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Etude de lois d'échelle et d'effets de frustration dans des réseaux supraconducteurs fractals

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IMPRIMATUR POUR LA THÈSE

Etude de lois d'échelle et effets de frustration dans des
réseaux supraconducteurs fractals

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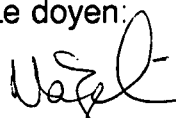
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Etude de lois d'échelle et d'effets de frustration dans des réseaux supraconducteurs fractals.

Il est bien établi que le désordre a une influence néfaste sur la supraconductivité, pouvant même mener à sa suppression. Ainsi les systèmes fractals dont nous avons étudié les propriétés de conduction permettent de nous familiariser avec des propriétés caractéristiques des supraconducteurs granulaires ou désordonnés. Dans notre approche, les aspects aléatoires sont évités grâce à une structure artificielle faite par lithographie, permettant l'étude expérimentale d'un système contenant des éléments d'inhomogénéité bien définis ainsi qu'une approche théorique analytique.

Les réseaux supraconducteurs de jonctions Josephson constituent une réalisation pratique du modèle XY, décrivant la transition de nombreux systèmes physiques (supraconductivité, fusion bidimensionnelle,...). Le système bidimensionnel (2D) représente un cas limite où une transition du type "ordre-désordre" peut se manifester, un état ordonné étant exclu à plus faible dimension. Les propriétés des vortex responsables de la transition changent avec la dimension du système et leur énergie ne diverge pas dans un milieu de dimension fractale ce qui modifie le comportement critique. Les propriétés dynamiques des vortex sont également différentes et nos mesures ont mis ces changements en évidence.

Du point de vue théorique, notre démarche a consisté à prévoir la pénétration du champ magnétique (réseau *frustré*) dans une structure fractale particulière appelée *Tamis de Sierpinski*. Sur la base de ces résultats, nous avons calculé la densité de superfluide correspondante. Grâce à une renormalisation exacte, l'approche analytique nous a permis des prédictions précises. Comme en pratique nos réseaux étaient de taille finie et présentaient une structure fractale périodiquement répétée, nous avons incorporé des effets de taille et proposé un scénario décrivant le changement de régime fractal-2D ("crossover" dimensionnel). Dans le cas *non-frustré*, la répétition périodique de fractales a pour effet de déplacer la transition BKT du réseau. En augmentant l'ordre de la fractale (en d'autres termes lorsqu'on s'approche du seuil de percolation), la température de transition est progressivement réduite. Le calcul de renormalisation nous a permis de prédire l'abaissement de la température de transition pour nos réseaux.

Nous avons expérimentalement vérifié ces prédictions en exposant les réseaux à une faible excitation électromagnétique (technique inductive). Nos observations ont nécessité des mesures d'une très grande précision (nous avons mesuré des impédances de l'ordre de $10^{-8} \Omega$ avec un courant par jonction de l'ordre de 10^{-7} A).

Dans le cas *non-frustré*, nous avons observé le déplacement prévu de la transition. Pour le système *frustré*, la comparaison entre la théorie et les mesures a également montré un accord excellent. En particulier, exploitant une vaste gamme de fréquences de mesure, nous avons expérimentalement vérifié un modèle brownien de la dynamique des vortex magnétiques dans un milieu inhomogène. Ce résultat original met en évidence les propriétés dynamiques inhabituelles des vortex dans la structure fractale.

En conclusion, ce travail a permis de nous familiariser avec des propriétés tout à fait particulières que l'on trouve de manière inhérente dans les systèmes désordonnés. L'avantage d'un système modèle permettant des comparaisons entre mesures et prédictions théoriques a pu être pleinement exploité et l'accord montre que les mécanismes sont bien compris.

Publications résumant la thèse

1. R. Meyer, J.L. Gavilano, B. Jeanneret, R. Théron, C. Leemann, H. Beck, and P. Martinoli.
Vortex dynamics in superconducting fractal networks.
Phys. Rev. Lett. **67**, 3022 (1991).
2. R. Meyer, P. Martinoli, J.L. Gavilano, and B. Jeanneret.
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Thermal vortices in superconducting fractals.
Physica A **191**, 458 (1992).
4. R. Meyer, B. Jeanneret, S.E. Koshunov, and P. Martinoli.
Observation of dilational symmetry breaking in a superconducting array of Sierpinski gaskets.
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Magnetoconductance of a superconducting Sierpinski gasket.
Phys. Rev. B **51**, 5914 (1995).

Vortex Dynamics in Superconducting Fractal Networks

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Superfluid and vortex dynamics in a fractal system were investigated by measuring the complex sheet impedance of a wire network of interconnected Sierpinski gaskets exposed to a weak magnetic field. By probing the network response over a range of length scales covering four stages of hierarchy in the gaskets, we find strong evidence for unusual scaling of the vortex energy U_h with the size r_h of the loops, $U_h \propto r_h^{-\zeta}$ with $\zeta = \ln \frac{5}{3} / \ln 2$, predicted by theory.

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Disorder in superconductors can assume a variety of forms and has been modeled in a number of ways. An elegant geometrical approach revealing some of the basic aspects of superconductivity in percolating materials is due to Alexander and Halevi (AH) [1]. Near the percolation threshold, such systems exhibit a natural self-similar structure with geometrical inhomogeneities occurring over a broad range of length scales. They can therefore be described by a family of scale-invariant fractal lattices, such as the Sierpinski gasket (SG) originally proposed by Gefen *et al.* [2] to mimic the topological features of the percolating cluster's backbone.

So far, studies of SG networks have focused on their superconducting-to-normal phase boundary $T_c(H)$ [3,4], which was found to agree with calculations [4,5] based on the Ginzburg-Landau (GL) theory. In this Letter we report the first study of superfluid and vortex dynamics in a SG network. A fundamental prediction emerging from the AH theory and its extensions [6] is that the energy required to create a vortex excitation in a loop of size r scales as $r^{-\zeta}$, where $\zeta = \ln \frac{5}{3} / \ln 2$. It follows that the pinning potential experienced by a vortex moving in a gasket has a marked hierarchical character which should become uniquely manifest in experiments probing the dynamics of vortices at different length (or time) scales. To this end, we have measured the complex sheet impedance Z of a network of interconnected gaskets exposed to a weak perpendicular magnetic field H over a wide range of driving angular frequencies ω . Close to $T_c(H)$, we observe richly structured Z -vs- H curves reflecting flux quantization in loops with a hierarchical distribution of sizes. Relying on thermally activated vortex motion to explain the evolution of the quantum structure with frequency, we find strong evidence for the scaling properties of vortices over four stages of hierarchy in the gasket. At high frequencies, where pinning and fluctuation effects are weak, the fine structure predominantly reflects the effect of frustration on the superfluid background.

The SG wire network was photolithographically patterned from a 1000-Å-thick granular aluminum film with a normal-state electrical resistivity of $14 \mu\Omega \text{ cm}$. The sample consisted of sixth-order gaskets sitting on the sites of a 32×32 triangular lattice and connected to each other

at the vertices. The length and width of the elementary links were, respectively, $a = 4.9 \mu\text{m}$ and $w \approx 1 \mu\text{m}$. Real and imaginary parts of $Z = R + i\omega L$ were extracted from measurements of the screening properties of the sample performed with an inductive drive-receive coil technique [7]. Excitation levels were chosen to ensure a linear response. Data were taken over 5 orders of magnitude in frequency, from 100 Hz to 10 MHz, using a lock-in technique above 1 kHz and a SQUID detector at lower frequencies.

Measurements of L^{-1} and R performed at the upper and lower limits of our spectral range are shown in Fig. 1 as a function of the frustration parameter f expressing the number of flux quanta in an elementary triangular

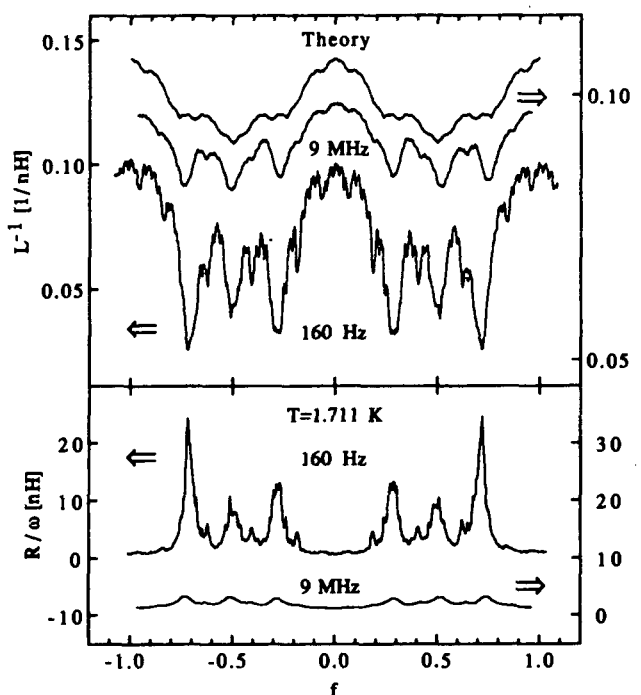


FIG. 1. Inverse sheet inductance and sheet resistance of the SG network at two different frequencies as a function of frustration. The theoretical prediction for a third-order gasket (shifted upward for clarity) should be compared with the 9-MHz $L^{-1}(f)$ data.

cell of the gaskets. The data, periodic in f with period 1, exhibit a complex fine structure arising from flux-quantization phenomena occurring in the loops of various size which generate the self-similar pattern of the gaskets. As expected, "dips" in $L^{-1}(f)$, reflecting a lower degree of superconducting phase coherence in the fractal lattice, correspond to "peaks" in the dissipative component $R(f)$. Quantum oscillations were observed only in a very narrow temperature range (~ 15 mK) just below the transition temperature $T_c(0) = 1.718$ K of the network. This dramatic temperature dependence as well as the observation that the fine structure becomes richer and sharper with decreasing frequency suggest that thermal fluctuations play a major role in the response of the system near the transition.

A theoretical description of superfluid and vortex dynamics in a fractal system including fluctuation effects does not exist so far. To get some insight into this complex problem, it is useful to consider the response at very high frequencies (VHF), where vortex motion is governed by viscous forces only. Since pinning effects, vortex-vortex interactions, and thermal fluctuations are negligible in this regime, Z can be written as [8] $Z = i\omega L_k + R_F$, where L_k is the bare kinetic inductance associated with nondissipative motion of the accelerated superfluid and R_F the flux-flow resistance due to viscous damping of otherwise freely moving vortices. Since R_F is a structureless (linear) function of f , in the VHF limit, quantum structures merely arise from the nonmonotonic effect of frustration on the superfluid background. To find the functional dependence of L_k on f , we recall that L_k^{-1} is proportional to the second derivative of the network ground-state energy with respect to the vector potential [9] and assume, for the sake of simplicity, that the network behaves as a Josephson-junction array (JJA) with a sinusoidal current-phase relation. Then, relying on the quadratic approximation of the cosine in the expression of L_k and using the AH recursive relation [1] for the gasket ground-state magnetic energy, we can express the frustration-induced change $\Delta L_k^{-1}(f) \equiv L_k^{-1}(0) - L_k^{-1}(f)$ of the inverse kinetic inductance as

$$\Delta L_k^{-1}(T, f) = L_k^{-1}(T, 0) \frac{4\pi^2}{18} \times \left[f^2 + \frac{1}{2} \sum_{h=0}^{n-1} \frac{(2^{2h+1}f - p_h)^2}{5^h} \right], \quad (1)$$

where n is the gasket order and the sum runs over the n families of loops in the gasket (h is the hierarchical index labeling a family of identical loops). The integers $\{p_h\}$, chosen to minimize the gasket energy, denote the total number of flux quanta threading a loop of the species h and one of the three gaskets (of order h) which surround it. As shown in Fig. 2(a) for $n=2$, $\Delta L_k^{-1}(f)$ is a fractal curve [even in f and symmetric with respect to $f = 0.5(\text{mod}1)$], whose parabolic segments correspond to

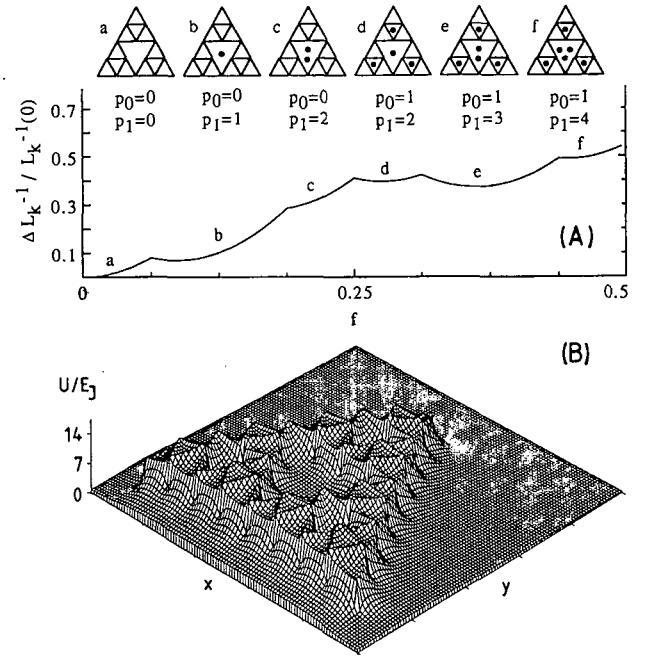


FIG. 2. (a) Relative change of the inverse kinetic inductance and corresponding vortex configurations for a second-order gasket as a function of frustration; (b) hierarchical vortex pinning potential for a third-order gasket.

vortex configurations specified by a well-defined set of $\{p_h\}$. Inspection [1] of Eq. (1) reveals that, in the interval $f_{c1} < f < 0.5$ ($f_{c1} = 4^{-n}$ is the frustration corresponding to the nucleation of the first vortex in the central loop of the gasket), $\Delta L_k^{-1}(f)$ scales as f^β with the anomalous exponent $\beta = \ln 5 / \ln 4$. Deviations from a sinusoidal current-phase relation depending on the relative size of the GL coherence length $\xi(T)$ to the link dimensions tend to depress the magnitude of $\Delta L_k^{-1}(f)$, which vanishes in the limit of a linear current-phase relation [$a \gg \xi(T)$] [10].

In Fig. 1 we compare the 9-MHz $L^{-1}(f)$ curve with a calculation based on Eq. (1). To agree with the data at $T = 1.711$ K (where $a/\xi \approx 5$), the amplitude of the superfluid depression at $f = 0.5$ predicted by Eq. (1) was reduced by a factor of ~ 3 . In order to reproduce the richness of the fine structure, it was sufficient to compute $L_k^{-1}(f)$ for $n=3$; features reflecting flux quantization in the higher hierarchical stages of the gaskets were beyond experimental resolution. Although the overall agreement tends to support the VHF interpretation, closer inspection of the data reveals a marked dip near $f = 0.25$ not predicted by theory. This suggests that, even at the highest frequencies studied in this work, vortex fluctuations were sufficiently important to weaken phase coherence in the system at some selected values of f . Additional evidence that the response at 9 MHz only partially fulfills the VHF requirement is provided by the presence of weak structures in $R(f)$, showing that pinning effects are still affecting vortex dynamics at 9 MHz.

A more crucial test of the VHF model is presented in

Fig. 3. To emphasize the low-field scaling properties of the response resulting from the self-similar nature of the gaskets, we show the frustration-induced relative variation $\Delta L^{-1}(f)/L^{-1}(0)$ of the network's inverse inductance at three different frequencies on a \log_5 - \log_4 plot. As shown in the upper part of Fig. 3, the 9-MHz data exhibit two well-resolved stages of dilational invariance in magnetic flux, in good agreement with the third-order calculation described above, and scale as $\Delta L^{-1}(f) \propto f^\beta$ with $\beta = 1.19 \pm 0.05$, a value consistent with that predicted by theory ($\beta = 1.16$).

With decreasing frequency, thermal fluctuations make the fine structure sharper and richer; the growing richness is demonstrated by the increasing number of self-similar stages (at least four at 160 Hz) emerging from the \log_5 - \log_4 plots of Fig. 3. This interpretation is consistent with similar behavior observed in renormalization-group calculations of the phase boundary [11]. Thermal fluctuations are believed to be responsible also for the deviations of the low-frequency data of Fig. 3 from the power-law scaling prediction of the AH theory.

While the inclusion of fluctuation-induced renormalization effects in a dynamical context appears to be an extremely difficult problem, the evolution of the fine-structure richness with frequency finds a natural explanation in terms of thermal activation of the vortices in the

potential-energy landscape $U(x,y)$ created by the gaskets. Continuing with the JJA analog, we have computed $U(x,y)$ using the arctan approximation to the vortex-phase configuration [12]. The result, shown in Fig. 2(b) for a third-order gasket, reveals a pronounced hierarchical structure; the energy U_h of the local minima of $U(x,y)$ located near the centers of the various loops scales with the loop size r_h as

$$U_h(T) \approx \pi^2 E_J(T) (a/r_h)^\zeta, \quad (2)$$

where $E_J(T)$ is the coupling energy of adjacent nodes [13] and $\zeta = \ln \frac{5}{3} / \ln 2$. This relation follows from the AH theory by calculating the energy of the first excited state (one single vortex in the central loop) of a gasket in zero field. Since the recursive procedure underlying the AH approach only applies to symmetric vortex configurations, Eq. (2) slightly underestimates U_h for vortex nucleation in off-center loops [6]. Notice that ζ is the ratio of the conductivity and correlation-length exponents in two-dimensional percolating clusters [2].

To proceed further in the description of thermally activated vortex motion in the fractal lattice, we need an estimate of the energy barrier Δ_h experienced by a vortex of species h . In this connection, the quantity of interest is the energy U_s of the saddle points of $U(x,y)$ located on the interconnecting links. In contrast with the strong hierarchical dependence of U_h , U_s turns out to be roughly independent of the link position in the gasket, a result allowing great simplification. Writing $U_s = sE_J$, we find that s varies from ~ 10 to ~ 12 as the form of the current-phase relation changes from sinusoidal to linear.

Assuming that vortices of species h behave as independent Brownian particles with an activation energy $\Delta_h = U_s - U_h$, we now argue that there will be detectable contributions to the fine structure arising from loops of species h only if the vortices have an appreciable probability to diffuse out of the corresponding potential wells on the characteristic time scale set by the measurement, i.e., only if the thermal escape rate $\Omega_h = \Omega_J \exp(-\Delta_h/kT)$ is larger than ω . Since the loops which dominate the response at any level of frustration in the low-field scaling regime are those enclosing one flux quantum [1], i.e., having a size $r_f \approx a/\sqrt{f}$, we can express the depinning condition $\Omega_h > \omega$ in the form $f > f_c(T, \omega)$ by setting $r_h = r_f$ in Ω_h . The critical frustration $f_c(T, \omega)$, the analog of the irreversibility line in high-temperature superconductors, is given by

$$[f_c(T, \omega)]^{\zeta/2} = A(T) + [kT/c_2 E_J(T)] \ln \omega, \quad (3)$$

where $A(T) = c_1 - [kT/c_2 E_J(T)] \ln \Omega_J(T)$ and c_1 and c_2 are constants of order 1 and 10, respectively. In the JJA approach, one expects $\Omega_J(T) \approx (2e/\hbar)^2 E_J(T) R_J$, where R_J is the normal-state resistance of an elementary link [8].

To test the logarithmic frequency dependence predicted by Eq. (3), we take advantage of the self-similar proper-

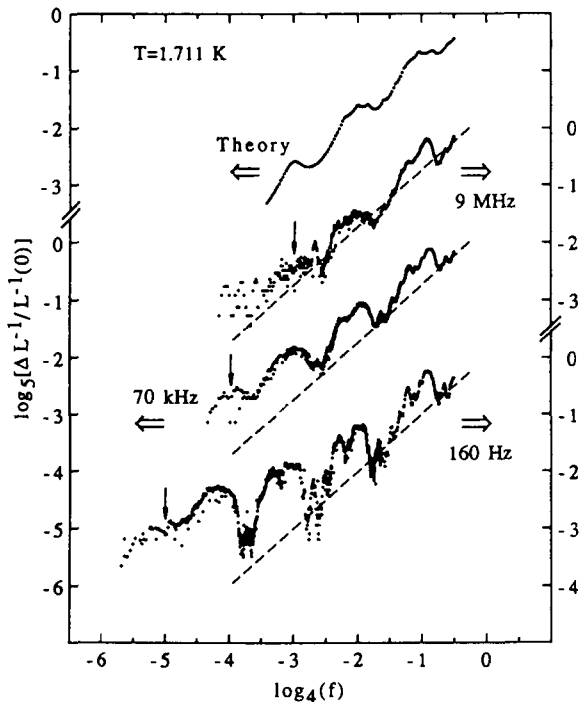


FIG. 3. \log_5 - \log_4 plot of the relative change of the inverse sheet inductance of the SG network at three different frequencies as a function of frustration. The theoretical prediction for a third-order gasket should be compared with the 9-MHz data. The arrows denote the critical frustration. The power-law scaling prediction of the Alexander-Halevi theory is indicated by the dashed lines.

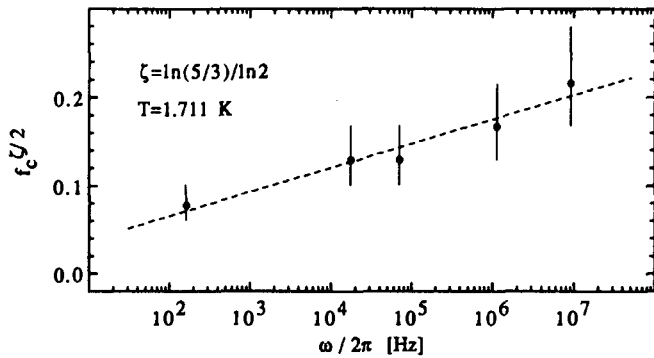


FIG. 4. Dependence of $f_c^{5/2}$, as calculated from the f_c values defined in Fig. 3, on the logarithm of the driving frequency. The dashed line is a fit based on Eq. (3). See text for the definition of the error bars.

ties of the data of Fig. 3 to define $f_c(\omega)$ as the lowest value of frustration at which we are still able to resolve traces of a hierarchical stage. Adopting this procedure, which allows one to locate $f_c(\omega)$ with an accuracy corresponding roughly to one hierarchical stage, we find the f_c values denoted by arrows in Fig. 3. A more elaborate method, based on a Fourier analysis of the data, led to similar results. Using $\zeta = \ln(5/3)/\ln 2$, in Fig. 4 we have plotted $f_c^{5/2}$ against $\log_{10}\omega$ as a check of Eq. (3). Taking $R_J = 7 \Omega$ to compute $E_J(T)$ [13] at $T = 1.711$ K, we find a good fit for a quite reasonable set of the adjustable parameters c_1 and c_2 ($c_1 = 0.3$ and $c_2 = 9.5$). It should be noticed, however, that the logarithmic frequency dependence of the data is rather insensitive to the value of ζ , although drastic deviations from the theoretical value would imply unreasonable values of the fitting parameters. Also notice that the attempt frequency $\Omega_J(T)/2\pi$ at $T = 1.711$ K is roughly an order of magnitude higher than the highest frequency (9 MHz) studied in this work, an observation lending further support to the conclusion reached above in connection with the VHF regime.

In conclusion, ac impedance measurements performed on a network of interconnected Sierpinski gaskets have provided novel insight into the dynamics of superfluids and vortices in a fractal system. In particular, by probing the network response over a range of length scales cover-

ing four levels of hierarchy in the gaskets, we have found strong evidence for unusual scaling properties of vortices in a fractal lattice emerging from the Alexander-Halevi theory.

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VORTEX DYNAMICS IN SUPERCONDUCTING FRACTAL NETWORKS

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Abstract. We report experiments measuring the ac response of a Sierpinski Gasket (SG) of Al wires to a weak driving ac field, as a function of dc magnetic field and driving frequency. Genuine fractal behavior becomes particularly manifest in the network's ac magnetoconductance, where a very rich fine structure reflecting flux quantization phenomena related to the multiconnected self-similar geometry of the gasket is observed. The analysis of the frequency dependence of the fine structure suggests that thermally activated fluctuations have to be taken into account in order to explain the dynamical properties of the system.

Current interests in superconductivity concentrate on granular and percolating materials. The Sierpinski Gasket (SG) is a system originally proposed by Gefen et al. [1] to model the geometrical features of the percolating cluster's backbone.

So far, experimental studies focused on the $T_c(H)$ phase boundary [2], which was found to agree well with spectrum calculations of the linearized Ginzburg-Landau (GL) theory [3].

Another approach based on the non-linear GL equation in the London limit ($|\psi| = cst$) was made by Alexander and Halevi (AH) [4]. Using the SG scaling properties, they expressed the magnetic energy of an n^{th} order gasket for different symmetrical vortex configurations (lowest energy solutions [5]). A major result of this study is that for a given vortex configuration, the dominant energy contribution comes from loops surrounding one flux quantum.

In a Josephson junction model, it can also be shown [6] that, due to the loops of various sizes forming the gasket, the pinning potential experienced by a vortex is highly hierarchical. This idea, combined with the AH result, makes the SG very attractive for studying vortex dynamics.

Using an inductive technique, we measured the complex ac response of a network of interconnected 6th order SG as a function of frustration f , expressing the number of flux quanta per elementary triangular cell of the SG, over a wide range of driving frequencies $\omega/2\pi$ (100Hz - 10MHz). We extracted the array's inverse kinetic inductance L_k^{-1} with a numerical method [7].

The sample was photolithographically patterned from a 1000Å thick Al film. Length and width of the elementary links were $a = 4.9\mu\text{m}$ and $w = 1\mu\text{m}$. All the measurements presented here were performed 7mK below the network's mean-field transition temperature $T_c(0) = 1.718\text{K}$, a region where the wire network can be considered a Josephson junction array (JJA) [8] and where vortex pinning is sufficiently weak [8] to observe the oscillations in L_k^{-1} shown in Fig. 1. As expected, the data are periodic (with period 1) and even in the frustration f . The fine structures reflect the change of phase coherence through the array due to flux redistribution in the different loops of the SG. This idea is illustrated by the prediction of the AH theory [4].

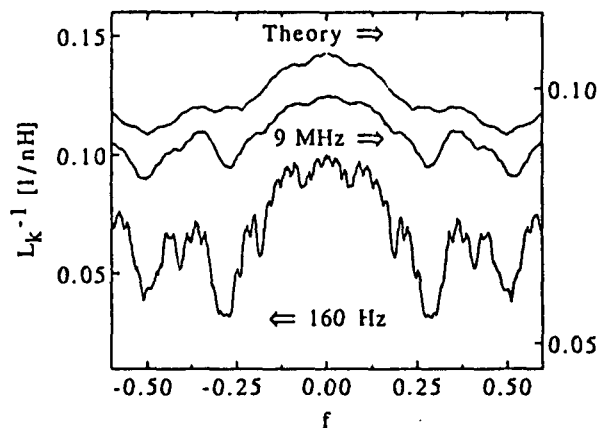


Figure 1: Inverse kinetic inductance at two different frequencies and prediction of the AH theory (slightly shifted for clarity) as a function of frustration.

We recall that in a JJA with sinusoidal current-phase relation, if we neglect pinning and fluctuations, the non-renormalized inverse kinetic inductance $L_{k_0}^{-1}$ is proportional to the second derivative of the array's ground-state energy with respect to the vector potential. Using the AH recursive relation for the SG ground-state energy [4] and relying on a quadratic expansion of the cosine, we obtain the following expression for the frustration induced change of the inverse kinetic inductance, $\Delta L_{k_0}^{-1}(T, f) = L_{k_0}^{-1}(0) - L_{k_0}^{-1}(f)$

$$\Delta L_{k_0}^{-1}(T, f) = L_{k_0}^{-1}(T, 0)(4\pi^2/18)[f^2 + (1/2) \sum_{h=0}^{n-1} (2^{2h+1} f - p_h)^2 / 5^h] \quad (1)$$

where the sum runs over the n families of loops forming the n^{th} order gasket. The integers p_h describe the vortex occupation and are chosen to minimize the energy.

An example for a 2^{nd} order gasket is shown in Fig. 2. Notice that this curve has to be imagined even in f , and symmetric with respect to $f=1/2$.

In order to compare this prediction with the experimental results, we made a calculation for a 3^{rd} order gasket which, as shown in Fig. 1, displays approximately the same structure as the 9 MHz measurement. At low frequency, we also see that the fine structure is much sharper and richer. The features of the 160 Hz curve cannot be reproduced with this simple fluctuation-free model, even with calculations of higher order gaskets. This observation, supported by the facts that the oscillations were observed only in a very narrow region ($\sim 15\text{mK}$) close to $T_c(0)$ and by a strong temperature dependence of the structure, suggests that thermally activated fluctuations play a dominant role. Other arguments supporting this idea will be presented elsewhere [6].

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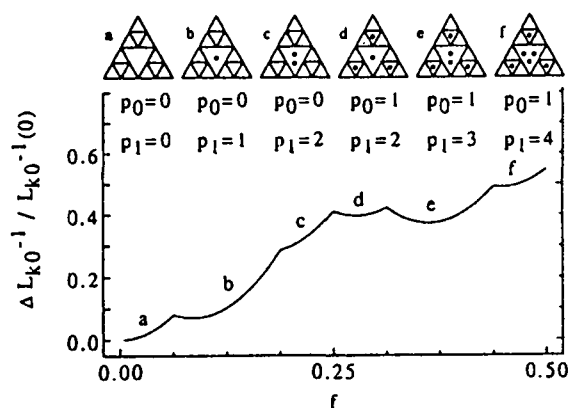


Figure 2: Relative change of kinetic inductance and corresponding vortex configurations for a 2^{nd} order gasket according to the AH theory.

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Thermal vortices in superconducting fractals

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A study of the zero-field ac response of a wire network of interconnected 6th-order Sierpiński gaskets as a function of temperature is presented. In a narrow temperature range below the superconducting-to-normal transition thermally nucleated vortices dominate the response. A theoretical model treating the vortices as Brownian particles moving about in the hierarchical potential-energy landscape of the fractal lattice is developed and its predictions are found to agree with the experimental observations.

1. Introduction

The concept of fractal structure provides a very useful geometrical tool to describe some of the features of random systems. In particular, fractal concepts have been successfully used to understand the physics of disordered granular superconductors near the percolation threshold [1–3]. Within the vast family of fractals, the Sierpiński gasket has proven to be a very convenient model to mimic the natural self-similar topology of such percolating materials [4].

Some of the most prominent features of disordered superconductors are intimately related to the dynamics of vortices in a fractal structure. In this connection, it is important to realize that the properties of vortices in fractals [5,6] are fundamentally different from those of vortices in Euclidian systems. In the Sierpiński gasket, for instance, the energy U_h required to create a vortex excitation in a (triangular) loop of a given species h (h is the “hierarchical” index labeling a family of identical loops) scales with the size r_h of the loop as $r_h^{-\zeta}$ with $\zeta = \ln(\frac{5}{3})/\ln 2$, a prediction in striking contrast with the logarithmic divergence of the vortex nucleation energy in a genuine two-dimensional (2D) system. By studying the ac response of a wire network of interconnected Sierpiński gaskets exposed to a weak magnetic field, we recently demonstrated the unusual scaling of U_h with the underlying hierarchical distribution of loop sizes [5]. In this paper we focus on the low-frequency response of the fractal network in *zero field* and show that, in the critical region close to the superconducting-to-normal transition, its complex sheet impedance Z is well

described by a model where thermally nucleated vortices behave as Brownian particles moving in the hierarchical potential-energy landscape provided by the gaskets.

2. Experimental details

The wire network studied in this work is the same as of the investigations reported in ref. [5]. It consists of 6th-order aluminum gaskets sitting on the sites of a 32×32 triangular lattice and connected to each other at the vertices. Length and width of the elementary links are, respectively, $a = 4.9 \mu\text{m}$ and $w \approx 1 \mu\text{m}$. Their normal-state resistance is $R_s = 6.9 \Omega$, as determined from the electrical resistivity of the unpatterned Al film. Real and imaginary parts of $Z = R + i\omega L$ were extracted from inductive measurements of the screening properties of the sample performed with a drive–receive coil technique [7]. Data were taken at a driving angular frequency of 10^3 rad/s using a SQUID detector. The excitation level, chosen to ensure a linear response, reached a maximum rms value of $\sim 10 \mu\text{G}$ at the center of the network, while ambient magnetic fields were suppressed to $\sim 1 \text{ mG}$.

3. Results and discussion

In zero magnetic field and at temperatures such that thermally generated vortices can be ignored the network behaves as a pure inductor ($R = 0$) with $L = L_{k0}$, the bare kinetic inductance per square given by [8]

$$L_{k0}^{-1}(T) = (\pi/\hbar R_N) \Delta(T) \tanh[\Delta(T)/2kT], \quad (1)$$

where $\Delta(T)$ is the BCS gap parameter. The fractal nature of the network enters eq. (2) through R_N , the normal-state sheet resistance of the sample. If one applies Stephen's result [1] for the resistance of a single gasket to our triangular lattice of interconnected gaskets, it is readily seen that $R_N = (R_s/\sqrt{3})(\frac{5}{3})^n$, where n is the order of the gaskets.

The network's inverse sheet kinetic inductance L_k^{-1} , proportional to the effective (i.e. renormalized by vortex excitations) superfluid density in the system, is shown in fig. 1 as a function of temperature. The data were fitted to eq. (1) using the mean-field transition temperature T_{c0} and R_N as adjustable parameters. With the exception of the very narrow ($\sim 15 \text{ mK}$) critical region discussed in more detail below, an excellent fit was found with $T_{c0} = 1.718 \text{ K}$

and $R_N = 89 \Omega$, a value in good agreement with that ($R_N = 85 \Omega$) computed from the expression relating R_N to R_s (for $n = 6$).

Let us now focus on the critical region just below T_{c0} . Because of the cross-over from the fractal to the homogeneous 2D regime at length scales r larger than the percolation correlation length $\xi_p \approx 2^n a$, a periodic network of interconnected gaskets should exhibit, in principle, a phase transition driven by the vortex unbinding mechanism of the Kosterlitz–Thouless (KT) theory. As shown by a renormalization procedure [6] which takes advantage of the self-similarity of the gaskets, for $r > \xi_p$ such a system is indeed equivalent to a 2D triangular lattice with a suitably rescaled coupling energy given by $(\frac{3}{5})^n E_J(T)$ when expressed in terms of the “elementary” coupling energy $E_J(T)$ of adjacent nodes in the gaskets. Therefore, a KT transition is predicted to occur at a temperature T_{KT} such that $(\frac{3}{5})^n E_J(T_{KT}) \approx (\frac{2}{3})(2/\pi)kT_{KT}$ or, more precisely, $L_k^{-1}(T_{KT}) = (2/\sqrt{3})(8\pi/\phi_0^2)kT_{KT}$ (the numerical prefactors $\frac{2}{3}$ and $2/\sqrt{3}$ are related to the triangular structure of the lattice underlying the gaskets [9]). Inspection of the temperature dependence of L_k^{-1} very close to T_{c0} (see fig. 2a) reveals that appreciable renormalization of the superfluid density sets in at a temperature consistent with the theoretical prediction for T_{KT} . However, in contrast with weakly coupled periodic networks [8], clear evidence for a KT

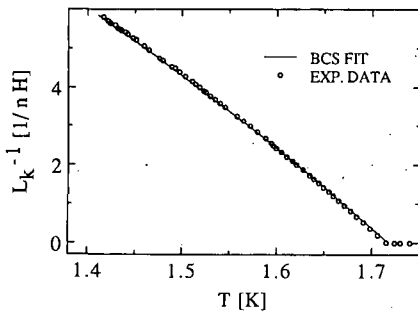


Fig. 1.

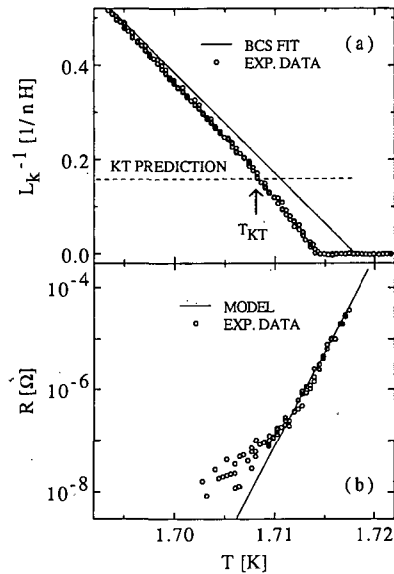


Fig. 2.

Fig. 1. Comparison of the temperature dependence of the gasket inverse kinetic inductance measured at 160 Hz with the BCS mean-field prediction.

Fig. 2. Comparison of the gasket ac response in the critical region with various theoretical predictions (see text for details): (a) inverse sheet kinetic inductance and (b) sheet resistance.

transition in the form of a sharp drop of $L_k^{-1}(T)$ near T_{KT} is missing. We attribute this absence of well defined KT features to the stronger pinning experienced by vortices in a fractal network. In fact, when measured in units of the relevant thermal energy kT_{KT} , the lowest energy barrier against vortex crossing created by the individual wires in an n th-order gasket turns out [10] to be at least a factor of $(\frac{5}{3})^n$ larger than that opposing vortex motion in a regular triangular lattice with the same “elementary” $E_J(T)$. The obvious conclusion emerging from these considerations is that vortex dynamics in a fractal network is, to a large extent, dominated by pinning phenomena which drastically reduce the mobility of vortex excitations and, therefore, inhibit a fully developed KT transition.

As shown in fig. 2b, the fluctuation-dominated critical region is marked by a rapid rise of dissipation, the network sheet resistance $R(T)$ growing exponentially with temperature in the low-resistance range accessible to our experimental method. As shown by the following analysis, the observed temperature dependence finds a natural explanation in terms of Brownian motion of thermally nucleated vortices in the hierarchical potential-energy landscape $U(x, y)$ created by the gaskets.

We begin by noticing that the contribution $R_h(T)$ of vortices of the species h to the total sheet resistance $R(T)$ is given by [8]

$$R_h(T) = \phi_0^2 n_h(T) \mu_h(T), \quad (2)$$

where $n_h(T)$ and $\mu_h(T)$ are, respectively, the areal density and the mobility of vortices of the family h and ϕ_0 the flux quantum. For thermally nucleated vortices, we expect $n_h(T)$ to be proportional to $\exp[-U_h(T)/kT]$, where $U_h(T)$ is the energy required to create a single vortex excitation in a loop of the species h from the vortex-free ground state of the gaskets. Thus, $U_h(T)$ is the energy of the local minima of $U(x, y)$ located near the centers of the various loops and, being proportional to $E_J(T)(a/r_h)^\xi$, has a pronounced hierarchical character [5]. On the other hand, at the low frequencies used in our experiments the vortex mobility has a simple thermally activated form $\mu_h(T) \sim \exp[-\Delta_h(T)/kT]$, where $\Delta_h(T)$ is the energy barrier experienced by vortices of the species h . An estimate of $\Delta_h(T)$ can be inferred from the energy $U_s(T)$ of the saddle points of $U(x, y)$, such that $\Delta_h = U_s - U_h$. Using the arctan approximation to the vortex phase configuration [5,11] to compute $U(x, y)$, it turns out that $U_s(T)$, in contrast with the strong hierarchical dependence of $U_h(T)$, is roughly independent of the saddle point positions in the gaskets. Writing $U_s(T) = \alpha(T) E_J(T)$, we find that $\alpha(T)$ depends on the relative size of the Ginzburg–Landau (GL) coherence length $\xi(T)$ to the link dimensions a and w [12]. In the temperature range of interest very close to T_{c0} , the ratios

$a/\xi(T)$ and $w/\xi(T)$ are such that 2D vortex core nucleation processes in the wires can be excluded, while 1D phase-slip events will presumably set in only near the lower limit ($T \approx 1.70$ K) of the critical region, where $a/\xi(T)$ becomes fairly large (~ 9). Thus, in most of the critical interval the network behaves essentially like a Josephson junction array for which α is a constant, independent of temperature. From the numerical calculations of $U(x, y)$, shown in fig. 3 for a particular vortex path visiting all the hierarchical stages of a 4th-order gasket, we find that α is of the order of 10.

The analysis of $U_h(T)$ and $\Delta_h(T)$ reveals that $R_h(T) \sim \exp[-\alpha E_j(T)/kT]$ is independent of the vortex species, a result which greatly simplifies the calculation of $R(T)$. Using the GL expression of $E_j(T)$ valid near T_{c0} [8], assuming that $R(T_{c0}) \approx R_N$ and setting $\Delta T \equiv T_{c0} - T$, $R(T)$ can be immediately written as

$$R(T) = R_N \exp[-3.68\alpha(\hbar/e^2 R_s) \Delta T/T_{c0}]. \quad (3)$$

Taking $R_N = 89 \Omega$, the data of fig. 2b were fitted to eq. (3) using T_{c0} and α as adjustable parameters. A good fit over more than three decades in resistance was obtained with $T_{c0} = 1.734$ K and $\alpha = 0.7$. If one realizes that the assumption $R(T_{c0}) \approx R_N$ tends to overestimate the mean-field transition temperature, the value of T_{c0} is in reasonable agreement with that inferred from the $L_k^{-1}(T)$ data. The origin of the large discrepancy between the α -value extracted from experiment and that predicted by the numerical simulations is not fully understood at present. It might be possibly traced back to the absence of any form of fluctuation-induced renormalization in our treatment. The inclusion of such effects would actually result in a depression of $E_j(T)$ in the critical region,

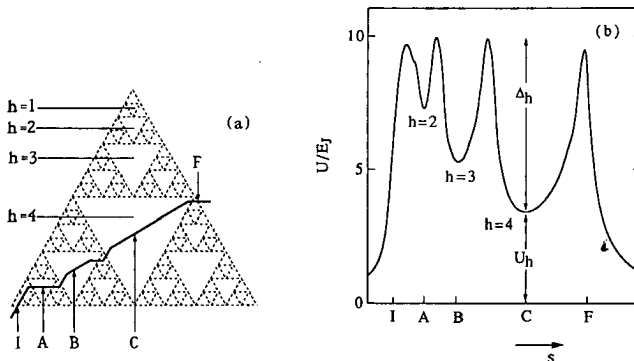


Fig. 3. (a) Vortex path visiting all the hierarchical stages of a 4th-order Sierpiński gasket. (b) Normalized pinning potential $U(s)/E_j$ experienced by a vortex moving along the path shown in (a).

thereby providing an explanation for the small value of α observed in experiment.

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Observation of dilational symmetry breaking in a superconducting array of Sierpinski gaskets

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The inverse sheet kinetic inductance L_k^{-1} of a periodic array of n^{th} order Sierpinski gaskets has been measured as a function of frustration f , f being the number of magnetic flux quanta in the elementary triangular cell of the fractal structure. The Josephson junction array shows prominent oscillations only for frustrations corresponding to multiples of $f_n=1/(2 \times 4^n)$. A simple model taking into account the interplay between fractal and two-dimensional (2D) régime has been developed to calculate L_k^{-1} for the gasket array. It is shown that the periodic boundary conditions imposed by the 2D lattice are responsible for the observed oscillations.

1. Introduction

Since the beginning of the high- T_c era, disorder in superconductors is an attractive and important subject. Geometrical disorder has been elegantly described by percolating models [1], where the concept of self-similarity plays a key role. Therefore, a thorough study of the basic physics of fractals is a useful step towards the understanding of disorder in superconductors.

In this paper, we study the ac linear response of a periodic array of Sierpinski gaskets (SG) exposed to a weak magnetic field H . The array is made of SNS (Pb/Cu/Pb) Josephson junctions. Similar systems have been recently investigated [2] to demonstrate the crossover from the fractal to the 2D régime as observed in the low-field phase boundary $T_c(H)$. In this paper we show that the interplay between the fractal and the 2D régime has a profound impact on the magnetic properties of the system below $T_c(H)$. A simple model, where the periodic boundary conditions imposed by the 2D lattice are essential, accounts for our observations.

2. Experimental results

The array structure is composed of interconnected 2nd order SG. A unit cell is shown in Fig.1. The zero-field superconducting-to-normal transition temperature is $T_c(0) = 2.8$ K.

The sample ac response at $T = 2$ K was measured as a function of the frustration f using our standard two-coil technique [3] (f defines the number of flux quanta in an elementary triangular cell of the gasket). The inverse sheet kinetic inductance L_k^{-1} was then extracted using a numerical inversion procedure [3]. The result is shown in Fig.1 for an excitation

frequency of 5 kHz. In striking contrast to the much smoother response of a single SG [4], $L_k^{-1}(f)$ of the periodic lattice of SG exhibits sharp oscillations. In the next section, we develop a simple model explaining these different behaviors.

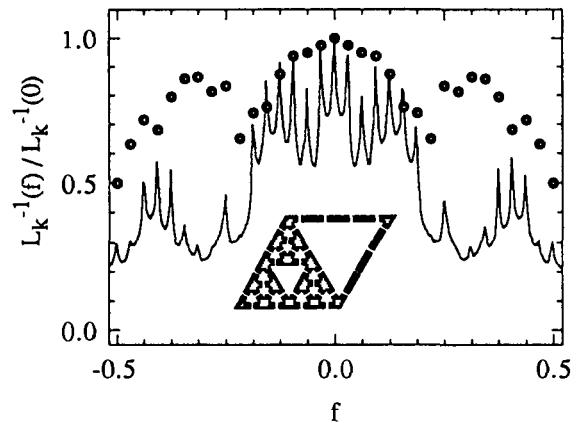


Fig.1: L_k^{-1} of the array of SG as a function of f . The experimental data (line) are compared to the model of Sec.3 (dots). The inset shows a unit cell of the array.

3. Theoretical model

The gauge invariant phase differences $\{\gamma_{ij}\}$ across the junctions are determined by current conservation at each node j , $\sum_i \sin \gamma_{ij} = 0$ and fluxoid quantization in the loops of various sizes formed by the array, $\sum_{\text{loop}} \gamma_{ij} = 2\pi (k_h - 4^h f)$. The index h labels the different families of loops ($h=0, \dots, n-1$) and k_h is the number of vortices in a loop of species h . For each loop and a given f , k_h has to be chosen in order to minimize the array's energy $E \propto -\sum_{\text{array}} \cos \gamma_{ij}$. The

integers $\{k_n\}$ for the ground state configurations correspond to highly symmetric vortex distributions [4,5]. Consequently, for a single n^{th} order SG, imposing both three-fold rotation and reflection symmetries reduces the number N of equations and variables $\{\gamma_{ij}\}$ to $N=(3^n+1)/2$. This system of equations can be solved to obtain the ground state energy $E(f)$ of a single n^{th} order SG. The result is shown in Fig.2a.

To calculate L_k^{-1} , we characterize each junction by its equivalent inductance $L_{ij} \propto 1/\cos\gamma_{ij}$, where the $\{\gamma_{ij}\}$ are those of the ground states computed above. Then, using Kirchhoff's laws as in electrical networks, we obtain $L_k^{-1}(f)$ of a single SG as shown in Fig.2b.

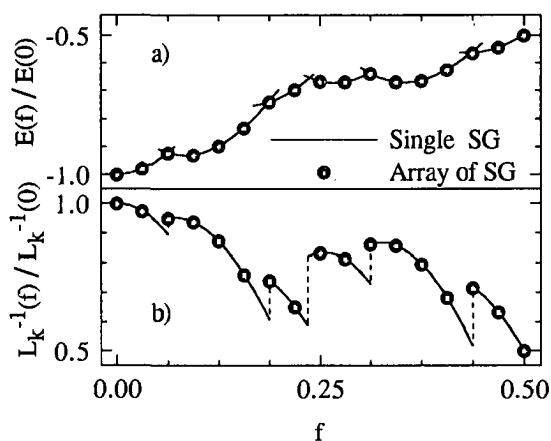


Fig.2: (a) Ground state energy E and (b) corresponding L_k^{-1} for a single 2^{nd} order SG and for an array of 2^{nd} order SG as a function of f .

Let's now come to the case of the periodic repetition of SG. Connecting the n^{th} order SG together creates triangular loops of linear size $2^n \times a$. Expressing flux quantization in these triangular loops introduces an additional equation. As a consequence, the system of $N+1$ equations becomes overdetermined for every value of f , except for multiples of $f_n=1/(2 \times 4^n)$, where the additional equation is trivially satisfied. The multiples of f_n are exactly the values emerging from the nesting property proven in Ref. [6]. Translated into our context, this property allows to assemble three n^{th} order SG together in one $(n+1)^{\text{th}}$ order SG without changing the energy (or L_k^{-1}). It follows that, for the discrete set of

multiples of f_n , periodic boundary conditions are readily satisfied and, accordingly, the energy (or L_k^{-1}) of the 2D array is determined by that of a single n^{th} order SG as illustrated in Fig.2. For all other frustrations, the ground state of the periodic array cannot be found from a single gasket calculation since it will have a larger elementary cell.

4. Discussion and conclusions

In Fig.1, the measured oscillations are exactly located at multiples of f_n (with $n=2$) showing that the array is in a phase coherent state for these values of f only. Nonetheless, the amplitudes of many peaks are depressed with respect to our $T=0$ calculations, showing that thermal fluctuations are important. It seems also natural that no peaks occur at fractional multiples of f_n , since they would correspond to ground states with larger elementary cells and, therefore, are more vulnerable to fluctuations. In contrast to the case of a genuine periodic array [7] in which a non-monotonic sequence of peaks results from the formation of superlattice states, in the present experimental situation, the peaks of different heights correspond to states with the same size of the unit cell and are therefore equally spaced along the f axis.

Finally we notice that, if f_n falls below experimental resolution, the discrete nature of $L_k^{-1}(f)$ (Fig.1) will be washed out. This situation is met in periodic arrays of higher-order SG, whose response was found to be similar to that of a single SG [4].

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Magnetoinductance of a superconducting Sierpinski gasket

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A study of the magnetoinductance $L(B)$ of a planar superconducting fractal lattice, the Sierpinski gasket (SG), exposed to a perpendicular magnetic field B is reported. Being inversely proportional to the superfluid density in the gasket, $L(B)$ provides a tool to appreciate how frustration effects created by B and characterized by a parameter $f \propto B$ affect phase coherence in a superconductor sharing essential geometrical elements with a truly percolating system near threshold. Both Josephson junction arrays (JJA) and superconducting wire networks (SWN) differing in their current-phase relations are considered and described in terms of interacting phase variables associated with the sites of the gasket. Relying on a mean-field approach, two central issues are addressed: the fine structure of $L(f)$ reflecting flux-quantization phenomena in loops with a hierarchical distribution of sizes and the low-field ($f \rightarrow 0$) scaling behavior of $L(f)$ resulting from the self-similar geometry of the gasket. It is shown that for a particular set of f values consistent with the requirement of fluxoid quantization in the central loop of a gasket generated by repeated juxtapositions of gaskets of lower order ($f = P/2 \cdot 4^N$, where N is the gasket order and P an integer) the problem of computing $L(f)$ reduces to a calculation on a finite gasket and can be solved exactly once its ground-state phase configuration is known. Considerable simplification is achieved by making use of the triangle-star transformation of electric networks. The amplitude of the fine structure is found to depend crucially on the degree of anharmonicity of the phase interaction function. It vanishes (thereby implying that L is independent of f) in weakly coupled SWN with a strictly harmonic interaction and reaches its maximum strength in JJA with a sinusoidal interaction. Using a perturbative decimation procedure which takes advantage of the self-similar structure of the SG, the frustration-induced inductance correction $\delta L(f)$ is predicted to scale as f^ν with $\nu = \ln(125/33)/\ln 4 \approx 0.96$ in the asymptotic limit ($f \rightarrow 0$). This exact result as well as other theoretical predictions emerging from the model are found to agree with high-resolution measurements of $L(f)$ performed on triangular arrays of periodically repeated gaskets of proximity-effect coupled Pb/Cu/Pb Josephson junctions.

I. INTRODUCTION

The concept of fractal structure provides a very useful geometrical tool to describe some of the features of random systems.¹ For instance, percolating materials exhibit, near the percolation threshold, a natural self-similar structure with geometrical inhomogeneities occurring over a broad range of length scales. They can therefore be described by a family of scale-invariant lattices, such as the Sierpinski gasket originally proposed by Gefen *et al.*² to mimic the topological properties of the percolating cluster's backbone.

With regard to superconductivity, fractal concepts have proven to be instructive in getting some insight into the physics of granular superconductors near percolation.³⁻⁶ These materials, usually conceived as arrays of randomly distributed superconducting grains weakly coupled by the Josephson effect,⁷ exhibit intriguing magnetic properties arising from the combined effect of disorder and frustration.⁸ Unfortunately, in most cases the structural aspects of randomness in real superconductors are poorly known making a detailed comparison of theory and experiment almost impossible. With the ad-

vent of modern microfabrication techniques, however, it has become possible to investigate model systems, such as Josephson junction arrays (JJA) and superconducting wire networks (SWN), where both the nature and the amount of disorder can be accurately controlled and the level of frustration continuously tuned via an external magnetic field B .

Within the vast family of systems with fractal features, the Sierpinski gasket (SG), because of its simple hierarchical structure deprived of the complexity resulting from randomness and its dilational symmetry, appears to be an excellent candidate to explore novel behavior emerging from fundamental ideas in statistical mechanics and condensed-matter physics. Early work on SG wire networks has focused on their mean-field superconducting-to-normal phase boundary $T_c(B)$ (Refs. 9 and 10) which was found to agree with calculations based on the Ginzburg-Landau theory.^{10,11}

More recently, it has been shown^{12,13} that the properties of vortices in fractals are fundamentally different from those of vortices in Euclidean systems. It turns out, in fact, that the energy required to create a vortex in a (triangular) loop of a given species h (h is the hierarchical

index labeling a family of identical loops) scales with the size r_h of the loop as $r_h^{-\zeta}$ with $\zeta = \ln(\frac{2}{3})/\ln 2$, in striking contrast with the logarithmic divergence of the vortex nucleation energy in a genuine two-dimensional (2D) system. Impedance measurements¹³ probing the dynamics of vortices in weakly frustrated SG networks over a range of length scales covering several levels of hierarchy in the gaskets were found to be consistent with the unusual scaling of the vortex energy in fractal lattices predicted by theory. This observation was confirmed in further work¹⁴ where the dynamic response of an unfrustrated ($B=0$) SG in the critical region close to the transition was interpreted in terms of thermally nucleated vortices moving like Brownian particles in the hierarchical potential-energy landscape provided by the gasket.

In this paper we investigate, both theoretically and experimentally, the magnetoinductance $L(B)$ of a superconducting SG exposed to a weak perpendicular field B . The interest of this quantity resides in the observation that, being inversely proportional to the (size-dependent) superfluid density in the gasket, it provides a tool to appreciate how the degree of superconducting phase coherence in the system changes with B . $L(B)$ is therefore of considerable importance to understand the magnetic properties of a superconductor sharing essential geometrical elements with a truly percolating system. From previous experimental work¹³ it is known that $L(B)$ exhibits a complex fine structure reflecting flux quantization phenomena in loops with a hierarchical distribution of sizes. Even more significant is the observation that, at very low fields, $L(B)$ exhibits scaling properties intimately related to the fractal structure of the gasket. Asymptotic scaling ($B \rightarrow 0$) and fine structure of $L(B)$ are the central issues we shall address in this paper. For simplicity, it will be assumed that screening currents are weak, so that B penetrates the gasket homogeneously. In this regime the system is uniformly frustrated, the degree of frustration being measured by a parameter f proportional to B .

The model we adopt throughout this paper relies on the usual description of JJA and SWN in terms of interacting phase variables associated with the sites of the gasket.¹⁵ An exhaustive calculation of $L(f)$ should incorporate renormalization effects due to thermal fluctuations which, on account of the reduced (nontrivial) dimensionality of the SG, are expected to have a profound effect on phase coherence. While vortex fluctuations were studied in detail in connection with the critical behavior of the unfrustrated gasket ($f=0$), where they were found to suppress the Berezinskii-Kosterlitz-Thouless (BKT) transition,¹² the inclusion of fluctuation-induced renormalization phenomena at nonvanishing arbitrary frustrations appears to be a prohibitive task and will be ignored in this paper. Within this mean-field approach, the magnetoinductance merely depends on the structure of the ground state of the system (more precisely, on the ground-state distribution of the gauge-invariant phase differences across the links of the gasket). In this connection, it turns out that the requirement of fluxoid quantization in the central loop of a gasket resulting from the juxtaposition of three gaskets of lower order selects a

particular set of frustrations for which the ground state of an infinite gasket can be constructed by replicating that of a finite gasket. Hence, for these f values the problem of calculating $L(f)$ reduces to a calculation on a finite gasket and can be solved exactly once its ground state is known. For computational purposes, it is useful to rely on an algorithm, the triangle-star transformations,¹⁶ well known in the theory of electric networks.

Although particularly efficient for numerical calculations, this "direct method" does not provide any insight into the characteristic low-field scaling properties of the magnetoinductance resulting from the fractal geometry of the gasket. To study this interesting asymptotic limit ($f \rightarrow 0$), we take advantage of the self-similar structure of the system to develop a perturbative decimation method which reduces the problem of calculating the inductance corrections $\delta L(f)$ at low frustrations for a given gasket to an identical problem at higher frustrations for a gasket of lower order whose ground state can be easily determined. A major exact result emerging from this treatment is that in the asymptotic limit $\delta L(f)$ is predicted to scale with frustration as f^ν with $\nu = \ln(\frac{125}{33})/\ln 4 \approx 0.96$. The fraction $\frac{33}{125}$ in the numerator of the exponent ν reflects the scaling of δL at each decimation step and is shown to arise from the first nonlinear (cubic) term in the current-phase relation of the individual junctions of the gasket (the argument of the logarithm in the denominator expresses the trivial renormalization of f resulting from the quadruplication of the cell areas at each decimation step).

To test the theoretical predictions, we have performed high-resolution magnetoinductance measurements on a periodic (triangular) array of fourth-order gaskets consisting of proximity-effect coupled Josephson junctions. At low temperatures, the asymptotic scaling of the data with frustration as well as the amplitude and the shape of the fine structure follow very closely the behavior predicted by theory. At higher temperatures, thermal fluctuations strongly enhance the fine structure, but have almost no influence on the mean-field scaling properties of $L(f)$ at low frustrations.

The paper is organized as follows. In Sec. II, we show that JJA and SWN behave as networks of inductances whose values are inversely proportional to the first derivative of the current-phase relation on the corresponding bond. In Sec. III, we describe the "direct method" to compute $L(f)$ using the triangle-star transformation. In Sec. IV, we introduce the concept of decimation and apply it to find the rules governing the scaling of the current-phase relation. In Sec. V, the decimation procedure is generalized to the case of a frustrated gasket and then used in a perturbative approach to find the asymptotic behavior of the inductance correction $\delta L(f)$. Magnetoinductance data are presented and discussed in Sec. VI.

II. INDUCTANCE OF NETWORKS AND ARRAYS

Quite generally, the inductance L of a conductor can be defined as a measure of the energy E required to drive a current I into the system:

$$E = \frac{L}{2} I^2. \quad (1)$$

This definition is still valid for nonlinear systems provided one considers only small currents. For a network of inductive elements $\{L_{jj'}\}$, Eq. (1) can be generalized to

$$E = \sum_{jj'} \frac{L_{jj'}}{2} I_{jj'}^2, \quad (2)$$

where the index j labels the nodes of the network and the sum runs over its links $\{jj'\}$. Current conservation at each node j requires

$$\sum_{j'} I_{jj'} = 0 \quad (I_{jj'} \equiv -I_{j'j}). \quad (3)$$

To calculate the total inductance of a network, one needs to know how the currents $\{I_{jj'}\}$, are distributed in the system for a given value of the total current I . This problem is completely equivalent to that of finding the current distribution in a resistor network. It can be solved by minimizing, for a given I , the total energy of the system (in a resistor network the corresponding quantity is the total energy loss) with the requirement of current conservation at each node. Thus, the calculation of the total inductance of an inductor network is entirely analogous to that of the total resistance of a resistor network.³ For a SG of order N (Fig. 1) consisting of identical resistances R it has been shown^{2,3} that the resistance R_N between the vertices of the gasket is given by

$$R_N = \left[\frac{5}{3} \right]^N \frac{2R}{3}, \quad (4)$$

$2R/3$ being the resistance of the zeroth-order gasket (simple triangle). Therefore, the inductance of a SG formed by identical inductances L is given by the same equation with R replaced by L .

Let us now consider a superconducting network (JJA or SWN) with arbitrary current-phase relation on each bond exposed to an applied perpendicular magnetic field B . Since at the low temperature of interest in our mean-field approach only the phase of the complex order parameter is a relevant degree of freedom, the energy of the system can be expressed by a Hamiltonian of the form

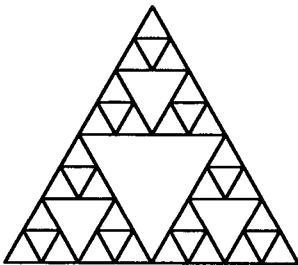


FIG. 1. Like other self-similar structures the Sierpinski gasket is defined by the recursive algorithm of its construction. A gasket of the order N is obtained by connecting three gaskets of order $(N-1)$ at their vertices. The construction starts from a simple triangle which plays the role of the zeroth-order gasket. The picture shows a third-order gasket.

$$H = \sum_{jj'} V_{jj'}(\varphi_j - \varphi_{j'} + A_{jj'}), \quad (5)$$

where φ_j is the phase of the order parameter in the superconducting grain (or node) at the site j , $A_{jj'}$ is proportional to the line integral of the vector potential along the path connecting j to j' , and $V_{jj'}(\theta)$ is an even periodic function of θ (with period 2π) depending on the form of the current-phase relation in the link jj' .

In the following, we shall neglect screening effects and assume that the $\{A_{jj'}\}$ are entirely determined by the vector potential of the external magnetic field B . Then, the sum of the $A_{jj'}$ along any closed path on the network is proportional to the area S enclosed by the path

$$\sum_{\square} A_{jj'} = \left[\frac{2\pi}{\phi_0} \right] BS, \quad (6)$$

where ϕ_0 is the superconducting flux quantum. If we introduce the notion of gauge-invariant phase difference $\theta_{jj'}$ across the bond jj' ,

$$\theta_{jj'} = \varphi_j - \varphi_{j'} + A_{jj'} \equiv -\theta_{j'j}, \quad (7)$$

condition (6) can be rewritten as

$$\sum_{\square} \theta_{jj'} = 2\pi \left[\frac{S}{S_0} f - m \right], \quad (8)$$

where f is the magnetic flux (in units of ϕ_0) threading some reference cell of area S_0 and m is an integer accounting for the fact that the phase is defined only modulo 2π . Equation (8) is nothing but a manifestation of fluxoid quantization in a multiply connected superconductor. In the gauge-invariant description it is natural to reduce the $\{\theta_{jj'}\}$ to the interval $-\pi < \theta_{jj'} < \pi$. Then, in the ground state the integers $\{m\}$ can only be zero or positive (for positive f) and can be interpreted as the number of vortices penetrating a given cell.

In networks of thin superconducting wires the periodicity of $V(\theta)$ results from phase-slip processes which are bound to occur if θ tries to escape from the interval $-\pi < \theta < \pi$. This allows one to describe both JJA and SWN by Hamiltonians having the same structure and differing only in the functional dependence of the periodic interaction V on θ .

Variation of Eq. (5) with respect to φ up to second order in $\delta\varphi$ gives

$$\delta E = \frac{1}{2} \sum_{jj'} V_{jj'}''(\theta_{jj'}) (\delta\varphi_j - \delta\varphi_{j'})^2, \quad (9)$$

where the contribution involving the linear terms vanishes because of current conservation at the nodes [Eq. (3)]. If we now relate $\delta\varphi_j$ to the variation $\delta I_{jj'}$ of the current $I_{jj'} = -(2e/\hbar)V_{jj'}'(\theta_{jj'})$ through the link jj' ,

$$\delta I_{jj'} = -(2e/\hbar)V_{jj'}''(\theta_{jj'}) (\delta\varphi_j - \delta\varphi_{j'}),$$

we find that δE can be expressed as

$$\delta E = \frac{1}{2} \sum_{jj'} \left[\frac{\hbar}{2e} \right]^2 \frac{1}{V_{jj'}''(\theta_{jj'})} (\delta I_{jj'})^2. \quad (10)$$

Comparison of Eq. (10) with Eq. (2) shows that, as seen from an external current source, JJA and SWN behave as networks of inductances whose values are given by

$$L_{jj'} = \left[\frac{\hbar}{2e} \right]^2 \frac{1}{V_{jj'}''(\theta_{jj'})}. \quad (11)$$

For a JJA the interaction functions $V(\theta)$ is determined by the energy-phase relation of a single junction and is of the form

$$V(\theta) = J(1 - \cos\theta), \quad (12)$$

where J is the Josephson coupling energy, the current across the junction being $I = (2e/\hbar)J \sin\theta$. It follows that for a JJA exposed to a perpendicular magnetic field the effective inductance of the junction jj' is given by

$$L_{jj'} = \left[\frac{\hbar}{2e} \right]^2 \frac{1}{J \cos(\theta_{jj'})}. \quad (13)$$

Thus, even if all the junctions are identical, their effective inductances in the frustrated system may differ substantially from each other on account of the nonuniform distribution of the $\{\theta_{jj'}\}$.

On the other hand, in SWN the energy of a link (a piece of superconducting wire) is almost harmonic in θ , the anharmonic corrections being of the order of $(\xi/a)^2$, where ξ is the coherence length and a the length of the link.¹⁷ Accordingly, in SWN the inductance modulation by a magnetic field should be much less pronounced than in JJA unless a becomes comparable to ξ . For simplicity, in the following we shall systematically omit the factor $(\hbar/2e)^2$ which should appear in all the explicit expressions of the inductance.

III. DIRECT CALCULATION OF THE INDUCTANCE OF SIERPINSKI GASKETS AND OF SIERPINSKI GASKET ARRAYS

In an ideal SG (Fig. 1) the area of the large cell located in the center of a gasket of order N is given by $4^{N-1}S_0$, where S_0 is the area of the smallest triangular cell. Since all the cell areas are multiples of S_0 , it is convenient to use the magnetic flux f threading the smallest cell as a measure of the frustration of the system. From the structure of the Hamiltonian (5), it is clear that H is invariant with respect to changes in the flux of each cell corresponding to an integer number of flux quanta and also to changes in the sign of the magnetic field. Therefore, it is sufficient to focus only on the interval $0 \leq f \leq \frac{1}{2}$.

For a particular set of frustrations, the structure of the ground state of the infinite gasket can be found by studying a finite gasket. We start with a gasket of order N at frustration f and assume that its ground state is symmetric (in terms of the gauge-invariant variables $\{\theta_{jj'}\}$) with respect to rotations of the system. Then, the sum Θ of the $\{\theta_{jj'}\}$ along each side of the gasket is given by

$$\Theta = \frac{2\pi}{3}(4^N f - M),$$

where M is the sum of the variables $\{m\}$ in Eq. (8) and can be interpreted as the total number of vortices thread-

ing the gasket. Now, suppose one tries to construct the ground state of the gasket of order $(N+1)$ by juxtaposition of three N th-order gaskets. One can then easily verify (see Fig. 2 for an illustration) that the constraint (8) imposed to the central cell of the new gasket is fulfilled only if $2 \cdot 4^N f$ is an integer, that is only if

$$f = \frac{P}{2 \cdot 4^N}, \quad (14)$$

where P is an integer.

For a SWN with a linear current-phase relation it has been shown by Ceccatto *et al.*³ that the ground state of a second-order gasket is always symmetric with respect to rotations of the system. However, even if the ground state of an N th-order gasket (with $N > 2$) were not symmetric at some frustration and, consequently, the variables $\{\Theta\}$ were different on the three sides of the gasket, the condition

$$\Theta_1 + \Theta_2 + \Theta_3 = 2\pi(4^N f - M)$$

should still be satisfied. Moreover, for frustrations satisfying Eq. (14) the juxtaposition of three gaskets with non-symmetric ground states (see Fig. 2) can be always performed in such a way that the constraint (8) will be also satisfied on the central loop of the resulting system. This means that for frustrations obeying Eq. (14) the condition (8) will be automatically fulfilled on the central loop of a gasket generated by repeated juxtapositions. This implies that the ground state of an infinite gasket will consist of "replicas" of the same state of a finite gasket. Thus, the structure of the ground state of an infinite gasket at frustrations given by Eq. (14) can be found by determining the ground state of a finite gasket. Having shown that the network inductance corresponding to any state is completely determined by the distribution of the $\{\theta_{jj'}\}$, we conclude that for frustrations satisfying Eq. (14) the problem of calculating the inductance reduces to a calcu-

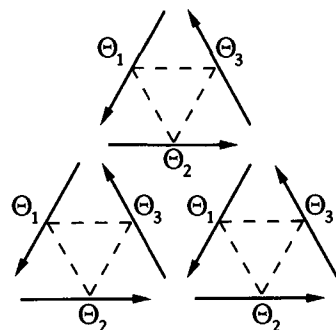


FIG. 2. In constructing a gaskets of order N by juxtaposition of three gaskets of order $(N-1)$, the arrows associated with the gauge-invariant phase differences on the central loop of the resulting system turn out to rotate in a direction opposite to that of the other loops. From the constraints imposed by fluxoid quantization [Eq. (8)] in the different loops it follows that the configuration shown in the figure is possible only if $2(\Theta_1 + \Theta_2 + \Theta_3)$ is a multiple of 2π . Obviously, this conclusion is still valid if the variables Θ_i ($i=1,2,3$) are equal to each other.

lation on a finite gasket and can be solved exactly once the structure of its ground state is known.

As an illustration, let us calculate the inductance of a SG at $f = \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}$. In this case it is sufficient to consider a first-order gasket formed by nine inductive elements whose inductances can take only two different values, L_1 and L_2 (Fig. 3). The current distribution in this simple network is easily determined so that, in terms of L_1 and L_2 , its total inductance (for a current entering at one vertex and leaving at another one) can be written as

$$L = \frac{2L_1}{3} \frac{3L_1 + 2L_2}{2L_1 + L_2}. \quad (15)$$

For a JJA the values of L_1 and L_2 at $f = \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}$ follow from Eq. (13) by substituting the values of $\theta_{jj'}$ deduced from simple symmetry considerations. For instance, for $f = \frac{1}{4}$ one has $\theta_{jj'} = \pi/2$ on the three internal bonds and accordingly $L_2 = \infty$, whereas $\theta_{jj'} = 0$ on the six external bonds and therefore $L_1 = 1$ (in units of $1/J$). Then, using Eq. (15), one obtains

$$\frac{L(1/4)}{L(0)} = \frac{6}{5}.$$

The situation is even simpler for $f = \frac{1}{2}$, since $\theta_{jj'} = \pi/3$ on all the bonds and therefore

$$\frac{L(1/2)}{L(0)} = 2.$$

To eliminate the trivial dependence on the gasket size, inductances are conveniently normalized to $L(0)$, the value for the unfrustrated system.

For frustrations f expressed by an irreducible fraction with denominators 16 or 32 it is sufficient to consider a second-order gasket. In this case one has first to solve a system of equations (one of which is, in general, non-linear) in order to find the structure of the ground state at the frustration of interest and then to calculate the inductance of a network of 27 inductors having five different values which is rather cumbersome. Considerable simplification of the second step of this program can be achieved by making use of the so-called triangle-star transformation which is well known in the theory of

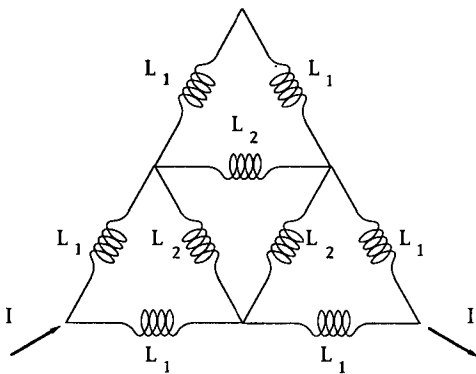


FIG. 3. To calculate the inductance of a Sierpinski gasket at frustrations corresponding to multiples of $f = \frac{1}{8}$, it is sufficient to consider a first-order gasket constructed with only two inductances L_1 and L_2 .

resistive networks.¹⁶ An elementary step of this transformation is shown in Fig. 4(a). The triangle formed by the inductances L_1 , L_2 , and L_3 can be replaced by the star formed by the inductances

$$l_i = \frac{L_1 L_2 L_3}{(L_1 + L_2 + L_3) L_i}, \quad (16)$$

where $i = 1, 2, 3$. The inverse transformation is given by

$$L_i = \frac{l_1 l_2 + l_2 l_3 + l_3 l_1}{l_i}. \quad (17)$$

For an N th-order gasket, $(N+1)$ successive applications of the triangle-star transformation (16) reduce the gasket [as shown in Fig. 4(b)] to a simple "star" whose total inductance (between the star vertices) can be trivially calculated, the inductive elements being now connected in series.

In Fig. 5, we present numerical calculations of the normalized ground-state energy (per elementary bond) and inverse inductance of a JJA with cosinusoidal interaction [Eq. (12)]. The calculations were performed on a third-order gasket at multiples of $f = \frac{1}{128}$ [Eq. (14)]. According to our previous discussion, these results are valid also for gaskets of larger order ($N > 3$), including the infinite SG.

In the experiments discussed later on in this paper (Sec. VI) the *sheet inductance* of a regular triangular lattice of identical gaskets of order N' connected at the vertices was investigated. For frustrations satisfying Eq. (14) with $N \leq N'$ the structure of the ground state of such a system is also determined by the structure of the ground state of a single gasket of order N . Since the sheet inductance of a regular gasket array is proportional to the inductance of its constituent gaskets, the results of the calculation performed on the third-order gasket (Fig. 5) are also valid for the normalized sheet inductance of the corresponding composite periodic system.

Although the direct inductance calculation presented in this section can in principle be performed for a gasket of arbitrary (finite) size, the problem of finding the structure of the ground state becomes more and more complex with increasing gasket order. In fact, the number of equations to be solved to determine the structure of a

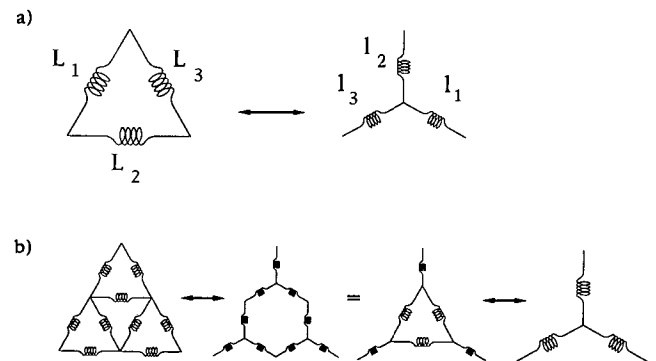


FIG. 4. (a) The triangle-star transformation and (b) its application to the calculation of the inductance of a first-order gasket.

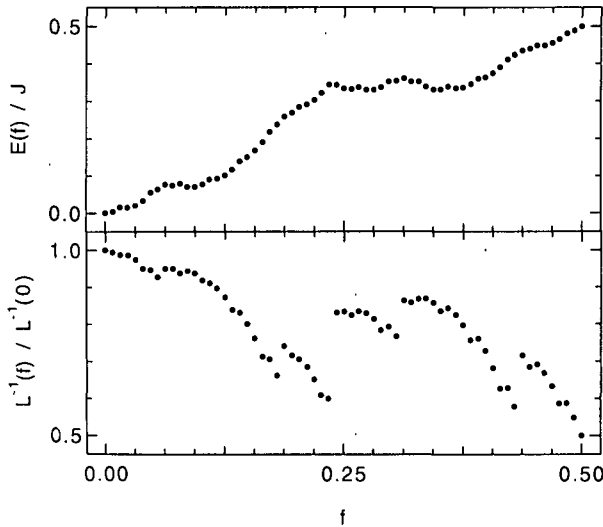


FIG. 5. Normalized ground-state energy (per bond) and inverse inductance of a Sierpinski gasket of Josephson junctions with a sinusoidal current-phase relation at frustrations corresponding to multiples of $f = \frac{1}{128}$.

symmetric ground state on an N th-order gasket is equal to $(3^N + 1)/2$, an expression showing very clearly why calculations become rapidly tedious with increasing N .

Even more significantly, numerical calculations prevent any insight into the asymptotic behavior of the inductance at small frustrations. To study the scaling properties of the inductance at small f , it is necessary to develop methods which take advantage of the self-similar nature of the SG. Such methods will be introduced in the next two sections.

IV. SCALING OF THE NONLINEAR CORRECTIONS TO THE CURRENT-PHASE RELATION

Like other fractals, the SG has a hierarchical self-similar structure. A second-order gasket can be thought of as a first-order gasket consisting of three first-order gaskets and so forth. This suggests that it should be possible to construct recursive relations allowing to express the properties of a given gasket in terms of those of a gasket of smaller order. In this section and in the following one we systematically develop such an approach and apply it to explore the asymptotic behavior of different quantities, including the frustration-dependent corrections of the inductance. For simplicity, in this section we start with the analysis of the unfrustrated system.

Let us consider a first-order SG. In the absence of a magnetic field it is described by the Hamiltonian

$$H = \sum_{\langle ij \rangle} V(\varphi_i - \varphi_j), \quad (18)$$

where the sum is taken over the nine pairs of nearest neighbors (Fig. 6). In the following we shall assume that, even if the gasket under consideration is part of a larger system (for instance, of a gasket of higher order or of a regular triangular lattice of first-order gaskets), it can be connected to the rest of the system only by its vertices (sites 1, 2, and 3 in Fig. 6). Therefore, since we are deal-

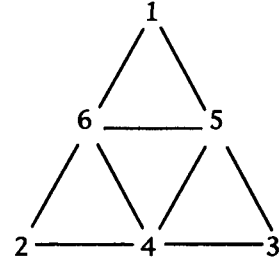


FIG. 6. Labeling of the nodes of the first-order gasket on which the decimation procedure is carried out. After decimation only the nodes 1, 2, and 3 survive.

ing with pairwise interactions, only the site variables φ_1 , φ_2 , and φ_3 can interact with the rest of the system whereas φ_4 , φ_5 , and φ_6 only interact with φ_1 , φ_2 , and φ_3 . This means that, if we find the values of φ_4 , φ_5 , and φ_6 which minimize the energy of the first-order gasket for given values of φ_1 , φ_2 , and φ_3 , we can then express its energy as a function of φ_1 , φ_2 , and φ_3 only. This procedure, which amounts to replacing the original first-order gasket by a zeroth-order gasket (an elementary triangle), is called decimation and can be applied to all the first-order gaskets composing a larger system. For example, decimation will transform an N th-order gasket into an $(N-1)$ th-order gasket.

For the case of a pure harmonic interaction

$$V(\theta) = \frac{J}{2} \theta^2 \quad (19)$$

the minimization of the energy of the first-order gasket with respect to the "internal" variables φ_4 , φ_5 , and φ_6 can be performed exactly.¹² Variation of Eq. (18) with respect to φ_4 , φ_5 , and φ_6 results in a system of three linear equations whose solution is

$$\begin{aligned} \varphi_4 &= \frac{\varphi_1 + 2\varphi_2 + 2\varphi_3}{5}, \\ \varphi_5 &= \frac{2\varphi_1 + \varphi_2 + 2\varphi_3}{5}, \\ \varphi_6 &= \frac{2\varphi_1 + 2\varphi_2 + \varphi_3}{5}. \end{aligned} \quad (20)$$

Then, substitution of Eq. (20) into the Hamiltonian leads to an expression of the form

$$H = V_R(\varphi_1 - \varphi_2) + V_R(\varphi_2 - \varphi_3) + V_R(\varphi_3 - \varphi_1) \quad (21)$$

showing that the energy is a sum of contributions associated with pairs of nearest neighbors on the decimated gasket. The renormalized interaction $V_R(\theta)$ differs from the original one [Eq. (19)] only in the coupling constant which is renormalized according to the rule¹²

$$J \Rightarrow J_R = \frac{3}{5} J.$$

To illustrate how decimation works in a specific system, let us consider a triangular lattice of N th-order gaskets as that studied in the experiments described in Sec. VI. By applying N times the decimation process, one

is finally left with a regular triangular lattice with a renormalized coupling constant

$$J_R^{(N)} = \left(\frac{2}{3}\right)^N J. \quad (22)$$

Since the inductance is inversely proportional to the coupling constant, Eq. (22) implies that the sheet inductance of the system scales with N as $\left(\frac{2}{3}\right)^N$. This agrees with the conclusion reached in Sec. II [Eq. (4)].

In real systems the interaction function is not, in general, harmonic. Let us explore what happens if we add the next-order term to the harmonic approximation:

$$V(\theta) = \frac{J}{2}\theta^2 - \frac{K}{24}\theta^4. \quad (23)$$

Differentiation of $V(\theta)$ shows that K sets the amplitude of the first nonlinear correction to the current-phase relation:

$$I(\theta) \propto \frac{dV}{d\theta} = J\theta - \frac{K}{6}\theta^3.$$

The equations for φ_4 , φ_5 , and φ_6 obtained by variation of the Hamiltonian (18) with the interaction (23) are nonlinear and can be solved only perturbatively, the first correction to the unperturbed solution [Eq. (20)] being of the form

$$\begin{aligned} \delta\varphi_4 &= (K/J)F(\varphi_1 - \varphi_3, \varphi_2 - \varphi_1, \varphi_3 - \varphi_2), \\ \delta\varphi_5 &= (K/J)F(\varphi_2 - \varphi_1, \varphi_3 - \varphi_2, \varphi_1 - \varphi_3), \\ \delta\varphi_6 &= (K/J)F(\varphi_3 - \varphi_2, \varphi_1 - \varphi_3, \varphi_2 - \varphi_1), \end{aligned} \quad (24)$$

where the function $F(\theta_1, \theta_2, \theta_3) \equiv F(\theta_2, \theta_1, \theta_3)$ is a polynomial of the third degree whose coefficients do not depend on the coupling constants J and K . Since the values of φ_i (with $i=4,5,6$) given by Eq. (20) are those minimizing the harmonic part of the Hamiltonian, substitution of $\varphi_i + \delta\varphi_i$ into this part of H only produces second-order corrections in K . First-order corrections in K only appear by substituting the solution into the fourth-order term of H and merely depend on the unperturbed solution (20). Thus, if we restrict our attention only to first-order terms in K , it is possible to derive the renormalized Hamiltonian simply by substituting Eq. (20) into H without knowing the explicit form of $F(\theta_1, \theta_2, \theta_3)$. It turns out that the renormalized Hamiltonian has the same form as the original one, but with the renormalized coupling constants:

$$J_R = \frac{2}{3}J, \quad K_R = \frac{99}{625}K. \quad (25)$$

Remarkably, the form of the fourth-order term remains the same, thereby leaving the pairwise nature of the interaction between the phases unaltered.

The perturbative treatment outlined above shows that a single decimation step of the SG reduces the relative amplitude $\gamma = K/J$ of the nonlinear term in the current-phase relation by a factor of $\frac{33}{125}$ which is significantly smaller than one. Therefore, even if the perturbative approach is not fully justified at the beginning of the decimation process, it becomes quantitatively correct after a few decimation steps.

Contributions to the Hamiltonian of higher order in θ are expected to decay with decimation even faster than the fourth-order term. For instance, it can be shown that, if two of the three variables φ_1 , φ_2 , and φ_3 are equal, the term of order $2n$ in $V(\theta)$ scales with decimation by a factor of

$$\lambda_{2n} = \frac{3^{2n} + 2^{2n} + 2}{5^{2n}} \quad (26)$$

which monotonically decreases with increasing n . Notice that for $n=1$ and $n=2$, Eq. (26) correctly predicts the scaling factors we found before [Eq. (25)] within the framework of the more general treatment.

Since $V(\theta)$ is periodic in θ , the fast decrease of the anharmonic corrections predicted by the perturbative treatment implies that, even if one starts from a cosinusoidal interaction [Eq. (12)], the shape of the effective interaction evolves with decimation towards a piecewise parabolic function which is significantly more "rigid" against large-amplitude thermal fluctuations than the cosinusoidal form. This means that in a triangular lattice consisting of periodically repeated gaskets of Josephson junctions fluctuations will be less relevant than in a triangular JJA with a coupling constant equal to the renormalized coupling constant of the gasket array [Eq. (22)].

As can be seen from Eq. (11), the nonlinear corrections to the current phase relation do not affect the inductance of an unfrustrated system. On the other hand, at nonvanishing frustrations currents are flowing in the system even in the absence of an external current. It is precisely the presence of nonharmonic terms in the Hamiltonian which provides the coupling between the externally driven and the internal currents. Since a single decimation step corresponds to a change in the effective frustration by a factor of 4, we expect that for $f \rightarrow 0$ the frustration-induced corrections to the inductance due to the presence of nonlinear terms in the current-phase relation should scale as f^ν with

$$\nu = \frac{\ln(125/33)}{\ln 4} \approx 0.96, \quad (27)$$

where the fraction $\frac{33}{125}$ reflects the scaling of the inductance corrections one expects at every decimation step. In the next section we generalize the decimation procedure to the case of a frustrated gasket and show that the power-law asymptotic behavior of the inductance corrections corresponds indeed to the exponent given by Eq. (27).

V. DECIMATION OF THE GASKET IN FRUSTRATED SYSTEMS AND ASYMPTOTIC BEHAVIOR OF THE INDUCTANCE

A. Transformation of the harmonic Hamiltonian

At nonvanishing frustrations the system can be described in terms of the gauge-invariant bond variables $\theta_{jj'} \equiv -\theta_{j'j}$ [Eq. (7)]. Once more, it is convenient to consider a first-order SG described by the Hamiltonian

$$H = \sum_{\langle ij \rangle} V(\theta_{jj'}) , \quad (28)$$

where the sum runs again over the nine pairs of nearest neighbors (Fig. 6). The $\{\theta_{jj'}\}$, however, should now obey constraints of the form (8):

$$\begin{aligned} \theta_{16} + \theta_{65} + \theta_{51} &= 2\pi f, & \theta_{24} + \theta_{46} + \theta_{62} &= 2\pi f, \\ \theta_{35} + \theta_{54} + \theta_{43} &= 2\pi f, & \theta_{45} + \theta_{56} + \theta_{64} &= 2\pi f. \end{aligned} \quad (29)$$

In writing these expressions we have assumed that there are no vortices inside the cells of the gasket [i.e., $m=0$ in Eq. (8)], which is certainly correct in the limit of low frustrations in which we are interested here. After decimation, it is necessary to introduce a set of new $\{\theta_{jj'}\}$ defined on the bonds of the decimated (zeroth-order) gasket and related to the old variables by (see Fig. 6):

$$\begin{aligned} \theta_{12} &= \theta_{16} + \theta_{62}, \\ \theta_{23} &= \theta_{24} + \theta_{43}, \\ \theta_{31} &= \theta_{35} + \theta_{51}. \end{aligned} \quad (30)$$

The new θ variables should also satisfy a constraint of the form (8):

$$\theta_{12} + \theta_{23} + \theta_{31} = 2\pi f_R ,$$

where we have introduced the "renormalized" frustration $f_R = 4f$.

In the harmonic approximation (19) the minimization of the Hamiltonian (28) under the constraints (29) and (30) can be performed exactly and leads to

$$\begin{aligned} \theta_{16} &= \frac{5\theta_{12} + \theta_{23} - \theta_{31}}{10}, \\ \theta_{65} &= \frac{-3\theta_{12} + \theta_{23} - 3\theta_{31}}{20}, \\ \theta_{51} &= \frac{-\theta_{12} + \theta_{23} + 5\theta_{31}}{10} \end{aligned} \quad (31)$$

and to analogous expressions for the other six bond variables of the original (first-order) gasket which follow from Eq. (31) by cyclic permutations of the indices on the rhs. Then, substitution of the solution in the harmonic Hamiltonian gives

$$\sum_{\langle ij' \rangle} \frac{J}{2} \theta_{jj'}^2 = \frac{J}{2} \left[\frac{3}{5} (\theta_{12}^2 + \theta_{23}^2 + \theta_{31}^2) - \frac{1}{5} (2\pi f)^2 \right]. \quad (32)$$

This expression exhibits the same scaling factor ($\frac{3}{5}$) of the coupling constant which was obtained in the previous section for the unfrustrated system. It also shows that, if $V(\theta)$ is harmonic, the presence of frustration does not change the functional form of the renormalized Hamiltonian (for a given frustration the second term on the rhs of Eq. (32) simply causes an overall shift of the energy). Since the inductive properties of the system are determined by the second derivative of the energy [see Eq. (11)] and are therefore insensitive to a global energy shift, this result confirms our previous conclusion that the inductance of a gasket characterized by a purely harmonic interaction is independent of frustration.

B. Dependence of the ground-state energy on frustration

Decimation transforms a gasket of order N at frustration f into a gasket of order $(N-1)$ at frustration $f_R = 4f$. For f low enough for no vortices to penetrate the first-order gaskets forming the N th-order gasket, Eq. (32) allows one to relate the ground-state energies of the two systems without knowing the structure of their ground states. Moreover, since an infinite SG is self-similar, we can also use Eq. (32) to find a relation between the ground-state energies of an infinite gasket at frustration differing from each other by a factor of 4:

$$E(f) = \frac{1}{5} \left[E(4f) - \frac{J}{6} (2\pi f)^2 \right], \quad (33)$$

where the rhs has been multiplied by a factor of $\frac{1}{5}$ to account for the ratio between the number of bonds in the decimated gasket and in the original one.

Iteration of the recursive relation (33) leads to the following expression for the ground-state energy of a gasket characterized by a linear current-phase relation:

$$\begin{aligned} E(f/4^N) &= \left[E(f) - \frac{2\pi^2 J f^2}{33} \right] \left[\frac{1}{5} \right]^N \\ &+ \frac{2\pi^2 J f^2}{33} \left[\frac{1}{16} \right]^N, \end{aligned} \quad (34)$$

where $0 < f < \frac{1}{2}$. The dependence on N in Eq. (34) shows that, for $f \rightarrow 0$, $E(f)$ obeys a power law, $E(f) \propto f^{\nu_E}$, with the exponent

$$\nu_E = \frac{\ln 5}{\ln 4} \approx 1.16 \quad (35)$$

and allows one to appreciate how fast the asymptotic regime is actually reached.

The value of the exponent ν_E was first calculated by Alexander and Halevi⁵ by studying an expression for an upper bound to $E(f)$. In contrast to their work, Eq. (34) is an exact result. The structure of Eq. (34) implies that in a log-log plot $E(f)$ should look, in the limit $f \rightarrow 0$, as a tilted periodic function with a slope corresponding to the exponent (35). In the work of Meyer *et al.*¹³ it was erroneously assumed that the same exponent describes the asymptotic scaling behavior of the frustration-dependent corrections to the inductance.

C. Transformation of the θ -dependent inductance corrections with gasket decimation

Let us now come back to a first-order gasket. At low frustrations (no vortices) the gauge-invariant phase differences $\{\theta_{jj'}\}$ are small on all the (nine) bonds. As a consequence, the corresponding inductances $\{L_{jj'}\}$ are almost equal and we can write

$$L_{jj'} = L_0 + \delta L_{jj'}, \quad |\delta L_{jj'}| \ll L_0 .$$

To calculate the small inductance corrections $\{\delta L_{jj'}\}$ caused by frustration, we rely on a perturbative approach which takes advantage of the decimation scheme. A

two-stage application of the triangle-star transformation [Fig. 4(b)] and, subsequently, of its inverse (star-triangle transformation) replaces the first-order gasket by a triangle (zeroth-order gasket) made up by the inductances:

$$\begin{aligned} L_{12} &= \frac{5}{3}L_0 + \frac{1}{9}[6(\delta L_{16} + \delta L_{62}) + 3(\delta L_{51} + \delta L_{24}) \\ &\quad - 2(\delta L_{43} + \delta L_{35}) + \delta L_{54}], \\ L_{23} &= \frac{5}{3}L_0 + \frac{1}{9}[6(\delta L_{24} + \delta L_{43}) + 3(\delta L_{62} + \delta L_{35}) \\ &\quad - 2(\delta L_{51} + \delta L_{16}) + \delta L_{65}], \\ L_{31} &= \frac{5}{3}L_0 + \frac{1}{9}[6(\delta L_{35} + \delta L_{51}) + 3(\delta L_{43} + \delta L_{16}) \\ &\quad - 2(\delta L_{62} + \delta L_{24}) + \delta L_{46}], \end{aligned} \quad (36)$$

where only first-order terms in δL were retained and the two-index notation is the same as the one introduced in Sec. V A (see Fig. 6).

In the perturbative treatment it is sufficient to keep only the first nonharmonic term in the interaction function [as in Eq. (23)]. Then, by expanding Eq. (11) to first order in K , the inductance of each bond *before* decimation can be related to the corresponding gauge-invariant phase difference θ by

$$L(\theta) = \frac{1}{J} + \delta L(\theta), \quad \delta L(\theta) \approx \frac{K}{2J^2} \theta^2. \quad (37)$$

Since we know how the $\{\theta_{jj'}\}$ transform in the decimation process [Eq. (31)], we can apply Eq. (36) to find the inductance *after* decimation without actually knowing how the $\{\theta_{jj'}\}$ are distributed in the ground state. In fact, by substituting Eq. (37) into Eq. (36) and using Eq. (31), it turns out that the inductance $L^{(1)}$ of each bond of the decimated gasket can be expressed as a function of the corresponding gauge-invariant phase difference θ as

$$L^{(1)}(\theta) = \frac{5}{3J} [1 + a_1 - b_1 \theta + \frac{c_1}{2} \theta^2], \quad (38)$$

where

$$\begin{aligned} a_1 &= \frac{161}{750} (2\pi f)^2 \gamma, \\ b_1 &= \frac{67}{375} (2\pi f) \gamma, \\ c_1 &= \frac{33}{125} \gamma, \end{aligned} \quad (39)$$

and $\gamma = K/J$ is the ratio of the unrenormalized coupling constants in Eq. (23).

Let us now turn to larger gaskets, as we did in the previous subsection. Having demonstrated that the inductance of each bond of a decimated (zeroth-order) gasket only depends on the value of the corresponding θ , we can iterate the decimation procedure as long as the renormalized frustration is low enough for no vortices to penetrate the constituent first-order gaskets participating in the decimation process. If the decimation procedure is repeated N times, we get

$$L^{(N)}(\theta) = \left[\frac{5}{3} \right]^N \frac{1}{J} \left[1 + a_N - b_N \theta + \frac{c_N}{2} \theta^2 \right], \quad (40)$$

where

$$\begin{aligned} a_N(f_R) &= \left[\frac{35461}{703638} \left(\frac{33}{125} \right)^N + \frac{1139}{2619} \left(\frac{7}{100} \right)^N - \frac{10561}{21762} \left(\frac{1}{16} \right)^N \right] (2\pi f_R)^2 \gamma, \\ b_N(f_R) &= \frac{67}{291} \left[\left(\frac{33}{125} \right)^N - \left(\frac{7}{100} \right)^N \right] (2\pi f_R) \gamma, \\ c_N(f_R) &= \left(\frac{33}{125} \right)^N \gamma, \end{aligned} \quad (41)$$

and $f_R = 4^N f$ is the renormalized frustration. For $N=1$, Eq. (41) reduces to Eq. (39), as it should.

Notice that on the decimated gasket $L(\theta)$ is no longer an even function of θ . Since $L(\theta)$ is related to $V(\theta)$ by Eq. (11), it immediately follows that the renormalized interaction function $V_R(\theta)$ also loses its inversion symmetry. This is quite natural in a frustrated system where the vector character of the magnetic field enters the decimation procedure. As a consequence, the sign of the linear term in $L(\theta)$ is determined by the direction of the field.

D. Asymptotic behavior of the inductance at small frustrations

In the previous subsection we have shown that the problem of calculating the inductance corrections at low frustrations for a given gasket can be reduced to an identical problem at higher frustrations for a gasket of lower order where the distribution of the $\{\theta_{jj'}\}$ in the ground state may be already known. For instance, if we start from $f = 1/2^{2N+1}$ and make N decimations, the effective frustration f_R of the decimated gasket will be $f_R = \frac{1}{2}$. At this frustration the ground state corresponds to $\theta = \pi/3$ on all the bonds of the gasket (see Sec. III), so that the inductance change $\delta L(f) \equiv L(f) - L(0)$ at $f = 1/2^{2N+1}$ can be easily calculated by setting $\theta = \pi/3$ in Eq. (40):

$$\frac{\delta L(f)}{L(0)} = \left[\frac{3425}{117273} \left(\frac{33}{125} \right)^N + \frac{1340}{2619} \left(\frac{7}{100} \right)^N - \frac{10561}{21762} \left(\frac{1}{16} \right)^N \right] \pi^2 \gamma, \quad (42)$$

where $\gamma = 1$ for a JJA with a cosinusoidal interaction [Eq. (12)], whereas for a SWN γ will be smaller (or even much smaller) than one. For $N=0$ ($f = \frac{1}{2}$) the numerical coefficient within the brackets on the rhs of Eq. (42) reduces to $\frac{1}{18}$ and for $N=1$ ($f = \frac{1}{8}$) to $\frac{19}{1440}$.

Quite similarly, if start from $f = 1/2^{2N+2}$ and perform again N decimations, we end up with a gasket with an effective frustration $f_R = \frac{1}{4}$. At this frustration the ground state corresponds to $\theta = \pi/2$ on the three internal and to $\theta = 0$ on the six external bonds of each constituent first-order gasket (see Sec. III). To calculate the relative inductance correction $\delta L(f)/L(0)$, we expand Eq. (15) for the inductance of a first-order gasket to first order in the corrections δL_i ($i=1,2$, see Fig. 3) and substitute Eqs. (40) and (41) obtaining, for $f = 1/2^{2N+2}$,

$$\begin{aligned} \frac{\delta L(f)}{L(0)} &= \frac{14}{15} \frac{\delta L(\theta=0)}{L} + \frac{1}{15} \frac{\delta L(\theta=\pi/2)}{L} \\ &= a_N \left(\frac{1}{4} \right) - b_N \left(\frac{1}{4} \right) \frac{\pi}{30} + c_N \left(\frac{1}{4} \right) \frac{\pi^2}{120} \\ &= \left[\frac{10024}{586365} \left(\frac{33}{125} \right)^N + \frac{1474}{13095} \left(\frac{7}{100} \right)^N - \frac{10561}{87048} \left(\frac{1}{16} \right)^N \right] \pi^2 \gamma. \end{aligned} \quad (43)$$

for $N=0$ ($f=\frac{1}{4}$) the numerical coefficient on the rhs of Eq. (43) reduces to $\frac{1}{120}$ and for $N=1$ ($f=\frac{1}{16}$) to $\frac{3463}{720000}$. Analogous calculations can be performed also for other families of frustrations whose values scale as 4^N with the result that the final expressions for the relative inductance variation have the same form as Eqs. (42) and (43), but with different coefficients in front of the powers of the scaling factors.

The asymptotic behavior of $\delta L(f)$ in the limit $f \rightarrow 0$ is determined by the first term in Eqs. (42) and (43). Thus, for $f \rightarrow 0$ the frustration-induced inductance correction obeys a power law, $\delta L(f) \propto f^\nu$, with the same exponent

$$\nu = \frac{\ln(125/33)}{\ln 4} \approx 0.96 \quad (44)$$

we found earlier in Sec. IV [Eq. (27)]. We would like to stress that the leading terms in Eqs. (42) and (43) provide an exact asymptotic description (prefactors included) of $\delta L(f)/L(0)$ for the two specific frustration families considered above. It is for this reason that the numerical coefficients in these equations were expressed in their exact fractional form. It should also be noticed that our approach is not perturbative in the parameter γ , but in the inductance variations of the individual bonds of the original (undecimated) gasket which quickly fall off with decreasing frustration.

An alternative method to calculate $\delta L(f)$ within a perturbation scheme consists in substituting into the Hamiltonian $\{\theta_{jj'}\}$ of the form

$$\theta_{jj'} = \theta_{jj'}(f) + I\theta_{jj'}^I, \quad (45)$$

where $\{\theta_{jj'}(f)\}$ and $\{\theta_{jj'}^I\}$ are, respectively, the distribution of the gauge-invariant phase differences in the ground state at a given frustration f and the distribution of the additional "phase twists" induced by an external current I flowing through the system. If Eq. (45) is substituted into the harmonic part of the Hamiltonian, mixed terms will never show up in the energy of the system as this quantity cannot be linear in I . As a consequence, the current contribution to the energy resulting from the harmonic part of the Hamiltonian is independent of frustration. However, when Eq. (45) is substituted into higher-order (nonharmonic) terms of the Hamiltonian, the internal currents of the frustrated gasket couple to the external current I , thereby providing a method to calculate $\delta L(f)$. After computing the $\{\theta_{jj'}(f)\}$ and the $\{\theta_{jj'}^I\}$ in the harmonic approximation, we have explicitly performed such a calculation for the fourth-order term in the Hamiltonian at some frustrations ($f=\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$) finding expressions for $\delta L(f)/L(0)$ identical to those given by Eqs. (42) and (43). Thus, the two perturbative methods developed in this work are essentially equivalent. However, the approach based on the coupling between internal and external currents shows more clearly why the asymptotic behavior of $\delta L(f)$ at small frustrations is entirely determined by the fourth-order term in the Hamiltonian. On the other hand, the decimation procedure is more appropriate to calculate $\delta L(f)$ at arbitrarily small frustrations.

Our results in the asymptotic regime imply that on a

log-log plot $\delta L(f)/L(0)$ looks, at small f , like a tilted periodic function with a slope corresponding to the exponent ν given by Eq. (44). This is valid for both JJA and SWN, the only difference consisting in an overall downward shift of the curve for a SWN reflecting its lower value of γ . In order to estimate the amplitude of the oscillations, it is convenient to combine the decimation procedure with numerical calculations. For frustrations satisfying $f=P/2^{2N+1}$ with $5 \leq P \leq 16$ and $N > 2$ we first proceed to $(N-2)$ decimation steps to reduce the problem to a second-order gasket whose inductance is then calculated with the direct method developed in Sec. III using Eqs. (40) and (41) to define the inductances of the bonds. The result, expressed in terms of the quantity $\Delta L^{-1} \equiv [L^{-1}(0) - L^{-1}(f)]$ measuring the change in superconducting phase coherence caused by frustration, is shown (open circles) in log-log form in Fig. 7 where it is compared with that obtained from a direct numerical calculation, based on Eq. (13), performed on a third-order gasket (solid circles). In order to appreciate how fast the asymptotic behavior is reached, we have subtracted the linear background by plotting, instead of $\log_{10}[\Delta L^{-1}/L^{-1}(0)]$, $\log_{10}[\Delta L^{-1}/L^{-1}(0)] - \nu \log_{10} f$ vs $\log_4 f$ [notice that $\Delta L^{-1}/L^{-1}(0) = \delta L/L(0)$ in the asymptotic limit].

It is seen that already in the second "decade" (or "hierarchical stage"), i.e., in the interval $\frac{1}{32} < f < \frac{1}{8}$, the behavior is very close to its asymptotic form. However, in the first "decade" ($\frac{1}{8} < f < \frac{1}{2}$) it is rather different from that observed in the other hierarchical stages. This is readily understood if one realizes that in this frustration range the $\{\theta_{jj'}\}$ on some bonds can be of the order of one. As a consequence, the exact value of the corresponding inductance, as given by Eq. (13), differs substantially from the approximation (37) which serves as a base for the

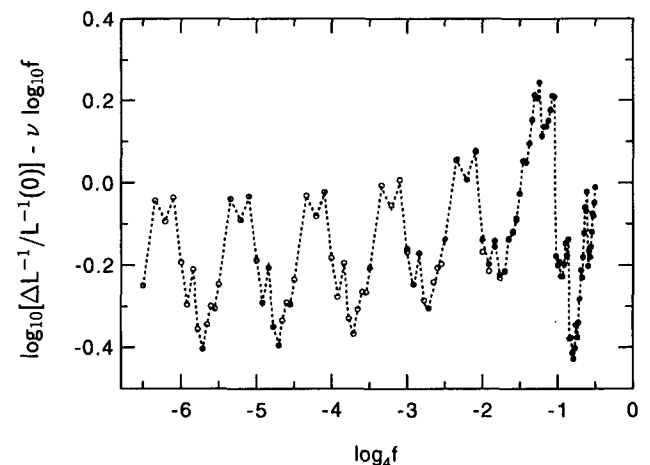


FIG. 7. log-log plot of the relative change of the inverse inductance of a Sierpinski gasket of Josephson junctions ($\gamma=1$) as a function of frustration (the linear background $\nu \log_{10} f$ has been subtracted). Solid circles: result of a direct numerical calculation performed on a third-order gasket. Open circles: result of a calculation based on the perturbative approach in which the problem is reduced to a calculation on a second-order gasket after performing one or more decimation steps. This result becomes asymptotically exact for $f \rightarrow 0$. The line connecting the circles is simply a guide to the eye.

asymptotic calculation. It should also be noticed that, already in the second hierarchical stage, the results of the asymptotic approach are very close to those emerging from the direct numerical calculation, thereby providing strong support for the reliability of the perturbation treatment developed in this section.

The peculiar behavior of the inductance in the first “decade” justifies our reluctance to comment on the self-similarity of $L(f)$ in Fig. 5. In fact, Fig. 7 clearly shows that, in order to make the self-similar nature of the results manifest, the numerical calculations should be performed at least on fourth- or fifth-order gaskets.

VI. COMPARISON WITH EXPERIMENT

In the previous sections we have developed various theoretical methods to calculate the mean-field inductance of a SG as a function of frustration. In particular, we have shown (see Fig. 7) that the hierarchical self-similar nature of the gasket becomes clearly manifest only in the asymptotic limit of very low frustrations where the inductance change is predicted to scale with f according to a well-defined power law reflecting the fractal geometry of the gasket. In this section we compare the theoretical predictions with high-resolution magnetoinductance measurements performed on triangular arrays of periodically repeated gaskets of proximity-effect coupled Josephson junctions. Under favorable experimental conditions, we have been able to resolve up to four hierarchical stages in the gasket magnetoinductance, thereby probing the inductive properties of the system in the low-frustration limit of interest.

A. Experimental details

The sample studied in this work is shown in Fig. 8. It consists of fourth-order gaskets sitting on the sites of a 78×78 triangular lattice and connected to each other at the vertices. Each gasket is an array of $3^4 = 81$ SNS junctions consisting of superconducting (S) Pb islands proximity-effect coupled to each other by an underlying

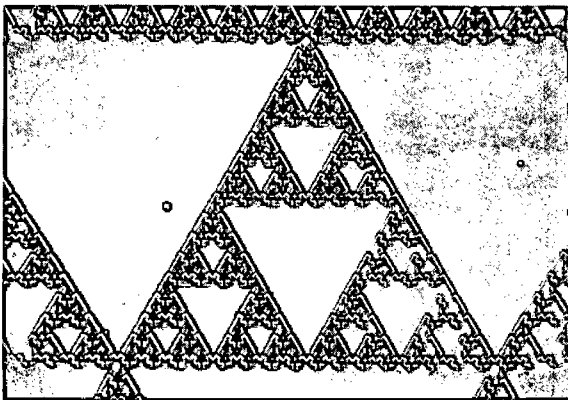


FIG. 8. Scanning electron micrograph showing a portion of a triangular array of periodically repeated fourth-order Sierpinski gaskets of proximity-effect coupled Pb/Cu/Pb Josephson junctions. The length of the elementary links of the gaskets is $8 \mu\text{m}$. Notice that, with the exception of those centered on the common vertices of three constituent gaskets, the superconducting Pb islands have an asymmetric “truncated-star” shape.

Cu normal (N) layer. Length and width of the elementary links of the gaskets are, respectively, $a = 8 \mu\text{m}$ and $w = 2 \mu\text{m}$. Since the ratio $D/\xi_n(T_{c0})$ between the junction gap D ($\approx 0.72 \mu\text{m}$) and the normal-metal coherence length $\xi_n(T_{c0})$ ($\approx 0.19 \mu\text{m}$) at the transition temperature T_{c0} ($\approx 7 \text{K}$) of the individual superconducting islands is of the order of 4, the system is expected to behave as a genuine JJA with a sinusoidal current-phase relation over a relatively wide temperature range, $V(\theta)$ deviating significantly from Eq. (12) only at low temperature.¹⁷ Because of its 2D nature at length scales larger than the gasket size, the unfrustrated array ($f=0$) exhibits a BKT transition at $T_c = 5.93 \text{K}$.

Relying on a numerical inversion procedure, the sheet magnetoinductance of the gasket array was extracted from measurements of the mutual inductance change of a coaxially mounted drive-receive coil system due to the supercurrents flowing in the sample in response to a weak ac field.¹⁸ The rms flux created by the driving ac current in an elementary triangular cell located just underneath the coils did not exceed $10^{-4}\phi_0$. This low-level excitation ensured a linear response and, combined with a suppression of ambient magnetic fields to $\sim 1 \text{mG}$, allowed f to be tuned with a precision better than 10^{-4} . Data were taken at 160 Hz with a sensitive SQUID-operated ac bridge. The major factor limiting the inductance resolution during our swept-frustration impedance measurements was the low-frequency noise generated by the solenoid providing the dc magnetic field. Typically, we have been able to resolve inductance changes of the order of 10 pH. The resolution was found to be somewhat better near the superconducting transition where, on account of the weaker screening effect provided by the sample, the gradiometer configuration of the receive coil suppressed external flux noise more efficiently than at lower temperatures.

The periodic arrangement of finite-order gaskets studied in this work ensured sample homogeneity over the macroscopic length scales set by the diameter ($\approx 2 \text{mm}$) of the detection coil. This is essential for the analysis of the diamagnetic response based on the inversion procedure to apply.¹⁸ Moreover, the gasket order ($N=4$) is sufficiently high for the set of frustrations $f = P/2 \times 4^N = P/512$ [see eq. (14)] at which theory can be compared to experiment to be dense enough.

In the following, temperatures will be expressed in terms of the reduced temperature $\tau \equiv kT/J(T)$ relevant for the statistical mechanics of the system. The temperature-dependent Josephson coupling energy $J(T)$ was inferred from measurements of the “bare” sheet (kinetic) inductance $L(T) = (\hbar/2e)^2 (5/3)^N / \sqrt{3} J(T)$ of the unfrustrated array at temperatures well below T_c . In terms of τ the BKT transition occurs at $\tau_c \approx 0.23$ in good agreement with the theoretical prediction¹² $\tau_c = (3/5)^4 \tau_{c0}$ based on Eq. (22), $\tau_{c0} \approx 1.5$ being the reduced BKT transition temperature of a regular triangular lattice¹⁹ with the same coupling energy $J(T)$.

B. Results and discussion

The inverse sheet inductance L^{-1} of our gasket array is shown in Fig. 9(a) as a function of the frustration pa-

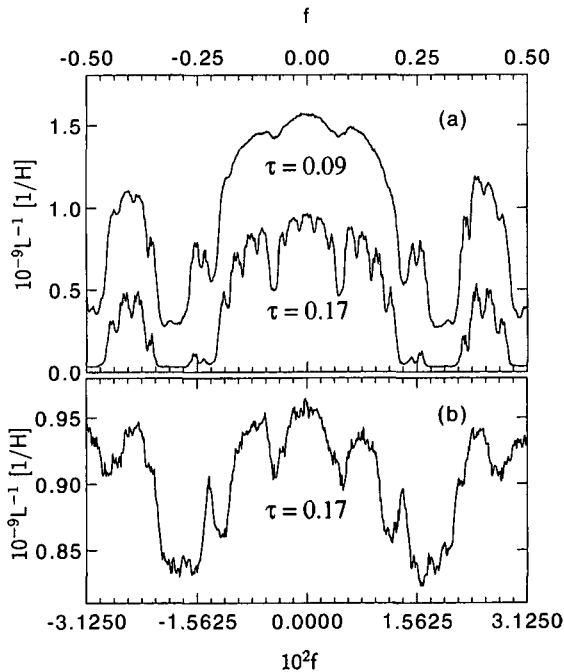


FIG. 9. (a) Inverse sheet inductance of the triangular lattice of fourth-order Sierpinski gaskets shown in Fig. 8 at two different reduced temperatures as a function of frustration. Both sets of data were taken at a driving frequency of 160 Hz. In (b) the flux axis has been expanded by a factor of 16 in order to highlight the scale-invariant behavior of the magnetoinductance resulting from the self-similar structure of the gaskets.

parameter f at two different reduced temperatures, $\tau=0.09$ ($T=5.51$ K) and $\tau=0.17$ ($T=5.81$ K). The data, periodic in f with period 1, exhibit a complex fine structure reflecting flux quantization in loops with a hierarchical distribution of sizes. This aspect is illustrated by comparing the inverse magnetoinductance data at $\tau=0.17$ of Fig. 9(a) with those shown in Fig. 9(b), where the flux axis was expanded by a factor of 16 to highlight the scale-invariant properties of the inductance resulting from the fractal geometry of the gaskets. The striking similarity of the two curves provides a clear demonstration of the self-similar structure of the gaskets, where the loop area S_h and, consequently, the magnetic flux scale as 4^{h-1} with the hierarchical index $h=1,2,\dots,N$ labeling the different families of loops.

Although the overall shape of the inverse magnetoinductance curves at $\tau=0.09$ and 0.17 is quite similar [Fig. 9(a)], the fine structure is found to become richer and sharper with increasing temperature suggesting that thermal fluctuations play a major role in the description of superfluid and vortex dynamics. Similar behavior was observed also in wire networks of interconnected gaskets¹³ and in regular triangular SNS arrays.²⁰ We interpret it as clear evidence that, at sufficiently high temperatures, phase coherence in the neighborhood of the “commensurate” ground states at $f=P/(2\times 4^N)$, where the vortex lattice is pinned, is drastically disrupted by vortex-lattice defects, created by excess or missing vortices, moving almost freely on the pinned vortex background. This sharpens the fine structure substantially, thereby enhancing the amplitude of the oscillations.

In order to verify the asymptotic prediction of Sec. V, the logarithm of the relative inverse sheet inductance change $\Delta L^{-1}/L^{-1}(0)$, as deduced from the data of Fig. 9, is plotted against $\log_4 f$ in Fig. 10 and compared with the result of a calculation identical to that we described in connection with Fig. 7, where the direct method developed in Sec. III was combined with the perturbative decimation procedure based on Eqs. (40) and (41). At the lowest temperature ($\tau=0.09$), where thermal fluctuations are expected to be almost irrelevant, the general scaling of the data with frustration as well as the amplitude and the shape of the oscillations follow quite nicely the behavior predicted by the mean-field theory. However, on account of the limited inductance resolution [$\delta L/L(0)\approx 1\%$] attainable in the ac measurements, at $\tau=0.09$ we have been able to resolve only two hierarchical stages and to observe incipient asymptotic behavior merely in the second one ($\frac{1}{32} < f < \frac{1}{8}$). In this respect, the experimental conditions are much more favorable at higher temperatures, where thermal fluctuations not only enhance the amplitude of the oscillations, but also promote a richer fine structure, as demonstrated by the larger number (four) of self-similar stages emerging from the data at $\tau=0.17$ in Fig. 10 (notice that $h=4$ is actually the largest number of stages compatible with the order of our gaskets). As shown in Ref. 13, the evolution of the fine-structure richness with temperature finds a natural explanation in terms of thermal activation of the vortices in the potential-energy landscape created by the gaskets.

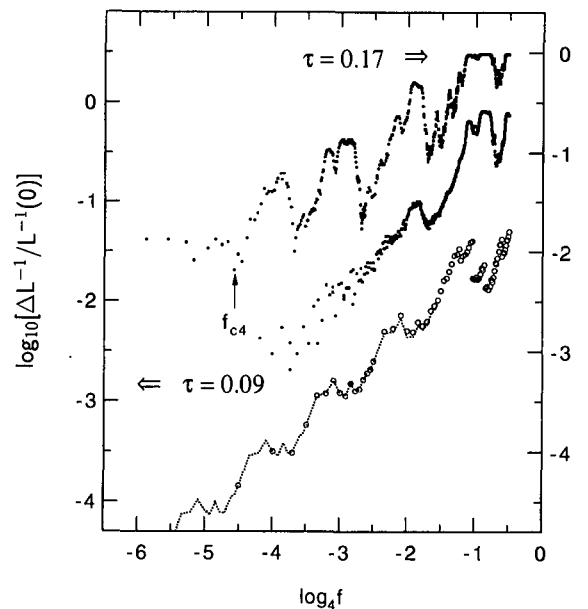


FIG. 10. log-log plot of the relative change of the inverse sheet inductance of the triangular lattice of fourth-order Sierpinski gaskets shown in Fig. 8 at two different reduced temperatures as a function of frustration. The theoretical curve (shifted downwards, for clarity, by one decade with respect to the vertical axis on the left) was computed by combining the direct method described in Sec. III (open circles) with the perturbative decimation procedure based on Eqs. (40) and (41) (dotted line). $f_{c4}=\frac{1}{512}$ is the frustration at which the crossover from the fractal ($f > f_{c4}$) to the Euclidean 2D ($f < f_{c4}$) regime occurs.

Remarkably, in spite of the indisputable evidence for fluctuation effects, the data at $\tau=0.17$ follow quite closely the asymptotic scaling prediction [Eq. (44)] of the mean-field theory. A weak upwards deviation sets in only in the last hierarchical stage ($\frac{1}{312} < f < \frac{1}{128}$) where the inductance measurements were carried out at the limit of our experimental resolution and should therefore be taken with caution. Failing a detailed theoretical description including fluctuations, we are unable to explain why their effect on the asymptotic scaling behavior turns out to be so weak.

Because of the 2D nature of our sample at length scales larger than the gasket size, at $f_{cN} = 1/(2 \times 4^N) = \frac{1}{312}$, the frustration defining the ground-state configuration in which each (rhombohedral) unit cell of the periodic gasket array contains just one single vortex, we expect a crossover from the fractal ($f > f_{cN}$) to the Euclidean 2D ($f < f_{cN}$) regime.²¹ The tendency of the high-temperature data of Fig. 10 to flatten out below f_{c4} does indeed provide some evidence for the occurrence of a dimensional crossover. This preliminary observation has been confirmed by recent high-resolution studies of the fine structure below f_{cN} and will be discussed in more detail elsewhere.

Closer inspection of the fine structure in the data of Fig. 10 reveals a discrepancy between theory and experiment. It consists in an almost temperature-independent shift of the structures which becomes particularly manifest in the first hierarchical stage for $f > \frac{1}{4}$, but is also present at lower frustrations. We attribute this effect to

the inhomogeneous frustration resulting from the change in the effective area of the different plaquettes caused by the asymmetric (with respect to the link direction) diamagnetic response of the "truncated-star-shaped" superconducting islands (see Fig. 8). Because of this particular geometrical form, the screening currents flowing in these grains create a distortion of the triangular current patterns associated with the individual loops which perturbs the self-similarity of the gaskets. Since at the temperatures of interest the magnetic penetration depth in the Pb islands is much less than their geometrical dimensions, the distortion is appreciable, thereby making f nonuniform. This interpretation is corroborated by the observation that the shift is more pronounced at higher frustrations where the loops providing the dominant contribution to the magnetoinductance, the smallest ones, turn out to be those exhibiting the largest distortion. A quantitative account of this sample-specific size effect will be published separately.

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