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Moments of standardized Fernandez–Steel skewed distributions: Applications to the estimation of GARCH-type models

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ABSTRACT

We provide general expressions for obtaining raw, absolute and conditional moments for a standardized version of the class of skewed distributions proposed by Fernandez and Steel (1998). We show that these expressions are readily programmable in addition of greatly reducing the computational cost. We discuss several applications that are relevant for the purpose of estimating asymmetric conditional volatility models under skewed distributions.

1. Introduction

The search for flexible conditional volatility models that accurately capture empirically observed features such as skewness, fat tails and the leverage effect is highly relevant to portfolio management, asset pricing and dynamic hedging. Engle and Ng (1993) provide the first systematic comparison of volatility models with the emphasis on Black (1976)'s leverage effect. To account for the presence of skewness and fat tails in financial market datasets, Fernandez and Steel (1998) propose a general framework for introducing skewness into symmetric distributions. This framework has been adopted by many researchers to model conditional volatility, a recent example being the Beta-Skew-*t*-EGARCH model of Harvey and Sucarrat (2014). This leads some of these researchers to develop a re-parametrization to standardize Fernandez–Steel skewed distributions so that they have zero mean and unit variance (e.g., see Lambert and Laurent, 2001).

Estimating GARCH-type models often requires to calculate absolute moments (e.g. EGARCH model) or conditional moments (e.g. GJR and TGARCH models). To our knowledge, there are no known general expressions which allow to do this analytically for the full class of standardized Fernandez–Steel skewed distributions. One thus have to rely on numerical integration methods (e.g., see Press et al., 2007, Chapter 4), which involve costly computations in addition of requiring a working knowledge of advanced mathematical methods and programming techniques. This issue is addressed in Section 2 of this note, where we present general analytical formulas that are readily programmable and greatly reduce the computational cost of estimating raw, absolute and conditional moments of standardized Fernandez–Steel skewed distributions. Other research articles have mainly focused on specific Skew-*t* distributions (e.g., see Zhu and Galbraith (2010)). In contrast, our

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analytical results hold for the full class of standardized Fernandez–Steel skewed distributions. In Section 3, we present several applications of these formulas that are relevant for the purpose of estimating asymmetric GARCH-type models in the presence of skewness.

2. Moments of standardized Fernandez–Steel skewed distributions

2.1. Definition of the skewed density

The skewed density is defined using the approach of Fernandez and Steel (1998), together with a re-parametrization to ensure that the distribution has zero mean and unit variance (e.g., see Lambert and Laurent (2001) and Castillo et al. (2011)). The resulting density can be written as:

$$f_\xi(z) \equiv \frac{2\,\sigma_\xi}{\xi + \xi^{-1}}\, f_1(z_\xi), \quad z_\xi \equiv \begin{cases} \xi^{-1}(\sigma_\xi z + \mu_\xi) & \text{if } z \geq -\mu_\xi/\sigma_\xi \\ \xi(\sigma_\xi z + \mu_\xi) & \text{if } z < -\mu_\xi/\sigma_\xi, \end{cases} \tag{1}$$

with:

$$\mu_\xi \equiv M_1(\xi - \xi^{-1}), \quad \sigma_\xi^2 \equiv (1 - M_1^2)(\xi^2 + \xi^{-2}) + 2M_1^2 - 1, \quad M_1 \equiv 2\int_0^\infty u f_1(u)du,$$

where $0 < \xi < \infty$ is the parameter describing the degree of asymmetry and $f_1(\,\cdot\,)$ can be any symmetric unimodal density with zero mean and unit variance.

2.2. Cumulative distribution function

The expression for the cumulative distribution function is straightforward to obtain using simple changes of variables:

$$\begin{aligned} F_\xi(x) &\equiv \int_{-\infty}^{x} f_\xi(z)dz \\ &= \begin{cases} \frac{2}{\xi+\xi^{-1}}\left(\xi F_1(\xi^{-1}[\sigma_\xi x + \mu_\xi]) + \xi^{-1}\right) - 1 & \text{if } x \geq -\mu_\xi/\sigma_\xi \\ \frac{2\,\xi^{-1}}{\xi+\xi^{-1}} F_1(\xi[\sigma_\xi x + \mu_\xi]) & \text{if } x < -\mu_\xi/\sigma_\xi, \end{cases} \end{aligned} \tag{2}$$

where $F_1(\,\cdot\,)$ is the cumulative distribution function of the symmetric density $f_1(\,\cdot\,)$.

2.3. Raw, absolute and conditional moments

General expressions for obtaining raw, absolute and conditional moments of the skewed density $f_\xi(\,\cdot\,)$ are presented below. The steps of the derivations are outlined in the Appendix.

The raw moments of order $n = 1, 2, \ldots$ are given by:

$$\mathbb{E}[z^n] = \frac{(-1)^n}{(\xi + \xi^{-1})\sigma_\xi^n} \sum_{k=0}^{n} \binom{n}{k} \mu_\xi^{n-k} M_k [\xi^{-(k+1)} + (-1)^k \xi^{k+1}], \tag{3}$$

where $\binom{n}{k} \equiv \frac{n!}{k!(n-k)!}$ is the binomial coefficient and:

$$M_k \equiv 2\int_0^\infty u^k f_1(u)du, \tag{4}$$

with $M_0 = 1$ by normalization and $M_2 = 1$ because $f_1(\,\cdot\,)$ has unit variance.

The absolute moments of order $n = 1, 2, \ldots$ are given by:

$$\mathbb{E}[|z|^n] = \frac{2}{(\xi + \xi^{-1})\sigma_\xi^n} \left(\frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} \mu_\xi^{n-k} M_k \left[\xi^{-(k+1)} + (-1)^{n-k} \xi^{k+1}\right] + \mathcal{I}_{\xi,n}[(-1)^n - 1] \right), \tag{5}$$

where:

$$\mathcal{I}_{\xi,n} \equiv \begin{cases} \xi^{n+1} \int_0^{\mu_\xi \xi^{-1}} (u - \mu_\xi \xi^{-1})^n f_1(u)du & \text{if } \xi \geq 1 \\ \frac{-1}{\xi^{n+1}} \int_{\mu_\xi \xi}^{0} (u - \mu_\xi \xi)^n f_1(u)du & \text{if } \xi < 1\,. \end{cases} \tag{6}$$

The negative-conditional moments of order $n = 1, 2, \ldots$ are given by:

$$\mathbb{E}[z^n \,|\, z \leq 0] = \frac{2}{F_\xi(0)(\xi + \xi^{-1})\sigma_\xi^n} \left(\frac{(-1)^n}{2\,\xi^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (\mu_\xi \xi)^{n-k} M_k + \mathcal{I}_{\xi,n} \right). \tag{7}$$

Table 1
Estimation of $\mathbb{E}\left[z^2 \mid z \leq 0\right]$ for different distributions and degree of asymmetry ξ. We use $\nu = 4$ degrees of freedom for the Student-t distribution and a shape parameter of $\nu = 5$ for the generalized error distribution (GED). All distributions are standardized to have zero mean and unit variance. The exact formula is computed using the composite Simpson's rule with 10^4 subintervals, whereas the approximated formula uses only four subintervals. Each Monte Carlo estimate is calculated from a random sample of size 10^7. For the approximated formula and the Monte Carlo estimation, the Error column is the absolute difference between the corresponding estimate and the value from the exact formula. The R/Rcpp codes to obtain these results are available from the authors upon request.

Distribution	Exact formula	Approximated formula		Monte Carlo estimation	
	$\mathbb{E}\left[z^2 \mid z \leq 0\right]$	Estimate	Error $[\times 10^{-5}]$	Estimate	Error $[\times 10^{-5}]$
Normal					
$\xi = 0.1$	1.5021569	1.5021790	2.2147	1.5025721	41.5226
$\xi = 0.5$	1.3902252	1.3902289	0.3680	1.3908974	67.2123
$\xi = 0.9$	1.0776386	1.0776386	0.0001	1.0778939	25.5288
Student-t					
$\xi = 0.1$	2.0377929	2.0378214	2.8496	2.0351234	266.9483
$\xi = 0.5$	1.8264110	1.8264166	0.5668	1.8294919	308.0958
$\xi = 0.9$	1.1635354	1.1635354	0.0003	1.1684867	495.1235
Laplace					
$\xi = 0.1$	1.9998000	1.9997627	3.7332	1.9967480	305.2017
$\xi = 0.5$	1.8823529	1.8823459	0.7019	1.8795628	279.0094
$\xi = 0.9$	1.2076565	1.2076565	0.0002	1.2060873	156.9278
GED					
$\xi = 0.1$	1.1528213	1.1528239	0.2508	1.1528903	6.8969
$\xi = 0.5$	1.1016381	1.1016382	0.0152	1.1019301	29.2070
$\xi = 0.9$	1.0183087	1.0183087	0.0001	1.0182125	9.6244

Finally, the positive-conditional moments of order $n = 1, 2, \ldots$ are given by:

$$\mathbb{E}[z^n \mid z \geq 0] = \frac{2}{(1 - F_\xi(0))(\xi + \xi^{-1})\sigma_\xi^n} \left(\frac{\xi^{n+1}}{2} \sum_{k=0}^{n} \binom{n}{k} (-\mu_\xi \xi^{-1})^{n-k} M_k \; - \; \mathcal{I}_{\xi,n} \right). \tag{8}$$

2.4. Efficient approximation of the derived formulas

We now show that the derived formulas can be approximated numerically at a very low cost, and we compare the accuracy and the computational speed to the Monte Carlo approach. The methods discussed below are all implemented in Rcpp on a 2.2 GHz Intel Core i7 processor.

First, note that for most distributions of interest in finance (e.g. Normal, Student-t, Laplace, GED), M_k in (4) is usually known in closed-form and can be computed efficiently. After this step, what remains is the estimation of the integral $\mathcal{I}_{\xi,n}$ in (6). The cost of numerically approximating this integral is however very low in general. The main reason for this is the small length of the domain of integration, which is bounded above by M_1 and tends to zero as $\xi \to 1$.

As shown in Table 1, which reports the estimation of $\mathbb{E}\left[z^2 \mid z \leq 0\right]$ for different skewed densities, approximating the integral $\mathcal{I}_{\xi,2}$ using the composite Simpson's rule with only four subintervals (i.e. nine integration points) provides an excellent accuracy. For comparison, Table 1 also reports the corresponding Monte Carlo estimate of $\mathbb{E}\left[z^2 \mid z \leq 0\right]$, which is computed from a random sample of size 10^7. We see that the Monte Carlo estimate is orders of magnitude less accurate than our scheme. Moreover, each Monte Carlo estimate requires several seconds of computation, whereas our proposed scheme requires about one microsecond. Apart from Monte-Carlo simulations, the other alternative is to directly approximate the mathematical expectation of interest by using an adaptive quadrature (e.g., see Press et al., 2007, Chapter 4) to integrate over the full domain. When estimating $\mathbb{E}\left[z^2 \mid z \leq 0\right]$ using the adaptive Simpson's rule, we find that several hundreds of integration points are required to reach the same accuracy as our proposed scheme. In general, this approach takes about 200 microseconds, which is orders of magnitude slower than our scheme.

3. Applications to the estimation of GARCH-type models

Here, we discuss some challenges associated with the estimation of asymmetric GARCH models under skewed distributions, and we then outline some applications of the formulas given above.

3.1. Some popular GARCH-type models

We consider a framework in which the log-returns $\{y_t\}_{t=1}^T$ can be expressed as $y_t \equiv \sigma_t z_t$ where σ_t is the conditional volatility at time t and the innovations $\{z_t\}$ are *i.i.d.* random variables distributed according to some standardized (zero mean and unit variance) skewed density $f_\xi(\,\cdot\,)$.

For the scedastic function σ_t, we consider the GJR specification (see Glosten et al., 1993), the EGARCH specification (see Nelson, 1990) and the TGARCH specification (see Zakoian, 1994). These models, which are well-known for their ability to reproduce the asymmetric behaviour of the conditional variance observed in equity markets, are summarized below together with their second-order stationarity condition (s.c.):[1]

$$\text{GJR} \quad \sigma_t^2 \equiv \omega + (\alpha + \gamma\, I_{\{y_{t-1}<0\}}) y_{t-1}^2 + \beta \sigma_{t-1}^2 \tag{9}$$

$$\text{s.c.:} \quad \alpha + \gamma\, \mathbb{E}\big[z^2 I_{\{z<0\}}\big] + \beta < 1,$$

$$\text{TGARCH} \quad \sigma_t \equiv \omega + \big(\alpha\, I_{\{y_{t-1}\geq 0\}} - \gamma\, I_{\{y_{t-1}<0\}}\big) y_{t-1} + \beta \sigma_{t-1} \tag{10}$$

$$\text{s.c.:} \quad \alpha^2 + \beta^2 - 2\beta(\alpha+\gamma)\mathbb{E}\big[z\, I_{\{z<0\}}\big] - (\alpha^2 - \gamma^2)\mathbb{E}\big[z^2 I_{\{z<0\}}\big] < 1,$$

$$\text{EGARCH} \quad \log(\sigma_t^2) \equiv \omega + \alpha(|z_{t-1}| - \mathbb{E}[|z_{t-1}|]) + \gamma z_{t-1} + \beta \log(\sigma_{t-1}^2)$$

$$\text{s.c.:} \quad |\beta| < 1, \tag{11}$$

where $I_{\{\,.\,\}}$ is the indicator function.

3.2. Estimation of asymmetric GARCH models

Estimating the above models using either maximum likelihood or Markov chain Monte Carlo requires to compute the log-likelihood over different values of the model's parameters $(\omega, \alpha, \gamma, \beta)$, the asymmetry parameter ξ and possibly additional parameters ν that control the shape of the distribution (e.g. degrees of freedom of the Student-t distribution). Moreover, the estimation process is usually carried under the constraint of second-order stationarity.

For the GJR model, the second-order stationarity condition can be computed efficiently by using the following formula:[2]

$$\mathbb{E}[z^2 I_{\{z<0\}}] = \frac{2}{(\xi + \xi^{-1})\sigma_\xi^2}\left(\frac{1 + M_1^2(\xi^4 - 1)}{2\,\xi^3} + \mathcal{I}_{\xi,2}\right). \tag{12}$$

To compute the second-order stationarity condition of the TGARCH model, we also use:[3]

$$\mathbb{E}[z\, I_{\{z<0\}}] = \frac{-2}{(\xi + \xi^{-1})\sigma_\xi}\left(\frac{M_1}{2} - \mathcal{I}_{\xi,1}\right). \tag{13}$$

For the EGARCH model, we use the formula below to compute the absolute moment $\mathbb{E}[|z|]$:[4]

$$\mathbb{E}[|z|] = \frac{2}{(\xi + \xi^{-1})\sigma_\xi}\big(M_1 - 2\,\mathcal{I}_{\xi,1}\big). \tag{14}$$

In practice, we find that using the above formulas yields a significant speed gain for the purpose of estimating the GJR, TGARCH and EGARCH models. For the case of maximum likelihood estimation, when compared to the use of an adaptive quadrature implemented in C++, we find the speed gain is at least a factor two and can be as high as a factor ten, depending on the model, the number of observations and the optimization algorithm used. More importantly, our simples formulas are much simpler to program than advanced numerical integration methods, which are highly sensitive to implementation details. These formulas are thus undoubtedly helpful for estimating the GJR, TGARCH and EGARCH models, which are widely used.

3.3. Other potential applications

Here, we outline other potential applications of the formulas presented in this work. First, we note that these formulas allow efficient target variance estimation of GARCH-type models such as the GJR and the TGARCH models. The formulas might also be useful to implement Krause and Paoella (2014)'s approach for predicting value-at-risk and expected shortfall.

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1 To prove the stationarity condition of the GJR model, take unconditional expectation $\mathbb{E}[\cdot]$ on both sides of (9), then isolate $\mathbb{E}\big[\sigma_t^2\big] = \mathbb{E}\big[\sigma_{t-1}^2\big]$ and find the necessary condition for $\mathbb{E}\big[\sigma_t^2\big] > 0$. To prove the stationarity condition of the TGARCH model, just generalize the formula presented in Francq and Zakoian (2010, Chapter 10.2). For the EGARCH model, the stationarity condition follows directly from Francq and Zakoian (2010, Theorem 10.1).

2 To prove (12), use (7) with $n = 2$ in $\mathbb{E}\big[z^2 I_{\{z<0\}}\big] = F_\xi(0)\mathbb{E}\big[z^2 \,|\, z < 0\big]$ and simplify using $\mu_\xi \xi = M_1(\xi^2 - 1)$.

3 To prove (13), use (7) with $n = 1$ in $\mathbb{E}\big[z\, I_{\{z<0\}}\big] = F_\xi(0)\mathbb{E}[z \,|\, z < 0]$ and simplify using $\mu_\xi \xi = M_1(\xi^2 - 1)$.

4 To prove (14), directly use (5) with $n = 1$ and simplify using $\mu_\xi = M_1(\xi - \xi^{-1})$.

For instance, its could be more efficient to use empirical moments instead of empirical quantiles when fitting the distribution of the innovations. This is because theoretical quantiles need to be pre-computed to create a lookup table (based on a tight grid of the parameter values) for this approach to be efficient. In contrast, we have shown that the theoretical moments can be computed efficiently. Finally, the formulas also provide additional moments equations that could be used to generalize Castillo et al. (2011)'s moment estimator, which was tested for the skewed Normal distribution.

Appendix A. Proofs of the formulas

This appendix outlines the steps to derive the formulas presented in Section 2.3.

A1. Derivation of the general formula for $\mathbb{E}[z^n \,|\, z \le 0]$

Under the skewed density (1), we first note that:

$$\mathbb{E}[z^n \,|\, z \le 0] = \frac{1}{F_\xi(0)} \int_{-\infty}^{0} z^n f_\xi(z) dz = \frac{2\sigma_\xi}{F_\xi(0)(\xi + \xi^{-1})} \int_{-\infty}^{0} z^n f_1(z_\xi) dz\,. \tag{15}$$

Let us first consider the case $\xi \le 1 \Leftrightarrow \mu_\xi \le 0$. We have:

$$\begin{aligned}\int_{-\infty}^{0} z^n f_1(z_\xi) dz &= \int_{-\infty}^{0} z^n f_1(\xi[\sigma_\xi z + \mu_\xi]) dz \\ &= \frac{1}{\xi^{n+1}\sigma_\xi^{n+1}} \int_{-\infty}^{\mu_\xi \xi} (u - \mu_\xi \xi)^n f_1(u) du \\ &= \frac{1}{\xi^{n+1}\sigma_\xi^{n+1}} \left(\int_{-\infty}^{0} (u - \mu_\xi \xi)^n f_1(u) du - \int_{\mu_\xi \xi}^{0} (u - \mu_\xi \xi)^n f_1(u) du \right),\end{aligned} \tag{16}$$

where the second line follows from the change of variable $u \equiv \xi[\sigma_\xi z + \mu_\xi]$. We next use the well-known binomial formula to show that the first integral in (16) can be written as:

$$\int_{-\infty}^{0} (u - \mu_\xi \xi)^n f_1(u) du = \sum_{k=0}^{n} \binom{n}{k} (-\mu_\xi \xi)^{n-k} \int_{-\infty}^{0} u^k f_1(u) du = \frac{(-1)^n}{2} \sum_{k=0}^{n} \binom{n}{k} (\mu_\xi \xi)^{n-k} M_k, \tag{17}$$

where $M_k \equiv 2\int_0^\infty u^k f_1(u) du$, with $M_0 = 1$ by normalization and $M_2 = 1$ for distributions with unit variance. We then combine (16) and (17) to write (15) as:

$$\mathbb{E}[z^n \,|\, z \le 0] = \frac{2}{F_\xi(0)(\xi + \xi^{-1})\sigma_\xi^n} \left(\frac{(-1)^n}{2\xi^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (\mu_\xi \xi)^{n-k} M_k - \frac{1}{\xi^{n+1}} \int_{\mu_\xi \xi}^{0} (u - \mu_\xi \xi)^n f_1(u) du \right),$$

which holds for the case $\xi \le 1$.

We now consider the case $\xi \ge 1 \Leftrightarrow \mu_\xi \ge 0$, for which we have:

$$\int_{-\infty}^{0} z^n f_1(z_\xi) dz = \int_{-\infty}^{-\mu_\xi/\sigma_\xi} z^n f_1(\xi[\sigma_\xi z + \mu_\xi]) dz + \int_{-\mu_\xi/\sigma_\xi}^{0} z^n f_1(\xi^{-1}[\sigma_\xi z + \mu_\xi]) dz\,. \tag{18}$$

For the first integral on the right-hand side, we use the change of variable $u \equiv \xi[\sigma_\xi z + \mu_\xi]$ together with the identity (17). For the second integral, we use the change of variable $u \equiv \xi^{-1}[\sigma_\xi z + \mu_\xi]$. We then carry the simplifications and use (15) to show that:

$$\mathbb{E}[z^n \,|\, z \le 0] = \frac{2}{F_\xi(0)(\xi + \xi^{-1})\sigma_\xi^n} \left(\frac{(-1)^n}{2\xi^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (\mu_\xi \xi)^{n-k} M_k + \xi^{n+1} \int_{0}^{\mu_\xi \xi^{-1}} (u - \mu_\xi \xi^{-1})^n f_1(u) du \right),$$

which holds for the case $\xi \ge 1$.

A2. General formula for $\mathbb{E}[z^n \,|\, z \ge 0]$

The derivation is similar to the previous one and can be obtained from the authors upon request.

A3. General formula for $\mathbb{E}[|z|^n]$

We use the following identity (which is easy to demonstrate):

$$\mathbb{E}[|z|^n] = (-1)^n F_\xi(0)\mathbb{E}[z^n \,|\, z \le 0] \;+\; (1 - F_\xi(0))\mathbb{E}[z^n \,|\, z \ge 0],$$

together with (7) and (8) and we simplify to obtain (5).

A4. General formula for $\mathbb{E}[z^n]$

We use the following identity (which is easy to demonstrate):

$$\mathbb{E}[z^n] = F_\xi(0)\mathbb{E}[z^n \mid z \le 0] \; + \; (1 - F_\xi(0))\mathbb{E}[z^n \mid z \ge 0],$$

together with (7) and (8) and we simplify to obtain (3).

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