is based on the one of Theorem I.5.27 in [MV03].

Proof. We want to understand the filtrations for K_i^G with $i = 0, 1$ which is graded by p, q where $0 \leq p \leq 2$ and $p + q = i$. Let $i = 0$. Consider the following filtration

$$F_{0,0} \subset F_{1,-1} \subset F_{2,-2} = \mathrm{K}_0^G(\underline{\mathrm{E}}G).$$

Due to Lemma 3.6.6 we have $\mathrm{E}^2_{1,-1} = \mathrm{E}^\infty_{1,-1}$, and moreover

$$\mathrm{E}^2_{1,-1} = \mathrm{H}_1^{\mathfrak{F}}(\underline{\mathrm{E}}G; \mathrm{K}_{-1}^G(\cdot)) = \mathrm{H}_1^{\mathfrak{F}}(\underline{\mathrm{E}}G; \mathrm{K}_1^G(\cdot)) = 0,$$

where the second equality holds by Bott periodicity for $\mathrm{K}_*^G(\cdot)$. This yields that

$$0 = \mathrm{E}^2_{1,-1} = \mathrm{E}^\infty_{1,-1} = F_{1,-1}/F_{0,0},$$

from which we get $F_{0,0} = F_{1,-1}$. Therefore the filtration reduces to

$$F_{0,0} = F_{1,-1} \subset F_{2,-2} = \mathrm{K}_0^G(\underline{\mathrm{E}}G).$$

Once again by definition we have

$$\mathrm{E}^{\infty}_{2,-2} \cong F_{2-2}/F_{1,-1} = F_{2,-2}/F_{0,0}.$$

In view of Lemma 3.6.6 we have $F_{0,0} = \mathrm{H}^{\mathfrak{F}}_0(G; \mathrm{K}^G_0(\cdot)) = \mathrm{E}^2_{0,0}$. Therefore we get a surjection $F_{2,-2} \to \mathrm{E}^{\infty}_{2,-2}$, with the kernel being $F_{0,0} = \mathrm{H}^{\mathfrak{F}}_0(G; \mathrm{R}_{\mathbb{C}})$. Using the identification

$$\mathrm{E}^{\infty}_{2,-2} = \mathrm{E}^2_{2,-2} = \mathrm{H}^{\mathfrak{F}}_2(G; \mathrm{K}^G_0(\cdot)) = \mathrm{H}^{\mathfrak{F}}_2(G; \mathrm{R}_{\mathbb{C}}),$$

and putting all these together we get the desired short exact sequence

$$0 \to \underbrace{\mathrm{H}^{\mathfrak{F}}_0(G; \mathrm{R}_{\mathbb{C}})}_{\cong F_{0,0}} \to \underbrace{\mathrm{K}^G_0(\underline{\mathrm{E}}G)}_{\cong F_{2,-2}} \to \underbrace{\mathrm{H}^{\mathfrak{F}}_2(G; \mathrm{R}_{\mathbb{C}})}_{\cong \mathrm{E}^{\infty}_{2,-2}} \to 0.$$

In order to verify the isomorphism for K^G_1, we consider the filtration

$$F_{0,1} \subset F_{1,0} \subset F_{2,-1} = \mathrm{K}^G_1(\underline{\mathrm{E}}G).$$

Due to Lemma 3.6.6 and the fact that q is odd, we conclude $F_{0,1} = 0$. To finish the proof we need to show $F_{1,0} = F_{2,-1}$. By the definition we have $\mathrm{E}^{\infty}_{2,-1} \cong F_{2,-1}/F_{1,0}$. Since the spectral sequence collapses on the second page, we have that $\mathrm{E}^{\infty}_{2,-1} = \mathrm{E}^2_{2,-1} = 0$, which implies $F_{1,0} = F_{2,-1} = \mathrm{K}^G_1(\underline{\mathrm{E}}G)$. Moreover,

$$\mathrm{K}^G_1(\underline{\mathrm{E}}G) = F_{1,0} \cong \mathrm{E}^{\infty}_{1,0} = \mathrm{E}^2_{1,0} = \mathrm{H}^{\mathfrak{F}}_1(G; \mathrm{K}^G_0(\cdot)) = \mathrm{H}^{\mathfrak{F}}_1(G; \mathrm{R}_{\mathbb{C}}).$$

□

3.6.2 Equivariant Lyndon-Hochschild-Serre spectral sequence

In this part, we discuss (a suitable version of) the Lyndon-Hochschild-Serre spectral sequence. The Atiyah-Hirzebruch spectral sequence introduced in the previous section can be considered as a particular case of this spectral sequence. Classically it was introduced in a topological setting where we have a fibration $F \to E \to B$, where the space B is the base space, E is the total space, and F is the fibre space. The algebraic version of this was introduced later where the above mentioned topological setting is replaced by a short exact sequence of groups. Here is the statement of this spectral sequence.

Theorem 3.6.8. *(Lyndon-Hochschild-Serre spectral sequence for homology) [W94, 6.8.2]*
Let $1 \to N \to G \to \bar{G} \to 1$ be a short exact sequence of groups and let M be a G-module. There is a first quadrant homological spectral sequence with the second page

$$\mathrm{E}^2_{p,q} = \mathrm{H}_p(\bar{G}; \mathrm{H}_q(N; M)) \Rightarrow \mathrm{H}_{p+q}(G; M)$$

An equivariant version of the Lyndon-Hochschild-Serre spectral in Bredon homology was introduced by Conchita Martínez-Pérez in [Mar02]. In that article, a spectral sequence is presented which relates the Bredon homology groups over different families of subgroups, in particular of the the normal subgroup and its quotient. We explain the setting here.
Let $\mathfrak{F}$ and $\mathfrak{H}$ be families of subgroups of G which are closed under intersections and conjugations. Let $P_* \to \underline{R}$ be a projective resolution of $\underline{R}$ in $\mathbf{mod}_{\mathfrak{F}}$-$G$. For $M \in G$-$\mathbf{mod}_{\mathfrak{F}}$ and $S \in \mathfrak{H}$, we have

$$\mathrm{H}^{\mathfrak{F} \cap S}_q(S; M) = \mathrm{H}_q(P_* \otimes_{\mathfrak{F} \cap S} M),$$

which is a module in S- $\mathrm{Mod}_{\mathfrak{F} \cap S}$.

Theorem 3.6.9. *[Mar02, Theorem 3.9]*
Let $\mathfrak{F}$ and $\mathfrak{H}$ be families of subgroups of G which are closed under intersection and conjugation. Then for any module M in G-$\mathbf{mod}_{\mathfrak{F}}$ there is spectral sequence whose second page is

$$\mathrm{E}^2_{p,q} = \mathrm{H}^{\mathfrak{H}}_p(G, \mathrm{H}^{\mathfrak{F} \cap -}_q(-; M)) \Rightarrow \mathrm{H}^{\mathfrak{F}}_{p+q}(G; M).$$

One application of this theorem is where we are given a group extension $1 \to N \to G \to \bar{G} \to 1$. Suppose $\mathfrak{F}$ be a family of finite subgroups of G and let $\bar{\mathfrak{H}}$ be a family of finite subgroups of $\bar{G} = G/N$. Consider the pull back of the family $\bar{\mathfrak{H}}$

$$\mathfrak{H} = \{S \leq G \colon N \leq S \,\mathrm{and}\, S/N \in \bar{\mathfrak{H}}\},$$

which provides us with a family of subgroups G which is closed under intersection and conjugation. There is *natural equivalence of categories* given by

$$G\text{-}\mathbf{mod}_{\mathfrak{H}} \to \bar{G}\text{-}\mathbf{mod}_{\bar{\mathfrak{H}}} \, : M \mapsto \bar{M},$$

where $\bar{M}(\bar{G}/\bar{S}) = M(G/S)$ for $\bar{S} = S/N \in \bar{\mathfrak{H}}$. This equivalence induces an isomorphism in homology

$$\mathrm{H}^{\mathfrak{H}}_*(G; M) \cong \mathrm{H}^{\bar{\mathfrak{H}}}_*(\bar{G}; \bar{M}).$$

Now under these assumptions there is a first quadrant spectral sequence whose E^2-term is

$$\mathrm{E}^2_{p,q} = \mathrm{H}_p^{\bar{\mathfrak{H}}}(\bar{G}, \overline{\mathrm{H}_q^{\mathfrak{F}\cap -}(-;M)}) \Rightarrow \mathrm{H}^{\mathfrak{F}}_{p+q}(G;M). \tag{3.6.3}$$

Note that $\mathrm{H}_q^{\mathfrak{F}\cap -}(-;M)$ is a module in $\bar{G}$-$\mathbf{mod}_{\bar{\mathfrak{H}}}$.
We finish this section by providing a short recipe on how to compute the value of such modules. Take an element $\bar{V} \leq \bar{G}$ in $\bar{\mathfrak{H}}$ and consider its pull-back $V \subset G$. Next take the intersection of the family $\mathfrak{F}$ with S which is the family of all finite subgroups of V. Now the value of the covariant functor $\overline{\mathrm{H}_q^{\mathfrak{F}\cap -}(-;M)}$ over $\bar{V}$ is $\mathrm{H}_q^{\mathfrak{F}\cap V}(V;M)$.

Remark 3.6.10. In the special case $\mathfrak{F} = \{1\}$, i.e. G is torsion free, spectral sequence in Theorem 3.6.9 simplifies to a spectral sequence relating Bredon homology to the ordinary homology

$$\mathrm{H}_p^{\mathfrak{H}}(G; \mathrm{H}_q(-;M)) \Rightarrow \mathrm{H}_{p+q}(G;M).$$

Chapter 4

The Baum-Connes conjecture and the trace conjecture

This chapter consists of two sections; the Baum-Connes conjecture and the (modified) trace conjecture. In the first section we give a short report on the current status of the conjecture and later we discuss the Baum-Connes assembly map. The second part provides a survey on the trace conjecture, its counterexample and its modification.

4.1 The Baum-Connes conjecture

Baum, Connes and Higson [BCH94] in 1994 polished and improved the Baum-Connes conjecture [BC00] of 1982. The statement of the conjecture reads as follows.

Conjecture 4.1.1. *(Baum-Connes conjecture)*
Let G be a group. The assembly map (or the index map)

$$\mu_i^G \colon \mathrm{K}_i^G(\underline{\mathrm{E}}G) \to \mathrm{K}_\mathrm{i}(\mathrm{C}_\mathrm{r}^*(G))$$

is an isomorphism of abelian groups for $i = 0, 1$.

The conjecture has been verified for large classes of groups including:

- discrete subgroups of $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$ (Kasparov [K83] and Julg-Kasparov [JK95]),
- discrete subgroups of amenable, connected Lie groups (Kasparov [K88]),
- cocompact lattices of $\mathrm{SL}_3(F)$, where F is $\mathbb{R}$, $\mathbb{C}$ or a p-adic fields (Lafforgue [La98]),

- one-relator groups (Beguin-Bettaieb-Valette [BBV99]),
- groups with the Haagerup property (or a-T-menable groups) (Higson-Kasparov [HK01]),
- Gromov hyperbolic groups and hence cocompact lattices of the simple Lie groups $\mathrm{SO}(n,1)$, $\mathrm{S}U(n,1)$, $\mathrm{S}p(n,1)$ and $\mathrm{F}_4(-20)$ (Lafforgue [La02]),
- subgroups of Gromov hyperbolic groups (Mineyev-Yu [MY02]),
- almost connected groups (Chabert-Echterhoff-Nest [CEN03]),
- certain Levi-decomposable linear algebraic groups over local function fields (Echterhoff-Li-Nest [ENL18]).

Nevertheless the conjecture is not yet known for $\mathrm{SL}_3(\mathbb{Z})$.

The conjecture is not only important because it suggests a conjectural model for the K-theory of $\mathrm{C}^*_\mathrm{r}(G)$ but it also admits profound applications in other fields. For instance, its injectivity implies the Novikov conjecture in topology. As of now the Baum-Connes assembly map is the only known method to prove this conjecture. Moreover, its surjectivity yields the Kadison-Kaplansky conjecture in analysis. We will discuss in more detail the relation between the surjectivity of the Baum-Connes conjecture and the Kadison-Kaplansky conjecture in the next section.

One of the distinct features of the Baum-Connes conjecture is the various descriptions of the left-hand side and of the assembly map. In [BCH94], the left-hand side was defined in the language of KK-theory and the Baum-Connes assembly map was interpreted as an index map associating to certain *cycles* belonging to the KK-class associated to $\underline{\mathrm{E}}G$ an abstract index lying in the K-theory of $\mathrm{C}^*_\mathrm{r}(G)$. It is to say that it took 12 years for Baum, Connes and Higson to come up with this description regarding the left-hand side. We refer to this assembly map as the *analytical assembly map*. Another description is due to Davis and Lück [DL98] from 1998. Their point of view is both functorial and topological. The crucial issue for them was to construct the K-homology functor K^G_* which satisfies

$$\mathrm{K}^G_*(\{pt\}) \cong \mathrm{K}_i(\mathrm{C}^*_\mathrm{r}(G)).$$

The *topological assembly map* is obtained then by applying this functor to a map. Formally speaking we have the following.

Conjecture 4.1.2. *(topological Baum-Connes conjecture)*
The assembly map induced by the G-map $\underline{\mathrm{E}}G \to \{pt\}$

$$\mu^G_i \colon \mathrm{K}^G_i(\underline{\mathrm{E}}G) \to \mathrm{K}^G_i(\{pt\}) \cong \mathrm{K}_i(\mathrm{C}^*_\mathrm{r}(G))$$

is an isomorphism for $i = 0, 1$.

Hambleton and Pedersen in [HP04] in 2004, identified the original assembly map expressed in Kasparov's KK-theory with the topological one due to Davis and Lück. Therefore we refer to these as *the* Baum-Connes assembly map or *the* Baum-Connes conjecture.

One motivation for Davis and Lück behind all these categorical formalities, as mentioned in their article, was "to set stage for explicit computations". And indeed their approach comes equipped with a topological tool, namely the Atiyah-Hirzebruch spectral sequence, discussed in Section 3.6.1.

Remark that although in principle the possibility to apply algebraic topology tools to compute the left-hand side of the Baum-Connes assembly map is provided, in practice the computations can get very hard as confirmed by Lück in [Lu05].

In Chapter 6, Chapter 7 and Chapter 8, we investigate the Baum-Connes assembly map for some groups with the Haagerup property. In doing so we want to answer, through concrete examples, the following type of questions which arises naturally.

Question 4.1.3. *How does the Baum-Connes assembly map transforms the topological and analytical information coming from the left- and right- hand side to each other? Alternatively, suppose the* K*-groups are equipped with natural sets of generators. Would satisfying the Baum-Connes conjecture mean that the assembly map sends these sets of generators to each other?*

4.2 Trace conjecture

As promised in the previous section, in this part we discuss the *trace conjecture* related to the Baum-Connes conjecture. This conjecture is tightly related with the well-known classical Kadison-Kaplansky conjecture in analysis. Both these conjectures implied by the surjectivity of the Baum-Connes conjecture.
The statement of the Kadison-Kaplansky conjecture reads as follows.

Conjecture 4.2.1. *(Kadison-Kaplansky, 1949)*
If G is a torsion free discrete group, then $\mathrm{C}^*_{\mathrm{r}}(G)$ has no idempotent other than 0 and 1.

We first explain the setting and then we clarify the link between the two. We start with a useful notion.

Definition 4.2.2. (*trace*)
Let A be a C*-algebra. A trace on A is a positive linear map $\tau\colon A \to \mathbb{C}$ satisfying $\tau(ab) = \tau(ba)$ for all $a, b \in A$.

Remark 4.2.3. Not all C*-algebras have a trace. A nonexample is the *Cuntz algebra $\mathcal{O}_2$.*

However, for a discrete group G, the reduced C*-algebra $\mathrm{C}^*_\mathrm{r}(G)$ *always* has a trace. This trace is defined as follows. We identify an element $g \in G$ with $u_g \in \mathbb{C}G \subset \mathrm{C}^*_\mathrm{r}(G) \subset \mathcal{B}(\ell^2(G))$, and associate to $g \in G$, the characteristic function $\delta_g \in \ell^2(G)$. The trace is then defined by

$$\tau\colon \mathrm{C}^*_\mathrm{r}(G) \to \mathbb{C}, \quad x \mapsto \langle x\delta_e, \delta_e\rangle.$$

Passing to K-theory, this trace induces a homomorphism of abelian groups, called the *induced trace*

$$\tau_*\colon\ \mathrm{K}_0(\mathrm{C}^*_\mathrm{r}(G)) \to \mathbb{R}, \quad [p] \mapsto \tau(p).$$

As G is assumed to be discrete, $\mathrm{C}^*_\mathrm{r}(G)$ is unital and hence $[1] \in \mathrm{K}_0(\mathrm{C}^*_\mathrm{r}(G))$. Clearly $\tau_*([1]) = 1$. Additivity of a trace implies that $\mathbb{Z} \subset \mathrm{Im}(\tau_*)$.
Suppose G is in addition torsion free. If the Baum-Connes assembly map is surjective then, thanks to the following result, we know that the above image consists only of integers.

Proposition 4.2.4. *[BCH94, Proposition 7.15]*
Let G be torsion free discrete group and assume that the assembly map $\mu_i^G\colon\ \mathrm{K}_0(BG) \to \mathrm{K}_0(\mathrm{C}^*_\mathrm{r}(G))$ is surjective. The image of the induced trace is precisely the integers $\mathbb{Z}$.

To see this consider the composition of the assembly map and the induced trace

$$\mathrm{K}_0(\mathrm{B}G) \xrightarrow{\mu_0} \mathrm{K}_0(\mathrm{C}^*_\mathrm{r}G) \xrightarrow{\tau_*} \mathbb{R},$$

where this composite map associates to each *elliptic* operator its *ordinary* Fredholm index which takes value in $\mathbb{Z}$ (this is an abstract version of an index theorem of Atiyah on covering spaces).
And finally we have the link with the Kadison-Kaplansky conjecture. This is Proposition 7.16 of [BCH94].

Proposition 4.2.5. *(Baum-Connes-Higson 1994)*
If G is a torsion free discrete group such that the Baum-Connes assembly map is surjective, then the Kadison-Kaplansky conjecture holds.

Proof. Let p be a projection. We have that $1-p$ is also a projection. Consider

$$1 = \tau_*([1]) = \tau_*([p]) + \tau_*([1-p]),$$

where the terms are non negative integers by Proposition 4.2.4. Therefore one of the terms $\tau_*(p)$ or $\tau_*(1-p)$ must be zero. As the induced trace is faithful, either p or $1-p$ must be zero. $\square$

Now suppose G has torsion. We would like to understand the image of the induced trace in this case. Let $g \in G$ with $g^k = e$. We may construct a projection

$$q = \frac{1}{k}\sum_{i=1}^{k} u_{g^i} \qquad \text{such that} \qquad \tau_*([q]) = \frac{1}{k}.$$

Moreover for any projection p, the image $\tau_*([p])$ belongs to $[0,1]$. To see this we again use the fact that $1 = \tau_*([1]) = \tau_*([p]) + \tau_*([1-p])$, now positivity of the trace implies that $\tau(p), \tau(1-p) \geq 0$.

Baum and Connes on page 20 of [BC00] conjectured that:
"$\tau_* \colon \mathrm{K}_0(\mathrm{C}^*_{\mathrm{r}}(G)) \to \mathbb{R}$ maps $\mathrm{K}_0(\mathrm{C}^*_{\mathrm{r}}(G))$ *onto* the *additive* subgroup of $\mathbb{Q}$ generated by all rational numbers of the form $\frac{1}{n}$, where n is the order of a finite subgroup of G".
We refer to the above quoted statement as the *trace conjecture.* In 1999, Ranja Roy [R99] constructed a counterexample to this conjecture by providing a group G with 3-torsion which does not have an element of order 9 however it contains an element p in $\mathrm{K}_0(\mathrm{C}^*_{\mathrm{r}}(G))$ with $\tau_*(p) \in \frac{1}{9}\mathbb{Z} \setminus \frac{1}{3}\mathbb{Z}$. A few years later in 2002, Lück [Lu02] suggested the *modified trace conjecture.*

Conjecture 4.2.6. *(modified trace conjecture)*
The image of the induced trace τ_* defined above is contained in the subring of $\mathbb{Q}$ obtained from $\mathbb{Z}$ extended by the inverse of the order of finite subgroups of G, mathematically speaking

$$\operatorname{Im}\tau_* \subseteq \mathbb{Z}[\{\frac{1}{|F|} \colon F \overset{\textit{finite}}{\leq} G\}] \overset{\textit{subring}}{\leq} \mathbb{Q}.$$

Note that this conjecture is still an implication of the surjectivity of the Baum-Connes assembly map, [Lu02].

In Chapter 6, Chapter 7 and Chapter 8 we will calculate the image of the induced trace directly.

Chapter 5

Overview of the main results

In this chapter we review the main results of this thesis. The results include computations of K-theory and K-homology, presentation of explicit models for the classifying spaces and commutations of the image of the induced trace. We use the notations as in the corresponding chapter.

The Baum-Connes conjecture for $\mathrm{BS}(1,n)$ **(Chapter 6)**

We consider the groups of the form

$$\mathrm{BS}(1,n) = \langle a, b \colon bab^{-1} = a^n \rangle \cong \mathbb{Z}[\frac{1}{n}] \rtimes_\alpha \mathbb{Z}.$$

This group is both solvable (hence amenable) and one-relator. The Baum-Connes conjecture holds true due to [HK01] or [BBV99].
The K-theory of $\mathrm{C}^*(\mathrm{BS}(1,n))$ for $n \neq 1$ is

- $\mathrm{K}_0(\mathrm{C}^*\mathrm{BS}(1,n)) = \mathbb{Z}[1]$,
- $\mathrm{K}_1(\mathrm{C}^*\mathrm{BS}(1,n)) = \mathbb{Z} \oplus \mathbb{Z}/|n-1|.\mathbb{Z}$ with generator $[b]$ of infinite order and $[a]$ of order $|n-1|$.

A model for the classifying space $\mathrm{B}\,\mathrm{BS}(1,n)$ is its presentation complex which is 2-dimensional. Its K-homology is

- $\mathrm{K}_0(\mathrm{B}\,\mathrm{BS}(1,n)) = \mathrm{H}_0(\mathrm{B}\,\mathrm{BS}(1,n); \mathbb{Z}) = \mathbb{Z}$,
- $\mathrm{K}_1(\mathrm{B}\,\mathrm{BS}(1,n)) = \mathrm{H}_1(\mathrm{B}\,\mathrm{BS}(1,n); \mathbb{Z}) = \mathrm{BS}(1,n)^{\mathrm{ab}} = \mathbb{Z} \oplus \mathbb{Z}/|n-1| \cdot \mathbb{Z}$.

The image of the induced trace is

$$\tau_*(\mathrm{K}_0(\mathrm{C}^*\mathrm{BS}(1,n))) = \mathbb{Z}.$$

The Baum-Connes conjecture for $F \wr \mathbb{Z}$ (chapter 7)

The group under consideration is

$$F \wr \mathbb{Z} = \bigoplus_{\mathbb{Z}} F \rtimes \mathbb{Z},$$

where F is a finite group.
This group is amenable hence it satisfies the Baum-Connes conjecture by [HK01].
The K-theory for $\mathrm{C}^*(F \wr \mathbb{Z})$ is

- $\mathrm{K}_0(\mathrm{C}^*(F \wr \mathbb{Z})) = \mathbb{Z}R$ with R a countable basis indexed by representatives for $\mathbb{Z}$-orbits in $(\mathrm{Min}(\mathbb{C}F))^{(\mathbb{Z})}$,
- $\mathrm{K}_1(\mathrm{C}^*(F \wr \mathbb{Z})) = \mathbb{Z}[u]$, where $[u]$ is the unitary coming from the $\mathbb{Z}$-action.

The classifying space $\underline{\mathrm{E}}F \wr \mathbb{Z}$ has a 2-dimensional model which is a mapping telescope. Its K-homology is

- $\mathrm{K}_0^L(\underline{\mathrm{E}}L) = \mathbb{Z}R'$ with R' a countable basis indexed by representatives for $\mathbb{Z}$-orbits in $\hat{F}^{(\mathbb{Z})}$.
- $\mathrm{K}_1^L(\underline{\mathrm{E}}L) = \mathbb{Z}[v]$, where $[v]$ is the canonical generator of $\mathrm{K}_1^{\mathbb{Z}}(\underline{\mathrm{E}}\mathbb{Z})$.

The image of the induced trace is

$$\tau_*(\mathrm{K}_0(\mathrm{C}^*(F \wr \mathbb{Z})) = \mathbb{Z}[\frac{1}{|F|}].$$

The Baum-Connes conjecture for $F \wr \mathbb{F}_n$ (Chapter 8)

We consider the wreath product

$$F \wr \mathbb{F}_n = \bigoplus_{\mathbb{F}_n} F \rtimes \mathbb{F}_n,$$

where F is a finite group.
This group is a-T-menable hence satisfies the Baum-Connes conjecture due to [HK01].
The K-theory of $\mathrm{C}^*_\mathrm{r}(F \wr \mathbb{F}_n)$ is

- $\mathrm{K}_0(\mathrm{C}^*_\mathrm{r}(F \wr \mathbb{F}_n)) = \mathbb{Z}R$ with R a countable basis indexed by representatives for $\mathbb{F}_n$-orbits in $(\mathrm{Min}\,(\mathbb{C}F))^{(\mathbb{F}_n)}$.
- $\mathrm{K}_1(\mathrm{C}^*_\mathrm{r}(F \wr \mathbb{F}_n)) = \mathbb{Z}[u_1] \oplus \ldots \oplus \mathbb{Z}[u_n]$, where $[u_i]$'s are the unitary coming from the $\mathbb{F}_n$-action.

The classifying space $\underline{\mathrm{E}}F \wr \mathbb{F}_n$ has a 2-dimensional model which is a mapping telescope. Its K-homology is

- $\mathrm{K}_0^{\mathrm{F}\wr\mathbb{F}_\mathrm{n}}(\underline{\mathrm{E}}F \wr \mathbb{F}_n) = \mathbb{Z}R'$ with R' a countable basis indexed by representatives for $\mathbb{F}_n$-orbits in $\hat{F}^{(\mathbb{F}_n)}$.
- $\mathrm{K}_1^{\mathrm{F}\wr\mathbb{F}_\mathrm{n}}(\underline{\mathrm{E}}F \wr \mathbb{F}_n) = \mathbb{Z}[v_1] \oplus \ldots \oplus \mathbb{Z}[v_n]$, where $[v_i]$ is the canonical generator of $\mathrm{K}_1^{\mathbb{Z}}(\underline{\mathrm{E}}\mathbb{Z})$.

The image of the induced trace is

$$\tau_*(\mathrm{K}_0(\mathrm{C}^*_\mathrm{r}(F \wr \mathbb{F}_n)) = \mathbb{Z}[\frac{1}{|F|}].$$

Chapter 6

The Baum-Connes conjecture for $\mathrm{BS}(1, n)$

In this chapter, we discuss the Baum-Connes conjecture for the solvable Baumslag-Solitar group $\mathrm{BS}(1, n)$. The conjecture holds for this group due to several reasons; being one-relator group [BBV99], [Tu99], [Oyo98] or being an amenable group [HK01]. We want to elucidate this isomorphism and describe it explicitly by presenting bases for all K-groups and identifying them via the assembly map. In order to draw a clear picture, we will present a model for its classifying space. This chapter is based on the article [PoVa16] and is partially taken from there.

6.1 Baumslag-Solitar groups

The Baumslag-Solitar groups are a particular class of two generator one-relator groups which proved useful in group theory. They were introduced by Gilbert Baumslag and Donald Solitar in [BS62] to provide some simple examples of non-Hopfian groups, defined bellow.

Definition 6.1.1. (*Hopfian group)*
A group G is Hopfian if it is not isomorphic to any of its proper quotients that means every epimporphism $G \to G$ is an isomorphism.

Before we continue with the presentation of these groups, let us introduce a few related notions.

Definition 6.1.2. (*solvable group)*
A group Γ is solvable if it has a finite derived series, that is

$$\Gamma = \Gamma_n \trianglerighteq \Gamma_{n-1} \trianglerighteq \cdots \trianglerighteq \Gamma_0 = 1$$

such that the quotients are all abelian. The smallest n *is called the derived length of* Γ. *If the derived length is 1 then the group is abelian and for* $n \leq 2$ *the group is metabelian.*

Definition 6.1.3. (*HNN extension*)
Let G be a group presented by $G = \langle S \mid R \rangle$. Let $\varphi\colon H \to K$ be an isomorphism of two subgroups of G. Let t be a new symbol not in S. The HNN extension of G relative to φ is

$$G*_{\varphi} = \big\langle S, t \mid R, tht^{-1} = \varphi(h), \forall h \in H \big\rangle.$$

The Baumslag-Solitar groups are defined as

$$\mathrm{BS}(m, n) = \langle a, b \mid ab^m a^{-1} = b^n \rangle$$

This is an HNN-extension where $\varphi\colon m\mathbb{Z} \to n\mathbb{Z}$ by $mb \mapsto nb$, where b is a generator of $\mathbb{Z}$.
Trivial examples that were known prior to Baumslag and Solitar's article are:

1. $\mathrm{BS}(\pm 1, \pm 1) = \mathbb{Z} \times \mathbb{Z}$
2. $\mathrm{BS}(\pm 1, \mp 1) = \mathbb{Z} \rtimes \mathbb{Z}$.
   Note that $\mathrm{BS}(1, -1)$ is the fundamental group of the Klein bottle.

Remark 6.1.4. The classification of the Baumslag-Solitar groups, up to isomorphism, was obtained in [Mo91]: $\mathrm{BS}(m, n) \cong \mathrm{BS}(p, q)$ if and only if $\{|m|, |n|\} = \{|p|, |q|\}$. In particular there are isomorphisms between $\mathrm{BS}(m, n)$, $\mathrm{BS}(-m, -n)$, $\mathrm{BS}(n, m)$ and $\mathrm{BS}(-n, -m)$.

One remarkable contrast between different Baumslag-Solitar groups stems from the distinction whether or not the parameters (m, n) satisfy $|m|$ or $|n| = 1$ or satisfy $|m|$ and $|n| \neq 1$. As an example we have the next proposition.

Proposition 6.1.5. *The group* $\mathrm{BS}(m, n)$ *is solvable if and only if* $|m| = 1$ *or* $|n| = 1$.

Proof. First assume $\mathrm{BS}(m, n) = \langle a, b \mid ab^m a^{-1} = b^n \rangle$ is solvable but neither $|m|$ nor $|n|$ equals 1. Consider the group generated by two elements a and bab^{-1}. Employing Britton's lemma, this group is isomorphic to the free group $\mathbb{F}_2$. This contradicts the fact that $\mathrm{BS}(m, n)$ is solvable. To show the converse, we assume $m = 1$ (cf. Remark 6.1.4). Using the HNN extension description or the universality of the presentation, we can identify $\mathrm{BS}(1, n)$ with the semidirect product $\mathbb{Z}[\frac{1}{n}] \rtimes_{\alpha} \mathbb{Z}$ (see Section 6.2 bellow). This implies immediately that the group is metabelian, hence solvable. $\square$

6.2 The C*-algebra of solvable Baumslag-Solitar groups

In this section we will discuss the C*-algebra associated to the solvable Baumslag-Solitar group $\mathrm{BS}(1,n)$. This group is presented by $\mathrm{BS}(1,n) = \langle a, b \mid aba^{-1} = b^n \rangle$. We assume $|n| > 1$.
There is a faithful homomorphism from $\mathrm{BS}(1,n)$ to the affine group of the real line, given by

$$\mathrm{BS}(1,n) \to \mathrm{Aff}_1(\mathbb{R})\colon \begin{cases} a \mapsto (x \mapsto nx) & \text{dilation by } n \\ b \mapsto (x \mapsto x+1) & \text{translation by} +1. \end{cases}$$

It realises an isomorphism

$$\mathrm{BS}(1,n) \cong \mathbb{Z}[\frac{1}{n}] \rtimes_\alpha \mathbb{Z},$$

where $\mathbb{Z}[\frac{1}{n}] = \{\frac{m}{n^l} \in \mathbb{Q} \mid m \in \mathbb{Z}, l \in \mathbb{N}\}$, viewed as an additive group, and α is multiplication by n.
In turn, its C*-algebra can be written as the crossed product $\mathrm{C}^*(\mathbb{Z}[\frac{1}{n}]) \rtimes_\alpha \mathbb{Z}$. The normal subgroup $N := \mathbb{Z}[\frac{1}{n}]$ can be viewed as the inductive limit of

$$\mathbb{Z} \xrightarrow{i_0} \mathbb{Z} \xrightarrow{i_1} \mathbb{Z} \xrightarrow{i_2} \dots,$$

where $i_k\colon \mathbb{Z} \to \mathbb{Z}$ (for $k \geq 0$) is multiplication by n. Therefore $\hat{i}_k\colon \mathbb{T} \to \mathbb{T}$ is raising to the power n, and the Pontryagin dual $\hat{N}$ is the projective limit of

$$\dots \xrightarrow{\hat{i}_2} \mathbb{T} \xrightarrow{\hat{i}_1} \mathbb{T} \xrightarrow{\hat{i}_0} \mathbb{T},$$

which we identify with the solenoid

$$X_n = \{z = (z_k)_{k\geq 0} \in \mathbb{T}^{\mathbb{N}} \mid z_{k+1}^n = z_k, \forall k \geq 0\}$$

The duality between X_n and $\mathbb{Z}[\frac{1}{n}]$ is given by

$$(z, m) = z_l^m, \qquad m \text{ belongs to the } l\text{-th copy of } \mathbb{Z},$$

which is well-defined as $(z, i_l(m)) = z_{l+1}^{n\cdot m} = z_l^m = (z, m)$.
For $\frac{m}{n^l} \in \mathbb{Z}[\frac{1}{n}]$, this corresponds to $(z, \frac{m}{n^l}) = z_l^m$ for $z = (z_k)_{k\geq 0} \in X_n$. Therefore, rephrasing the explanation above we have a pairing between the two given by

$$\begin{aligned} X_n \times \mathbb{Z}[\frac{1}{n}] &\to \mathbb{T} \\ (z, m) &\mapsto z_l^m \quad m \text{ belongs to the } l\text{-th copy of } \mathbb{Z}, \\ (z, \frac{m}{n^l}) &\mapsto z_l^m \qquad z = (z_k)_{k\geq 0} \in X_n. \end{aligned}$$

The automorphism α is given by $\alpha(m) = i_l(m)$, where m lies in the l-th copy of $\mathbb{Z}$. Thus $\hat{\alpha}$ is the automorphism of X_n given by the backward shift:

$$(\hat{\alpha}(z))_k = z_{k+1}.$$

Therefore, $\mathrm{C^*BS}(1,n) = \mathrm{C}(X_n) \rtimes_{\hat{\alpha}} \mathbb{Z}$. This crossed product can be viewed as the universal C*-algebra generated by two unitaries u and v satisfying $uvu^{-1} = v^n$, where $u \in \mathrm{C^*}\mathbb{Z}$ corresponds to $+1 \in \mathbb{Z}$ and $v \in \mathrm{C}(X_n)$ is given by the function $z \mapsto z_0$ on X_n.
Remark that this crossed product description of $\mathrm{C^*BS}(1,n)$ already appears in [BreJo91] and [IMSS11].

We now compute the K-theory of $\mathrm{C}(X_n)$.

Lemma 6.2.1. *Let X_n be the solenoid defined above. We have*

$$\mathrm{K_0}(\mathrm{C}(X_n)) = \mathbb{Z} \cdot [1],$$

the infinite cyclic group generated by the class of $1 \in \mathrm{C}(X_n)$, and that

$$\mathrm{K_1}(\mathrm{C}(X_n)) \cong \mathbb{Z}[\frac{1}{n}].$$

Proof. We have observed that $\mathrm{C}(X_n) = \mathrm{C^*}(\mathbb{Z}[\frac{1}{n}]) = \varinjlim(\mathrm{C^*}(\mathbb{Z}), i_k)$(where we also denote by i_k the *-homomorphism $\mathrm{C^*}\mathbb{Z} \to \mathrm{C^*}\mathbb{Z}$ associated with the group homomorphism i_k). Since the K-theory commutes with inductive limits, we get

$$\mathrm{K_i}(\mathrm{C}(X_n)) = \varinjlim(\mathrm{K_i}(\mathrm{C^*}\mathbb{Z}), (i_k)_*) \qquad i = 0, 1.$$

We know that $\mathrm{K_0}(\mathrm{C^*}\mathbb{Z}) = \mathbb{Z} \cdot [1]$ and i_k is a unital $*$-homomorphism, therefore we obtain that $\mathrm{K_0}(\mathrm{C}(X_n)) = \varinjlim(\mathbb{Z} \cdot [1], \mathrm{id})$.
On the other hand, let $u \in \mathrm{C^*}\mathbb{Z}$ be the unitary corresponding to the generator $+1 \in \mathbb{Z}$ such that $\mathrm{K_1}(\mathrm{C^*}\mathbb{Z}) = \mathbb{Z}[u]$. Then $i_k(u) = u^n$, i.e. $(i_k)_*([u]) = n[u]$, and the inductive system $(\mathrm{K_1}(\mathrm{C^*}\mathbb{Z}), (i_k)_*)$ is isomorphic to the original system $(\mathbb{Z}, i_k)$, hence they have the same limit $\mathbb{Z}[\frac{1}{n}]$. □

6.3 Analytical side: K-theory of $\mathrm{C^*BS}(1,n)$

In this section we describe the K-theory of $\mathrm{C^*BS}(1,n)$. As a result of viewing the C*-algebra of $\mathrm{C^*BS}(1,n)$ as a crossed product by $\mathbb{Z}$, the Pimsner-Voiculescu 6-term exact sequence 2.3.1 will be the main tool for computing its K-theory.

The next lemma describes the image of the unitary coming from the action under the boundary map $\partial_1\colon \mathrm{K_1}(A \rtimes_\alpha \mathbb{Z}) \to \mathrm{K_0}(A)$.

Lemma 6.3.1. *Let A be a unital C^*-algebra. Consider the crossed product $A \rtimes_\alpha \mathbb{Z}$ where α is implemented by the unitary $u \in \mathrm{C}^*\mathbb{Z} \subset A \rtimes \mathbb{Z}$. The boundary map ∂_1 sends the K_1 class of $[u]$ to the K_0 class $-[1]$ in A.*

Proof. Let $\mathrm{C}^*(S)$ be the C^*-algebra generated by a non-unitary isometry S and let $P = \mathrm{I} - SS^*$. The Toeplitz algebra for A and α, denoted by $\mathcal{T}_{A,\alpha}$, is the C^*-subalgebra of $(A \rtimes_\alpha \mathbb{Z}) \otimes \mathrm{C}^*(S)$ generated by $u \otimes S$ and $A \otimes \mathrm{I}$. Let $\mathcal{K}$ be the C^*-algebra of compact operators on a separable Hilbert space, with the corresponding canonical system of matrix units $(e_{ij})_{i,j\geq 0}$. Consider the Toeplitz extension associated with $A \rtimes_\alpha \mathbb{Z}$ as in [PV80]

$$0 \to A \otimes \mathcal{K} \xrightarrow{\varphi} \mathcal{T}_{A,\alpha} \xrightarrow{\psi} A \rtimes_\alpha \mathbb{Z} \to 0,$$

with

$$\begin{aligned} &\varphi(a, e_{ij}) = u^i a u^{*^j} \otimes S^i P S^{*j} \quad i,j \in \mathbb{N} \\ &\psi(u \otimes S) = u, \quad \psi(a \otimes \mathrm{I}) = a \qquad a \in A. \end{aligned}$$

The map $\partial_1\colon \mathrm{K}_1(A \rtimes_\alpha \mathbb{Z}) \to \mathrm{K}_0(A \otimes \mathcal{K})$ is the boundary map associated with the Toeplitz extension, and we compute $\partial_1([u])$ following the description given in the Definition 8.3.1 in [Bla86].
Consider the unitary matrix $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \in U_2(A \rtimes_\alpha \mathbb{Z})$. This matrix can be lifted via ψ to a matrix $N = \begin{pmatrix} u \otimes S & 1 \otimes P \\ 0 & u^* \otimes S^* \end{pmatrix} \in U_2(\mathcal{T}_{A,\alpha})$. For the matrix $p_1 := \begin{pmatrix} 1 \otimes \mathrm{I} & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_2(\mathcal{T}_{A,\alpha})$, we have

$$Np_1N^* - p_1 = \begin{pmatrix} 1 \otimes SS^* - 1 \otimes \mathrm{I} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 \otimes P & 0 \\ 0 & 0 \end{pmatrix}.$$

The pull back of this element via φ is

$$z := \begin{pmatrix} -1 \otimes e_{00} & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_2(A \otimes \mathcal{K}).$$

Therefore, $\partial_1([u]) = -[-z]$. Via the isomorphism $\mathrm{K}_0(A \otimes \mathcal{K}) \cong \mathrm{K}_0(A)$, the element $[-z] = [1 \otimes e_{11}]$ corresponds to $[1]$. Hence $\partial_1([u]) = -[1]$. □

Theorem 6.3.2. *For $n \neq 1$, K-theory of $\mathrm{C}^*\mathrm{BS}(1,n)$ is described as*

- $\mathrm{K}_0(\mathrm{C}^*\mathrm{BS}(1,n)) = \mathbb{Z}[1]$,

- $\mathrm{K}_1(\mathrm{C}^*\mathrm{BS}(1,n)) = \mathbb{Z} \oplus \mathbb{Z}/|n-1|\mathbb{Z}$ *with generator* $[a]$ *of infinite order and* $[b]$ *of order* $|n-1|$.

Proof. We may view $\mathrm{C}^*(\mathrm{BS}(1,n))$ as the crossed product $\mathrm{C}^*(\mathbb{Z}[\frac{1}{n}]) \rtimes_\alpha \mathbb{Z}$. Due to Lemma 6.3.1, we have

- $\mathrm{K}_0(\mathrm{C}^*(\mathbb{Z}[\frac{1}{n}])) = \mathbb{Z}[1]$
- $\mathrm{K}_1(\mathrm{C}^*(\mathbb{Z}[\frac{1}{n}])) \cong \mathbb{Z}[\frac{1}{n}]$.

Plug in these K-theoretic information to the Pimsner-Voiculescu exact sequence 2.3.1, it reduces to

$$\begin{tikzcd}
\mathbb{Z}\cdot[1] \arrow[r, "\text{Id-}\alpha_*"] & \mathbb{Z}\cdot[1] \arrow[r, "\iota_*"] & \mathrm{K}_0(\mathrm{C}^*\mathrm{BS}(1,n)) \arrow[d, "\partial_0"] \\
\mathrm{K}_1(\mathrm{C}^*\mathrm{BS}(1,n)) \arrow[u, "\partial_1"] & \mathbb{Z}[\frac{1}{n}] \arrow[l, "\iota_*"'] & \mathbb{Z}[\frac{1}{n}]. \arrow[l, "\text{Id-}\alpha_*"']
\end{tikzcd}$$

Since $\alpha(1) = 1$, the upper-left arrow is the zero map. The bottom-right arrow is given by multiplication by $1-n$ on $\mathbb{Z}[1/n]$, so it is injective, hence the right boundary map is zero. This shows that $\iota_* \colon \mathbb{Z}\cdot[1] \to \mathrm{K}_0(\mathrm{C}^*\mathrm{BS}(1,n))$ is an isomorphism.

In order to compute K_1, we observe that the relation $[b] = [aba^{-1}] = [b^n]$ implies that $(n-1)\cdot[b] = 0$, i.e. the order of $[b]$ divides $|n-1|$. To prove that this is exactly $|n-1|$, we look at the bottom line of the Pimsner-Voiculescu sequence. Since

$$\mathbb{Z}[1/n]/\mathrm{Im}\,(\text{Id-}\alpha_*) \cong \mathbb{Z}/|n-1|\mathbb{Z},$$

we get a short exact sequence

$$0 \to \mathbb{Z}/|n-1|\mathbb{Z} \to \mathrm{K}_1(\mathrm{C}^*\mathrm{BS}(1,n)) \xrightarrow{\partial 1} \mathbb{Z}\cdot[1] \to 0,$$

which splits to give $\mathrm{K}_1(\mathrm{C}^*\mathrm{BS}(1,n)) = \mathbb{Z} \oplus \mathbb{Z}/|n-1|\mathbb{Z}$, with $[b]$ a generator of order $|n-1|$. Due to Lemma 6.3.1, we have that $\partial_1([a]) = -[1]$, therefore $[a]$ is a generator of infinite order. □

6.4 Topological side: K-homology of $\underline{\mathrm{E}}\,\mathrm{BS}(1,n)$

In this section we describe the K-homology of $\underline{\mathrm{E}}\,\mathrm{BS}(1,n)$. The fact that this group does not have torsion, simplifies the computations.

For G a torsion free group, we know from [BCH94] that

$$\mathrm{K}_i^G(\underline{\mathrm{E}}G) = \mathrm{K}_i(\underline{\mathrm{E}}G/G) = \mathrm{K}_i(\mathrm{E}G/G) = \mathrm{K}_{\mathrm{i}}(\mathrm{B}G).$$

We should first understand this classifying space for the group $\mathrm{BS}(1,n)$. Here is a definition.

Definition 6.4.1. (*presentation complex*)
Let G be a group presented by $\langle s_1,\dots,s_m \,|\, r_1,\dots,r_n\rangle$. The presentation complex of G is a (at most) 2-dimensional cell complex X which has one 0-cell, m 1-cell for each generators s_i $(1 \le i \le m)$ and one 2-cell for each relations r_j $(1 \le j \le n)$.

Example 6.4.2. *For a free group* $\mathbb{F}_n$*, the presentation complex is a bouquet of* n *circles and for* $\mathbb{Z}^2$ *it is the torus.*

Remark 6.4.3. Due to a result by Lyndon, in fact a consequence of his Identity Theorem [Lyn50], for a torsion free one-relator group, its presentation complex is a 2-dimensional model for BG.

The following Lemma will be our main tool for computing the K-homology of this complex.

Lemma 6.4.4. *([BBV99], Lemma 4)*
Let X *be a finite CW-complex of dimension 2. We have*

$$\mathrm{K}_0(X) \cong \mathrm{H}_0(X) \oplus \mathrm{H}_2(X), \qquad \mathrm{K}_1(X) \cong \mathrm{H}_1(X).$$

Theorem 6.4.5. *Let* $n \neq 1$*. For* $\mathrm{BS}(1,n)$*, homology computation yields*

- $\mathrm{K}_0(\mathrm{BBS}(1,n)) = \mathbb{Z}$,
- $\mathrm{K}_1(\mathrm{BBS}(1,n)) = \mathbb{Z} \oplus \mathbb{Z}/|n-1|\cdot\mathbb{Z}$.

Proof. Let $\mathrm{BS}(1,n) = \langle S \,|\, R\rangle$. We know that its presentation complex has one vertex, two edges and one 2-cell for the relator $aba^{-1}b^{-n}$. A cellular chain complex for this model is

$$\mathrm{C}_2(\mathrm{BBS}(1,n)) \xrightarrow{\partial_2} \mathrm{C}_1(\mathrm{BBS}(1,n)) \xrightarrow{\partial_1} \mathrm{C}_0(\mathrm{BBS}(1,n)).$$

Let us first compute the H_2-term of this chain complex. Using the description of the complex, the boundary map ∂_2 coincides with

$$\partial_2 \colon \mathbb{Z}[R] \to \mathbb{Z}[S],$$

sending the 2-cell $[R]$ to its boundary obtained by running around $[R]$ and summing up the letters appearing in R with a sign equal to the exponent (which corresponds to their orientation as you run along the boundary of the 2-cell). Because $R = a^{-1}ba^nb^{-1}$ is not in the commutator subgroup of $\mathbb{F}(S)$, the free group on S, the sum of all exponents of R is non-zero. Hence ∂_2 is injective, from where

$$\mathrm{H}_2(\mathrm{B}\,\mathrm{BS}(1,n)) = \mathrm{Ker}\,\partial_2 = 0.$$

Lemma 6.4.4 then reduces to

$$\mathrm{K}_0(\mathrm{B}\,\mathrm{BS}(1,n)) \cong \mathrm{H}_0(\mathrm{B}\,\mathrm{BS}(1,n)), \quad \mathrm{K}_1(\mathrm{B}\,\mathrm{BS}(1,n)) \cong \mathrm{H}_1(\mathrm{B}\,\mathrm{BS}(1,n)).$$

Moreover, we have

$$\mathrm{H}_0(\mathrm{B}\,\mathrm{BS}(1,n)) = \mathbb{Z} = \langle \text{inclusion of the base point} \rangle.$$

For K_1, we use the fact that $\mathrm{H}_1(G) = G^{\mathrm{ab}}$, the abelianisation of the group.

$$\mathrm{H}_1(\mathrm{B}\,\mathrm{BS}(1,n)) = \mathrm{BS}(1,n)^{\mathrm{ab}} = \langle a, b \mid ab = ba, b^{|n-1|} = e \rangle = \mathbb{Z} \oplus \mathbb{Z}/|n-1| \cdot \mathbb{Z}.$$

□

6.5 Summary of computations

Here is a summary of the computations of this chapter for $\mathrm{BS}(1,n)$.

	topological side	analytical side
K_0	$\mathbb{Z}$	$\mathbb{Z}$
K_1	$\mathbb{Z} \oplus \mathbb{Z}/\|n-1\| \cdot \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/\|n-1\| \cdot \mathbb{Z}$

6.6 The Baum-Connes assembly map for $\mathrm{BS}(1,n)$

In the section we conclude the isomorphism of the Baum-Connes assembly map for $\mathrm{BS}(1,n)$ using our computations. We will use the following theorem.

Theorem 6.6.1. *([MV03], Theorem 1.4)*
Let G^{ab} be the abelianisation of the group G. Consider the natural homomorphism $\kappa_G\colon G^{ab} \to \mathrm{K}_1(\mathrm{C}^*_\mathrm{r}G)$. There exists a homomorphism

$$\beta_1\colon G^{ab} \to \mathrm{K}_1^{\mathrm{G}}(\underline{\mathrm{E}}G),$$

such that

$$\kappa_G = \mu_1^G \circ \beta_1.$$

Here is the conclusion of the K-theoretic computations.

Theorem 6.6.2. *Let $n \neq 1$. The assembly map*

$$\mu_i^{\mathrm{BS}(1,n)}\colon\ \mathrm{K_i}(\mathrm{B\,BS}(1,n)) \to \mathrm{K_i}(\mathrm{C}^*\mathrm{BS}(1,n)) \qquad i = 0,1,$$

is an isomorphism.

Proof. We verify the isomorphism for $i = 0, 1$.

$i = 0$ Thanks to computations in 6.3.2 and 6.4.5 we know that both groups are abstractly $\mathbb{Z}$. Due to Example 2.11 on page 97 of [MV03],

$$\mu_0^{\mathrm{BS}(1,n)}([\text{generator of}\,\mathrm{K}_0]) = [1] \in \mathrm{K}_0(\mathrm{C}^*(\mathrm{BS}(1,n)))$$

This proof is given in [PoVa16]. Alternatively, here we present another proof, where we use the Baum-Connes conjecture for the trivial group $\{e\}$! Consider the inclusion $\{e\} \hookrightarrow \mathrm{BS}(1,n)$.

$$\begin{tikzcd}
\mathrm{K}_0(\{pt\}) \arrow[r, "\cong"] \arrow[d, "\{pt\}\mapsto \mathrm{B\,BS}(1,n)"] & \mathrm{K}_0(\underline{\mathrm{E}}\{e\}) \arrow[r, "\mu_0^e"] \arrow[d] & \mathrm{K}_0(\mathrm{C}^*(\{e\})) \cong \mathrm{K}_0(\mathbb{C}) \arrow[d, "{[1]\mapsto[1]}"] \\
\mathrm{K}_0(\mathrm{B\,BS}(1,n)) \arrow[r, "\cong"] & \mathrm{K}_0(\underline{\mathrm{E}}\mathrm{BS}(1,n)) \arrow[r, "\mu_0^{\mathrm{BS}}"] & \mathrm{K}_0(\mathrm{C}^*(\mathrm{BS}(1,n))).
\end{tikzcd}$$

Thanks to functoriality of the assembly map the diagram is commutative. A diagram chasing confirms that

$$\mu_0^{\mathrm{BS}(1,n)}([\text{generator of}\,\mathrm{K}_0]) = [1] \in \mathrm{K}_0(\mathrm{C}^*(\mathrm{BS}(1,n))).$$

$i = 1$ Note that $G^{\mathrm{ab}} = \mathrm{H}_1(\mathrm{B}G)$. Consider the following map

$$\begin{aligned} \kappa_{\mathrm{BS}(1,n)} \colon \mathrm{BS}(1,n)^{\mathrm{ab}} &\to \mathrm{K}_1(\mathrm{C}^*(\mathrm{BS}(1,n))) \\ g &\mapsto [u_g]. \end{aligned}$$

Due to Theorem 6.6.1 and the fact that the group is torsion free, the map $\kappa_{\mathrm{BS}(1,n)}$ coincides with $\mu_1^{\mathrm{BS}(1,n)}(\mathrm{H}_1(\mathrm{BS}(1,n)))$ on the lowest-dimensional part of $\mathrm{K}_1(\mathrm{B}\,\mathrm{BS}(1,n))$, which is $\mathrm{H}_1(\mathrm{BS}(1,n)) = \mathrm{BS}(1,n)^{\mathrm{ab}}$. In other words, we have that

$$\mu_1^{\mathrm{BS}(1,n)} \colon\ \mathrm{K}_1(\mathrm{B}\,\mathrm{BS}(1,n)) \to \mathrm{K}_1(\mathrm{C}^*(\mathrm{BS}(1,n)))$$

coincides with

$$\kappa_{\mathrm{BS}(1,n)} \colon \mathbb{Z} \oplus \mathbb{Z}/|n-1| \cdot \mathbb{Z} \to \mathrm{K}_1(\mathrm{C}^*(\mathrm{BS}(1,n))),$$

which is an isomorphism by Theorem 6.3.2.

□

6.7 Trace conjecture for $\mathrm{BS}(1,n)$

In this short note we verify directly the trace conjecture [BC00] for $\mathrm{BS}(1,n)$.

Let $\tau \colon \mathrm{C}^*(\mathrm{BS}(1,n)) \to \mathbb{C}$ be the canonical trace on $\mathrm{C}^*(\mathrm{BS}(1,n))$. This induces the homomorphism $\tau_* \colon\ \mathrm{K}_0(\mathrm{C}^*(\mathrm{BS}(1,n))) \to \mathbb{R}$ as discussed in Section 4.2. Since $\tau(1) = 1$, we have that $\tau_*(\mathrm{K}_0(\mathrm{C}^*(\mathrm{BS}(1,n)))) = \mathbb{Z}$. Therefore as one expects for a torsion free group the image of this induced trace is the whole integers (cf. Proposition 4.2.4).

Chapter 7

The Baum-Connes conjecture for $F \wr \mathbb{Z}$

In this section we discus the Baum-Connes conjecture for the generalised lamplighter group $L = F \wr \mathbb{Z}$, where F is finite. This group is amenable therefore the Baum-Connes conjecture is satisfied due to work of Higson and Kasparov [HK01]. The aim is to shed light on this isomorphism and describe it explicitly by presenting generators for K-groups and identify them via the assembly map. We will describe a concrete model for the classifying space $\underline{E}F \wr \mathbb{Z}$. In the last part, we will directly calculate the image of the induced trace. This chapter is partially taken from [FPV16].

7.1 Wreath products

Wreath product have proved useful in group theory. Let us start with a history taken from [Ba17] on wreath products. Leo Kaloujnine understood the importance of wreath products in the early 1940's. It is said that he worked, during the second World War, in a uniform factory and observed a rivet machine which is made of a rotating ring containing many rotating disks on it. Identifying the movement of the ring with an action of G and each subdisk with an action of H, one sees that motions of the machine are described by wreath product elements. He did his thesis with Elie Cartan and he studied the Sylow subgroups of symmetric groups, and showed that they were iterated wreath products.
However, this description of maximal p-subgroups of symmetric groups already appear in the classical treatise by Camille Jordan [J89] who implicitly defined wreath products there. This is remarkable since Sylow's theorems were only published two years later [Sy72]!

Kaloujnine is known for the following important result on classification of group extensions; namely, that the wreath product is a universal object containing all extensions.

Theorem 7.1.1. *(Kaloujnine-Krasner [KK48])*
Let K and H be groups. If G is an extension of H by K, then G is isomorphic to a subgroup of the wreath product $H \wr K$. That is the wreath product contains isomorphic copies of every extensions of H by K.

A well-known example of wreath products is the *lamplighter* group. This group has numerous equivalent definitions.

1. $L = \{\pm 1\} \wr \mathbb{Z}$,
2. $L = (\mathbb{Z}/2\mathbb{Z})[x, x^{-1}] \rtimes \mathbb{Z}$, generator of $\mathbb{Z}$ acts by multiplication by x,
3. $L = \left\{ \begin{pmatrix} x^k & f \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z},\, f \in (\mathbb{Z}/2\mathbb{Z})[x, x^{-1}] \right\}$,
4. $L = \langle a, t \colon a^2, (at^n a t^{-n})^2, n \in \mathbb{Z} \rangle$.

Beside all these definitions, it is constructive to view the lamplighter group as follows. An infinite street with at each integer position a lamppost. The lamp there can be "on" or "off". Imagine also that the street is viewed from the perspective of the lamplighter, namely the person in charge of turning various lamps on and off. The generator s of $\mathbb{Z}$ means "move up the street", or equivalently "shift all lamps down" relatively to the lamplighter; and the generator t of $\mathbb{Z}_2$ means "change the status of the lamp" currently in front of the lamplighter", [Ba17].
This group is an example of a group with exponential growth which is still amenable. Furthermore, one interesting result of Baumslag [B61] confirms that wreath product $G \wr H$ is finitely presented if and only if G is trivial or H is finite. Therefore there are examples of finitely generated group but not finitely presented groups.

7.1.1 Invariants and coinvariants

In this part we recall the notions of invariants and coinvariants of a group action. These will be useful when it comes to describing the K-groups of $F \wr \mathbb{Z}$.

For a set X, we denote by $\mathbb{Z}X$ the free abelian group generated by X, which can be identified with the group of almost everywhere zero functions from X to $\mathbb{Z}$. Let M be a G-module. We denote by M^G the submodule of

G-invariants of the action and by $M_G = M/\langle m - g{\cdot}m \colon m \in M, g \in G\rangle$ the module of G-coinvariants. In fact, M^G and M_G are respectively the largest G-invariant submodule and G-invariant quotient. Here is a lemma identifying these two as free abelian groups.

Lemma 7.1.2. *Let G be a countable group, X be a countable G-set, and Y be the set of finite G-orbits in X. The space of invariants and coinvariants of $G \curvearrowright \mathbb{Z}X$ can be described as*

$$(\mathbb{Z}X)^G \cong \mathbb{Z}Y,$$
$$(\mathbb{Z}X)_G \cong \mathbb{Z}(G\backslash X).$$

Proof. Since a G-fixed function on X must be constant on G-orbits, the first claim is clear. For the second, let $\mathcal{R}$ be a system of representatives for G-orbits in X. Then the map

$$I \colon \mathbb{Z}X \to \mathbb{Z}(G\backslash X) \colon \phi \mapsto (r \mapsto \sum_{g \in G} \phi(g.r))$$

is onto with kernel precisely $\langle g\phi - \phi : g \in G, \phi \in \mathbb{Z}X\rangle$. $\square$

If R is a system of representatives for the G-orbits in X, as in the proof above, then

$$\mathbb{Z}X = \langle g\phi - \phi : g \in G, \phi \in \mathbb{Z}X\rangle \oplus \mathbb{Z}R.$$

Let F be a finite group. Take $\mathfrak{p} \in F$ and consider F as the pointed set $(F, \mathfrak{p})$. We denote by $F^{\mathbb{Z}}$ the countable set of maps $\mathbb{Z} \to F$ that are almost everywhere equal to $\mathfrak{p}$. In the notations of Lemma 7.1.2, we consider $X = F^{(\mathbb{Z})}$ with $G = \mathbb{Z}$ acting by the shift α. We describe a set of representatives for the orbits of α on X. For $g \in F$, we denote by χ_g the element in $F^{(\mathbb{Z})}$ taking the value g at $k = 0$ and $\mathfrak{p}$ everywhere else. For $n \geq 0$, $g, h \in F \setminus \{\mathfrak{p}\}$, and $\varepsilon \in F^n$, let $\chi_{g\varepsilon h}$ be the element of $F^{(\mathbb{Z})}$ defined by

$$\chi_{g\varepsilon h}(k) = \begin{cases} g & k = 0 \\ \varepsilon_k & k = 1, \dots, n \\ h & k = n+1 \\ \mathfrak{p} & \text{otherwise.} \end{cases}$$

Lemma 7.1.3. *A set $\mathcal{R}$ of representatives for the α-orbits on $F^{(\mathbb{Z})}$ is*

$$R = \{\chi_g \colon g \in F\} \cup \{\chi_{g\varepsilon h} \colon n \geq 0, \varepsilon \in F^n, g, h \in F \setminus \{\mathfrak{p}\}\}.$$

In particular, we have $\mathbb{Z}(F^{(\mathbb{Z})}) = \langle m - \alpha(m) \colon m \in \mathbb{Z}(F^{(\mathbb{Z})})\rangle \oplus \mathbb{Z}R$.

Proof. It is clearly seen that distinct elements of $\mathcal{R}$ belong to different orbits for the shift α. For $\phi \in F^{(\mathbb{Z})}$, let the support of ϕ, denoted by $\operatorname{supp}\phi$, be the set of integers k where $\phi_k \neq \mathfrak{p}$. If $\operatorname{supp}\phi$ is empty, then ϕ is the constant map with value $\mathfrak{p}$ (denoted by $\mathbf{1}_{\mathfrak{p}}$). If $\operatorname{supp}\phi$ has just one point, then ϕ is in the same orbit as some χ_g with $g \neq \mathfrak{p}$. If $\operatorname{supp}\phi$ has at least two points, then by shifting the minimum of $\operatorname{supp}\phi$ to 0 we get some $\chi_{g\varepsilon h}$. □

7.2 The Baum-Connes assembly map for finite groups

This short section is on the Baum-Connes conjecture for *finite* groups. Since this case will be the base for the identification we will do in the next two chapters, we spend a bit of time discussing this case.

Recall that a model for the classifying space of a finite group is a one-point space $\{pt\}$. Davis-Lück's description (DL) of the Baum-Connes assembly map for a finite group F, tautologically provides us with an isomorphism

$$\mathrm{K}_0^F(F/F) = \mathrm{K}_0^F(\{pt\}) \simeq \mathrm{K}_0(\mathrm{C}^*_\mathrm{r}(F)) = \mathrm{K}_0(\mathbb{C}F).$$

However this does not allow for further computations. In order to overcome this problem we need an identification of $\mathrm{K}_0(\mathrm{C}^*_\mathrm{r}(F))$ and the representation ring $\mathrm{R}_\mathbb{C}$. Let us explain it here. Take an irreducible representation $[\pi] \in \hat{F} \subset \mathrm{R}_\mathbb{C}(F)$, therefore $\pi\colon F \to \mathcal{U}(\mathbb{C}^n)$ is a unitary representation of dimension n for some $n \in \mathbb{N}$. Using the universal property we get a unique homomorphism $\mathbb{C}F \to \mathcal{B}(\mathbb{C}^n)$. This makes $\mathbb{C}^n$ a $\mathbb{C}F$-module. This is a semi-simple algebra, as $\mathbb{C}F$ is a direct sum of matrix algebras. Therefore $\mathbb{C}^n$ is a finitely generated projective $\mathbb{C}F$-module, which in turn provides us with a projection in K_0. Because we started with an *irreducible* representation π, the associated projection we obtained is *minimal.* In short we have $[\pi] \mapsto [e_\pi]$, where $[e_\pi]$ is a minimal projection. Putting these in a diagram, we get that

$$\begin{tikzcd}
\mathrm{K}_0^F(\{pt\}) \arrow[r, "="] \arrow[d] & \mathrm{K}_0^F(\{pt\}) \arrow[d, "\text{def. DL}"] \\
\mathrm{R}_\mathbb{C}(F) \arrow[r, "{[\pi]\mapsto[e_\pi]}"'] & \mathrm{K}_*(\mathrm{C}^*_\mathrm{r}(F)).
\end{tikzcd}$$

This way of describing the assembly map is compatible with the KK-picture of the assembly map. See [MV03, Example II.2.11] for an alternative description using KK-theory.

7.3 The Baum-Connes assembly map for locally finite groups

In this section we write down explicitly the Baum-Connes conjecture for a locally finite group B. Understanding of this case will be very useful. This is because the locally finite subgroup $B = \bigoplus_{\mathbb{Z}} F$ of $F \wr \mathbb{Z}$ which is the torsion subgroup of this group, plays a crucial role is our computations.

Definition 7.3.1. (*locally finite group*)
A group G is called locally finite if it is the direct limit of a system of finite groups for which all connecting maps are injective, see Section 2.1.1.

Let H be a subgroup of the finite group F, denote by $\iota\colon H \to F$ the natural inclusion. Due to functoriality of the assembly map we have a commutative diagram

$$\begin{tikzcd}
\mathrm{R}_{\mathbb{C}}(H) \arrow[r, "\mathrm{Ind}_H^F"] \arrow[d, "\mu_0^H"'] & \mathrm{R}_{\mathbb{C}}(F) \arrow[d, "\mu_0^F"] \\
\mathrm{K}_0(\mathrm{C}^*H) \arrow[r, "\iota_0"] & \mathrm{K}_0(\mathrm{C}^*H).
\end{tikzcd}$$

Let B be a (countable) locally finite group. Write it as $B = \operatorname{colim} B_n$ of an increasing sequences of finite subgroups B_n. Due to Corollary I.5.7 and Theorem I.5.2 in [MV03], both equivariant K-homology and the assembly map are continuous that means we have

$$\mathrm{K}_\mathrm{i}^\mathrm{B}(\underline{\mathrm{E}}B) = \operatorname{colim} \mathrm{K}_\mathrm{i}^{\mathrm{B_n}}(\underline{\mathrm{E}}B_n) \qquad \text{and} \qquad \mu_i^{B_n} = \operatorname{colim} \mu_i^{B_n}.$$

In particular, the Baum-Connes conjecture holds for B. We will elucidate the colimits appearing on both sides. But first we describe a model for $\underline{\mathrm{E}}B$ which is a tree, the so-called tree of filtration.
We consider the filtration of B over the natural numbers $\mathbb{N}$. This means that we associate to each natural number $n > 0$, the quotient group B/B_n. The group B acts via left-multiplication on each of these quotients. In each of these fibres, there exists a distinguished coset, namely eB_n. The stabiliser of B_n is itself hence finite and for all other cosets bB_n in the fibre, the stabiliser is conjugate to B_n, hence finite. Let $\mathbb{V}$ denote the set of vertices and $\mathbb{E}$ denote the set of edges of a tree.

- $\mathbb{V} = \mathbb{E} = \coprod_{n>0} B/B_n$
- The edge gB_n connects the vertices gB_n and gB_{n+1} for $g \in B$.

Being a tree obviously implies that it is contractible. Moreover, the group B acts on T via left-multiplication. In fact this action is proper, because the vertex stabilisers are finite, see Lemma 3.1.10. For a finite subgroup H of B, there is $m \in \mathbb{N}$ such that H is included in B_m. This implies that T^H is a subtree hence contractible. Therefore, this tree is a model for $\underline{E}B$. The following figure demonstrates part the tree when $[B_{n+1} \colon B_n] = 2$.

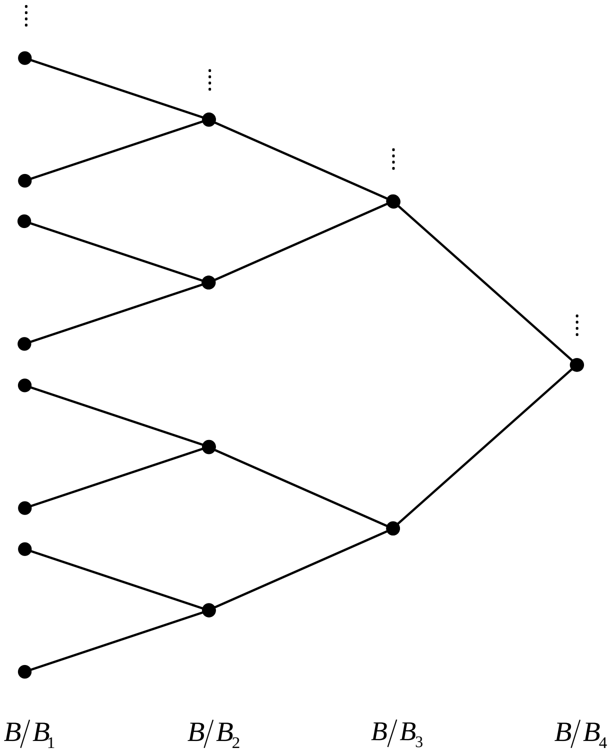

FIGURE 1 : $\underline{E}B$, for $[B_{k+1} : B_k] = 2$

Now that we understood the classifying space $\underline{\mathrm{E}}B$, we can describe its equivariant-K-homology. Here is a lemma that we will use.

Lemma 7.3.2. *([FlWi13, lemma 4.3])*
Let X be a G-CW-complex and let $(X_\alpha)_{\alpha\in I}$ be a filtration of X by G-invariant subcomplexes. Then the inclusions $X_\alpha \hookrightarrow X$ induce an isomorphism

$$\varinjlim_{\alpha} \mathrm{H}_i^{\mathfrak{F}}(X_\alpha, M) = \mathrm{H}_i^{\mathfrak{F}}(X, M),$$

for all left $\mathfrak{O}_{\mathfrak{F}}G$-modules M.

Proposition 7.3.3. *Let $(F_n)_{n>0}$ be collection of finite groups, let $B_n = \bigoplus_{k=1}^{n} F_k$, and let $B = \bigoplus_{k=1}^{\infty} F_k$. Consider $\hat{B}_{n+1} = \hat{B}_n \times \hat{F}_{n+1}$ and the embedding*

$$\begin{aligned} s_n\colon \hat{B}_n &\to \hat{B}_{n+1}\\ \pi_n &\mapsto \pi_n \otimes 1_{F_{n+1}}, \end{aligned}$$

where 1_F denotes the trivial one-dimensional representation of F_n. The free abelian group on the colimit of the system of sets $(\hat{B}_n, s_n)_{n>0}$ is naturally isomorphic to $\mathrm{K}_0^{\mathrm{B}}(\underline{\mathrm{E}}B)$.

Proof. The classifying space $\underline{\mathrm{E}}B$ can be described as an increasing union of models for $\underline{\mathrm{E}}B_n$, which are B-subcomplexes. Due to 7.3.2 we have that

$$\varinjlim \mathrm{H}_i^{\mathfrak{F}}(B_n; \mathrm{R}_{\mathbb{C}}) = \mathrm{H}_i^{\mathfrak{F}}(B; \mathrm{R}_{\mathbb{C}}).$$

But B_n are finite, therefore $\mathrm{H}_i^{\mathfrak{F}}(B; \mathrm{R}_{\mathbb{C}}) = 0$ for $i \geq 1$. By Theorem 3.6.7 then we have that $\mathrm{H}_0^{\mathfrak{F}}(B; \mathrm{R}_{\mathbb{C}}) = \mathrm{K}_0^{\mathrm{B}}(\underline{\mathrm{E}}B)$. Viewing B as the fundamental group of ray of groups (obtained by the quotient of the tree by B) and appealing to Theorem 3.5.4 we have that $\mathrm{H}_0^{\mathfrak{F}}(B; \mathrm{R}_{\mathbb{C}})$ is the co-kernel of

$$\begin{aligned} f\colon \bigoplus_{n>0} \mathrm{R}_{\mathbb{C}}(B_n) &\longrightarrow \bigoplus_{n>0} \mathrm{R}_{\mathbb{C}}(B_n)\\ (\pi_n) &\mapsto (\pi_n, -\mathrm{Ind}_{B_n}^{B_{n+1}} \pi_n). \end{aligned}$$

Let λ_{F_n} denote the regular representation of a F_n, by an application of *Frobenius reciprocity*

$$\mathrm{Ind}_{B_n}^{B_{n+1}} \pi_n = \pi_n \otimes \lambda_{F_n} = \sum_{\sigma\in\hat{F}_{n+1}} \dim\sigma \cdot (\pi_n \otimes \sigma).$$

As $\hat{B}_{n+1} = \hat{B}_n \times \hat{F}_{n+1}$, denoting by $r_n \colon \hat{B}_{n+1} \to \hat{B}_n$ the projection on the first factor, we may re-write

$$\operatorname{Ind}_{B_n}^{B_{n+1}} \pi_n = \sum_{\pi_{n+1} \in \hat{B}_{n+1} \colon r_n(\pi_{n+1}) = \pi_n} \frac{\dim \pi_{n+1}}{\dim r_n(\pi_{n+1})} \cdot \pi_n.$$

We may view $\bigoplus_{n>0} \mathrm{R}_{\mathbb{C}}(B_n)$ as $\mathbb{Z}(\coprod_{n>0} \hat{B}_{n+1})$, the group of integer-valued functions with finite support on the disjoint union $\coprod_{n>0} \hat{B}_{n+1}$. In this picture we have that for $\phi \in \mathbb{Z}(\coprod_{n>0} \hat{B}_{n+1})$

$$(f(\phi))(\pi_n) = \phi(\pi_n) - \phi(r_{n-1}(\pi_n)) \cdot \frac{\dim \pi_{n+1}}{\dim r_n(\pi_{n+1})} \cdot \pi_n, \tag{7.3.1}$$

where $\pi_n \in \hat{B}_n$. Consider now the subgroup

$$H =\colon \mathbb{Z}(\coprod_{n>0} (\hat{B}_n \setminus s_{n-1}(\hat{B}_{n-1}))).$$

The proposition then follow from the following claim.
Claim. $\operatorname{Im} f \oplus H = \mathbb{Z}(\coprod_{n>0} \hat{B}_n)$ *which implies* $H \cong \operatorname{Coker} f$.
To prove the claim, first we show that $\operatorname{Im} f + H = \mathbb{Z}(\coprod_{n>0} \hat{B}_n)$. To do so we use induction over n that every $\pi_n \in \hat{B}_n$, viewed as a basis element of $\mathbb{Z}(\coprod_{n>0} \hat{B}_n)$ is in $\operatorname{Im} f + H$.This is clear for $n = 1$, as $\mathbb{Z}(\hat{B}_1) \subset H$, it is also clear if $\partial_n \notin s_{n-1}(\hat{B}_{n-1})$ as then $\pi_n \in H$. So assume that $\pi_n = s_{n-1}(\pi_{n-1}) = \pi_{n-1} \otimes 1_F$, for some $\pi_{n-1} \in \hat{B}_{n-1}$. Then

$$f(\pi_{n-1}) = \pi_{n-1} - \pi_n - \sum_{\sigma \in \hat{F}_n, \sigma \neq 1_{F_n}} \dim \sigma \cdot (\pi_{n-1} \otimes \sigma),$$

or

$$\pi_n = \pi_{n-1} - f(\pi_{n-1}) - \sum_{\sigma \in \hat{F}_n, \sigma \neq 1_{F_n}} \dim \sigma \cdot (\pi_{n-1} \otimes \sigma).$$

Using the induction hypothesis for π_{n-1}, the right-hand side belongs to $\operatorname{Im} f + H$. It remains to show $\operatorname{Im} \cap H = \{0\}$. Let $\phi \in \mathbb{Z}(\coprod_{n>0} \hat{B}_n)$ be such that $f(\phi)$ vanishes on $\coprod_{n>0} s_n(\hat{B}_n)$. We prove that ϕ is identically zero. Using the identity $r_n \circ s_n = \mathrm{Id}_{\hat{B}_n}$ and the formula 7.3.1, for $\pi \in \hat{B}_n$ we have that

$$0 = f(\phi)(s_n(\pi_n)) = \phi(s_n(\pi_n)) - \phi(\pi_n),$$

that is $\phi(s_n(\pi_n)) = \phi(\pi_n)$. Iterating, we have for every $k > 0$

$$\pi(s_{n+k}(\cdots(s_n(\pi_n)))) = \phi(\pi_n).$$

Since ϕ has finite support, then for large enough k we get that $\pi(s_{n+k}(\cdots(s_n(\pi_n)))) = \phi(\pi_n) = 0$, hence $\phi(\pi_n) = 0$. □

Using this description of the colimit on the left-hand side of the assembly map for B, we can describe dually the colimit on the right-hand side. Let us first fix some notations. For a finite group F, we denote by $\mathrm{Min}(\mathbb{C}F)$ the set of Murray-von Neumann equivalence classes of minimal projections in the group ring $\mathbb{C}F$. If we write $\mathbb{C}F = \bigoplus_{\pi\in\hat{F}} \mathrm{M}_{\dim\pi}(\mathbb{C})$, then $\mathrm{Min}(\mathbb{C}F)$ can be realised by picking a rank one projection in each direct summand.
Note that if (S, s) is a pointed set, then we may define $S^{(\mathbb{Z})}$ as the set of maps $\mathbb{Z} \to S$ taking the value s for almost every integer, in turn $(S^{(\mathbb{Z})}, 1_s)$ is a pointed set with respect to the constant map with value s. When F is a finite group, the dual $\hat{F}$ is pointed by the trivial representation 1_F, and the set $\mathrm{Min}(\mathbb{C}F)$ by the minimal projection associated with the trivial representation, $p_F = \frac{1}{|F|}\sum_{f\in F} f$.
Consider the setting of the Proposition 7.3.3. At the level of group ring, analogously to the map s_n we consider the non-unital, injective homomorphism

$$\begin{aligned} j_n \colon \mathbb{C}B_n &\to \mathbb{C}B_n \otimes \mathbb{C}F_{n+1} = \mathbb{C}B_{n+1} \\ x &\mapsto x \otimes p_{F_{n+1}}, \end{aligned}$$

such that

$$(j_n)_* \circ \mu_0^{B_n} = \mu_0^{B_{n+1}} \circ s_n.$$

Then $j_n(\mathrm{Min}(\mathbb{C}B_n)) \subset \mathrm{Min}(\mathbb{C}B_{n+1})$ and $\mathrm{K}_0(\mathrm{C}^*B)$ appears as the free abelian group on the co-limit of the system of sets $(\mathrm{Min}(\mathbb{C}B_n), j_n)_{n>0}$.
When all groups F_n are equal to the same finite group F, our computations immediately yield:

Corollary 7.3.4. *Let F be a non-trivial finite group, and let $B = \oplus_{\mathbb{Z}} F$.*

1. *The free abelian group $\mathbb{Z}(\hat{F}^{(\mathbb{Z})})$ is isomorphic to $\mathrm{K}_0^{\mathrm{B}}(\underline{\mathrm{E}}B)$ via*

   $$\pi \in \hat{F}^{(\mathbb{Z})} \longmapsto \otimes_{k\in\operatorname{supp}\pi}\pi_k \in R_{\mathbb{C}}(\oplus_{k\in\operatorname{supp}\pi} F_k) \subset \mathrm{K}_0^{\mathrm{B}}(\underline{\mathrm{E}}B),$$

   (where $\pi = (\pi_k)_k$ and $\pi_k \in \mathrm{R}_{\mathbb{C}}(F_k)$).

2. *The free abelian group $\mathbb{Z}(\mathrm{Min}(\mathbb{C}F)^{(\mathbb{Z})})$ is isomorphic to $\mathrm{K}_0(\mathrm{C}^*B)$ via*

   $$p \in \mathrm{Min}(\mathbb{C}F)^{(\mathbb{Z})} \longmapsto [\otimes_{k\in\operatorname{supp} p} p_k] \in \mathrm{Min}(\mathbb{C}(\oplus_{k\in\operatorname{supp} p} F_k)) \subset \mathrm{K}_0(\mathrm{C}^*(B)),$$

   and in particular the trivial map $1_{p_F} \in \mathrm{Min}(\mathbb{C}F)^{(\mathbb{Z})}$ with the constant value p_F is mapped to $[1] \in \mathrm{K}_0(\mathrm{C}^*(B))$.

3. *For $\pi \in \hat{F}^{(\mathbb{Z})}$, we have $\mu_0^B(\pi) = \otimes_{k\in\operatorname{supp}\pi}\mu_0^F(\pi_k) \in \mathrm{K}_0(\mathrm{C}^*(B))$.*

7.4 Analytical side: K-theory of $\mathrm{C}^*(F \wr \mathbb{Z})$

In this section we describe the analytical side of the assembly map of $L = F \wr \mathbb{Z}$ by computing the K-theory of its C*-algebra. Let $B = \bigoplus_{\mathbb{Z}} F$. Viewing the group as a semi-direct product $B \rtimes \mathbb{Z}$, its C*-algebra is the crossed product $\mathrm{C}^*(B) \rtimes_\alpha \mathbb{Z}$. Having such a crossed product the Pimsner-Voiculescu 6-term exact sequence 2.3.1 is the main tool to describe the K-theory. Let us explain the induced action $\alpha\colon \mathbb{Z} \curvearrowright \mathrm{C}^*(B)$. The action $\alpha\colon \mathbb{Z} \curvearrowright B$ is defined by

$$(\alpha(x_n)_{n\in\mathbb{Z}}))_k = x_{k+1}, \qquad \forall k \in \mathbb{Z}, (x_n)_{n\in\mathbb{Z}} \in B.$$

Now this induces the following action on the unitaries spanning the group C*-algebra $\mathrm{C}^*(B)$

$$(\alpha((u_{x_n})_{n\in\mathbb{Z}}))_k = u_{x_{k+1}}, \qquad \forall k \in \mathbb{Z}, (u_{x_n})_{n\in\mathbb{Z}} \in \mathbb{C}B.$$

Note that both groups B and $B \rtimes \mathbb{Z}$ are amenable hence their reduced group C*-algebras coincide with the universal one. Therefore we may define $\mathrm{C}^*(L) = \mathrm{C}^*(B) \rtimes \mathbb{Z}$ abstractly as the universal C*-algebra generated by $\mathrm{C}^*(B)$ and u satisfying the relation $\alpha(x) = uau^{-1}$ for $x \in \mathrm{C}^*(B)$.

Theorem 7.4.1. *Let* $L = B \rtimes_\alpha \mathbb{Z}$*.*

$\mathrm{K}_0(\mathrm{C}^*(L)) = \mathbb{Z}R$ *with* R *a countable basis indexed by representatives for* $\mathbb{Z}$*-orbits in* $\mathrm{Min}(\mathbb{C}F)^{(\mathbb{Z})}$*,*

$\mathrm{K}_1(\mathrm{C}^*(L)) = \mathbb{Z}[u]$*, where* $[u]$ *is the unitary implementing the action* α*.*

Proof. We plug K-theoretic information of $\mathrm{C}^*(B)$ into the Pimsner-Voiculescu 6-term exact sequence; by Corollary 7.3.4, $\mathrm{K}_0(\mathrm{C}^*(B))$ is identified, α-equivariantly, with $\mathbb{Z}((\mathrm{Min}(\mathbb{C}F))^{(\mathbb{Z})})$. Since $\mathrm{C}^*(B)$ is an AF-algebra, its K_1-group is zero. Consider

$$\begin{tikzcd}
\mathbb{Z}(\mathrm{Min}(\mathbb{C}F)^{(\mathbb{Z})}) \arrow[r, "\mathrm{Id}-\alpha_*"] & \mathbb{Z}(\mathrm{Min}(\mathbb{C}F)^{(\mathbb{Z})}) \arrow[r, "\iota_*"] & \mathrm{K}_0(\mathrm{C}^*(L)) \arrow[d] \\
\mathrm{K}_1(\mathrm{C}^*(L)) \arrow[u, "\partial_1"] & 0 \arrow[l] & 0. \arrow[l]
\end{tikzcd}$$

By injectivity of ∂_1 and exactness of the sequence we have

$$\mathrm{K}_1(\mathrm{C}^*(L)) \simeq \mathrm{Ker}(\mathrm{Id} - \alpha_*) = \mathbb{Z}\,\mathbf{1}_{p_F},$$

where the constant map $\mathbf{1}_{p_F}$ corresponds to the class $[1] \in \mathrm{K}_0(\mathrm{C}^*(B))$, due to Corollary 7.3.4. Moreover, Lemma 6.3.1 yields that $\partial_1([u]) = -[1]$, hence we get $\mathrm{K}_1(\mathrm{C}^*(L)) = \mathbb{Z}[u]$. In order to compute $K_0(\mathrm{C}^*(L))$, we focus on the right-hand side of the diagram. Surjectivity of ι_* implies that

$$\mathrm{K}_0(\mathrm{C}^*(L)) \simeq \mathbb{Z}((\mathrm{Min}(\mathbb{C}F))^{(\mathbb{Z})})/\mathrm{Ker}\,\iota_*.$$

Exactness of the diagram yields that $\mathrm{Ker}\,\iota_* = \mathrm{Im}(\mathrm{Id} - \alpha_*)$. Hence $\mathrm{K}_0(\mathrm{C}^*(L))$ is the set of coinvariants of the shift in $\mathbb{Z}((\mathrm{Min}(\mathbb{C}F))^{(\mathbb{Z})})$, which is described by Lemma 7.1.2 and Lemma 7.1.3. □

7.5 Topological side: Equivariant K-homology

We describe the equivariant K-homology of $\underline{\mathrm{E}}F \wr \mathbb{Z}$ via computing the Bredon homology of this classifying space. In order to establish a clear picture of the left-hand side of the assembly map for this group, we provide a concrete model for the classifying space $\underline{\mathrm{E}}L$.

7.5.1 A 2-dimensional model for $\underline{\mathrm{E}}F \wr \mathbb{Z}$

Consider a group extension of H by $\mathbb{Z}$. Let $G = H \rtimes \mathbb{Z}$. M. Fluch in [Flu11] constructs a model for the classifying space of G in terms of the one for H such that if $\dim \underline{\mathrm{E}}H$ is n, then $\underline{\mathrm{E}}G$ admits an $(n+1)$-dimensional one. In Section 7.3 we described $\underline{\mathrm{E}}B$ as a tree, hence of dimensional one. Fluch's construction therefore provides us with a two-dimensional model for $\underline{\mathrm{E}}F \wr \mathbb{Z}$. Remark that Lück's result, Theorem 3.2.4, also provides us with a 2-dimensional model for this group. However, we prefer the one of Fluch as it is more visual. Let us explain this model here concretely. As before, let B be the colimit of an increasing sequence $(B_n)_{n>0}$ of finite groups. Recall from 7.3 that $\underline{\mathrm{E}}B$ is a tree X, with vertex and edge sets $\mathbb{V} = \mathbb{E} = \coprod_{n>0} B/B_n$ such that xB_n is an edge between cosets xB_n and xB_{n+1}. In general,

$$xB_n \sim yB_{n+1} \quad \text{iff} \quad xB_n \subset yB_{n+1}.$$

Let α be an automorphism of B such that that $\alpha^{-1}(B_n) \subset B_{n+1}$ for every n.

Example 7.5.1. *Let α be the shift on $B = \oplus_{\mathbb{Z}} F$, with F a finite group. Set $B_n =: \oplus_{k=-n}^{n} F$: then B is the co-limit of the B_n's, and $\alpha^{-1}(B_n) \subset B_{n+1}$.*

For $k \in \mathbb{Z}$, let X_k be the tree X with a twisted B-action:

$$g \cdot_k xB_n = \alpha^{-k}(g) xB_n \qquad g, x \in B.$$

Let $f_k : X_k \to X_{k+1}$ be defined by $f_k(xB_n) = \alpha^{-1}(x)B_{n+1}$. It follows from our assumption that f_k is well-defined, it is B-equivariant

$$f_k(g \cdot_k xB_n) = g \cdot_{k+1} f_k(xB_n),$$

and maps edges of X_k to edges of X_{k+1}.

Now we define the space Y as a "*mapping telescope*"

$$Y = (\coprod_{k\in\mathbb{Z}} (X_k \times [k, k+1]))/\sim,$$

where $(xB_n, k+1)$ in the component $X_k \times [k, k+1]$ is identified with $(f_k(xB_n), k+1)$ in the component $X_{k+1} \times [k+1, k+2]$. Since f_k is B-equivariant, the B-actions on the components glue together to yield a B-action on Y. We call the sub-complex X_k the *k-th level* of Y. We also have a "vertical" action of $\mathbb{Z}$ on Y defined by

$$\sigma(xB_n, t) = (xB_n, t+1).$$

It is easily checked that this action is well-defined, and free with fundamental domain $D =: X_0 \times [0, 1[$. Moreover, for $(xB_n, t) \in Y$ with $k \leq t \leq k+1$ and $g \in B$, we have

$$\begin{aligned} \sigma \circ g \circ \sigma^{-1}(xB_n, t) &= \sigma \circ g\,(xB_n, t-1) = \sigma(\alpha^{1-k}(g)xB_n, t-1) \\ &= (\alpha^{1-k}(g)xB_n, t) = \alpha(g)(xB_n, t). \end{aligned}$$

So $\sigma \circ g \circ \sigma^{-1} = \alpha(g)$, that is the B-action and $\mathbb{Z}$-action on Y combine into an action of $L = B \rtimes \mathbb{Z}$. Since vertex stabilizers are clearly finite, due to Lemma 3.1.10 the action of L on Y is proper.

Proposition 7.5.2. *The space Y just defined is a two-dimensional model for $\underline{\mathrm{E}}L$.*

Proof. Let H be a subgroup of L. Following Theorem 3.2.3, we must look at the fixed point set Y^H and prove that it is empty if H is infinite, and contractible if H is finite. There are two possibilities.

- H is not contained in B (implying that H is infinite): then $Y^H = \emptyset$.
- H is contained in B. Then a point $(xB_n, t) \in \gamma^k(D)$ is H-fixed if and only if $H \subset \alpha^k(xB_nx^{-1})$. This already implies that $Y^H = \emptyset$ if H is infinite.
  Assume that H be finite then. We follow the proof in [Flu11, Lemma 5.10]. As Y^H is a CW-complex, Whitehead's Theorem [W78, pp.219ff], guarantees that it suffices to show that it is weakly contractible, i.e. $\pi_n(Y^H)$ is trivial for every $n \geq 0$. The space X^H is a subtree of X hence contractible, and so is

$$X_k^H = X^{\alpha^k(H)} \quad \forall k \in \mathbb{Z}.$$

Therefore, Y^H is a mapping telescope of a collection of contractible spaces X_k^H. Now, every representative $S^n \to Y^H$ of an element on $\pi_n(Y^H)$ is contained, by compactness, in a finite subtelescope of Y^H, which retracts by deformation in X_k^H for $k \gg 0$. Hence, $\pi_n(Y^H)$ is trivial and Y^H weakly contractible, thus contractible.

□

7.5.2 Computations on the left-hand side

We describe the equivariant K-homology of $\underline{E}F \wr \mathbb{Z}$ via computing the Bredon homology of this classifying space with coefficients in $\mathrm{R}_{\mathbb{C}}$. As we are dealing with the semidirect product $L = \oplus_{\mathbb{Z}} F \rtimes \mathbb{Z}$, the equivariant Lyndon-Hochschild-Serre spectral sequence of Section 3.6.2 is our main tool for computing the Bredon homology. Respecting the notations of that section, N is the locally finite subgroup B, $\bar{G}$ is $\mathbb{Z}$ and M is $\mathrm{R}_{\mathbb{C}}$. As before $\mathfrak{F}$ and η are the family of finite subgroups of respectively G and $\mathbb{Z}$, where the latter one contains only the trivial subgroup. Therefore the preimage of $\bar{\eta}$ has only one element, namely B. The E^2-page of the spectral sequence in Theorem 3.6.9 with respect to this extension is

$$\mathrm{E}^2_{p,q} = \mathrm{H}_p(\mathbb{Z}; \mathrm{H}_q^{\mathfrak{F}\cap B}(B; \mathrm{R}_{\mathbb{C}})). \tag{7.5.1}$$

We should compute all these groups. Note that the dimension of $\mathbb{Z}$ is one as the real line is a model for $\underline{E}\mathbb{Z}$. This implies that $\mathrm{E}^2_{p,q}$ vanishes whenever $p \geq 2$. Moreover, due to Theorem 3.6.7 the Bredon homology of B with coefficient in $\mathrm{R}_{\mathbb{C}}$ is non trivial only in zero-degree, which in turn implies that $\mathrm{E}^2_{p,q} = 0$ for $q \geq 1$. As a matter of fact, the only non trivial terms of the second page are $\mathrm{E}^2_{0,0}$ and $\mathrm{E}^2_{1,0}$. In particular, the E^2-page is concentrated in the horizontal axis, therefore there are no differentials and the spectral sequence collapses in this page. This yields that

$$\mathrm{E}^\infty_{p,q} = \mathrm{E}^2_{p,q} \qquad p, q \geq 0.$$

Due to Theorem 3.6.9, the spectral sequence converges to $\mathrm{H}^{\mathfrak{F}}_{p+q}(G; \mathrm{R}_{\mathbb{C}})$. Now, as the only non-zero terms of the limit of a sequence are $\mathrm{E}^\infty_{0,0}$ and $\mathrm{E}^\infty_{1,0}$, again by the equation in 7.5.1 we have

$$\begin{aligned} \mathrm{H}^{\mathfrak{F}}_0(G; \mathrm{R}_{\mathbb{C}}) &\simeq \mathrm{E}^\infty_{0,0} \simeq \mathrm{E}^2_{0,0} \cong \mathrm{H}_0(\mathbb{Z}; \mathrm{H}_0^{\mathfrak{F}\cap B}(B; \mathrm{R}_{\mathbb{C}})), \\ \mathrm{H}^{\mathfrak{F}}_1(G; \mathrm{R}_{\mathbb{C}}) &\simeq \mathrm{E}^\infty_{1,0} \simeq \mathrm{E}^2_{1,0} \cong \mathrm{H}_1(\mathbb{Z}; \mathrm{H}_0^{\mathfrak{F}\cap B}(B; \mathrm{R}_{\mathbb{C}})). \end{aligned} \tag{7.5.2}$$

Therefore the Bredon homology computations reduce to computations in ordinary homology with coefficients. In view of Example 1 on page 58 of [Bro82] the homological groups can be realised as

$$\begin{aligned}\mathrm{H}_0(\mathbb{Z};\mathrm{H}_0^{\mathfrak{F}\cap B}(B;\mathrm{R}_{\mathbb{C}})) &= \text{coinvariants of the action } \mathbb{Z}\curvearrowright \mathrm{H}_0^{\mathfrak{F}\cap B}(B;\mathrm{R}_{\mathbb{C}}),\\ \mathrm{H}_1(\mathbb{Z};\mathrm{H}_0^{\mathfrak{F}\cap B}(B;\mathrm{R}_{\mathbb{C}})) &= \text{invariants of the action } \mathbb{Z}\curvearrowright \mathrm{H}_0^{\mathfrak{F}\cap B}(B;\mathrm{R}_{\mathbb{C}}).\end{aligned} \tag{7.5.3}$$

Next we exhibit the equivariant K-homological groups of L in term of the action.

Theorem 7.5.3. *Let $L = F \wr \mathbb{Z}$ be the lamplighter group over a (non-trivial) finite group F.*

$\mathrm{K}_0^L(\underline{\mathrm{E}}L) = \mathbb{Z}R'$ *with R' a countable basis indexed by representatives for $\mathbb{Z}$-orbits in $\hat{F}^{(\mathbb{Z})}$.*

$\mathrm{K}_1^L(\underline{\mathrm{E}}L) = \mathbb{Z}[v]$*, where $[v]$ is the canonical generator of* $\mathrm{K}_1^{\mathbb{Z}}(\underline{\mathrm{E}}\mathbb{Z})$*.*

Proof. By computations involving the equivariant Lyndon-Hochschild-Serre spectral sequence 3.6.2, we have that

$$\mathrm{H}_2^{\mathfrak{F}}(L;\mathrm{R}_{\mathbb{C}}) = 0.$$

Therefore applying Theorem 3.6.7 to our lamplighter group implies that $\mathrm{H}_{\mathrm{i}}^{\mathfrak{F}}(L;\mathrm{R}_{\mathbb{C}}) \simeq \mathrm{K}_i^L(\underline{\mathrm{E}}L)$ for $i = 0, 1$. By equations in 7.5.3, these groups can be viewed as invariants and coinvariants of the left shift on $\mathrm{H}_0^{\mathfrak{F}\cap B}(B;\mathrm{R}_{\mathbb{C}})$. This group is in turn isomorphic to $\mathbb{Z}\hat{F}^{(\mathbb{Z})}$ due to Corollary 7.3.4. Thanks to Lemma 7.1.3 K_0 is the free abelian group on the countable set of representatives for the orbits and K_1 is infinite cyclic group with the generator being the fixed point of the action. This fixed point is the constant map $\mathbf{1}_F$ which is the base point of $\hat{F}$. The last step is to identify this element with the canonical generator of $\mathrm{K}_1^{\mathbb{Z}}(\underline{\mathrm{E}}\mathbb{Z})$. To do so we need to consider several identifications.

$$\begin{aligned} K_1^{\mathbb{Z}}(\underline{\mathrm{E}}\mathbb{Z}) \simeq \mathrm{K}_1(\mathrm{B}\mathbb{Z}) \simeq \mathrm{K}_1(\mathrm{S}^1) &\qquad \mathbb{Z}\,\text{is torsion-free},\\ K_1^{\mathbb{Z}}(\underline{\mathrm{E}}\mathbb{Z}) \simeq \mathrm{H}_1(\underline{\mathrm{E}}\mathbb{Z};\mathbb{Z}\hat{F}^{(\mathbb{Z})}) &\qquad \text{Theorem}\,3.6.7,\\ \mathrm{H}_1(\underline{\mathrm{E}}\mathbb{Z};\mathbb{Z}) \to \mathrm{H}_1(\underline{\mathrm{E}}\mathbb{Z};\mathbb{Z}\hat{F}^{(\mathbb{Z})}) &\qquad \text{induced by} \quad \mathbb{Z}\to\mathbb{Z}\hat{F}^{(\mathbb{Z})}: 1\mapsto \mathbf{1}_F.\end{aligned}$$

Considering the composition

$$K_1^{\mathbb{Z}}(\underline{\mathrm{E}}\mathbb{Z}) \xrightarrow{\simeq} \mathrm{K}_1(\mathrm{S}^1) \xrightarrow{\simeq} \mathrm{H}_1(\underline{\mathrm{E}}\mathbb{Z};\mathbb{Z}) \longrightarrow \mathrm{H}_1(\underline{\mathrm{E}}\mathbb{Z};\mathbb{Z}\hat{F}^{(\mathbb{Z})}) \xrightarrow{\simeq} \mathrm{K}_1^{\mathrm{L}}(\underline{\mathrm{E}}L).$$

the canonical generator $[v]$ of $\mathrm{H}_1(\underline{\mathrm{E}}\mathbb{Z};\mathbb{Z})$ is mapped to $\mathbf{1}_F$. □

7.6 Summary of computations

Here is a summary of the computations of this chapter.
Computations for the assembly map of a finite group F.

	topological side	analytical side
K_0	$\hat{F}$	$\operatorname{Min}\mathbb{C}F$
K_1	0	0

Computations for the assembly map of the locally finite group $\oplus_{\mathbb{Z}}F$.

	topological side	analytical side
K_0	$\mathbb{Z}\hat{F}^{(\mathbb{Z})}$	$\mathbb{Z}(\operatorname{Min}(\mathbb{C}F))^{(\mathbb{Z})}$
K_1	$\mathbb{Z}$	$\mathbb{Z}$

Computations for the assembly map of the lamplighter group $F \wr \mathbb{Z}$.

	topological side	analytical side
K_0	coinv. of $\mathbb{Z} \curvearrowright \hat{F}^{(\mathbb{Z})}$	coinv. of $\mathbb{Z} \curvearrowright (\operatorname{Min}(\mathbb{C}F))^{(\mathbb{Z})}$
K_1	$\mathbb{Z}$	$\mathbb{Z}$

7.7 The Baum-Connes conjecture for $F \wr \mathbb{Z}$

In this chapter we assemble our K-theoretic computations in order to describe the assembly map of the lamplighter group $L = F \wr \mathbb{Z}$. In doing so, we reprove the Baum-Connes conjecture for this group.

Theorem 7.7.1. *Let F be a non-trivial finite group and let $L = F \wr \mathbb{Z}$ be the lamplighter group over group F.*

The assembly map $\mu_0^L \colon \mathrm{K}_0^L(\underline{\mathrm{E}}L) \to \mathrm{K}_0(\mathrm{C}^*(L))$ is an isomorphism between two countably generated free abelian groups.

The assembly map $\mu_1^L \colon \mathrm{K}_1^L(\underline{\mathrm{E}}L) \to \mathrm{K}_1(\mathrm{C}^*(L))$ is an isomorphism between two infinite cyclic groups.

Proof. *μ_0^L is an isomorphism.* We already showed in Theorem 7.4.1 and Theorem 7.5.3 that both K_0-groups are free abelian on countably many generators. As before let $B = \oplus_{\mathbb{Z}}F$ and consider the inclusion $\iota \colon B \hookrightarrow L$. Consider the diagram

$$\begin{tikzcd}
\mathrm{K}_0^{\mathrm{B}}(\underline{\mathrm{E}}B) \arrow[r, "\text{Id-}\alpha_*"] \arrow[d, "\mu_0^B"'] & \mathrm{K}_0^{\mathrm{B}}(\underline{\mathrm{E}}B) \arrow[r, "\iota_*"] \arrow[d, "\mu_0^B"'] & \mathrm{K}_0^{\mathrm{L}}(\underline{\mathrm{E}}L) \arrow[r] \arrow[d, "\mu_0^L"'] & 0 \\
\mathrm{K}_0(\mathrm{C}^*(B)) \arrow[r, "\text{Id-}\alpha_*"] & \mathrm{K}_0(\mathrm{C}^*(B)) \arrow[r, "\iota_*"] & \mathrm{K}_0(\mathrm{C}^*(L)) \arrow[r] & 0.
\end{tikzcd}$$

The diagram commutes due to functoriality of the assembly map. Theorem 7.4.1 together with Theorem 7.5.3 imply that the top and the bottom sequences are exact. Since μ_0^B is an isomorphism by Corollary 7.3.4, the five lemma then implies that the middle vertical map μ_0^L is an isomorphism.

μ_1^Γ *is an isomorphism.* Consider the comparison diagram

$$\begin{tikzcd}
\mathrm{K}_1^{\mathrm{L}}(\underline{\mathrm{E}}L) \arrow[r, "\mu_1^L"] & \mathrm{K}_1(\mathrm{C}^*(L)) \\
\mathrm{K}_1^{\mathbb{Z}}(\underline{\mathrm{E}}\mathbb{Z}) \arrow[r, "\mu_1^{\mathbb{Z}}"] \arrow[u, "\simeq"] & \mathrm{K}_1(\mathrm{C}^*(\mathbb{Z})). \arrow[u, "\simeq"']
\end{tikzcd}$$

Theorem 7.4.1 together with Theorem 7.5.3 imply that both groups are infinite cyclic. Therefore we need to identify the image of the canonical generator of $\mathrm{K}_1^{\mathrm{L}}(\underline{\mathrm{E}}L)$ under μ_1^L. In view of Theorem 7.5.3 the generator of $\mathrm{K}_1^{\mathrm{L}}(\underline{\mathrm{E}}L)$ is the one coming from $\mathrm{K}_1^{\mathbb{Z}}(\underline{\mathrm{E}}\mathbb{Z})$. Moreover, the generator of $\mathrm{K}_1(\mathrm{C}^*(L))$ is the one coming from $\mathrm{K}_1(\mathrm{C}^*(\mathbb{Z}))$. Combining these plus the commutativity of the digram imply that the assembly map μ_1^L is identified with the one for $\mathbb{Z}$ which is an isomorphism.

□

7.8 Modified trace conjecture for $\mathrm{F} \wr \mathbb{Z}$

This section addresses the modified trace conjecture for $\mathrm{F} \wr \mathbb{Z}$, see Section 4.2. Due to this conjecture which is implied by surjectivity of the Baum-Connes assembly map, the range of τ_* is the *subring* in $\mathbb{Q}$ generated by the inverses of the order of finite subgroups. We compute this image directly using our K-theory computations.
The trace conjecture for finite groups is well-known, nevertheless we give a proof for completeness.

Proposition 7.8.1. *For a finite group F, let $\tau \colon \mathbb{C}F \to \mathbb{C}$ be the canonical trace. The image of the induced homomorphism $\tau_* \colon \mathrm{K}_0(\mathbb{C}F) \to \mathbb{R}$ is*

$$\operatorname{Im} \tau_* = \frac{1}{|F|}\mathbb{Z}.$$

Proof. Consider the minimal projection $p_F = \frac{1}{|F|} \sum_{f \in F} f$ associated to the trivial representation of F. Since $\tau(p_F) = \frac{1}{|F|}$, we get that $\frac{1}{|F|}\mathbb{Z} \subset \tau_*(\mathrm{K}_0(\mathbb{C}F))$. To show the converse, for $\pi \in \hat{F}$ consider the character associated to π, that is $\chi_\pi(f) = \mathrm{Tr}(\pi(f))$ for $f \in F$. Consider the minimal central projection $p_\pi = \frac{\dim \pi}{|F|}\chi_\pi$ in $\mathbb{C}F$ such that

$$\mathbb{C}F = \oplus_{\pi \in \hat{F}} p_\pi \mathbb{C}F \cong \oplus_{\pi \in \hat{F}} \mathrm{M}_{\dim \pi}(\mathbb{C}).$$

This satisfies $\tau(p_\pi) = \frac{(\dim \pi)^2}{|F|}$. Every p_π is the sum of $\dim \pi$ minimal projections in $\mathrm{M}_{\dim \pi}(\mathbb{C})$. As all these have the same trace, the trace of a minimal projection in $\mathbb{C}F$ is $\frac{\dim \pi}{|F|} \in \frac{1}{|F|}\mathbb{Z}$. This finishes the proof. □

Proposition 7.8.2. *Let $L = F \wr \mathbb{Z}$, let τ be the canonical trace. The range of the induced homomorphism $\tau_* \colon \mathrm{K}_0(\mathrm{C}^*(L)) \to \mathbb{R}$ is precisely*

$$\operatorname{Im} \tau_* = \mathbb{Z}[\frac{1}{|F|}].$$

Proof. View L as $B \rtimes \mathbb{Z}$. For the subgroup B we consider the restriction of τ on $\mathrm{C}^*(B)$ and denote it by τ. This is the canonical trace on $\mathrm{C}^*(B)$. Due to computation in Theorem 7.4.1, $\iota_* \colon \mathrm{K}_0(\mathrm{C}^*(B)) \to \mathrm{K}_0(\mathrm{C}^*(L))$ is surjective hence we have

$$\tau_*(\mathrm{K}_0(\mathrm{C}^*(L))) = \tau_*(\mathrm{K}_0(\mathrm{C}^*(B))).$$

Therefore we need to show $\tau_*(\mathrm{K}_0(C^*B)) = \mathbb{Z}[\frac{1}{|F|}]$. Viewing B as the colimit of finite groups B_n's, we obtain

$$\mathrm{K}_0(\mathrm{C}^*(B)) = \varinjlim \mathrm{K}_0(\mathbb{C}B_n).$$

Similar to before the canonical trace τ on $\mathrm{C}^*(B)$ restricts to the canonical trace τ on $\mathbb{C}B_n$'s for $n \in \mathbb{N}$. Therefore we obtain the inductive system

$$\tau_*(\mathrm{C}^*(B)) = \varinjlim(\tau_*(\mathrm{K}_0(\mathbb{C}B_n)), \iota_n),$$

where the ι_n's are the inclusions $\iota_n \colon \tau_*(\mathrm{K}_0(\mathbb{C}B_n)) \to \tau_*(\mathrm{K}_0(\mathbb{C}B_{n+1}))$ for $n \in \mathbb{N}$. Proposition 7.8.1 immediately implies that

$$\tau_*(\mathrm{K}_0(\mathrm{C}^*(L)) = \varinjlim(\frac{1}{|F|^n}\mathbb{Z}, \iota_n) = \mathbb{Z}[\frac{1}{|F|}].$$

□

Chapter 8

The Baum-Connes conjecture for $F \wr \mathbb{F}_n$

This chapter deals with the Baum-Connes conjecture for the wreath product $F \wr \mathbb{F}_n$, where F is a finite group and $\mathbb{F}_n$ is free group. Thanks to [CSV12, Theorem 1.1] this group is a-T-menable, hence Higson-Kasparov's result [HK01] guarantees that the Baum-Connes conjecture holds for this group. We throw light on this isomorphism by providing bases for all K-groups and identifying them via the Baum-Connes map. We include a model for the classifying space $\underline{E}F \wr \mathbb{F}_n$. Further, we compute directly the range of the trace on $\mathrm{K}_0(\mathrm{C}^*_{\mathrm{r}}(F \wr F_n))$. This chapter generalises the results for $F \wr \mathbb{Z}$ and is based on [Po17].

Remark 8.0.1. In Chapter 2, we discussed that semidirect products do not inherit a-T-menability. A counterexample we gave there was $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$. However, wreath products do inherit a-T-menability. This reveals another nice behaviour of this family among semidirect products.

Before we proceed with computations, we state a lemma in analogy with Lemma 7.1.3 describing representatives of orbits of group action. First some notations. Let $(F, \mathfrak{p})$ be a finite pointed set. And let $F^{(\mathbb{F}_n)}$ denote the set of all finitely supported functions from $\mathbb{F}_n$ to F or alternatively the set of all sequences indexed by words of $\mathbb{F}_n$ with values in F taking almost everywhere the value $\mathfrak{p}$. We denote by $S = \{s_1, \dots, s_n\}$ a set of generators for the free group $\mathbb{F}_n$. Given $f \in F^{(\mathbb{F}_n)}$, we consider the convex hull of its support which is a finite subtree T' of the Cayley graph of $\mathbb{F}_n$. We present such f by $\chi^f_{T'} \in F^{(\mathbb{F}_n)}$ in order to emphasise the rôle of the subtree T' associated to its support. Therefore, for $w \in \mathbb{F}_n$

$$\chi^f_{T'}(w) = \begin{cases} f(w) & w \in \operatorname{supp} f \\ \mathfrak{p} & \text{otherwise.} \end{cases}$$

For a tree T, we define its barycentre as either a vertex or an edge remaining after removing successively the terminal vertices and the corresponding edges.

Definition 8.0.2. (*admissible tree*)
A tree T is admissible if it is a finite subtree of the Cayley graph of $\mathbb{F}_n$ on $\{s_1, \dots, s_n\}$ with its barycentre either e or an edge $[e, s_i]$ for some $i = 1, \dots, n$.

The next lemma generalises the facts in Section 7.1.1 and will help us to describe the K_0-groups as coinvariants of certain actions.

Lemma 8.0.3. *Let $\mathbb{F}_n$ be the free group on $\{s_1, \dots, s_n\}$ and let $(F, \mathfrak{p})$ be a finite pointed set. Consider the action $\mathbb{F}_n \curvearrowright F^{(\mathbb{F}_n)}$ by left multiplication. A countable set R of representatives for $\mathbb{F}_n$-orbits is*

$$R = \{\chi^f_{T'} \colon f \in F^{(\mathbb{F}_n)},\ T' \textit{ is admissible}\}.$$

In particular,

$$\begin{aligned}\mathbb{Z}(F^{(\mathbb{F}_n)}) &= \langle m - w \cdot m \colon m \in \mathbb{Z}(F^{(\mathbb{F}_n)}),\ w \in \mathbb{F}_n \rangle \oplus\ \mathbb{Z}R \\ &= \langle m - s_i \cdot m \colon m \in \mathbb{Z}(F^{(\mathbb{F}_n)}),\ 1 \le i \le n \rangle \oplus\ \mathbb{Z}R.\end{aligned}$$

Proof. One observes that distinct elements of R belong to the distinct orbits of the action. We should show that all elements of $F^{(F_n)}$ lie in the orbit of some element in R. Take $f \in F^{(\mathbb{F}_n)}$. Let T' be the subtree associated to its support. Its barycentre is either a vertex w or an edge $[w, ws_i]$ for $i \in \{1, \dots, n\}$. In either case, the subtree $\tilde{T}' := w^{-1}T'$ is admissible. Therefore, f belongs to the orbit of $\chi^{\tilde{f}}_{\tilde{T}'}$, where $\tilde{f} \in F^{(\mathbb{F}_n)}$. $\square$

8.1 Analytical side: K-theory of $\mathrm{C}^*_{\mathrm{r}}(F \wr \mathbb{F}_n)$

This section describes the right-hand side of the assembly map for $G = F \wr \mathbb{F}_n$ based on the K-theoretic computations for $\mathrm{C}^*_{\mathrm{r}}(G)$. Similar to the case with $F \wr \mathbb{Z}$, the normal subgroup $B = \bigoplus_{\mathbb{F}_n} F$ plays an important rôle since it contains all torsion of the group. In particular, as we will see, $\mathrm{C}^*(B)$ provides us with sufficiently many projections to generate $\mathrm{K}_0(\mathrm{C}^*_{\mathrm{r}}(G))$.

Let $\mathcal{B}_n = \{w \in \mathbb{F}_n \colon\ |w| \le n\}$ denote the ball of radius n (with respect to the word metric) on the Cayley graph of $\mathbb{F}_n$. Write $B = \bigoplus_{\mathbb{F}_n} F$. The subset $B_n = \{f \in B \colon\ \mathrm{supp} f \subset \mathcal{B}_n\}$ is isomorphic with $B_n = \bigoplus_1^{m_n} F$, where $m_n = |\mathcal{B}_n|$. We express the group B as the colimit of the increasing sequence of the B_n's.

Viewing the wreath product $F \wr \mathbb{F}_n$ as a semidirect product, its C*-algebra can be represented as the crossed product $\mathrm{C}^*(B) \rtimes \mathbb{F}_n$. Therefore the Pimsner-Voiculescu 6-term exact sequence 2.3.2 becomes the main tool to compute its K-theory. Consider

$$\begin{tikzcd}
\bigoplus_{i=1}^n \mathrm{K}_0(\mathrm{C}^*(B)) \arrow[r, "\sigma_*"] & \mathrm{K}_0(\mathrm{C}^*(B)) \arrow[r, "\iota_*"] & \mathrm{K}_0(\mathrm{C}^*(B) \rtimes_{\mathrm{r}} \mathbb{F}_n) \arrow[d, "\partial_0"] \\
\mathrm{K}_1(\mathrm{C}^*(B) \rtimes_{\mathrm{r}} \mathbb{F}_n) \arrow[u, "\partial_1"] & \mathrm{K}_1(\mathrm{C}^*(B)) \arrow[l, "\iota_*"'] & \bigoplus_{i=1}^n \mathrm{K}_1(\mathrm{C}^*(B)). \arrow[l, "\sigma_*"']
\end{tikzcd}$$

We computed the K-theory for a locally finite group B in Corollary 7.3.4. In view of that result

- $\mathrm{K}_0(\mathrm{C}^*(B)) = \mathrm{Min}(\mathbb{C}F)^{(\mathbb{F}_n)}$,
- $\mathrm{K}_1(\mathrm{C}^*(B)) = 0$.

Therefore, we need to understand the kernel of the homomorphism σ_* appearing above. We start with a lemma which is essential for the later computations.

Lemma 8.1.1. *Let $(F, \mathfrak{p})$ be a finite pointed set and let $\mathbb{F}_n$ be the free group on $\{s_1, \ldots, s_n\}$. Suppose $X = F^{(\mathbb{F}_n)}$, and consider*

$$\psi \colon \bigoplus_{i=1}^n \mathbb{Z}X \to \mathbb{Z}X, \qquad \psi((f_i)_{1 \le i \le n}) = \sum_{i=1}^n f_i - s_i \cdot f_i,$$

where $s_i \cdot f_i(x) = f_i(s_i^{-1}x)$. The kernel of ψ is

$$Ker(\psi) = \mathbb{Z}\mathbf{1}_{\mathfrak{p}} \oplus \ldots \oplus \mathbb{Z}\mathbf{1}_{\mathfrak{p}},$$

where $\mathbf{1}_{\mathfrak{p}}$ denotes the constant map with the value $\mathfrak{p}$.

Proof. Choose $(f_i)_{1 \le i \le n} \in \bigoplus_{i=1}^n \mathbb{Z}X$. If all f_i's are $\mathbb{F}_n$-invariant, then obviously $(f_i)_{1 \le i \le n} \in \mathrm{Ker}(\psi)$. In particular, $\mathbf{1}_{\mathfrak{p}}$ is $\mathbb{F}_n$-invariant, hence

$$\bigoplus_{i=1}^n \mathbb{Z}\mathbf{1}_{\mathfrak{p}} \subset \mathrm{Ker}(\psi)\,.$$

To conclude the statement of the lemma we need to show that these are in fact the only elements in the kernel.
Write $X = \bigcup_{x \in X/\mathbb{F}_n} \mathbb{F}_n \cdot x$. For $x \in X \setminus \{\mathbf{1}_{\mathfrak{p}}\}$, we have that $\mathbb{F}_n \cdot x \cong \mathbb{F}_n$, as $\mathbb{F}_n$ acts freely on $X \setminus \{\mathbf{1}_{\mathfrak{p}}\}$. For $f \in \mathbb{Z}X$ and $x \in X \setminus \{\mathbf{1}_{\mathfrak{p}}\}$, we define $\tilde{f}_x \colon \mathbb{F}_n \to \mathbb{Z}$

by $\tilde{f}_x(w) = f(w \cdot x)$. Note that for $0 \neq f \in \mathbb{Z}X$, if $\tilde{f}_x = 0$ for all $x \in X \setminus \{\mathbf{1}_\mathfrak{p}\}$, then $f \in \mathbb{Z} \cdot \mathbf{1}_\mathfrak{p}$. Therefore the refined statement to prove is

$$0 \neq (f_i)_{1 \leq i \leq n} \in \operatorname{Ker}(\psi) \quad \text{implies} \quad \forall x \in X \setminus \{\mathbf{1}_\mathfrak{p}\}, \quad (\tilde{f}_{ix})_{1 \leq i \leq n} = 0.$$

For $x \in X \setminus \{\mathbf{1}_\mathfrak{p}\}$, we define

$$\mathcal{S} = \bigcup_{1 \leq i \leq n} \operatorname{supp} \tilde{f}_{ix} \cup \bigcup_{1 \leq i \leq n} s_i \operatorname{supp} \tilde{f}_{ix}.$$

We assume that $\mathcal{S}$ is non-empty. Note that for $(f_i)_{1 \leq i \leq n} \in \operatorname{Ker}(\psi)$, there is some $w \in \mathcal{S}$ such that w has to belong to the supports of at least two functions, otherwise $(f_i)_{1 \leq i \leq n}$ can not belong to the kernel. Choose a word $w \in \mathcal{S} \subset \mathbb{F}_n$ with the maximum length. The requirements above lead us to consider the following possibilities for $i \neq j \in \{1, \cdots, n\}$:

1. $w \in \operatorname{supp} \tilde{f}_{i_x} \cap s_i \operatorname{supp} \tilde{f}_{i_x}$,
2. $w \in \operatorname{supp} \tilde{f}_{i_x} \cap \operatorname{supp} \tilde{f}_{j_x}$,
3. $w \in s_i \operatorname{supp} \tilde{f}_{i_x} \cap s_j \operatorname{supp} \tilde{f}_{j_x}$, and
4. $w \in \operatorname{supp} \tilde{f}_{i_x} \cap s_j \operatorname{supp} \tilde{f}_{j_x}$.

Claim: We can deduce contradictions from all these cases.
Accepting this claim immediately implies that $\tilde{f}_{i_x} = 0$ for $i = 1, \ldots, n$. As $x \in X \setminus \{1_\mathfrak{p}\}$ is chosen arbitrarily, we have that $(\tilde{f}_{ix})_{1 \leq i \leq n} = 0$ for all such x. Therefore $(f_i)_{1 \leq i \leq n} \in \bigoplus_{i=1}^n \mathbb{Z} \cdot \mathbf{1}_\mathfrak{p}$, which finishes the proof.
Proof of the claim: By an argument on the length of words in $\mathbb{F}_n$, we successively show that all of the above possibilities lead to a contradiction.

Case 1: If $w \in \operatorname{supp} \tilde{f}_{i_x}$, then $s_i w \in s_i \operatorname{supp} \tilde{f}_{i_x} \subset \mathcal{S}$ (†). Moreover, by the other assumption in 1, $w \in s_i \operatorname{supp} \tilde{f}_{i_x}$. Hence there exists $u \in \operatorname{supp} \tilde{f}_{i_x}$ such that $w = s_i u$, equivalently $u = s_i^{-1} w$ (‡). Now on the one hand, (†) forces w to start with s_i^{-1} (otherwise, $|s_i^{-1} w| > |w|$ which is impossible). On the other hand (‡) implies that w starts with s_i (if not, then $|u| = |s_i^{-1} w| > |w|$ which is a contradiction). Therefore this case can not happen.

Case 2: Let w be in the supports of $\tilde{f}_{i_x}$ and $\tilde{f}_{j_x}$. Denote by s_l the starting letter of w. Either $l \in \{i, j\}$ or $l \neq i, j$. If $l = i$ (respectively, $l = j$), then $s_j w \in s_j \operatorname{supp} \tilde{f}_{j_x}$ (respectively, $s_i w \in s_i \operatorname{supp} \tilde{f}_{i_x}$) provides a longer word in $\mathcal{S}$, which is impossible. Now if $l \neq i, j$, then $s_i w \in s_i \operatorname{supp} \tilde{f}_{i_x}$ provides a longer word in $\mathcal{S}$ which is impossible. Hence this case can not happen.

Case 3: If w is in the intersection of the shifted supports, then there exists $u \in \operatorname{supp} \tilde{f}_{i_x}$ and $v \in \operatorname{supp} \tilde{f}_{j_x}$ such that $s_i u = w = s_j v$. Equivalently, we have that $s_i^{-1} w = u$ and $s_j^{-1} w = v$. If w starts with either s_i or s_j, then v or u respectively gives us a longer word in $\mathcal{S}$. If w starts in any other letter than these two, then both u and v give a longer word in $\mathcal{S}$. That is a contradiction.

Case 4: Let $w \in \operatorname{supp} \tilde{f}_{i_x}$ and $w \in s_j \operatorname{supp} \tilde{f}_{j_x}$. Then there exists $u \in \operatorname{supp} \tilde{f}_{j_x}$ such that $s_j^{-1} w = u$. This implies that w has to start with s_j. Moreover, the assumption at the beginning that $w \in \operatorname{supp} \tilde{f}_{i_x}$ together with the fact that w starts with s_j guarantee that $s_i w \in s_i \operatorname{supp} \tilde{f}_{i_x} \subset \mathcal{S}$ has a longer length than the maximum. This is impossible.

□

The next proposition describes the image of the unitary $u_i \in A \rtimes_{\mathrm{r}} \mathbb{F}_n$, $i = 1, \ldots, n$, under the boundary map ∂_1 of the Pimsner-Voiculescu 6-term exact sequence 2.3.2. This generalises Lemma 6.3.1.

Proposition 8.1.2. *Let A be a unital* C**-algebra. For $i \in \{1, \ldots, n\}$, let $\alpha_i \in \operatorname{Aut}(A)$ define an action α of $\mathbb{F}_n$ on A. The boundary map*

$$\begin{aligned} \partial_1 \colon \mathrm{K}_1(A \rtimes_{\mathrm{r}} \mathbb{F}_n) &\to \bigoplus_{i=1}^{n} \mathrm{K}_0(A) \\ [u_i] &\mapsto (0, \ldots, 0, \underbrace{-[1]}_{i\text{–}th\ slot}, 0, \ldots, 0), \end{aligned}$$

where the unitaries $u_i \in \mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_n) \subset A \rtimes_{\mathrm{r}} \mathbb{F}_n$, for $1 \leq i \leq n$, come from the generators of $\mathbb{F}_n$.

Our proof is based on the the original proof of Pimsner and Voiculescu in [PV82].

Proof. As before $\{s_1, \ldots, s_n\}$ is a generating set for $\mathbb{F}_n$. Let $i \in \{1, \ldots, n\}$. Consider

$$\mathcal{W}_i = \{\text{reduced words in } \mathbb{F}_n \text{ that do not end with } s_i^{-1}\} \subset \mathbb{F}_n.$$

Note that for $i \neq j \in \{1, \ldots, n\}$, we have that $e \in \mathcal{W}_i$, $s_j \mathcal{W}_i = \mathcal{W}_i$ and $s_i \mathcal{W}_i = \mathcal{W}_i \setminus \{e\}$.

Assume $A \subset \mathcal{B}(H)$. Let $\mathcal{K} \subset \mathcal{B}(H)$ denote the C^*-algebras of the compact operators on H. Recall that for $j = 1, \ldots, n$ we have that $u_j \in \mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_n) \subset$

$A \rtimes_{\mathrm{r}} \mathbb{F}_n \subset \mathcal{B}(\ell^2(\mathbb{F}_n, H))$. Consider the compression of these unitaries to $\ell^2(\mathcal{W}_i, H) \subset \ell^2(\mathbb{F}_n, H)$. These provide us with $n-1$ unitaries U_j, for $i \neq j \in \{1, \dots, n\}$, and one non-unitary isometry S_i. We consider Toeplitz algebra

$$\mathcal{T}_{n,i} = \mathrm{C}^*(\{A, U_1, \dots, U_{i-1}, S_i, U_{i+1}, \dots, U_n\}) \subset \mathbb{B}(\ell^2(\mathcal{W}_i, H)).$$

Let $P_e = \mathrm{I} - S_i S_i^*$ be the projection to $\delta_e \otimes H$. The ideal generated by P_e in $\mathcal{T}_{n,i}$ is isomorphic to $A \otimes \mathcal{K}$. Now we consider the Toeplitz extension

$$0 \longrightarrow A \otimes \mathcal{K} \longrightarrow \mathcal{T}_{n,i} \xrightarrow{P_{n,i}} A \rtimes_{\mathrm{r}} \mathbb{F}_n \longrightarrow 0.$$

The surjection $P_{n,i} \colon \mathcal{T}_{n,i} \to A \rtimes_{\mathrm{r}} \mathbb{F}_n$ is defined by

$$\begin{aligned} a &\mapsto a, \quad a \in A \ , \\ U_j &\mapsto u_j, \quad j \in \{1, \dots, n\}, \ j \neq i, \\ S_i &\mapsto u_i. \end{aligned}$$

Let $B_n = \{(t_1, \dots, t_n) \in \bigoplus_{i=1}^n \mathcal{T}_{n,i} \colon P_{n,1}(t_1) = \ldots = P_{n,n}(t_n)\}$ be the fibred product of the C^*-algebras $\mathcal{T}_{n,i}$ over $A \rtimes_{\mathrm{r}} \mathbb{F}_n$. Consider the exact sequence introduced on page 153 in [PV82]

$$0 \longrightarrow (A \otimes \mathcal{K})^n \longrightarrow B_n \longrightarrow A \rtimes_{\mathrm{r}} \mathbb{F}_n \longrightarrow 0.$$

We denote the restriction of the action α to its i-th generator by α_i and identify $\langle s_i \rangle \cong \mathbb{Z}$. Together with the Toeplitz extension, this exact sequence fits into a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & (A \otimes \mathcal{K})^n \arrow[r] & B_n \arrow[r] & A \rtimes_{\mathrm{r}} \mathbb{F}_n \arrow[r] & 0 \\
0 \arrow[r] & A \otimes \mathcal{K} \arrow[r] \arrow[u, "\iota_i"] & \mathcal{T}_{n,i} \arrow[r] \arrow[u] & A \rtimes \mathbb{Z} \arrow[r] \arrow[u] & 0,
\end{tikzcd}$$

where ι_i is the embedding to the i-th component and the middle homomorphism maps $a \in \mathcal{T}_{n,i}$ to $(a, \dots, a)$ and maps S_i to $(U_i, \dots, U_i, S_i, U_i, \dots, U_i)$. Once we have such a commuting diagram the naturality of the 6-term exact sequence provides us with a commutative diagram in K-theory

$$\begin{tikzcd}
 & \mathrm{K}_0((A \otimes \mathcal{K})^n) \arrow[rr] & & \mathrm{K}_0(B_n) \arrow[rr] & & \mathrm{K}_0(A \rtimes_{\mathrm{r}} \mathbb{F}_n) \arrow[dd] \\
\mathrm{K}_0(A \otimes \mathcal{K}) \arrow[ur] \arrow[rr] & & \mathrm{K}_0(\mathcal{T}_{n,i}) \arrow[ur] \arrow[rr] & & \mathrm{K}_0(A \rtimes \mathbb{Z}) \arrow[ur] \arrow[dd] & \\
 & \mathrm{K}_1(A \rtimes_{\mathrm{r}} \mathbb{F}_n) \arrow[uu] & & \mathrm{K}_1(B_n) \arrow[ll] & & \mathrm{K}_1((A \otimes \mathcal{K})^n) \arrow[ll] \\
\mathrm{K}_1(A \rtimes \mathbb{Z}) \arrow[uu] \arrow[ur] & & \mathrm{K}_1(\mathcal{T}_{n,i}) \arrow[ll] \arrow[ur] & & \mathrm{K}_1(A \otimes \mathcal{K}). \arrow[ll] \arrow[ur] &
\end{tikzcd}$$

The left vertical square fits into the diagram

$$\begin{tikzcd}
\mathrm{K}_0(A) \arrow[r, "\cong"] & \mathrm{K}_0(A \otimes \mathcal{K}) \arrow[r] & \mathrm{K}_0((A \otimes \mathcal{K})^n) \arrow[r, "\cong"] & \bigoplus_{i=1}^n \mathrm{K}_0(A \otimes \mathcal{K}) \arrow[r, "\cong"] & \bigoplus_{i=1}^n \mathrm{K}_0(A) \\
& \mathrm{K}_1(A \rtimes \mathbb{Z}) \arrow[ul] \arrow[u] \arrow[r] & \mathrm{K}_1(A \rtimes_{\mathrm{r}} \mathbb{F}_n). \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

The triangles at the sides are the natural identifications in K-theory. Hence we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{K}_0(A) \arrow[r, "\iota_i"] & \bigoplus_{i=1}^n \mathrm{K}_0(A) \\
\mathrm{K}_1(A \rtimes \langle u_i \rangle) \arrow[u, "\partial_{1,i}"] \arrow[r] & \mathrm{K}_1(A \rtimes_{\mathrm{r}} \mathbb{F}_n). \arrow[u, "\partial_1"']
\end{tikzcd}$$

Remark that $\partial_{1,i}$ is the boundary map in the Pimsner-Voiculescu 6-term exact sequence for $\mathbb{Z}$, that is $n = 1$ in $\mathbb{F}_n$. Moreover, we have that $\partial_1 = \bigoplus_{i=1}^n \partial_{1,i}$ as mentioned in the statement of Theorem 3.5 in [PV82]. Due to Lemma 6.3.1, we have that $\partial_{1,i}([u_i]) = -[1] \in \mathrm{K}_0(A)$ which is mapped by ι_i to the i-th component of the n-tuple $(0, \ldots, 0, -[1], 0, \ldots, 0)$. This finishes the proof. ☐

Employing Lemma 8.1.1 and Proposition 8.1.2, we can describe explicitly the right-hand side of the Baum-Connes conjecture for $F \wr \mathbb{F}_n$.

Theorem 8.1.3. *Let $G = F \wr \mathbb{F}_n$ with F a non-trivial finite group.*

$\mathrm{K}_0(\mathrm{C}^*_{\mathrm{r}}(G)) = \mathbb{Z}R$ *with R a countable basis indexed by representatives for $\mathbb{F}_n$-orbits in* $\mathrm{Min}\,(\mathbb{C}F)^{(\mathbb{F}_n)}$.

$\mathrm{K}_1(\mathrm{C}^*_{\mathrm{r}}(G)) = \mathbb{Z}[u_1] \oplus \ldots \oplus \mathbb{Z}[u_n]$.

Proof. Let $B = \oplus_{\mathbb{F}_n} F$. We may write $\mathrm{C}^*_{\mathrm{r}}(G) = \mathrm{C}^*(B) \rtimes_{\mathrm{r}} \mathbb{F}_n$. Due to Proposition 7.3.3, we have that $\mathrm{K}_0(\mathrm{C}^*(B)) = \mathbb{Z}\,(\mathrm{Min}(\mathbb{C}F)^{(\mathbb{F}_n)})$. Moreover, $\mathrm{C}^*(B)$ is an AF-algebra as $\mathrm{C}^*(B) = \bigotimes_{\mathbb{F}_n} \mathbb{C}F$ hence $\mathrm{K}_1(\mathrm{C}^*(B)) = 0$. Substituting these K-groups in the Pimsner-Voiculescu 6-term exact sequence, it reduces to

$$\begin{tikzcd}
\bigoplus_{i=1}^n \mathbb{Z}\,(\mathrm{Min}\,\mathbb{C}F^{(\mathbb{F}_n)}) \arrow[r, "\sigma_*"] & \mathbb{Z}(\mathrm{Min}(\mathbb{C}F)^{(\mathbb{F}_n)}) \arrow[r, "\iota_*"] & \mathrm{K}_0(\mathrm{C}^*_{\mathrm{r}}\Gamma) \arrow[d] \\
\mathrm{K}_1(\mathrm{C}^*_{\mathrm{r}}(\Gamma)) \arrow[u, "\partial_1"] & 0 \arrow[l] & 0. \arrow[l]
\end{tikzcd}$$

We start from the left-hand side of the diagram. Injectivity of ∂_1 implies that $\mathrm{K}_1(\mathrm{C}^*_{\mathrm{r}}(G)) = \mathrm{Im}\,\partial_1$. By exactness of the diagram at $\bigoplus_{i=1}^n \mathbb{Z}\,(\mathrm{Min}(\mathbb{C}F)^{(\mathbb{F}_n)})$, we have that $\operatorname{Im}\partial_1 = \operatorname{Ker}(\sigma_*)$. In view of Lemma 8.1.1, for $X = (\operatorname{Min}(\mathbb{C}F))^{(\mathbb{F}_n)}$, $\mathfrak{p} = p_F$ and the operator $\psi = \sigma_*$, the kernel is generated by n copies of $\mathbf{1}_{p_F} \in (\operatorname{Min}\mathbb{C}F)^{(\mathbb{F}_n)}$. Thus we have

$$\operatorname{Ker}(\sigma_*) = \mathbb{Z}\mathbf{1}p_F \oplus \cdots \oplus \mathbb{Z}\mathbf{1}p_F.$$

Recall from Proposition 7.3.3 that the element $\mathbf{1}_{p_F}$ corresponds to $[1] \in \mathrm{K}_0(\mathrm{C}^*(B))$. Moreover, we observed in Proposition 8.1.2 that $\partial_1([u_i]) = (0, \ldots, -[1], 0, \ldots, 0)$, therefore

$$\mathrm{K}_1(\mathrm{C}^*_\mathrm{r}(G)) = \mathbb{Z}[u_1] \oplus \cdots \oplus \mathbb{Z}[u_n].$$

In order to compute $\mathrm{K}_0(\mathrm{C}^*_\mathrm{r}(G))$, we focus on the right-hand side of the diagram. Surjectivity of ι_* implies that $\mathrm{K}_0(\mathrm{C}^*_\mathrm{r}(G)) = \mathbb{Z}\,(\operatorname{Min}(\mathbb{C}F)^{(\mathbb{F}_n)})/\operatorname{Ker}\iota_*$. By exactness of the diagram at $\mathbb{Z}\,(\operatorname{Min}(\mathbb{C}F)^{(\mathbb{F}_n)})$, we have that $\operatorname{Ker}\iota_* = \operatorname{Im}(\sigma_*)$. Therefore

$$\mathrm{K}_0(\mathrm{C}^*_\mathrm{r}(G)) = \mathbb{Z}\,(\operatorname{Min}(\mathbb{C}F)^{(\mathbb{F}_n)})/\operatorname{Im}(\sigma_*)\,.$$

Furthermore, we may write

$$\operatorname{Im}(\sigma_*) = \langle f - s_i \cdot f \colon f \in \mathbb{Z}\,(\operatorname{Min}(\mathbb{C}F)^{(\mathbb{F}_n)}), 1 \leq i \leq n\rangle.$$

Thanks to Lemma 7.1.3, we can then express this quotient as the free abelian group on the (countable) set of representatives for $\mathbb{F}_n$-orbits, that is $\mathrm{K}_0(\mathrm{C}^*_\mathrm{r}(G)) = \mathbb{Z}R$ with the described basis. □

8.2 Topological side: Equivariant K-homology

This section presents the equivariant K-homology of $\underline{\mathrm{E}}F \wr \mathbb{F}_n$ from the computation of its Bredon homology. We include a concrete description of a 2-dimensional model for $\underline{\mathrm{E}}F \wr \mathbb{F}_n$. All these together describe the left-hand side of the Baum-Connes assembly map for this group.

8.2.1 A 2-dimensional model for $\underline{\mathrm{E}}\,F \wr \mathbb{F}_n$

The model that we will present here is in analogy with the model we demonstrated in Section 7.5.1 in the case with cyclic extensions. Similarly, this model will be constructed from the model of the normal subgroup.

Remark 8.2.1. Due to Theorem 3.2.4, there exists a 2-dimensional model for this classifying space. We will make this existence result concrete.

The normal subgroup of this wreath product is a locally finite group B. In Section 7.3 we described a model for the classifying space of B which was its filtration tree. Let $\mathbb{V}$ and $\mathbb{E}$ respectively denote the set of vertices and edges of T. The tree T can be described as follows.

- $\mathbb{V} = \mathbb{E} = \coprod_{n>0} B/B_n$
- For $b \in B$, the vertices bB_n, $bB_{n+1} \in V$ are connected via the edge labelled by $bB_n \in E$.

Consider the Cayley graph of $\mathbb{F}_n$. Intuitively speaking, the idea of construction is to install the tree T coming from a model for $\underline{E}B$, over all vertices of the Cayley graph of $\mathbb{F}_n$ and then identify certain subcomplexes of these trees in a compatible way that the resulting complex meets all requirements of being a model for $\underline{E}G$.

Denote by T_w the tree T over the word $w \in \mathbb{F}_n$, and by $bB_{m,w}$ a vertex on the m-th level of the filtration of the tree T_w. For $w \in \mathbb{F}_n$, we denote by $[w, ws_j]$ the edge from w to ws_j on the Cayley graph of $\mathbb{F}_n$.

We define a B-action on T_w: for $b, f \in B$ and $w \in \mathbb{F}_n$

$$f \cdot_w bB_{m,w} := (w^{-1} \cdot f)bB_{m,w}, \quad m \in \mathbb{N}.$$

Note that each T_w is a model for $\underline{E}B$ as well.

For $j = 1, \dots, n$, we define the gluing maps between neighbouring trees

$$\varphi_w^{s_j} : \mathrm{T}_w \to \mathrm{T}_{ws_j} : \quad bB_{m,w} \mapsto (s_j^{-1} \cdot b)B_{m+1,ws_j}.$$

It can easily be checked that the $\varphi_w^{s_j}$ are B-equivariant, that is

$$\varphi_w^{s_j}(f \cdot_w bB_{m,w}) = f \cdot_{ws_j} \varphi_w^{s_j}(bB_{m,w}).$$

For $w \in \mathbb{F}_n$, we may identify an edge $[w, ws_i]$ with the interval $[0, 1]$. We define

$$\tilde{\mathrm{Z}} := \bigcup_{j=1}^{n} \bigcup_{\substack{w \in \mathbb{F}_n \\ |ws_j| > |w|}} (\mathrm{T}_w \times [w, ws_j]) \cup \bigcup_{j=1}^{n} \bigcup_{\substack{w \in \mathbb{F}_n \\ |ws_j^{-1}| > |w|}} (\mathrm{T}_w \times [w, ws_j^{-1}]).$$

Each non-trivial word on the Cayley graph of $\mathbb{F}_n$ has $2n - 1$ possibilities of increasing its length. We therefore identify points on the boundaries of edges in $\tilde{\mathrm{Z}}$. More explicitly, if we assume that w ends with s_i, then for $k, j \in \{1, \dots, n\}$ with $k \neq j \neq i$ we identify

$$\begin{aligned}
\mathrm{T}_w \times [w, ws_j^{\pm 1}] \ni \quad & (bB_{m,w}, ws_j^{\pm 1}) \sim (\varphi_w^{s_j^{\pm 1}}(bB_{m,w}), ws_j^{\pm 1}) \quad && \in \mathrm{T}_{ws_j} \times [ws_j^{\pm 1}, ws_j^{\pm 1}s_k], \\
T_w \times [w, ws_i] \ni \quad & (bB_{n,w}, ws_i) \sim (\varphi_w^{s_i}(bB_{m,w}), ws_i) \quad && \in T_{ws_i} \times [ws_i, ws_i^2].
\end{aligned}$$

For the trivial word we hence consider all $2n$ identifications. We define the mapping telescope

$$\mathrm{Z} := \tilde{\mathrm{Z}}/\sim.$$

This quotient space is a candidate to be the desired model for $\underline{E}G$. The group G acts on it. Indeed, the actions of B and $\mathbb{F}_n$ on Z combine into the conjugation action β of G on Z. To see this, we define the following actions:

$$\begin{aligned} B \overset{\theta}{\curvearrowright} \mathrm{Z}\colon \theta(f)(bB_{m,w}, t_{[w,\, ws_i]}) &= ((w^{-1} \cdot f)bB_{m,w}, t_{[w,\, ws_i]}), \\ \mathbb{F}_n \overset{\eta}{\curvearrowright} \mathrm{Z}\colon \eta(s_j)(bB_{m,w}, t_{[w,\, ws_i]}) &= (bB_{m,s_jw}, t_{[s_jw,\, s_jws_i]}), \end{aligned}$$

where $t_{[w,ws_i]}$ denotes a point on the edge $[w, ws_i]$, and $t_{[s_jw,s_jws_i]}$ is the corresponding point on the shifted edge. Moreover,

$$\begin{aligned} \eta(s_j)\theta(f)\eta(s_j)^{-1}(bB_{m,w}, t_{[w,\, ws_i]}) &= \eta(s_j)\theta(f)(bB_{m,s_j^{-1}w}, t_{[s_j^{-1}w,\, s_j^{-1}ws_i]}) \\ &= \eta(s_j)(\theta(f)(bB_{m,s_j^{-1}w}), t_{[s_j^{-1}w,\, s_j^{-1}ws_i]}) \\ &= \eta(s_j)(((s_j^{-1}w)^{-1} \cdot f)(bB_{m,\, s_j^{-1}w}),\ t_{[s_j^{-1}w,\, s_j^{-1}ws_i]}) \\ &= \eta(s_j)((w^{-1} \cdot (s_j \cdot f))(bB_{m,\, s_j^{-1}w}),\ t_{[s_j^{-1}w,\, s_j^{-1}ws_i]}) \\ &= ((w^{-1} \cdot (s_j \cdot f))(bB_{m,\, w}),\ t_{[w,\, ws_i]}) \\ &= \beta_{s_j}(f)(bB_{m,\, w}, t_{[w,\, ws_i]}). \end{aligned}$$

Some observations are immediate.

- $\mathbb{F}_n$ acts freely.
- The action η has a fundamental domain $D = \bigcup_{j=1}^{n} \mathrm{T}_e \times [e, s_j)$.
- Vertex stabilisers are finite, hence G acts properly on Z due to Lemma 3.1.10.

Combining these we have the following proposition.

Proposition 8.2.2. *The topological space* Z *is a* 2*-dimensional model for* $\underline{E}F \wr \mathbb{F}_n$.

Proof. The proof is verbatim of the proof of Proposition 7.5.2. □

8.2.2 Computations on the left-hand side

Having discussed (the dimension of) the classifying space for $F \wr \mathbb{F}_n$ we now proceed with computations of its equivariant K-homology.

Remark 8.2.3. Let $e \in \mathbb{F}_n$ denote the neutral element. Consider the free right $\mathbb{F}_n$-module $\bigoplus_{i=1}^{n} \mathbb{Z}\mathbb{F}_n$. The elements $e_j = (0, \ldots, 0, \underbrace{e}_{j-th\ slot}, 0, \ldots, 0) \in \bigoplus_{i=1}^{n} \mathbb{Z}\mathbb{F}_n$, for $1 \leq j \leq n$, form the canonical basis for this free module. Example I.4.3 in [Bro82] provides us with the free resolution.

$$0 \longrightarrow \bigoplus_{i=1}^{n} \mathbb{Z}\mathbb{F}_n \xrightarrow{\delta} \mathbb{Z}\mathbb{F}_n \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0, \tag{8.2.1}$$

with the augmentation ϵ and the boundary map δ satisfying $j \in \{1, \ldots, n\}$,

$$\delta(e_j) = e - s_j.$$

Let M be a left $\mathbb{F}_n$-module. We recall that $\mathbb{Z}\mathbb{F}_n \bigotimes_{\mathbb{Z}\mathbb{F}_n} M \cong M$. Applying the functor $- \bigotimes_{\mathbb{Z}\mathbb{F}_n} M$ to the above resolution provides us with the chain complex

$$0 \longrightarrow \bigoplus_{i=1}^{n} M \xrightarrow{\delta} M \longrightarrow 0. \tag{8.2.2}$$

Note that in the complex above, by abusing the notation of δ, we have that

$$\delta(m_1, \ldots, m_n) = \sum_{j=1}^{n} m_j - s_j m_j.$$

Therefore, we can write the first two homology groups

$$\mathrm{H}_0(\mathbb{F}_n; M) = M/\mathrm{Im}(\delta) \text{ and } \mathrm{H}_1(\mathbb{F}_n; M) = \mathrm{Ker}(\delta) \leq \bigoplus_{i=1}^{n} M.$$

After this remark in ordinary homology we return to the setting of Bredon homology. Since the dimension of $\underline{E}G$ is 2, the equivariant Atiyah-Hirzebruch spectral sequence reduces to the exact sequences in Theorem 3.6.7. Therefore these exact sequences describe the K-homology groups. For computing these groups, as we are dealing with a group extension, we appeal to the equivariant Lyndon-Hochschild-Serre spectral sequence, introduced in Section 3.6.2. Consider the split exact sequence $0 \to B \to G \to \mathbb{F}_n \to 0$ associated to the group $G = B \rtimes \mathbb{F}_n$. We need to understand $\mathrm{H}_i^{\mathfrak{F}}(G, \mathrm{R}_{\mathbb{C}})$, for $i = 0, 1, 2$. As $\bar{\mathfrak{H}}$ is the family of finite subgroups of $\mathbb{F}_n$, it consists of the trivial group only and its pull-back consists only of the group B. Hence the second page of this spectral sequence is

$$E^2_{p,q} = \mathrm{H}_p^{\bar{\mathfrak{H}}}(\mathbb{F}_n; \mathrm{H}_q^{\mathfrak{F}\cap B}(B; R_{\mathbb{C}})) = \mathrm{H}_p(\mathbb{F}_n; \mathrm{H}_q^{\mathfrak{F}\cap B}(B; R_{\mathbb{C}})).$$

We recall that the dimension of models for $\underline{E}B$ and $\underline{E}\mathbb{F}_n$ is one. Hence on the one hand, $\mathrm{E}^2_{p,q}$ is trivial for $p \geq 2$, and on the other hand, by Theorem 3.6.7 and Lemma 7.3.2, Bredon homology group of locally finite group B is trivial for $q \geq 1$. Therefore, the only non-zero terms are $p = 0, 1$ and $q = 0$. In particular, $\mathrm{E}^2_{0,0}$ and $\mathrm{E}^2_{1,0}$ are the only non-trivial terms of the E^2-page. This means that the E^2-page is concentrated on the horizontal axis and the spectral sequence collapses on this page as there is no differential. Accordingly, we have $\mathrm{E}^\infty_{p,q} = \mathrm{E}^2_{p,q}$ for $p, q \geq 0$. We recall that by Martìnez's result in [Mar02], the spectral sequence converges to $\mathrm{H}^{\mathfrak{F}}_{p+q}(G; R_{\mathbb{C}})$. Together with the discussion at the beginning of this section, we may identify Bredon homology groups with homology groups of $\mathbb{F}_n$ with coefficient in the free abelian group $\mathrm{H}^{\mathfrak{F}_B}_0(B; R_{\mathbb{C}})$. In particular, we need to compute

$$\begin{aligned} \mathrm{H}^{\mathfrak{F}}_0(G; R_{\mathbb{C}}) = \mathrm{E}^\infty_{0,0} = \mathrm{E}^2_{0,0} = \mathrm{H}_0(\mathbb{F}_n; \mathrm{H}^{\mathfrak{F}\cap B}_0(B; R_{\mathbb{C}})) \\ \mathrm{H}^{\mathfrak{F}}_1(G; R_{\mathbb{C}}) = \mathrm{E}^\infty_{1,0} = \mathrm{E}^2_{1,0} = \mathrm{H}_1(\mathbb{F}_n; \mathrm{H}^{\mathfrak{F}\cap B}_0(B; R_{\mathbb{C}})). \end{aligned} \tag{8.2.3}$$

Theorem 8.2.4. *Let $G = F \wr \mathbb{F}_n$ with F a non-trivial finite group.*

$\mathrm{K}^{\mathrm{G}}_0(\underline{E}G) = \mathbb{Z}R'$ *with R' a countable basis indexed by representatives for $\mathbb{F}_n$-orbits in $\hat{F}^{(\mathbb{F}_n)}$.*

$\mathrm{K}^{\mathrm{G}}_1(\underline{E}G) = \mathbb{Z}[v_1] \oplus \ldots \oplus \mathbb{Z}[v_n]$, *where $[v_i]$ is the canonical generator of $\mathrm{K}^{\mathbb{Z}}_1(\underline{E}\mathbb{Z})$ via the identification $\mathbb{Z} = \langle s_i \rangle$. Indeed, the inclusions $\langle s_i \rangle \hookrightarrow \mathbb{F}_n \hookrightarrow G$ give rise to an inclusion $\mathrm{K}^{\mathbb{Z}}_1(\underline{E}\mathbb{Z}) \hookrightarrow \mathrm{K}^{\mathbb{F}_n}_1(\underline{E}\mathbb{F}_n) \cong \mathrm{K}^{\mathrm{G}}_1(\underline{E}G)$.*

Proof. Due to Theorem 3.6.7 and the equations in 8.2.3,

$$\mathrm{K}^G_i(\underline{E}G) \cong \mathrm{H}^{\mathfrak{F}}_i(G; R_{\mathbb{C}}) \cong \mathrm{H}_i(\mathbb{F}_n; \mathrm{H}^{\mathfrak{F}\cap B}_0(B; R_{\mathbb{C}})),$$

for $i = 0, 1$. In order to compute the homological groups with the appropriate coefficients, we tensor the free resolution 8.2.1, at the beginning of this section, with the $\mathbb{F}_n$-module $\mathrm{H}^{\mathfrak{F}\cap B}_0(B; R_{\mathbb{C}})$. In view of Theorem 3.6.7 (applied to B) and Proposition 7.3.3, we have that

$$\mathrm{H}^{\mathfrak{F}\cap B}_0(B; R_{\mathbb{C}}) \cong \mathrm{K}^{\mathrm{B}}_0(\underline{E}B) \cong \mathbb{Z}\hat{F}^{(\mathbb{F}_n)}.$$

Therefore, we have

$$\mathrm{K}^G_0(\underline{E}G) \cong \mathrm{H}^{\mathfrak{F}}_0(G; R_{\mathbb{C}}) = \frac{\mathbb{Z}\hat{F}^{(\mathbb{F}_n)}}{\mathrm{Im}(\delta)} = \frac{\mathbb{Z}\hat{F}^{(\mathbb{F}_n)}}{\langle f - s_i \cdot f : f \in \mathbb{Z}\hat{F}^{(\mathbb{F}_n)}, 1 \leq i \leq n \rangle}.$$

Due to Remark 8.2.3 and Lemma 7.1.3, $\mathrm{K}^G_0(\underline{E}G)$ is a free abelian group on the orbit space of the action $\mathbb{F}_n \curvearrowright \hat{F}^{(\mathbb{F}_n)}$ with the described basis.

For computing $\mathrm{K}_1^G(\underline{\mathrm{E}}G)$, we appeal to Lemma 8.1.1. In view of that lemma for $X = \hat{F}^{(\mathbb{F}_n)}$ and $\mathfrak{p} = \mathbf{1}_F$, the trivial representation of F the kernel is

$$\mathrm{K}_1^G(\underline{\mathrm{E}}G) \cong \mathrm{H}_1^{\mathfrak{F}}(G; R_{\mathbb{C}}) = \mathrm{Ker}(\delta) = \underbrace{\mathbb{Z}\mathbf{1}_{\mathbf{1}_F} \oplus \ldots \oplus \mathbb{Z}\mathbf{1}_{\mathbf{1}_F}}_{n \;\; terms}.$$

In order to identify $\mathbf{1}_{\mathbf{1}_F}$ in i-th copy with $[v_i]$, we make some observations. As the groups $\mathbb{Z}$ and $\mathbb{F}_n$ are torsion free, we have that

$$\mathrm{K}_1^{\mathbb{Z}}(\underline{\mathrm{E}}\mathbb{Z}) \cong \mathrm{K}_i(\mathrm{B}\mathbb{Z}) \cong \mathrm{K}_1(\mathrm{S}^1) \quad \text{and} \quad \mathrm{K}_1^{\mathbb{F}_n}(\underline{\mathrm{E}}\mathbb{F}_n) \cong \mathrm{K}_i(\mathrm{B}\mathbb{F}_n) \cong \mathrm{K}_1(\bigvee_n \mathrm{S}^1),$$

where $\mathrm{B}G$ stands for the classifying space and $\bigvee_n \mathrm{S}^1$ denotes the wedge product of n-circles. Note that $\mathrm{H}_1(\mathbb{F}_n) = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ (n-times), and by Theorem 3.6.7, $\mathrm{K}_1^{\mathbb{F}_n}(\underline{\mathrm{E}}\mathbb{F}_n) \cong H_1(\mathbb{F}_n; \mathbb{Z})$. Moreover, due to functionality of K-theory and homology theory we have that

$$\begin{aligned} \mathrm{S}^1 \hookrightarrow \bigvee_n \mathrm{S}^1 \quad &\text{induces} \quad \mathrm{K}_1(\mathrm{S}^1) \hookrightarrow \mathrm{K}_1(\bigvee_n S^1), \\ \mathbb{Z} \to \mathbb{Z}\hat{F}^{(\mathbb{F}_n)} \colon \; 1 \mapsto \mathbf{1}_{\mathbf{1}_F} \quad &\text{induces} \quad \mathrm{H}_1(\mathbb{F}_n; \mathbb{Z}) \to \mathrm{H}_1(\mathbb{F}_n; \mathbb{Z}\hat{F}^{(\mathbb{F}_n)}). \end{aligned}$$

Now consider the following composition

$$\mathrm{K}_1^{\mathbb{Z}}(\underline{\mathrm{E}}\mathbb{Z}) \xrightarrow{\cong} \mathrm{K}_1(\mathrm{S}^1) \longrightarrow \mathrm{K}_1(\textstyle\bigvee_n S^1) \xrightarrow{\cong} \mathrm{H}_1(\mathbb{F}_n; \mathbb{Z}) \longrightarrow \mathrm{H}_1(\mathbb{F}_n; \mathbb{Z}\hat{F}^{(\mathbb{F}_n)}) \xrightarrow{\cong} \mathrm{K}_1^{\Gamma}(\underline{\mathrm{E}}\Gamma).$$

Choose $s_i \in \mathbb{F}_n$, and identify $\langle s_i \rangle \cong \mathbb{Z}$. By Theorem 3.6.7 (applied to $\mathbb{Z}$) we have that $\mathrm{K}_1^{\mathbb{Z}}(\underline{\mathrm{E}}\mathbb{Z}) \cong \mathrm{H}_1(\mathbb{Z}; \mathbb{Z}) = \mathbb{Z}[v_i]$. According to the composition above, $[v_i] \mapsto (0, \ldots, 0, \mathbf{1}_{\mathbf{1}_F}, 0, \ldots, 0) \in \mathrm{K}_1^G(\underline{\mathrm{E}}G)$. This finishes the proof. $\square$

8.3 Summary of computations

Let us summarise our computations.
Computations for the assembly map of the locally finite group $\oplus_{\mathbb{F}_n} F$.

	topological side	analytical side
K_0	$\mathbb{Z}\hat{F}^{(\mathbb{F}_n)}$	$\mathbb{Z}(\mathrm{Min}\,(\mathbb{C}F)^{(\mathbb{F}_n)})$
K_1	$\mathbb{Z}$	$\mathbb{Z}$

Computations for the assembly map of the wreath product $F \wr \mathbb{F}_n$.

	topological side	analytical side
K_0	coinv. of $\mathbb{F}_n \curvearrowright \hat{F}^{(\mathbb{F}_n)}$	coinv. of $\mathbb{F}_n \curvearrowright (\mathrm{Min}\;(\mathbb{C}F))^{(\mathbb{F}_n)}$
K_1	$\mathbb{Z}^n$	$\mathbb{Z}^n$

8.4 The Baum-Connes conjecture for $F \wr \mathbb{F}_n$

In this section, we explicitly describe the Baum-Connes assembly map for $F \wr \mathbb{F}_n$, where F is a non-trivial finite group. In doing so we make an identification between the bases of the both sides.

Theorem 8.4.1. *Let F be a non-trivial finite group.*

The Baum-Connes assembly map $\mu_0^G \colon \mathrm{K}_0^G(\underline{\mathrm{E}}G) \to \mathrm{K}_0(\mathrm{C}^*_{\mathrm{r}}(G))$ is an isomorphism between two countably generated free abelian groups.

The Baum-Connes assembly map $\mu_1^G \colon \mathrm{K}_1^G(\underline{\mathrm{E}}G) \to \mathrm{K}_1(\mathrm{C}^*_{\mathrm{r}}(G))$ is an isomorphism between two free abelian groups of rank n.

Proof. *μ_0^G is an isomorphism.* Let $\mathbb{F}_n$ be generated by $\{s_1, \dots, s_n\}$. Let $\iota \colon B \hookrightarrow G$ be the natural inclusion. Considering the diagram

$$\begin{tikzcd}
\bigoplus_{i=1}^n \mathrm{K}_0^{\mathrm{B}}(\underline{\mathrm{E}}B) \arrow[r, "\sigma_*"] \arrow[d, "\bigoplus_{i=1}^n \mu_0^B"'] & \mathrm{K}_0^{\mathrm{B}}(\underline{\mathrm{E}}B) \arrow[r, "\iota_*"] \arrow[d, "\mu_0^B"'] & \mathrm{K}_0^{\mathrm{G}}(\underline{\mathrm{E}}G) \arrow[r] \arrow[d, "\mu_0^G"'] & 0 \\
\bigoplus_{i=1}^n \mathrm{K}_0(\mathrm{C}^*(B)) \arrow[r, "\sigma_*"] & \mathrm{K}_0(\mathrm{C}^*(B)) \arrow[r, "\iota_*"] & \mathrm{K}_0(\mathrm{C}^*_{\mathrm{r}}G) \arrow[r] & 0,
\end{tikzcd}$$

with exact rows, functoriality of the assembly map yields the commutativity of the whole diagram. Moreover by Proposition 7.3.3, μ_0^B and hence $\bigoplus_{i=1}^n \mu_0^B$ are isomorphisms. The five lemma then implies that μ_0^G is an isomorphism.

μ_1^G is an isomorphism. Consider the comparison diagram

$$\begin{tikzcd}
\mathrm{K}_1^{\mathrm{G}}(\underline{\mathrm{E}}G) \arrow[r, "\mu_1^G"] & \mathrm{K}_1(\mathrm{C}^*_{\mathrm{r}}(G)) \\
\mathrm{K}_1^{\mathbb{F}_n}(\underline{\mathrm{E}}\mathbb{F}_n) \arrow[r, "\mu_1^{\mathbb{F}_n}"] \arrow[u] & \mathrm{K}_1(\mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_n)) \arrow[u] \\
\mathrm{K}_1^{\mathbb{Z}}(\underline{\mathrm{E}}\mathbb{Z}) \arrow[r, "\mu_1^{\mathbb{Z}}"] \arrow[u] & \mathrm{K}_1(\mathrm{C}^*(\mathbb{Z})) \arrow[u]
\end{tikzcd}$$

Due to Theorem 8.1.3 and Theorem 8.2.4, we have that $\mathrm{K}_1^{\mathrm{G}}(\underline{\mathrm{E}}G) \cong \mathrm{K}_1(\mathrm{C}^*_{\mathrm{r}}(G)) \cong \mathbb{Z}^n$ and $\mathrm{K}_1^{\mathrm{G}}(\underline{\mathrm{E}}G) = \mathrm{K}_1^{\mathbb{F}_n}(\underline{\mathrm{E}}\mathbb{F}_n) \cong \bigoplus_{i=1}^n \mathrm{K}_1^{\mathbb{Z}}(\underline{\mathrm{E}}\mathbb{Z})$. For $i = 1, \dots, n$, take the generator $[v_i]$ of the i-th summand in $\mathrm{K}_1^{\mathrm{G}}(\underline{\mathrm{E}}G)$. Write $\langle s_i \rangle \cong \mathbb{Z}$. The n-tuple $(0, \dots, 0, [v_i], 0, \dots, 0) \in \mathrm{K}_1^{\mathrm{G}}(\underline{\mathrm{E}}G)$ maps to $[v_i] \in \mathrm{K}_1^{\mathbb{Z}}(\underline{\mathrm{E}}\mathbb{Z})$. By the explicit description in [[MV03], Section II.2.4], we know that the assembly map $\mu_1^{\mathbb{Z}}$ transfers the generator on one side to the other. Moreover, by functoriality of K-theory we have that the isomorphism $\mathrm{K}_1(\mathrm{C}^*_\mathrm{r}(\mathbb{F}_n)) \cong \bigoplus_{i=1}^n \mathrm{K}_1(\mathrm{C}^*(\mathbb{Z}))$ respects the natural set of generators meaning the generator $(0, \dots, 0, [v_i], 0, \dots, 0)$ is mapped to $(0, \dots, 0, [u_i], 0, \dots, 0)$. In other words the assembly map μ_1^G is identified with $\mu_1^{\mathbb{F}_n}$.

□

8.5 Modified trace conjecture for $F \wr \mathbb{F}_n$

We close this chapter by short computations related to the modified trace conjecture of Section 4.2 for $\mathrm{K}_0(\mathrm{C}^*(F \wr \mathbb{F}_n))$. Notations are as in Section 7.8.

Let $G = F \wr \mathbb{F}_n$. Thanks to the surjectivity of $\iota_* \colon \mathrm{K}_0(\mathrm{C}^*(B)) \to \mathrm{K}_0(\mathrm{C}^*_\mathrm{r}(G))$, we only need to consider $\operatorname{Im} \tau_*(\mathrm{C}^*(B)) = \operatorname{Im} \tau_*(\mathbb{Z}(\mathrm{Min}(\mathbb{C}F)^{(\mathbb{F}_n)}))$. Therefore the computations in Proposition 7.8.2 imply the predicted result

$$\operatorname{Im} \tau_*(\mathrm{K}_0(\mathrm{C}^*_\mathrm{r}(G))) = \mathbb{Z}\Big[\frac{1}{|F|}\Big].$$

Chapter 9

Conclusion and perspective

This final chapter of this thesis contains two parts. In the first part, we describe the similarities and differences between tools we employed to compute K-theory and equivariant K-homology of our groups. In the second part, we discuss some problems raised from our work.

9.0.1 Conclusion

The groups we considered in our work contained both torsion free groups and groups with huge torsion. The complexity of computation hence increased in the presence of torsion. All these groups could be decomposed as a semidirect product

$$G = N \rtimes \mathbb{F}_n,$$

hence their reduced group C*-algebras could be viewed as a crossed product

$$\mathrm{C}^*_{\mathrm{r}}(G) = \mathrm{C}^*_{\mathrm{r}}(N) \rtimes \mathbb{F}_n.$$

Thanks to this decomposition, the Pimsner-Voiculescu 6-term exact sequence appeared as the main tool to compute the K-theory of this C*-algebra.

Turning to the topological side, we illustrated a 2-dimensional model for the classifying space $\underline{\mathrm{E}}G$ of all our groups. In order to compute equivariant K-homology, we consider two cases, namely the torsion free case and the case with torsion.

In the torsion free case we employed the identification between homological K-groups and ordinary homology groups provided by

$$\mathrm{K}_0(\underline{\mathrm{E}}G) \cong \mathrm{H}_0(\underline{\mathrm{E}}G) \oplus \mathrm{H}_2(\underline{\mathrm{E}}G), \qquad \mathrm{K}_1(\underline{\mathrm{E}}G) \cong \mathrm{H}_1(\underline{\mathrm{E}}G).$$

As a result of our computations, it turned out that the group $\mathrm{H}_2(\underline{E}G)$ vanished realising

$$\mathrm{K}_0(\underline{E}G) \cong \mathrm{H}_0(\underline{E}G), \qquad \mathrm{K}_1(\underline{E}G) \cong \mathrm{H}_1(\underline{E}G).$$

In the case with torsion we appealed to the Davis-Lück's picture to the left-hand side of the assembly map. In view of this picture our computations evolved around spectral sequences. Thanks to existence of a 2-dimensional model for $\underline{E}G$, the Atiyah-Hirzebruch spectral sequence reduced to

$$0 \longrightarrow \mathrm{H}_0^{\mathfrak{F}}(G;\mathrm{R}_{\mathbb{C}}) \longrightarrow \mathrm{K}_0^G(\underline{E}G) \longrightarrow \mathrm{H}_2^{\mathfrak{F}}(G;R_{\mathbb{C}}) \longrightarrow 0,$$

$$K_1^G(\underline{E}G) \cong \mathrm{H}_1^{\mathfrak{F}}(G;\mathrm{R}_{\mathbb{C}}).$$

This meant that we had to extract information from non-spliting short exact sequence comparing to the previous case. Due to the fact that our groups decomposed as semidirect product, we were eligible to apply the Lyndon-Hochschild-Serre spectral sequence in Bredon homology to describe the Bredon homology groups $\mathrm{H}_i^{\mathfrak{F}}(G;R_{\mathbb{C}})$, $i \in \{0,1,2\}$ in terms of the Bredon homology groups of $\mathrm{H}_p(\underline{E}\mathbb{F}_n;\mathrm{H}_q^{\mathfrak{F}}(\underline{E}N, R_{\mathbb{C}})$ with $p+q \in \{0,1,2\}$. Taking advantage of the fact that both $\underline{E}\mathbb{F}_n$ and $\underline{E}N$ were of dimension one, $\mathbb{F}_n$ was torsion free and N was locally finite group, we had that the Bredon homology group $\mathrm{H}_2^{\mathfrak{F}}(\underline{E}G,\mathrm{R}_{\mathbb{C}})$ vanished. Combining all these we got

$$\mathrm{K}_0^G(\underline{E}G) \cong \mathrm{H}_0^{\mathfrak{F}}(G;R_{\mathbb{C}}), \quad \mathrm{K}_1^G(\underline{E}G) \cong \mathrm{H}_1^{\mathfrak{F}}(G;R_{\mathbb{C}}).$$

Therefore in both cases namely groups with or without torsion, the homology group K_0^G was expressed in terms of H_0 and the homology group K_1^G was expressed in terms of H_1.

9.0.2 Perspective

Several natural problems were raised by our work. Most of them were asked during talks were given, mainly, in front of operator algebraists. We would like to discus a few of them here.

Question 9.0.1. *In Chapter 7 and Chapter 8, we described the Baum-Connes assembly map for the group $F \wr \mathbb{Z}$ and $F \wr \mathbb{F}_n$. While $\mathbb{F}_n$ is the noncommutative generalisation of $\mathbb{Z}$ we could consider $\mathbb{Z}^n$, its commutative generalisation. Therefore the question is to describe the Baum-Connes assembly map for $F \wr \mathbb{Z}^n$. Recall that this group can be viewed as semidirect* *product $\oplus_{\mathbb{Z}^n} F \rtimes \mathbb{Z}^n$. It seems that using iteratively the Pimsner-Voiculescu, we would be able to describe the* K*-theory of this crossed product. To compute its equivariant* K*-homology, we could follow the steps in the computations of the case with $F \wr \mathbb{Z}$. Due to the fact that this groups has a semidirect product decomposition, the Lyndon-Hochschild-Serre spectral sequence in Bredon homology would be our main tool. Employing that spectral sequence, the Bredon homology groups of the* E^2*-page of the Atiyah-Hirzebruch spectral sequence can be expressed as*

$$\mathrm{H}_p(\mathbb{Z}^n; \mathrm{H}_q^{\mathfrak{F}}((\oplus_{\mathbb{Z}^n} F))), \qquad p \in \{0, 1, \cdots, n\},\ q = 0.$$

Therefore, we would need to compute these groups.

Question 9.0.2. *Another generalisation could be to replace, this time, the base group F in the wreath product with the infinite group $\mathbb{Z}$. Therefore, a question is to describe the explicit Baum-Connes assembly map for the group $\mathbb{Z} \wr \mathbb{Z}$. Similar to Question 9.0.1, applying Pimsner-Voiculescu 6-term exact sequence we would be able to compute its* K*-theory. Its equivariant* K*-homology can be computed in a similar way as in other cases with semidirect product decomposition. Unlike the situation above this group is torsion free and it has infinite cohomological dimension, therefore after applying the Lyndon-Hochschild-Serre spectral sequence, we would need to compute ordinary homology groups with coefficient*

$$\mathrm{H_p}(\mathbb{Z}; \mathrm{H_q}((\oplus_{\mathbb{Z}} \mathbb{Z}))), \qquad p \in \{0, 1\}, q \in \mathbb{N}.$$

Note that for computing $\mathrm{H_q}(\oplus_{\mathbb{Z}} \mathbb{Z})$ we may appeal to the Künneth formula.

Question 9.0.3. *In Chapter 8 we described* K*-theory and equivariant* K*-homology for $F \wr \mathbb{F}_n$, where $n \in \mathbb{N}$. Another question would be to describe* K*-theory and equivariant* K*-homology for $F \wr \mathbb{F}_\infty$. Viewing $\mathbb{F}_\infty$ as a direct limit of $\mathbb{F}_n$'s, we could profit from continuity of* K*-theory,* K*-homology and the assembly map with respect to direct limit. It seems then plausible that, up to verifying the compatibility with respect to all these direct limits, we would be able to describe the Baum-Connes assembly map for this group.*

Question 9.0.4. *Let $G = F \wr \mathbb{F}_n$ and denote $B = \oplus_{\mathbb{F}_n} F$. The* C^**-algebra of G can be viewed as $\mathrm{C}^*(B) \rtimes \mathbb{F}_n$, where $\mathrm{C}^*(B)$ is an AF-algebra. If we replace $\mathrm{C}^*(B)$ with some other AF-algebra, in how far would we be able to describe the Baum-Connes assembly map for* C^**-algebra using our computations in Chapter 8? Note that this kind of* C^**-algebras does not arise necessarily as group* C^**-algebra.*

Question 9.0.5. *To what extend we would be able to describe the left-hand side of the Baum-Connes assembly map for well-known* C^**-algebras, for instance,* $A_\theta \rtimes \mathbb{Z}$*, where* $\theta \in \mathbb{R} \setminus \mathbb{Q}$*. The* C^**-algebra* A_θ *is called irrational rotational algebra and proved useful in operator algebras (see Section 2.3).*

We close this chapter by two remarks. The first one is on our joint work in progress with Alain Valette and Ramon Flores, where we study an explicit Baum-Connes assembly map for the group $\mathbb{Z}^2 \rtimes \mathrm{S}L_2(\mathbb{Z})$. This group is not a-T-menable (see discussion after Remark 2.1.13), therefore result of Higson-Kasparov does not apply to it. However, the conjecture is known to be true due to several results for the Baum-Connes conjecture with coefficients, for instance [Oyo01]. In contrast to our results discussed in this thesis, the classifying space $\underline{\mathrm{E}}(\mathbb{Z}^2 \rtimes \mathrm{S}L_2(\mathbb{Z}))$ has a 3 dimensional model which made computations on the left-hand side more complicated, in particular the exact sequences of Theorem 3.6.7 will not apply any more. For the right-hand side, we needed another six-term exact sequence rather than the one of Pimsner-Voculescu 2.3.2, which is due to Pimsner [P86].

The second remark is about a very recent work of Xin Li [Li18] related to our work. In [CEL13], Cuntz, Echterhoff and Li, among other results, computed the K-theory of the reduced C^*-algebra of $F \wr H$, where F is an *abelian* finite group and H satisfies the Baum-Connes conjecture with coefficients. Lately, subsequence to our articles [FPV16] and [Po17], Li extended their results and computed the K-theory of $\mathrm{C}^*_{\mathrm{r}}(F \wr H)$ where F is finite group and H is a countable group satisfying the Baum-Connes conjecture with coefficients.

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