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# LOGICAL GROUNDS

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The view that some facts obtain *in virtue of* other facts is both natural and plausible. Causal examples come to mind: if an event $e$ caused an event $f$, then it sounds correct to say that the fact that $f$ occurred obtains in virtue of the fact that $e$ occurred—or, to put it in a less cumbersome way, that $f$ occurred because $e$ occurred. But many *noncausal* cases also come to mind, as illustrated, for example, by the following statements:

- These two apples resemble each other because they have the same shape and the same colour;
- Her belief that $\pi$ is an irrational number is justified because it is based on reliable testimony;
- You acted wrongly because your intention was to cause harm.

In each case, it is said that a fact (e.g., the fact that the two apples resemble each other) obtains in virtue of another fact (e.g., the fact that they have the same shape and the same colour), and the explanatory link involved is not—or so is it plausible to think—of the causal sort.

Noncausal explanatory links have widely been invoked in all domains of philosophy. Suitable generalisations of the previous examples are indeed important philosophical theses, and further illustrations include, for example, (generalisations of) Aristotle's oft cited claim that 'It is not because we think truly that you *are* white, that you are white, but because you are white we who say this have the truth',[1] versions of the Rationalists' Principle of Sufficient Reason, various "reduction" claims (e.g., that the mental reduces to the physical, the normative to the natural, or again the aesthetic to the subjective and/or cultural) and truth-making theory, which assumes that (at least some) truths are true in virtue of the existence of things.

One special sort of noncausal explanatory relation called 'grounding' has recently been the focus of much philosophical thinking.[2] As I see it, a distinctive trait of grounding is that it is an *objective* relation: whether it is true or not that a given fact is grounded in other facts is not relative to epistemic contexts or subjective standpoints (unless the obtaining of the facts in question is subject to such a relativity). In this respect, 'grounds' is unlike 'explains' in many of its uses (although, of course, the latter verb can be used in place of the former as long as it is properly understood). Many of the philosophical theses previously mentioned are best construed as involving grounding rather than some form of nonobjective noncausal explanatory connection.

Grounding is sometimes thought to come in various forms or types. In particular, there appears to be a threefold distinction between *metaphysical*, *conceptual* and *logical*

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1 *Metaphysics*, 1051b6-8.
2 Starting with Fine (2001). See Correia & Schnieder (2012) and Trogdon (2013).

grounding. Thus, it may be said that the fact that the glass is fragile is metaphysically grounded in some particular fact concerning its molecular structure, that the fact that the wall is red is conceptually grounded in the fact that it is scarlet, and that the fact that there are mountains is logically grounded in the fact that Mont Blanc is a mountain. Distinguishing between these three forms of grounding ties does not mean believing that they do not intersect. It is indeed plausible to hold that every case of logical grounding is a case of conceptual grounding (but not vice versa), and that every case of conceptual grounding is a case of metaphysical grounding (but not vice versa).[3]

This threefold distinction is somewhat akin to the more usual distinction between metaphysical, conceptual and logical necessity, whose elements are also standardly taken to exemplify the order of relative strength from logical to conceptual to metaphysical. The connection between the two distinctions is even arguably tighter, since it is plausible to hold that each type of grounding relation generates a necessary connection of the corresponding sort, that is, whenever some fact is grounded, in either of the three senses, in other facts, it is necessary, in the corresponding sense, that the former fact obtains if the latter do.[4]

Although I have some sympathy for the view that there are these two distinctions and that the notions distinguished have the features I have just mentioned, it will not be presupposed in its entirety in this paper. My focus will indeed be mainly on the notion of *logical grounding* and on its connections with those of truth and logical consequence. Other notions of grounding will appear in discussion only at some points, and the connection between grounding and necessity will not be properly discussed (even though, given that logical consequence is tightly connected with logical necessity, my discussion of the connection between logical grounding and logical consequence will have some bearing on the issue).

What I wish to do is the following:

- *Offer elements of clarification of the notion of logical grounding.* To this effect, I will give precise characterisations of logical grounding treated as a relation between formulas of certain formal languages, to wit propositional and first-order languages. The proposed characterisations will be proof-theoretic as opposed to semantic.
- *Offer elements of a theory of its "external" connections with the concept of truth.* I will show that, for languages of the sort described above, the concept of truth-according-to-a-valuation/model can be extensionally captured in terms of logical grounding (as previously characterised).
- *Offer elements of a theory of its connections with the notion of logical consequence.* Using the previous results, I will show that, again for languages of the sort described above, various types of well-known relations of logical consequence can be defined in terms of logical grounding (as previously characterised).
- *Offer elements of a theory of its "internal" connections with the concept of truth.* I will first extend the characterisation of logical grounding to first-order languages containing a predicate for truth and then use that characterisation to provide a syntactic companion to Kripke's (1975) characterisation of truth-in-a-model for such languages. This will in particular throw some syntactic light on the intuitive

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[3] In Correia (2005, chap. 3), I make a distinction between various types of grounding ties, including logical and metaphysical grounding. Fine (2012b) also distinguishes between various types of such ties, but his discussion mentions only the notions of metaphysical, normative and natural grounding. Correia (2008) and Correia & Schnieder (2012) mention the full, threefold distinction between metaphysical, conceptual and logical grounding.

[4] See the papers cited in the previous footnote, and the discussion in Trogdon (2013).

notion of "groundedness" at work in the literature on the paradoxes of truth, which is itself semantically characterised by Kripke.

As I see it, the main interest of this study is threefold. First, the study provides a precise account of a pretheoretic notion of logical explanation which, I take it, is of great intrinsic interest. Second, it shows that the concept of logical grounding can be used to provide a new angle of approach in logic, which is illuminating and possesses a certain power of unification. And third, it shows that the concept of logical grounding is not irremediably obscure or fruitless, thereby providing (i) a direct response to some forms of scepticism about this concept and (ii) an element of response to certain forms of scepticism about more general concepts of grounding (in particular, that of metaphysical grounding).[5,6,7]

**§1. Grounding on a propositional language.** We suppose given a standard propositional language $\mathcal{L}$ with atoms $p$, $q$, etc. and primitive connectives $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation).[8] Following standard terminology, we let the *literals* be the atoms of the language and their negations.

One main aim of this section is to characterise the relation of logical grounding on $\mathcal{L}$. I suggest that this be done in terms of the following rules of inference, which I call *the basic rules*:

$$(\wedge 1)\ \frac{\phi \quad \psi}{\phi \wedge \psi} \qquad (\wedge 2)\ \frac{\neg\phi}{\neg(\phi \wedge \psi)} \qquad (\wedge 3)\ \frac{\neg\psi}{\neg(\phi \wedge \psi)}$$

$$(\vee 1)\ \frac{\neg\phi \quad \neg\psi}{\neg(\phi \vee \psi)} \qquad (\vee 2)\ \frac{\phi}{\phi \vee \psi} \qquad (\vee 3)\ \frac{\psi}{\phi \vee \psi}$$

$$(\neg)\ \frac{\phi}{\neg\neg\phi}$$

5 For recent responses to scepticism about metaphysical grounding, see, for example, Rosen (2010), Audi (2012), and Raven (2012).

6 I have used interchangeably the predicate 'grounds', the sentential operator 'because' and the hybrid expression 'in virtue of' to express the notion of grounding. This should not be understood as indicating that I take these three modes of expression to be on a par. Quite the contrary, I think that canonical statements of ground should be made using a sentential connective like 'because'. (For discussions on these issues, see Correia, 2010; Correia & Schnieder, 2012; Fine, 2012b; Trogdon, 2013.) Likewise, in what follows I will treat logical grounding as expressed by a relational predicate over formulas of formal languages, but no conclusion on my favoured way of expressing the notion should be drawn from that fact.

7 Logical studies on grounding are few. To my knowledge they boil down to the following: Batchelor (2010), Correia (2010, 2011), Fine (2010, 2012a, 2012b), Litland (Manuscript) and Schnieder (2011). Leaving aside Fine (2010), none of these works specifically deals with the concept of *logical* grounding. The core of Fine (2010) is in this respect in line with these works—it is mainly concerned with a notion of (partial) grounding which is not specifically logical. Yet in the last section of Fine's paper, which discusses the connections between previous considerations and Kripke's theory of truth, suggestions are made about how to "read off" a relation of (partial) grounding from inference rules governing the behaviour of the logical constants, and the resulting relations well deserve the qualification 'logical'. The general idea suggested by Fine at the end of his paper (that of "reading off" a relation of logical grounding from appropriate inference rules) actually turns out to be the one I will follow here. However, my aims, as well as the results I will present, are significantly different from Fine's (see in particular §7.4.).

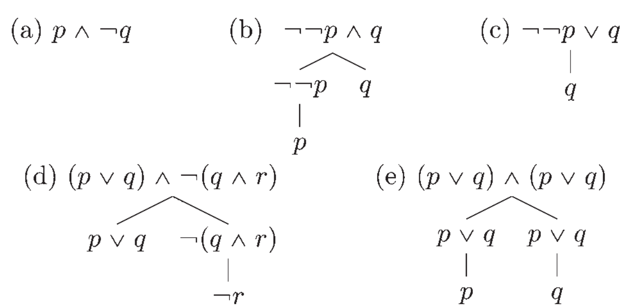
8 Having these as primitives is not necessary, but proves convenient for comparison with other works on the logic of grounding or on logic *tout court*.

The thought is the following. Say that a rule of inference for $\mathcal{L}$ is *ground-theoretically admissible* if it represents a rule which licences only inferences from grounding to grounded statements (in the intuitive sense). The suggestion is that (i) each basic rule for $\mathcal{L}$ is ground-theoretically admissible, (ii) each derived rule for $\mathcal{L}$ (i.e., each rule obtained by chaining basic rules) is also ground-theoretically admissible, and (iii) these basic and derived rules are the only rules for $\mathcal{L}$ which are ground-theoretically admissible.

Things can be made more precise in terms of trees. Let a TREE be a rooted tree whose nodes are occupied by formulas of $\mathcal{L}$, and whose transitions are given by the basic rules, in the sense that in a TREE (i) no parent node is occupied by a literal and (ii) every parent node has either one child or two children, in such a way that the principles depicted in the following table are satisfied:

| Node occupied by | Number of children | Child(ren) occupied by |
|---|---|---|
| $\phi \wedge \psi$ | 2 | $\phi$ and $\psi$ |
| $\phi \vee \psi$ | 1 | $\phi$ or $\psi$ |
| $\neg(\phi \wedge \psi)$ | 1 | $\neg\phi$ or $\neg\psi$ |
| $\neg(\phi \vee \psi)$ | 2 | $\neg\phi$ and $\neg\psi$ |
| $\neg\neg\phi$ | 1 | $\phi$ |

The following are examples of TREES:




Clearly, every branch of a TREE is of finite length (i.e., comprises only finitely many nodes), and a TREE has only finitely many branches.

A TREE *for* a formula is defined as a TREE whose root node is occupied by the formula itself, and a TREE for a formula is said to be *from* a set of formulas $\Delta$ iff $\Delta$ is the set of all the formulas which occupy leaves on the TREE. Thus, (a) above is a TREE for $p \wedge \neg q$ from $\{p \wedge \neg q\}$, (b) a TREE for $\neg\neg p \wedge q$ from $\{p, q\}$, etc. A TREE is said to be a G-TREE ('G' is mnemonic for 'grounding') iff it is not degenerate, that is, iff it does not consist of just one node. Thus, (b)–(e) are G-TREES whereas (a) is not.

Let us now say that a set $\Delta$ of formulas STRICTLY GROUNDS a formula $\phi$—in symbols: $\Delta \rhd \phi$—iff there is a G-TREE for $\phi$ from $\Delta$. The suggestion I previously made is simply that STRICT GROUNDING correctly characterises the intuitive notion of logical grounding relative to language $\mathcal{L}$.

For present purposes, it will often prove very convenient to work, not with STRICT GROUNDING itself, but with a corresponding "nonstrict" relation defined in terms of it—to wit, the relation of GROUNDING. A set $\Delta$ of formulas is said to GROUND a formula $\phi$—in symbols: $\Delta \unrhd \phi$—iff either $\phi \in \Delta$, or for some $\Gamma \subseteq \Delta$, $\Gamma \rhd \phi$. Obviously, $\rhd$ is strictly stronger than $\unrhd$. Also notice that GROUNDING could have been characterised directly in terms of TREES, since $\Delta \unrhd \phi$ iff for some $\Gamma \subseteq \Delta$, there is a TREE for $\phi$ from $\Gamma$.

Let us define the *degree of complexity* of a formula in the usual, recursive fashion:

- For $\phi$ atomic, degree($\phi$) = 0;
- degree($\phi \wedge \psi$) = degree($\phi \vee \psi$) = 1 + max{degree($\phi$), degree($\psi$)};
- degree($\neg\phi$) = 1 + degree($\phi$).

A quick inspection of the basic rules will convince us that STRICT GROUNDING invariably takes one from lower to higher degrees of complexity:

PROPOSITION 1.1. *For every set of formulas $\Delta$ and every formula $\phi$:*

> *If $\Delta \rhd \phi$, then $\phi$'s degree of complexity is strictly greater than the degree of complexity of any member of $\Delta$.*

Other important properties of STRICT GROUNDING are listed here:[9]

PROPOSITION 1.2.

1. *If $\Delta \rhd \phi$, then $\Delta \neq \varnothing$*
2. *If $\Delta \rhd \phi$, then $\phi$ is not a literal*
3. *Not: $\Delta, \phi \rhd \phi$* — Generalised Irreflexivity
4. *If $\Delta, \phi \rhd \psi$, $\phi \notin \Delta$ and $\Gamma \rhd \phi$, then $\Delta, \Gamma \rhd \psi$* — Restricted Cut

Notice the restricting clause '$\phi \notin \Delta$' in the Cut principle. The unrestricted principle indeed fails for STRICT GROUNDING. For instance, let $p$ be an atom. Then $\neg\neg p \rhd \neg\neg\neg\neg p$, and so, (a) $\neg\neg p, \neg\neg p \rhd \neg\neg\neg\neg p$. On the other hand, (b) $p \rhd \neg\neg p$. (a) and (b), together with unrestricted Cut, yield (c) $\neg\neg p, p \rhd \neg\neg\neg\neg p$. But (c) fails: there is no G-TREE for $\neg\neg\neg\neg p$ from $\{\neg\neg p, p\}$.

The relation of GROUNDING behaves differently in this respect. Here is a list of some of its properties:

PROPOSITION 1.3.

1. *If $\Delta \unrhd \phi$, then $\Delta \neq \varnothing$*
2. *If $\Delta \unrhd \phi$ and $\phi \notin \Delta$, then $\phi$ is not a literal*
3. $\Delta, \phi \unrhd \phi$ — Generalised Reflexivity
4. *If $\Delta, \phi \unrhd \psi$ and $\Gamma \unrhd \phi$, then $\Delta, \Gamma \unrhd \psi$* — Cut
5. *If $\Delta \unrhd \phi$, then $\Delta, \Gamma \unrhd \phi$* — Weakening

Of course, Weakening fails for STRICT GROUNDING.

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A number of remarks are in order at this point, in particular about alternative ways of characterising logical grounding on a language such as $\mathcal{L}$.

**A. Factivity.** It is commonplace to view grounding as *factive*. Taking grounding as a relation between statements, its being factive amounts to the truth of the following principle: if a statement is grounded in other statements, then both the grounded statement and the grounding statements must be true. STRICT GROUNDING is not intended to correspond to a

[9] These should be understood as closed under the corresponding universal quantifiers: following standard practice I will often omit to mention outer universal quantifiers. Also, here and below, I follow the standard notational convention of using $\Delta, \Gamma$ for $\Delta \cup \Gamma$ and $\Delta, \phi$ for $\Delta \cup \{\phi\}$, etc.

factive notion. In fact, where $\phi$ is any formula of language $\mathcal{L}$, $\{\phi, \neg\phi\}$ STRICTLY GROUNDS $\phi \wedge \neg\phi$, whereas on many (if not all) intended interpretations, there are formulas $\phi$ of the language such that $\phi$ and $\neg\phi$ are not both true. Thus, one might argue, the theory of logical grounding put forward above is defective.

In response to this objection, one option is to grant the premises but reject the conclusion—or rather, to temper it: yes, the theory as it has been formulated is defective; but it's almost correct. So far no semantics for $\mathcal{L}$ has been given. But suppose we provide $\mathcal{L}$ with a notion of truth, more precisely a notion of truth which is preserved by the basic rules. To fix ideas, let it be the classical notion of truth according to a valuation. Define STRICT GROUNDING*, a relation between sets of formulas and formulas *relative to a valuation*, as follows: $\Delta$ STRICTLY GROUNDS* $\phi$ relative to valuation $\upsilon$ iff$_{\text{df}}$ (i) all the members of $\Delta$ are true according to $\upsilon$, and (ii) $\Delta$ STRICTLY GROUNDS (in the old sense) $\phi$. STRICT GROUNDING* is factive: if $\Delta$ STRICTLY GROUNDS* $\phi$ relative to $\upsilon$, then both the members of $\Delta$ and $\phi$ are true according to $\upsilon$. The suggestion is then that logical grounding is correctly characterised by STRICT GROUNDING* (or by a notion defined in the same way but starting with some other concept of truth).[10]

Another reply to the objection is simply to deny that grounding is factive. More precisely, the view put forward here is that there are nonfactive notions of grounding as well as factive ones, and that STRICT GROUNDING is intended to characterise a nonfactive notion of logical grounding.[11]

I do not think there is a deep difference between the views put forward here. On the first view, STRICT GROUNDING characterises a nonfactive relation between statements, which is not a relation of logical grounding itself, but which nevertheless constitutes the "properly relational core" of such a relation. In contrast, on the second view, logical grounding comes in two species, one factive and the other not,[12] the nonfactive notion being (intended to be) characterised by STRICT GROUNDING. Both views thus admit the existence of a nonfactive relation intimately tied to the factive relation of logical grounding, and the central divide between these two views boils down to the fact that one takes the nonfactive relation to be itself a relation of logical grounding while the other denies it. It is then tempting to think that the difference between the two views is merely verbal. Be it as it may, this difference is immaterial as far as this paper is concerned, and I shall speak of the nonfactive relation intended to be captured by STRICT GROUNDING as of a relation of logical grounding.

**B. Grain.** For $\phi$ any formula of language $\mathcal{L}$, $\phi$ STRICTLY GROUNDS $\phi \wedge \phi$, as well as $\phi \vee \phi$ or again $\neg\neg\phi$. The notion of logical grounding intended to be characterised by STRICT GROUNDING is thus very fine-grained. It appears that most of those who thought about the interaction between grounding and truth-functions had a similarly fine-grained notion in mind.[13] In Correia (2010), I argue in favour of a more coarse-grained conception, but in later work (Correia, 2011) I admit that both conceptions are viable.

**C. Why these rules?** My choice of rules to play the role of the building blocks of the characterisation of logical grounding does not come out of the blue. Clearly, no literal of

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[10] Of course, this recipe for defining a factive notion of grounding in terms of a nonfactive, proof-theoretically defined notion is perfectly general, and can be applied for example to the notions introduced in sections D and F.

[11] See Fine (2012b) on factive versus nonfactive grounding.

[12] In the same way as, for instance, the concepts expressed by 'since $p$, $q$' and 'if $p$, then $q$' might be viewed as two species, one factive and the other not, of the same generic notion.

[13] See Correia (2005), Batchelor (2010), Rosen (2010), Fine (2010, 2012b) and Schnieder (2011).

language $\mathcal{L}$ can be taken to be logically grounded in a set of formulas of $\mathcal{L}$. A nonliteral must be either of one of the two positive forms $\phi \wedge \psi$ and $\phi \vee \psi$, or of one of the three negative forms $\neg(\phi \wedge \psi)$, $\neg(\phi \vee \psi)$ and $\neg\neg\phi$. The basic rules provides a partial answer to the question: Given a formula of any of these five forms, what logically grounds that formula? The complete answer to the question is then given by adding that a set $\Delta$ of formulas logically grounds a formula $\phi$ if and only if chaining the basic rules in some way yields a derived rule that validates the transition from the members of $\Delta$ (all of them) to $\phi$.

Why should all the basic rules—as opposed to only some of them, or even perhaps none—be taken on board? All I can say here is that they all strike me as correct once a fine-grained conception of grounding is taken for granted. Leaving the issue of factivity aside, there would actually seem to be something like a consensus, among the friends of a fine-grained conception, that the basic rules plausibly record genuine grounding ties, and I guess that those who do share that view would be happy to say that the ties in question are properly *logical* in character.[14]

**D. More rules?** Granted that the basic rules are acceptable, do they form a sufficient basis in the present context, or should further rules be countenanced in addition?

In the spirit of Fine (2012b), one might think that in addition to the basic rules for disjunction ($\vee$2) and ($\vee$3), rule ($\vee$4) should also be countenanced, and similarly that in addition to ($\wedge$2) and ($\wedge$3) we should countenance ($\wedge$4):

$$(\vee 4) \quad \frac{\phi \quad \psi}{\phi \vee \psi} \qquad\qquad (\wedge 4) \quad \frac{\neg\phi \quad \neg\psi}{\neg(\phi \wedge \psi)}$$

Of course, having these extra rules makes a difference. For example if $p$ and $q$ are any atoms, $p, q \rhd p \vee q$ does not hold in the original theory, but holds if the theory includes ($\vee$4).

It is not clear to me whether the new rules should be adopted. Fortunately, for the purposes of this paper it will not be very important to make a decision on this issue and, although I will work with the original theory throughout, I shall consider the richer theory as a serious alternative. The reason why it will not be important to choose between the two theories is, in a nutshell, that what will "do the real job" in what follows is "nonstrict" grounding, and that the two theories' nonstrict concepts of grounding are equivalent. To be more precise, let $\rhd'$ stand for the notion of STRICT GROUNDING defined as before but with ($\vee$4) and ($\wedge$4) as extra rules, and let $\unrhd'$ stand for the corresponding notion of GROUNDING. The point is that even though $\rhd'$ does not have the same extension as $\rhd$, $\unrhd'$ and $\unrhd$ are nevertheless equivalent.

**E. The basic rules as introduction rules.** The basic rules, which are the building blocks of the proposed theory of logical grounding for languages such as $\mathcal{L}$, all take simpler formulas to more complex formulas. This is why Proposition 1.1 holds. They can in fact all be viewed as *introduction rules*. ($\wedge$1) is known as the rule of conjunction introduction, and ($\vee$2) and ($\vee$3) as the rules of disjunction introduction. Following this standard terminology, ($\neg$) may also appropriately be called a rule of double negation introduction. The case of the remaining rules is a bit different. Yet these rules can naturally be described as introduction rules *in negative contexts*: each takes us from one or more

[14] There certainly is room for less than full consensus on the view that all the basic rules are acceptable. For instance, Fine (2010) contemplates the rejection of some of the basic rules as part of a solution to some puzzles about grounding. See §7.4.

negated formulas to a negated formula containing an extra connective (conjunction or disjunction, as the case may be).

Our set of basic rules is thus significantly different from the set of rules for a standard natural deduction system for (say) classical or intuitionistic logic, which comprises both introduction and elimination rules. This difference makes the set of basic rules rather weak indeed. For instance, whereas all the basic rules are intuitionistically and hence classically valid, some rules which are valid in these two logics are neither among the basic rule nor among the derived rules definable from them: the rule of conjunction elimination is a simple illustration. This does not mean that the basic rules cannot be used to characterise strong logics, though; as we shall see in what follows, they can be used to fully characterise classical logical consequence, as well as other important consequence relations.[15]

**F. Amalgamation.** Fine (2012a, 2012b) takes the concept of (strict) grounding to obey a principle he calls 'Amalgamation': if a statement is grounded in a plurality $X$ of statements, and also in another such plurality $Y$, then it is grounded in $X$ and $Y$ taken jointly.[16] Of course, STRICT GROUNDING does not obey such a principle: for $p$ and $q$ distinct atoms of $\mathcal{L}$, we have both $p \rhd p \vee q$ and $q \rhd p \vee q$, but not $p, q \rhd p \vee q$. Is this a problem?

Those who think that it is may wish to hold that logical grounding is properly characterised not by STRICT GROUNDING, but rather by the strictly stronger relation of STRICT GROUNDING$^{\times}$, which is defined as follows:

> $\Delta$ STRICTLY GROUNDS$^{\times}$ $\phi$ iff$_{\text{df}}$ there is a covering of $\Delta$ (i.e., a family of sets whose union is $\Delta$) such that for each $\Gamma$ in this covering, $\Gamma \rhd \phi$.

STRICT GROUNDING$^{\times}$ indeed obeys Amalgamation.

I have no firm intuition on the question whether logical grounding should be taken to obey Amalgamation. Yet, here as with the question whether rules ($\vee$4) and ($\wedge$4) should be countenanced, it will not be important to decide whether STRICT GROUNDING or STRICT GROUNDING$^{\times}$ should be taken to characterise logical grounding, and for just the same reason: the corresponding nonstrict notions are equivalent. (The details are straightforward.)

STRICT GROUNDING$'$, that is, the relation symbolised by $\rhd'$ in part D above, does not obey Amalgamation either. For let $p$ be an atom. Then $\neg\neg p \rhd' \neg\neg\neg\neg p$ and $p \rhd' \neg\neg\neg\neg p$. Yet it is not the case that $\neg\neg p, p \rhd' \neg\neg\neg\neg p$.[17] Those who think that ($\vee$4) and ($\wedge$4)

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[15] Gentzen (1969) took the natural deduction introduction rules for the logical constants to enjoy a privileged status over the elimination rules: the former rules provide the "definitions" (Gentzen himself uses scare quotes) of the logical constants, whereas the latter rules are (in some sense) just consequences of these definitions. Gentzen's view has given rise to a massive literature, both logical and philosophical (in particular on inferentialism in the theory of meaning and so-called proof-theoretic harmony), which certainly connects with the present work at several places. I hope to investigate these connections elsewhere. (Tatzel, Manuscript, is an interesting paper which deals with the connections between some of Gentzen's views and Bolzano's views about grounding, and is of particular relevance.)

[16] In these two papers, Fine distinguishes between strict and weak grounding. Fine's concept of weak grounding is significantly different from my notion of GROUNDING. He takes both strict and weak grounding as primitives, but indicates how one may be defined in terms of the other. His suggestion for defining the weak notion in terms of the strict one is very different from my definition of GROUNDING in terms of STRICT GROUNDING: for him, '$P$, $Q$, ... weakly ground $R$' can be defined as 'for every plurality of statements $X$ and every statement $S$, if $R$, $X$ strictly ground $S$, then $P$, $Q$, ..., $X$ strictly ground $S$'. One important formal difference between GROUNDING and Fine's weak grounding is that the former obeys Weakening whereas the latter does not.

[17] The same example could have been used to show that $\rhd$ infringes Amalgamation.

should be countenanced amongst the basic rules may alternatively wish to hold that logical grounding is rather to be characterised by STRICT GROUNDING$'^{\times}$, which is defined as follows and also obeys Amalgamation:

> $\Delta$ STRICTLY GROUNDS$'^{\times}$ $\phi$ iff$_{\text{df}}$ there is a covering of $\Delta$ such that for each $\Gamma$ in this covering, $\Gamma \rhd' \phi$.

Yet, again, the corresponding nonstrict notion is equivalent to GROUNDING, and consequently characterising logical grounding using this notion rather than STRICT GROUNDING would not make an essential difference.

**§2. Grounding and propositional truth.** The basic rules are intuitively connected with the concept of truth: they authorise only transitions from truths to truths. Yet the connection is stronger than that.

Let a *valuation* be a distribution of truth-values (True and False) over the atoms of language $\mathcal{L}$. It is not required that a valuation be *maximal* (i.e., that it assigns at least one truth-value to each atom) or *coherent* (i.e., that it assigns at most one truth-value to each atom).

Truth ($\models$) and falsity ($=\!\mid$) relative to a valuation $v$ are defined for arbitrary formulas of $\mathcal{L}$ in the following, natural way:

- $v \models p$ iff $p$ is True according to $v$
- $v =\!\mid p$ iff $p$ is False according to $v$
- $v \models \phi \wedge \psi$ iff $v \models \phi$ and $v \models \psi$
- $v =\!\mid \phi \wedge \psi$ iff $v =\!\mid \phi$ or $v =\!\mid \psi$
- $v \models \phi \vee \psi$ iff $v \models \phi$ or $v \models \psi$
- $v =\!\mid \phi \vee \psi$ iff $v =\!\mid \phi$ and $v =\!\mid \psi$
- $v \models \neg\phi$ iff $v =\!\mid \phi$
- $v =\!\mid \neg\phi$ iff $v \models \phi$.

In case valuation $v$ is maximal, $v \not\!=\!\mid \phi$ entails $v \models \phi$; in case $v$ is coherent, $v =\!\mid \phi$ entails $v \not\models \phi$; if $v$ is both maximal and coherent, $v =\!\mid \phi$ is equivalent to $v \not\models \phi$, and the definition above yields the standard characterisation of classical truth for formulas of $\mathcal{L}$.

The intuitive connection between the basic rules and truth mentioned above can now be given in a precise form: for any valuation $v$, each basic rule preserves the property of being true according to $v$. This yields the following preservation result:

PROPOSITION 2.1. *Let $v$ be a valuation, $\Delta$ a set of formulas and $\phi$ a formula. If both $v \models \Delta$ and $\Delta \unrhd \phi$, then $v \models \phi$.*

(By '$v \models \Delta$', I mean: $v \models \psi$ for all $\psi \in \Delta$.)

Let us now turn to establishing some lemmas which will lead us to the main result of this section, *Fundamental Connection*.

LEMMA 2.2. *Let $\Delta$ be any set of formulas. Then:*

1. *If $\Delta \unrhd \phi$ and $\Delta \unrhd \psi$, then $\Delta \unrhd \phi \wedge \psi$*
2. *If $\Delta \unrhd \neg\phi$ or $\Delta \unrhd \neg\psi$, then $\Delta \unrhd \neg(\phi \wedge \psi)$*
3. *If $\Delta \unrhd \phi$ or $\Delta \unrhd \psi$, then $\Delta \unrhd \phi \vee \psi$*
4. *If $\Delta \unrhd \neg\phi$ and $\Delta \unrhd \neg\psi$, then $\Delta \unrhd \neg(\phi \vee \psi)$*
5. *If $\Delta \unrhd \phi$, then $\Delta \unrhd \neg\neg\phi$.*

*Proof.* Let me just prove (1)—the proofs for (2)–(5) run in much the same way. Suppose that $\Delta \trianglerighteq \phi$ and $\Delta \trianglerighteq \psi$. Then there is a TREE $T$ for $\phi$ from some $\Gamma \subseteq \Delta$ and a TREE $T'$ for $\psi$ from some $\Gamma' \subseteq \Delta$. These two TREES may be represented as follows:

$$T:\ \begin{array}{c}\phi\\ |\\ \Gamma\end{array} \qquad\qquad T':\ \begin{array}{c}\psi\\ |\\ \Gamma'\end{array}$$

(Any of these TREES may be degenerate.) We can combine them into a new tree using rule ($\wedge$1) as follows:

$$U:\ \begin{array}{cc}\multicolumn{2}{c}{\phi \wedge \psi}\\ \phi & \psi\\ | & |\\ \Gamma & \Gamma'\end{array}$$

$U$ is obviously a TREE (it is indeed a G-TREE) for $\phi \wedge \psi$ from $\Gamma \cup \Gamma'$. Since $\Gamma \cup \Gamma' \subseteq \Delta$, it follows that $\Delta \trianglerighteq \phi \wedge \psi$. □

LEMMA 2.3. *Let $\Delta$ be any set of formulas. Then:*

1. *If $\Delta \triangleright \phi \wedge \psi$, then $\Delta \trianglerighteq \phi$ and $\Delta \trianglerighteq \psi$*
2. *If $\Delta \triangleright \neg(\phi \wedge \psi)$, then $\Delta \trianglerighteq \neg\phi$ or $\Delta \trianglerighteq \neg\psi$*
3. *If $\Delta \triangleright \phi \vee \psi$, then $\Delta \trianglerighteq \phi$ or $\Delta \trianglerighteq \psi$*
4. *If $\Delta \triangleright \neg(\phi \vee \psi)$, then $\Delta \trianglerighteq \neg\phi$ and $\Delta \trianglerighteq \neg\psi$*
5. *If $\Delta \triangleright \neg\neg\phi$, then $\Delta \trianglerighteq \phi$.*

*Proof.* Let me here only prove (1) and (3) (the other proofs are similar). (1) Suppose $\Delta \triangleright \phi \wedge \psi$. Then there is a G-TREE for $\phi \wedge \psi$ from $\Delta$. This G-TREE must have its root node occupied by $\phi \wedge \psi$, and this root node must have two children, one occupied by $\phi$ and the other one by $\psi$. It may be represented thus:

$$\begin{array}{cc}\multicolumn{2}{c}{\phi \wedge \psi}\\ \phi & \psi\\ | & |\\ \Gamma & \Gamma'\end{array}$$

where $\Gamma \cup \Gamma' = \Delta$, and

$$\begin{array}{c}\phi\\ |\\ \Gamma\end{array}$$

represents a (possibly degenerate) TREE for $\phi$ from $\Gamma$ and

$$\begin{array}{c}\psi\\ |\\ \Gamma'\end{array}$$

a (possibly degenerate) TREE for $\psi$ from $\Gamma'$. Clearly, then, both $\Delta \trianglerighteq \phi$ and $\Delta \trianglerighteq \psi$. (3) Suppose $\Delta \triangleright \phi \vee \psi$. Then there is a G-TREE for $\phi \vee \psi$ from $\Delta$. This G-TREE must have its root node occupied by $\phi \vee \psi$, and this root node must have one child, occupied either by $\phi$ or by $\psi$. Suppose it is occupied by $\phi$ (the other case is similar). The G-TREE may be represented thus:

$$\begin{array}{c}\phi \vee \psi \\ | \\ \phi \\ | \\ \Delta\end{array}$$

where

$$\begin{array}{c}\phi \\ | \\ \Delta\end{array}$$

represents a (possibly degenerate) TREE for $\phi$ from $\Delta$. Hence, $\Delta \trianglerighteq \phi$. □

DEFINITION 2.4. *A **situation** is a set of literals.*

Putting the previous two lemmas together yields:[18]

LEMMA 2.5. *Let $\Delta$ be a situation. Then:*

1. $\Delta \trianglerighteq \phi \wedge \psi$ *iff* $\Delta \trianglerighteq \phi$ *and* $\Delta \trianglerighteq \psi$
2. $\Delta \trianglerighteq \neg(\phi \wedge \psi)$ *iff* $\Delta \trianglerighteq \neg\phi$ *or* $\Delta \trianglerighteq \neg\psi$
3. $\Delta \trianglerighteq \phi \vee \psi$ *iff* $\Delta \trianglerighteq \phi$ *or* $\Delta \trianglerighteq \psi$
4. $\Delta \trianglerighteq \neg(\phi \vee \psi)$ *iff* $\Delta \trianglerighteq \neg\phi$ *and* $\Delta \trianglerighteq \neg\psi$
5. $\Delta \trianglerighteq \neg\neg\phi$ *iff* $\Delta \trianglerighteq \phi$.

*Proof.* The right-to-left directions are directly given by Lemma 2.2 (the assumption that $\Delta$ is a situation plays no role). For the other directions, let $\theta$ stand for either of $\phi \wedge \psi$, $\neg(\phi \wedge \psi)$, $\phi \vee \psi$, $\neg(\phi \vee \psi)$ and $\neg\neg\phi$. Suppose then $\Delta \trianglerighteq \theta$. Since $\Delta$ is a situation, $\theta \notin \Delta$. So there must be some $\Gamma \subseteq \Delta$ with $\Gamma \triangleright \theta$. Using Lemma 2.3 and Weakening for $\trianglerighteq$, we get the result. □

Thus, the notion of being-grounded-in-a-situation interacts with the truth-functional connectives exactly like the notion of being-true-according-to-a-valuation.

DEFINITION 2.6. *The **situation determined by a valuation** $\upsilon$—in symbols: $\mathcal{S}(\upsilon)$—is the set of all the literals which are true according to $\upsilon$.*

We have the following further connection:

THEOREM 2.7. *(Fundamental Connection) Let $\upsilon$ be a valuation and $\phi$ a formula. Then: $\upsilon \models \phi$ iff $\mathcal{S}(\upsilon) \trianglerighteq \phi$.*

*Proof.* It is possible to prove the theorem by induction on the degree of the formulas using Lemma 2.5 and Proposition 2.1. Another route, which I follow here, invokes Proposition 2.1 and Lemma 2.2.

The right-to-left direction of the Theorem is immediate given Proposition 2.1, since for every valuation $\upsilon$, $\upsilon \models \mathcal{S}(\upsilon)$. The other direction is proved by induction on the degree of the formulas—more precisely, what is proved by induction is the following proposition:

> For all formulas $\phi$: for all valuations $\upsilon$, (i) if $\upsilon \models \phi$, then $\mathcal{S}(\upsilon) \trianglerighteq \phi$ and (ii) if $\upsilon \dashv \phi$, then $\mathcal{S}(\upsilon) \trianglerighteq \neg\phi$.

[18] Fine (2012b) establishes a somewhat similar result for his concept of weak grounding.

(A) $\phi$ atomic. (i) Suppose $\upsilon \models \phi$. Then $\phi \in \mathcal{S}(\upsilon)$, and so $\mathcal{S}(\upsilon) \trianglerighteq \phi$. (ii) Suppose $\upsilon \dashv \phi$. Then $\upsilon \models \neg\phi$, and so $\neg\phi \in \mathcal{S}(\upsilon)$, and so $\mathcal{S}(\upsilon) \trianglerighteq \neg\phi$.

(B) $\phi = \neg\alpha$. (i) Suppose $\upsilon \models \neg\alpha$. Then $\upsilon \dashv \alpha$. By IH, then, $\mathcal{S}(\upsilon) \trianglerighteq \neg\alpha$. (ii) Suppose $\upsilon \dashv \neg\alpha$. Then $\upsilon \models \alpha$. By IH, then, $\mathcal{S}(\upsilon) \trianglerighteq \alpha$. By Lemma 2.2 it follows that $\mathcal{S}(\upsilon) \trianglerighteq \neg\neg\alpha$.

(C) $\phi = \alpha \wedge \beta$. (i) Suppose $\upsilon \models \alpha \wedge \beta$. Then both $\upsilon \models \alpha$ and $\upsilon \models \beta$. By IH, then, $\mathcal{S}(\upsilon) \trianglerighteq \alpha$ and $\mathcal{S}(\upsilon) \trianglerighteq \beta$. Then by Lemma 2.2, $\mathcal{S}(\upsilon) \trianglerighteq \alpha \wedge \beta$. (ii) Suppose $\upsilon \dashv \alpha \wedge \beta$. Then either $\upsilon \dashv \alpha$ or $\upsilon \dashv \beta$. Then by IH, $\mathcal{S}(\upsilon) \trianglerighteq \neg\alpha$ or $\mathcal{S}(\upsilon) \trianglerighteq \neg\beta$. By Lemma 2.2, it follows that $\mathcal{S}(\upsilon) \trianglerighteq \neg(\alpha \wedge \beta)$.

(D) The case $\phi = \alpha \vee \beta$ is similar to (C). □

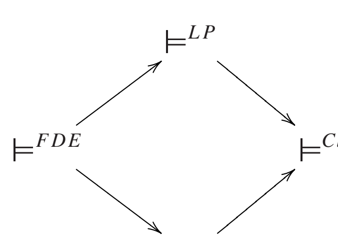
This result is noteworthy. It states that for a formula of language $\mathcal{L}$, being *true* according to a distribution of truth-values over the atoms of $\mathcal{L}$ is equivalent to being GROUNDED in the literals of $\mathcal{L}$ which are true according to that distribution. As a corollary, for a formula of $\mathcal{L}$, being *false* according to a distribution of truth-values over the atoms of $\mathcal{L}$ is equivalent to having its negation GROUNDED in the literals of $\mathcal{L}$ which are true according to that distribution. Thus, Fundamental Connection captures in a precise way the informal thought that the truth-value of any formula of $\mathcal{L}$ is ultimately explained by the truth-values of the atomic formulas of the language.

**§3. Grounding and propositional consequence.** The results of the previous section about the connections between GROUNDING and *truth* immediately yield interesting results about the connections between GROUNDING and *logical consequence*.

Four well-known consequence relations are naturally characterised in terms of valuations as defined above: classical consequence ($\models^{Cl}$), "first-degree entailment" consequence ($\models^{FDE}$), "logic of paradox" consequence ($\models^{LP}$) and "Kleene 3-valued logic" consequence ($\models^{K3}$).[19] Remember that a valuation is said to be *maximal* iff it assigns a truth-value to each atom of $\mathcal{L}$ and *coherent* iff it never assigns both truth-values to an atom. The characterisations run as follows:

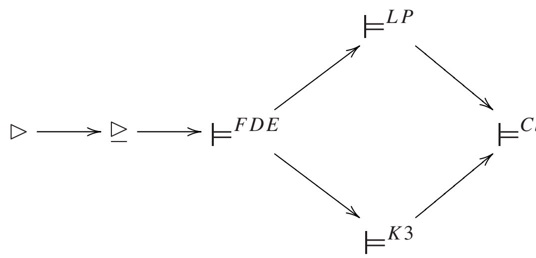
- $\Delta \models^{Cl} \phi$ iff for every maximal coherent valuation $\upsilon$, $\upsilon \models \Delta \Rightarrow \upsilon \models \phi$
- $\Delta \models^{FDE} \phi$ iff for every valuation $\upsilon$, $\upsilon \models \Delta \Rightarrow \upsilon \models \phi$
- $\Delta \models^{LP} \phi$ iff for every maximal valuation $\upsilon$, $\upsilon \models \Delta \Rightarrow \upsilon \models \phi$
- $\Delta \models^{K3} \phi$ iff for every coherent valuation $\upsilon$, $\upsilon \models \Delta \Rightarrow \upsilon \models \phi$.

$\models^{FDE}$ is strictly stronger than both $\models^{LP}$ and $\models^{K3}$, and in turn both are strictly stronger than $\models^{Cl}$. Using an arrow to represent strict inclusion, this fact can be depicted as follows:




[19] See for example Priest (2008, pp. 142–149).

By Proposition 2.1, GROUNDING is stronger than $\models^{FDE}$. In fact, it is strictly stronger, since, for example for $p$ an atom, $\neg\neg p \models^{FDE} p$ whereas $\neg\neg p$ does not GROUND $p$. Since in addition, STRICT GROUNDING is strictly stronger than GROUNDING, the diagram can be extended as follows:




The connections between GROUNDING and the four consequence relations do not boil down to this: Fundamental Connection straightforwardly suggests a syntactic characterisation of each consequence relation in terms of GROUNDING.

The function $\mathcal{S}$ which assigns situation $\mathcal{S}(v)$ to valuation $v$ is obviously a bijection from the set of all valuations to the set of all situations. Say that a situation $S$ is *maximal* iff for every atom $p$ of $\mathcal{L}$, either $p$ or $\neg p$ is in $S$, and that it is *coherent* iff for every atom $p$ in $\mathcal{L}$, not both $p$ and $\neg p$ are in $S$. For any valuation $v$, $\mathcal{S}(v)$ is maximal iff $v$ is maximal, and $\mathcal{S}(v)$ is coherent iff $v$ is coherent.

Let us define four relations of ground-theoretic derivability as follows ('$S \trianglerighteq \Delta$' means '$S \trianglerighteq \psi$ for all $\psi \in \Delta$'):

- $\Delta \Vdash^{Cl} \phi$ iff for every maximal coherent situation $S$, $S \trianglerighteq \Delta \Rightarrow S \trianglerighteq \phi$
- $\Delta \Vdash^{FDE} \phi$ iff for every situation $S$, $S \trianglerighteq \Delta \Rightarrow S \trianglerighteq \phi$
- $\Delta \Vdash^{LP} \phi$ iff for every maximal situation $S$, $S \trianglerighteq \Delta \Rightarrow S \trianglerighteq \phi$
- $\Delta \Vdash^{K3} \phi$ iff for every coherent situation $S$, $S \trianglerighteq \Delta \Rightarrow S \trianglerighteq \phi$.

We then have:

THEOREM 3.1. *Let $\Delta$ be a set of formulas and $\phi$ a formula. Then:*

- $\Delta \models^{Cl} \phi$ *iff* $\Delta \Vdash^{Cl} \phi$
- $\Delta \models^{FDE} \phi$ *iff* $\Delta \Vdash^{FDE} \phi$
- $\Delta \models^{LP} \phi$ *iff* $\Delta \Vdash^{LP} \phi$
- $\Delta \models^{K3} \phi$ *iff* $\Delta \Vdash^{K3} \phi$.

Like Theorem 2.7 (i.e., Fundamental Connection), this theorem is noteworthy: it states that our four semantic consequence relations can be fully characterised in ground-theoretic terms. Since GROUNDING is a proof-theoretic notion, Theorem 3.1 is, in effect, a soundness and completeness result.[20]

**§4. Grounding on a first-order language.** The aim is now to extend the previous results to first-order languages. The languages I will focus on have, like the previous

[20] Schnieder (Manuscript) has recently put forward a ground-theoretic characterisation of entailment. His approach is quite different from mine, but there are similarities, for instance regarding the conception of the interaction of grounding with the truth-functional connectives. It would be interesting to compare the two approaches.

propositional language $\mathcal{L}$, $\wedge$, $\vee$ and $\neg$ as primitive truth-functional connectives and have in addition both the universal quantifier $\forall$ and the existential quantifier $\exists$ as primitives. It will not be necessary to assume that these languages comprise a predicate for identity, but since for many applications identity is important I will nevertheless take them to have such a predicate, $=$. For the sake of simplicity I will suppose that these languages have individual constants but no functional terms. Also for the sake of simplicity, it will be understood that the ground-theoretic rules for these languages involve only *sentences*, that is, closed formulas.

In order to extend the previous study to a language of the sort under consideration, it is natural to make most of the analogies between universal quantification and conjunction on one hand, and existential quantification and disjunction on the other hand, and to introduce rules for $\forall$ and $\exists$ similar to the rules for $\wedge$ and $\vee$, respectively. A fairly natural suggestion is to adopt the following rules:

$$(\forall 1)\quad \frac{\phi(a)\quad \phi(b)\quad \ldots}{\forall x\phi(x)} \qquad\qquad (\forall 2)\quad \frac{\neg\phi(a)}{\neg\forall x\phi(x)}$$

$$(\exists 1)\quad \frac{\neg\phi(a)\quad \neg\phi(b)\quad \ldots}{\neg\exists x\phi(x)} \qquad\qquad (\exists 2)\quad \frac{\phi(a)}{\exists x\phi(x)}$$

where in ($\forall$1) and ($\exists$1), $a$, $b$, ... are all the constants of the language.[21]

Yet, in contrast with rules ($\forall$2) and ($\exists$2), ($\forall$1) and ($\exists$1) are problematic if—as this is the case in this study—we want to work with rules which are (at least) classically valid. That ($\forall$1) is not classically valid can be easily seen for example by taking a model whose domain comprises at least two objects $o$ and $o'$, such that all the constants of the language refer to $o$, and the extension of some monadic predicate $F$ of the language comprises $o$ but not $o'$. To show that ($\exists$1) is not classically valid either, it suffices to modify the example by changing the condition on predicate $F$, and require this time that its extension comprise $o'$ but not $o$.

Although ($\forall$1) and ($\exists$1) do not preserve classical truth-relative-to-a-model, they do preserve classical truth (and suitable generalisations thereof) relative to *full* models, that is, models such that every object in their domain is referred to by some constant of the language. This will be reason enough to take these two rules seriously, and I will do so in §5.

Yet it will also be interesting to work with an alternative set of rules, which relevantly differ in terms of truth-preservation. The formulation of these alternative rules requires that the languages under consideration be enriched with a special "totality" predicate $\mathcal{A}$.[22]

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[21] If our language is a language for arithmetic, and if $a$, $b$, ... are the numerals 0, 1, ..., then a standard name for ($\forall$1) is '$\omega$-rule'.

[22] Fine (2012b) advocates the use of such a predicate in the formulation of the principles governing the interaction of grounding with universal quantification. Notice that he also advocates the use of an existence predicate in the formulation of the principles governing the interaction of grounding with existential quantification, on grounds that would make him modify rules ($\forall$2) and ($\exists$2) by adding the condition '$a$ exists' on top of both. (He also suggests using totality statements instead of existence statements, but this is not a route I explore here.) The opposition between the original rules ($\forall$2) and ($\exists$2) and their suggested variants corresponds, of course, to the standard opposition between classical logic and free logic. I opted for the classical rules for the sake of simplicity, but it would not be difficult to adapt the forthcoming material starting with the alternative rules.

$\mathcal{A}$ is not a regular predicate: it is multigrade, capable of receiving any number of terms (one, finitely many, or even infinitely many) to make a formula. Where $t$, $u$, ... are terms of the language, $\mathcal{A}(tu...)$ is to be read '$t$, $u$, ... are all the objects that there are'.

The suggestion is thus to replace (∀1) and (∃1) by the following two sets of rules (again, involving only sentences):

$$(\forall 1*) \quad \frac{\phi(a) \quad \mathcal{A}(a)}{\forall x \phi(x)} \qquad \frac{\phi(a) \quad \phi(b) \quad \mathcal{A}(ab)}{\forall x \phi(x)} \qquad \cdots$$

$$(\exists 1*) \quad \frac{\neg\phi(a) \quad \mathcal{A}(a)}{\neg\exists x \phi(x)} \qquad \frac{\neg\phi(a) \quad \neg\phi(b) \quad \mathcal{A}(ab)}{\neg\exists x \phi(x)} \qquad \cdots$$

Given the intended interpretation of predicate $\mathcal{A}$, each of these new rules is, contrary to (∀1) and (∃1), intuitively truth-preserving. I will deal with these alternative rules in §6.

**§5. First-order languages: first set of rules.** Suppose given a first-order language $\mathcal{M}$ of the sort described at the beginning of the previous section. The basic rules for $\mathcal{M}$ are here taken to be the basic rules for the sentential connectives as given at the beginning of §1, plus rules (∀1), (∃1), (∀2) and (∃2) for the quantifiers. As I previously stressed, all these rules are taken to concern *sentences*, that is, closed formulas.

We define TREES as in §1, with the stipulation that the nodes of a TREE can only be occupied by sentences, and with the obvious modifications required by the presence of the new rules. Notice that the TREES so defined may have nodes with infinitely many children. In fact, any parent node in a TREE which is occupied by a sentence of type $\forall x \phi(x)$ or $\neg\exists x \phi(x)$ will have as many children as there are individual constants in the language, and these may be infinite in number. Hence, a TREE may have infinitely many branches. Yet, here as in the propositional case every branch of a TREE must be of finite length.

We then define, as before, a G-TREE as a nondegenerate TREE. The relations of STRICT GROUNDING and GROUNDING are also defined as in §1, but with the requirement that GROUNDING be a relation between sets of sentences and sentences (this condition is already satisfied by STRICT GROUNDING given the stipulation that the nodes of a TREE must be occupied by sentences). We finally define the *degree* of a sentence of $\mathcal{M}$ as we defined the degree of a formula of $\mathcal{L}$, but with the following condition for the quantifiers:

- degree($\forall x \phi(x)$) = degree($\exists x \phi(x)$) = 1 + degree($\phi(a)$) for $a$ an arbitrary constant.

(Which constant is chosen is actually immaterial.) Given these definitions, Propositions 1.1, 1.2 and 1.3 still hold.

Let us turn now to the semantics for $\mathcal{M}$. As in the case of language $\mathcal{L}$, we want to leave room for truth-value gaps and truth-value gluts. We accordingly define a *model* for $\mathcal{M}$ as a pair $\langle D, I \rangle$, where:

- $D$ is a nonempty set, and
- $I$ is a function which assigns:
  - To each constant of the language a member of $D$, and
  - To each $n$-place predicate $R$ distinct from $=$ an extension $I^+(R)$ and an anti-extension $I^-(R)$, both subsets of $D^n$.

A model is said to be *maximal* iff all $n$-place predicates $R$ are such that $I^+(R) \cup I^-(R) = D^n$, and *coherent* iff all predicates $R$ are such that $I^+(R) \cap I^-(R) = \varnothing$. The models which are both maximal and coherent correspond, of course, to the classical models for $\mathcal{M}$.

Exploiting a label already used in the previous section, say that a model $\langle D, I \rangle$ for $\mathcal{M}$ is *full* iff for every $o \in D$, there is a constant $a$ of $\mathcal{M}$ such that $I(a) = o$.

I will be especially interested in full models for languages such as $\mathcal{M}$, and so I first formulate the semantics for $\mathcal{M}$ relative to such models. Doing so allows one to avoid the use of assignments to variables (or the like) and to directly define truth and falsity for sentences relative to a model. Where $M = \langle D, I \rangle$ is a full model for $\mathcal{M}$, truth and falsity for sentences relative to $M$ are defined by the following clauses, plus the clauses for the truth-functional connectives corresponding to those used in §2:

- $M \models R(a_1, a_2, ...)$ iff $\langle I(a_1), I(a_2), ... \rangle \in I^+(R)$
- $M \dashv R(a_1, a_2, ...)$ iff $\langle I(a_1), I(a_2), ... \rangle \in I^-(R)$
- $M \models a = b$ iff $I(a) = I(b)$
- $M \dashv a = b$ iff $I(a) \neq I(b)$
- $M \models \forall x \phi(x)$ iff for all constants $a$, $M \models \phi(a)$
- $M \dashv \forall x \phi(x)$ iff for some constant $a$, $M \dashv \phi(a)$
- $M \models \exists x \phi(x)$ iff for some constant $a$, $M \models \phi(a)$
- $M \dashv \exists x \phi(x)$ iff for all constants $a$, $M \dashv \phi(a)$.

In case model $M$ is maximal, $M \not\dashv \phi$ entails $M \models \phi$; in case $M$ is coherent, $M \dashv \phi$ entails $M \not\models \phi$; if $M$ is both maximal and coherent, $M \dashv \phi$ is equivalent to $M \not\models \phi$, and the definition yields the standard characterisation of classical truth for sentences of $\mathcal{M}$.

Extending the terminology used in the propositional case, we adopt the following definitions:

DEFINITION 5.1. *A **situation** is a set of sentential literals of $\mathcal{M}$. The **situation determined by a model** $M$—in symbols: $\mathcal{S}(M)$—is the set of all the sentential literals of $\mathcal{M}$ which are true according to $M$.*

With these definitions in place, all the results of §2 about GROUNDING and truth can be shown to hold in the present context, *mutatis mutandis*. First, we have the following preservation result:

PROPOSITION 5.2. *Let $M$ be a full model for $\mathcal{M}$, $\Delta$ a set of sentences of $\mathcal{M}$ and $\phi$ a sentence of $\mathcal{M}$. If both $M \models \Delta$ and $\Delta \trianglerighteq \phi$, then $M \models \phi$.*

Then we have the following extensions of Lemmas 2.2, 2.3 and 2.5 for $\mathcal{M}$:

LEMMA 5.3. *Like Lemma 2.2, plus the following clauses:*

6. *If $\Delta \trianglerighteq \phi(a)$ for all constants $a$, then $\Delta \trianglerighteq \forall x \phi(x)$*
7. *If $\Delta \trianglerighteq \neg\phi(a)$ for some constant $a$, then $\Delta \trianglerighteq \neg\forall x \phi(x)$*
8. *If $\Delta \trianglerighteq \phi(a)$ for some constant $a$, then $\Delta \trianglerighteq \exists x \phi(x)$*
9. *If $\Delta \trianglerighteq \neg\phi(a)$ for all constants $a$, then $\Delta \trianglerighteq \neg\exists x \phi(x)$.*

LEMMA 5.4. *Like Lemma 2.3, plus the following clauses:*

6. *If $\Delta \triangleright \forall x \phi(x)$, then $\Delta \trianglerighteq \phi(a)$ for all constants $a$*
7. *If $\Delta \triangleright \neg\forall x \phi(x)$, then $\Delta \trianglerighteq \neg\phi(a)$ for some constant $a$*
8. *If $\Delta \triangleright \exists x \phi(x)$, then $\Delta \trianglerighteq \phi(a)$ for some constant $a$*
9. *If $\Delta \triangleright \neg\exists x \phi(x)$, then $\Delta \trianglerighteq \neg\phi(a)$ for all constants $a$.*

LEMMA 5.5. *Like Lemma 2.5, plus the following clauses:*

6. $\Delta \trianglerighteq \forall x\phi(x)$ *iff* $\Delta \trianglerighteq \phi(a)$ *for all constants a*
7. $\Delta \trianglerighteq \neg\forall x\phi(x)$ *iff* $\Delta \trianglerighteq \neg\phi(a)$ *for some constant a*
8. $\Delta \trianglerighteq \exists x\phi(x)$ *iff* $\Delta \trianglerighteq \phi(a)$ *for some constant a*
9. $\Delta \trianglerighteq \neg\exists x\phi(x)$ *iff* $\Delta \trianglerighteq \neg\phi(a)$ *for all constants a.*

And finally, the relevant version of Fundamental Connection holds:

THEOREM 5.6. *(Fundamental Connection) Let $M$ be a full model for $\mathcal{M}$ and $\phi$ a sentence of $\mathcal{M}$. Then: $M \models \phi$ iff $\mathcal{S}(M) \trianglerighteq \phi$.*

These results can be proved in much the same way as those of §2.

Contrary to Theorem 2.7, Theorem 5.6 cannot be used directly to provide characterisations of standard consequence relations in ground-theoretic terms, due to the fact that it concerns *full* models. Yet truth and falsity in an arbitrary (i.e., full or nonfull) model can be defined in terms of truth and falsity in a full model, and thanks to such a definition it is possible to characterise the first-order versions of the four consequence relations of §3 using Theorem 5.6. I will not go into details here, but I will nevertheless indicate how to define truth and falsity in an arbitrary model in terms of truth and falsity in a full model.[23]

Let $\mathcal{M}$ still be our target language. Let an $\mathcal{M}$-language be a language resulting from $\mathcal{M}$ by adding 0 or more constants. For every $\mathcal{M}$-language, truth and falsity relative to a full model are defined in the same way as above.

Let $M = \langle D, I\rangle$ be a model for $\mathcal{M}$. A pair $\langle \mathcal{M}^+, M^+\rangle$ is said to be a *full expansion* of the pair $\langle \mathcal{M}, M\rangle$ iff (i) $\mathcal{M}^+$ is an $\mathcal{M}$-language, (ii) $M^+$ is a model $\langle D, I^+\rangle$ for $\mathcal{M}^+$ which is full, and such that $I^+$ agrees with $I$ on the constants and predicates of $\mathcal{M}$. Thus, if $M$ is a full model for $\mathcal{M}$, $\langle \mathcal{M}, M\rangle$ is a full expansion of itself; and if $M$ is not a full model for $\mathcal{M}$, the full expansions are obtained by adding new constants to the language, at least until all the members of the domain have been named. Clearly, for every model $M$ for $\mathcal{M}$, $\langle \mathcal{M}, M\rangle$ has a full expansion (members of the domain can be used as names for themselves). Moreover, it can be shown that if $\langle \mathcal{M}^+, M^+\rangle$ and $\langle \mathcal{M}^{++}, M^{++}\rangle$ are any two full expansions of $\langle \mathcal{M}, M\rangle$, then for every sentence $\phi$ of $\mathcal{M}$, $M^+ \models \phi$ iff $M^{++} \models \phi$ and $M^+ \dashv \phi$ iff $M^{++} \dashv \phi$.

The suggestion is, then, to define truth and falsity for sentences of $\mathcal{M}$ relative to an arbitrary model by appealing to full expansions: a sentence $\phi$ of $\mathcal{M}$ is said to be true relative to model $M$ iff for some (equivalently: for every) full expansion $\langle \mathcal{M}^+, M^+\rangle$ of $\langle \mathcal{M}, M\rangle$, $M^+ \models \phi$—and similarly for falsity.

**§6. First-order languages: second set of rules.** Let us now focus on a language $\mathcal{N}$ which is just like $\mathcal{M}$ except that it contains the extra multigrade totality predicate $\mathcal{A}$. The basic rules for $\mathcal{N}$ are the same as those for $\mathcal{M}$ except for ($\forall$1) and ($\exists$1), which are replaced by sets of rules ($\forall 1*$) and ($\exists 1*$). We again take the basic rules to concern sentences, and the ground-theoretic notions, as well as the notion of the degree of a sentence, are defined as before.

[23] The method is used in Fine (1978).

The models for $\mathcal{N}$ are simply the models for $\mathcal{M}$. Truth and falsity relative to a full model $M = \langle D, I \rangle$ are defined as before, but with the following extra clauses for the totality predicate:

- $M \models \mathcal{A}(ab...)$ iff $D = \{I(a), I(b), ...\}$
- $M \dashv \mathcal{A}(ab...)$ iff $D \neq \{I(a), I(b), ...\}$.

Consider a sentence $\mathcal{A}(ab...)$ where $a, b, ...$ are *all* the constants of $\mathcal{N}$. Then by the truth-clause for $\mathcal{A}$, $\mathcal{A}(ab...)$ is true in any full model.

Truth and falsity for $\mathcal{N}$ have not been defined relative to any *arbitrary* model, but it is obvious how this should be done using assignments to variables or the like. Suppose, then, that this has been done. Then all the basic rules will preserve truth-in-a-model so defined, and so the version of Proposition 5.2 for language $\mathcal{N}$ without restriction to full models will hold. Yet the corresponding unrestricted version of Theorem 5.6 will nevertheless fail. For instance, let $M$ be a model for $\mathcal{N}$ which is not full, but which verifies $\forall x F(x)$ for some given monadic predicate $F$. Since $M$ is not full, no sentence of type $\mathcal{A}(ab...)$ is verified by $M$, and therefore no sentence of type $\mathcal{A}(ab...)$ belongs to $\mathcal{S}(M)$. As a consequence, $\mathcal{S}(M)$ cannot GROUND $\forall x F(x)$.

In contrast, the restricted version of Theorem 5.6 holds in the present context—and more generally, the results from the last section carry over to language $\mathcal{N}$ if we focus on full models, with only few differences.

First, all the basic rules for $\mathcal{N}$ preserve truth-in-a-full-model, and so Proposition 5.2 holds for $\mathcal{N}$. Second, Lemmas 5.3, 5.4 and 5.5 fail as they stand, but suitably modifying items 6 and 9 in each of them yields truths. In each case, the condition

$\Delta \trianglerighteq \phi(a)$ for all $a$

in item 6 should be replaced by:

for some $a, b, ...$, (i) $\mathcal{A}(ab...) \in \Delta$ and (ii) $\Delta \trianglerighteq \phi(a)$, $\Delta \trianglerighteq \phi(b)$, ...,

and the condition

$\Delta \trianglerighteq \neg\phi(a)$ for all $a$

in item 9 should be replaced by:

for some $a, b, ...$, (i) $\mathcal{A}(ab...) \in \Delta$ and (ii) $\Delta \trianglerighteq \neg\phi(a)$, $\Delta \trianglerighteq \neg\phi(b)$, ....

Finally, as announced, Theorem 5.6 holds in the present context:

THEOREM 6.1. *(Fundamental Connection) Let $M$ be a full model for $\mathcal{N}$ and $\phi$ a sentence of $\mathcal{N}$. Then: $M \models \phi$ iff $\mathcal{S}(M) \trianglerighteq \phi$.*

The proof for this latter result can be carried out as before, using the fact that for $M$ a full model for $\mathcal{N}$:

- $M \models \forall x\phi(x)$ iff for some constants $a, b, ...$, (i) $M \models \mathcal{A}(ab...)$ and (ii) $M \models \phi(a)$, $M \models \phi(b)$, ...;
- $M \dashv \exists x\phi(x)$ iff for some constants $a, b, ...$, (i) $M \models \mathcal{A}(ab...)$ and (ii) $M \dashv \phi(a)$, $M \dashv \phi(b)$, ....

(The assumption that $M$ is full is, of course, crucial here.)

Truth relative to an arbitrary model for $\mathcal{N}$ can, in the same way as before, be defined in terms of truth relative to a full model. Theorem 6.1 can then be used jointly with this definition to characterise consequence relations over the sentences of $\mathcal{N}$. I skip again the details.

**§7. Grounding and the liar.** Consider a first-order language $\mathcal{M}$ of the sort described in §5, and a full model $M = \langle D, I \rangle$ for that language, such that each sentence of the language belongs to the domain $D$. Since the model is full, each sentence of the language has a name within the language itself.

Could $\mathcal{M}$ possibly contain a monadic predicate $\mathsf{T}$ that behaves, relatively to model $M$, like a truth-predicate, that is, be such that:

(T) For every sentence $\phi$ of $\mathcal{M}$ and constant $a$ of $\mathcal{M}$ such that $I(a) = \phi$, $M \models \phi$ iff $M \models \mathsf{T}(a)$?

We know from the literature on the Liar Paradox that if we assume that $M$ is a classical model, then granted certain weak assumptions about $\mathcal{M}$ and $M$ the answer must be 'no'.

Assume $\mathcal{M}$ contains the two predicates $\mathsf{T}$ and $P$. Then $\mathcal{M}$ contains the sentence $\exists x\,(P(x) \wedge \neg\mathsf{T}(x))$, which we abbreviate as $\Lambda$, and a name $l$ for that sentence. Make the further semantical assumptions about $P$:

(i) $M \models P(l)$;
(ii) For every constant $b$, if $M \models P(b)$, then $I(b) = I(l)$.

Given these conditions, if $\mathsf{T}$ is interpreted as expressing truth, then the sentence $\Lambda$ effectively "says of itself" that it is not true. These assumptions about $\mathcal{M}$ and its model are easily satisfied: as Kripke (1975) nicely emphasised, whatever the interpretation of $\mathsf{T}$, the predicate $P$ in $\Lambda$ may be taken to express an ordinary, "empirical" property that is satisfied by $\Lambda$, and only by it, like for example the property of having a token written on the blackboard in FC's office at noon on October 23$^{\text{rd}}$, 2012. Be it as it may, the assumptions can be used to show that if $M$ is a classical model, then:

(P) $M \models \Lambda$ iff $M \not\models \mathsf{T}(l)$.

On these assumptions, thus, (T) leads to paradox, since (T) and (P) together entail: $M \models \mathsf{T}(l)$ iff $M \not\models \mathsf{T}(l)$.

That (P) holds granted that $M$ is classical and that the assumptions about predicate $P$ are satisfied can be shown as follows. (This is covered territory, but it will prove convenient to run through the proof in the way I do here.)

(u) Suppose $M \models \Lambda$, that is, $M \models \exists x(P(x) \wedge \neg\mathsf{T}(x))$. Then for some constant $b$, $M \models P(b) \wedge \neg\mathsf{T}(b)$. So, for some constant $b$, both $M \models P(b)$ and $M \models \neg\mathsf{T}(b)$. By assumption (ii) above, $M \models P(b)$ entails $I(b) = I(l)$. Therefore, $M \models \neg\mathsf{T}(l)$. HENCE, given that $M$ is classical, $M \not\models \mathsf{T}(l)$.

(v) Suppose now that $M \not\models \Lambda$, that is, $M \not\models \exists x(P(x) \wedge \neg\mathsf{T}(x))$. Then for all constants $b$, $M \not\models P(b) \wedge \neg\mathsf{T}(b)$. So, for all constants $b$, $M \not\models P(b)$ or $M \not\models \neg\mathsf{T}(b)$. So, $M \not\models P(l)$ or $M \not\models \neg\mathsf{T}(l)$. By assumption (i) above, $M \models P(l)$. It follows that $M \not\models \neg\mathsf{T}(l)$. HENCE, given that $M$ is classical, $M \models \mathsf{T}(l)$.

Notice that the only steps at which the assumption that model $M$ is classical is used are those indicated by 'HENCE'. The first 'HENCE' is justified if the model is coherent, and the second 'HENCE' if the model is maximal. Dropping any one of the two conditions on the model invalidates the reasoning.

Kripke (1975) famously showed how to consistently extend interpreted first-order languages without a predicate for truth into interpreted first-order languages that do contain such a predicate, in such a way that it semantically behaves like we expect it to do (a sentence $\phi$ of an extended language is true according to the associated model iff the sentence which says that $\phi$ is true is true in that same model). Kripke's models for the extended languages are, of course, not classical.

There is an interesting connection between the Kripkean constructions[24] and the theory of logical grounding developed in the previous sections, and it is the aim of the remaining part of this section to spell it out. I will first specify which languages and models we are interested in (§7.1.). I will then present the Kripkean constructions, in a somewhat more general form (§7.2.). Subsequently, I will spell out the connections between these constructions and the theory of logical grounding (§7.3.). A result which is close to the main result of that latter section, Theorem 7.11, has been stated in a somewhat different form by Fine (2010). In this and other respects, Fine's article is interestingly connected to the present paper, and I will accordingly devote the very last part of this section to a discussion of some of the relevant connections.[25]

***7.1. Set-up.*** We suppose given a first-order language $\mathcal{M}$ of the sort described in §5, and another first-order language $\mathcal{M}^{\mathsf{T}}$ which differs from $\mathcal{M}$ only by having an extra monadic predicate $\mathsf{T}$ (for truth).

We also suppose given a full model $M = \langle D, I \rangle$ for $\mathcal{M}$ and make the assumption that its domain $D$ contains all the sentences of the extended language $\mathcal{M}^{\mathsf{T}}$. We may assume that $M$ is classical, but this is not needed: $M$ may fail to be maximal, or to be coherent, or both.

Given that $M$ is a full model for $\mathcal{M}$, and given that its domain contains all the sentences of $\mathcal{M}^{\mathsf{T}}$, each of these sentences has a name in $\mathcal{M}$ (according to interpretation function $I$). We make the further inessential but simplifying assumption that each sentence $\phi$ of $\mathcal{M}^{\mathsf{T}}$ has only one name in $\mathcal{M}$, which we shall write $\ulcorner \phi \urcorner$. Importantly, such names are syntactically simple from the point of view of language $\mathcal{M}$.

To get a model for the extended language $\mathcal{M}^{\mathsf{T}}$ from model $M$, it suffices to extend interpretation function $I$ so that it assigns an extension and an anti-extension to predicate $\mathsf{T}$. However, it will be more convenient to adopt an alternative but equivalent characterisation of these models. Let $\mathfrak{D}$ be the set of all pairs $\langle E, A \rangle$ where both $E$ and $A$ are sets of sentences of $\mathcal{M}^{\mathsf{T}}$. A model for $\mathcal{M}^{\mathsf{T}}$ will be taken to be a pair $M[X] = \langle M, X \rangle$, where $X \in \mathfrak{D}$. If $X = \langle E, A \rangle$ is in $\mathfrak{D}$, then $E$ will play the role of the extension of $\mathsf{T}$ relative to $M[X]$, and $A$ the role of its anti-extension. This being said, truth and falsity for a sentence

[24] I have in mind those which occupy him for the most part of the paper, which assume what he and others call "Kleene's strong three-valued semantical scheme". He also discusses or mentions two other particular semantical schemes, that of supervaluations and that of Kleene's weak three-valued logic, but they will not concern us.

[25] There are various ways in which one can specify the languages and models to present the Kripkean constructions. For the sake of simplicity, and also to secure a straightforward continuity with the previous sections of this paper, I will make certain choices, some merely cosmetic, some others more substantial. My aim here is not to achieve maximal generality, but merely to illustrate how the theory of logical grounding can interestingly be connected to the Kripkean constructions. The material presented in §7.2 is rather familiar and fairly close to part of the material found in Kripke's paper, so I will not bother to mention connections with other relevant works. Nor will I attempt to connect the material in §7.3 with the rich post-Kripkean literature on "semantical groundedness" (see e.g., Yablo, 1982; Leitgeb, 2005). There certainly is much to say on this topic, but this is something I will have to leave for another occasion.

$\mathcal{M}^{\mathsf{T}}$ relative to model $M[X]$ is defined just like in §5, with the following special clauses for $\mathsf{T}$:

- $M[X] \models \mathsf{T}(a)$ iff $I(a) \in E$
- $M[X] \mathrel{=\!\!\mid} \mathsf{T}(a)$ iff $I(a) \in A$.

Notice that for any $X \in \mathfrak{D}$, a formula of the unextended language $\mathcal{M}$ is true (false) relative to $M$ iff it is true (false) relative to $M[X]$.

Let $M[X]$ be a model for $\mathcal{M}^{\mathsf{T}}$, with $X = \langle E, A \rangle$. $M[X]$ is said to be *down-adequate* iff the following two conditions are satisfied:

- For every sentence $\phi$ of $\mathcal{M}^{\mathsf{T}}$, if $\phi \in E$, then $M[X] \models \phi$
- For every sentence $\phi$ of $\mathcal{M}^{\mathsf{T}}$, if $\phi \in A$, then $M[X] \mathrel{=\!\!\mid} \phi$.

The model is said to be *up-adequate* iff the converse conditions hold, that is, iff:

- For every sentence $\phi$ of $\mathcal{M}^{\mathsf{T}}$, if $M[X] \models \phi$, then $\phi \in E$
- For every sentence $\phi$ of $\mathcal{M}^{\mathsf{T}}$, if $M[X] \mathrel{=\!\!\mid} \phi$, then $\phi \in A$.

And it is said to be *adequate* iff it is both down- and up-adequate. These definitions could be equivalently formulated by using '$M[X] \models \mathsf{T}(\ulcorner \phi \urcorner)$' instead of '$\phi \in E$' and '$M[X] \mathrel{=\!\!\mid} \mathsf{T}(\ulcorner \phi \urcorner)$' instead of '$\phi \in A$'. This explains why I chose the labels 'down-adequate' and 'up-adequate'.

Given the truth-clauses for $\mathsf{T}$, adequate models for $\mathcal{M}^{\mathsf{T}}$ are models relative to which $\mathsf{T}$ behaves like a truth-predicate. I now move on to Kripke-style constructions of such models.[26]

***7.2. Fixed points and adequate models.*** Define a binary relation $\leq$ on $\mathfrak{D}$, and a unary "jump" operation $J$ on $\mathfrak{D}$, as follows:

- For $X, Y \in \mathfrak{D}$, with $X = \langle E, A \rangle$ and $Y = \langle F, B \rangle$, $X \leq Y$ iff$_{\mathrm{df}}$ both $E \subseteq F$ and $A \subseteq B$.
- For $X \in \mathfrak{D}$, $J(X) =_{\mathrm{df}} \langle J_1(X), J_2(X) \rangle$, where:

$$J_1(X) =_{\mathrm{df}} \{\phi \in \mathcal{M}^{\mathsf{T}} : M[X] \models \phi\}$$
$$J_2(X) =_{\mathrm{df}} \{\phi \in \mathcal{M}^{\mathsf{T}} : M[X] \mathrel{=\!\!\mid} \phi\}.$$

$\leq$ is a partial ordering (a reflexive, antisymmetric and transitive relation) on $\mathfrak{D}$, and it can be shown that $J$ is monotonic relative to $\leq$:

LEMMA 7.1. *For all $X, Y \in \mathfrak{D}$, if $X \leq Y$ then $J(X) \leq J(Y)$.*

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[26] There are some differences between Kripke's constructions and the ones to be presented below which are worth highlighting. (1) Whereas I take the sentences of the extended language *themselves* to be members of the domain of the original model, Kripke takes these sentences to be represented ("coded") by numbers, themselves taken to be members of the domain of that model. (2) I require that, in a model for $\mathcal{M}^{\mathsf{T}}$, the anti-extension of the truth-predicate contain only sentences, whereas for Kripke the anti-extension of that predicate may contain both (codes of) sentences and objects of the domain which are not (codes of) sentences. (3) Kripke starts with a completely classical model for the initial language, and the models in his constructions allow for truth-value gaps but not for truth-value gluts. He nevertheless grants that the assumption that the initial model is classical is not necessary, so that gaps may be allowed. In contrast, I also allow both the initial model and the models for $\mathcal{M}^{\mathsf{T}}$ to be incoherent.

Let $X$ be in $\mathfrak{D}$. We adopt the following usual definitions:

- $X$ is *sound* iff$_{\text{df}}$ $X \leq J(X)$
- $X$ is *complete* iff$_{\text{df}}$ $J(X) \leq X$
- $X$ is *a fixed point* iff$_{\text{df}}$ $X$ is both sound and complete, that is, iff $J(X) = X$.

Clearly, then:

LEMMA 7.2. *For every $X \in \mathfrak{D}$:*

- *$X$ is sound iff $M[X]$ is down-adequate*
- *$X$ is complete iff $M[X]$ is up-adequate*
- *$X$ is a fixed point iff $M[X]$ is adequate.*

That there exists fixed points associated with any arbitrary sound member of $\mathfrak{D}$ can be established as follows. Let $X = \langle E, A\rangle \in \mathfrak{D}$ be sound, that is, such that $X \leq J(X)$. We define $X_\alpha = \langle E_\alpha, A_\alpha\rangle$ for all ordinals $\alpha$ as follows:

- $X_0 = X$; that is to say, $\langle E_0, A_0\rangle = \langle E, A\rangle$
- $X_{\alpha+1} = J(X_\alpha)$; that is to say, $\langle E_{\alpha+1}, A_{\alpha+1}\rangle = \langle J_1(X_\alpha), J_2(X_\alpha)\rangle$
- $X_\lambda = \langle \bigcup\{E_\alpha : \alpha < \lambda\}, \bigcup\{A_\alpha : \alpha < \lambda\}\rangle$ for $\lambda$ a limit ordinal.

Since $X \leq J(X)$, $\leq$ is a partial ordering and $J$ is monotonic relative to $\leq$, it can be shown that the series of $X_\alpha$s is constantly increasing:

LEMMA 7.3. *For all ordinals $\alpha$, $\beta$ such that $\alpha \leq \beta$, $X_\alpha \leq X_\beta$.*

The series of the $X_\alpha$s cannot be constantly *strictly* increasing, since otherwise all its members would be distinct, and hence as many as there are ordinals, which is not the case since they are all elements of the *set* $\mathfrak{D}$. So, there must be some member $X_\epsilon$ of the sequence such that $X_\epsilon = J(X_\epsilon)$, that is, which is a fixed point. Since $X$ is the first member of the sequence, such a fixed point extends $X$ (in the sense of $\leq$). Notice that once a fixed point $X_\epsilon$ is reached in the series of $X_\alpha$s, the series remains constant thereafter, that is, each subsequent element of the series is identical with $X_\epsilon$. We call this unique fixed point in the series $\widehat{X}$. It is easy to show that $\widehat{X}$ is the smallest fixed point extending $X$, that is, that for every fixed point $Y$ such that $X \leq Y$, $\widehat{X} \leq Y$.

Summing up, we can state:

THEOREM 7.4. *Let $X \in \mathfrak{D}$ be sound. Then there is a member $Y$ of $\mathfrak{D}$ with $X \leq Y$ which is a fixed point.*

*More specifically, there is a (unique) member $\widehat{X}$ of $\mathfrak{D}$ such that: (i) $\widehat{X}$ is a fixed point, (ii) $X \leq \widehat{X}$, and (iii) for every $Y \in \mathfrak{D}$ with $X \leq Y$ which is a fixed point, $\widehat{X} \leq Y$.*

Thanks to Lemma 7.2, this result can be equivalently formulated thus:

COROLLARY 7.5. *Let $X \in \mathfrak{D}$ such that $M[X]$ is down-adequate. Then there is a member $Y$ of $\mathfrak{D}$ with $X \leq Y$ such that $M[Y]$ is adequate.*

*More specifically, there is a (unique) member $\widehat{X}$ of $\mathfrak{D}$ such that: (i) $M[\widehat{X}]$ is adequate, (ii) $X \leq \widehat{X}$, and (iii) for every $Y \in \mathfrak{D}$ with $X \leq Y$ such that $M[Y]$ is adequate, $\widehat{X} \leq Y$.*

A particularly interesting sound member of $\mathfrak{D}$ is $\odot = \langle\varnothing, \varnothing\rangle$. The corresponding model $M[\odot]$ verifies no literals of $\mathcal{M}^{\mathsf{T}}$ containing the truth-predicate $\mathsf{T}$, and so makes true only the literals of $\mathcal{M}$ which are made true by $M$. $\widehat{\odot}$ is the smallest fixed point (*simpliciter*), and by the previous result, $M[\widehat{\odot}]$ is an adequate model.

Suppose $\mathcal{M}^{\mathsf{T}}$ contains monadic predicate $P$ and that $\exists x(P(x) \wedge \neg \mathsf{T}(x))$, which we again abbreviate as $\Lambda$, is a Liar sentence. We may take this to mean, in the present context, that $I^{+}(P)$ is $\{\Lambda\}$ and $I^{-}(P)$ is the whole domain minus $\Lambda$. Then for all $X \in \mathfrak{D}$:

- $M[X] \models \Lambda$ iff $M[X] \dashv T(\ulcorner \Lambda \urcorner)$
- $M[X] \dashv \Lambda$ iff $M[X] \models T(\ulcorner \Lambda \urcorner)$.

For $M[X]$ adequate, we thus have:

- $M[X] \models \Lambda$ iff $M[X] \dashv \Lambda$.

We then know that $M[\widehat{\odot}]$ cannot be classical, that is, both maximal and coherent. If $M$ is coherent (be it maximal or not), $M[\widehat{\odot}]$ will be coherent but not maximal: $\Lambda$ will be neither true nor false in $M[\widehat{\odot}]$. If on the other hand $M$ is incoherent (be it maximal or not), $M[\widehat{\odot}]$ will be incoherent but again not maximal: $\Lambda$ will still be neither true nor false in $M[\widehat{\odot}]$.[27]

As I previously stressed, Kripke (1975) starts his constructions by assuming that the model for the unextended language is classical; he notes that the assumption that the model be maximal can be dropped, but he does not envisage the case where the model is not coherent. The starting point of his first construction is a model for the extended language in which the truth-predicate has both an empty extension and an empty anti-extension, and the result of the construction is the model based on the smallest fixed point—that is, using our terminology, the starting point is the model $M[\odot]$ and the result the model $M[\widehat{\odot}]$. He defines a sentence of the extended language to be *grounded* iff the sentence has a truth-value in the model based on the smallest fixed point. As we shall see in the next section, Kripke's semantic notion of groundedness is connected in a definite and interesting sense to our syntactic notion of GROUNDEDNESS.

***7.3. Connection with logical grounding.*** Following a previously given definition, the situation determined by model $M$, $\mathcal{S}(M)$, is defined as the set of all the sentential literals of language $\mathcal{M}$ which are true according to $M$. Similarly, where $X \in \mathfrak{D}$, the situation determined by model $M[X]$, $\mathcal{S}(M[X])$, is defined as the set of all the sentential literals of $\mathcal{M}^{\mathsf{T}}$ which are true according to $M[X]$. In order to simplify notation I shall abbreviate $\mathcal{S}(M[X])$ to $\mathcal{S}(X)$. Notice that:

- $\mathcal{S}(M) = \mathcal{S}(\odot)$;
- For all $X \in \mathfrak{D}$, $\mathcal{S}(M) \subseteq \mathcal{S}(X)$;
- For all $X, Y \in \mathfrak{D}$, if $X \leq Y$, then $\mathcal{S}(X) \subseteq \mathcal{S}(Y)$.

Consider then $X \in \mathfrak{D}$ sound and the series of the $X_{\alpha}$s as defined in the previous section. This series starts with $X$, is constantly increasing, eventually reaches a fixed point, $\widehat{X}$, and remains constant thereafter. By the last point above, the series of the $\mathcal{S}(X_{\alpha})$s starts with $\mathcal{S}(X)$, is also constantly increasing, eventually reaches $\mathcal{S}(\widehat{X})$, and remains constant thereafter.

The Fundamental Connection result of §5 (Theorem 5.6) yields the following fact:

LEMMA 7.6. *Let $X \in \mathfrak{D}$. Then for all sentences $\phi$ of $\mathcal{M}^{\mathsf{T}}$: $M[X] \models \phi$ iff $\mathcal{S}(X) \trianglerighteq \phi$.*

[27] Interestingly, the pair $Z = \langle\{\Lambda\}, \{\Lambda\}\rangle$ is sound, and the adequate model based on the corresponding fixed point $\widehat{Z}$ makes $\Lambda$ both true and false.

This is one connection between the semantic language $\mathcal{M}^{\mathsf{T}}$ and logical grounding, but it is of limited interest since there is nothing special in this connection as opposed to the connection between logical grounding and nonsemantic languages.

Things become much more interesting if we modify our definition of GROUNDING by extending the basic rules from which the notion is formally defined. There is a plausible principle connecting truth and the notion of grounding, sometimes associated with the name of Aristotle (see the introduction to this paper), according to which every truth grounds the fact that it is true. There are several ways the principle might be formally expressed, and in the present context it is natural to render the principle as saying that if a statement is true, then it grounds the statement which says that it is true. A dual principle involving negation is equally plausible, namely the principle that if a statement is not true, then the negation of the statement grounds the statement which says that it is not true. These motivates the view that the following two rules for sentences of $\mathcal{M}^{\mathsf{T}}$ are ground-theoretically admissible:[28]

$$(\mathsf{T}1)\quad \frac{\phi}{\mathsf{T}(\ulcorner\phi\urcorner)} \qquad\qquad (\mathsf{T}2)\quad \frac{\neg\phi}{\neg\mathsf{T}(\ulcorner\phi\urcorner)}$$

Let us then define a new concept of strict grounding, just like STRICT GROUNDING has been defined in §5 but adding these two rules to the set of basic rules, and let us define a corresponding new notion of grounding in terms of that notion of strict grounding, just like GROUNDING has been defined in terms of STRICT GROUNDING. We may use 'STRICT GROUNDING$^{\mathsf{T}}$' and 'GROUNDING$^{\mathsf{T}}$' for these new notions, and $\rhd^{\mathsf{T}}$ and $\unrhd^{\mathsf{T}}$ for the corresponding abbreviations.

Importantly, contrary to the previous rules $\mathsf{T}1$ and $\mathsf{T}2$ do not systematically license transitions from the less complex to the more complex. In fact, by our definitions $\mathsf{T}(\ulcorner\phi\urcorner)$ always has degree of complexity 0 and $\neg\mathsf{T}(\ulcorner\phi\urcorner)$ degree of complexity 1, whereas $\phi$ can have any degree of complexity and $\neg\phi$ any degree of complexity greater than 0. Proposition 1.1 thus fails for $\rhd^{\mathsf{T}}$. Another effect of having rules $\mathsf{T}1$ and $\mathsf{T}2$ is that Proposition 1.2(2) and Proposition 1.3(2) fail in the present context, since for every sentence $\phi$, both $\mathsf{T}(\ulcorner\phi\urcorner)$ and $\neg\mathsf{T}(\ulcorner\phi\urcorner)$ are literals.[29]

Relatedly, STRICT GROUNDING$^{\mathsf{T}}$ is not irreflexive (and consequently, in the present context some TREES have some branches of infinite length). The following TREE shows that $\exists x\mathsf{T}(x)$ STRICTLY GROUNDS$^{\mathsf{T}}$ itself, one transition being given by $\exists 2$ and the other one by $\mathsf{T}1$:

$$\begin{array}{c} \exists x\mathsf{T}(x) \\ | \\ \mathsf{T}(\ulcorner\exists x\mathsf{T}(x)\urcorner) \\ | \\ \exists x\mathsf{T}(x) \end{array}$$

Some might take this as problematic: logical grounding is intuitively irreflexive, so the thought goes, therefore the view that STRICT GROUNDING$^{\mathsf{T}}$ corresponds to a genuine

[28] It might be questioned whether these rules capture links of ground which are properly *logical* in character, rather than just conceptual or even just metaphysical. I will not pursue this issue here.

[29] The assumption that the names for the sentences of the extended language are syntactically simple is clearly at work here. Suppose that, in contrast, $\ulcorner\phi\urcorner$ is understood as literally containing $\phi$. Then $\mathsf{T}(\ulcorner\phi\urcorner)$ and $\neg\mathsf{T}(\ulcorner\phi\urcorner)$ can in a very good sense be said to be more complex than $\phi$ and $\neg\phi$, respectively. Yet, given the corresponding alternative sense of 'complex', the rules for the quantifiers will fail to always take one from the less to the more complex.

relation of logical grounding over the sentences of language $\mathcal{M}^{\mathsf{T}}$ is wrong. But I do not see why this should be the appropriate reaction. The view that grounding (logical, or *simpliciter*) is irreflexive may be *prima facie* plausible, and I am prepared to grant that the putative examples of self-grounding we typically think of when wondering whether grounding is irreflexive turn out to be intuitively not cases of grounding at all. Yet we have two rules, $\exists 2$ and $\mathsf{T}1$, which are very plausibly ground-theoretically acceptable. These two rules, in conjunction with the equally plausible view that grounding is transitive, straightforwardly yield cases of self-grounding. Why give up one of the rules or the transitivity of grounding, rather than its irreflexivity? I personally take the proposed example involving $\exists 2$ and $\mathsf{T}1$ to provide a nice and convincing counterexample to irreflexivity.

Let us go back to the connections. We first have the following preservation result:

PROPOSITION 7.7. *Let $X \in \mathfrak{D}$. If $X$ is complete (in particular, if $X$ is a fixed point), then for all sets of sentences $\Delta$ of $\mathcal{M}^{\mathsf{T}}$ and all sentences $\phi$ of $\mathcal{M}^{\mathsf{T}}$: if both $M[X] \models \Delta$ and $\Delta \trianglerighteq^{\mathsf{T}} \phi$, then $M[X] \models \phi$.*

*Proof.* Suppose $X$ is a complete member of $\mathfrak{D}$. Then by Lemma 7.2, $M[X]$ is up-adequate, and so:

- For all sentences $\phi$ of $\mathcal{M}^{\mathsf{T}}$: $M[X] \models \phi \Rightarrow M[X] \models \mathsf{T}(\ulcorner \phi \urcorner)$, and
- For all sentences $\phi$ of $\mathcal{M}^{\mathsf{T}}$: $M[X] \models \neg\phi \Rightarrow M[X] \models \neg\mathsf{T}(\ulcorner \phi \urcorner)$.

It follows that the rules $\mathsf{T}1$ and $\mathsf{T}2$ preserve truth-relative-to-$M[X]$. Therefore, the same goes for GROUNDING$^{\mathsf{T}}$. □

It allows one to establish the following connection between language $\mathcal{M}^{\mathsf{T}}$ and GROUNDING$^{\mathsf{T}}$:

LEMMA 7.8. *Let $X \in \mathfrak{D}$. If $X$ is complete (in particular, if $X$ is a fixed point), then for all sentences $\phi$ of $\mathcal{M}^{\mathsf{T}}$: $M[X] \models \phi$ iff $\mathcal{S}(X) \trianglerighteq^{\mathsf{T}} \phi$.*

*Proof.* Take $X \in \mathfrak{D}$ complete, and let $\phi$ be a sentence of $\mathcal{M}^{\mathsf{T}}$. The members of $\mathcal{S}(X)$ are all true according to $M[X]$. So if $\mathcal{S}(X) \trianglerighteq^{\mathsf{T}} \phi$, then by the previous proposition, $M[X] \models \phi$. For the other direction, suppose that $M[X] \models \phi$. By Lemma 7.6, then, $\mathcal{S}(X) \trianglerighteq \phi$. But since $\trianglerighteq \subseteq \trianglerighteq^{\mathsf{T}}$ over the sentences of $\mathcal{M}^{\mathsf{T}}$, we get $\mathcal{S}(X) \trianglerighteq^{\mathsf{T}} \phi$. □

Now we have the following important fact about the interaction between GROUNDING$^{\mathsf{T}}$ and the jump operator $J$ defined previously:

LEMMA 7.9. *For all $X \in \mathfrak{D}$, $\mathcal{S}(X) \trianglerighteq^{\mathsf{T}} \mathcal{S}(J(X))$.*

*Proof.* Let $X \in \mathfrak{D}$. By Lemma 7.6, $\mathcal{S}(X) \trianglerighteq J_1(X)$. Given that $\trianglerighteq \subseteq \trianglerighteq^{\mathsf{T}}$, it follows that $\mathcal{S}(X) \trianglerighteq^{\mathsf{T}} J_1(X)$. The lemma will follow if we establish that $J_1(X) \trianglerighteq^{\mathsf{T}} \mathcal{S}(J(X))$.

To this effect, let $\phi \in \mathcal{S}(J(X))$. Then $\phi$ is a sentential literal true according to $M[J(X)]$. If $\phi$ does not contain $\mathsf{T}$, then it is true according to any model of type $M[-]$, and so it is true according to $M[X]$, that is, $\phi \in J_1(X)$. Then trivially, $J_1(X) \trianglerighteq^{\mathsf{T}} \phi$. Suppose now that $\phi$ contains $\mathsf{T}$. Then $\phi$ is either (a) $\mathsf{T}(\ulcorner \psi \urcorner)$ or (b) $\neg\mathsf{T}(\ulcorner \psi \urcorner)$ for some sentence $\psi$ of $\mathcal{M}^{\mathsf{T}}$.

(a) Suppose $\phi$ is $\mathsf{T}(\ulcorner \psi \urcorner)$. That $\phi$ is true relative to $M[J(X)]$ then means that $\psi \in J_1(X)$. Now $\psi \trianglerighteq^{\mathsf{T}} T(\ulcorner \psi \urcorner)$. Consequently, $J_1(X) \trianglerighteq^{\mathsf{T}} \phi$.

(b) Suppose $\phi$ is $\neg\mathsf{T}(\ulcorner \psi \urcorner)$. That $\phi$ is true relative to $M[J(X)]$ then means that $\psi \in J_2(X)$, and this implies that $\neg\psi \in J_1(X)$. Now $\neg\psi \trianglerighteq^{\mathsf{T}} \neg T(\ulcorner \psi \urcorner)$. Consequently, $J_1(X) \trianglerighteq^{\mathsf{T}} \phi$. □

Consider now a sound member $X$ of $\mathfrak{D}$, and consider again the series of the $X_\alpha$s. As I previously emphasised, the corresponding series of the $\mathcal{S}(X_\alpha)$s is constantly increasing in the sense of set-theoretic inclusion. Using the previous lemma and transfinite induction, it can be shown that it is also constantly increasing in the sense of GROUNDING$^{\mathsf{T}}$:

LEMMA 7.10. *For all $\alpha$, $\beta$, if $\alpha \leq \beta$, then $\mathcal{S}(X_\alpha) \trianglerighteq^{T} \mathcal{S}(X_\beta)$.*

We are now in a position to establish the main result of this section:

THEOREM 7.11. *Let $X$ be a sound member of $\mathfrak{D}$, and $\widehat{X}$ the smallest fixed point extending $X$. For all sentences $\phi$ of $\mathcal{M}^{\mathsf{T}}$: $M[\widehat{X}] \models \phi$ iff $\mathcal{S}(X) \trianglerighteq^{T} \phi$.*

*Proof.* Let $X$, $\widehat{X}$ and $\phi$ be as stated. Suppose $M[\widehat{X}] \models \phi$. Then by Lemma 7.8, $\mathcal{S}(\widehat{X}) \trianglerighteq^{\mathsf{T}} \phi$. By the previous lemma, we know that $\mathcal{S}(X) \trianglerighteq^{\mathsf{T}} \mathcal{S}(\widehat{X})$. So, $\mathcal{S}(X) \trianglerighteq^{\mathsf{T}} \phi$. Conversely, suppose $\mathcal{S}(X) \trianglerighteq^{\mathsf{T}} \phi$. Since $X \leq \widehat{X}$, $\mathcal{S}(X) \subseteq \mathcal{S}(\widehat{X})$, and so $\mathcal{S}(\widehat{X}) \trianglerighteq^{\mathsf{T}} \phi$. But then by Lemma 7.8, $M[\widehat{X}] \models \phi$. □

Consider the special case where $X$ is $\odot = \langle \varnothing, \varnothing \rangle$. By the theorem, for a sentence $\phi$ of the extended language $\mathcal{M}^{\mathsf{T}}$ to be true (false) in the model $M[\widehat{\odot}]$ based on the smallest fixed point is for it (its negation) to be GROUNDED$^{\mathsf{T}}$ in situation $\mathcal{S}(M)$ (which comprises no semantic sentences, that is, no sentence containing the truth-predicate). Thus, a sentence of $\mathcal{M}^{\mathsf{T}}$ is grounded in Kripke's sense iff either it or its negation is GROUNDED$^{\mathsf{T}}$ in $\mathcal{S}(M)$.

***7.4. A note on Fine's "Some Puzzles of Ground".*** Fine's (2010) article "Some Puzzles of Ground" (2010) is interestingly connected to the present paper. Given the particular line of thought followed by Fine in the article, I found it more appropriate to discuss it in a separate section rather than making piecemeal remarks in the previous parts. My aim here is not to run through all connections in details, but rather to make some comments on some important points.

The article is concerned with certain puzzles concerning the concept of *partial grounding*. The qualification 'partial' is understood in a relaxed sense: what fully grounds, as well as what merely helps to ground, is taken to be a partial ground. The concept of grounding of interest to Fine is not specifically *logical* in character, although many of the connections of grounding Fine takes into consideration may appropriately be understood to be connections of logical grounding.

Fine's puzzles arise when it is assumed that partial grounding is irreflexive and that a number of other plausible assumptions—regarding partial grounding, logical validity and other notions like truth and existence—are in place. The puzzles arise because these assumptions can be shown to lead to inconsistency. Fine discusses ways of avoiding inconsistency and eventually retains some options which all stick to the view that partial grounding is irreflexive. Barring a radical move he calls 'predicativism', the retained options either reject both a certain set GP of ground-theoretic principles and a certain set LP of classically valid logical principles (compromise position), or reject one set of principles while keeping the other (extremist positions).[30]

[30] See Fine (2010, pp. 109–110). GP comprises the rules of inference ($E$ is the existence predicate and $\prec$ the partial grounding operator):

- $\forall x \phi(x),\ E(y) \,/\, \phi(y) \prec \forall x \phi(x)$
- $\phi(y),\ E(y) \,/\, \phi(y) \prec \exists x \phi(x)$

Each position, Fine argues, "corresponds" to a Kripkean approach to the Liar paradox (see footnote 24): the compromise position corresponds to the "strong Kleene" approach, one of the extremist position to the "weak Kleene" approach, and the other extremist position to the supervaluational approach. Since I have so far focused on the first Kripkean approach, I will simply ignore the other two.

Fine proposes a proof-theoretic approach to Kripke's (strong Kleene) fixed point constructions which turns out to be very close to the one I have put forward in the previous section. Putting things in terms we previously used, he assumes that the model $M$ for the unextended language $\mathcal{M}$ is classical, and he ends up, in effect, stating that a sentence of the extended language $\mathcal{M}^{\mathsf{T}}$ is true in the model based on the smallest fixed point, $M[\widehat{\odot}]$, iff it is GROUNDED$^{\mathsf{T}}$ in the set of all the sentences of $\mathcal{M}$ which are true according to $M$.[31] To be completely accurate, Fine does not invoke the relation of GROUNDING$^{\mathsf{T}}$ but rather directly the basic rules from which that relation is defined.[32]

Importantly, unlike me, Fine does not take mere chaining of these basic rules to always give rise to links of partial ground. For, as we saw, chaining the rules $\exists 2$ and $\mathsf{T}1$ yields a connection between $\exists x \mathsf{T}(x)$ and itself, and Fine maintains that partial grounding is irreflexive (he actually uses our very example when discussing the point). He rather proposes an alternative characterisation of partial grounding in terms of these very same rules, according to which partial grounding turns out to be irreflexive.

Now the correspondence which Fine sees between the compromise position on the puzzles of ground and the Kripkean (strong Kleene) approach to the Liar boils down to this: the set GP of ground-theoretic principles and the set LP of classical logical principles which are both rejected by the compromise position are also rejected on the Kripkean approach (granted the alternative characterisation of partial grounding).

My rejection of the irreflexivity of logical grounding (see §7.3) brings with it a rejection of the irreflexivity of the broader notion of grounding that Fine has in mind, as well as a rejection of the irreflexivity of the corresponding partial notions. It allows me to take mere chaining of the basic rules to systematically give rise to links of partial ground, and indeed it allows me to view relation $\rhd^{\mathsf{T}}$ as corresponding to genuine connections of full

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- $\phi \,/\, \phi \prec \phi \vee \psi$
- $\psi \,/\, \psi \prec \phi \vee \psi$,

and LP the theses ($E$ is, again, the existence predicate):

- $\forall x(\phi(x) \vee \neg\phi(x))$
- $\exists x(\phi(x) \vee \neg\phi(x))$
- $\forall x E(x)$
- $\exists x E(x)$.

When I talk of rejection of either of these two sets, I mean rejection of at least some of its members, not necessarily rejection of all of them.

31 The instance of Theorem 7.11 for $X = \odot$ and $M$ a classical model is a slightly "stronger" result, in so far as it mentions the set of all the *sentential literals*, rather than the set of all the sentences, of $\mathcal{M}$ which are true according to $M$.

32 Davis (1979) already established a similar result. (See also Hazen, 1981.) The rules in action in Davis' constructions are somewhat different, though: instead of stating that one can move from one or more formulas to a formula, they state that one can move from the truth or falsity of one or more formulas to the truth or falsity of a formula. On Davis' approach, the truth of $\neg\phi$ is to be distinguished from the falsity of $\phi$. In contrast, it is in the spirit of the approach I followed to identify the two.

grounding. On this account, the Kripkean approach to the Liar will *not* correspond to a compromise position in Fine's sense: the ground-theoretic principles which cannot jointly be accepted on a compromise position, namely those constituting the set GP, will all hold.

Finally, let me stress that rejecting the irreflexivity of partial grounding will have a direct impact on Fine's puzzles of ground, since the arguments proposed by Fine which lead to inconsistency all explicitly invoke irreflexivity. What exactly remains of the puzzles once irreflexivity is abandoned is an interesting question, but one I will not pursue here.[33]

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[33] Versions of parts of this paper have been presented on a number of occasions: Logic Tea, ILLC, Amsterdam (March 2012), Colloque de recherche, Neuchâtel (October 2012), Institutskolloquium, Philosophisches Seminar, Hamburg (October 2012), Colloquium in Mathematical Philosophy, Munich Center for Mathematical Philosophy, LMU (November 2012), Departmental Talks, Philosophisches Seminar, Tübingen (November 2012) and *eidos* Seminar, Geneva (December 2012). I am grateful to the audiences of these events for many helpful comments and discussions and also to a referee of this journal for helpful suggestions. This work was carried out while I benefited from a part-time Swiss National Science Foundation professorship (project PP00P1-135262), and was in charge of the Swiss National Science Foundation sinergia project 'Intentionality as the Mark of the Mental' (project CRSI11-127488). The research leading to these results has also received funding from the European Community's Seventh Framework Programme FP7/2007-2013 under grant agreement no. FP7-238128.

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