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ON THE MATHEMATICAL FOUNDATIONS OF NETWORK THEORY

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On the Mathematical Foundations of Network Theory

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autorise l'impression de la présente thèse sans exprimer d'opinion sur les propositions qui y sont contenues.

Neuchâtel, le 22 mai 1975

Le doyen :

Willy Form

Willy Form

INTRODUCTION

In this work we have tried to give a rigorous approach to theorems of existence and uniqueness of solutions of networks in the steady state. As in the original works by Kirchhoff (1), Hermann Weyl (2), Dolezal (3) and Seshu (4), the basic mathematical structure involved is graph theory. The first two works mentioned above only consider networks of resistors and voltage sources; the third work considers networks of resistors, capacitors, coils and voltage sources but no current sources; the fourth considers networks of resistors, capacitors, coils and voltage and current sources but excludes perfect magnetic couplings and alternating current networks.

At first sight the theorems obtained in these works seem to be of little or no value when linear networks without the restrictions mentioned above are considered. (See the examples in chapter 3.) It is only when a distinction is made between series element networks (NS) and parallel element networks (NP) (5) that these theorems can be generalized for further use. Moreover, this distinction becomes essential since without it the concept of network is ambigious.

In order to be able to treat completely symmetrically networks NS and NP we introduce the shrinking process as the "dual" of the demolishing process (2). As far as we know the shrinking process and the distinction between networks NS and NP are new and they allow us to improve the existing theorems and formulate new ones.

The essential feature of the shrinking process is the following. If in a graph we shrink any arc and all the arcs in parallel with it we obtain a graph whose coloops (cut-sets) are also coloops of the original graph. (The demolishing process consists in demolishing (removing) an arc of a graph to obtain a subgraph whose loops are also loops of the original graph.)

The demolishing and shrinking processes can be used to localize incompatibilities (non-existence of a unique solution in the general case) and singularities (non-existence of a unique solution of alternating current networks). We also give a method to eliminate incompatibilities due to loops of voltage sources and coloops of current sources in consistent networks (with at least one solution).

In chapter 3, a few selected examples should illustrate the theory. They were constructed to give counterexamples to "natural" extentions of the theorems proved and point out some unexpected behavior.

Our point of view is not the only one possible. For instance, the work of Belevitch (6) also gives conditions for the existence and uniqueness of solutions of linear networks based on the systematic study of n-port networks. This approach has an obvious physical root whereas ours emphasizes the mathematical relation between Kirchhoff's laws and the graph of the network.

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0. NOTATIONS AND BASIC PRINCIPLES

(0.1) Cycles and loops.

A (directed) graph G is a triple (H,E,T) where $H = \{h_1,\dots,h_r\}$ is a finite set of elements called arcs, $E = \{u_1,\dots,u_s\}$ a finite set of elements called vertices and T is a function from H into $E \times E$. If $T(h_i) = (u_{i_1},u_{i_2})$, u_{i_1} is called the initial (u_{i_2} the terminal) vertex of h_i and we say h_i is directed from u_{i_1} to u_{i_2}. Both vertices are the end vertices of h_i. We shall consistently denote by r the number of arcs, s the number of vertices and p the number of connected components (pieces) of a directed graph.

Formal expressions $K = \sum c_i h_i$ with complex numbers c_i as coefficients are called 1-chains. Similarly, we define 0-chains to be formal sums $\sum d_j u_j$ with complex coefficients d_j. We say that an arc h_i belongs to (or appears in) K if its coefficient c_i does not vanish (and similarly for a vertex u_j in a 0-chain). With respect to componentwise addition and scalar multiplication, the set of 1-chains (resp. 0-chains) is a complex vector space of dimension r (resp. s). The boundary mapping ∂ from 1-chains to 0-chains, is defined by linearity from $\partial h_i = u_b - u_a$ if $T(h_i) = (u_a,u_b)$. A cycle is a 1-chain with 0 boundary: $\partial K = 0$. Thus the space of cycles is the kernel of the boundary operator (it is a linear subspace of the space of 1-chains).

Let $K = \sum c_i h_i$ be a 1-chain with coefficients $c_i = 0$ or ± 1 and denote by G_K the subgraph of G formed by the arcs appearing in K (and their end vertices), G_K^* the graph deduced from G_K by changing the orientation of h_i whenever $c_i = -1$. Then K is called a path if the arcs of G_K^* can be ordered in a sequence

$$h_{i_1},\dots, h_{i_q}$$

with terminal vertex of h_{i_k} being the initial vertex of $h_{i_{k+1}}$ $(k = 1,\dots,q-1)$. A path K is elementary if every vertex of G_K (or of G_K^*) is end vertex of at most two arcs of G_K. An elementary path which is simultaneously a cycle is a loop (q-loop if it contains q arcs). The following result is easily proved by induction.

Theorem 1. The subset of loops generates the subspace of cycles. More precisely, if $K = \Sigma c_i h_i$ is a cycle, then it is a linear combination of loops $K_k = \Sigma e_{kj} h_j$ with coefficients $e_{kj} \neq 0$ only when $c_j \neq 0$.

A matrix X (having r lines) whose columns consist of the components of a complete set of linearly independent loops of G is called "the" loop matrix of G.

(0.2) Forests and trees.

A forest of G is a subgraph with no loops. The connected components of a forests are trees. Consider a connected component G_1 of G. A connected subgraph of G_1 which contains all the vertices of G_1 but does not contain any loop is a maximal tree of G. If in each component of G we take a maximal tree, we obtain a maximal forest. We shall only consider maximal forests and maximal trees and therefore shall omit the word maximal. The arcs of the trees of a forest are called branches and the arcs not in the forests, chords.

Fix a tree G_1 of a connected graph G. Each chord of G_1 determines a unique fundamental loop as follows: there is a unique elementary path of G_1 going from the terminal to the initial vertex of the chord, which we close in a loop by adding the chord. (By convention, a 1-loop is also a fundamental loop.)

For later reference we list the following results.

Theorem 2. Every connected graph contains a tree.

Theorem 3. A subgraph G_1 of a connected graph G can be made part of a tree if and only if it contains no loop.

Theorem 4. Let G be a connected graph, H' and H" disjoint subsets of the set H of arcs of G such that H' is a set of chords of a tree G_1 and H" a set of branches of a tree G_2. Then there exists a tree G_3 for which H' is a set of chords and H" a set of branches.

(0.3) Coloops and cocycles.

Unless otherwise specified, vectors are always column vectors, and we systematically use the notation ${}^tA$ for the transpose and A^* for the adjoint (transpose of conjugate) of a matrix A. For example, if $c = (c_i)$ is an r-tuple, written as column vector, the corresponding 1-chain $K = \Sigma c_i h_i$ will also be written $K = {}^tc.h$ (or simply ${}^tch$):

matrix product of the line matrix ${}^tc = (c_1,\ldots,c_r)$ with the formal column matrix $h = (h_i)$ consisting of the arcs of G.

Let G be a graph, E' a subset of vertices of G. If h_i is an arc of G with end vertices $T(h_i) = (u_a,u_b)$ we define

$d_i = +1$ if $u_b \in E'$ and $u_a \notin E'$ (i.e. h_i goes into E'),

$d_i = -1$ if $u_a \in E'$ and $u_b \notin E'$ (i.e. h_i goes out of E'),

$d_i = 0$ otherwise (i.e. stays in or out of E').

Then the 1-chain $\omega(E') = {}^td\cdot h$ constructed by means of the line vector ${}^td = (d_1,\ldots,d_r)$ is the copath of E'. Clearly

$$\omega(E') = \sum_{u \in E'} \omega(u) \ .$$

A copath ω (= $\omega(E')$) is called a coloop when there exist disjoint connected subgraphs G_1 , G_2 with the following properties

i) each arc belonging to ω has one end vertex in G_1 and the other in G_2 ,

ii) G_1 , G_2 and the arcs belonging to ω make up a connected component of G.

(Intuitively, coloops describe links between two subgraphs.)

A matrix A (having r lines) whose columns consist of the components of a complete set of independent coloops of G is called "the" coloop matrix of G.

If G_1 is a (maximal) tree of G, the removal of any branch h of G_1 partitions it in two pieces

$$G_1' = (H_1',E_1',T_1') \ , \ G_1'' = (H_1'',E_1'',T_1'') \ .$$

If h has end vertices $T(h) = (u_a,u_b)$, the fundamental coloop associated with h (in G_1) is $\omega(E_1')$ if $u_b \in E_1'$ and $\omega(E_1'')$ if $u_b \in E_1''$.

Let $V = C^r$ be the (complex) vector space of column vectors with r components. Its dual V^* is the space of r-tuples written as line vectors. We identify V with the space of 1-chains and call W its subspace of cycles. The annihilator W^0 (or orthogonal) of W in V^* is called the space of cocycles. More precisely the 1-chain ${}^td\cdot h$ is a cocycle if for every $c = (c_i) \in W$, ${}^td\cdot c = \sum d_ic_i = 0$.

Theorem 5. A coloop is a cocycle. More explicitely, if $\omega(E') = {}^{t}d \cdot h$ *is a coloop, then* ${}^{t}d$ *is in the annihilator* W^{o} *of the space of cycles* $W \subset V = \mathbb{C}^{r}$.

(0.4) *The shrinking and demolishing processes.*

Let $G = (H,E,T)$ be a connected graph with no 1-loops. A vertex of G is, then, of *degree* n if it is an end vertex of exactly n arcs. Two arcs are *in series* if they have one common vertex of degree 2. The arcs $h_{i_1},\dots,h_{i_n}$ are *in parallel* between the vertices u,v, if they are incident at both vertices. If no other arcs are in parallel between u,v,

where $H_1 = H - \{h_{i_1},\dots,h_{i_n}\}$, $E_1 = E - \{v\}$ and

$$T_1(h_i) = T(h_i) \text{ if } u_{i_1} \neq v , \quad u_{i_2} \neq v$$

$$T_1(h_i) = (u,u_{i_2}) \text{ if } u_{i_1} = v$$

$$T_1(h_i) = (u_{i_1},u) \text{ if } u_{i_2} = v$$

is obtained from G *by shrinking* v *into* u.

Theorem 6. A coloop $\omega(E')$ *of* G_1 *is equal to a coloop of* G.

Proof: Two cases are possible: $u \in E'$, $u \notin E'$. In the first case $\omega(E')$ is equal to the coloop $\omega(E' \cup \{v\})$ of G. In the second case $\omega(E')$ is also a coloop of G.

We now give the shrinking process. Let $\omega(E')$ be a coloop of G; let $u \in E'$ and $v \in E - E'$ be the end vertices of an arc of $\omega(E')$. Let G_1 be obtained from G by shrinking v into u. If G_1 contains coloops consider one of them and proceed as above, etc. Since at each step we eliminate one and only one vertex, after $s - 1$ steps we obtain a subgraph consisting of one vertex.

Theorem 7. The shrinking process applied to G *gives a set of* $s - 1$ *linearly independent coloops of* G.

Proof: If $s = 2$ the theorem clearly holds. Beginning the shrinking process in G we obtain a coloop $\omega(E')$ and a subgraph G_1 with $s - 1$ vertices. By induction hypothesis, the shrinking process gives $s - 2$ linearly independent coloops of G_1 which, by the previous theorem, are also coloops of G. Since $\omega(E')$ contains one arc not contained in the other coloops, the $s - 1$ coloops are linearly independent.

An arc h_i is *essential* when it is not a coloop. If h_{i_1} is an essential arc of G, construct the loop ${}^tc^1.h$, say, containing h_{i_1}. Let G_1 be obtained from G by demolishing (removing) h_{i_1}. If h_{i_2} is an essential arc of G_1, construct the loop ${}^tc^2h$ in G_1 containing h_{i_2} and demolish it to obtain G_2. Continuing in this manner we obtain a subgraph G which is a tree. Obviously ${}^tc^1h,..,{}^tc^{\mu}h$ are loops of G and they are linearly independent since each one contains an arc not contained in the others. Since a tree has $s-1$ branches, we have:

Theorem 8. *G contains $\mu = r-(s-1)$ linearly independent loops*.

Observe that $\dim W + \dim W^0 = r$ (See (0.3)). Therefore we can summarize:

Theorem 9. *Let G be any graph*. *Then*

i) *the demolishing process gives a complete set of $\mu = r-(s-p)$ linearly independent loops*. *This set is a basis for the subspace of cycles*.

ii) *the shrinking process gives a complete set of $\rho = s-p$ linearly independent coloops*. *This set is a basis for the subspace of cocycles.*

We shall consistently denote by μ and ρ the number of linearly independent loops and coloops, respectively, of a graph G.

If X is "the" loop matrix and A is "the" coloop matrix of any graph G, then we clearly have ${}^tXA = {}^tAX = 0$.

Let G_1 be a tree of the connected graph G. The removal of any branch h_i from G_1 partitions it in two pieces, say, $G_2 = (H_2,E_2,T_2)$, $G_3 = (H_3,E_3,T_3)$. If u_{i_2} is in E_2 consider the coloop $\omega(E_2)$ in G, otherwise consider $\omega(E_3)$. Such a coloop is a *fundamental coloop*.

We now introduce a new concept: *shrinkage of arcs*.

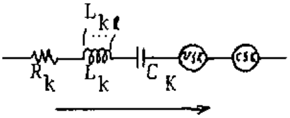
Let h_i be an arc of the connected graph G such that $T(h_i) = (u_a,u_b)$, $u_a \neq u_b$. If we remove h_i and coalesce u_a and u_b, the resulting graph, say G_1, is said to be obtained from G by shrinking the arc h_i. If there are no arcs in parallel with h_i, to shrink u_a into u_b is equivalent to shrinking h_i.

The graph G_1 contains one vertex less and thusly one coloop less than G. However, since the shrinkage of h_i decreases the number of arcs and the number of vertices of G by exacly one, G_1 and G have the same number of loops.

If we apply the same process to G_1, etc., after $\rho = s - 1$ steps we obtain a (daisy-shaped) connected graph, say G_ρ, with no coloop and with the same number of loops as G. Each of these loops is a 1-loop (since otherwise G_ρ would contain at least one coloop).

(0.5) <u>Kirchhoff laws</u>: <u>loop and coloop equations</u>.

Consider a <u>general series element</u> of the form



$k, \ell = 1, .., r.$

The resistor, the coil and the condenser are the <u>basic passive elements</u> (of the general series element), the voltage and current sources are the <u>basic active elements</u>. In a general series element some but not all of the basic elements may be absent; if the sources are absent the element is passive.

If with every arc of a graph G we associate a general series element we obtain a network NS. The arrow attached to every general series element, which coincides with the direction of the corresponding arc will serve as a reference for both the voltage drop V_k and the current I_k through the element. V_k and I_K are related by

$$V_k = R_k I_k + S_k D^{-1} I_k + \sum_\ell L_{k\ell} D I_\ell + V_{vsk} + V_{csk}$$

where D is the differential operator. We suppose that the matrix $(L_{k\ell})$ of the mutual impedances is positive semidefinite and that R_k , L_k, S_k are positive (if present).

If we define $Z_{kk} = R_k + L_k D + S_k D^{-1}$

$$Z_{k\ell} = L_{k\ell} D \qquad k, \ell = 1, .., r.$$

we can write $V_k = \sum Z_{k\ell} I_\ell + V_{vsk} + V_{csk}$. If $Z = (Z_{k\ell})$, $V = (V_i)$, $I = (I_i)$, $V' = (V_{vsi})$, $V'' = (V_{csi})$, then

$$(a)\quad V = ZI + V' + V''$$

$$(b)\quad {}^tXV = 0$$

$$(c)\quad {}^tAI = 0$$

are the equations of NS.

For topological purposes we shall identify a given network NS (below also an NP) with the corresponding graph. Thus we shall speak of the arcs, vertices, loops, etc. of NS.

Clearly the equations of NS are consistent only if the following conditions are satisfied:

i) if a loop of NS consists only of voltage sources, the sum of the voltage drops through them around the loop is equal to zero.

ii) if every arc of a coloop $\omega(E')$ of NS contains a current source, the sum of the currents through them and into E' is equal to zero.

We shall only consider networks NS for which the above conditions are satisfied. A network NS is <u>consistent</u> if its equations have at least one solution.

Consider the change of variables

$$(d)\quad I = XJ$$

where J is a vector of dimension $r - (s - p)$ whose elements are called loop currents. Since ${}^tA(XJ) = ({}^tAX)J = 0$ the change of variables is consistent. From (a), (b), (d) we obtain the system

$$(e)\quad {}^tXZXJ + {}^tXV'' = -{}^tXV'$$

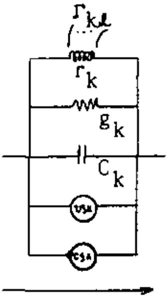
of $r - (s - p)$ equations with $r - (s - p) + c$ unknowns, where c is the number of current sources. Yet, (d) gives c more equations and no more unknowns.

Let $\overline{X}$ (resp. $\tilde{X}$) be the matrix obtained from X by eliminating the rows corresponding to the arcs not containing current (resp. voltage) sources. Let $\overline{V}$ (resp. $\overline{I}$) be obtained from V" (resp. I) by eliminating the rows corresponding to the arcs not containing current sources. Let $\tilde{V}$ be obtained from V' by eliminating the rows corresponding to the arcs not containing voltage sources. Then (d) and (e) give the loop equations of NS:

$$\begin{bmatrix} {}^tXZX & {}^t\overline{X} \\ \overline{X} & 0 \end{bmatrix} \begin{bmatrix} J \\ \overline{V} \end{bmatrix} = \begin{bmatrix} -{}^t\tilde{X}\tilde{V} \\ \overline{I} \end{bmatrix}$$

where ${}^tXZX$ is the well known loop impedance matrix.

If with every arc of a graph G we associate a general parallel element of the form shown in the figure, where some but not all of the basic elements may be absent, we obtain a network NP.



The attached arrow, which coincides with the direction of the corresponding arc, serves as a reference direction for both the voltage drop V_k and the current I_k through the element. They are related by

$$I_k = g_kV_k + C_kDV_k + \sum_\ell \Gamma_{k\ell}D^{-1}V_\ell + I_{csk} + I_{vsk} \qquad k,\ell = 1,..,r.$$

We suppose that the matrix $(\Gamma_{k\ell})$ of the invertances is positive semidefinite and g_k, C_k, Γ_k are positive (if present). If we define

$$Y_{kk} = g_k + C_kD + \Gamma_kD^{-1}$$
$$Y_{k\ell} = \Gamma_{k\ell}D^{-1}$$

we can write

$$I_k = \Sigma Y_{k\ell}V_\ell + I_{csk} + I_{vsk} .$$

If $Y = (Y_{k\ell})$, $I = (I_i)$, $V = (V_i)$, $I' = (I_{vsi})$, $I'' = (I_{csi})$ the equations of NP are:

(a) $I = YV + I' + I''$

(b) ${}^tXV = 0$

(c) ${}^tAI = 0$.

Clearly the above equations are consistent only if the following conditions are satisfied:

i) if a coloop $\omega(E')$ of NP consists only of current sources, the sum of the currents through them and into E' is equal to zero.

ii) if every arc of a loop of NP contains a voltage source, the sum of the voltage drops through them around the loop is equal to zero.

Compare these conditions with the corresponding ones for a network NS. The reader should keep in mind that an arc of a network NS is a general series element while an arc in a network NP is a general parallel element.

We shall only consider networks NP for which the above conditions are satisfied.

A network NP is *consistent* if its equations have at least one solution.

Consider the (consistent) change of variables

(d) $V = AU$

where U is a vector of dimension $s - p$ whose elements are called *coloop voltages* (*or potentials*). Then from (a), (c), (d), we obtain the system

(e) ${}^{t}AYAU + {}^{t}AI' = -{}^{t}AI''$

of $s - p$ equations with $(s - p) + f$ unknowns, where f is the number of voltage sources of NP. Yet, (d) gives f more equations and no more unknowns.

Let $\tilde{A}$ (resp. $\bar{A}$) be the matrix obtained from A by eliminating the rows corresponding to the arcs not containing voltage (resp. current) sources. Let $\tilde{V}$ (resp. $\bar{I}$) be the vector obtained from V (resp. I") by eliminating the rows corresponding to the arcs not containing voltage (resp. current) sources. Let $\tilde{I}$ be obtained from I' by eliminating the rows corresponding to the arcs not containing voltage sources. Then (d) and (e) give the *coloop equations* of NP:

$$\begin{bmatrix} {}^{t}AYA & {}^{t}\tilde{A} \\ \tilde{A} & 0 \end{bmatrix} \begin{bmatrix} U \\ \tilde{I} \end{bmatrix} = \begin{bmatrix} -{}^{t}\bar{A}\bar{I} \\ \tilde{V} \end{bmatrix}$$

where ${}^{t}AYA$ is the well known coloop admittance matrix.

Notice that by appropriately grouping the basic elements, any network NS (resp. NP) can be considered as an NP (resp. NS) if and only if the matrix $(L_{k\ell})$ (resp. $(\Gamma_{k\ell})$) is positive definite. However, the corresponding graphs will in general be distinct.

We shall denote by NSP (or simply N) a network that can be considered either as an NS or as an NP.

Observe that for a network NSP the following conditions are equivalent:

i) NSP as an NS (resp. NP) contains a coloop (resp. loop) such that every arc has a current (resp. voltage) source.

ii) NSP as an NP (resp. NS) contains a coloop (resp. loop) consisting only of current (resp. voltage) sources.

It is important to realize that we have not defined the concept of network. Instead we have defined two concepts: that of network NS and that of network NP.

1. COMPATIBILITY

(1.1) *Compatible and incompatible networks*.

Let NS (resp. NP) be a network of general series (resp. parallel) elements and let

$$\Delta_1 = \det \begin{bmatrix} {}^t XZX & {}^t\bar{X} \\ \bar{X} & 0 \end{bmatrix} \qquad \Delta_2 = \det \begin{bmatrix} {}^t AYA & {}^t\tilde{A} \\ \tilde{A} & 0 \end{bmatrix} .$$

We know that if $\Delta_1 = 0$ (resp. $\Delta_2 = 0$), that is if it is the zero polynomial, the loop (resp. coloop) equations of NS (resp. NP) cannot have a unique solution. More precisely,

Lemma 1. *A necessary condition that a network* NS (resp. NP) *have a unique solution is that* Δ_1 (resp. Δ_2) *not be the zero polynomial*.

Definition 1. *A network* NS (resp. NP) *of general series* (resp. *parallel*) *elements is compatible if* $\Delta_1 \neq 0$ (resp. $\Delta_2 \neq 0$).

Proposition 1. *For any network* N, *the following conditions are equivalent*:

a) N *contains a forest* G_1 *such that the voltage sources are branches and the current sources are chords*.

b) *no loop of* N *consists only of voltage sources and no coloop of N consists only of current sources*.

Proof: Without loss of generality we may suppose that N is connected and every basic element is an arc.

Suppose that (a) holds. Since a tree contains no loops, no loop consists only of voltage sources. Now, every coloop contains at least one branch of every tree, since otherwise the removal of the arcs of the coloop would not split the network. Thus, no coloop consists only of current sources.

Suppose that (b) holds. Since no loop consists only of voltage sources, by (0.2) the voltage sources can be made part of a tree G_1. Remove all the current sources from N; the network, say N', thus obtained contains all vertices of N and is connected. By (0.2) there is a tree G_2 of N', which obviously is also a tree of N, such that the current sources are chords. (0.2) proves that there is a tree satisfying (a).

(1.2) *Some conditions for compatibility*.

Proposition 2. *A necessary condition that a network* NS (resp. NP) *be compatible is that it satisfy the equivalent conditions of Proposition 1.*

Proof: Suppose that NS (resp. NP) does not satisfy (b). Then, either (i) NS (resp. NP) contains a loop, say ${}^t ch$, consisting only of voltage sources, or (ii) NS (resp. NP) contains a coloop, say ${}^t dh$, consisting only of current sources. We have to show that either (i) or (ii) implies that $\Delta_1 = 0$ (resp. $\Delta_2 = 0$).

If NS (resp. NP) satisfies (i) (resp. (ii)), choose a complete set of linearly independent loops (resp. coloops) containing ${}^t ch$ (resp. ${}^t dh$). Then Δ_1 (resp. Δ_2) contains a row of zeros. If NS (resp. NP) satisfies (ii) (resp. (i)), the only elements different from zero in ${}^t d$ (resp. ${}^t c$) are those corresponding to the arcs of the coloop (resp. loop) consisting only of current (resp. voltage) sources. Since ${}^t dX = 0$ (resp. ${}^t cA = 0$), we see that the rows of $\overline{X}$ (resp. $\tilde{A}$) are linearly dependent.

Theorem 1. *Consider a network* NS (resp. NP) *such that the arcs containing current* (resp. *voltage*) *sources are chords* (resp. *branches*) *of a forest*. *Let* $\tilde{N}$S (resp. $\overline{N}$P) *be the network obtained from* NS (resp. NP) *by removing* (resp. *shrinking one end vertex into the other end vertex of*) *the arcs containing current* (resp. *voltage*) *sources*. *Then* NS (resp. NP) *is compatible if and only if* $\tilde{N}$S (resp. $\overline{N}$P) *is compatible*.

Proof: By the demolishing (resp. shrinking) procees we obtain a fundamental loop (resp. coloop) for each arc containing a current (resp. voltage) source. Considering these loops (resp. coloops) first, the loop (resp. coloop) equations can be written in the form $(_*)$ (resp. $(_{**})$):

$$(_*)\quad \left[\begin{array}{ccccc}
z_{11}\cdots & z_{1f} & z_{1f+1}\cdots & z_{1\mu} & 1..0 \\
\hdashline\hdashline
z_{1f}\cdots & z_{ff} & z_{ff+1}\cdots & z_{f\mu} & 0..1 \\
z_{f+11}\cdots & z_{f+1f} & z_{f+1f+1}\cdots & z_{f+1\mu} & 0..0 \\
\hdashline\hdashline
z_{\mu 1}\cdots & z_{\mu f} & z_{\mu f+1}\cdots & z_{\mu\mu} & 0..0 \\
1\ .....\ & 0 & 0\ ....\ & 0 & 0..0 \\
\hdashline\hdashline
0\ .....\ & 1 & 0\ ....\ & 0 & 0..0
\end{array}\right]
\left[\begin{array}{c}
J_1 \\ \hdashline\hdashline J_f \\ J_{f+1} \\ \hdashline\hdashline J_\mu \\ V_{cs1} \\ \hdashline\hdashline V_{csf}
\end{array}\right]
=
\left[\begin{array}{c}
-\tilde{V}_1 \\ \hdashline\hdashline -\tilde{V}_f \\ -\tilde{V}_{f+1} \\ \hdashline\hdashline -\tilde{V}_\mu \\ I_{cs1} \\ \hdashline\hdashline I_{csf}
\end{array}\right]$$

$$(**)\quad \left[\begin{array}{llllll} y_{11}\cdots & y_{1g} & y_{1\,g+1} & \cdots\; y_{1\rho} & 1..0 \\ \hdashline \hdashline y_{g1}\cdots & y_{gg} & y_{gg+1} & \cdots\; y_{g\rho} & 0..1 \\ y_{g+1\,1}\cdot\cdot & y_{g+1\,g} & y_{g+1\,g+1} & \cdot\cdot\; y_{g+1\rho} & 0..0 \\ \hdashline \hdashline y_{\rho 1}\cdots & y_{\rho g} & y_{\rho g+1} & \cdots\; y_{\rho\rho} & 0..0 \\ 1\;\ldots\ldots & 0\;\ldots & 0\;\ldots\ldots & 0 & 0..0 \\ \hdashline \hdashline 0\;\ldots\ldots & 1\;\ldots & 0\;\ldots\ldots & 0 & 0..0 \end{array}\right] \left[\begin{array}{l} U_1 \\ \hdashline \hdashline U_g \\ U_{g+1} \\ \hdashline \hdashline U_\rho \\ I_{vs1} \\ \hdashline \hdashline 1_{vsg} \end{array}\right] = \left[\begin{array}{l} -\bar{I}_1 \\ \hdashline \hdashline -\bar{I}_g \\ -\bar{I}_{g+1} \\ \hdashline \hdashline -\bar{I}_\rho \\ V_{vs1} \\ \hdashline \hdashline V_{vsg} \end{array}\right]$$

where f (resp. g) is the number of current (resp. voltage) sources and $\tilde{V}_i$ (resp. $\bar{I}_i$) is the sum of the voltages (resp. currents) of the voltage (resp. current) sources in the loop (resp. coloop) i. Clearly the determinant of the coefficient matix of (*) (resp. (**)) is given by

$$\begin{bmatrix} z_{f+1f+1} & \cdots & z_{f+1f+1} \\ z_{\kappa f+1} & \cdots & z_{\mu\kappa} \end{bmatrix} \qquad \begin{bmatrix} y_{g+1g+1} & \cdots & y_{g+1g+1} \\ y_{\rho\, g+1} & \cdots & y_{\rho\rho} \end{bmatrix}$$

which is the coefficient matrix of the loop (resp. coloop) equations of $\tilde{N}S$ (resp. $\bar{N}P$).

(1.3) <u>Compatibility conditions for networks</u> NS (resp. NP) <u>with no current</u> (resp. <u>voltage</u>) <u>sources</u>.

The following lemma follows immediately from the definition of compatibility.

<u>Lemma 2</u>. <u>A network</u> NS (resp. NP) <u>with no current</u> (resp. <u>voltage</u>) <u>sources is compatible if and only if</u>

$\det({}^tXZX) \neq 0$ (resp. $\det({}^tAYA) \neq 0$).

<u>Lemma 3</u>. <u>If R is a square symmetrical real matrix, then for every vector</u> c <u>of the proper dimension the number</u> c^*Rc <u>is real</u>.

Proof: Let $c = a + ib$, where a and b are real vectors. Then $c^*Rc = ({}^ta - i\,{}^tb)R(a + ib) = {}^taRa + {}^tbRb + i({}^taRb - {}^tbRa)$. Since R is symmetrical ${}^taRb = {}^t({}^taRb) = {}^tbRa$. Therefore c^*Rc is real.

<u>Proposition 3</u>. <u>A network</u> NS (resp. NP) <u>with no current</u> (resp. <u>voltage</u>) <u>sources is compatible if for every cycle</u> (resp. <u>cocycle</u>) ${}^tch \neq 0$ (resp, ${}^tdh \neq 0$), $c^*Zc \neq 0$ (resp. $d^*Yd \neq 0$).

Proof: Suppose that $\det({}^tXZX) = 0$ (resp. $\det({}^tAYA) = 0$). Then the system of equations ${}^tXZXy = 0$ (resp. ${}^tAYAy = 0$) has a non-zero solution. Yet, if y_1 is such a solution, $y_1^{*}{}^tXZXy_1 = 0$ (resp. $y_1^{*}AYAy_1 = 0$) which implies that ${}^tch$ (resp. ${}^tdh$), with $c = Xy_1$ (resp. $d = Ay_1$) is a non-zero cycle (resp. cocycle) for which $c^*Zc = (Xy_1)^*Z(Xy_1) = y_1^{*}{}^tXZXy_1 = 0$ (resp. $d^*Yd = (Ay_1)^*Y(Ay_1) = y_1^{*}{}^tAYAy_1 = 0$).

Theorem 2. *A network* NS, *with no current sources, is compatible if it contains a resistor or a condenser in every loop.*

Proof: Let $K = {}^tch = \sum c_ih_i \neq 0$ be a cycle. By (0.1) there exist loops $K_j = \sum e_{ij}h_i$, $j = 1,..,q$, say, with $e_{ij} \neq 0$ only if $c_i \neq 0$, such that $K = \sum \alpha_j K_j$. Since NS contains a resistor or a condenser in every loop, for every $j \in \{1,..,q\}$ there exists $k \in \{1,..,r\}$ such that $R_k + D^{-1}S_k \neq 0$ and $e_{kj} \neq 0$. Yet this implies that there exists $k \in \{1,..,r\}$ such that $c_k \neq 0$ and $R_k + D^{-1}S_k \neq 0$. If $c^* = {}^ta - i\,{}^tb = [a_1 \dots a_r] - i\,[b_1 \dots b_r]$, by lemma 3, $c^*(R + D^{-1}S)c = {}^taRa + D^{-1}aSa + {}^tbRb + D^{-1}bSb = \sum a_k^2R_k + D^{-1}\sum a_k^2S_k + \sum b_k^2R_k + D^{-1}\sum b_k^2S_k$. Since at least one of the sums is different from zero, $c^*Zc \neq 0$.

Theorem 3. *Let* NS *be a network with no current sources. Suppose that the matrix* $(L_{k\ell})$ *is positive definite. Then* NS *is compatible if*

(i) *it contains a resistor, a condenser or a coil in every loop, or equivalently,*

(ii) *no loop consists only of voltage sources.*

Proof: Let ${}^tch = \sum c_ih_i \neq 0$ be a cycle. As in the proof of Theorem 2 we can show that there exists $k \in \{1,..,r\}$ such that $c_k \neq 0$, $R_k + DL_k + D^{-1}S_k \neq 0$. If $c = a + ib$, by Lemma 3, $c^*Zc = {}^taRa + D^{-1}aSa + D\,{}^taLa + {}^tbRb + D^{-1}bSb + D\,{}^tbLb$. Let a_L, b_L, be the vectors formed by the elements of a, b, respectively, corresponding to the rows of L such that $L_k \neq 0$. It is easy to see that ${}^taLa = {}^ta_L(L_{k\ell})a_L$, ${}^tbLb = {}^tb_L(L_{k\ell})b_L$. Therefore, $c^*Zc \neq 0$, since at least one of the sums ${}^taRa, \dots, {}^tbLb$, is different from zero.

(1.4) *The main theorems of compatibility*.

Theorem 4. *Let* NS *be a network such that the matrix* $(L_{k\ell})$ *is positive definite*. *Then* NS *is compatible if it contains no loop consisting only of voltage sources and no coloop consisting only of current sources*.

Proof: Suppose NS satisfies the conditions of the theorem. Remove every arc containing a current source. Then, by Theorem 1, the network $\widetilde{NS}$ thus obtained is compatible if and only if NS is compatible. Yet, by Theorem 3, $\widetilde{NS}$ is compatible.

Theorem 5. *Let* NS *be a network such that the matrix* $(L_{k\ell})$ *is positive definite*. *Then*, *if* NS *is incompatible, it contains either a loop consisting only of voltage sources or a coloop consisting only of current sources*.

Proof: Suppose NS contains no coloop consisting only of current sources. Remove the arcs containing current sources. Then, by Theorem 1, the network $\widetilde{NS}$ thus obtained is incompatible. Therefore $\widetilde{NS}$ contains a loop consisting only of voltage sources, since otherwise, by Theorem 3, it would be compatible. Clearly such a loop is also a loop of NS.

It is easy to see that the dual theorems of theorems 2,3,4,5, are true for planar networks. More precisely,

Theorem 6. *A planar network* NP *with no voltage sources is compatible if it contains a resistor or a condenser in every coloop*.

Theorem 7. *Let* NP *be a planar network with no voltage sources*. *Suppose that the matrix* $(\Gamma_{k\ell})$ *is positive definite*. *Then* NP *is compatible if*

(i) *it contains a resistor, a condenser or a coil in every coloop, or equivalently*

(ii) *it contains no coloop consisting only of current sources*.

Theorem 8. *Let* NP *be a planar network such that the matrix* $(\Gamma_{k\ell})$ *is positive definite*. *Then* NP *is compatible if it contains no loop consisting only of voltage sources and no coloop consisting only of current sources*.

Theorem 9. *Let* NP *be a planar network such that the matrix* $(\Gamma_{k\ell})$ *is positive definite*. *Then*, *if* NP *is incompatible it contains a loop consisting only of voltage sources or a coloop consisting only of current sources*.

The following theorem is clear.

Theorem 10. *If a network* NSP *is compatible* (resp. *incompatible*) *as an* NP (resp. NS), *then it is also compatible* (resp. *incompatible*) *as an* NS (resp. NP).

The next theorem follows from Theorem 9 of (0.4).

Theorem 11. *The subset of coloops generates the subspace of cocycles*. *More precisely*, *if* $L = \Sigma d_i h_i$ *is a cocycle*, *then it is a linear combination of coloops* $L_k = \Sigma f_{kj} h_j$.

If in the last theorem we could add (as in Theorem 1 of (0.1)): having coefficients $f_{kj} \neq 0$ only when $d_j \neq 0$, then Theorems 6, 7, 8, and 9 would be valid for non-planar networks as can be easily verified by reviewing the proofs of the dual theorems.

We finish this section with the next obvious result.

Theorem 12.

i) *a planar network* NSP *is compatible as an* NS *if and only if it is compatible as an* NP.

ii) *if a planar network* NSP *is incompatible*, *then it contains a loop consisting only of voltage sources or a coloop consisting only of current sources*.

(1.5) *Removal of incompatibilities*.

Consider a network NS (resp. NP) and suppose it has loops (resp. coloops) consisting only of voltage (resp. current) sources. By the demolishing (resp. shrinking) process obtain as many linearly independent loops (resp. coloops) as possible consisting only of voltage (resp. current) sources and construct the loop matrix X (resp. coloop matrix A) so that its first, say n, columns correspond to these loops (resp. coloops). We suppose that $n < r - (s - p)$ (resp. $n < s - p$). Observe that at each of the first n demolishing (resp. shrinking) steps we remove (resp. shrink) an arc consisting of a single voltage (resp. current) source. For this choice of loops (resp. coloops), the loop (resp. coloop) equations take the form:

$$\begin{bmatrix} O_{11} & O_{12} & O_{13} \\ O_{21} & E_{22} & E_{23} \\ O_{31} & E_{32} & O_{33} \end{bmatrix} \begin{bmatrix} J^1 \\ J^2 \\ \bar{V}^3 \end{bmatrix} = \begin{bmatrix} O_1 \\ V^2 \\ \bar{I}^3 \end{bmatrix} \qquad \begin{bmatrix} O_{11} & O_{12} & O_{13} \\ O_{21} & F_{22} & F_{23} \\ O_{31} & F_{32} & O_{33} \end{bmatrix} \begin{bmatrix} U^1 \\ U^2 \\ \tilde{I}^3 \end{bmatrix} = \begin{bmatrix} O_1 \\ 1^2 \\ \tilde{V}^3 \end{bmatrix}$$

where O_1 and the O_{ij} are null matrices. In particular O_{11} is the null matrix of order n, the number of linearly independent loops (resp. coloops) consisting only of voltage (resp. current) sources. E_{22} (resp. F_{22}) is the impedance (resp. admittance) matrix of the network, say NS' (resp. NP'), obtained from NS (resp. NP) by removing from NS (resp. shrinking in NP) the n voltage (resp. current) sources we demolished (resp. shrank) when we obtained the matrix X (resp. A). Moreover, the other matrices are such that the loop (resp. coloop) equations of NS' (resp. NP') are given by

$$\begin{bmatrix} E_{22} & E_{23} \\ E_{32} & O_{33} \end{bmatrix} \begin{bmatrix} J^2 \\ \bar{V}^3 \end{bmatrix} = \begin{bmatrix} V^2 \\ \bar{I}^3 \end{bmatrix} \qquad \begin{bmatrix} F_{22} & F_{23} \\ F_{32} & O_{33} \end{bmatrix} = \begin{bmatrix} U^2 \\ \tilde{I}^3 \end{bmatrix} \begin{bmatrix} 1^2 \\ \tilde{V}^3 \end{bmatrix} .$$

Comparing the loop (resp. coloop) equations of NS and NS'(resp. NP an NP') we see that the solutions of the former can be obtained from the solutions of the latter by assigning arbitrary values to the loop currents $J_1,..,J_n$, (resp. coloop voltages $U_1,..,U_n$) of the loops of NS (resp. coloops of NP) consisting only of voltage (resp. current) sources.

Observe that $J_1,..,J_n$, (resp. $U_1,..,U_n$) give the currents (resp. voltage drops) through the voltage (resp. current) sources we removed from NS (shrank in NP) to obtain NS' (resp. NP') and that these loop currents (resp. coloop voltages) are not needed to determine the currents (resp. voltages) through the arcs of NS (resp. NP) other than those belonging to the loops (resp. coloops) consisting only of voltage (resp. current) sources.

The following theorem follows immediately from the above discussion.

Theorem 13. *From any incompatible network* NSP *satisfying the conditions of section* (0.5) *about sums of voltages* (resp. *currents*) *in loops* (resp. *coloops*) *consisting only of voltage* (resp. *current*) *sources, we can always obtain a compatible network* NSP' *applying the following process*. *If* NSP *contains a loop consisting only of voltage sources, remove any one of these voltage sources*. *If the resulting network contains a loop consisting only of voltage sources, proceed as above and continue the process until we obtain a network, say* NSP_1, *with no such a loop*. *Either* NSP_1 *is compatible or it contains at least one coloop consisting only of current sources*. *If* NSP_1 *contains a coloop consisting only of current sources, shrink any one of these current sources*. *If the resulting network contains a coloop consisting only of current sources, proceed as above and continue the process until we obtain the compatible network* NSP' *with neither loops consisting only of voltage sources nor coloops consisting only of current sources*. *Clearly, proceeding as above, solutions of* NSP' *furnish solutions of* NSP.

2. SINGULARITY

(2.1) Alternating current networks.

Consider the set S of all sine functions or sinusoids of the form $f(t) = A \sin(\omega t + \alpha)$ for a fixed real angular frequency $\omega > 0$ and for all values of the amplitude A and the phase angle α.

Any sinusoid $f(t) = A \sin(\omega t + \alpha)$ can be expressed in the form $f(t) = a \sin \omega t + b \cos \omega t$, where $a = A \cos\alpha$, $b = A \sin\alpha$. If

$f_1(t) = A_1 \sin(\omega t + \alpha_1) = a_1 \sin \omega t + b_1 \cos \omega t$

$f_2(t) = A_2 \sin(\omega t + \alpha_2) = a_2 \sin \omega t + b_2 \cos \omega t$, we define

$f_1(t) + f_2(t) = a_3 \sin \omega t + b_3 \cos \omega t$, where $a_3 = a_1 + a_2$, $b_3 = b_1 + b_2$.

Thus, $f_1(t) + f_2(t) = A_3 \sin(\omega t + \alpha_3)$, where $A_3 = (a_3^2 + b_3^2)^{\frac{1}{2}}$, $\alpha_3 = \tan^{-1}(b_3/a_3)$. We also define the product $rf(t)$, for any real number r, as $rf(t) = (rA) \sin(\omega t + \alpha)$.

It is clear that S with the operations of addition and scalar multiplication, as defined above, is a 2-dimensional vector space. Since the set $\mathbb{C}$ of all complex numbers is also a 2-dimensional real vector space, S and $\mathbb{C}$ are isomorphic.

Consider the mapping $\varphi: S \longrightarrow \mathbb{C}$ defined as follows: if $f(t) \in S$, $\varphi(f) = f(\pi/2\omega) - (i/\omega) f'(\pi/2\omega)$, where f' is the derivative of f. It is easy to see that φ is an isomorphism of S onto $\mathbb{C}$. The reciprocal (inverse) isomorphism ψ of φ is defined as follows: if $z = a + ib$, then $\psi(z) = \mathrm{Re}\,(-ize^{i\omega t})$.

If D is the differential operator, we can easily verify that $\varphi(Df) = i\omega\varphi(f)$, $\varphi(D^{-1}f) = (i\omega)^{-1}\varphi(f)$.

A network (NS or NP) all of whose voltage and current sources are sinusoids of the form $f(t) = A \sin(\omega t + \alpha)$, where $\omega > 0$ is fixed, is an alternating current network. The main problem of alternating current theory is to find functions of the same sinusoidal form representing the voltage drop and the current in some or all of the arcs such that they satisfy the equations of the network. We solve this problem, when possible, as follows: we transform, by means of the isomorphism φ, the system of equations of the alternating current network into an algebraic system for which we find the solutions, then finally we obtain the voltage drop and the current in the arcs by using the isomorphism φ. Since we shall usually be concerned only with the existence and uniqueness of the solutions, we shall focus our attention on the algebraic transformed system.

The transformed relations between the voltage drop and the current for the basic elements are shown below.

Resistor	$V = RI$	$I = g/V$
Condenser	$V = (S/i\omega)I$	$I = i\omega CV$
Isolated coil	$V = i\omega LI$	$I = (\Gamma/i\omega)V$
System of n coils	$V_k = \Sigma i\omega L_{k\ell} I_\ell$ $k = 1,..,n.$	$I_k = \Sigma(\Gamma_{k\ell}/i\omega)\ V_\ell$ $k = 1,..,n.$
Voltage source	$V = V_{vs}$	Current source $I = I_{cs}$
Current source	$V = V_{cs}$	Voltage source $I = I_{vs}$

In what follows, the networks to be considered will be alternating current networks, thus the word network will mean alternating current network.

For a network NS we have the transformed equations

$V_k = R_k I_k + i\omega \Sigma L_{k\ell} I_\ell + (S_k/i\omega) I_k + V_{vsk} + V_{csk} \quad k = 1,..,r.$

${}^tXV = 0 \ , \quad {}^tAI = 0 \ .$

Similarly, for a network NP we have

$I_k = g_k V_k + i\omega C_k V_k + (i\omega)^{-1} \Sigma \Gamma_{k\ell} V_\ell + I_{vsk} + I_{csk} \quad k = 1,..,r.$

${}^tXV = 0 \ , \quad {}^tAI = 0 \ .$

Thus, we see that the loop and coloop equations of an alternating current network can be obtained from the corresponding ones in the general case by simply replacing D by $i\omega$. In particular, the proper and mutual impedances of a general series element of a network NS are given by

$$\begin{aligned} Z_{k\ell} &= R_k + i(\omega L_k - (S_k/\omega)) && k = \ell \\ &= i\omega L_{k\ell} && k \neq \ell \ . \end{aligned}$$

The proper and mutual admittances of a general parallel element of a network NP are given by

$$
\begin{aligned}
Y_{k\ell} &= g_k + i(\omega C_k - (\Gamma_k/\omega)) && k=\ell \\
&= \Gamma_{k\ell}/i\omega && k \neq \ell\ .
\end{aligned}
$$

(2.2) ω-*regular networks*.

Definition 1. For a given ω, a network NS (resp. NP) is ω-*regular* if it has a unique solution. If the network is not ω-regular, then it is ω-*singular*. A network NS (resp. NP) is *regular* if it is ω-regular for every ω.

Lemma 1. *A network* NS (resp. NP) *is* ω-*regular if and only if* (a) (resp. (b)) *holds*.

$$
\text{(a)}\quad \det\begin{bmatrix} {}^{t}XZX & {}^{t}\overline{X} \\ \overline{X} & 0 \end{bmatrix} \neq 0 \qquad\qquad \text{(b)}\quad \det\begin{bmatrix} {}^{t}AYA & {}^{t}\tilde{A} \\ \tilde{A} & 0 \end{bmatrix} \neq 0\ .
$$

Notice that the elements of the matrix Z (resp. Y) are now complex numbers.

The next theorem is obvious.

Theorem 1. i) *a compatible network can be* ω-*singular only at a finite number of values of* ω.

ii) *a network is incompatible if and only if it is* ω-*singular for every* ω.

iii) *if a network is* ω-*regular for one value of* ω, *then it is compatible*.

As in (1.2) we can prove

Theorem 2. *Consider the network* NS (resp. NP) *such that the arcs containing current* (resp. *voltage*) *sources are chords* (resp. *branches*) *of a forest*. *Let* $\tilde{N}$S (resp. $\bar{N}$P) *be the network obtained from* NS (resp. NP) *by removing* (*shrinking one end vertex into the other end vertex of*) *the arcs containing current* (resp. *voltage*) *sources*. *Then* NS (resp. NP) *is* ω-*regular if and only if* $\tilde{N}$S (resp. $\bar{N}$P) *is* ω-*regular*.

(2.3) *The concept of nullifying sources*.

Definition 2. To nullify a source means to replace it by a similar one of value zero.

Theorem 3. *Let* NS (resp. NP) *be a network with no current* (resp. *voltage*) *sources*; *let* NS' (resp. NP') *be the network obtained from* NS (resp. NP) *by nullifying the voltage* (resp. current) *sources*. *Then any solution of* NS' (resp. NP') *is such that the currents* (resp. *voltage drops*) *through the resistors are equal to zero.*

Proof: Since a cycle (resp. cocycle) is a linear combination of loops (resp. coloops), from the equations of NS' (resp. NP') (see section (0.5)) we see that for any cycle ${}^t ch$ (resp. cocycle ${}^t dh$) ${}^t cZI = 0$ (resp. ${}^t dYV = 0$). Suppose that $I = (I_i)$ (resp. $V = (V_i)$) is a solution of NS' (resp. NP'). Since ${}^t AI = 0$ (resp. ${}^t XV = 0$), we see that I^*h (resp. V^*h) is a cycle (resp. cocycle). Thus $I^*ZI =$ $I^*(R + i(\omega L - (S/\omega))I = 0$ (resp. $V^*YV = V^*(g + i\omega C - (\Gamma/\omega)V = 0)$, which implies that $I^*RI = \Sigma R_k |I_k|^2 = 0$ (resp. $V^*gV = \Sigma g_k |V_k|^2 = 0$). Therefore, $R_k \neq 0$ (resp. $g_k \neq 0$) implies $I_k = 0$ (resp. $V_k = 0$).

The next theorem is an obvious consequence of Lemma 1.

Theorem 4. *Let* NS' (resp. NP') *be the network obtained from* NS (resp. NP) *by nullifying some or all the sources*. *Then* NS (resp. NP) *is* ω-*regular if and only if* NS' (resp. NP') *is* ω-*regular*.

(2.4) *Removing and shrinking resistors*.

Theorem 5. *Let* NS (resp. NP) *be a network with no coloop* (resp. *loop*) *consisting only of current* (resp. *voltage*) *sources*. *Let* NS' (resp. NP') *be the network obtained from* NS (resp, NP) *by removing* (resp. *shrinking one end vertex into the other end vertex of*) *some or all the arcs containing resistors*. *Then if* NS (resp. NP) *is* ω-*singular*, NS' (resp. NP') *is also* ω-*singular*; *or equivalently*, *if* NS' (resp. NP') *is* ω-*regular*, *then* NS (resp. NP) *is also* ω-*regular*.

Proof: By Theorems 2 and 4, we may suppose that NS (resp. NP) has no current (resp. voltage) sources and that all the voltage (resp. current) sources are nullified. Clearly it is enough to prove the theorem when we remove (resp. shrink one end vertex into the other end vertex of) only one arc containing a resistor.

Suppose that NS (resp. NP) has n linearly independent loops (resp. coloops); suppose that the arc h_k has a resistor. Choose a complete set of linearly independent loop currents (resp. coloop voltages) in such a manner that J_1 (resp. U_1) is equal to I_k (resp. V_k). The demolishing (resp. shrinking) process shows that this is always possible. We now show that two different solutions of NS (resp. NP) give rise to two different solutions of NS' (resp. NP').

Suppose that the loop (resp. coloop) equations of NS (resp. NP) are

$$\begin{array}{ll}
z_{11}J_1 + z_{12}J_2 + \cdots + z_{1n}J_n = 0 & y_{11}U_1 + y_{12}U_2 + \cdots + y_{1n}U_n = 0 \\
z_{21}J_1 + z_{22}J_2 + \cdots + z_{2n}J_n = 0 & y_{21}U_1 + y_{22}U_2 + \cdots + y_{2n}U_n = 0 \\
\hdashline
\hdashline
z_{n1}J_1 + z_{n2}J_2 + \cdots + z_{nn}J_n = 0 & y_{n1}U_1 + y_{n2}U_2 + \cdots + y_{nn}U_n = 0 .
\end{array}$$

If (J_i') and (J_i'') (resp. (V_i') and (V_i'')), $i = 1,2,\ldots,n$, are two different solutions of NS (resp. NP), then, by Theorem 3, $J_1' = J_1'' = 0$ (resp. $V_1' = V_1'' = 0$). Therefore (J_i') and (J_i'') (resp. (V_i') and (V_i'')), $i = 2,3,\ldots,n$, are two different solutions of

$$\begin{array}{ll}
z_{22}J_2 + \cdots + z_{2n}J_n = 0 & y_{22}U_2 + \cdots + y_{2n}U_n = 0 \\
\hdashline
z_{n2}J_2 + \cdots + z_{nn}J_n = 0 & y_{n2}U_2 + \cdots + y_{nn}U_n = 0 .
\end{array}$$

Yet this last system gives the loop (resp. coloop) equations of NS' (resp. NP').

(2.5) The main theorems of regularity.

Theorem 6. Let NS (resp. NP) be a network such that no coloop (resp. loop) consists only of current (resp. voltage) sources. Then NS (resp. NP) is regular if it contains a resistor or a current (resp. voltage) source in every loop (resp. coloop).

Proof: By theorem 2, we may suppose that NS (resp. NP) contains no current (resp. voltage) sources. Clearly, if the net-NS (resp.NP) contains only one loop (resp. coloop), it is regular. Let us now suppose that the theorem is true for all networks with $n-1$ linearly independent loops (resp. coloops) and let NS (resp. NP) be a network with n linearly independent loops (resp. coloops) containing a resistor in every loop (resp. coloop). If we remove (resp. shrink one end vertex into the other end vertex of) an arc containing a resistor, we obtain a network, say NS' (resp. NP'), with $n-1$ linearly independent loops (resp. coloops) containing a resistor in every loop (resp. coloop) which by hypothesis is regular. Then, by Theorem 5, NS (resp. NP) is regular.

The next theorem follows from Lemma 3 of (1.3).

Theorem 7. A network (NS or NP) containing only condensers and resistors is regular. A network (NS or NP) containing only coils and resistors is regular if the matrix $(L_{k\ell})$ is positive definite.

(2.6) Regular arcs.

Let NS (resp. NP) be a consistent network with no current (resp. voltage) sources and let (I^1,V^1), (I^2,V^2) be two solutions of NS (resp. NP), that is, pairs of vectors such that

$$V^1 = ZI^1 + V', \quad V^2 = ZI^2 + V' \quad (\text{resp. } I^1 = YV^1 + I'', \ I^2 = YV^2 + I'')$$

$$^tXV^1 = 0, \ ^tAI = 0 \quad ; \quad ^tXV^2 = 0 \ .$$

Substracting we obtain

$$V^2 - V^1 = Z(I^2 - I^1) \quad (\text{resp. } I^2 - I^1 = Y(V^2-V^1) \ )$$

$$^tX(V^2 - V^1) = 0, \quad ^tA(I^2 - I^1) = 0 \ ,$$

which implies that $(I,V) = (I^2 - I^1, \ V^2 - V^1)$ is a solution of the network, say NS' (resp. NP'), obtained from NS (resp. NP) by nullifying the voltage (resp. current) sources. By Theorem 3, if the arc k of NS' (resp. NP') contains a resistor we have $I_k = I_k^2 - I_k^1 = 0$ (resp. $V_k = V_k^2 - V_k^1 = 0$), which proves the following result.

Theorem 8. In a consistent network NS (resp. NP) with no current (resp. voltage) sources the currents (resp. voltage drops) through the arcs containing a resistor are uniquely determined.

We now generalize the above theorem to include networks NS (resp. NP) with current (resp. voltage) sources such that no subset of the set of, say n, current (resp. voltage) sources form a coloop (resp. loop). By the demolishing (resp. shrinking) process choose a complete set of linearly independent loop currents $J_1,..,J_{n+1},..,J_\mu$ (resp. coloop voltages $U_1,..,U_{n+1},..,U_\rho$) such that the first n coincide with the currents (resp. voltages) of the current (resp. voltage) sources. In the loop equations of NS (resp. coloop equations of NP) eliminate the first n equations and transpose to the right hand side the terms containing $J_1,..,J_n$ (resp. $U_1,..,U_n$). The resulting system gives the loop (resp. coloop) equations of the network, say NS' (resp. NP'), obtained from NS (resp. NP) by removing (resp. shrinking one end vertex into the other end vertex of) the arcs containing a current (resp. voltage) source and inserting in each of the remaining loops (resp. coloops) a voltage (resp. current) source of the apropriate value in the arc we demolished (resp. shrunk) in the demolishing (resp. shrinking) process. By the above theorem, the currents (resp. voltage drops) through the arcs of NS' (resp. NP') containing a resistor are uniquely determined as a linear combination of the loop currents $J_{n+1},..,J_{\mu}$ (resp. coloop voltages $U_{n+1},..,U_{\rho}$). Therefore the currents (resp. voltage drops) through the arcs of NS (resp. NP) containing a resistor are uniquely determined as a linear combination of the loop currents $J_1,..,J_{\mu}$ (resp. coloop voltages $U_1,..,U_{\rho}$).

Finally, if the consistent network NSP has loops consisting only of voltage sources and coloops consisting only of current sources, by the method used in (1.5) we can reduce it to a network with no such a loop and no such a coloop whose solutions can be enlarged to solutions of NSP. In particular, the current (resp. voltage drop) through any general series (resp. parallel) element containing a resistor is the same in both networks.

Definition 3. *An arc of a network* NS (resp. NP) *is* ω-*regular if the current and the voltage drop through it are uniquely determined*.

The following result is a consequence of the above discussion.

Theorem 9. i) *in any consistent network the resistors are* ω-*regular*.

ii) *in any consistent network* NS (resp. NP) *a magnetically isolated arc is* ω-*regular if it contains a resistor but no current* (resp. *voltage*) *source*.

We now give an important application of the concept of regular arc.

Definition 4. *Consider an interconnection of basic passive elements with two available terminals*. *We suppose that the terminals are mutually accessible through the basic elements*. *Such an interconnection is called a two terminal structure or simply box*.

We represent a typical box by the symbol

The reference direction serves for both the current I and the voltage drop V from a to b.

Let us connect a non-zero voltage source or current source between a and b. If the voltage drop V and the current I from a to b are uniquely determined and if $I \neq 0$, the ratio V/I, denoted by Z, is the impedance of the box; if $V \neq 0$, the ratio I/V, denoted by Y, is the admittance of the box. In practice, when the box together with the source is considered as an NS, (for which it is necessary that $\det(\Gamma_{k\ell}) \neq 0$ or that $(L_{k\ell})$ be given) it is convenient to choose the loop currents in such a manner that I coincide with, say J_1. Similarly, when the box and the source are considered as an NP it is convenient to choose the coloop voltages in such a manner that V coincide with , say U_1.

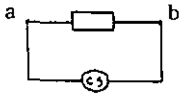


Fig. (a). N_1.

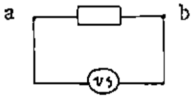


Fig. (b). N_2.

Theorem 10. Suppose that the network N_1 in Fig. (a) can be considered as an NS. If the loop equations have no solution for $I_{cs} \neq 0$, the loop equations of N_2, obtained from N_1 by replacing the current source by a voltage source, have a solution. Moreover, every solution of these equations is such that $I_{vs} = 0$.

Proof: The loop equations of N_1 and N_2 are, respectively,

$$\begin{bmatrix} z_{11}\cdots z_{1n} & 1 \\ z_{21}\cdots z_{2n} & 0 \\ \hdashline z_{n1}\cdots z_{nn} & 0 \\ 1 \quad \cdots 0 & 0 \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \\ \hdashline J_n \\ V_{cs} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \hdashline 0 \\ I_{cs} \end{bmatrix} \qquad (*)$$

$$\begin{bmatrix} z_{11}\cdots z_{1n} \\ z_{21}\cdots z_{2n} \\ \hdashline z_{n1}\cdots z_{nn} \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \\ \hdashline J_n \end{bmatrix} = \begin{bmatrix} V_{vs} \\ 0 \\ \hdashline 0 \end{bmatrix} . \qquad (**)$$

Let us suppose that $(*)$ has no solution for $I_{cs} \neq 0$. Then every solution of $(**)$ is such that $J_1 = 0$, since if (J_i), with $J_1 \neq 0$, were a solution of $(**)$, then

$$\begin{bmatrix} J_1 \\ --- \\ J_n \\ -V_{vs} \end{bmatrix}$$

would be a solution of $(*)$ with $I_{cs} = J_1 \neq 0$.

We now show that $(**)$ has a solution. Consider the mapping $(z_{ij}) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, where (z_{ij}) is the matrix in $(**)$. If (x_i) is in Ker (z_{ij}), then $x_1 = 0$, since otherwise

$$\begin{bmatrix} x_1 \\ -- \\ x_n \\ 0 \end{bmatrix}$$

would be a solution of $(*)$ with $I_{cs} = x_1 \neq 0$.

For any w in Ker (z_{ij}) we have that $(z_{ij})w = 0$, or equivalently ${}^t w(z_{ij}) = 0$. If u is in Im (z_{ij}) then $u = (z_{ij})v$ for some v in $\mathbb{C}^n$ and ${}^t wu = {}^t w(z_{ij})v = 0$. Therefore

$$\text{Im } (z_{ij}) \subset \text{orthogonal of } {}^t\text{Ker } (z_{ij}) = ({}^t\text{Ker } (z_{ij}))^{\perp}.$$

We know that $\dim \text{Im } (z_{ij}) + \dim \text{Ker } (z_{ij}) = n$

$\dim \text{Ker } (z_{ij}) + \dim \text{Ker } ((z_{ij}))^{\perp} = n$. Thus

$\dim({}^t\text{Ker } (z_{ij}))^{\perp} = \dim \text{Im } (z_{ij})$, which implies that

Im $(z_{ij}) = ({}^t\text{Ker } (z_{ij}))^{\perp}$. Therefore

$$\begin{bmatrix} V_{vs} \\ 0 \\ --- \\ 0 \end{bmatrix}$$

is in Im (z_{ij}), which proves that $(**)$ has a solution for any V_{vs}.

By using the coloop method we can similarly prove the next result.

Theorem 11. Suppose that the network N_2 *in Fig. (b) can be considered as an NP. If the coloop equations have no solution for* $V_{vs} \neq 0$, *then the coloop equations of* N_1, *obtained from* N_2 *by replacing the voltage source by a current source, have a solution. Furthermore, every solution of these equations is such that* $V_{cs} = 0$.

Now we can prove the next important theorem.

Theorem 12. Either the impedance or the admittance of any passive box whatsoever is uniquely determined.

Proof: We first prove the theorem when N_1 (and N_2) can be considered as NS. By theorem 10, it is enough to show that if the system (*) is consistent, all the solutions give the same V_{cs}. If (*) has the solution $(J_1,..,J_n,V_{cs})$, then

$$(***) \qquad \begin{bmatrix} 1 & z_{12} \cdots z_{1n} \\ 0 & z_{22} \cdots z_{2n} \\ \vdots & \vdots \\ 0 & z_{n2} \cdots z_{nn} \end{bmatrix} \begin{bmatrix} V_{cs} \\ J_2 \\ \vdots \\ J_n \end{bmatrix} = -I_{cs} \begin{bmatrix} z_{11} \\ z_{21} \\ \vdots \\ z_{n1} \end{bmatrix}$$

has the solution $(V_{cs},J_2,..,J_n)$. Let

$$x = \begin{bmatrix} V_{cs} \\ J_2 \\ \vdots \\ J_n \end{bmatrix}, \quad b = -I_{cs} \begin{bmatrix} z_{11} \\ z_{21} \\ \vdots \\ z_{n1} \end{bmatrix}$$

and let A be the n by n matrix in (***). Since $y \in \operatorname{Ker}({}^tA)$ implies that ${}^tby = ({}^tx\,{}^tA)y = {}^tx({}^tAy) = 0$, b is in the orthogonal of $\operatorname{Ker}({}^tA)$. Let us consider two solutions of (***), $(V'_{cs},J'_2,..,J'_n)$ and $(V''_{cs},J''_2,..,J''_n)$. Their difference $(V_{cs},J_2,..,J_n)$ is a solution of $Ax = 0$. We must show that $V_{cs} = 0$, or equivalently that $z_{12}J_2 + .. + z_{1n}J_n = 0$. Thus, we have to show that

$\begin{bmatrix} 0 \\ J_2 \\ \vdots \\ J_n \end{bmatrix}$ is orthogonal to $\begin{bmatrix} z_{11} \\ z_{12} \\ \vdots \\ z_{1n} \end{bmatrix} = -(1/I_{cs})b$. Yet, since b is orthogonal to $\mathrm{Ker}({}^tA)$, it is enough to verify that

$\begin{bmatrix} 0 \\ J_2 \\ \vdots \\ J_n \end{bmatrix} \in \mathrm{Ker}({}^tA)$. This condition is satisfied since (***) and

$z_{ij} = z_{ji}$ imply that

$${}^tA \begin{bmatrix} 0 \\ J_2 \\ \vdots \\ J_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ z_{12} & z_{22} & \dots & z_{n2} \\ \vdots & & & \vdots \\ z_{1n} & z_{2n} & \dots & z_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ J_2 \\ \vdots \\ J_n \end{bmatrix} = 0.$$

Finally, if N_1 (and N_2) are NP and cannot be considered as NS, we can similarly show that the theorem holds.

3. EXAMPLES

Example 1. Consider the network NP in Fig. 1. Suppose that $\Gamma=\Gamma_M$. The coloop equations are

$$\begin{bmatrix} g + (\Gamma/i\omega) & -g + (\Gamma_M/i\omega) \\ -g + (\Gamma_M/i\omega) & g + (\Gamma/i\omega) \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since the determinant of the matrix on the left is equal to $(4g/i\omega)\Gamma$ the network is regular.

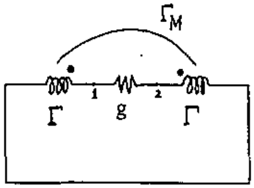


Fig. 1

One could think that this is a natural result since the network contains a resistor in every loop. However, if we change the sense of the winding of one of the coils the network thus obtained, say NP', is incompatible, as can be verified by replacing Γ_M by $-\Gamma_M$ in the coloop equations of NP.

By (1.2) we see that the insertion of a voltage source in the loop of NP or NP' does not change the regularity properties of the networks.

If we insert current sources instead, NP becomes incompatible and NP' remains incompatible. To see this consider the network in Fig. 2 whose coloop equations are

$$\begin{bmatrix} \Gamma/i\omega & -\Gamma/i\omega & -\Gamma_M/i\omega \\ -\Gamma/i\omega & g + (\Gamma/i\omega) & -g + (\Gamma_M/i\omega) \\ -\Gamma_M/i\omega & -g + (\Gamma_M/i\omega) & g + (\Gamma/i\omega) \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} I_{cs} \\ 0 \\ 0 \end{bmatrix} .$$

Since the determinant of the matrix on the left is equal to $(g/\omega^2)\ (\Gamma_M^2 - \Gamma^2)$, the network is incompatible if $\Gamma=\Gamma_M$ or $\Gamma=-\Gamma_M$. Moreover, it can be verified that if $1_{cs} \neq 0$, the network is inconsistent when $\Gamma=\Gamma_M$ but it is consistent when $\Gamma = -\Gamma_M$.

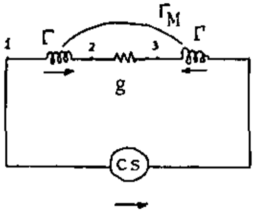


Fig. 2

Example 2. Similarly, as in Example 1, we can show that

i) the network NS in Fig. 3 is regular.

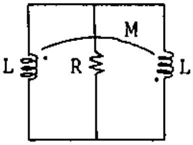


Fig. 3

ii) if we change the sense of the winding of one of the coils the network thus obtained, say NS', is incompatible.

iii) if $L = M$ or $L = -M$, the network NS in Fig. 4 is incompatible.
This means that if we connect a voltage source in parallel with the network NS in Fig. 3, the network becomes incompatible. NS' remains incompatible in such a connection.

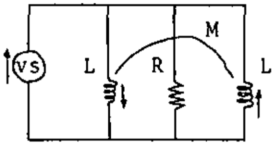


Fig. 4

iv) if we connect a current source instead, the regularity properties of NS and NS' in (i), (ii), remain unchanged.

Notice that the above examples do not contradict the fundamental Theorem 6 in (2.5) since the networks in Example 1 cannot be considered as NS and those in Example 2 cannot be considered as NP: in the first, det $(\Gamma_{k\ell}) = 0$; in the second, det $(L_{k\ell}) = 0$. The next example will clarify even further this fact.

Example 3. The fundamental Theorem 6 in (2.5) shows that the network NS in Fig. 5 (resp. NP in Fig. 6), where the source is either a voltage or a current source, is regular, for all possible values of L_1, L_2, M (resp. Γ_1, Γ_2, Γ_M) .

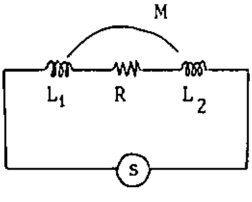


Fig. 5

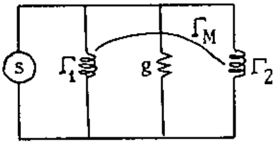


Fig. 6

By now it should be clear that the distinction between networks NS and NP is crucial as far as regularity and compatibility are concerned.

Example 4. Consider the network NS in Fig. 7, where $L_1 = |M_1|$, $L_2 = |M_2|$.

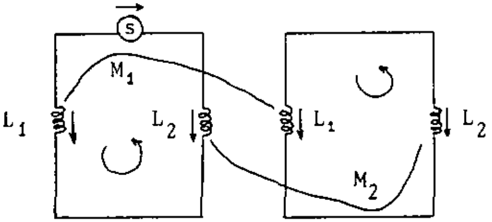


Fig. 7

Suppose first that the source is a voltage source. Thus the loop equations are

$$\begin{bmatrix} (L_1 + L_2)i\omega & (M_1 + M_2)i\omega \\ (M_1 + M_2)i\omega & (L_1 + L_2)i\omega \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} = \begin{bmatrix} V_{vs} \\ 0 \end{bmatrix}.$$

The determinant of the system is equal to $-2\omega^2(L_1L_2 - M_1M_2)$. Hence if M_1, M_2 have the same sign the network is incompatible. If they have opposite signs the network is regular. Clearly, if $V_{vs} \neq 0$, the network is inconsistent when M_1, M_2 have the same sign.

Now suppose that the source is a current source. Thus the loop equations are

$$\begin{bmatrix} (L_1 + L_2)i\omega & (M_1 + M_2)i\omega & 1 \\ (M_1 + M_2)i\omega & (L_1 + L_2)i\omega & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \\ V_{cs} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -I_{cs} \end{bmatrix}.$$

Since the determinant of the system is equal to $-(L_1 + L_2)i\omega$ the network is regular.

Example 5. Since the network in Fig. 8 contains a loop consisting only of voltage sources it is incompatible. (See section (1.2)). The networks in Figs. 9, 10 are incompatible since each contains a coloop consisting only of current sources.

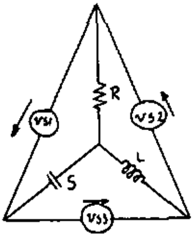


Fig. 8

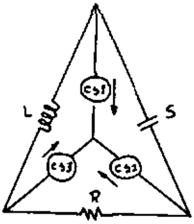


Fig. 9

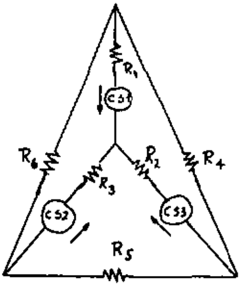


Fig. 10

It is important to realize that the first network is incompatible whether or not the condition of consistency

$$V_{vs1} + V_{vs2} + V_{vs3} = 0$$

is satisfied. Similarly, the other two are incompatible whether or not the condition of consistency

$$I_{cs1} + I_{cs2} + I_{cs3} = 0$$

is satisfied. Also notice that the last one contains a resistor in every arc.

In the next example we apply Theorem 2 in (2.2) .

Example 6. Consider the network NS in Fig. 11. The chords of a tree are shown in dashed lines. Hence NS is ω-regular if and only if the network $\tilde{NS}$ in Fig. 12 is ω-regular. Since the determinant of the coefficient matrix of $\tilde{NS}$ is

$$\begin{vmatrix} R_3 + S_4/i\omega & 0 \\ 0 & (L_5 + L_6 + 2L_{56})i\omega \end{vmatrix}$$

we see that $\tilde{NS}$ is regular if and only if $L_5 + L_6 + 2L_{56} \neq 0$. Therefore NS is regular, if and only if this condition is satisfied. Recall that $L_{k\ell}^2 - L_k L_\ell \leq 0$ implies $L_k + L_\ell - 2|L_{k\ell}| \geq 0$.

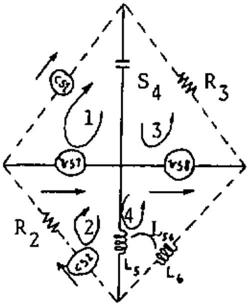


Fig. 11

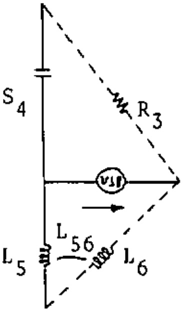


Fig. 12

Similarly, the network NP in Fig. 13 is ω-regular if and only if the network $\overline{\mathrm{N}}\mathrm{P}$ in Fig. 14 is ω-regular.

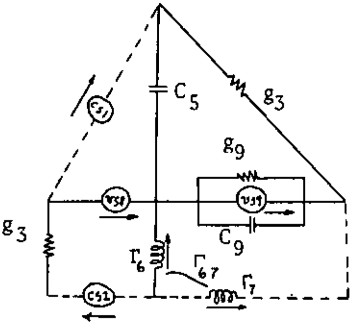


Fig. 13

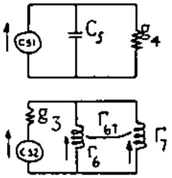


Fig. 14

Since the determinant of the coloop equations of $\overline{\mathrm{N}}\mathrm{P}$ is

$$\begin{vmatrix} g_4 + i\omega C_5 & 0 & 0 \\ 0 & g_3 & 0 \\ 0 & 0 & (\Gamma_6 + \Gamma_7 + 2\Gamma_{67})/i\omega \end{vmatrix}$$

we see that $\overline{\mathrm{N}}\mathrm{P}$ is regular if and only if $\Gamma_6 + \Gamma_7 + 2\Gamma_{67} \neq 0$. Therefore NP is regular if and only if this condition is satisfied.

All the ω-singular networks thus far considered are incompatible. In the examples that follow, all the networks are compatible.

Example 7. The network in Fig. 15 (resp. Fig. 16) is ω-singular for $\omega^2 = S/L$ (resp. $\omega^2 = \Gamma/C$) since its loop (coloop) equation is $0J = V_{vs}$ (resp. $0U = I_{cs}$) . If $V_{vs} = 0$ (resp. $I_{cs} = 0$), the network has an infinite number of solutions; if $V_{vs} \neq 0$ (resp. $I_{cs} \neq 0$), the network is inconsistent.

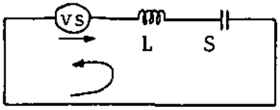


Fig. 15

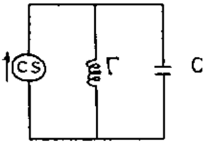


Fig. 16

By Theorem 6, (2.5), we see that the networks in Figs. 17, 18 are ω-regular for all possible values of L, S, Γ, C, ω .

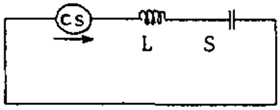


Fig. 17

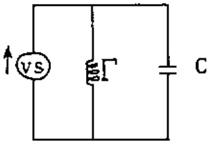


Fig. 18

Example 8. If $\omega^2 = S_1/L_1 = S_2/L_2$ (resp. $\omega^2 = \Gamma_1/C_1 = \Gamma_2/C_2$) the loop (resp. coloop) equations of the network in Fig. 19 (resp. Fig. 20) are

$$\begin{bmatrix} R & -R \\ -R & R \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} = \begin{bmatrix} -V_{vs} \\ V_{vs} \end{bmatrix} \qquad \begin{bmatrix} g & -g \\ -g & g \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} -I_{cs} \\ I_{cs} \end{bmatrix} .$$

Therefore the network is ω-singular at the given frequency.

Notice that the network NS (resp. NP) in Fig. 19 (resp. Fig. 20) is ω-singular even though it has a resistor in every loop (resp. coloop) of a complete set of linearly independent loops (resp. coloops).

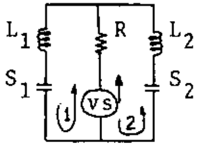


Fig. 19

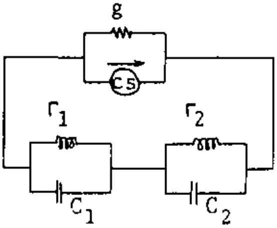


Fig. 20

In the examples that follow we repeatedly use Theorem 2, (2.2) and Theorem 5, (2.4). Since the networks in these examples have no magnetic couplings they can be considered either as NS or NP.

<u>Example 9</u>. Consider the network N in Fig. 21. By removing the arc containing R_1 and shrinking R we obtain the network in Fig. 22 which obviously is regular. In particular, N is ω-regular for $\omega^2 = S_1/L_1 = S_2/L_2$. Let us consider this frequency.

By direct computation or by recalling the well known property of the Wheatstone bridge we can easily verify that the current through the resistor R is equal to zero.

Since the voltage drop through R is also equal to zero we could conclude that its presence is irrelevant for the performance of the network. Yet if we remove it we obtain the ω-singular network in Fig. 19.

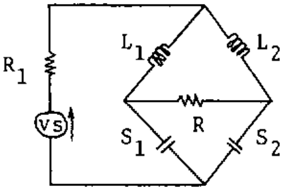


Fig. 21 N

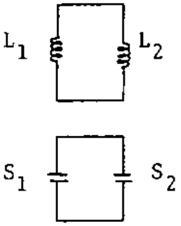


Fig. 22

The network N_1 in Fig. 23 is obtained from N in Fig. 21 by replacing R by a current source. By removing the current source we see that N_1 is ω-singular.

We could think that the replacement results in an ω-singular network because in N the current through R is equal to zero while in N_1 the current from a to b is given by the value of the current source. However, this reasoning is not correct since the singularity of N_1 does not depend on the value of the current source which we may take as zero. Yet, it can be verified that if (and only if) $I_{cs} \neq 0$, N_1 is inconsistent.

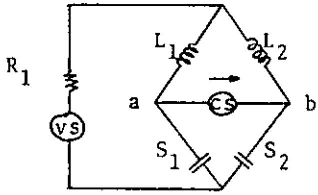


Fig. 23 N_1

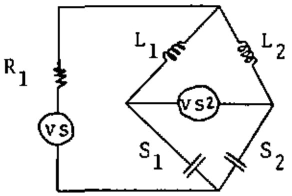


Fig. 24 N_2

By analogy we could think that the network N_2 in Fig. 24, obtained from N by replacing R by a voltage source, is ω-singular. However, by shrinking vs2 and removing the arc containing R_1, N_2 becomes the network in Fig. 22. Therefore N_2 is ω-regular.

Suppose that for the network in Fig. 25, $\omega^2 = S/L$. Thus it is easy to see that the current I from a to b is equal to zero for all possible values of the voltage source. In particular, $I = 0$ when $V_{vs} = 0$.

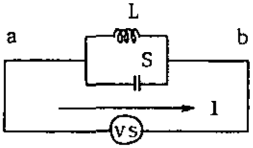


Fig. 25

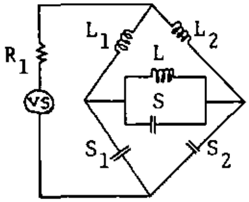


Fig. 26 N_3

One could expect that the network in Fig. 26, obtained from N in Fig. 21 by replacing R by the box between a and b in Fig. 25, is ω-regular. However, as we prove below, N_3 is ω-singular.

If we shrink in N_3 the voltage source and then R_1, we obtain the network in Fig. 27, which is ω-singular as can be immediately seen by applying the coloop method.

Notice that the shrinking process we have just applied to N_3 cannot be applied to N_2. (See Theorem 5, (2.4)).

Observe that N and N_3, considered as NS, have different graphs, but considered as NP, they have the same graph. Also observe that the network in Fig. 27 is still ω-singular if $\omega^2 \neq S/L$. Yet, N_3 is ω-singular only if $\omega^2 = S/L$. We now prove this fact.

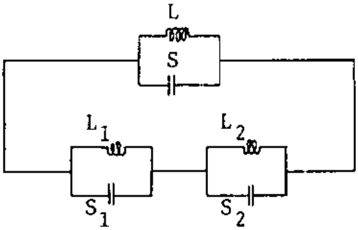


Fig. 27

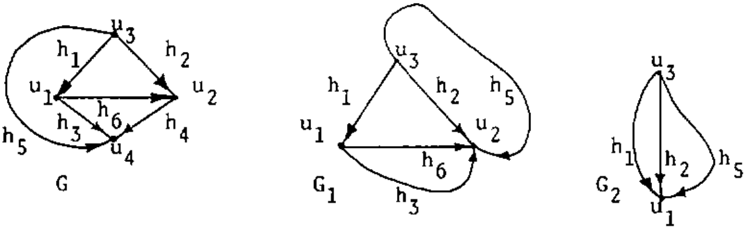


Fig. 28

The graph G of N_3 (as an NP) with the voltage source shrunk is shown in Fig. 28. Take the coloop $\omega(\{u_1, u_2\}) = h_1 + h_2 - h_3 - h_4$ and shrink u_4 into u_2 to obtain G_1. In G_1 take the coloop $\omega(\{u_1\}) = h_1 - h_3 - h_6$ and shrink u_2 into u_1 to obtain G_2. Finally take $\omega(\{u_1\}) = h_1 + h_2 + h_5$ in G_2. In this manner we obtain a complete set of linearly independent coloops of G. Since the determinant of the corresponding coloop equations is

$$\begin{vmatrix} 0 & 0 & \Gamma_1/i\omega \\ 0 & i(\omega C - \Gamma/\omega) & \Gamma_1/i\omega \\ \Gamma_1/i\omega & \Gamma_1/i\omega & g + (\Gamma_1 + \Gamma_2)/i\omega \end{vmatrix} = i\ (\Gamma_1^2/\omega^2)\ (\omega C - \Gamma/\omega)$$

our assertion is true.

It is obvious that the regularity properties of N, N_1, N_2, N_3 and the networks obtained from them by shrinking the voltage source in series with R_1 are the same, respectively.

Example 10. Consider the network N in Fig. 29, in which the boxes 1, 2, may be either a voltage source, a resistor or a current source. Suppose that $\omega^2 = S_1/L_1 = S_2/L_2 \neq S_1/L_2$.

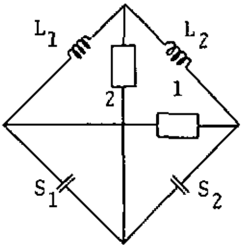


Fig. 29 N

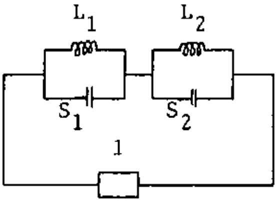


Fig. 30

If box 2 is a voltage source, by shrinking it we obtain the network in Fig. 30 which is ω-singular for any choice of box 1. Therefore, in these three cases N is ω-singular. We express this fact by putting - signs in the first column of Table 1 below.

1 \ 2	vs	R	cs
vs	-	+	+
R	-	+	+
cs	-	-	-

Table I N

If box 2 is a current source, by removing it we obtain the network in Fig. 31. Clearly this network has the same regularity properties as the network in Fig. 32, where $L = L_1 + L_2$ and $C = C_1 + C_2$. By (2.5), this network is regular if box 1 is a voltage source or a resistor; if box 1 is a current source, by Example 7 the network is ω-singular. Thus we put +, +, - in the last column of Table 1.

If box 2 is a resistor, by example 9 we see that N is regular when box 1 is a voltage source or a resistor and ω-singular when it is a current source. We put +, +, - in the second column of Table 1.

Similarly we can construct Tables 2 and 3 from the networks N_1 and N_2 in Figs. 33 and 34, where $\omega^2 = S_1/L_1 = S_2/L_2 \neq S_1/L_2$.

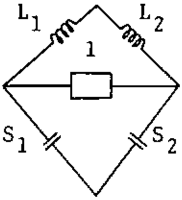


Fig. 31

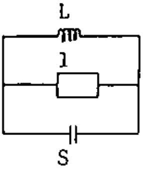


Fig. 32

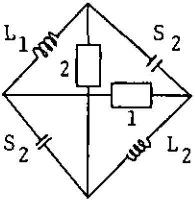


Fig. 33

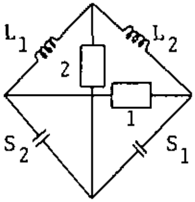


Fig. 34

1 \ 2	vs	R	cs
vs	-	-	-
R	+	+	-
cs	+	+	-

Table 2 N_2

1 \ 2	vs	R	cs
vs	-	+	+
R	+	+	+
cs	+	+	-

Table 3 N_3

Observe that the tables show that for an active box (that is, a box containing sources), it may happen that neither the impedance nor the admittance is defined. (See Theorem 12, (2.6).)

Example 11. Consider the network N in Fig. 35. Suppose that $\omega^2 = ((3+\sqrt{5})/2)(S/L)$ (or $\omega^2 = ((3-\sqrt{5})/2)(S/L)$). Since the determinant of the loop equations is

$$\begin{vmatrix} i(2\omega L - (S/\omega)) & -i\omega L \\ -i\omega L & i(\omega L - (S/\omega)) \end{vmatrix} = -(L^2/\omega^2)(\omega^2 - ((3+\sqrt{5})/2)(S/L))(\omega^2 - ((3-\sqrt{5})/2)(S/L))$$

N is ω-singular. It is easy to verify that no loop of N has zero self-impedance, and that no coloop of N has zero self-admittance.

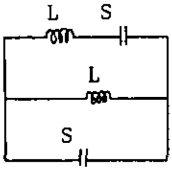


Fig. 35

Notice that all compatible ω-singular networks, before the last one, have a loop with zero self-impedance or a coloop with zero self-admittance at the frequency of singularity. Thus, our last example shows that this condition is not necessary for a network to be ω-singular; obviously the condition is not sufficient either.