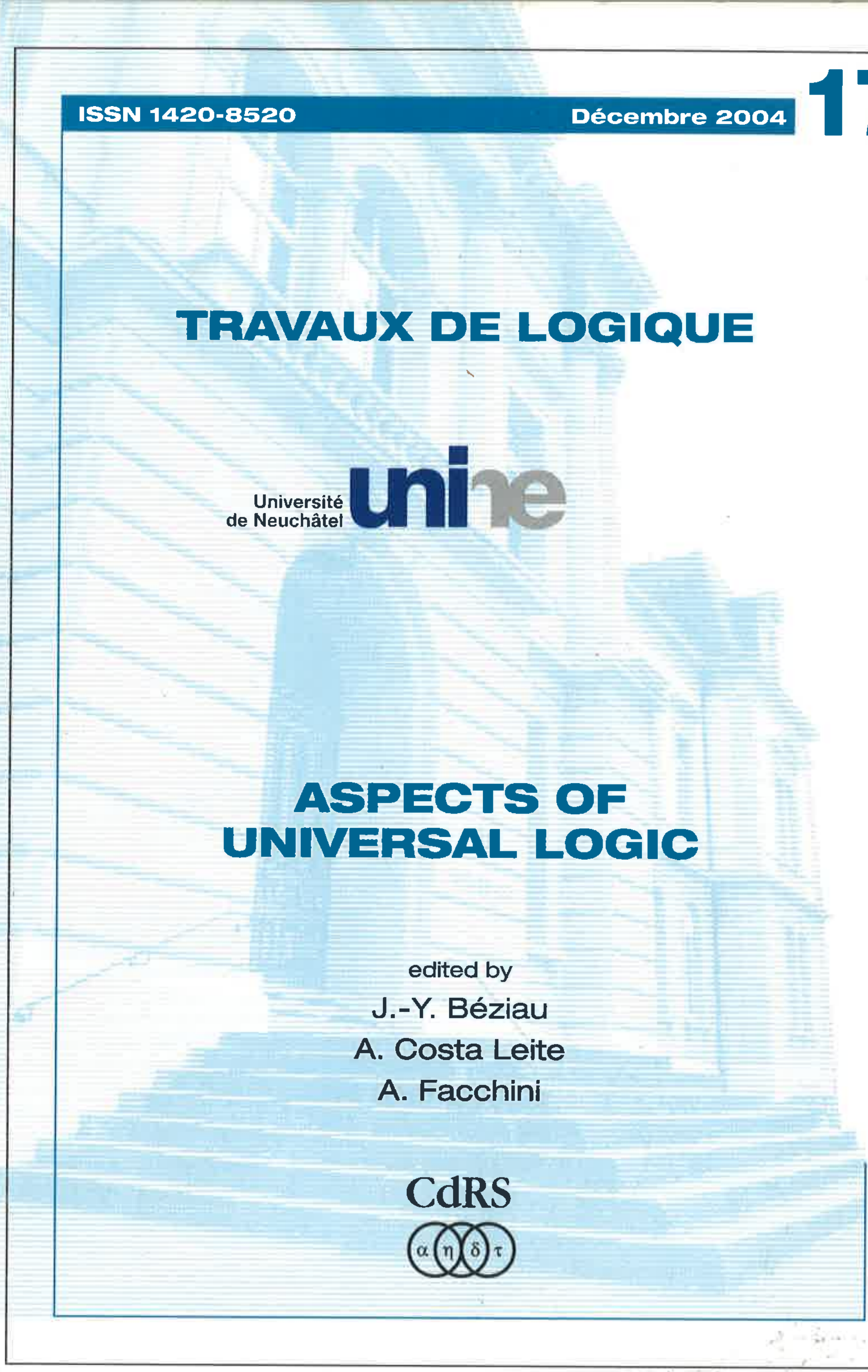

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TRAVAUX DE LOGIQUE

Université
de Neuchâtel

ASPECTS OF UNIVERSAL LOGIC

edited by
J.-Y. Béziau
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A. Facchini

CdRS

Centre de Recherches Sémiologiques
Travaux de logique
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Avant-propos

La logique manifeste depuis plus de deux millénaires des humeurs, des doutes et des aventures intellectuelles qui l'ont forgée à l'image d'un édifice respectable, remarquablement complexe, exaltant et parfois épistémologiquement contradictoire. Elle a été l'apanage de bien des communautés scientifiques : des écoles philosophiques aux cercles mathématiques, des élans informatiques et linguistiques aux projets de la science cognitive et à ceux de l'intelligence artificielle, elle a un peu perdu de son âme sans perdre le respect de ceux qui la courtisent. Aujourd'hui, dans le concert d'une mondialisation de la science pliée aux contraintes de la rentabilité à très court terme, la logique traverse un temps à la fois maussade et stimulant. Faisant partie des disciplines qualifiées de petites, elle risque les conséquences du couperet porté par le concept de la masse critique. Eclatée et diversifiée, elle a perdu un peu la nécessité de se penser en termes de paradigme global et cohérent. Cristallisée dans les perspectives formalistes du XXème siècle, elle a de la peine à se débarrasser d'un style et de formes qui prenaient pleinement leur sens en fonction des finalités d'alors.

Par rapport à ce constat, l'Institut de logique de l'Université de Neuchâtel a pu jouir de la très grande liberté qui lui est encore offerte pour conduire ses travaux ; il contribue ainsi à poursuivre cette quête incessante consistant à comprendre toujours davantage les fondements des processus inférentiels, à saisir les couleurs si subtiles de la vérité et à appréhender de quelle manière se meut la connaissance.

La communauté scientifique qui anime cet Institut a les qualités de la diversité, une diversité des questionnements qui s'est enrichie par la venue à Neuchâtel du Professeur du Fonds National Suisse de la Recherche Scientifique J.-Y Béziau et de son équipe. Ainsi, à la manière très systématique de développer et de faire usage d'un langage formel évolutif au service d'analyses logique, méthodologique ou conceptuelle, l'Institut connaît aussi aujourd'hui le temps d'une réflexion fondamentale induite par la multiplicité des logiques que le XXème siècle a commencé à engendrer. Cette réflexion pourrait s'intituler à la recherche de l'unité perdue ou à la conquête d'une théorie générale des logiques ; on a préféré la nommer Logique Universelle. La chose est ambitieuse et donc des plus stimulantes. Les résultats engendrés méritaient bien le temps d'un bilan, un bilan qui a pris la forme d'un colloque présidé par le professeur Béziau. La collection des *Travaux de logique* de l'Université de Neuchâtel est particulièrement heureuse d'en publier les actes.

Denis Miéville
Directeur de l'Institut de logique de l'Université de Neuchâtel

Foreword

In October 2003 an International Workshop on Universal logic was organized in Neuchâtel, this book mostly reflects this gathering. The coming of three Brazilian guests (Arthur Buchsbaum, Sheila and Paulo Veloso) was made possible by the LOCIA project (CNPq - Brazil) directed by Tarcisio Pequeno, Head of the Laboratory of Artificial Intelligence of the Federal University of Ceará (Fortaleza, Brazil). The Workshop was also sponsored by the University of Neuchâtel and the Swiss National Science Foundation (Universal Logic Research Project)

Neuchâtel, December 15th 2004
J.-Y.B.

J.-Y.Beziau, A.Costa-Leite and A.Facchini (eds)
Aspects of Universal Logic, 1-31

The Geometry of Knowledge

Johan van Benthem and Darko Sarenac

Abstract

The most widely used attractive logical account of knowledge uses standard epistemic models, i.e., *graphs* whose edges are indistinguishability relations for agents. In this paper, we discuss more general *topological* models for a multi-agent epistemic language, whose main uses so far have been in reasoning about space. We show that this more geometrical perspective affords greater powers of distinction in the study of common knowledge, defining new collective agents, and merging information for groups of agents.

1 Epistemic logic in its standard guise

1.1 Basic epistemic logic

Epistemic logic is in wide use today as a description of knowledge and ignorance for agents in philosophy [14], computer science [13], [22], game theory [12], and other areas. In this paper, we assume familiarity with the basic language of propositional epistemic logic, interpreted over multi-agent **S4** models whose accessibility relations are reflexive and transitive. Alternative model classes occur, too, such as equivalence relations for each agent in multi-agent **S5**–but our discussion is largely independent from such choices. The key semantic clause about an agent's knowledge of a proposition says that $K_i\phi$ holds at a world x if and only if ϕ is true in all worlds y accessible for i from x. That is, the epistemic knowledge modality is really a modal box $\Box_i\phi$. For technical convenience, we will use the latter notation

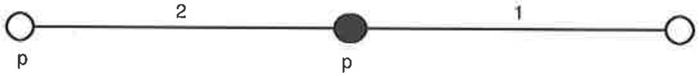

Figure 1: In the black central world, 1 does not know if p, while 2 does know that p. In the world to the left, 1 does know that p, so in the central world, 2 does not know if 1 knows that p.

for knowledge in the rest of this paper. The main modern interest in epistemic logic has to do with analyzing iterated knowledge of agents about themselves and what others know, for purposes of communication and interaction. Cf. [4], [9] on systems that combine epistemic logic and dynamic logic to describe information update in groups of agents. A simple example of how the basic logic works is the model in Figure 1.

The universally valid principles in our models are those of multi-agent **S4**. In an epistemic setting, the usual modal axioms get a special flavor. E.g., the iteration axiom $\Box_1\phi \rightarrow \Box_1\Box_1\phi$ now expresses 'positive introspection': agents who know something know that they know it. More precisely, we have **S4**-axioms for each separate agent, but no valid further 'mixing axioms' for iterated knowledge of agents, such as $\Box_1\Box_2\phi \rightarrow \Box_2\Box_1\phi$. Indeed, the latter implication fails in the above example. For instance, in the world on the left, 1 has no uncertainties, and so 1 knows that 2 knows that p. But 2 does not know there that 1 knows that p, because the latter assertion is false in the central world. Another way of describing the set of valid principles is as a *fusion* **S4**$\oplus$**S4** of separate logics **S4** for each agent, a perspective of 'merging logics' to which we will return below. In what follows, we shall mostly work with two-agent groups, $G = \{1, 2\}$, since most phenomena of interest can be studied there. Generalizations to finite k-agent cases are straightforward.

1.2 Group knowledge

Perhaps the most interesting topic in an interactive epistemic setting has been the discovery of various notions of what may be called *group knowledge.* Two well-known examples are as follows:

1. $E_G\phi$: every agent in group G knows that ϕ,
2. $C_G\phi$: ϕ is *common knowledge* in the group G.

The latter notion of group knowledge is much stronger than the former. It has been proposed in the philosophical, economic and linguistic literature as a necessary precondition for coordinated behavior between agents, cf. [16]. The usual semantic definition of common knowledge runs as follows:

$$M, x \models C_{1,2}\phi \text{ iff for all } y \text{ with } x\ (R_1 \cup R_2)^* y,\ M, y \models \phi$$

where $x(R_1 \cup R_2)^* y$ if there is a finite sequence of successive steps from either of the two accessibility relations connecting x to y. This relation is the reflexive transitive closure of the union of the relations for both agents. The key valid principles for common knowledge are the following additional axiom and rule:

Equilibrium Axiom: $C_{1,2}\phi \leftrightarrow (\phi \wedge (\Box_1 C_{1,2}\phi \wedge \Box_2 C_{1,2}\phi))$
Induction Rule: $\frac{\vdash p \to (\Box_1(q \wedge p) \wedge \Box_2(q \wedge p))}{\vdash p \to C_{1,2}q}$

This logic is known as $\mathbf{S4_2^C}$. It has been shown to be complete and decidable in [13] via a simple variation on similar proofs for propositional dynamic logic.

But there are still further interesting notions of knowledge for a group of agents. A prominent one is so-called *implicit knowledge*, $D_G\phi$, which describes what a group would know if its members decided to merge their information:

$$M, x \models D_{1,2}\phi \text{ iff for all } y \text{ with } xR_1 \cap R_2 y,\ M, y \models \phi$$

where $R_1 \cap R_2$ is the intersection of the accessibility relations for the separate agents. This new notion is technically somewhat different from the earlier two in that, unlike universal and common knowledge, it is not invariant under modal *bisimulations* of epistemic models. It also involves a new phenomenon of independent epistemic interest: viz. merging the information possessed by different agents. The latter topic will return throughout this paper.

1.3 Agents as epistemic accessibility relations

We can also think of new notions of group knowledge as introducing *new agents*. E.g., C_G defines a new kind of **S4**-agent, since $R_{(1\cup 2)^*}$ was again a pre-order. Note that $R_1 \cup R_2$ by itself is not a pre-order, so the new 'agent' corresponding to the fact that 'everybody knows' would have different epistemic properties. In particular, it would lack positive introspection as to what it knows. In contrast, the relation $R_1 \cap R_2$ for D_G is again an **S4**-agent as it stands, since Horn conditions like transitivity and reflexivity are preserved under intersections of relations. So, given a group of individual agents, our logical models suggest new agents. In particular, with two **S4**-agents $1, 2$, two additional ones supervene on these, one weaker, one stronger:

$$\begin{tikzcd}
 & R_1 \cap R_2 & \\
R_1 \arrow[ur] & & R_2 \arrow[ul] \\
 & (R_1 \cup R_2)^* \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

All this seems quite rich as an account for epistemic agents. And yet, there are indications that this framework is not yet flexible enough for its tasks.

1.4 Alternative views of common knowledge

Despite the success of the standard epistemic logic framework, there are still doubts about its expressive power and sensitivity. Some recurrent complaints seem endemic to logical approaches as such, like the vexing problem of logical omniscience: agents automatically know all laws of the system. But a more serious concern is the lack of epistemic distinctions in the standard modal setting. Notably, in his well-known critical paper [6], Barwise claimed that a proper analysis of common knowledge must distinguish three different approaches, that we may label

1. countably *infinite iteration* of individual knowledge modalities,
2. the *fixed-point view* of common knowledge as 'equilibrium',
3. agents' having a *shared epistemic situation.*

He then showed how to distinguish all three in a special situation-theoretic framework. As we will see below, however, Barwise's distinctions make sense in mainstream logic too–provided that we move to a broader topological semantics for the epistemic language involving products of models for individual agents. But before we do that, let us first analyze the reason why standard epistemic logic fails to distinguish the first two options. The third notion of 'shared understanding' is somewhat more mysterious, and harder to grasp in a standard relational modal setting. We will have a stab at it in the richer topological models of Section 2.

1.5 Computing epistemic fixed-points

The above Equilibrium Axiom for the common knowledge operator $C_G\phi$ shows how it may be viewed as defining a fixed-point of an epistemic operator $\lambda X.\phi \wedge \Box_1 X \wedge \Box_2 X$. In conjunction with the Induction Rule, it may even be seen to be a *greatest fixed-point* definable in the standard modal *μ-calculus* as:

$$C_G\phi := \nu p.\phi \wedge \Box_1 p \wedge \Box_2 p.$$

With a perhaps more familiar modal μ-operator, its existential variant would be defined as a smallest fixed-point

$$\Diamond_G^C\phi := \mu p.\phi \vee \Diamond_1 p \vee \Diamond_2 p.$$

As usual, a greatest fixed-point is defined as the fixed-point of a descending approximation sequence defined over the set of ordinals. We write $[|\phi|]$ for the truth set of ϕ in the relevant model where evaluation takes place:

$$C_{1,2}^0\phi := [|\phi|],$$

$C^{\kappa+1}_{1,2}\phi := [|\phi \wedge \Box_1(C^{\kappa}_{1,2}\phi) \wedge \Box_2(C^{\kappa}_{1,2}\phi)|]$,
$C^{\lambda}_{1,2}\phi := [| \bigwedge_{\kappa<\lambda} C^{\kappa}_{1,2}\phi|]$, for λ a limit ordinal.

Finally, we let $C_{1,2}\phi := C^{\kappa}_{1,2}\phi$ where κ is the least ordinal for which the approximation procedure halts: i.e., $C^{\kappa+1}_{1,2}\phi = C^{\kappa}_{1,2}\phi$. This approximation procedure must stop at some ordinal because the operator F applied is *monotonic*, a fact which is guaranteed by the positive occurrence of the propositional variable p in the body of F's definition. As a result, the approximation sequence for a greatest fixed-point operator always descends to subsets, and hence it must stop eventually. In general μ-calculus, reaching this stopping point may take any number of ordinal stages. A standard example is the least-fixed-point formula $\mu p.\Box p$ which computes the so-called 'well-founded part' of the binary accessibility relation for the modality. But in certain cases, stabilization is guaranteed to occur by the first infinite stage.

Fact 1.1 *In every relational epistemic model, the approximation procedure for the common knowledge modality stabilizes at $\kappa \leq \omega$.*

This simple behavior is most easily understood by observing that knowledge modalities $\Box_i$ distribute over any infinite conjunction. Thus, $\Box_i(\bigwedge_{n<\omega} C^{n}_{1,2}\phi)$ is simply $\bigwedge_{n<\omega} \Box_i C^{n}_{1,2}\phi$ which is equivalent to $\bigwedge_{n<\omega} C^{n}_{1,2}\phi$. More generally, stabilization for a formula $\nu p.\phi(p)$ is guaranteed by stage ω in any model just in case the syntax defining the monotone approximation operator is constrained as follows [10]. The formula $\phi(p)$ must be a disjunction whose members are constructed using only

1. arbitrary literals $(\neg)q$,
2. any epistemic formulas that do not contain q at all,
3. conjunctions and universal modalities.

The preceding Fact says that the fixed-point approach to common knowledge and that with countably infinite conjunctions of repeated knowledge modalities are equivalent in the standard setting, as $\nu p.\phi \wedge \Box_1 p \wedge \Box_2 p$ is equivalent to

$$K_{1,2}p := \phi \wedge \Box_1\phi \wedge \Box_2\phi \wedge \Box_1\Box_2\phi ...$$

This equivalence is often considered a technical convenience. But it may also indicate that our standard models are too weak to make a relevant distinction, and that more general models are needed. As we shall see, these two definitions of common knowledge are different in a *topological* modelling for epistemic logic— and even stronger ones can then be modelled, resembling Barwise's use of 'shared situations'.

1.6 Merging Information

Many further interesting issues are raised by a multi-agent epistemic setting. In particular, multi-agent models will often arise by *merging* models for separate agents, or groups of agents, so that common knowledge for the whole group becomes possible at all. One natural way of combining models for two or more agents emphasized in the recent literature on combining modal logics employs *products* of their underlying frames. More precisely,

Definition 1.2 *The* product *of two frames* $\mathcal{F}_1 = (W_1, R_1)$ *and* $\mathcal{F}_2 = (W_2, R_2)$ *is the frame* $\mathcal{F}_1 \times \mathcal{F}_2 = (W_1 \times W_2, R_1, R_2)$ *with* R_1 *defined as*

$$(x, y)R_1(z, w) \textit{ iff } xR_1z \,\&\, y = w$$

and the relation R_2 *defined likewise.*

Sometimes one also adds the direct product relation $R_{1,2}$ which requires successor steps in both components. But in the present setting, this is definable as the relational composition of R_1 and R_2 in any order.

This way of combining modal logics is explored in detail in [15]. The separate logics of the component frames are preserved in the product, as is easy to see. But the really interesting question is what happens in the joint language containing both modalities $\Box_1$ and $\Box_2$, which can express interaction between epistemic agents. As it turns out, by a simple argument, product frames automatically validate the following two axioms:

(*com*) $\Box_1\Box_2 p \equiv \Box_2\Box_1 p$
(*chr*) $\Diamond_1\Box_2 p \rightarrow \Box_2\Diamond_1 p$

[15] contains much more information on these principles, including general results on when they suffice for axiomatizing the complete logic of frame products over the merge of the component logics. But note that these two

principles were not valid in the general fusion logic $\mathbf{S4} \oplus \mathbf{S4}$ of epistemic agents, as we saw earlier. Figure 1 provided a formal counterexample to *com*. To put such a scenario in words: a student may know that the teacher knows the answer to questions on the test, while the teacher does not know if the student knows the answer. Moreover, if *com* does become valid, common knowledge trivializes, since any finite sequence of knowledge modalities will be equivalent to one of $\Box_1, \Box_2$ or $\Box_1\Box_2$.

Now there are other notions of merge for epistemic models, and the preceding collapse of common knowledge need not occur with other operations. Often, merging information for single agents or groups of agents is more naturally viewed as an operation on *models*, rather than frames. And in that case, the necessity of obtaining a consistent atomic valuation on pairs of worlds may complicate the above product construction, and thereby block *com* and *chr*. We discuss this issue briefly in Section 2.7. But for our purposes later on with analyzing common knowledge, frame products are important, provided we generalize them, again, to a wider topological setting. In that case, the two undesirable epistemic interaction laws no longer hold, and the above trivialization of common knowledge goes away.

We have now accumulated enough motivation for looking into broader alternative semantics for a multi-agent language, which should be fine-grained enough to distinguish different notions of common knowledge, while being sufficiently robust to still provide a plausible version of epistemic logic. We find this in the following mathematical generalization of relational models.

2 Epistemic Models in Topological Semantics

2.1 From graphs to topological spaces.

One of the major alternatives to relational semantics for modal logics, and historically even the earlier approach, employs *topological* models. Before going into our main epistemic concerns, we present this semantics here with its usual interpretation. Topology is an abstract mathematical theory of space, emphasizing qualitative notions of open environment, closure, boundary, or connectedness.

Definition 2.1 *A* topological space $\mathcal{X}$ *is a pair* (X, τ) *where* X *is a set of* *'points', and the set of 'opens'* $\tau \subseteq \wp(X)$ *contains* $X, \emptyset$, *and is closed under finite intersections and arbitrary unions.*

Example 2.2 *A typical example is the structure of the rationals with* $\mathbb{Q}$ *for the set* X *and the standard metric topology generated by closing the set of bounded open intervals* $\{p \,|\, q < p < q'\}$ *for* $p, q, q' \in \mathbb{Q}$ *under arbitrary unions. The standard topology on the reals* $\mathbb{R}$ *is obtained in the same fashion.*

The language $\mathcal{L}$ of propositional modal logic is just as before, with a countable set of propositional variables At, and the formulae defined recursively:

$$\phi := p \,|\, \neg\phi \,|\, \phi \wedge \psi \,|\, \Diamond p \,|\, \Box p$$

On the topological interpretation, Booleans are interpreted as the corresponding set operations, $\Box p$ as the topological interior of the set of points assigned to p, and $\Diamond p$ as the closure of the set assigned to p. More precisely, a topological model $\mathcal{M} = \langle X, \tau, V \rangle$ consists of a topological space $\langle X, \tau \rangle$ with a valuation function $V : At \to \wp(X)$. The key clauses of the truth definition then read:

$$\begin{array}{lll} \mathcal{M}, x \models \Box\phi & \text{iff} & (\exists U \in \tau)(x \in U \text{ and } (\forall y \in U)(\mathcal{M}, y \models \phi)), \\ \mathcal{M}, x \models \Diamond\phi & \text{iff} & (\forall U \in \tau)(x \in U \Rightarrow (\exists y \in U)(\mathcal{M}, y \models \phi)). \end{array}$$

All topological modalities in this paper satisfy the axioms of the modal logic **S4**, which reflect key properties of the topological interior operation. The interesting epistemic details then lie in the interaction among such modalities. For the moment, we cite two well-known results from [17]:

Theorem 2.3 **S4** *is a complete axiomatization of modal* $\Box$ *interpreted over arbitrary topological spaces.*

More striking, and much deeper, is the following result.

Theorem 2.4 **S4** *is a complete axiomatization of modal* $\Box$ *on any metric space that is dense-in-itself.*

This theorem shows that **S4** is the complete logic of $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}^2$, and many other interesting topologies close to our ordinary understanding of space.

Topological semantics generalizes standard modal model theory. A basic example is *bisimulation* for relational models (cf. [2]). Its pervasive invariance properties generalize to topological models.

Definition 2.5 *(*Topological Bisimulation*) A* topo-bisimulation *between two topological models $\langle \mathcal{X}, \tau, V\rangle$ and $\langle \mathcal{X}', \tau', V'\rangle$ is a nonempty relation $E \subseteq \mathcal{X} \times \mathcal{X}'$ such that, whenever xEx', then:*

1. *$x \in V(p)$ iff $x' \in V'(p)$, for every proposition letter p,*
2. *(forth condition) $x \in U \in \tau$ implies that there is a $U' \in \tau'$, $x' \in U'$ and for every $y' \in U'$ there is a $y \in U$ with yEy'.*
3. *(back condition) $x' \in U' \in \tau'$ implies that there is a $U \in \tau$, $x \in U$ and for every $y \in U$ there is a $y' \in U'$ with yEy'.*

Proposition 2.6 *(*Invariance for Bisimulation*) Let $\mathcal{M} = \langle \mathcal{X}, \tau, V\rangle$ and $\mathcal{M}' = \langle \mathcal{X}', \tau', V'\rangle$ be models with points x and x' related by some topo-bisimulation. Then, $\mathcal{M}, x \models \phi$ iff $\mathcal{M}', x' \models \phi$ for all modal formulas ϕ.*

The following special case of this result is the topological counterpart of the 'generated submodels' in relational semantics. Truth values only depend on what happens in arbitrarily small open neighbourhoods.

Proposition 2.7 *(*Topological Locality*) Let $\mathcal{X} = \langle X, \tau\rangle$ be a topological space, with $x \in U \in \tau$ and ν some valuation on X. Then, for any formula ϕ, $(\mathcal{X}, \nu), x \models \phi$ iff $(\mathcal{X}|U, \nu|U), x \models \phi$, where $\mathcal{X}|U$ is the topology obtained by taking U as the universe, letting the opens be all sets $U \cap U' \in \tau$, while $\nu|U = \nu(p) \cap U$ for all p.*

Topo-bisimulations are closely related to a more standard topological notion.

Definition 2.8 *Let $\mathcal{X} = (X, \tau_1), \mathcal{Y} = (Y, \tau_2)$ be two topological spaces. A map $f : X \to Y$ is said to be*

1. open*, if the f-image of any open set in τ_1, is open in τ_2,*
2. continuous*, if the f-inverse image of any open set in τ_2 is open in τ_1.*

It is easy to show that open continuous maps preserve modal theories of topological spaces, just as 'modal p-morphisms' preserve theories of relational frames.

This is a good point for stating the general connection between the two classes of models for modal or epistemic languages. Standard relational models can be viewed as a special kind of topological spaces through the following notion.

Definition 2.9 *A topological space $\mathcal{X}$ is* Alexandroff *if every intersection of open sets of $\mathcal{X}$ is again open.*

Any Alexandroff topology $\mathcal{X} = \langle X, \tau \rangle$ induces a standard relational frame $\langle X, R \rangle$ with a reflexive transitive relation Rxy iff $y \in \bigcap\{U \in \tau \,|\, x \in U\}$. Conversely, any reflexive transitive relational frame $\langle X, R \rangle$ induces an Alexandroff topology by taking the sets $U_x = \{y \,|\, Rxy\}$ for each $x \in X$ as a basis for τ. It is easily shown that topological interpretation of modal formulas in a relational model yields the same results as in their associated Alexandroff spaces, and vice versa. In this way, modal logics of relational models describe special sets of topological models. But in general, topological models include settings without a clear relational counterpart. E.g., the standard topologies on $\mathbb{Q}$ and $\mathbb{R}$ are clearly not Alexandroff: any singleton set (a non-open) is the intersection of the open intervals containing it.

There is a recent revival of interest in modal **S4** interpreted over topological spaces, because of its applications to spatial reasoning. [1] and [2] survey the expressive power of **S4** and its extensions for this purpose. We will use a few results from this spatial line later on. But before we cite them, let us make a connection with our major concern of what agents know.

2.2 Topology and information

Dating back to the 1930s, there has also been a more epistemic use of topological models, viz. for *intuitionistic* logic, cf. [20]. In that case, open sets are rather interpreted as 'pieces of evidence', e.g., about the location of a point, reflecting the intuitionistic idea of truth-as-provability. We can generalize this idea to epistemic logic, reading the above truth condition for a knowledge modality $\Box_i p$ as saying that there exists a piece of evidence for agent i (viz. an open set in i's topology) which validates the proposition p. Alternatively, we could also think of the topology as a collection of theories or data bases that an agent has at its disposal. [21] contains more abstract versions of this idea. As we will see, one of the side benefits of this information-based interpretation of the epistemic language is that common knowledge arises in a group of agents precisely when they share the same piece of information. But first, we explore the new handle that we get on the issue of merging information structures for different agents.

2.3 Combination of agents in topological products

To deal with epistemic merges, we need some results from recent work on products of topological spaces developed originally in the setting of spatial reasoning in [11].

Products of topological spaces $\mathcal{X}, \mathcal{Y}$ occur quite often, and they support a variety of new topologies. We start with a particularly simple way of 'lifting' the two components to *one-dimensional topologies* on the grid space $X \times Y$, which we sometimes visualize as 'horizontal' and 'vertical' directions in a plane.

Definition 2.10 *Let* $\mathcal{X} = \langle X, \eta \rangle$ *and* $\mathcal{Y} = \langle Y, \theta \rangle$ *be two topological spaces. Suppose* $A \subseteq X \times Y$. *We say that* A *is* horizontally open (H-open) *if for any* $(x, y) \in A$ *there exists* $U \in \eta$ *such that* $x \in U$ *and* $U \times \{y\} \subseteq A$. *Similarly, we say that* A *is* vertically open (V-open) *if for any* $(x, y) \in A$ *there exists* $V \in \theta$ *such that* $y \in V$ *and* $\{x\} \times V \subseteq A$. *If* A *is both* H*- and* V*-open, then we call it* HV-open. *Dual closed sets are defined as usual.*

We can now interpret the modal operators $\Box_1$ and $\Box_2$ of the combined language $\mathcal{L}_{\Box_1\Box_2}$ in product models $\langle X \times Y, \tau_1, \tau_2 \rangle$ with some arbitrary valuation for proposition letters. The two key clauses will read as follows:

$$(x, y) \models \Box_1\phi \text{ iff } (\exists U \in \eta)(x \in U \; \& \; \forall u \in U : (u, y) \models \phi)$$

$$(x, y) \models \Box_2\phi \text{ iff } (\exists V \in \theta)(y \in V \; \& \; \forall v \in V : (x, v) \models \phi)$$

In order to visualize this semantics, it helps to think of 'grids' of ordered pairs where one topology runs along horizontal lines, and the other along vertical ones. Next, we say that a formula ϕ of the language $\mathcal{L}_{\Box_1\Box_2}$ is *valid at* (x, y) in a product space $X \times Y$ if for every valuation on that space $(x, y) \models \phi$. The following proposition then tells us that the structural theories of component topologies (or agents' knowledge) 'lift' to the product

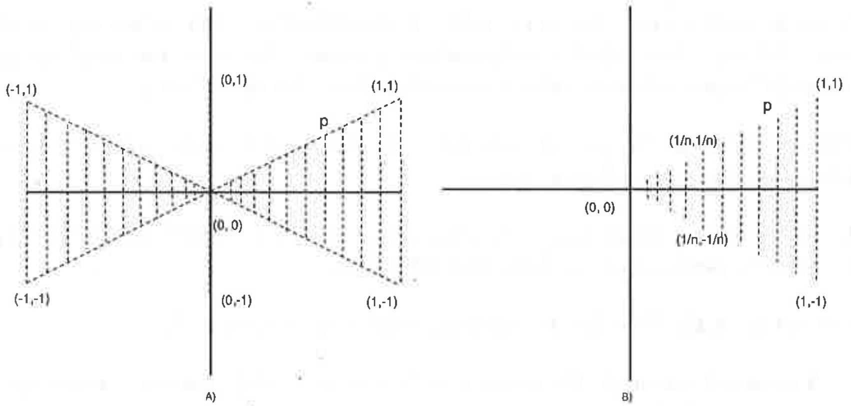

Figure 2: In A, the valuation $\nu(p) = (\bigcup_{x \in (-1,0)} \{x\} \times (x, -x)) \cup (\{0\} \times (-1,1)) \cup (\bigcup_{x \in (0,1)} \{x\} \times (-x, x))$ falsifies *com* at $(0,0)$. In B, $\nu'(p) = \bigcup \{\{\frac{1}{n}\} \times (-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ falsifies *chr* at $(0,0)$.

space without any additions. Unlike the case of products of relational frames in Section 1.6, topological product does not automatically enforce new interaction principles between agents.

Proposition 2.11 *A formula ϕ constructed from atoms, Booleans and the modal operator $\Box_1$ is valid at a point $(x, y) \in \langle X \times Y, \tau_1, \tau_2 \rangle$ iff ϕ is valid at x in $\mathcal{X}$. The same is true for the language with $\Box_2$ only, by taking the right projection.*

This was a result for the separate sublanguages of the agents. Moving to the joint language Let $\mathcal{L}_{\Box_1 \Box_2}$, it can be shown that the earlier product interaction principles *chr* and *com* fail on topological products. Figure 2 shows graphically how these failures occur for suitable valuations ν, ν' on the two-dimensional real plane:

$$(\mathbb{R} \times \mathbb{R}, \nu), (0,0) \not\models \Diamond_V \Box_H p \to \Box_H \Diamond_V p$$

$$(\mathbb{R} \times \mathbb{R}, \nu'), (0,0) \not\models \Box_H \Box_V p \to \Box_V \Box_H p$$

Next, we turn to matters of complete axiomatization. The following result from [11] says that topological products perform the most minimal merge of modal logics, without interactive side-effects for modalities.

Theorem 2.12 *The fusion logic* $\mathbf{S4} \oplus \mathbf{S4}$ *is complete with respect to products of arbitrary topological spaces.*

As in the single-agent case, one can prove stronger results for particular structures, and in fact, we have the following:

Theorem 2.13 $\mathbf{S4} \oplus \mathbf{S4}$ *is complete with respect to* $\mathbb{Q} \times \mathbb{Q}$.

A detailed proof of this result can be found in [11]. For later reference, we give a sketch here.

The first major observation to be made is that $\mathbf{S4}\oplus\mathbf{S4}$ is complete for the *infinite quaternary tree* $T_{2,2}$, using a standard modal unravelling procedure for countable relational models. To transfer modal counter-examples from that tree to topological products, we need to make a second step, showing that $T_{2,2}$ is the image of an HV-open subset of the 'rational plane' $\mathbb{Q} \times \mathbb{Q}$ under some HV-continuous and HV-open map. Such a map is constructed in stages via the following procedure, which is easily visualized. Let $T^n_{2,2}$ be the nodes of $T_{2,2}$ of R-depth n. Now, iteratively label a sequence of growing subsets of $\mathbb{Q} \times \mathbb{Q}$ with nodes of $T_{2,2}$ as follows:

Stg 0: Label $(0, 0)$ with the root r of the tree $T_{2,2}$.

Stg 1: Label $(-1, 0)$ with the immediate left R_1-successor, and $(1, 0)$ with the immediate right R_1-successor of r; also label $(0, -1)$ with the immediate left R_2-successor, and $(0, 1)$ with the immediate right R_2-successor of r. Call these four points *environmental points at the distance* $\frac{1}{3^0}$.

Stg n: The environmental points labelled at Stage $n-1$ are at the distance no smaller than $\frac{1}{3^{n-1}}$. Now for each of labelled points we create four environmental points at the distance $\frac{1}{3^n}$–two at the vertical distance $\frac{1}{3^n}$ and two at the horizontal distance $\frac{1}{3^n}$–and label them with respective immediate R_1- and R_2-successors in the tree.

This procedure labels a subset of $\mathbb{Q} \times \mathbb{Q}$ which can be contracted, modulo isomorphism, to an HV-open subset of $\mathbb{Q} \times \mathbb{Q}$. Moreover, there is an obvious map f taking labelled points in this set to nodes in the tree $T_{2,2}$.

A straightforward verification shows that this map is both HV-continuous and HV-open. Obviously, we can copy any valuation on the tree to one on $\mathbb{Q} \times \mathbb{Q}$ backward along the map f. Thus, if some modal formula is refuted in the root of the tree under some valuation, we get a topo-bisimulation with a model whose domain is a HV-open subset of the rational plane. By the above Locality Lemma 2.7, this counter-example can be lifted to the whole model $\mathbb{Q} \times \mathbb{Q}$, which is what we wanted.

Thus the fusion $\mathbf{S4} \oplus \mathbf{S4}$ is the logic of two epistemic agents combined into one framework using topological products, without any dramatic interaction enforced as in the case of products of relational frames. This result gives us the technical means to analyze different versions of common knowledge in a concrete setting of merged multi-agent models.

2.4 Common knowledge in product spaces

The earlier definitions of common knowledge still make sense in topological models. For instance, countably infinite iteration of all finite sequences of alternating knowledge modalities for the individual agents 1, 2 is as before:

$K_{1,2}p := \bigwedge_n^\omega K_{1,2}^n p$,

with $K_{1,2}^n p$ defined inductively as follows:

$$K_{1,2}^0 p := p$$
$$K_{1,2}^{n+1} p := \Box_1(K_{1,2}^n p) \wedge \Box_2(K_{1,2}^n p)$$

And the same is true for the fixed-point definition

$$C_{1,2}\phi := \nu p.\phi \wedge \Box_1 p \wedge \Box_2 p,$$

provided we make the appropriate adjustments in computing fixed points. In particular, the monotone operations generated by formulas positive in p now work a bit differently from before. In relational models, the operator $\Box_i$ applied to a set X yielded $\Box_i(X) = \{y \mid \forall x(R_i yx \to x \in X)\}$, making the modality a bounded universal quantifier. In topological semantics, however, the relevant operator is

$$\Box_i(X) = \{y \mid \exists U \in \tau_i \,\&\, \forall x(x \in U \rightarrow x \in X)\}$$

This reads a modality as an existential quantifier over open sets followed by a universal quantifier over elements of those sets. This two-quantifier combination complicates matters when approximating greatest or smallest fixed-points. Indeed, the definitions of common knowledge by fixed-points and by countably infinite iteration will now diverge. Here is a first indication why this may happen. The topological semantics validates the finitary logic **S4**, but it diverges from the relational validities in its infinitary behaviour.

Fact 2.14 *Topological interior does not distribute over infinite conjunctions:*

$$\Box_i \bigwedge_n p_n \text{ is not always equivalent to } \bigwedge_n \Box_i p_n$$

Take the standard topology on $\mathbb{Q}$. Define a valuation ν with, for all n, $\nu(p_n) = (-\frac{1}{n}, \frac{1}{n})$. Note that the intersection of these open sets is the singleton 0. Then $\bigwedge_n \Box_i p_n$ is true at 0, whereas $\Box_i \bigwedge_n p_n$ is not true anywhere. This result, though suggestive, is not yet a proof that the two definitions of common knowledge diverge. To do that, we will show that given a set p, the operator $K_{1,2}p$ does not always define a horizontally and vertically open set. Since the fixed-point version of $C_{1,2}p$ is always open in both these senses, the two cannot be the same.

We construct the relevant example by choosing a countable sequence of points in the rational plane $\mathbb{Q} \times \mathbb{Q}$ horizontally converging to the origin $(0,0)$. The first point in the sequence makes $\Box_1 p$ true but not $\Box_2\Box_1 p$, the second $\Box_1\Box_2 p, \Box_2\Box_1 p$ but not $\Box_2\Box_1\Box_2 p$, etc. This is possible by Theorem 2.12 for the logic of $\Box_1, \Box_2$: no finite iteration level of knowledge implies the next in the fusion logic **S4** $\oplus$ **S4**, and hence situations as described must exist in suitable models over $\mathbb{Q} \times \mathbb{Q}$. In particular, at each point of the sequence, $K_{1,2}$ will be false, and hence $\Box_1 K_{1,2}p$ is false at the origin $(0,0)$. It then remains to show that $K_{1,2}p$ itself does hold at $(0,0)$, but this will happen because of a well-chosen total valuation $\nu(p)$ for p on $\mathbb{Q} \times \mathbb{Q}$. To make this work, we make a number of more precise observations– while also slightly changing the formulas involved:

Theorem 2.15 *$K_{1,2}p \to \Box_1 K_{1,2}p$ is not valid on topological product spaces.*

Let ψ_n be the formula $\Box_1(K^n_{1,2}p) \to \Box_2(K^n_{1,2}p)$.

Fact 2.16 *(a) For all n, ψ_n is not a theorem of the fusion logic* **S4** $\oplus$ **S4**.
(b) There is a model M_n on $\mathbb{Q} \times \mathbb{Q}$ such that $M_n, (0,0) \not\models \Box_2(K^n_{1,2}p)$, and for all $q \in \mathbb{Q}, M_n, (q,0) \models K^n_{1,2}p$.

As for (a), one can easily construct finite fusion frames invalidating any given principle ψ_n.
(b) Since **S4** $\oplus$ **S4** is complete for $\mathbb{Q} \times \mathbb{Q}$, by (a) there is a model M'_n such that $M'_n, (0,0) \not\models \psi_n$, that is,

$$M'_n, (0,0) \models \Box_1(K^n_{1,2}p)$$

as well as

$$M'_n, (0,0) \not\models \Box_2(K^n_{1,2}p).$$

It follows that there is an open interval $((-q,0),(q,0))$ and every $(q',0)$ in this interval satisfies $K^n_{1,2}p$. By Locality (Proposition 2.7), in $(-q,q) \times \mathbb{Q}$ with the valuation from M'_n restricted to this space it is still true that $\Box_2(K^n_{1,2}p)$ fails at $(0,0)$ and that $K^n_{1,2}p$ holds at each point $(q',0)$. But $(-q,q) \times \mathbb{Q}$ is homeomorphic to $\mathbb{Q} \times \mathbb{Q}$ itself, and hence the valuation of M'_n transfers to $\mathbb{Q} \times \mathbb{Q}$ via the homeomorphism.

Fact 2.17 *There is a sequence of positive irrational numbers converging to 0 such that for any two adjacent numbers r, r' in the sequence, the distance $r - r'$ is a rational number.*

Take for instance $\sqrt{2}, \sqrt{2}-1, \sqrt{2}-1.4, \sqrt{2}-1.41$, etc. Next, for each rational interval, we form squares $S_1, S_2, \ldots$ of decreasing sizes over these intervals bounded by the separating irrationals [see Figure 3]. In the above example, the first square would be $(\sqrt{2}, \sqrt{2}-1) \times (-\frac{1}{2}, \frac{1}{2})$, the second $(\sqrt{2}-1, \sqrt{2}-1.4) \times (-0.2, 0.2)$, etc. Each of these squares is still homeomorphic to the rational plane $\mathbb{Q} \times \mathbb{Q}$ with some valuation for the proposition letter p.

Now, we create a new big model M over $\mathbb{Q} \times \mathbb{Q}$ as follows. In the sequence of squares S_n, we embed the earlier counter-examples M_n into S_n in such a way that its horizontal axis becomes the horizontal axis of the square S_n. This ensures that $K^n_{1,2}p$ holds everywhere on S_n's X−axis while $\Box_2(K^n_{1,2}p)$ fails somewhere on it. Outside of the squares, we put every point of the total rational plane in $V(p)$. Now we can prove the earlier informal assertion.

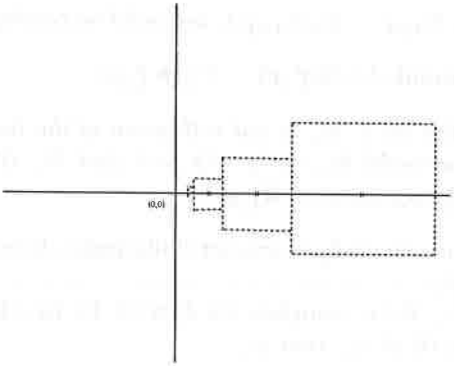

Figure 3:

Claim 2.18 *(a)* $M,(0,0) \models K_{1,2}p$
(b) $M,(0,0) \not\models \Box_1 K_{1,2}p$.

(a) We will prove that for all n, $K^n_{1,2}p$ holds at $(0,0)$. The proof is by induction. First note that any point on the y axis or to the left of it (except $(0,0)$) sits in an open circle interior in which p is true everywhere. Inside such a circle, these points evidently satisfy all formulas $K^n_{1,2}p$, and hence by Locality again, they also satisfy all these formulas in the whole model M.

Now we consider the origin $(0,0)$. The base step is simple: $K^0_{1,2}p$ is true by the definition of $\nu(p)$. Next consider the inductive step $K^n_{1,2}p \Rightarrow K^{n+1}_{1,2}p$, where $K^{n+1}_{1,2}p$ is $\Box_1(K^n_{1,2}p) \wedge \Box_2(K^n_{1,2}p)$. We show that the two conjuncts hold separately. To see that $\Box_2(K^n_{1,2}p)$ holds at $(0,0)$ we need an open set $((0,y),(0,-y))$ with $K^n_{1,2}p$ true at each point in this set. Evidently, this formula holds at $(0,0)$ itself by the inductive hypothesis. And it holds at any other point on the Y axis by the preceding observation about open p-circles.

Next we show that $\Box_1(K^n_{1,2}p)$ holds at $(0,0)$. This time we need an interval of the form $((-y,0),(x,0))$ with $K^n_{1,2}p$ true at every point in the interval. Here, points in $((y,0),(0,0))$ are covered by the observation about open p-circles again, and the origin itself by the inductive hypothesis. Then, looking toward the right, by the construction of the squares S_n, we know that $K^n_{1,2}p$ holds everywhere at the horizontal axis of S_n, and the same obviously remains true for S_m with $m > n$. Thus, for the desired right end-point $(x, 0)$ we can take any point on the horizontal axis of the square S_n. Since every point in $((0,0),(x,0))$ is in some S_m for $m \geq n$, we have the desired interval, and hence $\Box_1(K^n_{1,2}p)$ is true at the origin. In this connection, the idea behind our 'gluing' the squares at irrationals was that inside $\mathbb{Q} \times \mathbb{Q}$, there are then no boundary points to consider.

(b) To see that $\Box_1 K_{1,2}p$ fails at $(0,0)$, we observe that in any horizontal open interval I around $(0,0)$ there is a point where $K_{1,2}p$ fails. Note that for some n, the horizontal axis of S_n is a subset of I, by our construction of ever smaller squares S_n, and hence there is a point inside our interval where $\Box_2(K^n_{1,2}p)$ fails, and hence also $K_{1,2}p$, as desired.

Corollary 2.19 *$K_{1,2}p$ is not equivalent to $C_{1,2}p$ in topological models.*

Corollary 2.20 *Stabilization of the fixed-point version of $C_{1,2}X$ may occur later than ordinal stage ω.*

Thus, the topological setting achieves a natural separation between the first two definitions of common knowledge that Barwise distinguished. Moreover, our method raises further issues. First, it is rather 'logicky', and one might want a concrete independently motivated set of points in the rational plane for which the separation occurs. Also, it would be of interest to determine the exact (countable) ordinals at which epistemic fixed-point definitions do stabilize in this model.

This still leaves Barwise's third account of common knowledge in terms of 'shared situations'. We shall return to this matter in Section 2.6.

2.5 Complete logic of common knowledge on topo-products

Now what is the basic logic of the greatest fixed-point common knowledge modality $C_{1,2}$ on topological models? Perhaps surprisingly, the general answer is: 'the same as that for relational **S4**-models'. The reason is that the usual system $\mathbf{S4}^{\mathbf{C}}_{\mathbf{2}}$ already has principles for common knowledge that are satisfied by the fixed-point definition. Moreover, that system is complete w.r.t. relational models [13], and the latter are Alexandroff topological models at the same time. More interesting is what happens in our topological product models. In fact, the logic does not change here either, but this time, the argument takes a little more thought.

Theorem 2.21 *$S4_2^C$ is complete for products of arbitrary topologies. In fact it is even the complete logic of $\mathbb{Q} \times \mathbb{Q}$.*

The completeness argument runs along the lines of the earlier one for the language without common knowledge: this is why we sketched the main proof steps for Theorem 2.12 in some detail. By the usual completeness proof with respect to relational models, any non-theorem of $\mathbf{S4_2^C}$ fails on some finite rooted modal model. Next, such a model can be unravelled via a bisimulation into the double-binary branching tree $T_{2,2}$ with an appropriate valuation. Now we do the labelling construction described in the proof of Theorem 2.12. In the end, this procedure produced a topo-bisimulation between the given model on $T_{2,2}$ and some model on the rational plane $\mathbb{Q} \times \mathbb{Q}$. Now the only thing we need to observe is that topo-bisimulations do not just preserve truth values of ordinary modal formulas. They also evidently preserve truth values of formulas in any modal language allowing *infinite* conjunctions and disjunctions of formulas. And, the latter observation gives us exactly what we need to transfer counterexamples to formulas in the epistemic language with common knowledge viewed as a fixed-point operator.

Fact 2.22 *Topological bisimulations preserve arbitrary fixed-point formulas.*

In any given model M, any modal fixed-point formula ϕ is equivalent to some modal formula $\phi(\alpha)$ which has no fixed-point operators any more, but which uses infinite conjunctions and disjunctions up to a size determined by the ordinal α to 'unwind' approximation sequences. What this α is depends on the size of the model M. Moreover, it does not matter if we unwind up to any higher ordinal. Now, suppose that some fixed-point formula ϕ is true at M, s, and E is a bisimulation connecting s to t in a model N, t. Let α^* be the maximum of the unwinding ordinals for ϕ in the two models M, N. Then $\phi(\alpha^*)$ is true at s in M, and therefore also true at t in N. It follows that the original fixed-point formula ϕ is true in N, t.

Even so, given the difference between $C_{1,2}\phi$ and $K_{1,2}\phi$ that we have now found, a new completeness question arises, yet to be solved:

Question:

What is the complete logic of $K_{1,2}\phi$?

Given all this emphasis on geometrical models like the rational plane, can we really claim that they are also epistemically relevant? Our discussion only shows their use as visualizations of abstract distinctions. Whether there is any deeper *informational* meaning to $\mathbb{Q} \times \mathbb{Q}$ still remains to be seen.

In the remainder of this paper, we discuss some further aspects of the topological semantics for knowledge, analogous to those raised in Section 1.

2.6 More on epistemic agents as topologies

In relational semantics, agents were really just accessibility relations. Likewise, in our topological models, agents are topologies! As was explained in Section 2.2, what the agent knows in a world of some model is what holds there according to the box modality of its topology. Let us now draw some comparisons with the situation in Section 1.3., where two agents $1, 2$ generated at least two further 'introspective collective agents', one being their supremum $R_{(1\cup 2)^*}$ leading to common knowledge, and the other their infimum $R_1 \cap R_2$ leading to 'implicit knowledge' for the group. The topological semantics gives us interesting counterparts to these operations.

Remark.

Introspection principles If we are less strict in our logic, without requiring positive introspection, then many further options arise, just as with relational models. If we are more strict, as in relational **S5**-models with negative introspection, then we must only use topologies that do satisfy the axiom $\phi \to \Box\Diamond\phi$. It is easy to see that, on T_0 spaces in which all singletons are closed, imposing this principle makes the topology discrete, trivializing the epistemic logic. But then, even a weak separation axiom like T_0 is not plausible epistemically. On general spaces, $\phi \to \Box\Diamond\phi$ corresponds to the property that every set is a subset of the interior of its closure. Unpacked further this says that:

$$\forall x, \exists U \in \tau : x \in U \,\&\, \forall y \in U, y \in V \in \tau : x \in V$$

This means the space is a union of open sets whose points have the same open neighbourhoods – which is a topological counterpart of relational **S5** models.

Our favorite setting for studying new collective agents are the product models that we used so far. We start with a simple but perhaps surprising observation. Common knowledge as a greatest fixed-point corresponds to taking the following very natural operation on the given topologies for the individual agents. Consider the *intersection* $\tau_{1\cap 2}$ of the earlier topologies τ_1 and τ_2 on a product space. It is easy to see that this is again a topology: all closure conditions are satisfied. Now we observe the following connection:

Fact 2.23 $\forall M \forall x, M, x \models C_{1,2}\phi$ *iff* $M, x \models [1 \cap 2]\phi$

We will show that the truth sets $[|C_{1,2}\phi|]$ and $[|[1 \cap 2]\phi|]$ are identical in all models. First, $[|C_{1,2}\phi|] \in \tau_i$ for $i \in \{1,2\}$ since the truth set is a fixed-point of $\nu p.\phi \wedge \Box_1 p \wedge \Box_2 p$. But then $[|C_{1,2}\phi|] \in \tau_{1\cap 2}$ by the definition, and so $[|C_{1,2}\phi|] \subseteq [|[1\cap 2]\phi|]$. Next, $[|[1\cap 2]\phi|]$ satisfies $[|\Box_i[1\cap 2]\phi|] = [|[1\cap 2]\phi|]$ for $i \in \{1,2\}$. Hence $[|[1 \cap 2]\phi|]$ is a fixed-point. Since $[|C_{1,2}\phi|]$ is the greatest fixed-point, $[|[1 \cap 2]\phi|] \subseteq [|C_{1,2}\phi|]$.

It is worth observing that this argument holds in general, for any two given topologies on some space, not just the vertical and horizontal ones in products. In fact, intersection of topologies is the counterpart, under the model-to topology transformation sketched earlier, of taking the reflexive transitive closure of given accessibility relations.

Thus, we also expect a topological counterpart for the earlier operation of *relational intersection*, which modelled implicit group knowledge D_G. This should be the union of two topologies, and then closing off in the minimal way that produces a topology again. The result is the *sum topology* $\tau_1 + \tau_2$ which takes all pairwise intersections of opens of the two topologies as a basis. The latter topology need not always be of great interest. E.g., on our recurrent topo-product $\mathbb{Q} \times \mathbb{Q}$, it will just be the discrete topology, making every point an open. From an informational perspective, this means that merging the information that we get about points in the horizontal and vertical directions fixes their position uniquely.

The result of all this is again an inclusion diagram:

$$\begin{tikzcd}
 & \tau_1 + \tau_2 & \\
\tau_1 \arrow[ur] & & \tau_2 \arrow[ul] \\
 & \tau_{1\cap 2} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Let us now return to the three distinctions made in [6]. So far, we have separated the countably infinite conjunction view from the greatest fixed-point view of common knowledge. What about the third view of having a 'shared situation'? In some ways, using the intersection topology seems to model this. Its opens are precisely those information pieces that are accepted by both agents. But if that is the case, then we have not separated the second and third notions. Fact 2.23 tells us precisely that the two amount to the same thing. But topological product models have further resources! In particular, so far, we have not discussed what topologists would call the real *product topology* τ on spaces $X \times Y$. This topology is defined by letting the sets $U \times V$ form a basis, where U is open in $\mathcal{X}$ and V is open in $\mathcal{Y}$. An example is the natural metric topology on the plane $\mathbb{Q} \times \mathbb{Q}$, used briefly in the argument for Claim 2.18, with open circles around points as neighbourhoods. The agent corresponding to this new group concept τ only accepts very strong collective evidence for any proposition. Here are two relevant results from [11]:

Theorem 2.24 *The epistemic box modality for the true product topology is not definable in the language of the separate modalities* $\Box_1, \Box_2$*, even when we add fixed-point operators.*

Theorem 2.25 *The complete logic including the true product topology is the smallest normal modal logic in the language of three modalities* $\Box, \Box_1, \Box_2$ *that contains (i) the* **S4** *axioms for* $\Box_1, \Box_2$ *and* $\Box$*, (ii)* $\Box p \to \Box_1 p$ *and* $\Box p \to \Box_2 p$.

Thus, we have found an even stronger notion of common knowledge that might be said to model Barwise's third stage. Nevertheless, there are some difficulties with this identification. For instance, unlike the preceding two operations of intersection and union closure, true product topology has no general definition on arbitrary models for our language, as it exploits the product structure essentially. This makes it rather specialized, and this same fact is also reflected in the poverty of the complete logic given above. Nevertheless, there are also interesting logical aspects to this situation. In contrast with the *sequential quantification* embodied in the greatest fixed-point reading of common knowledge, the true product modality reads more like a *branching quantifier* as defined in [7]. We do not know what to make epistemically of this tantalizing analogy at this stage.

2.7 Operations that are safe for topo-bisimulation

To illustrate the preceding notions of knowledge and agency a bit further, we add a brief digression on simulations between topological models.

In relational semantics for modal languages, most natural operations $f(R_1, R_2)$ have the property of being *safe for bisimulation*, that is,

- any given bisimulation between two models w.r.t. the relations R_1, R_2 is also a bisimulation for the relation $f(R_1, R_2)$.

This says that the new operation stays at the same level of model structure as the old. The regular operations of composition, union, and iteration on binary relations are all safe in this sense, while a typical non-safe operation is *intersection.* Safety is a natural extension of invariance for static formulas to dynamic transition relations ([10] has a complete characterization of all first-order definable safe operations). Safety constrains the repertoire of definable transition relations within one given model. In general process theories, new relations can also be constructed out of old while forming a new model at the same time, as happens with products for concurrent processes in Process Algebra. In that setting, safety for operations generalizes to *respect for bisimulation*, e.g., if we let $\cong$ signify bisimulation:

- if $M \cong M'$ and $N \cong N'$, then $f(M, N) \cong f(M', N')$.

Most natural product operations show respect for bisimulation. As a check on our new notions, we can also look at operations on topologies in the same way, substituting the above topological bisimulations for the usual relational ones.

Of the repertoire of regular operations, only a small part matters in our perspective. When working only with reflexive transitive relations, composition and union by themselves do not qualify as operations, and we

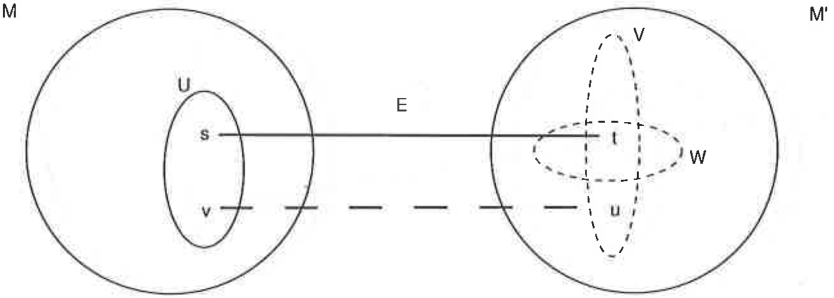

Figure 4:

need to take *-closures. And for reflexive-transitive $R_1, R_2, (R_1 \cup R_2)^*$ and $(R_1; R_2)^*$ yield even the same relation. The topological counterpart for the latter operation was *intersection of topologies* $\tau_1 \cap \tau_2$, as noted above. Fact 2.23 expressed the observation that the modality for this is the same as the common knowledge fixed-point modality for the modal operators $[\tau_1]$, $[\tau_2]$. The latter is invariant for topological bisimulations by earlier observations. Indeed we have the following

Fact 2.26 *Intersection of topologies is safe for topological bisimulation.*

Let E be a relation between topological models M, N which is a topological bisimulation for their two separate topologies, as in Figure 4.

For a start, let sEt, and $s \in U$ with U in $\tau_1 \cap \tau_2$. Since E is a bisimulation w.r.t. τ_1, there is a τ_1-open set V in M' such that every point $v \in V$ is E-related to some point u in U. Likewise, there is an τ_2-open set W in M' such that every point $v \in W$ is E-related to some point u in U. Now, it may be tempting to take the intersection of V and W at t for the required matching neighbourhood of U, but this need not be open in either topology. Instead, we consider every E-link between points u in U and points v in the union $V \cup W$. Using the bisimulation properties again, there are again both τ_1 and τ_2-open neighbourhoods for all such points u, which satisfy the backward zigzag condition toward U. Continuing this procedure countably many times, the union of all these successively produced subsets of M' is both τ_1- and τ_2 open, and moreover, it still satisfies the correct backward zigzag condition w.r.t. the original open neighbourhood U of s in M. The argument in the opposite direction is similar.

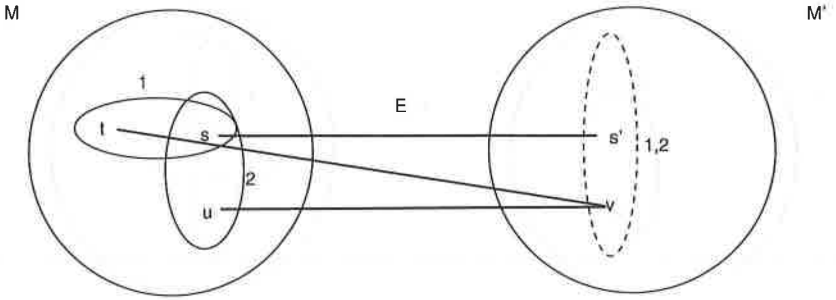

Figure 5:

This result may sound strange because intersection of binary relations led to non-invariance for bisimulation. But the topological counterpart of this operation was the sum topology $\tau_1 + \tau_2$ defined above, and its behaviour is indeed unsafe.

Fact 2.27 *Taking the sum of topologies is not safe for topological bisimulation.*

The counterexample is the same as for the relational case. Consider the two three-point models of Figure 5, with their topologies plus a binary relation E between their points as indicated.

Note that the sum topology on the left-hand side has the singleton set $\{s\}$ as an open, whereas the sum topology on the right has only the whole two-element space for a non-empty open. Also, the relation E is a bisimulation for both topologies τ_1 and τ_2. Next, consider the link sEt, with the open subset $\{s\}$ on the left. The only matching open set on the right can be $\{s', v\}$, but this fails to satisfy the backward zigzag condition, as sEv does not hold.

Finally, more general operations may produce new topologies over combined spaces. Our characteristic example was topological product as in Definition 2.10.

Fact 2.28 *Topological products $\tau_1 \times \tau_2$ respect topological bisimulation.*

Let E_1 be a bisimulation w.r.t. τ_1 between models M, M', and likewise E_2 a bisimulation w.r.t. τ_2 between models N, N'. Now define a bisimulation E between $M \times N$, $M' \times N'$ by setting:

$$(s,t)E(s',t') \text{ iff } sE_1s' \text{ and } tE_2t'.$$

Given Definition 2.10, it is completely straightforward to check that E is a bisimulation w.r.t both topologies on the product.
In contrast to this, taking a product of two topological spaces with the true product topology τ introduced a little while ago does not respect topological bisimulation. The reason is the earlier fact that the true product modality $\Box$ is not invariant for topological bisimulations w.r.t. the two component topologies.

2.8 Merging information revisited

Finally, we make a few comments on the issue of merging epistemic situations. We have shown that products of topological spaces are a natural setting for combining knowledge by different agents, and for distinguishing various forms of knowledge in the group of all agents. But as in Section 1, there is a broader question behind this. Our topological products are just one way of merging information models. The general subject of merging epistemic models goes far beyond the scope of this paper (cf. [8] for more on this topic). We only make one general point here which seems relevant to our move from relational semantics to topological models.

In general, we need to *specify* what we want to happen with existing knowledge and ignorance of agents when merging their information. Suppose we are given two epistemic models M for group G_1 and N for G_2, where G_1, G_2 overlap. In that case, we may want to require that the intersection group does not learn anything new in the 'merge model' $M * N$, at least w.r.t. formulas in its old language. This situation is reminiscent of the process of *amalgamation* of relational models in semantic proofs of the interpolation theorem for the basic modal language (cf. [3] for an elementary exposition). Such proofs often start with a $G_1 \cap G_2$ *bisimulation* between models M, s and N, t, which serves as an initial connection between the two different settings. The relevant merge $M * N$ then turns out to be a *submodel of the full product* $M \times N$, viz. just those pairs which stand in that bisimulation. One then shows that the projections from pairs to the original models M, N are bisimulations for the separate languages. Hence, formulas in the intersection of the two languages retain one unambiguous truth value: the one they had before under the bisimulation. In the case of interpolation theorems for shared modalities, this amalgamation construction has to be complicated, but the point remains the same. General merging of models for groups of agents may presuppose some initial connection, and its effects on modal formulas can be prescribed to some extent. In particular, we need not accept all pairs in a product as members of a merge model. Once we do this, the connection between topological models and relational models becomes more complicated, as we could also try to get the results of this paper with sub-product constructions on relational models. We refer to [18] for details.

3 Conclusion

Topological semantics for epistemic logic is a natural extension of the usual relational modelling. It provides distinctions that can be used to differentiate between various notions of common knowledge, and define various sorts of collective agents. Also, using product spaces, topological semantics suggests 'low-interaction' merges for epistemic models for separate groups of agents. Thus, we believe that there are good reasons for further development of this currently still marginal perspective.

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Selfextensional Logics in Abstract Algebraic Logic: a Brief Survey [1]

Ramon Jansana

1 Introduction

The name 'selfextensional logic' is due to Ryszard Wójcicki, who was the first to study the class of selfextensional logics in general in the papers [48, 49, 50] and the monographs [51] and [52]. Since 1996 the interest in selfextensional logics among researchers working in the field of Abstract Algebraic Logic (AAL) has been increasing, for the class of selfextensional logics cuts across the widely studied Leibniz hierarchy of logics which constitutes the core of the present day AAL. Therefore, the study of selfextensional logics can provide interesting new insights in this field.

A *selfextensional logic* $\mathcal{S}$ is any propositional logic with the following replacement property: if two formulas φ and ψ are interderivable (i.e. $\varphi \dashv_{\mathcal{S}}\vdash \psi$), then for every formula δ and every propositional variable p, the formulas $\delta(p/\varphi)$ and $\delta(p/\psi)$ are also interderivable (i.e. $\delta(p/\varphi) \dashv_{\mathcal{S}}\vdash \delta(p/\psi)$), where $\delta(p/\varphi)$ and $\delta(p/\psi)$ are the formulas obtained by substituting φ for p and ψ for p in δ respectively. In algebraic terms this means that the interderivability relation between formulas is a congruence relation of the formula algebra. Typical examples of selfextensional logics are classical propositional logic and intuitionistic propositional logic. These two logics have a stronger replacement property: for any set of formulas Γ, any formulas φ,

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ψ, δ and any variable p

$$\text{if } \Gamma, \varphi \vdash_{\mathcal{S}} \psi \text{ and } \Gamma, \psi \vdash_{\mathcal{S}} \varphi, \text{ then } \Gamma, \delta(p/\varphi) \vdash_{\mathcal{S}} \delta(p/\psi) \text{ and } \Gamma, \delta(p/\varphi) \vdash_{\mathcal{S}} \delta(p/\psi).$$

The logics with this stronger replacement property are called Fregean logics. Accordingly, selfextensional logics can also be called weakly Fregean logics, but we will use Wójcicki's terminology in this paper. For information on Fregean logics and the reasons why they are so named we refer the reader to [14, 17, 18, 39] and the survey [25]. Many selfextensional logics are not Fregean; one example is the deducibility relation defined by the Hilbert-style calculus whose axioms are the theorems of a normal modal logic L and modus ponens is taken as the sole inference rule. The logics defined in this way from the theorems of a modal logic are frequently called the local consequence relations of the modal logics, and we will use this term in this paper.

In AAL the concept of logic that has proven fruitful is the following. A *logic* (or deductive system) is a pair $\mathcal{S} = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}} \rangle$ where $\boldsymbol{Fm}$ is the algebra of the formulas of some set of connectives (or algebraic similarity type) and a denumerable set of variables $\mathcal{L}_{\mathcal{S}}$, and $\vdash_{\mathcal{S}}$ is a consequence relation on the universe Fm of $\boldsymbol{Fm}$, i.e. a relation between subsets of the set of formulas Fm and elements of Fm with the following three properties

1. if $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathcal{S}} \varphi$,

2. if $\Gamma \vdash_{\mathcal{S}} \varphi$ for all $\varphi \in \Delta$ and $\Delta \vdash_{\mathcal{S}} \psi$, then $\Gamma \vdash_{\mathcal{S}} \psi$,

3. if $\Gamma \subseteq \Delta$ and $\Gamma \vdash_{\mathcal{S}} \varphi$, then $\Delta \vdash_{\mathcal{S}} \varphi$,

that in addition is *substitution invariant*[2], that is, it satisfies that for every substitution σ (i.e. every endomorphism σ of $\boldsymbol{Fm}$)

4. if $\Gamma \vdash_{\mathcal{S}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathcal{S}} \sigma(\varphi)$.

A logic is *finitary* if the consequence relation $\vdash_{\mathcal{S}}$ is finitary, i.e. if for every Γ, φ

5. $\Gamma \vdash_{\mathcal{S}} \varphi$ iff there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\mathcal{S}} \varphi$.

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[2] Frequently this condition is called the structurality condition, and a consequence relation that satisfies it is said to be structural. Due to the increasing interest in the substructural logics, we prefer to use the term 'substitution invariant' introduced by Pigozzi in order to avoid misleading mental associations.

This concept of logic stems from Tarski's notion of consequence operation given in [43]. The algebra $\boldsymbol{Fm}$ is called the *algebra of formulas* of type $\mathcal{L}_{\mathcal{S}}$, or the $\mathcal{L}_{\mathcal{S}}$-algebra of formulas.

We will say that a pair $\langle \Gamma, \varphi \rangle$, where Γ is a set of formulas and φ is a formula, is a *rule* of $\mathcal{S}$ if $\Gamma \vdash_{\mathcal{S}} \varphi$. Thus, if $\langle \Gamma, \varphi \rangle$ is a rule of $\mathcal{S}$, then for every substitution σ, $\langle \sigma[\Gamma], \sigma(\varphi) \rangle$ is also a rule of $\mathcal{S}$.

Notice that the notion of logic just introduced encompasses both logics defined by syntactic means and logics defined by semantic means, and it departs from the notion of logic common in some quarters like the modal logic field where a logic is usually taken to be merely a set of formulas containing a set of axioms and closed under certain inference rules ([2, 10, 37]).

Besides classical logic, intuitionistic logic and the local consequences of the normal modal logics, other examples of selfextensional logics recently studied[3] are Visser's logic (called Basic logic by Visser) [45, 9], the strict implication fragments of the local consequences of the normal modal logics, in particular some subintuitionistic logics, [9], positive modal logic, [20, 8], Belnap's four-valued logic, [22], and the system of relevance logic WR, [27].

Some examples of non-selfextensional logics are the global consequences of the normal modal logics (they are defined in the same way as the local consequences but now, in addition to modus ponens, the rule of necessity is also taken as an inference rule), the system R of relevance logic, the classical and the intutitionistic linear logics without exponentials, and Łukasiewicz infinite-valued logic.

2 Generalities on AAL

Abstract Algebraic Logic can be described by saying that it studies the process of algebraization of logics rather than the algebraization of the particular logics in which one is interested. It intends to find the right concepts and discover the theorems that best explain the connections between properties of the algebras (or other algebra-related structures) associated with particular logics and their metalogical properties. For an overview of AAL we refer the reader to [25], where an extensive list of references is also given; for a detailed presentation of its main core, [14] is the best source.

As we will see in this paper, the study of selfextensional logics brings in new perspectives for the development of the theory of the algebraization of

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[3] The list is not exhaustive.

logics that AAL seeks.

One of the main goals in the AAL agenda is to find general and useful criteria to select in a canonical way and for any arbitrary logic $\mathcal{S}$ the class of algebras that best encodes or reflects the metalogical properties of $\mathcal{S}$. We shall start by describing the class of algebras that nowadays is taken to fulfill this role.

2.1 The class of algebras of a logic

The set of connectives of a logic $\mathcal{S}$ can be regarded as an algebraic similarity type. As usual in this context we adopt the convention of identifying formulas with terms.

In AAL some consensus has emerged on the canonical class of algebras that should be associated with a given logic. Different researchers have arrived at the same class despite starting from different perspectives. This gives some stability to the notion and provides some ground for the claim that we are on the right track. One example of this stability will be shown in a theorem below. The class of algebras of a logic can, therefore, be defined in several ways; the one we present here is the best suited for our purposes in this paper.

Let $\boldsymbol{A}$ be an algebra of type $\mathcal{L}_{\mathcal{S}}$. A set $F \subseteq A$ is an *$\mathcal{S}$-filter* if it is "closed under the rules of $\mathcal{S}$": formally speaking, if for every valuation v from the algebra of formulas into $\boldsymbol{A}$, whenever $\Gamma \vdash_{\mathcal{S}} \varphi$ and $v[\Gamma] \subseteq F$, $v(\varphi) \in F$. We denote the set of all $\mathcal{S}$-filters of $\boldsymbol{A}$ by $\mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}$. Notice that the $\mathcal{S}$-filters of the formula algebra are just the theories of $\mathcal{S}$, that is, the sets of formulas closed under the rules of $\mathcal{S}$. We denote the set of theories of $\mathcal{S}$ by $\mathrm{Th}(\mathcal{S})$.

A congruence θ of an algebra $\boldsymbol{A}$ is said to be *compatible* with a subset F of its universe if F is a union of equivalence classes, i.e. if $a\theta b$ and $a \in F$, then $b \in F$. There always exists the greatest congruence of $\boldsymbol{A}$ compatible with F; this is denoted by $\boldsymbol{\Omega}_{\boldsymbol{A}}F$ and it is called the *Leibniz congruence* of F. The operator that maps each subset of $\boldsymbol{A}$ to its Leibniz congruence is called the *Leibniz operator*.

Much of the work on AAL before 1996 centred on the behaviour of the Leibniz operator on $\mathcal{S}$-filters. A *local perspective* on $\mathcal{S}$-filters was taken and conditions such as monotonicity, continuity, commutation with arbitrary inverse homomorphisms, etc. of the Leibniz operator on the $\mathcal{S}$-filters of the algebras have been very useful in building the general theory of the algebraization of logics. But a *global perspective* on the family of $\mathcal{S}$-filters turns out to be even better suited for the development of this general theory.

This perspective in AAL was systematically explored for the first time in [23].

Let us associate with a given algebra $\boldsymbol{A}$ the greatest congruence which is compatible with all the $\mathcal{S}$-filters of $\boldsymbol{A}$; it is denoted by $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}\boldsymbol{A}$ and it is called the *Tarski congruence* of the pair $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\rangle$. It can be described as the intersection of the Leibniz congruences of the $\mathcal{S}$-filters of $\boldsymbol{A}$, i.e.

$$\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}\boldsymbol{A} = \bigcap_{F \in \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}} \boldsymbol{\Omega}_{\boldsymbol{A}}F.$$

The class of algebras which from the global perspective on $\mathcal{S}$-filters is associated canonically with a logic $\mathcal{S}$ can then be defined as follows

$$\mathsf{Alg}\mathcal{S} = \{\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}\boldsymbol{A} \text{ is the identity}\}.$$

This class of algebras is the class that is considered in [23] to be the canonical class of algebras of $\mathcal{S}$. It does not always coincide with the class of algebras which from the local perspective on $\mathcal{S}$-filters has been standardly associated with a logic $\mathcal{S}$. This last class of algebras is the class

$$\mathsf{Alg}^{*}\mathcal{S} = \{\boldsymbol{A} : (\exists F \in \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A})\ \boldsymbol{\Omega}_{\boldsymbol{A}}(F) \text{ is the identity}\}.$$

It is not difficult to see that $\mathsf{Alg}\mathcal{S}$ is the closure of $\mathsf{Alg}^{*}\mathcal{S}$ under subdirect products.

A class of algebras K is an *algebraic semantics*, or provides a complete algebraic semantics, for a logic $\mathcal{S}$ if there is a set of equations in one variable $E(p)$ (for many logics it is the singleton of the equation $p \approx 1$) such that, if we denote for any algebra $\boldsymbol{A}$ the set of solutions on $\boldsymbol{A}$ of the equations in $E(p)$ by $E(\boldsymbol{A})$, that is, if

$$E(\boldsymbol{A}) = \{a \in A : \delta^{\boldsymbol{A}}(a) = \varepsilon^{\boldsymbol{A}}(a), \forall \delta(p) \approx \varepsilon(p) \in E(p)\},$$

then

$$\Gamma \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad \text{for every} \boldsymbol{A} \in \mathsf{K} \text{ and every valuation } v \text{ on } \boldsymbol{A}, \qquad (1)$$

$$\text{if } v[\Gamma] \subseteq E(\boldsymbol{A}), \text{ then } v(\varphi) \in E(\boldsymbol{A}),$$

Many logics have an algebraic semantics in this sense and in particular for many logics the classes of algebras $\mathsf{Alg}\mathcal{S}$ or $\mathsf{Alg}^{*}\mathcal{S}$ are algebraic semantics, but for other logics no algebraic semantics exists. Thus to obtain a uniform way of giving a completeness theorem for every logic $\mathcal{S}$ using the algebras in $\mathsf{Alg}\mathcal{S}$ or in $\mathsf{Alg}^*\mathcal{S}$, the consideration of the algebras is not enough. The problem can be overcome by adding more structure to the algebras. We will discuss the two approaches that are mainly considered in AAL: the addition to the algebras in $\mathsf{Alg}\mathcal{S}$ or $\mathsf{Alg}^*\mathcal{S}$ of a set of distinguished elements that basically plays the role of the set $E(\boldsymbol{A})$ but it is not necessarily a definable subset of the algebra, and the addition of a non-empty family of such sets. The first approach gives us completeness theorems closer in their form to the algebraic completeness theorems of type (1), when these exist, and it can be considered directly inspired by them, while the second is not based on these theorems.

2.2 Logical matrices and atlases

Before expounding the details of these two approaches for adding more structure to the algebras in the classes $\mathsf{Alg}\mathcal{S}$ and $\mathsf{Alg}^*\mathcal{S}$ of any given logic $\mathcal{S}$ in order to obtain (at least) a correct and complete semantics for $\mathcal{S}$, we should make two introductory remarks.

First, notice that the class $\mathsf{Alg}^*\mathcal{S}$ is obtained from pairs of the form $\langle \boldsymbol{A}, F\rangle$ where $\boldsymbol{A}$ is an algebra and F is one of its $\mathcal{S}$-filters. The objects of this form, namely an algebra and a subset of its domain, called the set of distinguished elements, are known as *logical matrices*. The local perspective on $\mathcal{S}$-filters leads immediately to the idea that the logical matrices are the natural candidates for providing any propositional logic with a complete semantics. This was the approach mainly used by the Polish logicians to attain this goal; the books [52] and [14] together constitute a compendium of this approach. The second book also expounds the developments of AAL that are grounded on the local perspective on $\mathcal{S}$-filters.

By contrast, the class of algebras $\mathsf{Alg}\mathcal{S}$ is obtained from pairs of the form $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\rangle$. From the global perspective on $\mathcal{S}$-filters these objects, which we call *basic full models* of $\mathcal{S}$, are essentially the natural candidates for a complete semantics for $\mathcal{S}$. This is one of the main ideas of [23]. For reasons that will be explained later it is useful to move to structures of the form $\langle \boldsymbol{A}, \mathcal{B}\rangle$, where $\boldsymbol{A}$ is an algebra and $\mathcal{B}$ a family of $\mathcal{S}$-filters closed under intersections of arbitrary subfamilies. These structures are the *abstract logics* of Brown and Suzko [7], and they are used in [23] to develop a general algebraic semantics for propositional logics and to obtain some results on selfextensional logics. Sometimes it is advisable to move to an even more general kind of structures of this form where $\mathcal{B}$ is just a family of $\mathcal{S}$-filters.

They are called *generalized matrices* by R. Wójcicki in [47] and *atlases* by J.M. Dunn and G.M. Hardegree in [21]. We will use the latter terminology in this paper. Notice that one can associate an abstract logic with every atlas $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B} \rangle$ by closing $\mathcal{B} \cup \{A\}$ under intersections of arbitrary non-empty families.

Before 1996 the approach of considering atlases or abstract logics to obtain correct and complete semantics for logics was less frequently and sytematically explored than the logical matrix approach. Even so several authors, coming from different perspectives, did consider abstract logics either in the form defined above or in the equivalent forms of a closure operation or of a consequence operation on the domain of an algebra (or even simply on a set without any algebraic structure, in some of the most abstract approaches) in their logical studies; in addition to those mentionad in the previous paragraph we should cite (the list is not exhaustive) Beziau, Cleave, Koslow, Magari, Mangani, Martin and Pollard. Clearly, from the purely mathematical point of view, atlases encompass both logical matrices (identify a matrix with the atlas with the same algebra and the singleton of the set of distinguished elements of the matrix) and basic full models.

One of the advantages of abstract logics and atlases over logical matrices is that abstract logics, and therefore atlases, are good structures for modelling metalogical properties. The reason is that an abstract logic can be described by considering the dual closure operator (or the dual consequence operation) instead of the closure system. Thus the properties of the consequence relation of a logic can be translated into properties of abstract logics. We will come to this in detail in Section 5.

In the rest of this subsection we introduce the basic concepts of both the semantics of logical matrices and the semantics of atlases.

A logical matrix $\langle \boldsymbol{A}, F \rangle$ is a *model* of a logic $\mathcal{S}$ when $\boldsymbol{A}$ is an $\mathcal{L}_{\mathcal{S}}$-algebra and F is an $\mathcal{S}$-filter; that is, if whenever $\Gamma \vdash_{\mathcal{S}} \varphi$, then for every valuation v on $\boldsymbol{A}$ such that $v[\Gamma] \subseteq F$, $v(\varphi) \in F$. A logical matrix $\langle \boldsymbol{A}, F \rangle$ is said to be *reduced* if $\boldsymbol{\Omega}_{\boldsymbol{A}}(F)$ is the identity relation on A.

Every logic $\mathcal{S}$ is complete relative to the class of its logical matrix models and with respect to the class of its reduced logical matrix models. That is, if $\mathsf{M}(\mathcal{S})$ is the class of the logical matrices which are a model of $\mathcal{S}$, then

$$\Gamma \vdash_{\mathcal{S}} \varphi \quad \text{iff} \quad \forall \langle \boldsymbol{A}, F \rangle \in \mathsf{M}(\mathcal{S})\, \forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A}),\ \text{if } v[\Gamma] \subseteq F, \text{ then } v(\varphi) \in F,$$

where $\mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ denotes the set of valuations on $\boldsymbol{A}$, namely the set of homomorphisms from $\boldsymbol{Fm}$ into $\boldsymbol{A}$, and similarly we have a completeness theorem when instead of $\mathsf{M}(\mathcal{S})$ we take the class of reduced logical matrix models of $\mathcal{S}$ in the statement displayed above.

Let $\mathcal{S}$ be a logic. An atlas $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B} \rangle$ of type $\mathcal{L}_{\mathcal{S}}$ is a *model* of $\mathcal{S}$ if $\mathcal{B} \subseteq \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}$; that is, if for every $F \in \mathcal{B}$, the logical matrix $\langle \boldsymbol{A}, F \rangle$ is a model of $\mathcal{S}$. Since the intersection of any family of $\mathcal{S}$-filters is an $\mathcal{S}$-filter, an atlas is a model of a logic iff its associated abstract logic is also a model.

The *Tarski congruence* of an atlas $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B} \rangle$ is the greatest congruence compatible with all the elements of $\mathcal{B}$; it will be denoted by $\widetilde{\Omega}\mathbb{A}$. Thus,

$$\widetilde{\Omega}\mathbb{A} = \bigcap_{F \in \mathcal{B}} \Omega_{\boldsymbol{A}} F.$$

An atlas $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B} \rangle$ is *reduced* if its Tarski congruence is the identity on A. If a basic full model $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A} \rangle$ is reduced (as an atlas) we will say that it is a *reduced basic full model* of $\mathcal{S}$. Given an atlas $\mathbb{A}$, the quotient by its Tarski congruence, which is defined in the natural way[4], is a reduced atlas and it is called the *reduction* of $\mathbb{A}$; it will be denoted by $\mathbb{A}^*$.

An atlas $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B} \rangle$ of type $\mathcal{L}$ induces a *substitution invariant consequence relation* on the $\mathcal{L}$-algebra of formulas (i.e. a logic), the *consequence of* $\mathbb{A}$, denoted $\vdash_{\mathbb{A}}$, and defined by

$$\Gamma \vdash_{\mathbb{A}} \varphi \quad \text{iff} \quad \forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}), \forall F \in \mathcal{B} \text{ if } v[\Gamma] \subseteq F, \text{ then } v(\varphi) \in F.$$

Similarly a class of atlases K induces a substitution invariant consequence relation on the $\mathcal{L}$-algebra of formulas, the *consequence of* K, denoted $\vdash_{\mathsf{K}}$, and defined by

$$\Gamma \vdash_{\mathsf{K}} \varphi \quad \text{iff} \quad \forall \mathbb{A} \in \mathsf{K} \;\; \Gamma \vdash_{\mathbb{A}} \varphi.$$

The consequence relation induced by an atlas and the consequence relation induced by the associated abstract logic are the same. Thus from the logical point of view an atlas and its associated abstract logic behave in a similar way. Therefore, for many purposes to consider a semantics of atlases or a semantics of abstract logics makes no essential difference.

The logic induced by a class of atlases and the logic induced by the class of the reductions of its members are the same; in particular the logic induced by an atlas and the logic induced by its reduction coincide. Since reduced atlases encode the necessary logical information and their algebras do not have logically redundant elements, the reduced atlases and the reduced

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[4] It is defined as the atlas whose algebra is $\mathcal{A}/\widetilde{\Omega}\mathbb{A}$ and whose family of sets is the family of the sets of the form $\{a/\widetilde{\Omega}\mathbb{A} : a \in X\}$ for an $X \in \mathcal{B}$.

abstract logics are the objects that play a privileged role in the global perspective on $\mathcal{S}$-filters on the semantics of propositional logics.

Every logic $\mathcal{S}$ is complete relative to the class of its atlas models and with respect to the class of its reduced atlas models. Moreover, every logic is complete relative to the class of its basic full models and with respect to the class of its reduced basic full models. That is, if K is any of the classes of atlases just mentioned, then $\vdash_{\mathsf{K}} = \vdash_{\mathcal{S}}$.

The connection between the semantics for a logic $\mathcal{S}$ given so far with the classes of algebras $\mathsf{Alg}\mathcal{S}$ and $\mathsf{Alg}^*\mathcal{S}$ is the following: on the one hand $\mathsf{Alg}\mathcal{S}$ is both the class of the algebras of the reduced basic full models of $\mathcal{S}$ and the class of algebras of the reduced atlas models of $\mathcal{S}$, and on the other hand $\mathsf{Alg}^*\mathcal{S}$ is the class of the algebras of the reduced logical matrix models of $\mathcal{S}$.

2.3 The Leibniz hierarchy

The local perspective on the logical filters taken by many AAL works has been very fruitful. It has enabled AAL to arrived at what is known as the *Leibniz hierarchy of logics*. This hierarchy maps the landscape of logics according to the behaviour of the Leibniz operator on the $\mathcal{S}$-filters of the algebras in $\mathsf{Alg}\mathcal{S}$. First of all it divides the landscape of logics into two disjoint classes: protoalgebraic and non-protoalgebraic logics. Then it divides the class of protoalgebraic logics into several subclasses. The main ones that have proven useful to isolate and study are the regularly algebraizable logics, the algebraizable logics, the weakly algebraizable logics and the equivalential logics. We will describe them briefly in this subsection.

A logic $\mathcal{S}$ is said to be *protoalgebraic* if for every $\mathcal{L}_{\mathcal{S}}$-algebra $\boldsymbol{A}$ the Leibniz operator is $\subseteq$-monotone on the $\mathcal{S}$-filters of $\boldsymbol{A}$. Protoalgebraic logics were introduced by Blok and Pigozzi in [3], and in a different but (essentially) equivalent way by Czelakowski in [12]. Protoalgebraic logics can be characterized as the logics with a generalized implication connective. They are the logics $\mathcal{S}$ with a set of formulas in two variables $(p \Rightarrow q)$ that satisfies the generalized modus ponens:

$$p, (p \Rightarrow q) \vdash_{\mathcal{S}} q,$$

and the generalized identity:

$$\vdash_{\mathcal{S}} \varphi(p, p), \text{ for every formula } \varphi(p, q) \in (p \Rightarrow q).$$

A set of formulas with these two properties is called a *set of implication formulas* for $\mathcal{S}$.

For any protoalgebraic logic $\mathcal{S}$ the classes of algebras $\mathsf{Alg}\mathcal{S}$ and $\mathsf{Alg}^*\mathcal{S}$ coincide; thus the global and the local perspectives on $\mathcal{S}$-filters coincide as to the class of algebras they associate with a protoalgebraic logic. Moreover the semantics of logical matrices works mathematically very well for protoalgebraic logics in the sense that many of the results of universal algebra on varieties and quasivarieties generalize to results for the classes of logical matrices of these logics.

A logic $\mathcal{S}$ is *equivalential* if it is protoalgebraic and the Leibniz operator commutes with inverse homomorphisms between $\mathcal{L}_\mathcal{S}$-algebras, which means that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are $\mathcal{L}_\mathcal{S}$-algebras, then for every homomorphism $h : \boldsymbol{A} \to \boldsymbol{B}$ and every $\mathcal{S}$-filter F of $\boldsymbol{B}$, $\langle a, b\rangle \in \boldsymbol{\Omega}_{\boldsymbol{A}}(h^{-1}[F])$ iff $\langle h(a), h(b)\rangle \in \boldsymbol{\Omega}_{\boldsymbol{B}}(F)$, for any $a, b \in A$. The original definition of equivalential logic given by Prucnal and Wroński in [40] is syntactical. A logic $\mathcal{S}$ is equivalential iff there is a set of formulas in two variables $(p \Leftrightarrow q)$ such that it satisfies the generalized modus ponens: $p, (p \Leftrightarrow q) \vdash_\mathcal{S} q$, and the generalized congruence rules, namely, for every formula $\varphi(p, q) \in (p \Leftrightarrow q)$

1. $\vdash_\mathcal{S} \varphi(p, p)$
2. $(p \Leftrightarrow q) \vdash_\mathcal{S} \varphi(q, p)$
3. $(p \Leftrightarrow q) \cup (q \Leftrightarrow r) \vdash_\mathcal{S} \varphi(p, r)$
4. $(p_0 \Leftrightarrow q_0) \cup \ldots \cup (p_{n-1} \Leftrightarrow q_{n-i}) \vdash_\mathcal{S} \varphi(\star(p_0, \ldots, p_{n-1}), \star(q_0, \ldots, q_{n-1}))$, for every connective $\star$ of $\mathcal{L}_\mathcal{S}$, where n is its arity.

A set of formulas with these properties is called a *set of equivalence formulas* for $\mathcal{S}$. If $\mathcal{S}$ has a finite set of equivalence formulas it is said to be *finitely equivalential.* After the definition was given by Prucnal and Wroński in 1974, Czelakowski was the first to study equivalential logics in depth in [11]; to some extent this 1981 publication can be considered one of the starting points of AAL.

A logic $\mathcal{S}$ is *weakly algebraizable* iff it is protoalgebraic and the Leibniz operator is injective on the $\mathcal{S}$-filters of every $\mathcal{L}_\mathcal{S}$-algebra. Weakly algebraizable logics are studied in [16]. If in addition $\mathcal{S}$ is equivalential it is said to be *algebraizable.* The notion of algebraizable logic for finitary and finitely equivalential logics is due to Blok and Pigozzi [4], who presented it in 1989, and it was extended to arbitrary equivalential logics by Herrman in [31, 32] and by Czelakowski in [13, 14].

Algebraizable logics can be characterized syntactically in a similar way to equivalential logics, but with the extra condition that, in addition to the set of equivalence formulas $(p \Leftrightarrow q)$, there is a set of pairs of formulas, or equations, in one variable $E(r) = \{\langle \delta_i(r), \varepsilon_i(r) \rangle : i \in I\}$, called a *set of defining equations*, such that for every $i \in I$ and every $\varphi(p, q) \in (p \Leftrightarrow q)$,

5. $p \vdash_{\mathcal{S}} \varphi(\delta_i(p), \varepsilon_i(p))$ and $\{(\delta_i(p) \Leftrightarrow \varepsilon_i(p)) : i \in I\} \vdash_{\mathcal{S}} p$.

Finally, a logic $\mathcal{S}$ is *regularly algebraizable* if it is algebraizable and

6. $p, q \vdash_{\mathcal{S}} \varphi(p, q)$, for every $\varphi \in (p \Leftrightarrow q)$.

Every algebraizable logic $\mathcal{S}$ is complete with respect to its class of algebras $\mathsf{Alg}\mathcal{S}$, in the sense mentioned in Subsection 2.2, using any of the sets $E(p)$ of defining equations for $\mathcal{S}$, that is, if $\mathcal{S}$ is algebraizable and $E(p)$ is one of these sets, then

$$\begin{aligned} \Gamma \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad & \text{for every } \boldsymbol{A} \in \mathsf{Alg}\mathcal{S} \text{ and every valuation } v \text{ on } \boldsymbol{A}, \\ & \text{if } v[\Gamma] \subseteq E(\boldsymbol{A}) \text{ then } v(\varphi) \in E(\boldsymbol{A}). \end{aligned}$$

Moreover, they also have a kind of *inverse completeness theorem*. Let Π be any set of equations and let $\varphi \approx \psi$ be any equation. Define the consequence relation between sets of equations and equations modulo $\mathsf{Alg}\mathcal{S}$ as follows:

$$\begin{aligned} \Pi \models_{\mathsf{Alg}\mathcal{S}} \varphi \approx \psi \quad \text{iff} \quad & \text{for every } \boldsymbol{A} \in \mathsf{Alg}\mathcal{S} \text{ and every valuation } v \text{ on } \boldsymbol{A}, \\ & \text{if } \boldsymbol{A} \models \Pi[v], \text{ then } \boldsymbol{A} \models \varphi \approx \psi[v]. \end{aligned}$$

If $\mathcal{S}$ is algebraizable and $(p \Leftrightarrow q)$ is any one of its sets of equivalence formulas, then

$$\Pi \models_{\mathsf{Alg}\mathcal{S}} \varphi \approx \psi \quad \text{iff} \quad \bigcup\{(\delta \Leftrightarrow \varepsilon) : \delta \approx \varepsilon \in \Pi\} \vdash_{\mathcal{S}} \gamma(\varphi, \psi), \forall \gamma \in (p \Leftrightarrow q)$$

Moreover, if $E(r)$ is any one of its sets of defining equations, for every $\delta \approx \varepsilon \in E(r)$ and every $\gamma \in (p \Leftrightarrow q)$, then

$$p \approx q \models_{\mathsf{Alg}\mathcal{S}} \delta(\gamma(p, q)) \approx \varepsilon(\gamma(p, q)),$$

and

$$\{\delta(\gamma(p, q)) \approx \varepsilon(\gamma(p, q)) : \delta \approx \varepsilon \in E(r), \gamma \in (p \Leftrightarrow q)\} \models_{\mathsf{Alg}\mathcal{S}} p \approx q.$$

In fact the existence for a logic $\mathcal{S}$ of a set of formulas $(p \Leftrightarrow q)$ in two variables and a set of equations $E(r)$ in one variable with the completeness and inverse completeness theorems above and with the two last properties characterize the algebraizable logics.

If an algebraizable logic $\mathcal{S}$ is finitary and one of its sets of equivalence formulas is finite, then its class of algebras $\mathsf{Alg}\mathcal{S}$ is a quasivariety. The algebraizable logics with these two properties are the logics called algebraizable by Blok and Pigozzi in [4]. For these logics $\mathsf{Alg}\mathcal{S}$ is known as its *equivalent quasivariety semantics*.

We now give some examples of logics in each class of the Leibniz hierarchy. There are protoalgebraic logics which are not equivalential, such as the local consequence of the classical modal logic E. There are equivalential logics which are not algebraizable such as the local consequences of the normal modal logics; some, such as the local consequence of the modal logic K, are not finitely equivalential, but others, such as the local consequence of $S4$, are finitely equivalential. There are algebraizable logics such as classical linear logic without the exponentials or the relevance logic R, which are algebraizable but not regularly algebraizable. Finally, there are regularly algebraizable logics such as classical logic, intuitionistic logic and the global consequences of the normal modal logics.

3 Selfextensional logics

The class of selfextensional logics is orthogonal to the Leibniz hierarchy in the sense that there are interesting selfextensional logics in each hierarchy's class. Some examples of selfextensional but non-protoalgebraic logics are: positive modal logic ([34]), some subintuitionistic logics ([9]) and the system of relevance logic WR ([27]). Among the selfextensional and protoalgebraic logics we find protoalgebraic but not equivalential selfextensional logics, such as the logic $\mathcal{G}_1$ studied in [6] and defined by a restricted version of the deduction theorem. Also we find equivalential but not algebraizable selfextensional logics, such as the local consequences of the normal modal logics, and algebraizable selfextensional logics, such as classical logic and intuitionistic logic.

My view is that the study of selfextensional logics deserves to be developed in full for at least the following three reasons:

1. The most developed part of the general theory of the algebraization of logics is the theory of the protoalgebraic logics, but a truly general theory

should also encompass the non-protoalgebraic logics. Since several of the selfextensional logics are non-protoalgebraic, the study of selfextensional logics provides insights for the development of the desired theory. The monograph [23] is one of the first attempts to build such theory and in that work the results on selfextensional logics are central.

2. The study of selfextensional logics introduces a point of view into the study of the algebraization of logics which differs from the one taken by the more standard studies in AAL. Thus it can highligth some hitherto unnoticed phenomena even in the protoalgebraic logics family whose study can help to articulate the general theory.

3. Selfextensional logics can be used in the study of non-selfextensional logics. An example of this use will be given in the last part of this survey where we address one of the important open questions in AAL: Why does the class of algebras $\mathsf{Alg}\mathcal{S}$ of many algebraizable logics in the sense of Blok and Pigozzi turn out to be a variety while, in general, according to Blok and Pigozzi's theory of the algebraizable logics, one can only say that it is a quasivariety? Some results on selfextensional logics explain why for some algebraizable and selfextensional logics $\mathcal{S}$, $\mathsf{Alg}\mathcal{S}$ is a variety. By associating a selfextensional companion with the same class of algebras to some algebraizable but non-selfextensional logics we can also explain why for some algebraizable but non-selfextensional logics $\mathcal{S}$, $\mathsf{Alg}\mathcal{S}$ is a variety.

3.1 Selfextensional logics and referential semantics

R. Wójcicki characterized the selfextensional logics as the logics which admit a local referential semantics ([48, 50, 51, 52]). Referential semantics is an abstraction of the different Kripke style semantics encountered in the literature and we will expound in the subsequent subsection how it is connected with the semantics of atlases for selfextensional logics. Here we expound the tools of referential semantics and Wójcicki's characterization.

Let $\mathcal{L}$ be an algebraic similarity type. A *$\mathcal{L}$-referential algebra* is a structure $\mathcal{F} = \langle W, \mathcal{A} \rangle$ where W is a non-empty set, whose elements are called points, reference points, indices or states, and $\mathcal{A}$ is an algebra of type $\mathcal{L}$ whose universe is a set of subsets of W that we denote by A. We say that $\langle W, \mathcal{A} \rangle$ is a referential algebra on W. Referential algebras are an abstraction of the relational general frames of modal logic and similar models for other kinds of logics, not necessarily distributive, like orthologics. The labels 'referential algebra' and 'referential semantics' are due to the fact that each formula φ is interpreted in a referential algebra $\langle W, \mathcal{A} \rangle$ as an element

X of A, which can be considered as its intension, and that then the formula obtains at each point $w \in W$ a reference, which is its truth value at the point: `true`, if $w \in X$ and `false`, if $w \notin X$. The elements of W can therefore be called reference points because it is on them that the interpreted formulas obtain their reference. Precisely speaking, given a $\mathcal{L}$-referential algebra $\mathcal{F} = \langle W, \mathcal{A} \rangle$, an *interpretation* is an homomorphism from the algebra of $\mathcal{L}$-formulas to $\mathcal{A}$, and given a point $w \in W$, a formula φ is *true at* w under the interpretation h if $w \in h(\varphi)$; otherwise it is *false at* w.

An $\mathcal{L}$-referential algebra $\mathcal{F} = \langle W, \mathcal{A} \rangle$ induces two very natural substitution invariant consequence relations, $\vdash^l_{\mathcal{F}}$ and $\vdash^g_{\mathcal{F}}$, on the $\mathcal{L}$-algebra of formulas, called respectively the *local* and the *global* consequence induced by $\mathcal{F}$; they are defined for every set of formulas Γ and every formula φ by

$$\Gamma \vdash^l_{\mathcal{F}} \varphi \text{ iff } \forall h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A}), \bigcap_{\psi \in \Gamma} h(\psi) \subseteq h(\varphi)$$

and

$$\Gamma \vdash^g_{\mathcal{F}} \varphi \text{ iff } \forall h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A}), \text{ if } \bigcap_{\psi \in \Gamma} h(\psi) = W, \text{ then } h(\varphi) = W.$$

Thus $\Gamma \vdash^l_{\mathcal{F}} \varphi$ iff in any interpretation φ is true at a point whenever all the formulas in Γ are true at that point, and $\Gamma \vdash^g_{\mathcal{F}} \varphi$ iff whenever all the formulas in Γ are true at every point, φ is also true at every point. Similarly, a class F of $\mathcal{L}$-referential algebras induces two substitution invariant consequence relations on the $\mathcal{L}$-algebra of formulas, the *local consequence*, denoted by $\vdash^l_{\mathsf{F}}$, and the *global consequence*, denoted by $\vdash^g_{\mathsf{F}}$. They are defined as follows:

$$\Gamma \vdash^l_{\mathsf{F}} \varphi \text{ iff } \forall \mathcal{F} \in \mathsf{F}\ \Gamma \vdash^l_{\mathcal{F}} \varphi \quad \text{ and } \quad \Gamma \vdash^g_{\mathsf{F}} \varphi \text{ iff } \forall \mathcal{F} \in \mathsf{F}\ \Gamma \vdash^g_{\mathcal{F}} \varphi,$$

for every set of formulas Γ and every formula φ.

We say that an $\mathcal{L}$-referential algebra $\mathcal{F} = \langle W, \mathcal{A} \rangle$ is a *local model* of a logic $\mathcal{S}$ if $\vdash_{\mathcal{S}} \subseteq \vdash^l_{\mathcal{F}}$, i.e., if whenever $\Gamma \vdash_{\mathcal{S}} \varphi$ and $h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A})$, then $\bigcap_{\psi \in \Gamma} h(\psi) \subseteq h(\varphi)$. Also, we say that $\mathcal{F} = \langle W, \mathcal{A} \rangle$ is a *global model* of a logic $\mathcal{S}$ if $\vdash_{\mathcal{S}} \subseteq \vdash^g_{\mathcal{F}}$, i.e. if whenever $\Gamma \vdash_{\mathcal{S}} \varphi$, $h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A})$ and $\bigcap_{\psi \in \Gamma} h(\psi) = W$, then $h(\varphi) = W$.

A class of $\mathcal{L}$-referential algebras $\mathcal{F}$ is a *complete local referential semantics* for a logic $\mathcal{S}$ if $\vdash_{\mathcal{S}} = \vdash^l_{\mathcal{F}}$.

Theorem 1 (Wójcicki) *A logic $\mathcal{S} = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}} \rangle$ of type $\mathcal{L}$ is selfextensional iff it is the local consequence relation of some class of $\mathcal{L}$-referential algebras, i.e. iff it admits a local referential semantics.*

3.2 Duality between referential algebras and atlases

For selfextensional logics the semantics of referential algebras and the semantics given by atlases are essentially dual to each other. The details of this duality were worked out in the M.A. thesis of Alessandra Palmigiano and also appear in [35]. We present its main traits here.

We will consider two categories which are dually equivalent, one whose objects are referential algebras and the other whose objects are atlases. We assume throughout this subsection that the atlases and the referential algebras are all of the same algebraic similarity type.

Frege-reduced atlases and atlas morphisms

An atlas $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B} \rangle$ is said to have *the congruence property* if the relation $\Lambda_{\boldsymbol{A}}(\mathcal{B})$ that relates two elements if and only if they belong to the same sets in $\mathcal{B}$, formally

$$\langle a, b \rangle \in \Lambda_{\boldsymbol{A}}(\mathcal{B}) \quad \text{iff} \quad (\forall X \in \mathcal{B})(a \in X \Leftrightarrow b \in X),$$

is a congruence relation of $\boldsymbol{A}$. If this is the case it coincides with $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{B})$. We call the relation $\Lambda_{\boldsymbol{A}}(\mathcal{B})$ the *Frege relation* of the atlas $\langle \boldsymbol{A}, \mathcal{B} \rangle$. If $\Lambda_{\boldsymbol{A}}(\mathcal{B})$ is the identity relation on A we say that the atlas $\mathbb{A}$ is *Frege-reduced.* It is easy to see that an atlas $\mathbb{A}$ has the congruence property iff its reduction $\mathbb{A}^*$ is Frege-reduced. Thus the reduced atlases with the congruence property are exactly the Frege-reduced atlases. Notice that a logic $\mathcal{S}$ is selfextensional iff the atlas $\langle \boldsymbol{Fm}, \mathrm{Th}\mathcal{S} \rangle$ has the congruence property.

Let $\mathbb{A}_1 = \langle \boldsymbol{A}_1, \mathcal{B}_1 \rangle$ and $\mathbb{A}_2 = \langle \boldsymbol{A}_2, \mathcal{B}_2 \rangle$ be two $\mathcal{L}$-atlases. A map $h : A_1 \to A_2$ is an *atlas morphism* from $\mathbb{A}_1$ into $\mathbb{A}_2$ if

1. h is an (algebra) homomorphism from $\boldsymbol{A}_1$ into $\boldsymbol{A}_2$;
2. $\{h^{-1}[Y] : Y \in \mathcal{B}_2\} \subseteq \mathcal{B}_1$.

An atlas morphism h from $\mathbb{A}_1 = \langle \boldsymbol{A}_1, \mathcal{B}_1 \rangle$ into $\mathbb{A}_2 = \langle \boldsymbol{A}_2, \mathcal{B}_2 \rangle$ is said to be *strict* if in addition it satisfies

3. $\mathcal{B}_1 = \{h^{-1}[Y] : Y \in \mathcal{B}_2\}$.

A typical example of strict atlas morphism is the projection of an atlas onto its reduction.

Proposition 2 *Let h be an atlas morphism from $\mathbb{A}_1$ into $\mathbb{A}_2$.*

1. *If h is a strict and $\mathbb{A}_1$ is Frege-reduced, then h is injective.*
2. *if h is strict, then $\vdash_{\mathbb{A}_2} \subseteq \vdash_{\mathbb{A}_1}$;*
3. *if h is onto, then $\vdash_{\mathbb{A}_1} \subseteq \vdash_{\mathbb{A}_2}$;*
4. *if h is strict and onto, then $\mathbb{A}_1$ is a model of a logic $\mathcal{S}$ iff $\mathbb{A}_2$ is a model of $\mathcal{S}$.*

Reduced referential algebras and referential algebra morphisms

Given a referential algebra $\mathcal{F} = \langle W, \mathcal{A} \rangle$ we can consider the relation $\theta(\mathcal{F})$ defined by

$$\langle u, v \rangle \in \theta(\mathcal{F}) \quad \text{iff} \quad (\forall X \in \mathcal{A})(u \in X \Leftrightarrow v \in X)$$

This relation is clearly an equivalence relation. If it is the identity, that is if every pair of different points are separated by some set in the universe of $\mathcal{A}$, the referential algebra is said to be *reduced*, *differentiated*, or simply T_0.

Each referential algebra $\mathcal{F} = \langle W, \mathcal{A} \rangle$ has its reduction $\mathcal{F}^*$ which is defined by identifying the elements of W that belong to the same elements of $\mathcal{A}$. It is the referential algebra $\mathcal{F}^* = \langle W/\theta(\mathcal{F}), \mathcal{A}/\theta(\mathcal{F}) \rangle$ where the universe of the algebra $\mathcal{A}/\theta(\mathcal{F})$ is the set $\{\pi[X] : X \in \mathcal{A}\}$, where $\pi[X] = \{w/\theta(\mathcal{F}) : w \in X\}$, and the operations are the naturally induced ones by the operations of $\mathcal{A}$. Formally speaking, if $f \in \mathcal{L}$ is n-ary,

$$f^{\mathcal{A}/\theta(\mathcal{F})}(\pi[X_1], \ldots \pi[X_n]) = \pi[f^{\mathcal{A}}(X_1, \ldots, X_n)],$$

for every $X_1, \ldots, X_n \in \mathcal{A}$. This definition is sound because from the definition of $\theta(\mathcal{F})$ it easily follows that for every $X, Y \in \mathcal{A}$, $\pi[X] = \pi[Y]$ iff $X = Y$.

It is easy to see that:

1. The local consequence of $\mathcal{F}$ and the local consequence of its reduction $\mathcal{F}^*$ are the same.
2. $\theta(\mathcal{F}^*)$ is the identity, that is $\mathcal{F}^*$ is reduced.

From the logical point of view, it is natural then to restrict ourselves to referential algebras with the last property.

Let $\mathcal{F}_1 = \langle W_1, \mathcal{A}_1 \rangle$ and $\mathcal{F}_2 = \langle W_1, \mathcal{A}_1 \rangle$ be two $\mathcal{L}$-referential algebras. A map $f : W_1 \to W_2$ is a *morphism (of referential algebras)* from $\mathcal{F}_1 = \langle W_1, \mathcal{A}_1 \rangle$ into $\mathcal{F}_2 = \langle W_1, \mathcal{A}_1 \rangle$ if

1. $f^{-1}[Y] \in A_1$ for every $Y \in A_2$;
2. the map $f^{-1} : A_2 \to A_1$ is an homomorphism from $\mathcal{A}_2$ into $\mathcal{A}_1$.

A morphism f from $\mathcal{F}_1 = \langle W_1, \mathcal{A}_1 \rangle$ into $\mathcal{F}_2 = \langle W_1, \mathcal{A}_1 \rangle$ is said to be *strict* if in addition it satisfies

3. $A_1 = \{f^{-1}[Y] : Y \in A_2\}$.

A typical example of a strict morphism of referential algebras is the projection π from a referential algebra $\mathcal{F} = \langle W, \mathcal{A} \rangle$ onto its reduction $\mathcal{F}^*$.

Proposition 3 *Let f be a morphism from $\mathcal{F}_1 = \langle W_1, \mathcal{A}_1 \rangle$ into $\mathcal{F}_2 = \langle W_1, \mathcal{A}_1 \rangle$.*

1. *If f is a strict and $\mathcal{F}_1$ is differentiated, then f is injective.*
2. *if f is strict, then $\vdash_{\mathcal{F}_2} \subseteq \vdash_{\mathcal{F}_1}$;*
3. *if f is onto, then $\vdash_{\mathcal{F}_1} \subseteq \vdash_{\mathcal{F}_2}$;*
4. *if f is strict and onto, then $\mathcal{F}_1$ is a model of a logic $\mathcal{S}$ iff $\mathcal{F}_2$ is a model of $\mathcal{S}$.*

The functor $(.)^+$ from referential algebras to atlases

Let $\mathcal{F} = \langle W, \mathcal{A} \rangle$ be a referential algebra. The dual atlas of $\mathcal{F}$ is defined as the pair $\mathcal{F}^+ = \langle \mathcal{A}, W^+ \rangle$ with

$$W^+ = \{\varepsilon(v) : v \in W\}$$

where ε is the map from W to the powerset of the universe A of $\mathcal{A}$ defined by

$$\varepsilon(v) = \{Y \in A : w \in Y\}$$

for every $v \in W$.

For every referential algebra $\mathcal{F}$, its dual $\mathcal{F}^+$ is a Frege-reduced atlas. Moreover the local consequence relation of $\mathcal{F}$ and the consequence relation of $\mathcal{F}^+$ are the same. Thus, for every referential algebra $\mathcal{F}$, $\mathcal{F}$ is a local model of a logic $\mathcal{S}$ iff its dual $\mathcal{F}^+$ is a model of $\mathcal{S}$. These facts show that to find the dual category of referential algebras given by the construction $(.)^+$ we have to restrict ourselves to Frege-reduced atlas.

The dual of a referential algebra morphism is defined as follows. Let f be a referential algebra morphism from $\mathcal{F}_1 = \langle W_1, \mathcal{A}_1 \rangle$ into $\mathcal{F}_2 = \langle W_2, \mathcal{A}_2 \rangle$. The dual of f is the function $f^+ : A_2 \to A_1$ defined by

$$f^+(Y) = f^{-1}[Y]$$

for every $Y \in A_2$.

Proposition 4 *Let f be a referential algebra morphism from $\mathcal{F}_1 = \langle W_1, \mathcal{A}_1 \rangle$ into $\mathcal{F}_2 = \langle W_2, \mathcal{A}_2 \rangle$. Then, f^+ is an atlas morphism from $(\mathcal{F}_2)^+$ into $(\mathcal{F}_1)^+$, which is strict if f is onto, and which is onto if f is strict.*

As a corollary we have:

Corollary 5 *Let $\mathcal{F} = \langle W, \mathcal{A} \rangle$ be a referential algebra and let $\mathcal{F}^*$ be its reduction, which is differentiated. Then $\mathcal{F}^+$ and $(\mathcal{F}^*)^+$ are isomorphic.*

The corollary shows that we can concentrate on reduced referential algebras. Thus the category of referential algebras we will consider has the reduced referential algebras as objects and the referential algebra morphisms between them as arrows. It is easy to check that it is indeed a category. We denote it by **RRA**.

The functor $(.)_+$ from Frege-reduced atlases to referential algebras

Let $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B} \rangle$ be a Frege-reduced atlas. We define the dual referential algebra of $\mathbb{A}$ as the structure $\mathbb{A}_+ = \langle \mathcal{B}, \overline{\boldsymbol{A}} \rangle$ that we describe below. Let us first define the map η from A into the power set of $\mathcal{B}$ by

$$\eta(a) = \{X \in \mathcal{B} : a \in X\}$$

for every $a \in A$. Then $\mathbb{A}_+$ is defined by defining the algebra $\overline{\boldsymbol{A}}$ as follows:

1. the universe of the algebra $\overline{\boldsymbol{A}}$ is the set $\overline{A} = \{\eta(a) : a \in A\}$
2. for every n-ary symbol $f \in \mathcal{L}$ we define the n-ary operation on $\overline{A}$ by declaring

   $$f^{\overline{\boldsymbol{A}}}(\eta(a_1), \ldots, \eta(a_n)) = \eta(f^{\boldsymbol{A}}(a_1, \ldots, a_n))$$

   for every $a_1, \ldots, a_n \in A$.

The fact that $\mathbb{A}$ is Frege-reduced guarantees that for every $f \in \mathcal{L}$ the operation $f^{\overline{\boldsymbol{A}}}$ is well defined. Moreover, the definition of the algebra $\overline{\boldsymbol{A}}$ guarantees that η is an homomorphism from $\boldsymbol{A}$ onto $\overline{\boldsymbol{A}}$.

If $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B} \rangle$ is a Frege-reduced atlas, its dual $\mathbb{A}_+$ is a reduced referential algebra. Moreover, the consequence relation of $\mathbb{A}$ and the local consequence relation of $\mathbb{A}_+$ are the same; therefore $\mathbb{A}$ is a model of a logic $\mathcal{S}$ iff $\mathbb{A}_+$ is a local model of $\mathcal{S}$. Finally, the map η is an isomorphism between $\boldsymbol{A}$ and $\overline{\boldsymbol{A}}$.

Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be Frege-reduced atlases and let h be an atlas morphism from $\mathbb{A}_1$ into $\mathbb{A}_2$. The dual function $h_+ : \mathcal{B}_2 \to \mathcal{B}_1$ is defined by

$$h_+(Y) = h^{-1}[Y]$$

for every $Y \in \mathcal{B}_2$.

Proposition 6 *Let h be an atlas morphism from an atlas $\mathbb{A}_1$ into an atlas $\mathbb{A}_2$, both Frege-reduced. Then, h_+ is a referential algebra morphism from $(\mathbb{A}_2)_+$ into $(\mathbb{A}_1)_+$ which is strict if h is onto and onto if h is strict.*

The duality theorem

Let **FRAt** be the category whose objects are the Frege-reduced atlases of type $\mathcal{L}$ and whose arrows are the atlas morphisms between them, and let **RRA** be the category whose objects are the reduced referential algebras of type $\mathcal{L}$ and whose arrows are the referential algebra morphism between them. We consider the functors $(.)_+$ from **FRAt** into **RRA** and $(.)^+$ from **RRA** into **FRAt** defined above.

Theorem 7 (Palmigiano) *The functors $(.)_+$ and $(.)^+$ establish a dual equivalence between the category* **FRA** *and the category* **RRA**.

4 Fully selfextensional logics

For every selfextensional logic $\mathcal{S}$ it holds (by definition) that the Frege relation of the atlas $\langle \boldsymbol{Fm}, \mathrm{Th}(\mathcal{S}) \rangle$, namely $\Lambda_{\boldsymbol{Fm}}(\mathrm{Th}(\mathcal{S}))$ (which is the interderivability relation of $\mathcal{S}$), is a congruence relation. One of the open questions in [23] was whether for every selfextensional logic $\mathcal{S}$ and every algebra $\boldsymbol{A}$, the atlas $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A} \rangle$ has the congruence property.

Questions such as the above are typical of AAL. They are called *transfer problems*. Given a metalogical property Φ that is applicable to logics and can be abstractly formulated for basic full models, we say that Φ transfers for a logic $\mathcal{S}$ with the property Φ if any one of its basic full models $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\rangle$ also has the property Φ. We say that Φ transfers (in general) if it transfers for every logic with the property Φ.

A logic $\mathcal{S}$ is said to be *fully selfextensional* if for every algebra $\boldsymbol{A}$, the atlas $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\rangle$ has the congruence property, that is, if the "selfextensionality" property transfers for $\mathcal{S}$.

The transfer problem for selfextensionality has been solved in the negative by Babyonishev [1]. Nevertheless the only logic which is known to be selfextensional but not fully selfextensional was defined by Babyonishev in an ad hoc way just to show that the concepts selfextensional and fully selfextensional are not coextensive and also that the concepts of Fregean logic and fully Fregean logic (stronger than selfextensional and fully selfextensional respectively), which we do not discuss here, are not coextensive either.

In [23] it is proved that the selfextensional logics with a conjunction and the selfextensional logics with an implication that satisfies the modus ponens and the deduction theorem are fully selfextensional. These results show clearly that it will be hard to find natural selfextensional logics which are not fully selfextensional. To be precise, a logic $\mathcal{S}$ has the *property of conjunction* (PC) if there is a binary term $x \wedge y$ such that the three rules

$$\varphi, \varphi \vdash \varphi \wedge \psi \qquad \varphi \wedge \psi \vdash \varphi \qquad \varphi \wedge \psi \vdash \psi$$

are rules of $\mathcal{S}$. In this situation we say that $\mathcal{S}$ has (PC) relative to $\wedge$. A logic $\mathcal{S}$ has the *uniterm deduction-detachment property* (u-DDP) if there is a binary term $x \rightarrow y$ such that:

$$\Gamma, \varphi \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} \varphi \rightarrow \psi.$$

In this situation we say that $\mathcal{S}$ has the u-DDP relative to $\rightarrow$.

Theorem 8 (Font, Jansana) *If $\mathcal{S}$ is selfextensional with (PC) or the (u-DDDT), then $\mathcal{S}$ is fully selfextensional and moreover* $\mathsf{Alg}\mathcal{S}$ *is a variety.*

Fully selfextensional logics can be characterized inside the class of selfextensional logics by properties of their complete local referential semantics of reduced referential algebras. We expound briefly two characterizations given in [35]. One of them shows that the fully selfextensional logics are the selfextensional logics $\mathcal{S}$ for which the duality between the categories **RRA** and **FRAt** specializes in a very good way to a duality between the category with objects the elements of $\mathsf{Alg}\mathcal{S}$ and with arrows the homomorphisms between the elements of $\mathsf{Alg}\mathcal{S}$ and a subcategory of the reduced referential algebra models of $\mathcal{S}$.

Let $\mathcal{S}$ be a logic and let us consider the full subcategories of **RRA** and **FRAt** whose objects are respectively the reduced referential algebras which are a model of $\mathcal{S}$ and the Frege-reduced atlases which are a model of $\mathcal{S}$. We denote these two categories by $\mathbf{RRAM}_{\mathcal{S}}$ and $\mathbf{FRAt}_{\mathcal{S}}$. Moreover let us consider the category of atlases whose objects are the atlases of the form $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\rangle$ with $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$ (i.e., the basic full models of $\mathcal{S}$ whose algebra is in $\mathsf{Alg}\mathcal{S}$) and whose arrows are the atlas morphisms between them. If $\mathcal{S}$ is fully selfextensional the objects of this category are all Frege-reduced atlases and the category is a full subcategory of $\mathbf{FRAt}_{\mathcal{S}}$. We denote it by $\mathbf{FRBFM}_{\mathcal{S}}$.

Let $\mathcal{S}$ be a fully selfextensional logic. Since the inverse image of an $\mathcal{S}$-filter by an algebra homomorphism is an $\mathcal{S}$-filter, the category $\mathbf{FRBFM}_{\mathcal{S}}$ is isomorphic to the category $\mathbf{ALG}_{\mathcal{S}}$ whose objects are the elements of $\mathsf{Alg}\mathcal{S}$ and whose arrows are the algebra homomorphisms. The duality theorem implies that $\mathbf{FRBFM}_{\mathcal{S}}$ is dually equivalent to a full subcategory of the category $\mathbf{RRAM}_{\mathcal{S}}$. Thus the category $\mathbf{ALG}_{\mathcal{S}}$ is dually equivalent to a full subcategory $\mathbf{C}$ of $\mathbf{RRAM}_{\mathcal{S}}$. This statement hides important information given by its proof. If we move from the subcategory $\mathbf{C}$ of $\mathbf{RRAM}_{\mathcal{S}}$ by the functor $(.)^{+}$ to its dually equivalent category $\mathbf{FRBFM}_{\mathcal{S}}$ of Frege-reduced atlases, then the category $\mathsf{Alg}\mathcal{S}$ is obtained by the "forgetful" functor that maps each Frege-reduced atlas $\langle \boldsymbol{A}, \mathcal{B}\rangle$ to its algebra $\boldsymbol{A}$. The statement can be turned into a characterization of the fully selfextensional logics.

Theorem 9 *A logic $\mathcal{S}$ is fully selfextensional iff the category $\mathbf{ALG}_{\mathcal{S}}$ is dually equivalent to a full subcategory $\mathbf{C}$ of the category $\mathbf{RRAM}_{\mathcal{S}}$ and the composition of the functor $(.)^{+}$ with the forgetful "functor" is the functor of the equivalence from $\mathbf{C}$ onto $\mathsf{Alg}\mathcal{S}$.*

Another interesting characterization of the selfextensional logics which are fully selfextensional is given by the theorem below. The direction from left to right is an abstract *representation theorem* for the class of algebras $\mathsf{Alg}\mathcal{S}$ of any fully selfextensional logic $\mathcal{S}$. It can be considered as the abstract general framework for most of the well-known representation theorems for the classes of algebras associated with specific logics because many of them, even if not selfextensional, have an associated fully selfextensional logic companion with the same class of algebras. In the last section of the paper we will define this companion for a certain class of algebraizable logics.

Let

$$\mathsf{AlgRef}\mathcal{S} = \{\mathcal{A} : (\exists W)\ \langle W, \mathcal{A}\rangle \text{ is a reduced referential algebra model of } \mathcal{S}\},$$

that is, $\mathsf{AlgRef}\mathcal{S}$ is the class of the algebraic reducts of the reduced referential algebras which are a model of $\mathcal{S}$. In general it holds that $\mathsf{AlgRef}\mathcal{S} \subseteq \mathsf{Alg}\mathcal{S}$.

Theorem 10 *For any logic $\mathcal{S}$, $\mathcal{S}$ is fully selfextensional iff* $\mathsf{Alg}\mathcal{S} = \mathbf{I}(\mathsf{AlgRef}\mathcal{S})$.

The theorem above provides one of the reasons to consider $\mathsf{Alg}\mathcal{S}$ as the canonical class of algebras of a logic. Selfextensional logics are the logics with a local referential semantics. Among them, fully selfextensional logics are the logics whose class of algebras coincides up to isomorphisms with the class of algebras provided by its local referential semantics. Thus they are the selfextensional logics for which there is a good match between the two approaches to their semantics: the purely algebraic and the referential algebraic.

5 Abstract logics and Gentzen style rules

One of the features of abstract logics, and more generally of atlases, that make the global perspective on $\mathcal{S}$-filters fruitful is that they serve, as we already mentioned, as models of metalogical properties, in particular of the metalogical properties that can be encoded in Gentzen-style rules.

The perspective on atlases that enables one to consider them as possible models of Gentzen-style rules arises when we associate a closure operation with them. Let $\langle \boldsymbol{A}, \mathcal{B}\rangle$ be an atlas. Consider the closure system $\mathcal{C}_{\mathcal{B}}$ generated by $\mathcal{B}$, that is the set $\{\bigcap X : X \subseteq \mathcal{B}\}$, where $\bigcap \emptyset = A$. As a closure system $\mathcal{C}_{\mathcal{B}}$ has its corresponding closure operation $C_{\mathcal{B}}$, which is the map $C_{\mathcal{B}} : \mathcal{P}(A) \to \mathcal{P}(A)$ defined by

$$C_{\mathcal{B}}(X) = \bigcap \{Z \in \mathcal{C} : X \subseteq Z\}$$

for every $X \subseteq A$.

The properties of $C_{\mathcal{B}}$ that make it a closure operation are

1. $X \subseteq C_{\mathcal{B}}(X)$;

2. if $X \subseteq Y$, then $C_{\mathcal{B}}(X) \subseteq C_{\mathcal{B}}(Y)$;
3. $C_{\mathcal{C}}(C_{\mathcal{B}}(X)) \subseteq C_{\mathcal{B}}(X)$.

Moreover, we have the consequence relation $\vdash_{\mathcal{B}}$ (between subsets of A and elements of A) associated with $C_{\mathcal{B}}$. It is defined by

$$X \vdash_{\mathcal{B}} a \quad \textit{iff} \quad a \in C_{\mathcal{B}}(X)$$

for every $X \subseteq A$ and every $a \in A$. This relation will be called the consequence relation of the atlas $\langle A, \mathcal{B}\rangle$. The properties that make $\vdash_{\mathcal{B}}$ into a consequence relation are

1. if $a \in X$, $X \vdash_{\mathcal{B}} a$ (identity);
2. if $X \subseteq Y$ and $X \vdash_{\mathcal{B}} a$, then $Y \vdash_{\mathcal{B}} a$ (monotonicity);
3. if for every $a \in Y$, $X \vdash_{\mathcal{B}} a$ and $Y \vdash_{\mathcal{B}} b$, then $X \vdash_{\mathcal{B}} b$ (Abstract General Cut)

These conditions are the *abstract versions* of the conditions that define a logic, which for finitary logics are usually encoded together in the three Gentzen style rules

$$\frac{}{\Gamma, \varphi \vdash \varphi} \qquad \frac{\Gamma \vdash \varphi}{\Gamma, \Delta \vdash \varphi} \qquad \frac{\Gamma \vdash \varphi, \quad \Gamma, \varphi \vdash \psi}{\Gamma \vdash \psi}$$

A *sequent* is a pair (Γ, φ) where Γ is a finite set of formulas and φ is a formula. A sequent is usually written in the form

$$\Gamma \vdash \varphi$$

A *Gentzen style rule*, G-rule for short, is a pair $(\overline{\Pi}, \Gamma \vdash \varphi)$, where $\overline{\Pi}$ is a finite set of sequents, the sequents to which the rule applies, and (Γ, φ) is a sequent, the sequent produced by the rule. If $\overline{\Pi} = \{\Gamma_i \vdash \varphi_i : i \leq n\}$, the Gentzen rule is usually written as

$$\frac{\Gamma_i \vdash \varphi_i : i \leq n}{\Gamma \vdash \varphi} \tag{2}$$

We can generalize the notion of Gentzen style rule to admit rules where $\overline{\Pi}$ is an infinite set.

Given a Gentzen style rule (2), a substitution instance of (2) is a rule

$$\frac{\sigma[\Gamma_i] \vdash \sigma[\varphi_i] : i \leq n}{\sigma[\Gamma] \vdash \sigma[\varphi]}$$

for an arbitrary substitution σ.

An atlas, or an abstract logic, $\langle \boldsymbol{A}, \mathcal{B}\rangle$ will be said to be a *model of a Gentzen style rule* (2) if for every valuation v on $\boldsymbol{A}$, whenever $v(\varphi_i) \in C_{\mathcal{B}}(v[\Gamma_i])$ for every $i \leq n$, $v(\varphi) \in C_{\mathcal{B}}(v[\Gamma])$. We will also say that the rule holds in the atlas or the abstract logic.

In particular, given a logic $\mathcal{S}$ we say that a Gentzen style rule (2) is *valid of* $\mathcal{S}$ if $\langle \boldsymbol{Fm}, \mathrm{Th}(\mathcal{S})\rangle$ is a model of (2), that is, if for every substitution σ such that $\sigma[\Gamma_i] \vdash_{\mathcal{S}} \sigma(\varphi_i)$ for every $i \leq n$, $\sigma[\Gamma] \vdash_{\mathcal{S}} \sigma(\varphi)$.

6 Full Models

From the global perspective on the $\mathcal{S}$-filters of a logic $\mathcal{S}$ we are interested mainly in the properties of the basic full models of $\mathcal{S}$. Nevertheless each basic full model is "logically indiscernible" from a proper class of abstract logics related to it by the relativeness relation we describe below. Therefore for many purposes it is better to deal with this wider class.

Let $\langle \boldsymbol{A}, \mathcal{C}\rangle$ and $\langle \boldsymbol{B}, \mathcal{D}\rangle$ be abstract logics. A *bilogical morphism* between $\langle \boldsymbol{A}, \mathcal{C}\rangle$ and $\langle \boldsymbol{B}, \mathcal{D}\rangle$ is any strict atlas morphism from $\langle \boldsymbol{A}, \mathcal{C}\rangle$ onto $\langle \boldsymbol{B}, \mathcal{D}\rangle$.

An abstract logic $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is said to be a *relative* of an abstract logic $\langle \boldsymbol{B}, \mathcal{D}\rangle$ if it belongs to the smallest class of abstract logics that contains $\langle \boldsymbol{B}, \mathcal{D}\rangle$ and is closed under images and inverse images by bilogical morphisms.

The property of being a model of a Gentzen style rule is preserved under images and inverse images by bilogical morphisms, that is, if h is a bilogical morphism from $\langle \boldsymbol{A}, \mathcal{C}\rangle$ onto $\langle \boldsymbol{B}, \mathcal{D}\rangle$ and $\mathcal{G}$ is a Gentzen rule, then $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a model of $\mathcal{G}$ iff $\langle \boldsymbol{B}, \mathcal{D}\rangle$ is a model of $\mathcal{G}$. This fact shows that if we are interested in a class of abstract logics that are models of a logic $\mathcal{S}$, it seems reasonable that if we accept one model we must also accept all its relatives.

Since, as we already said, our main candidates as models of $\mathcal{S}$ are the basic full models of $\mathcal{S}$, we shall be interested in the class of abstract logics whose elements are the basic full models of $\mathcal{S}$ and all their relatives. We call the elements of this class *full models* of $\mathcal{S}$. In fact it can be shown that an abstract logic $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a full model of $\mathcal{S}$ iff there is a bilogical morphism from $\langle \boldsymbol{A}, \mathcal{C}\rangle$ onto a basic full model of $\mathcal{S}$.

By the preservation of the property of being a model of a Gentzen style rule by bilogical morphisms and inverses of bilogical morphisms, we have that for any Gentzen style rule R, the full models of a logic $\mathcal{S}$ are models of R iff the basic full models of $\mathcal{S}$ are models of R. In particular, since every logic is complete relative to the class of its basic full models, every logic is complete relative to the class of its full models and with respect to the class of its reduced full models. Moreover, to study which Gentzen rules hold in every full model of a logic $\mathcal{S}$ it is enough to see which ones hold in every one of its reduced basic full models.

From the perspective of the closure operators we can also say that an atlas $\langle \boldsymbol{A}, \mathcal{B}\rangle$ is *finitary* if the associated closure operator $C_{\mathcal{B}}$ is finitary, that is, if it holds that for every $X \subseteq A$ and every $a \in A$, such that $a \in C_{\mathcal{B}}(X)$, then there is a finite set $Y \subseteq X$ such that $a \in C_{\mathcal{B}}(Y)$. The basic full models of a finitary logic are all finitary and the property of being finitary is preserved by images and inverse images by bilogical morphisms. Thus every full model of a finitary logic is a finitary abstract logic.

7 Fully adequate Gentzen systems

From an abstract point of view we define a *Gentzen calculus* as just a set of Gentzen style rules. In the same way that a Hilbert-style axiomatic system defines a consequence relation on the set of formulas that is substitution invariant (namely, the relation $\vdash$ defined by: $\Gamma \vdash \varphi$ iff there is a proof of φ in the calculus using premises in Γ) a Gentzen calculus defines a consequence relation between sets of sequents and sequents which is invariant under substitutions in the natural way. Formally, given a Gentzen calculus $\mathbf{G}$, we denote by

$$\mid\!\sim_{\mathbf{G}}$$

the relation between sets of sequents and sequents generated by $\mathbf{G}$. That is,

$$\{\Gamma_i \vdash \varphi_i : i \in I\} \mid\!\sim_{\mathbf{G}} \Gamma \vdash \varphi \text{ iff there is a proof of } \Gamma \vdash \varphi \text{ from the sequents}$$
$$\text{in } \{\Gamma_i \vdash \varphi_i : i \in I\} \text{ using the substitution instances of the rules of } \mathbf{G}.$$

The consequence relations defined in this way by Gentzen calculus are called *Gentzen systems*, see [42, 30, 25]. Since atlases and abstract logics serve as models of Gentzen style rules, they also serve as models of Gentzen calculus and of Gentzen systems.

It is easy to show that an atlas $\langle \boldsymbol{A}, \mathcal{B}\rangle$ is a *model of* a Gentzen calculus $\mathbf{G}$ iff for every set of sequents $\{\Gamma_i \vdash \varphi_i : i \in I\}$ and every sequent $\Gamma \vdash \varphi$ such that $\{\Gamma_i \vdash \varphi_i : i \in I\} \mid\!\sim_{\mathbf{G}} \Gamma \vdash \varphi$ it holds that $\langle \boldsymbol{A}, \mathcal{B}\rangle$ is a model of the corresponding 'infinite' Gentzen style rule

$$\frac{\Gamma_i \vdash \varphi_i : i \in I}{\Gamma \vdash \varphi}.$$

Given a finitary logic $\mathcal{S}$ the following question was investigated in [23]: Is there a Gentzen calculus such that its finitary abstract logic models are exactly the full models of $\mathcal{S}$? If such a Gentzen calculus exists then its Gentzen system captures exactly all the Gentzen style rules which hold in every full model of $\mathcal{S}$, and therefore these rules are the Gentzen style rules which are valid of $\mathcal{S}$ and transfer to every one of its full models.

Given a finitary logic $\mathcal{S}$ with theorems we say that a Gentzen system is *fully adequate* if its finitary abstract logic models are exactly the full models of $\mathcal{S}$; if $\mathcal{S}$ does not have theorems we say that a Gentzen system is *fully adequate* if its finitary abstract logic models $\langle \boldsymbol{A}, \mathcal{C}\rangle$ with the property that $\emptyset \in \mathcal{C}$ are exactly the full models of $\mathcal{S}$. If a finitary logic $\mathcal{S}$ has a fully adequate Gentzen system, it is unique. Thus the question above is the question whether any finitary logic has a fully adequate Gentzen system. In [23] the following result is obtained:

Theorem 11 *If a finitary selfextensional logic $\mathcal{S}$ has (PC) or the (u-DDDT), then it has a fully adequate Gentzen system.*

It is also known that not every finitary logic has a fully adequate Gentzen system. In [24] it is shown that a finitary and weakly algebraizable logic has a fully adequate Gentzen system iff it has the multiterm deduction detachment-theorem (which is like the u-DDT but instead of a single formula $p \to q$ one has a set of formulas in two variables that collectively behave as the formula $p \to q$). There are algebraizable logics without the multiterm deduction-detachment theorem, for instance the global consequence of the least normal modal logic K; thus there are logics without a fully adequate Gentzen system. Moreover, there are examples of selfextensional logics without a fully adequate Gentzen system, for instance the $\Box$-fragment of the local consequence of the normal modal logic K.

8 Selfextensional logics and algebraizable logics

One of the important questions that it is not still fully answered in AAL is this: why do the class of algebras $\mathsf{Alg}\mathcal{S}$ of many finitary algebraizable logics turn out to be a variety when in general, according to the theory of algebraizable logics of Blok and Pigozzi one can only say that it is a quasivariety?

The results in [23] provide a partial answer, given in the theorem below which is implied by Theorem 8.

Theorem 12 *If $\mathcal{S}$ is an algebraizable logic which is finitary and selfextensional and has (PC) or has (u-DDT), then its equivalent algebraic semantics* $\mathsf{Alg}\mathcal{S}$ *is a variety.*

This result can be used to explain why other algebraizable logics wich have(PC) but are non-selfextensional have a variety as their equivalent algebraic semantics. We report some results of [36].

Let K be any class of algebras and $\wedge$ a binary term that defines a meet-semilattice operation on each algebra in K. Define the finitary logic $\mathcal{S}^{\leq}_{\mathsf{K}}$ as follows: for every $\varphi_0, \ldots, \varphi_{n-1}, \varphi$,

$$\varphi_0, \ldots, \varphi_{n-1} \vdash_{\mathcal{S}^{\leq}_{\mathsf{K}}} \varphi \quad \text{iff} \quad \forall \boldsymbol{A} \in \mathsf{K}\ \forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$$

$$v(\varphi_0) \wedge^{\boldsymbol{A}} \ldots \wedge^{\boldsymbol{A}} v(\varphi_{n-1}) \leq^{\boldsymbol{A}} v(\varphi)$$

and

$$\vdash_{\mathcal{S}^{\leq}_{\mathsf{K}}} \varphi \quad \text{iff} \quad \forall \boldsymbol{A} \in \mathsf{K}\ \forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})\ \forall a \in A, a \leq^{\boldsymbol{A}} v(\varphi),$$

where $\leq^{\boldsymbol{A}}$ is the order associated with the meet operation $\wedge^{\boldsymbol{A}}$. We say that a finitary logic $\mathcal{S}$ is *semilattice-based* if there is a class of algebras K and a binary term that defines a meet-semilattice operation on every element of K such that $\mathcal{S} = \mathcal{S}^{\leq}_{\mathsf{K}}$. Every semilattice-based logic is selfextensional.

Let $\mathcal{S}$ be from now on a finitary algebraizable logic with (PC) relative to $\wedge$ such that for every algebra $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, $\langle A, \wedge^{\boldsymbol{A}} \rangle$ is a semilattice. We define the *semilattice-based companion* of $\mathcal{S}$ as the logic $\mathcal{S}^{\leq}_{\mathsf{Alg}\mathcal{S}}$, which we simply denote by $\mathcal{S}^{\leq}$.

Under these conditions:

Proposition 13

1. *The logic* $\mathcal{S}$ *is an extension of* $\mathcal{S}^{\leq}$*;*
2. $\mathcal{S}^{\leq}$ *is fully selfextensional;*
3. $\mathsf{Alg}\mathcal{S} \subseteq \mathsf{Alg}^{*}\mathcal{S}^{\leq} \subseteq \mathsf{Alg}\mathcal{S}^{\leq}$.

Assume moreover that $(p \Leftrightarrow q)$ is a set of equivalence formulas and $E(p)$ a set of defining equations for $\mathcal{S}$ (see Section 2.3). Recall that we denote by $E(\boldsymbol{A})$ the set of solutions of the equations in $E(p)$ in the algebra $\boldsymbol{A}$.

Theorem 14 *If* $\mathcal{S}$ *has an implication set of formulas* $(p \Rightarrow q)$ *(see Section 2.3) such that*

1. *for every* $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, $a \leq^{\boldsymbol{A}} b$ *iff* $(a \Rightarrow^{\boldsymbol{A}} b) \subseteq E(\boldsymbol{A})$,

and in every algebra $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}^{\leq}$,

2. *the set* $E(\boldsymbol{A})$ *is an* $\mathcal{S}$*-filter;*
3. *for every* $a, b \in A$, *if* $(a \Leftrightarrow^{\boldsymbol{A}} b) \subseteq E(\boldsymbol{A})$, *then* $a = b$,

then $\mathsf{Alg}\mathcal{S} = \mathsf{Alg}\mathcal{S}^{\leq}$ *and, therefore,* $\mathsf{Alg}\mathcal{S}$ *is a variety.*

Notice that condition (1) says that the semilattice order of each algebra in $\mathsf{Alg}\mathcal{S}$ is definable in the stated way using the implication set of formulas.

Theorem 14 applies to several important algebraizable logics like the systems of relevance logic R and R_t, Lukasiewicz's infinite-valued logic and the different systems of linear logic without the exponentials, which are algebraizable but non-selfextensional, and explains why they have a variety as its equivalent algebraic semantics. It is worth mentioning that for the relevance systems R and R_t as well as for the linear logics, their semilattice-based companions do not have theorems, and so they are non-protoalgebraic. This is not the case for the Lukasiewicz infinite-valued logic, whose semilattice-based companion has theorems but is also non-protoalgebraic.

When the semilattice-based companion is protoalgebraic we have a simpler theorem.

Theorem 15 *Let* $\mathcal{S}$ *be an algebraizable logic with (PC) relative to* $\wedge$ *such that for an implication set* $(p \Rightarrow q)$ *for* $\mathcal{S}$ *and a set* $E(p)$ *of defining equations,*

1. *for every* $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, $\langle A, \wedge^{\boldsymbol{A}}\rangle$ *is a semilattice, whose ordering we denote by* $\leq^{\boldsymbol{A}}$*;*
2. *for every* $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, $a \leq b$ *iff* $(a \Rightarrow^{\boldsymbol{A}} b) \subseteq E(\boldsymbol{A})$.

Assume moreover that in any algebra $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}^{\leq}$ *the least* $\mathcal{S}^{\leq}$*-filter is an* $\mathcal{S}$*-filter and assume also that* $\mathcal{S}^{\leq}$ *is protoalgebraic. Then* $\mathsf{Alg}\mathcal{S} = \mathsf{Alg}\mathcal{S}^{\leq}$ *and therefore* $\mathsf{Alg}\mathcal{S}$ *is a variety.*

This theorem is applicable to the several logics of a modal type known in the literature. In particular if we restrict ourselves to the standard modal language, we have the local consequence relation of a normal modal logic L and its global consequence relation. Usually the global consequence relation $\mathcal{S}^g_L$ is a non-selfextensional and algebraizable logic with a variety as its equivalent algebraic semantics. But the local consequence relation $\mathcal{S}^l_L$ is selfextensional, non algebraizable but protoalgebraic and $(\mathcal{S}^g_L)^{\leq} = \mathcal{S}^l_L$. This, together with the theorem, explains why the equivalent algebraic semantics of $\mathcal{S}^g_L$ is a variety.

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A Framework for Maximality and Interpolation in Abstract Logics with and without Negation

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1 Introduction

In the beginning of the study of extensions of first-order logic, the interest in developing such extensions was grounded on the need of studying various mathematical concepts not definable in first-order logic that appear especially in some new fields in mathematics -such as being a countable, a well ordered, or a measurable set. That way, mathematicians studying those fields could benefit from the methods of the model theory of the particular logic adequate to the concepts they handed, just as algebraists benefited from the model theory of first-order logic. But while these extensions have a richer expressive power, it turned out that they lacked interesting properties present in first-order logic. In 1969, Lindström [14] proved a very important maximality theorem: *'First-order logic is a maximal logic satisfying Compactness and Löwenheim-Skolem theorems'.* That is, any logic expressing more things than first-order logic, will loose at least one of those two mentioned valuable model theoretic properties. This phenomenon is interesting by itself; it makes the study of model theoretic languages depart from its original aim, and give rise to abstract model theory, a new field in model theory that will study these languages concentrating in its model theoretic properties. In this field we are not interested anymore in designing a particular language being able to describe a model as having

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this or that porperty. Instead, we are interested in constructing logics with interesting properties, such as satisfying compactness, Löwenheim-Skolem property, Craig's interpolation theorem [7] [2]. We are happier if we find a logic to be maximal with respect to these properties. We even are satisfied with just proving the existence of such logics, without ever glancing at how do they look like.

This paper is devoted to the study of the relations between interpolation and maximality in first-order logic and its extensions. The framework for this study is hinted, although not fully exploited, in [9]. Caicedo [3], [4], [5] presented several results for extensions of first-order logic with generalized quantifiers that fit within this framework. As he reckons, under very weak assumptions any such extension can be expressed in the form $\mathcal{L}_{\omega\omega}(\bar{Q})$ (where $\mathcal{L}_{\omega\omega}$ denotes first-order logic), for $\bar{Q} = \{Q^i : i \in I\}$ any set of quantifiers. This is also the case for all logics with interpolation. All these logics can be provided with a back-and-forth system [6], so we can find a back-and-forth system for any logic worth to explore[3]. Likewise, all proofs in the above papers concerning interpolation made heavy use of this feature. We therefore use back-and-forth systems as the substrate for our investigation in interpolation.

Section 3 scarcely contains new results. It is more a compilation of several independent results presented under the same framework. That way, we are able to answer a question of Barwise and van Benthem [2] regarding an alternative proof of interpolation for $\mathcal{L}_{\omega_1\omega}$.

Since Section 3 links the proof of interpolation to maximality, in Section 4 we analyze what individual model theoretic properties give rise to orderings of logics with maximal points. We also investigate how this maximality translates in the case we do not have negation. No proof of maximality with or without negation uses back-and-forth methods, and although all these logics have a weaker form of interpolation, proving whether they have interpolation is a very difficult matter: we seem to find ourselves without tools to determine it.

The conclusions of the paper are, in the first place, that for interpolation

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[2] We give their definitions later, by now it suffices to know these properties can be regarded as a measure of a logic as being natural and easy to be treated.

[3] Although it is true that any extension of first-order logic can be written in the form $\mathcal{L}_{\omega\omega}(\bar{Q})$, these extensions are generally divided into two essential kinds: Infinitary logics, $\mathcal{L}_{\kappa\lambda}$ with less than κ conjunction or disjunctions, and strings of existential or universal quantifiers of size less than λ; and extensions by generalized quantifiers, presented for the first time by Mostoski [18], and Lindstrom [13]. Infinitary logics have back-and-forth systems [8] which are described independently of their representation as $\mathcal{L}_{\omega\omega}(\bar{Q})$

to be related to maximality, one should understand interpolation theorem as: "If $\mathbf{K}_1, \mathbf{K}_2$ are disjoint classes belonging to (a certain R-invariant fragment of) $PC(\mathcal{L})$, then they can be separated by an elementary $\mathcal{L}$-class." -one recovers the usual interpolation when he ignores the parenthesis; in the second place, that the proof of interpolation by back-and-forth arguments is only possible when the logic is maximal with respect to some model theoretic properties, and when the invariance under R is among these characterizing properties. These conclusions break down in the case the logic is not closed under negation. Basically what happens is that maximality and interpolation theorems can be stated as corollaries for a so called *separation theorem*, and the proofs of these corollaries have different sensibilities to the lack of negation.

2 Preliminaries

A *vocabulary* τ is a nonempty set that consists of finitary relation symbols $P, R, \ldots$, and constant symbols $c, d, \ldots$ A τ-*structure* $\mathfrak{A}$ is a pair $\langle A, \nu \rangle$, where A, called the domain of $\mathfrak{A}$, is a nonempty set and ν is a map that assigns to every n-ary relation symbol R in τ, an n-ary relation on A^n, and to every constant symbol in τ an element in A. For any symbol $T \in \tau, T^{\mathfrak{A}}$ denotes the interpretation of T on $\mathfrak{A}$. We denote structures by $\mathfrak{A}[\tau] = \langle A, T_i^{\mathfrak{A}} \rangle_{T_i \in \tau}$. Let $Str[\tau]$ denote the class of structures of vocabulary τ. A *logic* is a pair $\langle \mathcal{L}, \models_{\mathcal{L}} \rangle$ where $\mathcal{L}$ is a map $\mathcal{L} : \tau \mapsto \mathcal{L}[\tau]$ such that $\{\mathcal{L}[\tau] : x \in \tau\}$ is a class called the class of $\mathcal{L}$-*sentences* of vocabulary τ, and $\models_{\mathcal{L}}$ (the $\mathcal{L}$-*satisfaction relation*) is a relation between structures and $\mathcal{L}$-sentences such that conditions $1 - 10$ of Definition 1 hold. For φ some $\mathcal{L}$-sentence, and $\mathfrak{A}$ some structure, $\mathfrak{A} \models_{\mathcal{L}} \varphi$ is read "$\mathfrak{A}$ is a model of φ". By $Mod_{\mathcal{L}}^{\tau}(\varphi)$ we denote the class of τ-structures that of $\mathfrak{A} \models_{\mathcal{L}} \phi$.[4] If it's clear from the context which logic we are talking about, we omit the subscript $\mathcal{L}$. If K is the class of models of a sentence φ, then $\bar{K}$ is the class of models of $\neg\varphi$. A *renaming* is a map $\rho : \tau \mapsto \sigma$ that is a bijection from a vocabulary τ to a vocabulary σ, that maps relation symbols to relation symbols of the same arity, and constants to constants. Given a renaming and a structure $\mathfrak{A}$ of vocabulary τ, $\mathfrak{B} = \mathfrak{A}^{\rho}$ is a structure with $B = A$ and $\rho(T)^{\mathfrak{B}} = T^{\mathfrak{A}}$, for all symbols $T \in \tau$. Let $\sigma \subseteq \tau$, and let $\mathfrak{A} \in Str[\tau]$. We define $\mathfrak{A} \upharpoonright \sigma$,

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[4] Then $Mod^{\tau}(\varphi) = \varphi$, and we talk about formulas and model classes indistinctly.

the *reduct* of $\mathfrak{A}$, to be the structure $\mathfrak{B} = \mathfrak{A} \restriction \sigma = (A, T^{\mathfrak{A}})_{T \in \sigma}$, where $T^{\mathfrak{A}} = T^{\mathfrak{B}}$ for $T \in \sigma$. If K is a class of models of vocabulary τ, then $K \restriction \sigma = \{\mathfrak{A} \restriction \sigma : \mathfrak{A} \in K\}$. Let φ be a sentence of vocabulary $\tau \cup \{a\}$ and $\mathfrak{A}$ a structure of vocabulary τ. Then $\varphi^{\mathfrak{A}} = \{a \in A : (\mathfrak{A}, a) \models \varphi\}$. Given a class of models $\mathbf{K}$ in a vocabulary τ, and $\sigma \subseteq \tau$ we denote by $\mathbf{K} \restriction \sigma$ the class $\{\mathfrak{A} \restriction \sigma : \mathfrak{A} \in \mathbf{K}\}$.

Definition 1 (Closure Properties)

1. *Inclusion property. If $\sigma \subseteq \tau$, then $\mathcal{L}[\sigma] \subseteq \mathcal{L}[\tau]$.*
2. *Isomorphism property. If $\mathfrak{A} \models_{\mathcal{L}} \varphi$, and $\mathfrak{A} \cong \mathfrak{B}$, then $\mathfrak{B} \models_{\mathcal{L}} \varphi$.*
3. *Reduct Property. If $\varphi \in \mathcal{L}[\tau]$ and $\tau \subseteq \tau_{\mathfrak{A}}$, then $\mathfrak{A} \models_{\mathcal{L}} \varphi$ iff $\mathfrak{A} \restriction \tau \models_{\mathcal{L}} \varphi$.*
4. *Renaming Property. If $\rho : \sigma \to \tau$ is a renaming , then for each $\varphi \in \mathcal{L}[\sigma]$ there is a sentence ψ^{ρ}, from $\mathcal{L}[\tau]$ such that for all σ-structures $\mathfrak{A}$, $\mathfrak{A} \models_{\mathcal{L}} \varphi$ iff $\mathfrak{A}^{\rho} \models_{\mathcal{L}} \varphi^{\rho}$;*
5. *Substitution Property. Suppose $\sigma \subseteq \tau$, and $\varphi \in L[\tau]$, and for all $R_i \in \tau \backslash \sigma$, we have a sentence $\varphi_i(d_1^i, ..., d_{k_i}^i) \in \mathcal{L}[\sigma \cup \{d_1^i, ..., d_{k_i}^i\}], k_i$ the arity of R_i, and $d_1^i, ..., d_{k_i}^i$ new constants. For any structure $\mathfrak{A} \in Str_{\mathcal{L}}[\sigma]$, let $\mathfrak{A}^* \in Str_{\mathcal{L}}[\tau]$be such that $\mathfrak{A}^* \restriction \sigma = \mathfrak{A}$, and for all $R_i \in \tau \backslash \sigma, R_i^{\mathfrak{A}^*} = \{\langle a_1, ..., a_{k_i} \rangle : (\mathfrak{A}, a_1, ..., a_{k_i}) \models_{\mathcal{L}} \varphi_i(d_1^i, ..., d_{k_i}^i)\}$. Then there exists a sentence $\varphi^* \in \mathcal{L}[\sigma]$ such that for all $\mathfrak{A} \in Str_{\mathcal{L}}[\sigma]$, $\mathfrak{A} \models_{\mathcal{L}} \psi \leftrightarrow \mathfrak{A}^* \models_{\mathcal{L}} \varphi$. We say that ψ is obtained from φ by simultaneously replacing each $R_i \in \tau \backslash \sigma$ by $\varphi_i(d_1^i, d_2^i, ..., d_{k_i}^i)$;*
6. *Atom Property. For all τ and atomic $\varphi \in \mathcal{L}_{\omega\omega}[\tau]$ there is a sentence $\psi \in \mathcal{L}[\tau]$ such that $Mod_{\mathcal{L}}^{\tau}(\psi) = Mod_{\mathcal{L}_{\omega\omega}}^{\tau}(\varphi)$;*
7. *Conjunction Property. For all τ and all $\varphi, \psi, \in \mathcal{L}[\tau]$ there is $\theta \in \mathcal{L}[\tau]$ such that $Mod_{\mathcal{L}}^{\tau}(\varphi) \cap Mod_{\mathcal{L}}^{\tau}(\psi) = Mod_{\mathcal{L}}^{\tau}(\theta)$;*
8. *Disjunction Property. For all τ and all $\varphi, \psi, \in \mathcal{L}[\tau]$ there is $\theta \in \mathcal{L}[\tau]$ such that $Mod_{\mathcal{L}}^{\tau}(\varphi) \cup Mod_{\mathcal{L}}^{\tau}(\psi) = Mod_{\mathcal{L}}^{\tau}(\theta)$;*
9. *Particularization Property. If $c \in \tau$, then for any $\varphi \in \mathcal{L}[\tau]$ there is a sentence $\psi \in \mathcal{L}[\tau \backslash \{c\}]$ such that for all $[\tau \backslash \{c\}]$-structures $\mathfrak{A}, \mathfrak{A} \models_{\mathcal{L}} \psi$ iff $(\mathfrak{A}, a) \models_{\mathcal{L}} \varphi$ for some $a \in A$. In a context of a logic with free variables we write ψ as $\exists x \varphi(x)$;*

10. *Universalization. If* $c \in \tau$*, then for any* $\varphi \in \mathcal{L}[\tau]$ *there is a sentence* $\psi \in \mathcal{L}[\tau\backslash\{c\}]$ *such that for all* $[\tau\backslash\{c\}]$*-structures* $\mathfrak{A}, \mathfrak{A} \models_{\mathcal{L}} \psi$ *iff* $(\mathfrak{A}, a) \models_{\mathcal{L}} \varphi$ *for all* $a \in A$*. In a context of a logic with free variables we write* ψ *as* $\forall x \varphi(x)$*;*

11. *Q-Projection. If* R_Q *is a class of models of vocabulary* $\sigma = \{S_1, ..., S_m\}$ *disjoint from* τ*, then for any* $\varphi_i \in \mathcal{L}[\tau \cup \{c_0, c_1, ..., c_{k_i-1}\}]$ *there is* $\psi_i \in \mathcal{L}[\tau]$ *such that* $\mathfrak{A} \in Mod^{\tau}_{\mathcal{L}}(\psi - i)$ *iff*
    - *There is a structure* $\mathfrak{C} \in R_Q$ *with* $C = A$*, and*
    - *For all* k_i *and all* k_i*-ary* $S_i \in \sigma$*,* $S_i^{\mathfrak{C}} = \{(a_0, ..., a_{k_i-1}) \in C^n : (\mathfrak{A}, a_0, a_1, ..., a_{k_i-1} \models \varphi_i\}$*.*

    In a context of a logic with free variables we write ψ *as*

$$Qx_0^1, ..., x_{k_1-1}^1, ..., x_0^m, ..., x_{k_m-1}^m \varphi_1(x_0^1, ..., x_{k_1-1}^1)... \\ ...\varphi_m(x_0^m, ..., x_{k_m-1}^m).$$

12. *Negation Property. For all* τ *and all* $\varphi \in \mathcal{L}[\tau]$ *there is a sentence* $\psi \in \mathcal{L}[\tau]$ *such that* $Mod^{\tau}_{\mathcal{L}}(\varphi) = Str[\tau] \setminus Mod^{\tau}_{\mathcal{L}}(\psi)$*;*

13. *Relativization Property. If* $c \notin \tau \cup \sigma, \xi \in \mathcal{L}[\tau \cup c]$ *and* $\varphi \in \mathcal{L}[\tau]$*, then there is a sentence* $\psi \in \mathcal{L}[\tau \cup c]$ *such that for any* $(\tau \cup \sigma)$*-structure* $\mathfrak{B}$*, if the set* $\xi^{\mathfrak{B}} = \{b \in B : (\mathfrak{B}, b) \models \xi\}$ *is* τ*-closed, then* $\mathfrak{B} \models \psi$ *iff* $(\mathfrak{B} \restriction \tau)|\xi^{\mathfrak{B}} \models \varphi$*.*

14. PC_{Δ}*-operation. If* $S = \{\varphi_n : n \in \omega\}$ *is a set of* $\mathcal{L}[\tau]$*-sentences, and if* $\sigma \subseteq \tau$*, then there is a sentence* $\psi \in \mathcal{L}[\sigma]$ *such that*

$$Mod^{\sigma}_{\mathcal{L}}(\psi) = (\bigcap_n Mod^{\tau}_{\mathcal{L}}(\varphi_n)) \restriction \sigma.$$

    If S contains just one sentence, the operation is called PC.

If a logic satisfies condition 12, it is called *regular.* Any regular logic contains $\mathcal{L}_{\omega\omega}$. If a logic satisfies condition 13, it is called *relativizing.* **K** is an $\mathcal{L}$-elementary class, in symbols **K** is $EC(\mathcal{L})$, if there is a sentence $\theta \in \mathcal{L}[\tau]$ such that $\mathbf{K} = Mod_{\tau}(\theta)$. **K** is an $\mathcal{L}$-pseudo elementary class, in symbols **K** is $PC(\mathcal{L})$ if there is a vocabulary $\tau' \supseteq \tau$ and a sentence $\theta \in \mathcal{L}[\tau']$ such that $\mathbf{K} = Mod_{\tau'}(\theta) \restriction \tau$.

Let $\mathcal{L}, \mathcal{L}^*$ be logics. We say that $\mathcal{L}^*$ *(properly) extends* (written $\mathcal{L} \leq \mathcal{L}^*$) $\mathcal{L}$, if for every EC-class in $\mathcal{L}[\tau]$, there is an EC-class in $\mathcal{L}^*[\tau]$ with the same models (and also there is an EC-class in $\mathcal{L}^*[\tau]$, such that no EC-class in $\mathcal{L}[\tau]$ has the same models).

A *partial isomorphism* between two models $\mathfrak{A}, \mathfrak{B}$ is a function p from $X \subseteq A$ to $Y \subseteq B$ such that the following holds;

1. For all $n \geq 1$, n-ary $R \in \tau$ and $a_0, ..., a_{n-1} \in X : R^{\mathfrak{A}}(\vec{a})$ iff $R^{\mathfrak{B}}(p(\vec{a}))$;
2. For all $c \in \tau$ and $a \in X : c^{\mathfrak{A}} = a$ iff $c^{\mathfrak{B}} = p(a)$.

$Part(\mathfrak{A}, \mathfrak{B})$ denotes the set of partial isomorphisms between $\mathfrak{A}$ and $\mathfrak{B}$. A *back-and-forth* system for $(\mathfrak{A}, \mathfrak{B})$ is a decreasing sequence $I = (I_\beta)_{\beta \leq \alpha}$ of subsets of $Part(\mathfrak{A}, \mathfrak{B})$ that satisfies the following conditions:

(i) Each I_i is a set of partial isomorphisms.

(ii) $\emptyset \in I_\alpha$

(iii) For $m < \alpha$, if $p \in I_{m+1}$ and $a \in A$, then there is $b \in B$ such that $p \cup \{\langle a, b\rangle\} \in I_m$.

(iv) For $m < \alpha$, if $p \in I_{m+1}$ and $b \in B$, then there is $a \in A$ such that $f \cup \{\langle a, b\rangle\} \in I_m$.

We can generalize this concept for other functions f than partial isomorphisms. The nature of these functions will depend on the modifications on the closure under isomorphism of $\mathcal{L}$. We call the functions f *partial F-isomorphisms*, where F-isomorphism is the new notion of isomorphism. An F-back-and-forth system is a back-and-forth system of partial F-isomorphisms. All notation for partial isomorphisms would be obtained just by ignoring F and f in the following definitions. $Part(\mathfrak{A}, \mathfrak{B})$ has an obvious generalization to $PartF(\mathfrak{A}, \mathfrak{B})$ Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are α-F-isomorphic via I, written $I : \mathfrak{A} \cong^F_\alpha \mathfrak{B}$, iff $I = (I_\beta)_{\beta \leq \alpha}$ is an F-back-and-forth system. $I \subseteq PartF(\mathfrak{A}, \mathfrak{B})$ has the back (forth) property if for each $p \in I$ and $b \in B(a \in A)$ there is $q \in I, p \subseteq q$ with $b \in rg(q)(a \in dom(p))$. Two models are partially F-isomorphic $\mathfrak{A} \cong^F_p \mathfrak{B}$ if in there is $I \subseteq PartF(\mathfrak{A}, \mathfrak{B})$ with the back-and-forth property.

Two models $\mathfrak{A}, \mathfrak{B}$ are $\mathcal{L}$-equivalent, in symbols $\mathfrak{A} \equiv_{\mathcal{L}} \mathfrak{B}$ iff they satisfy the same $\mathcal{L}$-sentences.

Definition 2 (Model Theoretic Properties)

1. *$\mathcal{L}$ is* compact *iff for all countable $\Phi \subseteq \mathcal{L}[\tau]$, if each finite subset of Φ has a model, then Φ has a model. Substitutes of compactness are:*

  - Small well ordering number, *where the well ordering number of $\mathcal{L}$ is defined as the supremum of all ordinals α such that for any $\mathcal{L}$-sentence $\phi(<, ...)$ having only models with well ordered $<$, there is a model of ϕ that is a well-ordering of type α.*
  - Boundedness *$\mathcal{L}$ is* bounded *if for any $\mathcal{L}$-sentence $\phi(<, ...)$ having only models with well ordered $<$, there is an ordinal α such that the order type of $<$ is always less than α.*

2. *$\mathcal{L}$ has the* Downward Löwenhein-Skolem property *iff each satisfiable sentence has a model of size $\leq \aleph_0$. Substitutes of the Löwenhein-Skolem property are:*
   - *$l(\mathcal{L})$, the* Löwenheim number *of $\mathcal{L}$ is the least cardinal μ such that any satisfiable sentence has a model of power $\leq \mu$. Any logic $\mathcal{L}$ with a set number of classes has $l(\mathcal{L}) = \lambda$ for λ some ordinal. Such logics are called* small *or* set *logics. Otherwise, $l(\mathcal{L}) = \infty$, and $\mathcal{L}$ is called a* big *or* class *logic.*
   - *$l_\Sigma(\mathcal{L})$, the* Löwenheim number for countable sets of sentences *of $\mathcal{L}$ is the least cardinal μ such that any countable satisfiable set of sentences has a model of power $\leq \mu$.*

3. *$\mathcal{L}$ has the* Karp property *if any two partially isomorphic structures are $\mathcal{L}$-equivalent. Karp property can be generalized to partial f-isomorphisms.*

4. *$\mathcal{L}$ satisfies the* Craig interpolation theorem *if given $\varphi, \psi \in \mathcal{L}$, such that $\varphi \models \psi$, there is $\theta \in \mathcal{L}$ such that $\tau(\theta) \subseteq \tau(\varphi) \cap \tau(\psi)$ and $\varphi \models \theta$ and $\theta \models \psi$. Craig's theorem can also be stated as: "Any two disjoint PC-classes in $\mathcal{L}_{\omega\omega}$ can be separated by an EC-class in $\mathcal{L}_{\omega\omega}$.". A substitute of Craig theorem is*
   - *$\mathcal{L}$ satisfies the* Souslin-Kleene *separation property if for each class of models $K \in \mathcal{L}$, if K and $\overline{K}$ are PC in $\mathcal{L}$, then they are EC in $\mathcal{L}$.*

3 Interpolation and maximality

Logics differ from each other in their expressive power, and model-theoretic properties bound this expressive power. For instance, Löwenheim-Skolem property says that any sentence with a model can have a countable model, hence ruling out the possibility of any logic with that property to declare a model uncountable. At the light of this example, one would think that each logic is characterized by its exclusive model theoretic properties, in the sense that if we try to overcome its expressive power, we are in the domains of other logic, with therefore different model theoretic properties. This is the content of Lindström's Theorem. However, it is not true in all logics -the difference between expressive powers not always corresponds to model theoretic properties A close examination of the ingredients of Lindström's proof reveals that it is actually an application of some basic properties of back-and-forth systems. When the proof is broken into its parts, a proof of the interpolation theorem emerges. This has been known in the folklore of the subject [9], but has not been systematically exploited. As I said in the introducion, this paper tries to present this connection over the substrate of back-and-forth systems. Two recent works [2], [20] study the relations between back and forth systems and interpolation. The strategy for proving interpolation theorems begins by finding an appropiate back and forth system for the logic. However, the existence of back and forth systems does not guarantee the success in finding interpolation theorems. As a list of negative results, in the case of extensions with generalized quantifiers, the main result of Caicedo says no extension of first-order logic by means of an arbitrary number of monadic quantifiers satisfies interpolation. Mostowski have a similar result in the case on a finite number of generalized quantifiers of arbitrary type. In this section we address the case of infinitary logics, although the conclusions extend to any logic. $L_{\infty\kappa}$ for $\kappa \geq \omega$ has both a have a back-and-forth system, a Lindström's type theorem, but not interpolation. On the other hand, $\mathcal{L}_{\omega_1\omega}$ has interpolation but no Lindström's type characterization. We try to give a framework that connects interpolation and maximality and yet it is able to explain all these "anomalies".

We start by deriving Lindström's and interpolation theorems for first-order logic from a common source theorem called

Theorem 3 (Separation Theorem) *Let $\mathcal{L}^*$ be a compact logic with Downward Löwenhein-Skolem property. Let K_1 and K_2 be two disjoint $\mathcal{L}^*$-classes. Then there is a first-order sentence θ in the common language of K_1 and K_2 that separates them.*

Sketch of proof Suppose there is no such sentence. We extend the vocabulary with some predicate symbols in order to construct a sentence that says that for all m there are disjoint models $\mathfrak{A}_m$ and $\mathfrak{B}_m$ that satisfy the same sentences of quantifier rank m, and a function from A to B that is a partial isomorphism of length m. By compactness, there are two models that satisfy the same sentences, and between which we can construct a partial isomorphism of length ω. By Downward Löwenhein-Skolem property, we can get these models countable, getting an isomorphism. But this is a contradiction. □

Corollary 4 (Lindström's maximality theorem) *$\mathcal{L}_{\omega\omega}$ is a maximal compact logic satisfying the Löwenheim-Skolem property.*

Proof. Let $\mathcal{L}^*$ be closed under negation. □

Corollary 5 (Craig's interpolation theorem) *[7] Any two disjoint $PC(\mathcal{L}_{\omega\omega}$ classes can be separated by an $EC(\mathcal{L}_{\omega\omega})$-class.*

Proof. Take $\mathcal{L}^*$ to be the logic of $PC(\mathcal{L}_{\omega\omega})$ classes. □

At this point, we notice how important is the fact that first-order logic is closed under negation, for not both above corollaries behave equally if we give negation up. As an illustration, if we take a fragment of first-order logic not closed under negation, a theorem analogous to Theorem 3 [10] gives as corollary an interpolation theorem, but no characterization theorem exists for the logic. Specifically, $\mathcal{L}^P_{\omega\omega}$, the fragment of $\mathcal{L}_{\omega\omega}$ in which a given predicate P appears only positively, has interpolation, but is not maximal with respect to compactness and Löwenheim-Skolem property. Moreover, in next section we will prove no logic without negation is maximal with respect these two properties.

Now we try to generalize these ideas for logics other than $\mathcal{L}_{\omega\omega}$. The following table contains a number of logics and their respective satisfaction of interpolation and generalized forms of Lindström's type maximality theorems. The model theoretic properties in the box for LT are the characterizing ones:

Logic	Lindström's theorem	Interpolation theorem
$\mathcal{L}_{\omega\omega}$	Compactness and Löwenheim-Skolem.	YES
$\mathcal{L}_{\infty\omega}$	Boundedness and Karp property.	NO
$\mathcal{L}_{\kappa,\omega}$ $(\kappa = \beth_\kappa)$	Well ordering number $\leq \kappa$ and Karp property.	NO
$\mathcal{L}_{\omega_1\omega}$	NONE	YES
L^1_κ	Löwenheim-Skolem and strong form of undefinability of well order.	YES
$L^P_{\omega,\omega}$	NONE	YES

Next Definition introduces a relation R involved in the description of the back-and-forth system of a logic.

Definition 6 *Let τ be a vocabulary, R a binary relation between structures, and φ a sentence of a logic $\mathcal{L}$.*

1. *We say that φ is R-invariant if*

$$\mathfrak{A}R\mathfrak{B} \text{ and } \mathfrak{A} \models \varphi \text{ imply } \mathfrak{B} \models \varphi$$

Denote the class of R-invariant sentences as $\mathcal{L}^R$. In case $\mathcal{L} = \mathcal{L}^R$, we say $\mathcal{L}$ is a logic of R-invariant sentences.

2. *We say that φ entails ψ along R, written $\varphi \models_R \psi$, iff for all τ-structures $\mathfrak{A}$ and $\mathfrak{B}$, if $\mathfrak{A}R\mathfrak{B}$, and $\mathfrak{A} \models \varphi$ then $\mathfrak{B} \models \psi$.*

Next theorem is introduced with aims fo being a generalisation of the separation theorem 3.

Theorem 7 *[9] Suppose there is given for any vocabulary τ a set $\Phi^\tau \subseteq \mathcal{L}[\tau]$ and let $\mathcal{R}^\tau = Mod(\Phi^\tau)$. Assume that R is a binary relation between structures such that $\mathfrak{A}R\mathfrak{B}$ implies $\mathfrak{A}, \mathfrak{B} \in \mathcal{R}$ for some τ. Suppose that*

1. *R restricted to τ-structures is an equivalence relation.*

2. *R is invariant under renamings.*

3. *Given τ, for some $\tau' \supseteq \tau$, there are $\mathcal{L}$-sentences $\varphi_0, \varphi_1, ...$ such that for any τ-structures $\mathfrak{A}, \mathfrak{B}$, the following hold:*

   a. *$\mathfrak{A}R\mathfrak{B}$ iff $(\mathfrak{A}, \mathfrak{B}, ...) \models \{\varphi_i : i \in \omega\}$ for some choice of ..., and* *b. for $n \in \omega$ the relation R_n on $\mathcal{R}$ given by*

$$\mathfrak{A} R_n \mathfrak{B} \textit{ iff } (\mathfrak{A}, \mathfrak{B}, ...) \models \{\varphi_i : i \leq n\} \textit{ for some ...}$$

has the following two properties:

a.1. R_n is an equivalence relation on $\mathcal{R}^\tau$;

a.2. For $\mathfrak{A} \in \mathcal{R}^\tau$, there is $\psi^n_{\mathfrak{A}} \in \mathcal{L}[\tau]$ such that for $\mathfrak{B} \in \mathcal{R}^\tau$:

$$\mathfrak{A} R_n \mathfrak{B} \textit{ iff } \mathfrak{B} \models \psi^n_{\mathfrak{A}}.$$

The above are the general properties we requiere from R to have. There are some further properties that depend on the particular logic $\mathcal{L}$ we are working with.

Then

Let $\mathcal{L}$ be a logic with any given generalized compactness property. If $\mathcal{L}^* \geq \mathcal{L}$ is a logic of R-invariant sentences with the same generalized compactness property, and not necesarily closed under negation, then any two $\mathcal{L}^*$-classes can be separated by an $\mathcal{L}^R$-class. □

Corollary 8 (Lindström's maximality theorem) *$\mathcal{L}_{\omega\omega}$ is a maximal compact logic with the Löwenheim-Skolem property.*

Proof. Let $\mathcal{L}$ be $\mathcal{L}_{\omega\omega}$, and R be the relation of partial isomorphism $\cong_p$. Let $\mathcal{L}^*$ be compact and closed under negation. It suffices to show that any logic with Löwenheim-Skolem property has the Karp property, as proves the following

Proposition 9 *If $\mathcal{L}$ has the Löwenheim-Skolem property, then $\mathcal{L}$ has the Karp property.*

Proof. By contradiction, suppose $\mathcal{L}[\tau]$ has the Löwenheim-Skolem property, but does not have Karp property. Then, for some $\mathcal{L}$-sentence ϕ we have

$$\mathfrak{A} \cong_p \mathfrak{B}, \ \mathfrak{A} \models \phi \text{ and } \mathfrak{B} \models \neg\phi$$

If A and B are countable we are done, since partially isomorphic countable structures are isomorphic. So, let A and B be uncountable. Let I, V, W be new unary predicates; and G be one new ternary relation. Set $\tau' = \tau \cup \{I, V, W.G\}$. Let ψ be the conjunction of the following $\mathcal{L}[\tau']$-sentences:

"V and W are disjoint"

$\phi^{\{x:V(x)\}}$,

$\neg\phi^{\{x:W(x)\}}$,

"each $p \in I$ is a mapping from V to W" that is,

$$\forall p(I(p) \to \forall x\forall y(Gpxy \to (V(x) \wedge W(y)))$$

"each $p \in I$ is a partial injective mapping" that is,

$$\forall p(I(p) \to \forall x\forall y\forall u\forall v(G(p,x,u) \wedge G(p,y,v) \to (x = y \leftrightarrow u = v))),$$

"each $p \in I$ preserves all the symbols in τ" for example, for binary $T \in \tau$,

$$\forall p(I(p) \to \forall x\forall y\forall u\forall v(G(p,x,u) \wedge G(p,y,v) \to (T(x,y) \leftrightarrow T(u,v))),$$

"the set I is not empty"

"the set I has the forth property", that is,

$$\forall p(I(p) \to \forall x\exists q\exists y(I(q) \wedge G(q,x,y) \wedge \forall z\forall w(G(p,z,w) \to G(q,z,w)))),$$

"the set I has the back property."

Then a model whose relativizations to V and W are isomorphic to $\mathfrak{A}$ and $\mathfrak{B}$, respectively, is a model of ψ, since by hypothesis $\mathfrak{A}$ and $\mathfrak{B}$ are partially isomorphic, $\mathfrak{A} \models \phi$ and $\mathfrak{B} \models \neg\phi$. By Downward Löwenhein-Skolem property ψ has a countable model $\mathfrak{C}$. But then we obtain two countable structures $\mathfrak{A}' = \mathfrak{C}^V$ and $\mathfrak{B}' = \mathfrak{C}^W$, that are partially isomorphic, and therefore isomorphic, such that $\mathfrak{A}' \models \phi$ and $\mathfrak{B}' \models \neg\phi$, a contradiction. □

Note that in this case, we cannot derive as corollary Craig's interpolation theorem, since $PC(\mathcal{L}_{\omega\omega})$ does not preserve Karp property, that is, is not invariant under $\cong_p$[5]. Instead, interpolation theorem holds in $\mathcal{L}_{\omega\omega}$ because $PC(L_{\omega\omega})$ preserves compactness and Löwenheim-Skolem property.

Let's see the case for $\mathcal{L}_{\infty\omega}$. Adding for each ordinal α, a relation R_α with set many equivalence classes, we can prove in a similar way

Corollary 10 (Barwise's characterization for $\mathcal{L}_{\infty\omega}$) *$\mathcal{L}_{\infty\omega}$ is a maximal bounded logic with the Karp property.*

Proof. Let $\mathcal{L} = \mathcal{L}_{\infty\omega}$, and model-theoretic substitute of compactness be boundedness. Let R be the relation of partial isomorphism, and $\mathcal{L}^*$ be closed under negation. □

The following is the associated interpolation theorem. It is not a full interpolation theorem, for $PC(\mathcal{L}_{\infty\omega})$ does not preserve Karp property. Contrary to what happens in the case of first-order logic, $\mathcal{L}_{\infty\omega}$ cannot be characterized by any property preserved by the PC operation, so we don't have any hope of getting the full interpolation in this case.

Corollary 11 (Barwise-van Benthem interpolation theorem for $\mathcal{L}_{\infty\omega}$) *[2] Given $\psi \in \Sigma^1_1(\mathcal{L}_{\infty,\omega})$, and $\phi \in \Pi^1_1(\mathcal{L}_{\infty,\omega})$, the following are equivalent:*

1. *ψ entails ϕ along R.*
2. *There is a sentence θ of $\mathcal{L}_{\infty,\omega}$, such that $\psi \models \theta$ and $\theta \models \phi$.*

The proof of this theorem as given by they authors is very similar to the proof of maximality for $\mathcal{L}_{\infty\omega}$. We give the proof treating this theorem as a corollary of Theorem 7.

Proof. (From the generalized separation theorem) Let R be the relation of partial isomorphism. Let $\mathcal{L}^*$ be the R-invariant fragment of $PC(L_{\infty\omega})$. □

This interpolation theorem says that the only possible $\psi, \phi \in PC(\mathcal{L}_{\infty\omega})$ such that $\psi \models \phi$ has an interpolant are those in which ϕ and ψ are in the R-invariant fragment. On these grounds, Barwise and van Benthem argue interpolation theorem should be understood as:

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[5] Indeed, let $\mathfrak{A}$ be the unary structure with uncountable universe and P a predicate which has countable many elements and its complement is uncountable. Let $\mathfrak{B}$ be the unary structure with ω_1 as universe in which the predicate P and its complement are both uncountable. Clearly $\mathfrak{A} \cong_p \mathfrak{B}$. Let φ be the $PC(\mathcal{L}_{\omega\omega})$-sentence "there is a one-one mapping from the complement of P into P". Then $\mathfrak{B} \models \varphi$, but $\mathfrak{A} \not\models \varphi$.

(*) "A logic $\mathcal{L}$ has interpolation If $\mathbf{K}_1, \mathbf{K}_2$ are $PC(\mathcal{L})$-classes invariant under R, then they can be separated by an $EC(\mathcal{L})$-class."

Similar 'interpolation theorems' could be proved for $\mathcal{L}_{\kappa,\omega}, \kappa = \beth_\kappa$ and $\mathcal{L}_{\omega_1\omega}$. From López-Escobar [17], we know the later has the full interpolation, while the result provided by an adaptation of Theorem 11 for $\mathcal{L}_{\omega_1\omega}$ is only partial. Barwise and van Benthem [2] asked whether it would be possible to make some change in Theorem 11, so that we get the full interpolation theorem for $\mathcal{L}_{\omega_1\omega}$. We see here this is not possible, for separation is essentially Lindström's theorem, and $\mathcal{L}_{\omega_1\omega}$ does not have any Lindström's type characterization[6]. That is, Theorem 11 can be proved for $\mathcal{L}_{\omega_1\omega}$ per se, but it cannot be understood as a corollary from Theorem 7.

Is there any logic besides first-order with a maximallity and a full interpolation theorem? In [21], Shelah and Väänänen construct a new infinitary logic $\mathcal{L}^1_\kappa$ between $\mathcal{L}_{\kappa,\omega}$ and $\mathcal{L}_{\kappa,\kappa}$ characterized by Löwenheim-Skolem property and a substitute of compactness. Both properties are preserved by the PC operation. They achieve this way the new logic satisfying Lindström's and Craig's theorems.

4 Other maximality results

We have seen several cases of logics with model theoretic characterizations. We have seen the importance of Karp properties in the relation between interpolation and maximality. We can now ask what model theoretic properties are able to characterize a logic. That is, we can study the orderings of the family of all logics with respect to every model theoretic property and look for maximal points.

In the literature, there are already some examples. Sgro [16] proved that in the ordering of logics with the Łos' ultraproduct porperty, there is a maximal logic with this property; Lipparini [15] proved that there is a maximal logic that extends a given logic $\mathcal{L}$ and has the same complete extensions as $\mathcal{L}$; Wacławeck [22] proved there is a maximal logic with Löwenheim-Skolem property over any logic with this property. All these maximal logics enjoy

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[6] In the case of propositional extensions of $\mathcal{L}_{\omega_1,\omega}$, (see [11], [12]), Harrington [12] proved there are such extensions that continue to have the same model theoretic properties as $\mathcal{L}_{\omega_1,\omega}$, if we restrict to admissible fragments. Gostanian and Hrbacek proved in [11], not restricting to admissible fragments, that among propositional extensions of $\mathcal{L}_{\omega_1,\omega}$, only $\mathcal{L}_{\omega_1,\omega}$ itself warrants interesting model theoretic properties.

Souslin-Kleene separation theorem, which is a weakening of interpolation theorem. It is an open problem whether there is a maximal logic in the ordering of compact logics.

Of particular interest would be the existence of a maximal logic with Karp property[7]. Since Löwenheim-Skolem property implies Karp property, and the converse is true for logics with interpolation (cf. [9] p. 95), it is also of interest the ordering of logics with the Löwenheim-Skolem property. For this reason, as well as for methodological reasons, I reproduce here the proof of Wacławeck on the existence of a maximal logic with respect to the Löwenheim-Skolem property. Then we will prove that it has Souslin-Kleene separation theorem. We will be able to appreciate the essencial differences between these proofs and those of the previous section, as well as to asses the great complexity that a proof of interpolation would mean in this case.

In the next two theorems a logic is considered to be close under negation and conjunction only.

Theorem 12 *Let* $(LS, \leq)$ *be the ordering of logics that satisfy the Löwenheim-Skolem property. For any logic* $\mathcal{L} \in (LS, \leq)$*, there is a maximal logic* $\mathcal{L}' \in (LS, \leq)$ *such that* $\mathcal{L} \leq \mathcal{L}'$*.*

Proof.

Each sentence in a logic $\mathcal{L}$ with the Löwenheim-Skolem property is determined by its countable models, i.e. two different sentences in $\mathcal{L}$ do not have the same countable models, by definition.

Claim 2: There are at most $2^{\aleph_0}$ countable nonisomorphic models of finite vocabulary.

Let $\tau = \{T_1, ..., T_n\}$ be a vocabulary, and let m_i be the arity of T_i. Let A be a countable set, and let $c_i = |\{f : f \text{ is a function from } A^{m_i} \text{ to } \{0,1\}\}|$. Then the number s of models of vocabulary τ and domain A is $s = \Pi_{i=1,...,n} c_i$ but $c_i = 2^{\aleph_0}$ for all i, so $s = (2^{\aleph_0})^n = 2^{\aleph_0}$.

So there are at most $2^{2^{\aleph_0}}$ possible classes of countable nonisomorphic models, and hence every well-ordered chain in $(LS, \leq)$ has length smaller than $(2^{2^{\aleph_0}})^+$.

Claim 3: The union of an increasing sequence of logics $\mathcal{L}'$ with $\mathcal{L} \leq \mathcal{L}'$ is a supremum of this family of logics.

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[7] However, this is an open problem. $\mathcal{L}_{\infty\omega}$ is a maximal logic with respect to Karp property if we chose the relation of extension between two logics to be:"$\mathcal{L} <_{\equiv} \mathcal{L}'$ iff for all τ and all $\mathfrak{A}, \mathfrak{B} \in Str[\tau]$, if $\mathfrak{A} \equiv_{\mathcal{L}'}$ then $\mathfrak{A} \equiv_{\mathcal{L}}$

Let $\mathcal{L}^* = \bigcup_\alpha \mathcal{L}_\alpha$, where $L_\alpha \in (LS, \leq)$, and $\mathcal{L}_\delta \leq \mathcal{L}_\gamma$ for $\delta \leq \gamma$, and suppose $\mathcal{L}^* \notin (LS, \leq)$. Then there is some $\theta \in \mathcal{L}^*$ with a model but no countable models. But $\theta \in \mathcal{L}_\alpha$ for some α, a contradiction.

By Zorn's Lemma, there is a maximal logic $\mathcal{L}^{**} \in (LS, \leq)$ extending $\mathcal{L}$. □

Theorem 13 *Any maximal logic in with respect to the Löwenheim-Skolem property has the Souslin-Kleene separation property.*

Proof. Let $\mathcal{L}$ be a maximal logic with the Löwenheim-Skolem property, and let K, and $\bar{K}$ be two PC-classes in $\mathcal{L}$. Suppose K is not an $EC(\mathcal{L})$-class. Then we can add K as a new $EC(\mathcal{L})$-class, and get the extension $\mathcal{L}'$ closing under negation and intersection. We prove that $\mathcal{L}'$ satisfies Löwenheim-Skolem theorem, contradicting the hypothesis that $\mathcal{L}$ is a maximal logic in $(LS, \leq)$.

K has countable models, because it is a class of reducts of models of a sentence of $\mathcal{L}$, and similarly for $\bar{K}$, $M \cap K$, and $M \cap \bar{K}$.

In case $\mathcal{L}$ has negation we can also prove the following

Corollary 14 *Any logic has an extension with the same Löwenheim number, and the Souslin-Kleene separation theorem.*

Proof. Let $\mathcal{L}$ be a logic with Löweheim number κ. Consider the ordering of logics with Löwenheim number κ. By an argument similar to the proof of Theorem 12, there is a maximal logic $\mathcal{L}'$ extending $\mathcal{L}$. By an argument similar to that of the proof of Theorem 13, $\mathcal{L}'$ satisfies the Souslin-Kleene separation property. □

4.1 Maximality results for logics not closed under negation

In this section we study the orderings of logics not necessarily closed under negation, with respect to compactness and and Löwenheim-Skolem properties. We see there is no maximal set logic (cf. Definition 2.5) with respect to compactness and Löwenheim-Skolem properties.

Theorem 15 *If we do not assume negation, no small logic is maximal with respect to compactness and Löwenheim-Skolem properties.*

In oder to prove theorem 15, we need some preliminary definitions and theorems.

Definition 16 *Suppose K is a class of models. Suppose $\mathcal{L}$ and $\mathcal{L}$' are logics. We say that $\mathcal{L}$ is partially K-reducible to $\mathcal{L}$' if for any $\phi \in \mathcal{L}$ there is $\phi^* \in \mathcal{L}'$, such that $\phi \to \phi^*$ in all models, and $\phi \leftrightarrow \phi^*$ in models in K.*

In particular, a generalized quantifier Q is partially K-reducible to first-order logic if for any $\phi(\vec{x}) \in \mathcal{L}_{\omega\omega}$ there is $\phi^*(\vec{x}) \in \mathcal{L}_{\omega\omega}$, such that $Q(\vec{x})\phi(\vec{x}) \to \phi^*(\vec{x})$ in all models, and the converse is true in models in K.

Lemma 17 *Let K be a class of models containing all models of cardinality at most λ. Suppose a logic $\mathcal{L}$ is partially K-reducible to a compact logic $\mathcal{L}'$ with $l_\Sigma = \lambda$. Then $\mathcal{L}$ is countably compact.*

Proof. Let Φ be a finitely satisfiable set of $\mathcal{L}$-sentences. Let Φ^* be the set of $\mathcal{L}'$ -sentences that are the reductions of each sentence in Φ. Φ^* is finitely satisfiable, for take any finite set $\Sigma^* \subset \Phi^*$, and look at the corresponding set of sentences $\Sigma \subset \Phi$. By hypothesis, Σ has a model, and it is a model of each of the sentences in Σ^*, so Φ^* is finitely satisfiable. By compactness, it has a model, and as the Löwenheim number for countable sets of sentences of $\mathcal{L}'$ is λ, it has a model $\mathfrak{M}$ of that cardinality. As $\mathfrak{M}$ is a model with cardinality λ of each sentence in Φ^*, it is a model of each sentence in Φ. □

Theorem 18 *Let $\mathcal{L}$ be a compact logic. If we do not assume negation, there is no maximal compact set logic that extends $\mathcal{L}$.*

Proof. Let $\mathcal{L}$ be a compact logic not necessarily closed under negation. Let λ be its Löwenheim number. Add to $\mathcal{L}$ all finite classes plus the class K_κ of models of the generalized quantifier $Q^{\leq\kappa}x(x = x)$, whose interpretation is "there are at most κ elements". We can find $\kappa > \lambda$ such that $K_\kappa \notin \mathcal{L}$, because otherwise $\mathcal{L}$ would be a proper class. Then $\mathcal{L}(Q^{\leq\kappa})$ makes a proper extension of $\mathcal{L}$.

Claim: $\mathcal{L}(Q^{\leq\kappa})$ is compact.

Let K be the class of models of cardinality λ. Then $\mathcal{L}(Q^{\leq\kappa})$ is K-reducible to $\mathcal{L}$ by the sentece $\exists x(x = x)$. □

Lemma 19 *Let K be ta class of models containing all countable models. Suppose $\mathcal{L}$ is partially K-reducible to a logic $\mathcal{L}'$ with the Downward Löwenhein-Skolem property . Then $\mathcal{L}$ satisfies Downward Löwenhein-Skolem property.*

Proof. Let ϕ be an $\mathcal{L}$-sentence with an infinite model. Let ϕ^* be a K-partial reduction of ϕ to $\mathcal{L}'$. By Downward Löwenhein-Skolem theorem, ϕ^* has a countable model, and by definition of partial reduction, ϕ has also a countable model. □

Theorem 20 *Let $\mathcal{L}$ be a logic with with Löweneheim-Skolem theorem. If we do not assume negation, there is no maximal small logic with respect to Löweneheim-Skolem theorem that extends $\mathcal{L}$.*

Proof. Let $\mathcal{L}$ be an the Löwenheim-Skolem property logic not necessarily closed under negation. Add to $\mathcal{L}$ all finite classes plus the class K_κ of models of the generalized quantifier $Q^{\leq\kappa}x(x = x)$, whose interpretation is "there are at most κ elements". We can find $\kappa > \omega$ such that $K_\kappa \notin \mathcal{L}$, because otherwise $\mathcal{L}$ would be a proper class. Then $\mathcal{L}(Q^{\leq\kappa})$ makes a proper extension of $\mathcal{L}$. *Claim:* $\mathcal{L}(Q^{\leq\kappa})$ is the Löwenheim-Skolem property.

Let K be the class of models of cardinality ω. Then $\mathcal{L}(Q^{\leq\kappa})$ is K-reducible to $\mathcal{L}$ by the sentece $\exists x(x = x)$. □

From theorems 17 and 20, we conclude that there is no maximal compact logic with the Löwenheim-Skolem property.

5 Conclusions and further research

Although some of the ideas of this paper were floating around in the literature, we think we have given a framework from which we can understand better the relations between interpolation and maximality. As Prof. Väänänen put it, Lindström's is more than a single theorem, is a phenomenon occuring throughout logics, and it is good to understand it in a framework common to the logics of interest, in this case, logics satisfying also the interpolation theorem.

Understanding the role of negation, we can appreciate that the connections between interpolation and maximality results fade away in case of logics with interpolation. The natural continuation of this investigation, hence, would be to explore how the negative results in interpolation theorems for logics with generalized quantifiers (in particular monadic generalized quantifiers) translates for logics without negation.

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A Note on Definitions in Propositional Calculi [1]

Pierre Joray

1 Implicit *versus* explicit definitions

If we call *definition* every symbolic procedure wich enables us to introduce a symbol α in a logical system and gives to the symbol an accurate and determined position in the deductive framework, then it is obvious that two main kinds of definitions occur in axiomatisations of propositional calculi: explicit definitions and implicit ones. Formaly speaking, explicit definitions are those which are constituted of two separate expressions, (respectively called *definiendum* and *definiens*) stated to be in a certain relation which underwrites their inferential equivalence and fulfil the following general conditions:

1. The *definiendum* is an expression including $\alpha(v_1 v_2 \cdots v_n)$, where α is the symbol to be introduced and $v_1, v_2, \cdots, v_n$ are variables indicating its argument places (if any).
2. The symbol α to be introduced is the only new symbol in the *definiendum* and it does not occur more than once.
3. No variable in the *definiendum* occur more than once.

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4. The *definiens* is a *wff* of the language in which the definition is stated, it contains only symbols previously introduced.
5. Every variable occuring in the *definiens* also occurs in the *definiendum.*

The aim of explicit definitions is then to introduce in a certain language L a new symbol α, by the way of an expression which is still available in L before the introduction of α. Clearly explicit definitions cannot be used for the introduction of all the symbols of the logic to be axiomatised, for the first explicit definition requiers the use of symbols which are to be introduced by a different method, namely by implicit definitions.

Implicit definitions can also be used in order to introduce non primitive symbols and, as we know (in particular from mathematical examples), implicit definitions involves a big variety of subtypes. It is not my aim to present a complete picture of them; it will suffice for the present purpose to remind that they are all based on the same idea that it is possible to characterize a symbol α by adding to the language in question, with the status of thesis, an expression (or plurality of expressions) containing α. As implicit definitions can notoriously lead to important difficulties which cannot be simply eliminate by some given general conditions on their construction (cf. e.g. Horwich [2] and Hale & Wright [1]), their use is usualy restricted to the introduction of symbols which cannot be explicitly defined.

The way primitive terms of a system are introduced is an important example of the necessity of implicit definitions. But the existence of different equaly valuable axiomatisations or formal systems for the same logic (or theory) based on different sets of primitive terms leads to questions I would like to examine and which have been often neglected in modern logic.

2 Traditional and modern conceptions

In the old tradition of the *more geometrico* method, dominated by the canonical example of Euclidian geometry, the choice of such or such set of primitive terms was conditioned by epistemic considerations: primitive terms had to be the simplest intuitively understandable ones which can be characterised by axioms true by clear evidence. The search of the smallest set of primitive terms fulfilings these conditions and adequate for the full axiomatisation of the intended theory was naturaly also a central goal.

The discovery of non-Euclidian systems of geometry was the point of departure of a complete re-examination of this traditional position: as different systems using the same vocabulary were accepted unless they were based on incompatible sets of axioms, it was no more possible to consider primitive terms to be related to already given intuitively clear notions and axioms to express true jugments concerning these notions. After this modification in the conception of the axiomatic method, primitive terms had to be considered as implicitly defined by the axiomatic basis and the traditional relationship between the axiomatic systems and natural intuition was (at least theoreticaly) cut.

One of the consequences of this revolution was that the choice between different axiomatisations of the same theory based on different repartitions of the terms into primitive and non-primitive ones was no more considered to be a matter of epistemic considerations (exepted maybe in pedagogical situations). The only still available choice criteria were the adequacy of the set of primitive terms and the search of the most economical solution. This latter criteria is of course not necessary but it has at least two important justifications. The first justification has often been seen purely aesthetic[2]; it consists in considering that the search of the smalest number of primitive terms is involved in the axiomatic method itself. The second justification is more interesting for my purpose: the choose of a smaller set of primitive terms is a way to restrict the use of implicit definitions, in favor of explicit ones.

At this point, it seems clear that the best axiomatisation of a theory will be that which can capture the whole intended theory using the smallest possible number of primitive terms. But I will show, through the example of classical propositional calculus, that the problem is far from beeing closed by this simple view.

3 Classical propositional calculus

According to the classical theory of adequate sets of propositional connectives, the following examples are perfectly possible choices for the ax-

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[2] For an example of this position, see Sobociński [21], an exellent and quite unknown paper in which a long list of criteria are examined, which can be used in comparing different axiomatic sytems.

iomatisation of the full classical calculus[3]: $\{\wedge, \vee, \sim\}, \{\wedge, \sim\}, \{\vee, \sim\}, \{\supset, \sim\}, \{|\}, \{\downarrow\}$. On the other hand, it is easy to prove that the following ones are not adequate: $\{\equiv, \sim\}, \{\supset\}, \{\supset, \vee\}, \{\equiv\}$. Several comments can be made concerning these examples: 1. the adequacy of $\{\wedge, \sim\}$ and $\{\vee, \sim\}$ shows clearly that $\{\wedge, \vee, \sim\}$ must be rejected by those who requier for their axiomatisation a set of independant connectives; 2. the only adequate sets with only one connective are $\{|\}$ and $\{\downarrow\}$[4]; 3. non-adequate sets can be used for the study of certain important fragments of the full classical calculus, as for example $\{\supset\}$-calculus or $\{\equiv\}$-calculus[5]; 4. even non-adequate, $\{\equiv, \sim\}$ gives rise to a wider fragment than $\{\equiv\}$ and $\{\supset, \vee\}$ is not stronger than $\{\supset\}$ for $\vee$ can be expressed in terms of $\supset$ alone.

According to the criteria discussed at the end of the last section the best axiomatisations of the full classical calculus have to be based either on $\{|\}$ or on $\{\downarrow\}$. Such an axiomatisation has been proposed in 1917 by Nicod [17], it consists in one single axiom and a single inference rule:

$$\text{Axiom: } (P|(Q|R))|((S|(S|S))|((T|Q)|((P|T)|(P|T))))$$
$$\text{Rule: } \{P, P|(Q|R) \rightarrow R\}$$

The other connectives of the calculus can then be introduced without difficulty by explicit definitions such as the following:

$$\begin{aligned}
&\text{D1. } \sim P &&=_{df} P|P\\
&\text{D2. } P \vee Q &&=_{df} (P|P)|(Q|Q)\\
&\text{D3. } P \wedge Q &&=_{df} (P|Q)|(P|Q)\\
&\text{D4. } P \supset Q &&=_{df} \sim P \vee Q\\
&\text{D5. } P \equiv Q &&=_{df} (P \supset Q) \wedge (Q \supset P)
\end{aligned}$$

On a purely theoretical point of view, this kind of axiomatisation seems to be the best. Nevertheless most of modern logicians were not completly satisfied by it and did work with other adequate sets containing usualy two or even three connectives. The search for a clearer and more readable construction (two notions which are important in pedagogical presentations of logic) is not in my opinion sufficient to explain this situation. A more important reason is certainly that logicians try to find a system in which primitive terms are directly related to what is generaly conceived as central

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[3] Exepted for conjunction that I symbolise $\wedge$, I use in this paper the Peano-Russellian symbols for propositional connectives.

[4] For a proof, see for example Mendelson [13] (27-28).

[5] Classical studies of these calculi are to be found in Łukasiewicz [10] and [11].

theoretical notions of logic. In this respect, it seems to me that three connectives stand out against the whole picture, beeing in a special position: the conditional $\supset$ comes first for it is strongly related to the notion of deductibility (as deduction theorem shows); then comes the negation $\sim$, due to its central role in *reductio ad absurdum*; at last, the biconditional $\equiv$ has also to be mentioned for it is directly linked to the relation of logical equivalence. If I am right in this interpretation of logicians' inclinations, it becomes easy to understand that the most successfull set of primitive connectives is $\{\supset, \sim\}$ for none of the three above mentioned connectives is adequate alone and $\{\equiv, \sim\}$ is not adequate either (this fact giving to the biconditional the worst position).

This reason for the rejection of $\{|\}$ (or $\{\downarrow\}$) as the best choice in axiomatisation of the full propositional calculus is clearly of epistemic nature and cannot then be used officialy in an orthodox conception of pure axiomatic logic. The fact is that the use of $\{\supset, \sim\}$ gives more room to implicit definition than it is the case with $\{|\}$, with which only one connective is implicitly defined. This would be unproblematic if there would have been no difficulty in the increase of implicit definition in the case of $\{\supset, \sim\}$ comparing to $\{|\}$. Unfortunately, there is a difficulty which is quite wellknown. This difficulty is not easy to see in every axiomatisation of the $\{\supset, \sim\}$-calculus as for example the very compact Meredith's one [16]:

MerAx:
$((((p \supset q) \supset (\sim r \supset \sim s)) \supset r) \supset t) \supset ((t \supset p) \supset (s \supset p))$
Rules: Det, Sub

But it appears clearly in the following one based on three axioms and the two classical inference rules *Det* and *Sub*:

$A1 : p \supset (q \supset p)$
$A2 : (p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$
$A3 : (\sim p \supset \sim q) \supset ((\sim p \supset q) \supset p)$
Rules: Det, Sub

Negation only occurs in the third axiom and is completely absent from the other axioms and from the inference rules. Thanks to this peculiarity it is possible to understand this axiomatisation either as the full $\{\supset, \sim\}$-calculus, or as the following $\{\supset\}$-calculus where the former axiom $A3$ is now an implicit definition of the non primitive symbol $\sim$:

$A1 : p \supset (q \supset p)$
$A2 : (p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$
Rules: Det, Sub

$$Def_{\sim} : (\sim p \supset \sim q) \supset ((\sim p \supset q) \supset p)$$

These two constructions are of course perfectly equivalent in the sense that both determine the same set of theses. But the difficulty with the introduction of $\sim$ is made clear by the second one, for $\supset$-tautology (known as Peirce's law)

$$Peirce: ((p \supset q) \supset p) \supset p)$$

is not a consequence of $A1$ and $A2$ by the use of the rules, but necessitates the introduction of $Def_{\sim}$ in order to be proved. This shows that $Def_{\sim}$ is a *creative* implicit definition, for it allows to prove an expression which does not include the new symbol $\sim$ and which is not provable whitout the definition. It is even *strongly creative* in the sense that the primitive notion which is modified by the addition of $Def_{\sim}$ is precisely the connective ($\supset$) which is used in the definition in order to characterise the new symbol $\sim$.

For this difficulty is a consequence of the introduction by implicit definitions of more than one connective, it shows, in this perspective, the superiority of axiomatisations based on a single term, like Nicod's one. Nevertheless, it has to be noticed that the whole picture cannot be given without paying attention to the way explicit definition is used; its importance regarding the notion of *adequacy* (a set of connectives is said to be *adequate* when every possible classical connective is *explicitly defininable* on its basis) cannot be neglegted in the present discussion.

4 Explicit definitions

A great majority of contemporary logicians seem to share the conception of explicit definition I will call here the *external* conception. In this conception the aim of a definition is the introduction of a new conventional way of designating certain expressions or range of symbols of the official object language of the system in question. In Nicod's system, for example, the introduction of $\sim$ by the way of

D1. $\sim P =_{df} P|P$

does not increase officialy the object language by the addition of the new symbol $\sim$, but gives only the possibility to designate in the metalanguage expressions of the form $P|P$ by the way of the convenient and shorter metalinguistic expression $\sim P$. It has then to be noticed that D1 can hardly be said to be a genuine definition. As a matter of fact, the metalinguistic symbol $=_{df}$ cannot state that the *definiendum* and the *definiens* are inferentialy equivalent, for the *definiendum* is not an expression of the system. Stricly speaking, it follows from this remark that no formal system (in the usual sense of these words) can be said to constitute the full classical propositional calculus, for such a system would never include the tautologies involving non primitive connectives. With the external conception of explicit definitions, saying that $\{|\}$ is *adequate* signifies that, in an ideal system containing *all* the classical connectives as primitive terms, every expression E would be provably equivalent with an expression E^* in which $|$ occurs as the only connective.

At first sight, these considerations seems to be of purely proof-theoretical interest, with no consequence on semantical or structural conceptions of logic. In the rest of this paper, I will show however that the adoption of a different conception of explicit definition, that I will call the *internal* conception, leads to an important re-evaluation of the notion of *adequate set of connectives.*

5 Internal explicit definitions

Originated in the logical work of S. Leśniewski, the central idea of the internal conception is to consider definitions as theses of the system[6]. As every thesis, they should be introduced on the basis of axioms and inference rules. Definitions are then to be *wff* of the object language. Aside from the symbol to be defined, they should not involve symbols which do not belong to the system, as for example $=_{df}$. So as to state that *definiendum* and *definiens* are inferentialy equivalent expressions, it is then only possible to use the primitive terms of the system. For that purpose the most natural choice is no doubt the biconditional $\equiv$. The choice of $\equiv$ is nevertheless only possible in systems including it as one of the primitive connectives.

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[6] Cf. Lesńiewski [8] and Rickey [18].

In the twenties, S. Leśniewski and A. Tarski showed that it is possible to obtain a extended propositional system (called Protothetic[7]), from wich full classical calculus is a part, on the basis of $\{\equiv, \forall_s, \forall_{s/s}\}$: biconditional and the universal quantifier binding propositional and unary connective variables (variables of category s and s/s)[8]. Apart from (more or less) usual inference rules, Protothetic includes also a special rule for the introduction of definition-theses; depending on what the system already contains, this rule allows to consider as theses closed biconditional expressions of the form

$$Def_\alpha : (\forall v_1 v_2 \cdots v_n)(\alpha(v_1 v_2 \cdots v_n) \equiv E_{v_1, v_2, \cdots, v_n})$$

where the first argument of $\equiv$ is the *definiendum* and the second one the *definiens*, both fulfiling the general conditions presented at the begining of this paper.

Of course this solution requiers the use of a quantified propositional logic, but Leśniewski and Tarski also examined simpler possibilities in the framework of classical unquantified systems. Using, for example a complete $\{\equiv, \sim, \vee\}$-calculus, a definitional rule can be added which is based on the definitional frame

$$\text{(DF1)}: \ D_{um} \equiv D_{ens}$$

the following thesis beeing an example:

$$Def_\supset : (p \supset q) \equiv (\sim p \vee q)$$

Even definitions stated with $\equiv$ are the most natural, the authors of the Warsaw School examined also possibilities using other primitive connectives. On the basis of the provable inferential equivalence of the following expressions in the complete $\{\equiv, \sim, \supset, |\}$-calculus

$$p \equiv q$$
$$\sim ((p \supset q) \supset \sim (q \supset p))$$
$$(p|q)|((p|p)|(q|q))$$

the authors showed that a definitional $\{\supset, \sim\}$-calculus is possible with the definitional frame

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[7] For a modern presentation of Lesniewski's Protothetic, see Miéville [14] and [15].

[8] Definition of $\sim$ and $\wedge$ on the basis of $\{\equiv, \forall_s, \forall_{s/s}\}$ is to be found in Tarski [22]; see also Miéville [14] (164-174). Other interseting solutions based on $\{\equiv, \forall_s, \forall_{s/ss}\}$ are presented in Sobociński [20].

(DF2): $\sim((D_{um} \supset D_{ens}) \supset \sim(D_{ens} \supset D_{um}))$

and also a definitional $\{|\}$-calculus with the frame

(DF3): $(D_{um}|D_{ens})|((D_{um}|D_{um})|(D_{ens}|D_{ens}))$

At this point a particular attention must be paid to the $\{\supset\}$- and $\{\supset, \equiv\}$-fragments of the full classical calculus, for it can be shown in the latter that the following expressions are inferentialy equivalent provided v is a propositional variable which occurs neither in A nor in B:

$A \equiv B$

$((A \supset B) \supset ((B \supset A) \supset v)) \supset v$

This result shows that a definitional $\{\supset\}$-calculus is possible with the following definitional frame

(DF4): $((D_{um} \supset D_{ens}) \supset ((D_{ens} \supset D_{um}) \supset v)) \supset v$

where v is any variable wich does not occur either in D_{um} or in D_{ens}.

6 Lejewski's $\supset$-definitions

As in the classical calculi using external explicit definitions stated with $=_{df}$, the difference between $\{|\}$ and $\{\supset\}$ still remains here, for it is still impossible in the definitional $\{\supset\}$-calculus to define all the classical connectives, at least without introducing quantification[9].

Nevertheless, Lejewski [7] had the idea to use definitional frame (DF4) with an other kind of *definiendum*. As it is illustrated by several mathematical accepted definitions, like for example the following one, which introduces the new symbol $\circ$ on the basis of the primitive functor S (successor):

$Def_{\circ} : \circ(n) = m \quad =_{df} \quad S(S(n)) = m$

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[9] As it was already shown by Russell [19], $\sim P$ can be defined in $\{\supset, \forall_s\}$-calculus using the *definiens*: $P \supset (\forall v)v$.

the notion of explicit definition does not requier the *definiendum* (here: $\circ(n) = m$) to be identical with the function to be defined ($\circ(n)$). Using this possibility for the *definiendum* to be different from the *function to be defined* (F_{tbd}), Lejewski proposed to replace in (DF4) both occurences of D_{um} (up to then identical with F_{tbd}) by a *definiendum* of the form

$$D_{um} : F_{tbd} \supset w$$

where w is a propositional variable which does not occur in F_{tbd}. Apart from this peculiarity of Lejewski's $\supset$-defininitions, the general conditions concerning the constitution of the *definiendum* and the *definiens* are still the usual ones.

The two following $\supset$-definitions using (DF4) and the new sort of D_{um} are examples given by Lejewski. The first example is a definition of classical negation:

$$Def_{\sim} : (((\sim p \supset w) \supset ((p \supset w) \supset w)) \supset ((((p \supset w) \supset w) \supset (\sim p \supset w)) \supset v)) \supset v$$

where

$$D_{um} = \sim p \supset w \quad ; \quad F_{tbd} = \sim p$$
$$D_{ens} = (p \supset w) \supset w$$

Second example (Sheffer's stroke):

$$Def_{|} : ((((p|q) \supset w) \supset ((p \supset (q \supset w)) \supset w)) \supset ((((p \supset (q \supset w)) \supset w) \supset ((p|q) \supset w)) \supset v)) \supset v$$

where

$$D_{um} = (p|q) \supset w \quad ; \quad F_{tbd} = p|q$$
$$D_{ens} = (p \supset (q \supset w)) \supset w$$

It must be precised that these definitions are stated in a definitional $\{\supset\}$-system called by Lejewski S_1 which is a version of a Łukasiewicz's complete $\{\supset\}$-calculus including a special rule for the use of definitions[10].

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[10] For a description of S_1 rules, see Lejewski [7], 195.

System S_1:
ŁukAx: $((p \supset q) \supset r) \supset ((r \supset p) \supset (s \supset p))$
Rules: Det, Sub, Def

At last, Lejewski [7] proved that the above written examples of $\supset$-definitions are adequate for the introduction of the classical connectives $\sim$ and $|$ and also that there exist in his system S_1 adequate similar $\supset$-definitions for every classical connective[11]. S_1 is then a complete axiomatisation of the full classical propositional calculus.

7 Conclusion

Due to a long tradition which culminated in Whitehead and Russell's *Principia Mathematica*, it has been widely thought in logic that definitions are only to introduce "mere typographical conveniences (...) theoretically superfluous" ([24], I.11). This view is essentialy related to the external conception of explicit definitions. Russell was nevertheless one of the first to underline the strange status of external definitions:

> It is a curious paradox, puzzling to the symbolic mind, that definitions, theoretically, are nothing but statments of symbolic abreviations, irrelevant to the reasoning and inserted only for practical convenience, while yet, in the development of a subject, they always require a very large amount of thought, and often embody some of the greater achievements of analysis ([19], 63).

On the other hand, it is perfectly possible to reject this standard position and the view that definitions are "irrelevant to the reasoning". This leads to the adoption of an internal conception and the idea that definitions have to be considered as theses of the formal system, whose introduction is justified by the use of a special inference rule. This, of course, requiers that we change our idea of what a formal system is: due to the fact that new symbols are officially introduced in the object language, the system can no

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[11] In the case of negation, Lejewski shows that ŁukAx + $Def_{\sim}$ associated with *Det* and *Sub* constitute a complete classical $\{\supset, \sim\}$-calculus, ([7], 197-198).

more be viewed as a (closed) set of theses. It is rather a formal machinery in which theses have to be ordered, following step by step the inscriptions of definitions.

My aim in this note was not to argue in favor or against this non-standard approach of formal systems, but only to show that certain central notions of logic are dependant on our view on explicit definitions. Lejewski's important (and quite unknown) result that $\{\supset\}$ is adequate for the construction of the full classical propositional calculus goes clearly in that direction. It shows that the notion of *adequate set of connectives* is wider in the internal conception than it is in the standard external one[12].

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[12] For a more complete discussion on internal definitions, see also my [4] and [5].

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Geometry for Modalities? Yes: Through n-Opposition Theory

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Abstract

We present here some new results about the fundamental relations between modal logic and geometry. The key of our approach is a general renewed theory of "logical n-opposition", a strong geometrical generalisation of Aristotle's classical "opposition theory". The question of the possible relations between (modal) logic and geometry, in some sense brightly foreseen by the russian logician N.A. Vasil'ev around 1910, happens to be still quite a mysterious one and can be shown to be of great relevance for contemporary research in many fields (for instance, but not only, in cognitive science). The most famous (seminal) entanglement between logic and geometry is given by Aristotle's "logical square of oppositions", a very poor structure in terms of further mathematical developments. Concrete applications of this structure, although not uninteresting, have been rather mean as well (cf. J. Piaget, J.A. Greimas and J. Lacan). Some recent results, however, have clearly shown that this sad and poor field (opposition theory as a limited and unclear encounter of modal logic and elementary geometry) is in fact *much bigger* than it was thought. This is strongly and mainly suggested, as we will recall, by the surprising (and almost unnoticed) discovery in 1953 - inside "opposition theory" - of a "logical hexagon" (Blanché), and then by that, in 2003, of a 3-dimensional "logical tetradecahedron" (Béziau and Moretti), which both are logical "implicational" structures (expressing opposition relations) furnished with a notably high degree of geometrical symmetries. These discoveries are clearly open to further development and

generalization, as we will briefly sketch and demonstrate (we will present here many new such structures). The problem thus arising nowadays is a lack of comprehension of the kind of (new) modal "language" we are speaking while developing such structures. Haven't we entered a radically new logical field? The bulk of this paper will consist in establishing that, in fact, this mysterious emerging field is really a totally *new* one, and that it happens to be huge (it has to do with science, not with history), and (at least partly) already precisely shaped : there are noticeable guidelines, or trends, of it - the principal one consisting in the discovery of a geometrical treatment of 'contradiction' in terms of poly-dimensional symmetries. To make such a shape the more explicit we can, as a new research framework in modal logic, we develop here a general logical-geometrical theory of "n-opposition", consisting mainly in a theory of geometrical "$\alpha n(m)$-structures" and in a theory of modal "$n(m)$-graphs" (the two working closely together). Both theories are based on the notion of "simplex of dimension n". This framework succeeds in systematising and in explaining all previous results and, more than this, it leads to infinitely many new ones, as we will detail in the paper. After sketching the proof of a useful general theorem, we end by showing that an interesting (by now difficult) problem (a conjecture), very promising if solved, is left open, hopefully solvable by a suited theory still to come, which we call the problem of a possible theory of $\beta n(m)$-structures.

1 Previous results : from the square to the 4 hexagons and, finally, to the tetradecahedron

We recall here Aristotle's basic doctrine. "**Opposition**" consists in a complex ordering, expressed geometrically by the "logical square" (of oppositions), of four different ways for two terms to be "opposed" one to the other[1]. These ways are : (1) *contradiction*, defined for two terms as, simultaneously, the impossibility to be both true *and* the impossibility to be both false ; (2) *contrariety*, defined for two terms as, simultaneously, the impossibility to be both true *but* the possibility to be both false ; (3) *sub-contrariety*, defined for two terms as, simultaneously, the possibility to be both true *but* the impossibility to be both false ; (4) *sub-alternation* (or

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[1] In some sense, opposition theory can be said to be the logical theory of "difference".

implication), defined for an ordered couple of terms as the impossibility of having the first without having the second (so that, in some sense, it *contains* the fourth combinatorial case, i.e., *simultaneously*, the possibility of being both true *and* the possibility of being both false - plus the possibility that the first is false while the second is true).

As we see, the 4 kinds of oppositions exhaust the combinatorial possibilities of combined truth and/or falsity of two simultaneous terms. In the square, these 4 kinds constituting the concept of opposition are represented not by the 4 points (the vertices, the corners of the square) but by the lines (the square's 4 edges and 2 diagonals). For simplicity's sake, we will use the following convention : contradiction will generally be represented in red, contrariety will be in blue, sub-contrariety will be in green, sub-alternation (i.e. implication) will be in grey (sometimes in black) (sorry for colour-blind people!). The four points (corners) delimiting the square are *variables*, empty places to be fulfilled with modalities. **Classical opposition theory** is thus modal logic (at least the "core" of it, the universal Lewis system $S5$) entangled in some (simple) way, as we saw, with geometry (the square with its 4 vertices, 4 edges and 2 diagonals, cf. figure 1).

Many parts of Aristotle's logic have been abandoned or strongly revised during the "logical turn" of the second half of the nineteenth century (from Boole to Russell) : not the logical square[2]. This structure, in fact, if poor, seems nevertheless impressive by its incredible generality : it expresses graphically the fundamental quantificational relations (holding for $\forall$, $\exists$, $\neg\forall$, $\neg\exists$) and thus - modal logic being related to quantification theory, as we know now through "possible worlds semantics" - it expresses also the fundamental modal relations, at least those of the 4 "non-naked" modalities among the 6 basic ones of $S5$ (cf. figure 1).

But scholars in the history of logic have soon remarked the presence, in Aristotle's theory of modality, of two incompatible theoretical positions : the aforementioned logical square (AEOI) and the less known "logical triangle" (of contrarieties, AEY). The two do not fit together. Where does this unresolved ambiguity come from? It has been shown that Aristotle oscillates between two different notions of "possibility" : a "unilateral" one (expressed by the square of oppositions) and a "bilateral" one (expressed by

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[2] For instance, Frege keeps it (with a mistake concerning subcontrariety, unduly identified to contrariety !) in his 1879 masterwork *Begriffsschrift*, where he feels obliged to give his own, reformulated version of it (*Begriffsschrift*, §12, p. 24 of the original edition ; cf. figure 1).

the triangle of contrarieties, here in blue).[3] They are different in this sense that the "unilateral possible" ("I") is incompatible with the impossibility ("E") but compatible with necessity ("A") - it is its consequence -, whereas the "bilateral possible" ("Y") is incompatible both with impossibility *and* with necessity (cf. figure 2).

After almost 2500 years this almost unnoticed historical and logical-geometrical riddle (two "possibles" are possible, the square and the triangle being seemingly doomed to remain nastily unrelated) was elegantly solved, from a logical point of view, by R. Blanché (1953) : by adding in the logic and geometric plan of opposition theory a "triangle of sub-contrarieties" ("IUO", here in green) - dual of the triangle of contrarieties -, totally unknown to Aristotle, and thus getting a new geometrical-logical figure, the "logical hexagon"[4]. Effectively, the square emerges then, in a very elegant way, as a simple "by-product" of the two triangles, and in fact we have now not one ("AEOI"), but 3 symmetric logical squares (according to the 3 symmetry axes furnished by the 3 red diagonals of the hexagon) : AEOI, EYIU and YAUO (cf. figure 2). Aristotle's aporia is then solved : the two distinct notions of possibility (bilateral and unilateral) are now explained nicely and interrelated (the "bilateral possible" is equivalent to the conjunction of the "unilaterally possible that" and the "unilaterally possible that not" : $Y \leftrightarrow I \wedge O$). This new solution, the logical hexagon, is mathematically much more interesting than the old strange "logical square" (the symmetries are now much richer). From a philosophical point of view, there is a reason, apart from his aversion for geometry and mathematics in general, why Aristotle, as a logician, missed one of the 6 relevant positions (the topmost "U") of the hexagon, thus being prevented from discovering this last general structure : this position ($U =$ "$A \vee E$", "$\Box \vee \neg\Diamond$", "$\forall \vee \neg\exists$") reflects in fact the necessitarian position of Diodoros Cronos ("necessary or impossible, no third way"), the determinist position of the Megarian school, excluding the possibility of an open future (this last Aristotelian indeterminist position being represented by the dual position, "Y") : on a philosophical basis, Aristotle fights precisely against this option (by his doctrine of the "contingent futures", cf. *De Interpretatione*, 9). So the logical square alone is incomplete. One needs to consider the logical hexagon. However, this strange story of the (up to now cautious) intercourse of logic and geometry does not rest here.

Fifty years after Blanché, in 2003 J.-Y. Béziau discovers two further

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[3] Cf. [10], p. 16-17.

[4] Cf. [5], [6], [7], [8] and [10].

possible simple modal decorations of the abstract geometrical-implicational structure of the hexagon : he claims that one is “paracomplete” (i.e. intuitionist), the other “paraconsistent”[5], all this in addition to Blanché’s solution which Béziau calls retrospectively ”classical” (classical, paracomplete and paraconsistent are said with respect to the modal properties of the respective negations, cf. [2] and [3]). This result, the existence of more than one hexagon, is possible by virtue of the fact that even modally “naked” propositions (say α), i.e. propositions without modal modifiers ($\Box$, $\Diamond$, $\neg\Box$, $\neg\Diamond$), as well as their negations (say $\neg\alpha$), still are modalities.[6]

Following Béziau’s then unpublished (but shared) new results, A. Moretti and H. Smessaert discover (independently from each other) a fourth possible modal decoration of the abstract oppositional hexagon, here in blue (which I propose to call “emergent”). And, finally, Béziau and Moretti discover together a three-dimensional ordering of the four hexagons, the “logical tetradecahedron” (or “cube-octahedron”), a 3-dimensional polyhedron with 12 vertices and with 14 sides, 8 triangular and 6 square sides : the four hexagons are displayed according to an (invisible) inner tetrahedron, so that each vertex of the tetradecahedron lays at the intersection of 2 hexagons and no vertex of no hexagon remains non-intersected. This structure is geometrically striking (cf. figure 4).

This will be the starting point of the present paper.[7] Past discoveries seem to lay a strange problem : how comes that the beautiful logical order expressed geometrically by the “logical tetradecahedron” is so far related to nothing else similar in modal logic ? Is this really an isolated meaningless result ? Before any further consideration on the (mysterious) nature of this geometrical-logical structure, we want to quickly recall how the question of the possible uses of the tetradecahedron in (modal) logic has already been partly tackled.

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[5] A formal system is said to be “paraconsistent” if it is inconsistent but not trivial.

[6] They are “zero degree modalities”, according, for instance, to W.A. Carnielli and C. Pizzi, *Modalità e multimodalità*, FrancoAngeli, Milano, 2001 p.11.

[7] Note that in what follows the metalanguage will be classical, and we will not study paracomplete or paraconsistent aspects of the logic of the hexagons.

2 Possible uses of these start(l)ing results : the tetradecahedron as a translation rule between modal logic and geometry

For reasons to appear later, we will start by calling the implicitely well known graphs exhibiting the relations between basic modalities in the usual systems of modal logic (as, for instance, in [9], p.149-157 ; cf. figure 5 here) "**modal graphs**".

The discovery of the tetradecahedron makes additional sense when we see it as a **translation rule** between modal logic and some kind of geometry. Such a translation is rather precise : it relates the modal graph of $S5$ to the arrowed tetradecahedron (we use here the modal variables x_1, x_2, x_3, $\ldots$, x_6, cf. figure 6). Still for reasons to appear later, we call this last an instance of a "β-structure", whereas we will call the four hexagons it orders instances of "α-structures" (for short, a β-structure is something n-dimensional organising nicely some (n-1)-dimensional α-structures).

Now, the point is that this translation rule, which relates, modality by modality, the modal graph of $S5$ and the logical tetradecahedron containing the four hexagons seen before (cf. figure 6), is not a "dead horse". First, it belongs to an infinite (fractal) series of such translation rules, the series of the n-hyper-tetradecahedra (cf. [11]). And second, it allows to study many other classical modal graphs from a geometrical, tetradecahedron-based point of view (for such a study concerning the system $S4$, cf. [4] and [13]). We give here a hint to the simple technique allowing (given transitivity of the arrows in $S4$) to start a geometrical analysis of $S4$ through the translation rule (cf. fig. 7)[8].

As just mentioned (about the existence of an infinite fractal series of translation rules), a spectacular result concerns what we will call the "linear modal graphs" (as $S5$, modal graphs without branchings or outer unarrowed points). We show in [11] a general "fractal" property, determining in an algorithmic way the number, quality and structure of the geometrical n-dimensional figures ruling the logical space of linear modal graphs (it is a

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[8] This technique allows the discovery of 17 inner tetradecahedra of $S4$. Then a further examination starts, allowing to discover, from these, some further hexagons ("emergent" ones) and, collecting them, further tetradecahedra (emergent ones), and so on (5-dimensional "hyper-tetradecahedra" ...) until closure, i.e. until the full determination of the complex shape of the geometrical structure of the logical space of $S4$ (cf. [4] and [13]).

fractal generalisation of the translation rule - going into bigger and bigger spatial dimensions - of the logical tetradecahedron). As such, this result seems to justify this kind of fertile investigations.

Then, the geometrical study of branching modal graphs (as the ones of $S4$ and of the $K5$-systems, cf. figure 5) will be another topic. It can be shown that one could bring up this study systematically on the basis of the aforementioned examination of linear graphs (a branching modal graph can be seen as a combination of two or more linear graphs ; we study this in [12]).

But we can say no more in this place about such specialized branches of the arising theory of the interrelations between modal logic and geometry. For our goal, in the present paper, is (more fundamentally) to give a general framework to the oppositional figures seen so far (square, hexagon and tetradecahedron) from the point of view of the problem which originated them all, that is to say the problem handled by opposition theory.

3 Which comprehensive frame ? The viable idea of a generalized "theory of n-opposition"

So we have this strange logical tetradecahedron. Said with some humor, two options seem possible : (1) the tetradecahedron is just an accident, a mind-pitfall for stupid Platonists (I am one), opposition theory (the essence of transcendental logic) does not change ("long live Aristotelianism !") ; or (2) the tetradecahedron (and the hexagon) is the sign of a major change to come inside good old opposition theory, it means some kind of primacy of infinite mathematics over "transcendental logic" ("down with Aristotelianism !")[9]. Our bet will be that this beautiful structure is not an "hapax legomenon" (i.e. an isolated, meaningless event related to nothing determinable). On the contrary, we take it as the sign of an underlying complex theory (whose first elements we will try to bring here into light), a theory comprising it as a fragment.[10] This conjecture of ours, predicting a

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[9] For a philosophical position of this kind, we rely mainly on A. Badiou's proposal of a (atheistic) "Platonism of multiplicity", as in A. Badiou, *L'être et l'événement*, Seuil, Paris, 1988 ; *Conditions*, Seuil, Paris, 1991 ; *Court traité d'ontologie transitoire*, Seuil, Paris, 1998.

[10] For the very idea of "daring" look for extensions from "3-terms" oppositions (hexagonal case) to mysterious x-ones (with four terms, and perhaps more !), I wish to thank J.-Y. Béziau, whose interest in square-related "boring" stuff is pioneering and unusual among professional logicians, and from which I benefited, besides constant personal sup-

considerable extension of opposition theory, is not trivial : since Aristotle until now, opposition theory is considered a stupid fragment of classical mathematical logic (an object for history, not for scientific research), and concrete applications of this structure have been rather mean.[11] Our conjecture, if verified, would falsify this judgement and would, more than this, presumably bring new light on the fundamental relations between logic and geometry. Such relations, quite mysterious until now, are a central issue in contemporary thought (cf. for instance P. Gärdenfors' theory of "conceptual spaces" in cognitive science, opposing very convincingly geometry to logic as a paradigm for conceptual modelling ; as well as I. Matte Blanco' theory, a logical-geometrical approach to the "unconscious" features of the human mind).[12] And our bet (our conjecture) happens to be won (proved) : *there is* a global theory superseding Aristotle's one and explaining (and containing) the emergence of such strange logical structures as the tetradecahedron. Which we are now going to explain.

Considering that classical opposition theory (as emended by Blanché) is 3-opposition theory (it deals with two *triangles*, relating *three* contrary blue terms, and *three* subcontrary green terms), we want to explore the idea of a possible generalization of it by some kind of "n-opposition theory". By analogy with what we know already ($n = 3$, the hexagon's case), it will show up that the global n-theory has one geometrical side and one modal side. Effectively, one problem is that of looking for a good geometrical model of n-opposition. But the attempt to decorate with modalities such

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port, several funny and passionating discussions about syrens, dragons, monsters and the like.

[11] Cf., for instance, Piaget's theory of the "I.N.R.C. group" as a cognitive model of children intelligence growth, or Greimas' theory of the "semiotic square" as a general structuralist theory of meaning as such, not to mention Lacan's psychanalytical "sexuation formulas" on the complex structure of human sexual identity (cf. J. Piaget, *Six études de psychologie*, Gonthier, Genève, 1964, chapters 3, 5, 6 ; *Le structuralisme*, PUF, Paris, 1968 ; *L'épistémologie génétique*, PUF, Paris, 1970, p. 51-58 ; A.J. Greimas and F.R. Rastier, "Le jeu des contraintes sémiotiques" (1968) and Greimas, A.J., "Éléments d'une grammaire narrative" (1969), both in : Greimas, J.A., *Du sens. Essais sémiotiques*, Seuil, Paris, 1970 ; "Carré sémiotique" in : A.J. Greimas and J. Courtés, *Sémiotique. Dictionnaire raisonné de la théorie du langage*, Hachette, Paris, 1993 ; Dor, J., *Introduction à la lecture de Lacan. 2. La structure du sujet*, Denoël, Paris, 1992, chapters 12-15 ; Lacan, J., *Le séminaire - livre XX. Encore*, Seuil, Paris, 1975, chapter 7).

[12] P. Gärdenfors, *Conceptual Spaces. The Geometry of Mind*, The MIT Press, Cambridge MA and London, 2000 ; I. Matte Blanco, *The Unconscious as Infinite Sets. An Essay on Bi-logic*, Duckworth, London, 1975 ; E. Rayner, *Unconscious Logic. An Introduction to Matte Blanco's Bi-Logic and its Uses*, Routledge, London, 1995.

a n-oppositional model is yet another problem. Both must be tackled. We will first face the *geometrical* problem (how to express n-opposition geometrically beyond the hexagon). Then, armed with new formal tools, we will face the *logical* problem, that of decorating modally the geometrical expression of opposition.

3.1 First result (geometrical) : the notion of oppositional "n-structure" (the realm of αn-structures)

For simplicity, we begin by focussing on contrariety alone (we forget momentarily the other 3 kinds of oppositions). The starting point is a project of ours of generalizing the expression, by a (blue) triangle, of the relation of contrariety. The equilateral triangle expressed the fact that the 3 contrary terms under examination are "on a same plan", each one equally distant from the other two (they are, two by two, equally different). How to express the same thing with 4 terms (say : 4-contrariety) ? The obvious geometrical answer is, if we want to keep our geometrical equidistancy metaphor : "with a (blue) tetrahedron". Each point is then equally distant from the other 3, we are rescued by the use of a 3-dimensional, instead of a 2-dimensional, space. But then, how to express contrariety for 5 terms (5-contrariety) ? It can be shown that no 3-dimensional figure allows a distribution in the space of 5 points so that each point is equally distant from the other 4. Are we obliged to drop our equidistancy-criterium for contrariety, if we want to use geometry ? We are not : we only need to pursue our n-dimensional ascent. Here comes, in fact, the first important result in our "quest" : n-dimensional geometry allows a geometrical representation of n-contrariety (not yet n-opposition) - as "being n different guys at the same distance one from the other", or n-equidistancy -, by taking into account the mathematical series of the "simplexes of dimension n-1" (each "simplex of dimension n-1" is a (n-1)-dimensional structure whose n vertices have the same distance between any 2 of them, cf. [1] ; cf. figure 8).

Now, in order to have opposition (conceived as a combinatorially exhaustive theory, in the sense sketched above), we have to add to contrariety the three other opposition relations : sub-contrariety, contradiction and subalternation (i.e. implication). And this is geometrically possible ! Our geometrical leading remark (the existence of the series of the simplexes of dimension n) *does allow* the construction of the required opposition relations. It suffices, by analogy with the case of the hexagon, to combine, according to a suited symmetry, two simplexes (one blue for contrarieties, and one green for subcontrarieties) so that each vertex of the blue one is contradictory to the vertex of the green one symmetric to it (cf. figures 9-11).

We call "logical ***bi-simplex*** of dimension n-1" the blue-green (n-1)-dimensional structure thus obtained (having $2n$ vertices). Contrariety is assumed by construction (the blue simplex). Subcontrariety is obtained easily by symmetry (the green simplex). Contradiction is expressed by the n biggest diagonals (in red, when expressed) of the bi-simplex (as it was in the hexagon by the 3 red diagonals, and in the square by the 2 red diagonals, cf. figure 9). And subalternation can be easily expressed by adding a grey arrow between each couple of non-contradictory blue and green vertices (from the blue one to the green one).

This constructive device works out neatly even at the next stage (we move from $n = 3$ to $n = 4$, cf. figure 10). And this leads to the discovery of a new "logical oppositional structure", the "logical bi-tetrahedron" (or, more classically, "stella octangula") which, once the arrows expressing its subalternation relations (from blue to green vertices) are drawn (in grey), reveals itself to be a "**logical cube**" of oppositions.

And the algorithmic magic of this is that we can go further. If we step now from 4-opposition (the logical cube) to 5-opposition, combining two symmetric simplexes of dimension 4 (one blue for 5-contrariety, the other green for 5-subcontrariety), we get a new oppositional figure, which we call for simplicity (the geometry of a 4-dimensional space becoming slightly counter-intuitive) the logical "**α5-structure**" (cf. figure 11). Retrospectively, the logical hexagon and the logical cube will be named respectively **α3-structure** and **α4-structure**.

It seems to be rather easy to give a proof (by induction) of the generality of this geometric method, with a recursion based on the notion of simplex of dimension n-1 (such simplexes never stop, they are available for any n). However, we omit here to give such a proof. We move instead to the question of the graphical (geometrical) expression of the general case. If we look for a graphical expression of this efficient general algorithm, we can compare graphically the oppositional αn-structures, say for $n = 3$, 4, 5, 6, 7 (cf. figure 12). The drawing suggests a (very) small proviso to this general algorithm, according to the fact that n is odd or even. When n is odd it will be easy to lay the two simplexes, blue and green, "face to face", in order to let appear clearly in a 2-dimensional drawing the appropriate symmetry of the contradictory vertices (related by a red diagonal). We call this circular representation, which simply alternates blue and green terms, duly arrowed, "doughnut" or "big wheel" - in honour to Bart.

When n is even, however, the symmetry of the contradictory vertices cannot be expressed as easily in a 2-dimensional drawing (put in a circle, $2n$ points alternatively blue and green, with n even, do not allow blue and green points to be opposite - each blue point will be centrally symmetric to a blue point, each green to a green one, which is not geometrically satisfactory). So we adopt the convention of representing it in the way depicted in the lower part of our schema (of figure 12, or at the right end of figure 13), that is with 4 niveaux, the topmost one with a green term (as we will see, it will be called "head"), than a row of n-1 blue terms, then a row of n-1 green terms (as we will see, these two rows constitute the "body"), and finally, at the bottom, a blue term (as we will see, it will be called the "tail"), all this duly arrowed (we call this representation "hamburger" - or "merry-go-round", in honour to Homer).

The **general graphical algorithm** (for n odd or even) is finally the following : for each **αn-structure** - a "bi-simplex of dimension (n-1)", resulting from the superposition of two simplexes of dimension (n-1) (one blue, the other green) - all blue points are contrary to each other, all green points are subcontrary to each other, each blue point is contradictory to one green point and one only (and reciprocally), each blue point implies n-1 green points (all except its contradictory), each green point is implied by n-1 blue points (all except its contradictory). Geometrically speaking, we are done!

3.2 Second result (logical) : the notion of modal "$n(m)$-graph" (from αn-structures to $\alpha n(m)$-structures)

As we have now fulfilled the task of expressing *geometrically*, in a way generalized to any finite integer n, the logical relations constituting opposition (i.e. contradiction, contrariety, sub-contrariety and sub-alternation - viz implication), we want now to give "modal flesh" to our "geometrical skeleton". In other words, the important thing now is, once a decoration with modalities is applied to an αn-structure (i.e. once each vertex of the geometrical structure receives a modality), to check the structure's validity by checking the validity of the (grey) arrows (the checked oppositional αn-structure is **valid** iff *all* its arrows do obtain). One has to check them one by one, to see if the implication (subalternation) they mean of a modality (the arrow's ending point) by another (the arrow's starting point) obtains (i.e. it is true), and this depends on the context given by the chosen "modal graph". As we will see, outside the known case of the hexagons (in fact, the case $n = 3$), this question of the "checking" is not yet straightforward. As it happens, some difficulties must be tackled, something new must be created at this point in order to solve 2 new simultaneous tasks : (1) attributing modalities to the vertices (i.e. to the variables, to the empty places) of the αn-structures in a non-trivial way (there is no use in attributing modalities in a way that never - or always - works, there must be some working and some non-working decorations, as for the square and for the hexagon) ; (2) and then - inside a given decoration - judging, for each of the arrows, if it does obtain for that decoration (if the implication it expresses is valid for that decoration).

The case $n = 3$: "3-opposition theory". We will first examine the already known case of $n = 3$ (that is to say, how to decorate modally the logical hexagon). For reasons to appear later, we introduce the use of **"alphabetical variables"** for modalities : instead of using usual modal operators ($\Box$, $\Diamond$, $\neg\Box$, $\neg\Diamond$, ...), we will adopt Greek letters for the left side (positive modalities) and Arabic letters for the right side (negative modalities, cf. figure 14). The unique rule to be observed here is that the alphabetical order common to the two alphabets (A, B, G, D, E, F, K, L, M, N, ...) is implicative, i.e. each term of each of the two series implies the term, inside the same family, next to him alphabetically (A → B, B → G, etc.)[13].

We need now to make the already mentioned notion of **modal graph** more precise. A *modal graph* is an arrowed structure furnished with some symmetries. In this starting case (the classical modal graphs, as in [9], p. 149-157) the symmetry seems to be a left-right one, holding between a left side where "positive modalities" (i.e. without negations before them) lay and a right side where "negative modalities" (i.e. modalities preceded by a negation) lay.[14] This classical symmetry has 2 main features : (1) it

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[13] Clearly, our alphabetical order is a special one, resulting from some sort of a compromise between different diverging classical alphabets (we have, for instance, G instead of C, etc.).

[14] Truly speaking, even for arrows it is a *central* symmetry (the only restriction to such a centrality : the arrows always follow a top-down direction, i.e. the symmetry does not reverse the arrows' orientation). But given the fact that, in standard modal logic, each side of the modal graphs (left as right) has, for shapes, a further inner *vertical* symmetry (a top-bottom one) the central symmetry seems to reduce, for shapes, to a simpler horizontal symmetry (the mentioned left-right one). However, the central

seems to work like a mirror as for arrows' geometrical concatenations (the same concatenation at both sides via a left-right symmetry) ; (2) it ties each element (i.e. each modality, each arrow's extremity) of one side to an element of the other side, its contradictory negation (this last symmetry being central - left-top corresponds to right-bottom, left-center corresponds to right-center, and so on cf. figure 15).

Additionally, modal graphs have ***layers*** (that is, the number of terms "A", "B", …, "M", adopted for that graph in each alphabetical family), parametred by an integer m. For $m = 2$, you have A (alpha and alif) and B (beta and ba) ; for $m = 3$ you have A (alpha and alif), B (beta and ba) and G (gamma and jim) ; and so on (cf. figures 15 and 16). For reasons to appear later on, we call the linear, non-branching modal graphs of classical modal logic (think of $S5$) "**modal 3-graphs**" and, in order to take into account the parameter m, we call them still more precisely "**modal 3(m)-graphs**".

All this can also be expressed symbolically (i.e. without arrows) by sets of "**3(m)-relations**", saying, for each element, which element is its (contradictory) negation. The implications previously expressed by the arrows in the modal 3-graph are here encoded by the alphabetical order relative to each alphabetical family (A→B, B→G, etc. ; we assume here transitivity of the arrows, cf. figures 15 and 16).

So, modal graphs constitute a **decoration *method*** in the following sense. To decorate (modally) an hexagon you procede as follows : you give it a Greek value for a blue vertex, an Arabic value for another blue vertex ; the rest of the decoration follows automatically : their respective negations are given to two precise green vertices (the ones centrally symmetric to the previous blue vertices) ; as its value the remaining green vertex receives the disjunction of the two first blue vertices ; as its value the remaining blue vertex receives the conjunction of the first two green vertices (i.e. the conjunction of the negations of the first 2 blue vertices). This can be tried for every possible combination of 2 elements of the modal graph (one Greek, the other Arabic).[15] This method thus suffices : (1) for decorating vertices

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symmetry remains anyway observable as far as contradictory negations are concerned (e.g., if "$\Box\alpha$" is left-top, "$\neg\Box\alpha$" is right-bottom, and so on), and such a central symmetry could be brought back into study concerning the arrows themselves (i.e. breaking the mentioned inner vertical symmetry), as we study elsewhere, cf. [12].

[15] Nevertheless, **this decoration method is restrictive**: it always *forces* one green term of the hexagon (and thus, in a dual way, its contradictory, centrally symmetric blue term) to *be identical* to the disjunction of the two blue terms adjacent to it (we have in this case "strong hexagons"). This is coherent with what did Blanché and

(we will see) in a non-useless way and (2) for deciding the truth (or triviality, or falsity) of any such finite decoration of $\alpha3$-structures (their number is m^2), going far beyond the 4 hexagons of the "logical tetradecahedron". This is not a trivial result, because in this way we can build many more hexagons, true, trivial or false, relative to each of the $3(m)$-graphs - the four hexagons (of figures 3 and 4) were just the "by-product" of the 3(3)-graphs. We need to introduce the following terminology : an $\alpha3$-structure decorated via a modal $3(m)$-graph will be called an $\boldsymbol{\alpha3(m)}$**-structure**. Of course, this being an abstract generalization of modal logic (we do not care yet about boxes, diamonds, indexed boxes, indexed diamonds and the like), a further work, not even touched in this paper, will be to interpret with "concrete" modalities the alphabetical variables (but this can be done with no harm, cf. [4] [13]). In this theory we offer thus a very abstract, modally uninterpreted framework.

We give in figure 17 a sample of nine instances of $\alpha3(m)$-structures, three (one true, one trivial, one false) for each value of m ($m = 3$, 4, 5). Red $\top$ means that the so-denoted formula is a tautology, red $\bot$ means contradiction, red arrows mean false implications. Remark that the true hexagon of $\alpha3(3)$ in figure 17 (the one formed with "beta" and "alif") corresponds in fact to Béziau's paracomplete hexagon mentioned before (cf. figure 3)[16]. One important feature to be remarked is that the value of a possible decoration of an $\alpha3(m)$-structure "evolves" through the values of m, thus meaning that it is relative to the context of evaluation given by the modal $3(m)$-graph. Remark in this sense that the hexagon formed with "gamma" and "ba" is false in $\alpha3(3)$, trivial in $\alpha3(4)$ and true in $\alpha3(5)$. Similarly, the hexagon formed with "alpha" and "jim" is trivial in $\alpha3(3)$

Béziau. But you could as well accept hexagons where one green term (and, in a dual way, its contradictory blue term) *is implied by*, but *is different from*, the disjunction of the two blue terms adjacent to it (in such case, not taken into account in this paper, we would have "weak hexagons"). Such an implicative but not identical green term (and its blue dual) would still satisfy the logical constraints of the hexagon's arrows. In other terms, strong hexagons are defined by two parameters, weak hexagons by three. Such unrestricted view, studied by Pellissier in [16], leads to many interesting results. In particular, Pellissier proves that strong and weak hexagons together constitute *all* hexagons (which applies to higher αn-structures). In generalizing Blanché and Béziau's intuitive restriction to the strong hexagons by keeping it ourselves, we show, as we will see, that this move, even if restrictive, leads to a viable and fertile theory of the modal decoration of graphs, the "head-body-tail" theory. We argue that this sub-theory, which alone makes possible to elaborate the notion of modal $n(m)$-graph, is the backbone of the general complex one, and is necessary to understand the latter's inner geometrical structure (cf. [11]).

[16] By "$\alpha n(m)$" we mean the logical space of the $\alpha n(m)$-structures.

but true in $\alpha3(4)$. In the same way, the hexagon formed with "delta" and "ba" is false in $\alpha3(4)$ but true in $\alpha3(5)$ (for a richer study of $3(m)$-graphs and $\alpha3(m)$-structures and some very nice results, cf. [11]).

It would be interesting to extend this kind of examination to the case of modal 3-graphs containing branchings (why restrict ourselves to linearity?). This task will not be undertaken here (cf. [12]). The important point to be handled now is rather this one : how to generalize to 4-opposition what was seen so far for 3-opposition ? How to step from the decoration of the $\alpha3$-structure to that of the $\alpha4$-structure (the "logical cube") ? It will show up that the answer to this involves the two following new steps.

First step towards the extended idea of modal $n(m)$-graph ($n \geq 4$) : from "bi-dimensional contradiction" to "multi-dimensional contradiction". The problem is now the following : if we want to use such $3(m)$-graphs to decorate modally the $\alpha4$-structure (the bi-tetrahedron, or cube) we generally fail : such classical $3(m)$-graphs, do not seem to be an adequate tool in order to decorate modally the geometrical $\alpha4$-structure in a sensible (i.e. non trivial) way.[17] Such decorations will generally appear to be violating some of the logical constraints expressed geometrically by the $\alpha4$-structure : no clear model seems possible, the structure does not seem to hold in this way (we omit to give here a proof - combinatorial in nature - of this affirmation, the question is treated fully in [16]). The problem seems to remain even if you add to the $3(m)$-graph complications such as the ones of the modal graphs of $S4$ or of the $K5$-systems (i.e. even if you add branching, instead of linearity). It is very hard to find non-trivial decorations this way (if only possible). This simply means that standard modal logic (as long as it is identified, as it usually is, to the use of 3-graphs, linear or branching) is not fit to decorate "naturally" $\alpha4$-structures. In other words, something new must be done here, if we are to decorate it properly (i.e. non-trivially and easily). And indeed we must : if we do not, the $\alpha4$-structure (and a fortiori any αn-structure, $n \geq 4$) could be thrown away as modally useless.

Now, our new leading remark to solve this problem will be that, despite the fact that we are used to it, in modal graphs contradictions *need not*

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[17] For a very deep and clear discussion of this complex point, duly complicating (and clarifying) our present statement, by means of a powerful set-theoretical decorating procedure, cf. Pellissier [16]. This opens to a larger, *in fieri* scope of n-opposition theory, here untouched.

always relate points to points (why should they ?) : they could relate points to lines, points to surfaces, points to 3-dimensional volumes (cf. figure 18), ..., points to n-dimensional volumes, etc., so as to open modal logic to n-dimensional geometry (provided that we interpret such geometrical entities as sets of logically related points - in fact *disjunctions* of such points). And this is precisely all we need to do.

Hence we get the required idea concerning how to change usefully the shape of the modal graphs. We open classical modal graphs to **multi-dimensional contradiction**, by "expanding" the dimensional symmetry of their frame (i.e. by complexifying the simple left-right symmetry). First, we add an additional alphabet (so that there are three : Greek, Arabic and now Hebrew). Then we take a stack of m vertically parallel (black) *triangles* (m is the number of "layers", i.e. the number of triangles). The stack of triangles is entangled with three independent vertical columns of m-1 arrows each (each arrow relates 2 elements belonging to 2 adjacent layers or triangles). To decorate the points (joints) of each column (points which are triangle's vertices), we chose one different alphabet for each column, so that in every triangle, so to say, one vertex is Greek, another is Arabic and the third is Hebrew. Finally, for each term of each alphabet (and, which is the same, for each vertex in a triangle) its contradiction will be defined as the disjunction of the two terms (i.e. a triangle's edge relating them) - of the other alphabets - corresponding to it (this edge happens to be centrally symmetric to the starting term - a vertex -, all modal 4(m)-graphs being geometrically obtained so that the central symmetry relates each vertex to an edge and each edge to a vertex, cf. figure 21).

So, presumably, in the general case (n = any integer) it suffices to generalize these two moves : (1) generalize the use of the "alphabetical variables" (one alphabet for each n-1 oppositional family of terms : Greek, Arabic, Hebrew, Indian, Japanese, Russian, ...), a simple device that will reveal itself useful in building the new theory (alphabetical terms must be seen as abstract modalities, each family being equally opposed to all the others as, in classical modal logic, positive left-handed modalities are "opposed" to negative right-handed modalities) ; (2) and generalize the use of (black) triangles : for $n = 5$ it will be (black) tetrahedra, ... , etc. (for each n, it will be a black simplex of dimension n-2). This reflects our idea of an extended treatment of contradiction. We pass from a 1-dimensional (left-right) "point-point" treatment of contradiction to a poly-dimensional one. And this brings us back, as we will see later in more detail, to the now familiar series of the simplexes of dimension n (cf. figure 18).

Second step towards the extended $n(m)$-graphs ($n \geq 4$) : opposition terms can be "heads", "bodies" or "tails". We made previously a *geometrical* model of n-opposition theory (cf. section 3.1 above) without spending a word about the number and shape of the "composite modalities" (as seen for $n = 3$). Effectively, for $n = 3$, we saw that each hexagon has one "or" term and one "and" term (think of Blanché's discovery : the U and the Y vertices). The crucial point is then : how to generalize this ? Given that (1) it can be shown that in each αn-structure there must be at least 1 composite term (this is what Blanché has implicitely shown against Aristotle : not the square but 2 triangles forming a hexagon) and given that (2) it can't be shown that there can be no more than 1 such composite term in each αn-structure (as testified by Moretti and Smessaert's fourth emerging hexagon, and more deeply by Pellissier, cf. [16]), it seems that there is room for the taking of decisions. At the present, the most natural solution seems to consist in keeping, in each αn-structure, one and one only such "or" term and one and one only such "and" term (for any n, the composite "or-term" will be a green disjunction of n-1 blue simple terms, whereas the composite "and-term" will be a blue conjunction of n-1 green simple terms).[18] This solution works. We call it the "**head-body-tail theory**".

So, in each n-structure, among the $2n$ opposite terms (n contraries and, contradictory to them, n subcontraries), 2(n-1) should constitute some kind of basis (so they must be named with n-1 different alphabets), while the n-th couple of contradictory (i.e. centrally symmetric) terms is just the composition of the previous n-1 (more precisely, the singular green composite term is the disjunction of the n-1 basic - or "pure" - blue contraries, and the singular blue composite term is the conjunction of the n-1 basic - or "pure" - green sub-contraries). We call the singular blue composite term "**tail**", the singular green composite term "**head**", the rest of the blue terms "**(blue) body**", and the rest of the green terms "**(green) body**". Each αn-structure has, finally, 1 head (green), 2(n-1) terms in the body (n - 1 being blue, n - 1 being green) and 1 tail (blue) (cf. figure 19).

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[18] This choice of ours results in concentrating on a simpler family of logical hexagons (and further αn-structures), the "purest" in some sense, among all the possible ones ("strong hexagons" instead of "weak hexagons", according to a terminology we owe to Pellissier, cf. [16]). This restrictive choice, as we will see, allows to find beautiful orderings, leaving for further investigations more complex explorations of the general field, where the present prima facie articulations will reveal themselves very useful to structure the "peripheral" knowledge.

The case $n = 4$: there is room for a "4-opposition theory". In order to decorate the $\alpha 4(m)$-structures we want to consider now the set of possible modal $4(m)$-graphs (m is the number of "layers"). First we introduce a new alphabetical family, Hebrew (cf. figure 20).

According to what was previously said, each $4(m)$-graph is a column composed of m triangles. Each triangle is made out of 3 terms (its vertices) belonging to the three opposed modal families (here : one "Greek", one "Arabic', one "Hebrew", cf. figure 21). In each triangle each edge relating two vertices will be read as the (inclusive) logical disjunction of these two vertices.

The big change is that now contradiction, for each term X on the modal graph, is defined as the *disjunction* of the two terms Y and W most far from it (the truth of Y or the truth of W suffice to make X false ; X is true *iff* both Y and W are false). As we will now illustrate by some examples, each of such $4(m)$-graphs suffices to decorate the $\alpha 4$-structure, but each does it in a different way, thus specifying it in an $\alpha 4(m)$-structure (cf. figure 21).

The relations depicted graphically by the $4(m)$-graphs can be expressed symbolically by sets of "$4(m)$-relations", the implications previously depicted by the arrows being implicitely contained in the alphabetical order inside each oppositional family : A implies B, which implies G, which implies D, ..., which implies M (more particularly, inside each alphabetical family : alpha implies beta, alif implies ba, aleph implies beth, etc., cf. figure 22).

As already said, the use of a modal $4(m)$-graph relatively to an $\alpha 4(m)$-structure is twofold. First, it shows which decorations (with modalities) are possible from a purely combinatorial point of view : all possible triples of terms so that one is Greek, another is Arabic and the other is Hebrew (all other terms of the decoration being then mechanically determined : the 3 respective green negations of the 3 blue elements constituting the starting triple, the green disjunction of three mentioned blue terms, and the blue conjunction of the three green negations). The number of such possibilities in $\alpha 4(m)$ is m^3. Second, it provides a criterium as to which decorations, among all possible ones, are logically viable (there will be true, trivial and false decorations). In this respect one can see modal $4(m)$-graphs as some kind of 3-dimensional modal oppositional "truth tables", on which to rely in order to check the validity of oppositional modal formulas.

We give here two examples of such truth-value calculations via the modal graphs. In the first calculation we want to see if "alpha or alif or beth" is a tautology inside $\alpha 4(3)$. Using the modal 4(3)-graph we see that a counter-example of it is possible ("not alpha and not alif and not beth").[19] So the starting formula is not a tautology of $\alpha 4(3)$ (cf. figure 23).

In the second calculation we propose here, we want to check the validity of the formula "beta or ba or guimel" inside the logical space of $\alpha 4(4)$. Using the modal 4(4)-graph corresponding to it, we see that it is impossible to draw a counter-model of it without contradiction ("not beta and not ba and not guimel"), hence the starting formula is valid, it is a tautology of $\alpha 4(4)$ (cf. figure 24).[20]

Enjoying this new modal tool, we can check all possible cases inside $\alpha 4(m)$. As an example of this, we will give, first, some instances of possible decorations of the $\alpha 4(3)$-structure by the modal 4(3)-graph, that is, the graph formed by a stack of 3 black triangles (the 3 "layers"). Among the 27 possible decorations one can find three kinds of issues : an $\alpha 4(3)$-structure can be true, trivial or false. It is ***true*** when all arrows obtain *and* when the head is not a tautology (equivalently : the tail is not a contradiction). It is ***trivial*** when all arrows obtain *but* the head is a tautology (equivalently : the tail is a contradiction). It is false when some arrows are false (cf. figure 25).

If we consider now a further layer ($m = 4$) in the modal $4(m)$-graph, the reasoning is the same. There will be 64 possible cases here, some true, some trivial and some false. We give, in the figure, an instance of each type within the possible decorations of the $\alpha 4$-structure by the modal 4(4)-graph. Remark that one decoration which was trivial in $\alpha 4(3)$ ("beta or ba or beth") is true in $\alpha 4(4)$. Remark also that not all false decorations inside a given $\alpha n(m)$ make the head tautological (equivalently : the tail contradictory) : we see here that the decoration with "alpha", "jim" and "guimel" makes the $\alpha 4(4)$-structure false without making its head ("alpha or jim or guimel") tautological (equivalently : without making its tail "not alpha and not jim and not guimel" contradictory, cf. figure 26).

Same story, again, for a modal 4-graph with now five layers (i.e. a stack of 5 black triangles), the modal 4(5)-graph necessary to decorate the $\alpha 4(5)$-

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[19] Explanation of the graphical deductions depicted in figure 23. One passes from (3) to (4) because something false (as in (3)) cannot be implied by something true (as would be in (4)); then "(1) and (2)" imply (7), "(2) and (4)" imply (5), and "(4) and (1)" imply (6) because of the definition of contradictory negation inside 4-graphs.

[20] Explanation of the graphical deductions in figure 24. (1) and (2) imply (4) by the definition of contradictory negation in 4-graphs; but (4) and (3) lead to contradiction.

structure. Among the 125 possible decorations (i.e. 125 possible instances of $\alpha 5(5)$-structures) a check of the arrows and of the heads via the 4(5)-graph will give the three usual kinds of results (true ones, trivial ones and false ones, cf. figure 27).[21] Here, as well, we remark that a decoration (the A G G)[22] which was bad (false) with a trivial head in $\alpha 4(3)$ and bad (false) with a non trivial head in $\alpha 4(4)$ is fine (true) in $\alpha 4(5)$ (cf. figure 27).

So we take act that $\alpha 4(m)$-graphs constitute an adequate (i.e. non-trivial) tool to decorate the $\alpha 4(m)$-structures with modalities. The $\alpha 4$-structure (the "logical cube" presented in this paper) is thus neither trivial nor useless. In this way, 4-opposition theory works. But can opposition theory be developped further, considering the case $n = 5$?

The case $n = 5$: there is room for 5-opposition theory. [23]

What happens with $n = 5$? Again, it can be shown that in some sense neither 3-graphs nor 4-graphs can decorate the $\alpha 5$-structure. This means that if we want to go to the next step ($n = 5$), we just need to be able to deal with *four* (instead of three) oppositional families of abstract modal terms (four families of concatenated arrows, if m is bigger than 1 - for $m = 1$ there are no arrows) : the fifth term is a head-tail pair (we still change the geometrical quality of contradiction, we have, this time, a **point-surface contradictory negation**, cf. figure 29).

So we introduce $5(m)$-graphs in the usual way, by the adjunction (to the previous case) of a fourth term (welcome to the Indian-Sanskrit guys), passing thus from a (black) triangle to a (black) tetrahedron. The number of such (black) tetrahedra constituting (in a "column" or stack) the $5(m)$-graph is m (there are m "layers", so to speak, cf. figure 29). As we will show by some examples, each of such modal $5(m)$-graphs suffices to decorate the $\alpha 5$-structure with modalities.

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[21] Truly speaking, inside $\alpha 4$-structures and higher, there can be more than 3 issues, if inside the set of false decorations one makes an inner distinction according to the number of false arrows and the quality of the head, tautological or not. But this bears no consequences at our level, results thereupon will be given elsewhere.

[22] By "A G G" we mean, of course, the ordered triple consisting in the A-like Greek term, the G-like Arabic term and the G-like Hebrew term (the same kind of lecture ruling, naturally, also when other capitals are available, or when we deal with a different number of them).

[23] This case could have been omitted (as boringly similar to the previous one). We give its explicit development in order to familiarize the reader with handling α-structures more than 3-dimensional (as the 4-dimensional $\alpha 5(m)$-structures here, and in order to familiarize her/him with the handling of modal graphs of increasing geometrical complexity.)

Here as well, the relations depicted geometrically by the $5(m)$-graphs (as in figure 29) can also be expressed symbolically by a set of "$5(m)$-relations" (cf. figure 30, where we omit, however, the expression of the implications - we could state them one by one -, implicitely encoded in the alphabetical order of each series of variables).

As in the previous case ($n = 4$), the use of a modal $5(m)$-graph is double. First, it shows which decorations (with modalities) are possible from a combinatorial point of view. That is to say, simply all possible 4-tuples of terms such that one is Greek, another is Arabic, another is Hebrew, a fourth is Indian (these first 4 will be blue ; all other terms of the decoration being then automatically determined : the 4 green negations of the first mentioned 4 blue, their green disjunction and the blue conjunction of the green negations). The number of the possible combinations in $\alpha 5(m)$ is m^4. Second, it provides a criterium as to which decorations, among all possible ones, are logically viable (as before, there will be true, trivial and false decorations). In this respect one can see a modal $5(m)$-graph as some kind of 4-dimensional modal oppositional "truth table", on which to rely in order to check the validity of oppositional modal formulas (easy to use in its 2-dimensional paper projection, cf. figures 29, 31, 32).

As before, we give here two examples of such truth-value calculations via the modal graphs. In the first calculation we want to see if "alpha or alif or beth or ba" is a tautology inside $\alpha 5(3)$. Using the modal 5(3)-graph we see that a counter-example of it is possible ("not alpha and not alif and not beth and not ba"). So the starting formula is not a tautology of $\alpha 5(3)$ (cf. figure 31).[24]

In the second calculation proposed, we want to check the validity of the formula "alpha or ba or ba or da" inside the logical space of $\alpha 5(4)$. Using the modal 5(4)-graph corresponding to it we see that it is impossible to draw a counter-model of it without contradiction ("not alpha and not ba and not ba and not da"), hence the starting formula is valid, it is a tautology of $\alpha 5(4)$ (cf. figure 32)[25].

Enjoying this new modal tool, we can check all possible cases inside

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[24] Explanation of figure 31. (3) implies (5), and (4) implies (6) because something false cannot be implied by something true. "(1), (2) and (5)" imply (10), "(2), (5) and (6)" imply (7), "(5), (6) and (1)" imply (8), "(6), (1) and (2)" imply (9) because of the definition of contradictory negation in 5-graphs.

[25] Explanation of figure 32. (2) implies (5), and (3) implies (6) because something false cannot be implied by something true; "(1), (5) and (6)" imply (7) because of the definition of contradictory negation in 5-graphs; but (7) and (4) lead to contradiction.

$\alpha5(m)$. As an example, we will first give some instances of possible decorations of the $\alpha5(3)$-structure by the modal 5(3)-graph, that is, the graph formed by a stack of 3 black tetrahedra (the 3 "layers"). Among the 81 possible decorations here one can find three kinds of issues : an $\alpha5(3)$-structure can be true, trivial or false. As before, it is *true* when all its arrows obtain *and* when its head is not a tautology (equivalently : its tail is not a contradiction). It is *trivial* when all arrows obtain *but* the head is a tautology (equivalently : the tail is a contradiction). It is *false* when some arrows are false (cf. figure 33).

If we consider now a further layer ($m = 4$) in the modal graph, the reasoning is the same. There will be 256 possible cases here, some true, some trivial and some false. We give, in the figure, an instance of each type within the possible decorations of the $\alpha5$-structure by the modal 5(4)-graph. Remark that the decoration with "alpha", "ba", "beth" and "ga", which was false in $\alpha5(3)$, is now true (cf. figure 34).

We give, thirdly, some instances of possible decorations of the $\alpha5$-structure by the modal 5(5)-graph. We have here 625 possible cases, among which true ones, trivial ones and false ones. Remark that the "gamma", "jim", "guimel" and "ga" decoration, which was false in $\alpha5(4)$, is now trivial (cf. in figure 35).

We stop here this list of modal $n(m)$-graphs. It is easy to see that it can go on with no limit (however, we give here no proof of this sentence - we will give it fully elsewhere).

So there is room for generalized n-opposition theory, $n \geq 4$. It seems that all which we have seen can be generalized to any finite n. A **modal $n(m)$-graph** (i.e. a modal n-graph with m layers) is defined as a stack of m (black) simplexes of dimension n-2 (the "gems", cf. figures 36 and 37), each containing n-1 terms belonging to n-1 different alphabetical families ; between each couple of adjacent black simplexes lays a set of n-1 arrows, each arrow relating each upper alphabetical term to the lower one corresponding alphabetically to it. Each αn-structure deserves a n-graph in order to be decorated usefully, and each $n(m)$-graph specifies the αn-structure into a multiplicity of **$\alpha n(m)$-structures** (same geometrical shape, but different truth-value of the same decorations). And $n(m)$-graphs do work nicely : they are some kind of (n-1)-dimensional modal oppositional "truth-tables" (easy to use in their 2-dimensional paper projection). And given that for any n there is an αn-structure and a modal n-graph, for any n there is an adequate n-opposition : this is "**n-opposition theory**", the general framework we were looking for. This double-sided algorithm works for any finite integer values of n and m. It uses twice the series of the simplexes (of dimension n-1 and of dimension n-2, respectively) and one can imagine a rather simple proof, by recursion, of this generality relying (twice) precisely on the notion of simplex of dimension N (however, we must omit here to give the precise proof of this).

3.3 Some theorems of n-opposition theory

Before concluding, we give here some simple and intuitive definitions and theorems, useful to make quicker calculations for $\alpha n(m)$-structures within the frame of modal $n(m)$-graphs.

Simple theorems concerning the graphic treatment of modal $n(m)$-graphs in terms of their "gems". Inside modal $n(m)$-graphs it is useful to introduce the general notion of "gem" (i.e. the black structure - a simplex - in each layer of a $n(m)$-graph).

Definition 1 : we call "**$n(m)$-gem**" (for short, in what follows: "gem") each of the m "simplexes of dimension (n-2)" characterising a modal $n(m)$-graph.[26]

Definition 2 : a gem is "**lower**" than another *iff* it can be reached from this last by means of the oriented arrows of the $n(m)$-graph, i.e. *iff* each element of the second gem implies, by a finite series of concatenated arrows, one and only one element of the first gem, leaving no element of the first not implied.

Definition 3 : a gem is "**higher**" than another *iff* it is neither lower than it nor identical to it (there are no unordered gems inside a modal $n(m)$-graph).

Definition 4 : a gem is "**symmetric**" with respect to another gem *iff*, for each of the two gems, each of its n-1 terms is defined as the negation of the disjunction of the other n-2 terms of the other gem which are not of the same alphabetical family as the first term (the truth or falsity of an

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[26] Differently from a blue simplex (of dimension n-1) of contrariety, which has n terms and is (n -1)-dimensional (resp. differently from a green simplex of sub-contrariety ...), a $n(m)$-gem has (n -1) terms and is (n -2)-dimensional : it just has simple terms, i.e. "mono-alphabetical" ones, without propositional binary connectives : it does not have the composite term "tail" "$\neg a \wedge \neg b \wedge \ldots \wedge \neg(n-1)$" (resp. it does not have the composite term "head" "$a \vee b \vee \ldots \vee (n\text{-}1)$").

element of a given gem is always to be checked in the symmetric gem, as being the negation of the disjunction of the elements - of the families other than the first - of this last).

Definition 5 : we call "**descendant**" of an element of a given gem every element of every gem lower than the first one such that that element belongs to the same alphabet as the first element (i.e. it comes from the first by a finite series of implications).

Definition 6 : we call "**ancestor**" of an element of a given gem every element of every gem higher than the first one such that that element belongs to the same alphabet as the first element (i.e. it leads to the first by a finite series of implications).

Definition 7 : we have a "**true gem**" (green) *iff* all its elements are true (green).

Definition 8 : we have a "**false gem**" (red) *iff* all its elements are false (red).

Definition 9 : we have a "**normal gem**" *iff* some of its elements are true (green) and some of its elements are false (red).

Theorem 1 : if an element of a gem is true (green) then all its descendants are true (green).
(Proof : something true cannot imply something false).

Theorem 2 : if an element of a gem is false (red) then all its ancestors are false (red).
(Proof : something false cannot be implied by something true).

Theorem 3 : if a gem is true (green) then all lower gems are true (green).
(Proof : suppose that, with respect to a green gem, some lower gem is not green. Then at least one of the elements of this gem is false. But then all ancestors of this element must be false, including the one belonging to the starting green gem, which is impossible).

Theorem 4 : if a gem is false (red) than all higher gems are false (red).
(Proof : suppose that, with respect to a red gem, some higher gem is not red. Then at least one of the elements of this gem is true. But then all descendants of this element must be true, including the one belonging to the starting red gem, which is impossible).

Definition 10 : a gem is "**central**" *iff* one of the following equivalent conditions obtains : (1) when it lays at the same distance from the first and from the last gem of the modal $n(m)$-graph (m has to be odd) ; (2) when each of its elements is defined as the negation of the disjunction of the other elements (of the same gem) ; (3) when it is symmetrical to itself.

Definition 11 : a gem is "**peripheral**" when it is not central.

Theorem 5 : if a gem is central, it contains one and only one true element (green), all other elements of that central gem being false (red).
(Proof : (a) suppose it contains no true elements : then it is impossible to satisfy the constraint of poly-dimensional contradiction, defining, for every element of the gem, the falsity of that element (there is no contradictory green element available for that), which leads to contradiction. (b) Suppose the gem contains only true elements : than it is impossible to satisfy the constraint defining, for every element of the gem, the truth of that element : there are no red elements available for that, which again leads to overt contradiction. (c) Suppose there is, in that central gem, at least one false (red) element and at least two true (green) elements : by definition of truth inside modal $n(m)$-graphs, no element can be true (for an element of a gem, in order to be true, the disjunction of all other elements of its symmetric gem should be false), which leads, once more, to open contradiction).

Definition 12 : a gem is "**superior**" when it is not central and it belongs to the higher half of the stack (i.e., if there is a central gem, when it is higher with respect to the central gem).

Definition 13 : a gem is "**inferior**" when it is not central and belongs to the lower half of the stack (i.e., if there is a central gem, when it is lower with respect to the central gem).

Theorem 6 : if a gem is superior, then it is not true (not all green).
(Proof : if that superior gem were true (green), all implied elements (and thus all lower gems) should be true (green). But then there would be no false (red) elements in the symmetric gem, which must be inferior, and therefore no element of the first gem could be true (green), which leads to contradiction).

Theorem 7 : if a gem is inferior, then it is not false (not all red).
(Proof : suppose an inferior gem is red ; then all its ancestors - and thus all the gems higher than this first - should be false (red). But then there would be no true (green) elements in the symmetric gem which must be superior, and therefore no element of the first gem could be false (red), which leads to contradiction).

Theorem 8 : if a superior gem is false (red), then its symmetric gem is true (green).
(Proof : suppose a superior gem is red and its symmetric inferior gem is not green, i.e. it is red or normal. Then at least one element of the inferior gem is false (red), which is impossible, because there are no green elements available in the superior red gem to support this).

Theorem 9 : if an inferior gem is true (green), then its symmetric is

false (red).
(Proof : suppose an inferior gem is green while its symmetric superior gem is not red - i.e. it is green or normal. Then at least one element of that superior gem is true (green), which is impossible, because there are no red elements in the inferior green gem to support that).

Theorems and terminology of this kind allow quicker calculations and deductions with modal $n(m)$-graphs. As examples of applications of theorem 8, cf. figures 23 and 31. As examples of applications of theorem 2, cf. figures 23, 31 and 32.

A more general theorem concerning modal oppositional implications. There is one big useful theorem ruling all simple implications of one alphabetical term by another belonging to a different alphabetical family.

General Theorem : inside each $n(m)$-graph (that is, inside each $\alpha n(m)$-structure) the value (true or false) of all possible simple implications of one simple alphabetical modality by another one (the two belonging to two different alphabets) is given by a matrix (as the one given in figure 38) having m rows and m columns, such that in the place determined by the i^{th} row and the j^{th} column it contains the implication of a consequent "$\neg J$" by an antecedent "I". Such an implication is true *iff* it belongs to the "upper left" triangular half of the matrix (diagonal included), otherwise it is false.

Explanation : We give a graphical expression of it, as a matrix with m rows and m columns (in each place of the matrix there is a simple implication of two simple alphabetical modalities, cf. figure 38). The prime signs suffixing the consequent mean that this last term belongs to an alphabetical family different from the one of the antecedent (anyone different from it). So, for instance, "A $\rightarrow \neg B$' " means that any term "A" (i.e. alpha, alif, aleph, ...) implies the negation of any term "B" of the families other than the first one (the Greek "A" implies the negation of Arabic, Hebrew, Indian, ... "B" ; the Arabic "A" implies the negation of Greek, Hebrew, Indian, ... "B", and so on). The theorem covers all possible simple cases of implications. For each $n(m)$-graph, in order to check simple implications, one has to draw the adequate matrix (one with m rows and m columns) and then just read it! The proof of this theorem is simple but a bit tedious, we will give the full version of it elsewhere. Here we will just give a sketch of it (almost all steps are done by recursion).

Sketch of the proof : First, one has to prove that for all $\alpha n(m)$ we have

$A \rightarrow \neg M$'. Then, one generalizes that result by proving that for all $\alpha n(m)$ and for all p (with $0 \leq p \leq [m\text{-}1]$) we have $(A+p) \rightarrow \neg(M\text{-}p)$'.[27] This last proves that in each adequate matrix the terms on the left-bottom/right-top diagonal are true (true simple implications). The last two steps consist then in proving that in each row of the matrix all the implications *preceding* (i.e. at the left side of) the one on the diagonal are true and, lastly, that for each row of the matrix all the implications *following* (i.e. at the right side of) the true one in the diagonal are false, which ends the proof of the theorem. The first is done, as usual, by recursion (in fact, a finite series of embedded recursions). First we prove that for all $\alpha n(m)$ we have $(A \rightarrow \neg M') \rightarrow (A \rightarrow \neg K')$ (with $[K] \leq [M]$) and thus, by modus ponens, that for all $\alpha n(m)$ we have $A \rightarrow \neg K$' (for all K such that $K \leq M$). We generalize that by proving that for all $\alpha n(m)$, $(A+p) \rightarrow \neg K$' (with $[K] \leq [M\text{-}p]$ and $0 \leq p \leq [M\text{-}1]$) (the left-top triangular half of the matrix contains the true simple implications). Similarly, for the last point, we demonstrate that for all $\alpha n(m)$ $\neg((B+p) \rightarrow \neg(M\text{-}p)')$ (with $0 \leq p \leq [M\text{-}2]$), which we generalize by proving that for all $\alpha n(m)$ $\neg((M\text{-}p) \rightarrow \neg K')$ (with $0 \leq p \leq [M\text{-}2]$) (the "right-bottom" triangular half of the matrix contains the false simple implications).

4 Conclusion and perspectives

It is difficult to judge a theory which is new, especially if you are the author. Nevertheless it is time for us to try to draw some general guidelines. Our theory of n-opposition offers (or reveals) a possible geometrical side to modal logics. But is it the only possible - or the best - one ? As such this question is too wide, we won't be able to treat it here. In order to sum up about the general question of the relations between (modal) logic and geometry *in as much the present theory is concerned and can bring some lights*, we will briefly recall what has been done here, then we will evoke what should or could be done next, ending with some more philosophical considerations.

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[27] By "(A+p)" we mean the alphabetical letter whose (numerical) rank in the alphabet is the one of "A" plus the integer "p". So, for instance, in the alphabet used here (cf. figure 14) "(B+3)" is "E" ; "(C+1)" is "D", and so on. By "[M+p]" we mean the the number given by the sum of the rank of "M" (in our chosen alphabet of figure 14) and the number "p".

What has been achieved here. We indicated in which way such a new theory commands a necessarily two-folded approach of opposition, in terms of geometrical "$\alpha n(m)$-structures" first, to cope with the more geometrical part of the treatment of opposition, and then in terms of modal linear "$n(m)$-graphs", in order to cope with the more logical side of the job, consisting in decorating with abstract modalities the open series of geometrical multi-dimensional αn-structures. The key ingredient, in both parts of the theory, happens to be the mathematical notion of "simplex of dimension n" (giving the "bi-simplexes of dimension n-1" in the first case, the "$n(m)$-gems" in the second).

The theory produces then two noticeable novelties in the logical knowledge. First, it shows that the number of instances of logical squares, hexagons and tetradecahedra (the already known structures) is potentially infinite, inside 3-opposition, with respect to the possible modal decorations (this is generalized $3(m)$-opposition theory). This result is perhaps more spectacular than it seems, it produces for instance an infinite series of $\beta 3(m)$-structures of which the beautiful tetradecahedron is just the first and simplest element (we develop this in [11]). Second, it shows that there are, outside the two known α-structures ($\alpha 2$ and $\alpha 3$, i.e. square and hexagon), an infinity of other αn-structures ($\alpha 4$ or "cube", $\alpha 5$, $\alpha 6$, ..., αn, ..., we showed here the first three new ones, but gave the general intuitive law).

It must be noticed that in the present paper **we restricted ourselves quite much** by considering only "strong" hexagons (i.e. by imposing our "head-body-tail" construction principle over the decoration of the α-structures), instead of considering all possible "weak" hexagons (we owe this terminology to Pellissier, cf. [16]). This was a necessary move in order to elaborate and highlight the useful notion of "modal n-graph". But after reading a pre-final draft of the present paper, and relying on it, Pellissier investigated the more general case (weak hexagons) - by means of a set-theoretical decorating technique he elaborated - and found very interesting and very strong results, generalising ours. In particular, he has shown that, in $\alpha 3(3)$ *all* hexagons (including ones with modal terms composed of several binary propositional connectors - the 2 first of which had been discovered independently from Pellissier by Hans Smessaert) do collect themselves in a 3-dimensional "logical tetraisocahedron" (which, according to us, is composed of three tetradecahedra, among which Beziau and Moretti's one). This result is very important, because this elegant figure is some kind of real closure of the field we described for n=3 and m=3. Besides, Pellissier has found more general results, concerning the whole of modal graphs theory (cf. [16]).

Again, the **principal known application so far** seems to be the establishment of a series of translation rules (between modal logic and solid geometry), by now all inside 3-opposition (in this paper we saw just the simplest one and discussed some possible issues, but in [11] we expand it considerably). This result is already nice (it can, for instance, be suitably adapted to branching inside standard modal logic), but its lack of generality - we could have, but we still have not, translation rules based upon βn-structures with $n > 3$ - brings us to the following questioning remark.

What should or could be done next. There are two further crucial distinctions, one inside the structures, the other inside the graphs. Inside the geometrical structures we distinguish between α-structures and β-structures (as already said, β-structures are higher-dimensional structures gathering together nicely a multiplicity of α-structures, as the tetradecahedron does with respect to the four hexagons). Inside the modal graphs we distinguish between *linear* $n(m)$-graphs and *branching* $n(m)$-graphs (branching graphs, as the one of $S4$, can be obtained as combinations of partially different linear graphs, as the one of $S5$, these last ones being translatable into β-structures, as $S5$ is translatable into the tetradecahedron).[28]

The theory would be almost "perfect" in its architectural harmony if it was not for a persisting lack, the fact that we still know of no equivalent of the tetradecahedron (which belongs to 3-opposition) for 4-opposition (the case of the logical cubes) and beyond. In other terms, the **open question** sounds : is there, as well, an open (infinite) series of βn-structures (each term of this series presumably "fractalized", itself, by m into an infinite series of $\beta n(m)$-structures) ? A positive answer to this question would, according to us, definitely assess the theory and the novelty and legitimacy of this "strange new field" of modal logic. But such a positive answer is not yet available. The first step in answering this, the question of the existence of a 4-dimensional $\beta 4$-structure ordering nicely the 3-dimensional logical cubes (or $\alpha 4$-structures) is still open (the problem is more difficult in this case than in the tetradecahedron's case, because contradiction is now defined as a conjunction of negations, instead of as an unique negation, and this makes things a bit harder). If the answer to this general further question were "no", meaning that the tetradecahedron (and its infinite fractal

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[28] The geometry of branching graphs is yet another topic, cf. [12] and [13].

series inside $3(m)$-opposition, with a varying m) is a poor lone boy (or girl), then n-opposition theory would seem much less elegant and balanced. If not the tetradecahedron itself, its whole family of $(p+3)$-dimensional "p-hyper-tetradecahedra" (as we call them in [11]) would be, again, an "hapax legomenon", a strange geometrical fragment lost in a not so geometrical world of modal logics. We tend to believe that, despite the difficulties up to now in establishing general "βn-structure theory", there is probably one such theory not far, which someone will bring someday into light.

Among **more distant questions** we can mention:

1) the question whether there is a meaning in extending our present theory of n-opposition to the case where n belongs to Z (can "negative n-opposition" be meaningful?);

2) the question wether it is possible to conceive a similar theory with a number of oppositions different from 4 (i.e. contradictions, contrarieties, sub-contrarieties and sub-alternations);

3) the question wether we can conceive some non standard version of the present theory, that is, taking in the metalevel (or metalanguage) some non standard logic, instead of classical logic (these last two questions I owe to discussions with J.-Y. Béziau);

4) the question of the possible relations of n-opposition theory to n-categories theory. As it seems, n-opposition (modal graphs) is contained into 2-categories, but can be developed so as to become n-categorical (Moretti and Pellissier, joint paper to come);

5) the question of the relation of this theory to other "logical-geometrical" issues, such as linear logic or multi-dimensional modal logic (these last reflection I owe to Alexandre Costa-Leite).

Perspectives. It is now legitimate to go back to the principal question which originated our paper: do we have, when dealing with the logical "squares", "hexagons" and "tetradecahedron", a new field in (modal) logic (aren't such new structures just curious but irrelevant cases ?) ? Our paper was an attempt (we believe successful) to answer by a strong "YES" to that question. We testimony the birth of a new field of (modal) logic, strongly interrelated to geometry, where strange entities as the logically arrowed squares, hexagons and tetradecahedra do pullulate happily. We tried to show that these three emerging structures really need a reformulation of old opposition theory and that such a reformulation, in terms of our own theory of n-opposition, is possible and, by now, effective.

Again, the other important general open question concerns the status of the relations between (modal) logic and geometry (i.e. what can be done maximally in this direction).

More philosophical remarks From a philosophical point of view, the relevant question is, according to us, twofold:

1) does n-opposition theory bring new insights on the foundations of logic? In some sens it seems it does: it changes Aristotle's (largely shared) views (think of H. Slater's "sacred" use of Aristotle for criticizing paraconsistent logics); it shows some constitutive links to the problem of the meaning of negation (in particular paraconsistent negation) and thus to Béziau's elaboration of a "universal logic" (cf. [2], [3]); it is a very abstract version of modal logic; it incorporates some of Vasil'ev's most essentials ideas about the relations between logic and geometry (cf. [15]); it has important links to category theory and thus to topos theory (thus being possibly related to the ambitious and impressive contemporary philosophical project of A. Badiou).

2) Does n-opposition theory open a viable answer to P. Gardenfors radical objection to the modeling pretensions of logic over concepts? This question, at this stage, remains open, but it will be interesting, in the future, to see if our theory, somehow extended, will be able - being logical *and* geometrical - to express the logically untractable "conceptual spaces".

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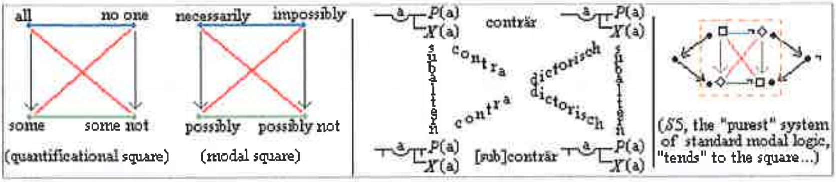


Figure 1. Instances of the "logical square" by Aristotle, Frege and C.I. Lewis

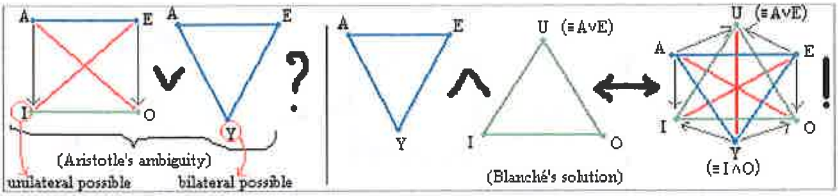


Figure 2. Aristotle's aporia and Blanché's solution (the "logical hexagon")

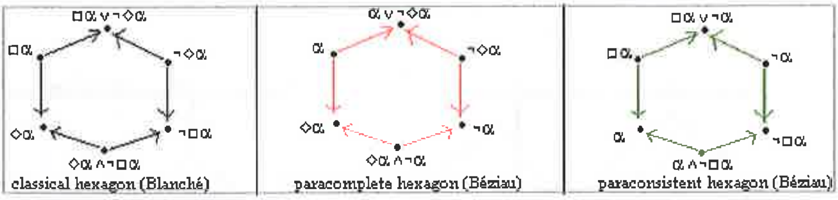


Figure 3. Béziau's two new modal hexagons (paracomplete and paraconsistent)

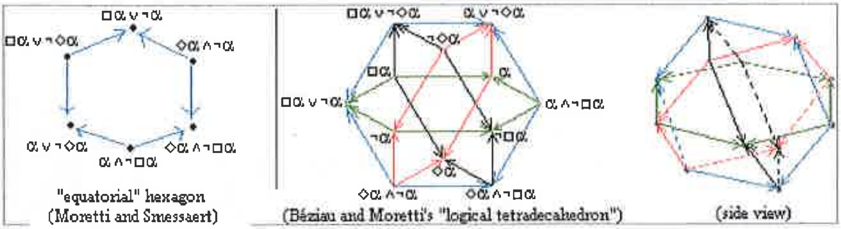


Figure 4. Moretti and Smessaert's fourth hexagon (in blue), Béziau and Moretti's "logical tetradecahedron" (ordering the four hexagons)

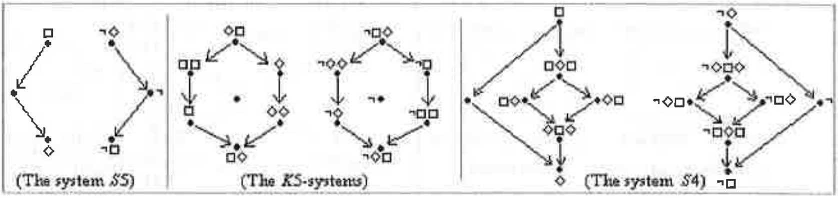


Figure 5. Three examples of classical "modal graphs" (from Chellas modified)

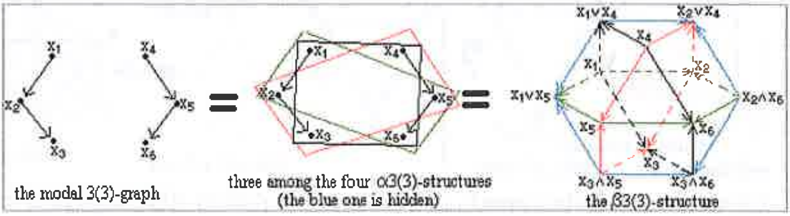


Figure 6. Our translation rule between the modal and the geometrical spaces

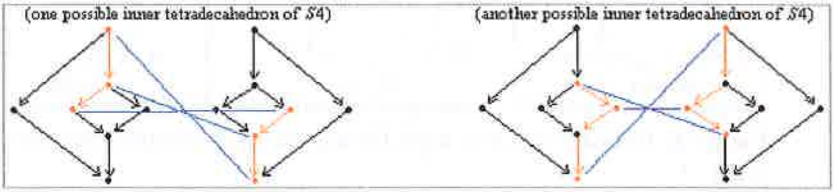


Figure 7. How many "non-emergent" tetradecahedra in $S4$? Just count them!

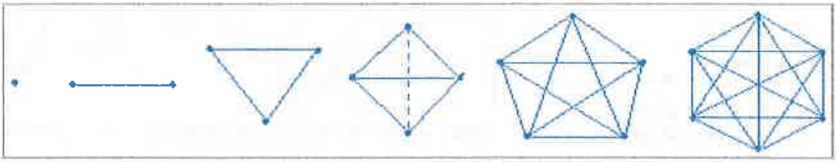


Figure 8. The series of the simplexes of dimension n ($n = 0, 1, 2, 3, 4, 5$)

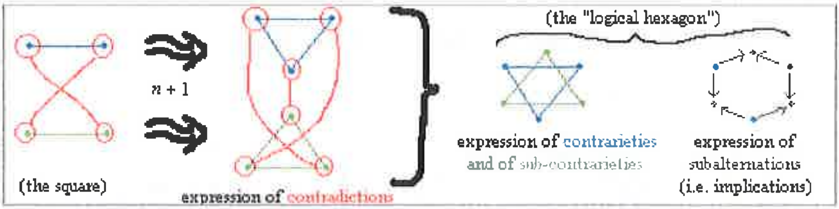


Figure 9. The opposition relations inside the "$\alpha 3$-structure" (or logical hexagon)

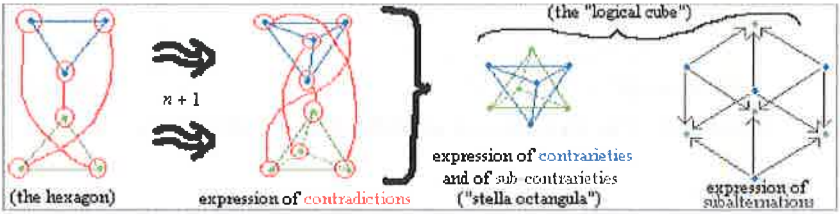


Figure 10. The opposition relations inside the "$\alpha 4$-structure" (or "logical cube")

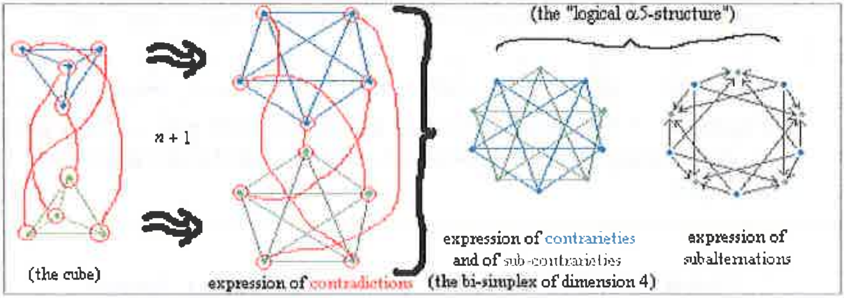


Figure 11. The opposition relations inside the "$\alpha 5$-structure"

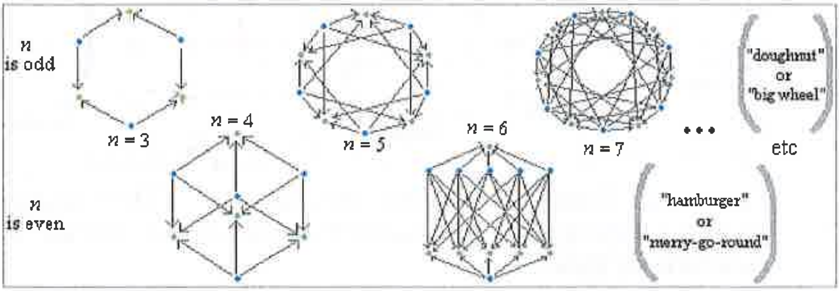


Figure 12. Some oppositional αn-structures ($n = 3, 4, 5, 6, 7$)

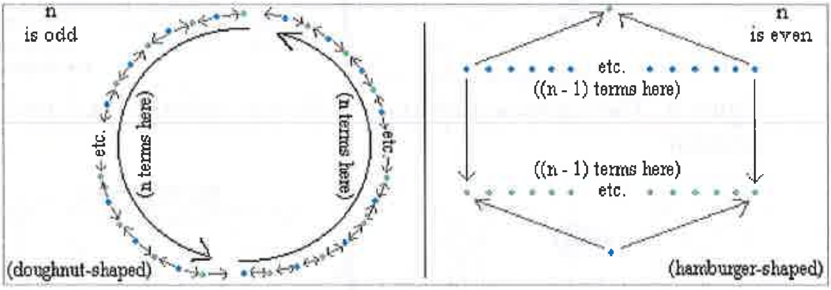
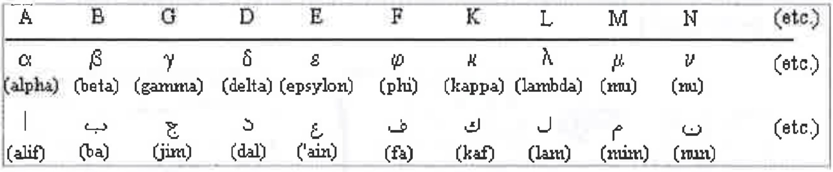


Figure 13. The two general models of the n-structures : for n is odd or even



A	B	G	D	E	F	K	L	M	N	(etc.)
α (alpha)	β (beta)	γ (gamma)	δ (delta)	ε (epsylon)	φ (phi)	κ (kappa)	λ (lambda)	μ (mu)	ν (nu)	(etc.)
ا (alif)	ب (ba)	ج (jim)	د (dal)	ع ('ain)	ف (fa)	ك (kaf)	ل (lam)	م (mim)	ن (nun)	(etc.)

Figure 14. The two alphabetical families (Greek and Arabic) used here to deal abstractly with modalities (left side and right side of the modal 3-graph)

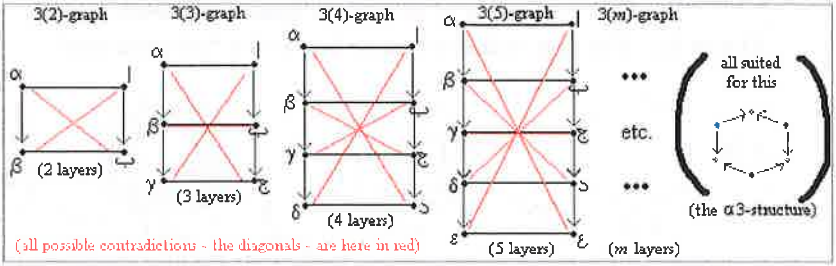


Figure 15. Examples of modal “3(m)-graphs”. They can support geometrical decoration for α3-structures (hexagons), but not for α4-structures or higher

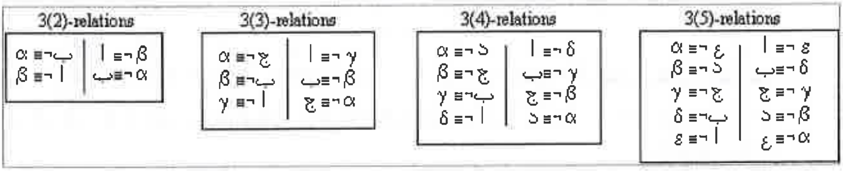


Figure 16. The $3(m)$-relations corresponding to the modal $3(m)$-graphs

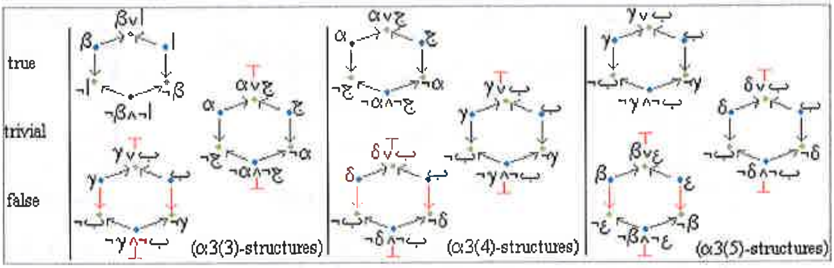


Figure 17. A sample of the possible $\alpha 3(m)$-structures, $m = 3, 4, 5$

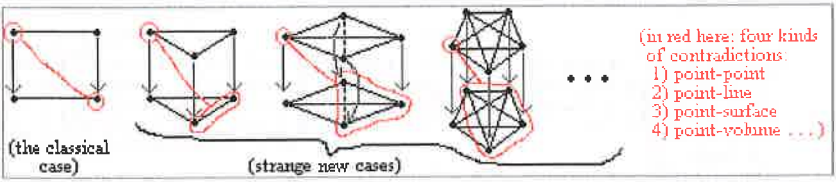


Figure 18. Some multi-dimensional alternatives in the ways to express "contradiction" geometrically (inside a $n(2)$-modal graph, $n = 3, 4, 5, 6, \ldots$)

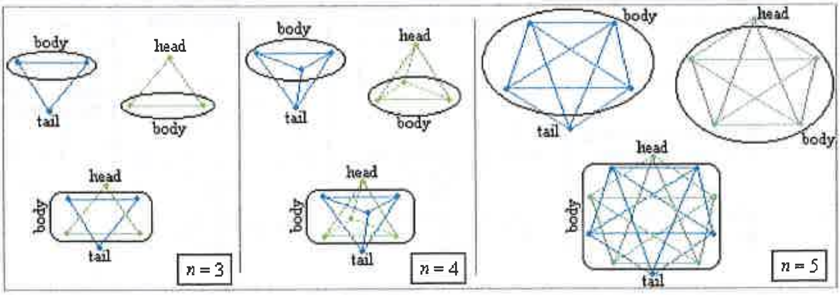
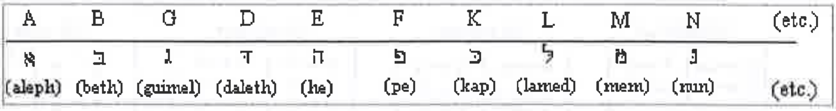


Figure 19. "Anatomy" of the αn-structures (or bi-simplexes of dimension n-1)



A	B	G	D	E	F	K	L	M	N	(etc.)
א	ב	ג	ד	ה	פ	כ	ל	מ	נ	
(aleph)	(beth)	(guimel)	(daleth)	(he)	(pe)	(kap)	(lamed)	(mem)	(nun)	(etc.)

Figure 20. The third alphabetical family, Hebrew, used to deal with 4-opposition

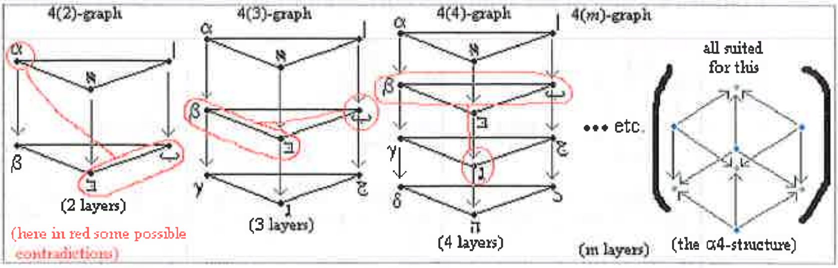
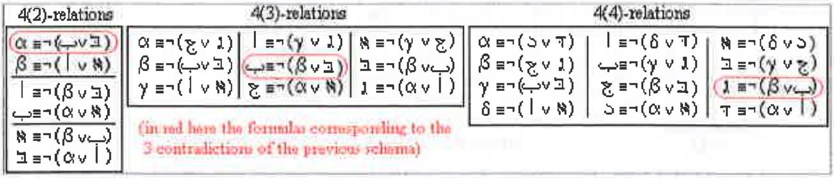


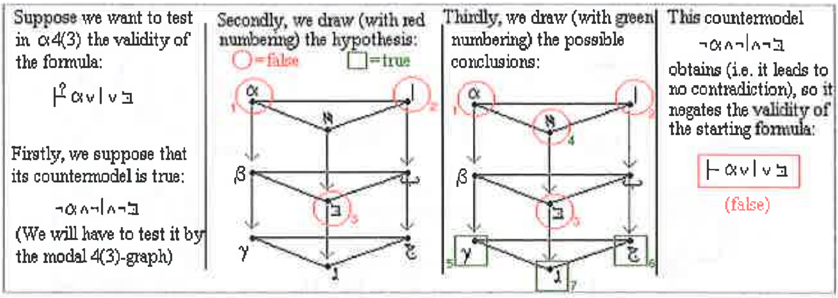
Figure 21. Examples of modal "4(m)-graphs". They can support modal decoration for the oppositional α4-structure, but not for the α5-structure or higher



4(2)-relations	4(3)-relations			4(4)-relations		
α ≡¬(ب∨ב)	α ≡¬(ج∨ג)	| ≡¬(γ∨ג)	א ≡¬(γ∨ج)	α ≡¬(כ∨ד)	| ≡¬(δ∨ד)	א ≡¬(δ∨כ)
β ≡¬(|∨א)	β ≡¬(ب∨ב)	ب ≡¬(β∨ב)	ב ≡¬(β∨ب)	β ≡¬(ج∨ג)	ب ≡¬(γ∨ג)	ב ≡¬(γ∨ج)
| ≡¬(β∨ב)	γ ≡¬(|∨א)	ج ≡¬(α∨א)	ג ≡¬(α∨|)	γ ≡¬(ب∨ב)	ج ≡¬(β∨ב)	ג ≡¬(β∨ب)
ب ≡¬(α∨א)				δ ≡¬(|∨א)	כ ≡¬(α∨א)	ד ≡¬(α∨|)
א ≡¬(β∨ب)						
ב ≡¬(α∨|)						

(in red here the formulas corresponding to the 3 contradictions of the previous schema)

Figure 22. The 4(m)-relations corresponding to the 4(m)-graphs, m = 2. 3. 4



Suppose we want to test in α4(3) the validity of the formula:

$\overset{?}{\vdash}$ α ∨ | ∨ ב

Firstly, we suppose that its countermodel is true:

¬α ∧ ¬| ∧ ¬ב

(We will have to test it by the modal 4(3)-graph)

Secondly, we draw (with red numbering) the hypothesis:
○=false ☐=true

Thirdly, we draw (with green numbering) the possible conclusions:

This countermodel

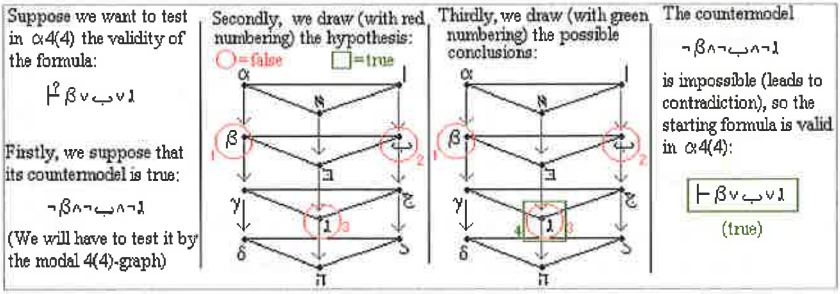
¬α ∧ ¬| ∧ ¬ב

obtains (i.e. it leads to no contradiction), so it negates the validity of the starting formula:

⊢ α ∨ | ∨ ב

(false)

Figure 23. A possible calculation on an α4(3)-structure via the modal 4(3)-graph



Suppose we want to test in α4(4) the validity of the formula: $\vDash^{?}$ β ∨ ب ∨ ג

Firstly, we suppose that its countermodel is true: ¬β ∧ ¬ب ∧ ¬ג

(We will have to test it by the modal 4(4)-graph)

Secondly, we draw (with red numbering) the hypothesis: ○=false ☐=true

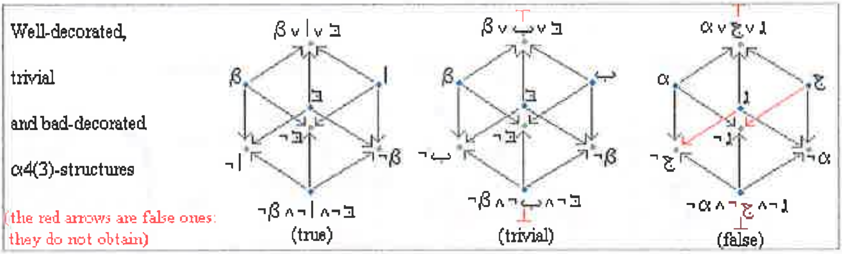
Thirdly, we draw (with green numbering) the possible conclusions:

The countermodel ¬β ∧ ¬ب ∧ ¬ג is impossible (leads to contradiction), so the starting formula is valid in α4(4):

⊢ β ∨ ب ∨ ג

(true)

Figure 24. A possible calculation on a α4(4)-structure via the modal 4(4)-graph



Well-decorated,

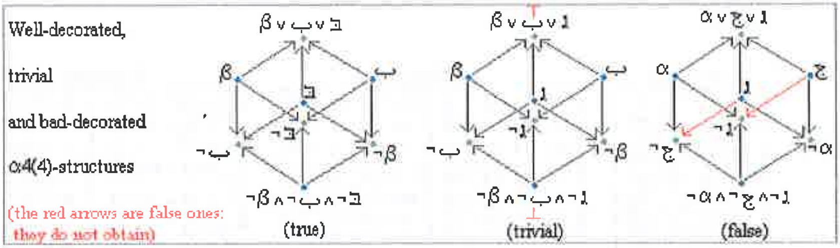
trivial

and bad-decorated

α4(3)-structures

(the red arrows are false ones: they do not obtain)

Figure 25. Some instances of possible decorations of the α4(3)-structure by the modal 4(3)-graph



Well-decorated,

trivial

and bad-decorated

α4(4)-structures

(the red arrows are false ones: they do not obtain)

Figure 26. Some instances of possible decorations of the α4(4)-structure by the modal 4(4)-graph

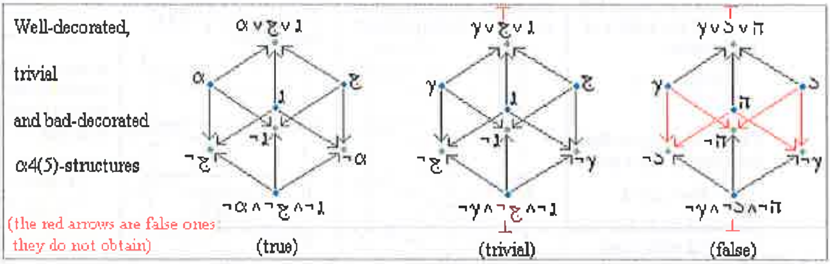
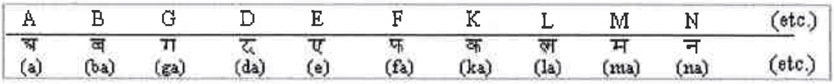


Figure 27. Some instances of possible decorations of the $\alpha4(5)$-structure by the modal 4(5)-graph



A	B	G	D	E	F	K	L	M	N	(etc.)
अ	ब	ग	द	ए	फ	क	ल	म	न	
(a)	(ba)	(ga)	(da)	(e)	(fa)	(ka)	(la)	(ma)	(na)	(etc.)

Figure 28.The fourth alphabetical family, Sanskrit (used for 5-opposition)

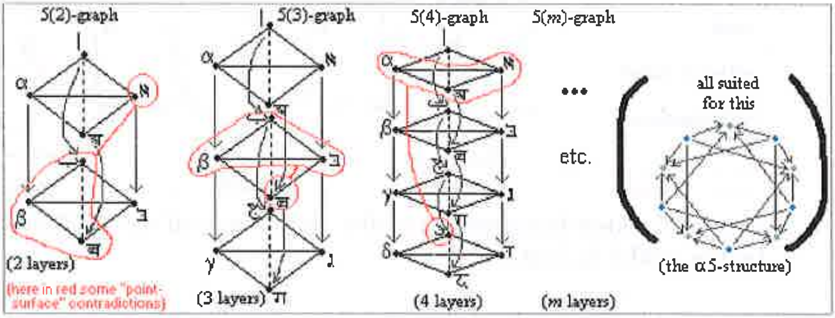
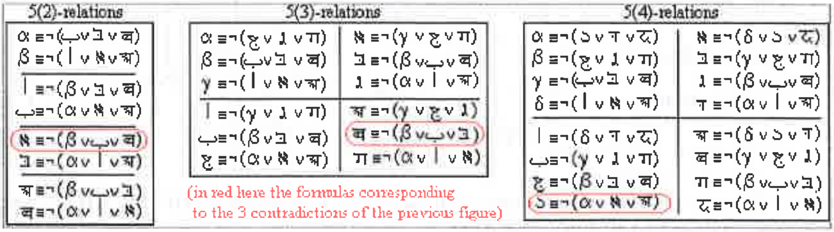


Figure 29. Examples of $5(m)$-graphs. They can support modal decorating for the oppositional $\alpha5$-structure, but not for the $\alpha6$-structure or higher



5(2)-relations

α ≡ ¬(ب ∨ ב ∨ ब) β ≡ ¬(ا ∨ ℵ ∨ अ)
ا ≡ ¬(β ∨ ב ∨ ब) ب ≡ ¬(α ∨ ℵ ∨ अ)
ℵ ≡ ¬(β ∨ ب ∨ ब) ב ≡ ¬(α ∨ ا ∨ अ)
अ ≡ ¬(β ∨ ب ∨ ב) ब ≡ ¬(α ∨ ا ∨ ℵ)

5(3)-relations

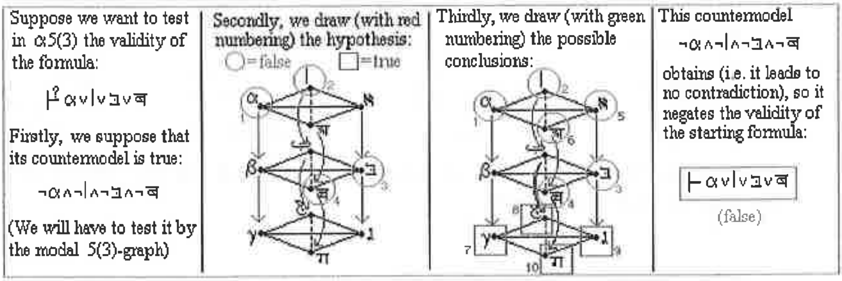
α ≡ ¬(ج ∨ ג ∨ ग) β ≡ ¬(ب ∨ ב ∨ ब) γ ≡ ¬(ا ∨ ℵ ∨ अ)	ℵ ≡ ¬(γ ∨ ج ∨ ग) ב ≡ ¬(β ∨ ب ∨ ब) ג ≡ ¬(α ∨ ا ∨ अ)
ا ≡ ¬(γ ∨ ג ∨ ग) ب ≡ ¬(β ∨ ב ∨ ब) ج ≡ ¬(α ∨ ℵ ∨ अ)	अ ≡ ¬(γ ∨ ج ∨ ג) ब ≡ ¬(β ∨ ب ∨ ב) ग ≡ ¬(α ∨ ا ∨ ℵ)

(in red here the formulas corresponding to the 3 contradictions of the previous figure)

5(4)-relations

α ≡ ¬(د ∨ ד ∨ द) β ≡ ¬(ج ∨ ג ∨ ग) γ ≡ ¬(ب ∨ ב ∨ ब) δ ≡ ¬(ا ∨ ℵ ∨ अ)	ℵ ≡ ¬(δ ∨ د ∨ द) ב ≡ ¬(γ ∨ ج ∨ ग) ג ≡ ¬(β ∨ ب ∨ ब) ד ≡ ¬(α ∨ ا ∨ अ)
ا ≡ ¬(δ ∨ ד ∨ द) ب ≡ ¬(γ ∨ ג ∨ ग) ج ≡ ¬(β ∨ ב ∨ ब) د ≡ ¬(α ∨ ℵ ∨ अ)	अ ≡ ¬(δ ∨ د ∨ ד) ब ≡ ¬(γ ∨ ج ∨ ג) ग ≡ ¬(β ∨ ب ∨ ב) द ≡ ¬(α ∨ ا ∨ ℵ)

Figure 30. The $5(m)$-relations corresponding to the $5(m)$-graphs, $m = 2, 3, 4$



Suppose we want to test in α5(3) the validity of the formula:

$\vdash^{?} \alpha \vee | \vee ב \vee ब$

Firstly, we suppose that its countermodel is true:

$\neg\alpha \wedge \neg | \wedge \neg ב \wedge \neg ब$

(We will have to test it by the modal 5(3)-graph)

Secondly, we draw (with red numbering) the hypothesis:

○=false ☐=true

Thirdly, we draw (with green numbering) the possible conclusions:

This countermodel

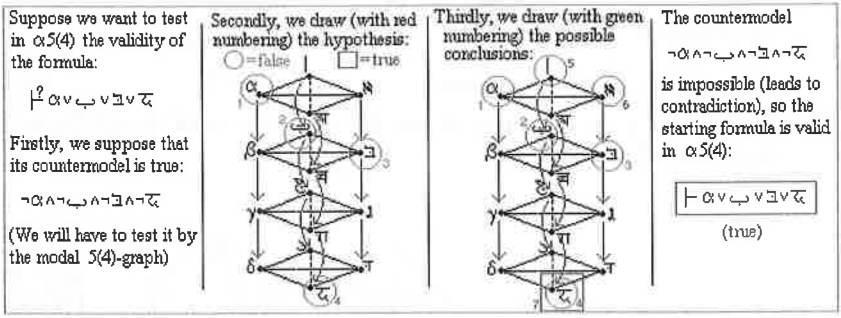
$\neg\alpha \wedge \neg | \wedge \neg ב \wedge \neg ब$

obtains (i.e. it leads to no contradiction), so it negates the validity of the starting formula:

$\boxed{\vdash \alpha \vee | \vee ב \vee ब}$

(false)

Figure 31. A possible calculation on an $\alpha 5(3)$-structure via the modal 5(3)-graph



Suppose we want to test in α5(4) the validity of the formula:

$\vdash^{?} \alpha \vee ب \vee ב \vee ک$

Firstly, we suppose that its countermodel is true:

$\neg\alpha \wedge \neg ب \wedge \neg ב \wedge \neg ک$

(We will have to test it by the modal 5(4)-graph)

Secondly, we draw (with red numbering) the hypothesis:

○=false | ☐=true

Thirdly, we draw (with green numbering) the possible conclusions:

The countermodel

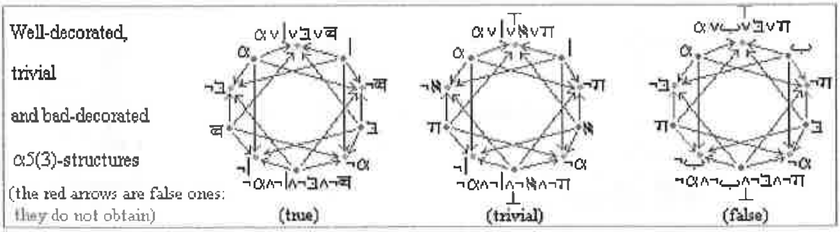
$\neg\alpha \wedge \neg ب \wedge \neg ב \wedge \neg ک$

is impossible (leads to contradiction), so the starting formula is valid in α5(4):

$\boxed{\vdash \alpha \vee ب \vee ב \vee ک}$

(true)

Figure 32. A possible calculation on an $\alpha 5(4)$-structure via the modal 5(4)-graph



Well-decorated,

trivial

and bad-decorated

α5(3)-structures

(the red arrows are false ones: they do not obtain)

Figure 33. Some instances of possible decorations of the $\alpha 5(3)$-structure by the modal 5(3)-graph

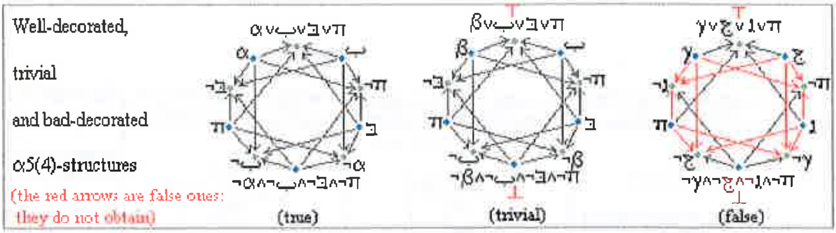


Figure 34. Some instances of possible decorations of the $\alpha 5(4)$-structure by the modal 5(4)-graph

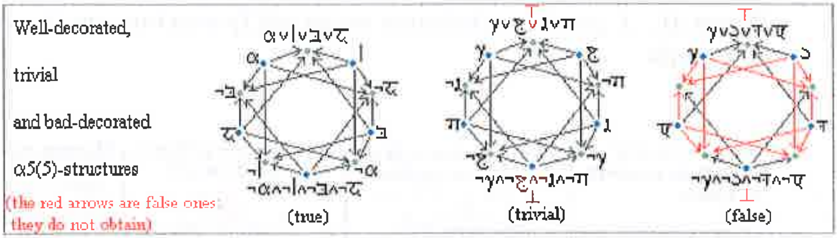


Figure 35. Some instances of possible decorations of the $\alpha 5(5)$-structure by the modal 5(5)-graph

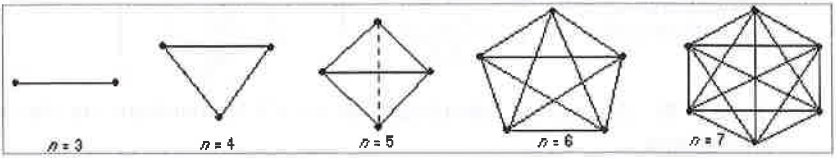


Figure 36. The series of the of the (black) "n-gems" ($n = 3, 4, 5, 6, 7$)

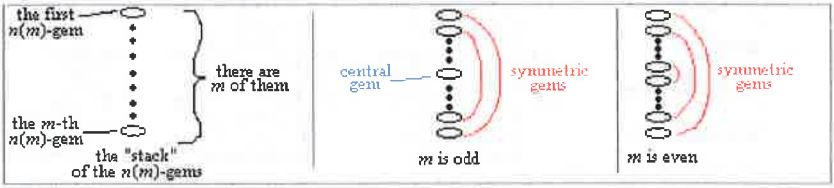


Figure 37. The shape of the modal $n(m)$-graphs in terms of the $n(m)$-gems

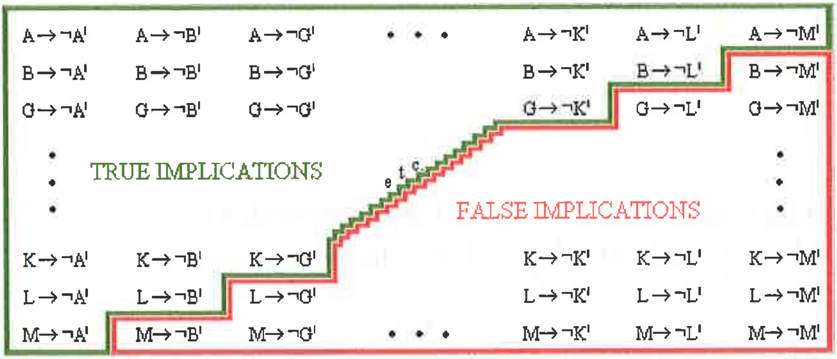


Figure 38. The general theorem detailing which simple implications are valid inside the modal $n(m)$-graph (the matrix must have m rows and m columns)

J.-Y.Beziau, A.Costa-Leite and A.Facchini (eds)
Aspects of Universal Logic, 146-168

On Modulated Logics for 'Generally': Some Metamathematical Issues[1]

Sheila R.M.Veloso and Paulo A.S.Veloso [2]

Abstract

Logics for 'generally' are intended to express assertions with some vague notions, such as 'generally', by means of new generalised quantifiers, and to reason about them. Here, we review such logical systems and examine some issues about them: axiomatisation, behaviour of the quantifiers, as well as deductive and expressive powers.

Keywords: Vague notions, generally, several, many, most, logics for vague notions, generalised quantifiers, families of sets, metamathematics, axiomatisation, oppositions, inference, expressivity.

Outline

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1 Introduction

In this paper, we review some logics for 'generally' and examine some issues about them: axiomatisations, behaviour of the quantifiers, as well as deductive and expressive powers.

Logics for 'generally' are intended to express assertions with some vague notions, such as 'generally', by means of new generalised quantifiers, and to reason about them (important issues in qualitative reasoning). The primary motivation is a precise treatment of some vague notions (such as 'generally', 'several', 'many', 'most', etc.), which appear often in ordinary language and in some branches of science.[3]

This paper is structured as follows. The next section provides some motivations and ideas underlying logics for 'generally'. In section 3 we examine logical systems for expressing and reasoning about assertions involving (some versions of) 'generally', with their syntax and semantics as well as sound and complete axiomatisations. In section 4 we examine some metamathematical properties of these logics for 'generally', comparing them to classical first-order logic: deductive and expressive powers, and the behaviour of the new quantifier. Section 5 contains some concluding remarks about on-going and related work.

2 Preliminaries

We will now review some motivations and ideas underlying logics for 'generally'.

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[3] We would like to have logics for some vague notions, much as one has logics embodying some mathematical notions ([B+F'85], p. 3).

2.1 Basic ideas

We first examine motivations underlying logics for 'generally'

Assertions and arguments involving some vague notions appear often, both in ordinary language and in some branches of science, where "modifiers", such as 'generally', 'rarely', 'several', 'few', 'many', 'most', 'typical', 'generic', etc., occur. For instance, one frequently encounters assertions such as "Many bodies expand when heated", "Most birds fly" and "Few metals are liquid under ordinary conditions".[4] The assertions "Whoever likes sports watches Sports-TV" and "Boys generally like sports" appear to lead to "Boys generally watch Sports-TV". Such qualitative arguments involving these vague notions appear to be quite widespread.[5]

Considering a universe of birds, we can express some assertions within classical first-order logic.[6]; but, what about vague assertions like "Birds generally fly"? We wish to express such assertions and reason about them in a formal manner; so we need precise meanings for these vague notions. Now, the intended meaning of "objects generally have property φ" can be given directly as "the set of objects having φ is important", or in terms of the set of exceptions as "the set of objects failing to have φ is negligible".[7]

2.2 Families for 'rarely' and 'generally'

We will now indicate how some notions of 'generally' and 'rarely' can be described by means of families of important and negligible sets. We actually have various notions of 'generally' and 'rarely', but some of them may be expected to share properties, which can be used to characterise these vague notions by means of the corresponding families of important and negligible sets [Vel'99, Vel'02]. To describe the important and negligible subsets, we may use common properties of their families $\mathcal{K}$ and $\mathcal{N}$. For instance, the above argument about boys and sports seems correct because of the intuitive feeling that if a set L has several objects and $L \subseteq T$, then set T will also have several objects: the family $\mathcal{K}$ of important sets (those having

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[4] Such notions may also be useful in reporting experimental set-ups and results. More elaborate expressions involving 'propensity' are often used as well: a physician may say that a patient's background indicates a certain propensity, making him (or her) prone to some ailments.

[5] A medical doctor usually prescribes a treatment considering it appropriate to a typical patient with such symptoms.

[6] For instance, "All birds fly" and "Some birds fly" by $\forall v F(v)$ and $\exists v F(v)$.

[7] One may understand "Eagles generally fly" as "The flying eagles form an important set" or "The non-flying eagles form a negligible set".

several objects) is closed under supersets. Families corresponding to other notions, such as (very) few, may be closed under union or intersection.[8]

For non-triviality, the family $\mathcal{K}$ of important subsets of a universe V should be proper ($\emptyset \notin \mathcal{K}$ and $V \in \mathcal{K}$). Some interesting classes of such families of important subsets are: up-closed (closed under supersets), lattices (closed under union and intersection), filters (closed under supersets and intersection), and ultrafilters (maximal filters).[9]

3 Logics for 'generally'

We shall now examine how one can set up logical systems for expressing and reasoning about assertions involving (some versions of) 'generally'. The goal is having logics for some vague notions, much as we have "logics embodying mathematical concepts" [B+F'85]. In this section we briefly review some of these logics: syntax, semantics and axiomatics as well as soundness and completeness.

Our logics for 'generally' add to classical first-order logic [End'72, Sho'67] a (non-standard) generalised quantifier, with intended interpretation "forming an important set of objects of the universe of discourse" [Gra'99, V+C'01].[10]

3.1 Syntax of ∇

The syntax of our logics is obtained by extending the usual first-order syntax by the new quantifier ∇.

Given a signature ρ, we let $\mathrm{L}(\rho)$ be the usual first-order language (with equality $\approx$) of signature ρ. We will use $\mathrm{L}^{\nabla}(\rho)$ for the extension of $\mathrm{L}(\rho)$ by the new operator ∇. The formulae of language $\mathrm{L}^{\nabla}(\rho)$ are built by

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[8] If one accepts the assertions "Few naturals are below fifteen" and "Few naturals divide twelve", then one would probably accept also the assertions: "Few naturals are below fifteen and even" and "Few naturals are below fifteen or divide twelve".

[9] For instance, the sets having more than, say, 70 % of the elements form an up-closed family (corresponding to a notion of 'several'); both the finite unions of intervals of the reals and the cofinite open subsets of an infinite topological space form lattices (corresponding to notions of 'many'); the subsets including a given set as well as the cofinite subsets of an infinite universe form filters (corresponding to notions of 'most'); and the subsets having a given element form an ultrafilter. The dual classes of such families of negligible subsets are: down-closed (closed under subsets), lattices (closed under $\cup$ and $\cap$), ideals (closed under subsets and $\cup$), and maximal (prime) ideals.

[10] With such new quantifiers we can handle assertions, such as "Birds generally fly" and "Metals generally are solid", as well as properties like "animals generally fear x".

the usual formation rules and a new variable-binding formation rule giving *generalised formulae* :

for each variable v, if φ is a formula in $\mathrm{L}^{\nabla}(\rho)$ then so is $\nabla v \varphi$.

Other syntactic notions, such as *substitution* ($\varphi[v/t]$ or $\varphi(t)$) and substitutable, can be easily adapted.

As a example, consider a signature ρ with binary predicate L (on persons). If we let $L(x, y)$ stand for 'x loves y', then $\forall x \nabla y L(x, y)$ expresses "everybody loves people in general", $\exists x \nabla y L(x, y)$ expresses "somebody loves people in general" and "people generally love each other" can be expressed by $\nabla x \nabla y L(x, y)$. If $L(x, y)$ stands for 'y is taller than x', then "people are generally taller than x" can be expressed by the formula $\nabla y L(x, y)$.

3.2 Semantics of 'generally'

The semantic interpretation for 'generally' is provided by enriching first-order structures with families of subsets and extending the definition of satisfaction to ∇. For this purpose, we resort to modulated structures.

A *modulated structure* $\mathcal{A}^{\mathcal{K}} = \langle \mathcal{A}, \mathcal{K} \rangle$ for signature ρ consists of a usual structure $\mathcal{A}$ for signature ρ together with a *complex*: a proper family $\mathcal{K}$ of subsets of the universe A of $\mathcal{A}$.

We extend the usual definition of *satisfaction* of a formula φ in a structure under an assignment $s : V \to \mathsf{A}$ to variables as follows

for a formula $\nabla v \varphi$, we define

$$\mathcal{A}^{\mathcal{K}} \models \nabla z \varphi[\, s] \text{ iff } \{b \in \mathsf{A} : \mathcal{A}^{\mathcal{K}} \models \varphi[\, s(v \mapsto b)]\} \text{ belongs to the complex } \mathcal{K}.$$

where, as usual, $s(v \mapsto b)$ is the assignment agreeing with s on every variable but v and $s(v \mapsto b)(v) = b$.[11]

Satisfaction of a formula hinges only on the realisations assigned to its symbols.[12]

A convenient notion is that of extension with respect to a variable: the *v-extension* of formula φ under assignment s is the set $\mathcal{A}^{\mathcal{K}}[\varphi(\, s|v)] := \{b \in \mathsf{A} : \mathcal{A}^{\mathcal{K}} \models \varphi[\, s(v \mapsto b)]\}$.[13] With this notation, satisfaction of a generalised

---

[11] Thus, the propositional connectives as well as the classical quantifiers $\forall$ and $\exists$ will keep their familiar interpretations.

[12] Thus, satisfaction for first-order formulae (without ∇) does not depend on the complex: for a formula φ of $\mathrm{L}(\rho)$, we have $\mathcal{A}^{\mathcal{K}} \models \varphi[\, s]$ iff $\mathcal{A} \models \varphi[\, s]$. We can also use the familiar notation $\mathcal{A}^{\mathcal{K}} \models \varphi[\underline{a}]$ for an assignment $\underline{a}$ to the free variables of formula φ.

[13] We similarly have the extension $\mathcal{A}^{\mathcal{K}}[\varphi(\underline{a}|v)] := \{b \in \mathsf{A} : \mathcal{A}^{\mathcal{K}} \models \varphi[\underline{a}, b]\}$.

formula becomes $\mathcal{A}^{\mathcal{K}} \models \varphi[\,s]$ iff the extension $\mathcal{A}^{\mathcal{K}}[\varphi(\,s|v)]$ belongs to the to the complex $\mathcal{K}$.[14] Other semantic notions, such as reduct and model ($\mathcal{A}^{\mathcal{K}} \models \Gamma$) are as usual.

We will modulate our structures by their complexes: a class of complexes will be called a *module.* The *basic module* $\underline{\mathcal{B}}$ consists of the proper complexes. We will be mainly interested in some classes of proper complexes: the modules $\underline{\mathcal{C}}$, $\underline{\mathcal{L}}$, $\underline{\mathcal{F}}$, and $\underline{\mathcal{U}}$ consisting of the proper up-closed complexes, of the lattices, of the filters and of the ultrafilters, respectively.[15] This gives rise to notions of *modulated consequence* as expected: consequence under module $\underline{\mathcal{M}}$ is defined by $\Gamma \models^{\underline{\mathcal{M}}} \tau$ iff $\mathcal{A}^{\mathcal{K}} \models \tau$, for every model $\mathcal{A}^{\mathcal{K}} \models \Gamma$ with $\mathcal{K}$ in $\underline{\mathcal{M}}$[16], likewise for *(modulated) validity.*

3.3 Axiomatics for 'generally'

We now formulate deductive systems for our logics for 'generally' by adding schemata to a calculus for classical first-order logic [Vel'98].

To set up our deductive systems for logics of 'generally', we take a sound and complete deductive calculus for classical first-order logic, with Modus Ponens (MP) as the sole inference rule (as in [End'72]), and extend its set Φ of axiom schemata by adding a set Φ_M of new axiom schemata (coding properties of the module), to form an axiomatisation for 'generally'.[17] We find convenient to divide our schemata into groups, namely

- syntactic schemata: related to invariance under syntax;
- common schemata: fundamental to the notions of 'generally';
- specific schemata: shared only by some versions of 'generally'.

The syntactic schemata aim to capture the idea that satisfaction hinges only on extension of a formula, and not on its syntactic form.

A syntactic schema handles extensionality: formulae with the same extension must be indistinguishable under 'generally'.

$$[\leftrightarrow \nabla] \quad \forall z(\psi \leftrightarrow \theta) \rightarrow (\nabla z\psi \leftrightarrow \nabla z\theta)$$

---

[14] Notice that $\mathcal{A}^{\mathcal{K}} \models \exists v\varphi[\,s]$ iff the extension $\mathcal{A}^{\mathcal{K}}[\varphi(\,s|v)]$ belongs to the family $\wp(\mathsf{A}) - \{\emptyset\}$ of the non-empty subsets of A.

[15] These modules are clearly related, e.g. $\underline{\mathcal{C}} \subseteq \underline{\mathcal{F}} \subseteq \underline{\mathcal{U}}$.

[16] For the module $\underline{\mathcal{F}}$ of filters: $\Gamma \models^{\underline{\mathcal{F}}} \tau$ iff $\mathcal{A}^{\mathcal{K}} \models \tau$, for every filter model $\mathcal{A}^{\mathcal{F}} \models \Gamma$. These modulated consequences are related and others can be similarly introduced.

[17] These schemata depend on the signature ρ, but we will prefer to use the simpler notations Φ and Φ_M rather than $\Phi(\rho)$ and $\Phi_M(\rho)$.

Another syntactic schema covers, in a similar manner, alphabetic variants.

$$[\nabla\nu] \quad \nabla v\varphi \leftrightarrow \nabla w\varphi[v/w] \quad \text{for a new variable } w$$

The *syntactic schemata* consist of these two schemata:

$$\Phi_I := [\leftrightarrow \nabla] \cup [\nabla\nu]$$

The common schemata code properties of the proper complexes.

$$\begin{array}{lll} [\forall\nabla] & \forall v\varphi \to \nabla v\varphi & [V \in \mathcal{K}] \\ [\nabla\exists] & \nabla v\varphi \to \exists v\varphi & [\emptyset \notin \mathcal{K}] \end{array}$$

The *basic axiomatisation* extends the syntactic schemata by these two common schemata:

$$\Phi_B := \Phi_I \cup [\forall\nabla] \cup [\nabla\exists].$$

The specific schemata code closure properties of special modules.

$$\begin{array}{lcr} [\to \nabla] & \forall v(\psi \to \theta) \to (\nabla v\psi \to \nabla v\theta) & [\text{up-closure}] \\ [\nabla\vee] & (\nabla v\psi \wedge \nabla v\theta) \to \nabla v(\psi \vee \theta) & [\cup\text{-closure}] \\ [\nabla\wedge] & (\nabla v\psi \wedge \nabla v\theta) \to \nabla v(\psi \wedge \theta) & [\cap\text{-closure}] \\ [\neg\nabla] & \neg\nabla v\varphi \to \nabla v\neg\varphi & [\text{prime}] \end{array}$$

We thus have some specific axiomatisations as follows.

- Up-closed logic: $\Phi_C := \Phi_B \cup [\to \nabla]$[18]
- Lattice logic: $\Phi_L := \Phi_B \cup [\nabla\vee] \cup [\nabla\wedge]$
- Filter logic: $\Phi_F := \Phi_C \cup [\nabla\wedge]$[19]
- Ultrafilter logic: $\Phi_U := \Phi_F \cup [\neg\nabla]$[20]

Now, each one of these axiomatisations for 'generally' gives a *derivability relation* $\vdash^M$, axiomatised by $\Phi^M := \Phi_B \cup \Phi_M$. Derivations are first-order derivations from the schemata

$$\Gamma \vdash^M \tau \text{ iff } \Gamma \cup \Phi_F \vdash \tau \qquad (\vdash^\star)$$

In fact, each set $\Xi \subseteq \Phi_F$ of axioms for 'generally' gives a derivability relation $\vdash^\Xi$, axiomatised by $\Phi^\Xi := \Phi \cup \Phi_B \cup \Xi$.

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[18] In up-closed logic we have $\vdash^C (\nabla v\psi \vee \nabla v\theta) \to \nabla v(\psi \vee \theta)$ and $\vdash^C \nabla v(\psi \wedge \theta) \to (\nabla v\psi \wedge \nabla v\theta)$ (by $[\to \nabla]$).

[19] In filter logic we have $\vdash^F \nabla v\neg\varphi \to \neg\nabla v\varphi$ (by $[\nabla\vee]$ and $[\nabla\exists]$) and ∇ distributes over $\wedge$: $\vdash^F \nabla v(\psi \wedge \theta) \leftrightarrow (\nabla v\psi \wedge \nabla v\theta)$ (by $[\nabla\wedge]$ and $[\to \nabla]$).

[20] In ultrafilter logic, ∇ commutes with negation ($\vdash^U \neg\nabla v\varphi \leftrightarrow \nabla v\neg\varphi$) and distributes over the binary propositional connectives (we have, for instance, $\vdash^U \nabla v(\psi\vee\theta) \leftrightarrow (\nabla v\psi\vee \nabla v\theta)$). We thus have prenex normal form.

3.4 Soundness and completeness

We shall now establish soundness and completeness of our deductive systems for the corresponding logics for 'generally'.

Soundness ($\vdash^M \subseteq \models^{\underline{\mathcal{M}}}$) is easy to establish as usual: the axioms in each axiomatisation Φ^M code properties of the complexes in the module $\underline{\mathcal{M}}$.

Completeness ($\models^{\underline{\mathcal{M}}} \subseteq \vdash^M$) is not so immediate, but, we can extend Henkin's familiar method of witnesses [Hen'49, Sho'67, C+K'73, End'72]. The crucial point here is obtaining an appropriate complex, which we can do by using the witnesses. We proceed to outline how this can be done [Vel'98, V+C'01]. To fix ideas, we will focus on filter logic and later indicate how to adapt these ideas to the other cases.

Consider a set Γ of sentences of $\mathrm{L}^{\nabla}(\rho)$ that is filter-consistent: $\Gamma \nvdash^F \perp$. We will show how to obtain a filter model $\mathcal{H}^{\mathcal{K}_\Sigma} \models \Gamma$ (with cardinality at most that of $\mathrm{L}^{\nabla}(\rho)$).

We first extend set $\Gamma \subseteq \mathrm{L}^{\nabla}(\rho)$ to a maximally consistent set Σ with witnesses for the existential sentences of $\mathrm{L}^{\nabla}(\rho \cup C)$ in set C of new constants (with cardinality $|C| \leq |\mathrm{L}^{\nabla}(\rho)|$).[21] We form the canonical structure $\mathcal{H}$, for signature $\rho \cup C$ as usual.[22]

We provide a complex, by considering the formulae of $\mathrm{L}^{\nabla}(\rho \cup C)$, having a single variable free, as follows. For each formula φ of $\mathrm{L}^{\nabla}(\rho \cup C)$ with single free variable v, let $\Sigma[\varphi|v] := \{c^{\mathcal{H}} \in \mathsf{H} : \varphi[v/c] \in \Sigma\}$, and form the family $\Sigma_{\nabla} := \{\Sigma[\varphi|v] \subseteq \mathsf{H} : \nabla v\varphi \in \Sigma\}$.[23] By our axioms, this family Σ_{∇} is proper and has the finite intersection property.[24] So, its closure $\mathcal{K}_\Sigma := \Sigma_{\nabla}^{\supseteq}$ under supersets is a filter, with the property $\Sigma[\varphi|v] \in \Sigma_{\nabla}$ iff $\Sigma[\varphi|v] \in \mathcal{K}_\Sigma$.[25] We can now show, by induction $\mathcal{H}^{\mathcal{K}_\Sigma} \models \tau$ iff $\tau \in \Sigma$, for each sentence τ of $\mathrm{L}^{\nabla}(\rho \cup C)$.[26]

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[21] The properties of conservative extensions by the addition of witnesses and Lindenbaum extensions for our deductive systems can be established as in classical first-order logic, by relying on the connection ($\vdash^\star$) in 4.3.

[22] The canonical structure $\mathcal{H}$ has universe $\mathsf{H} := C/\sim^{\Sigma}$, where $c' \sim^{\Sigma} c$" iff $\Sigma \vdash Fc' \approx c$".

[23] One can view $\Sigma[\varphi|v]$ as the set v-represented within Σ by formula φ and Σ_{∇} as the family of provably important represented subsets.

[24] Family Σ_{∇} is proper by the basic schemata [$\forall\nabla$] and [$\nabla\exists$], and its closure under finite intersection follows from the schemata [$\nabla\wedge$] and [$\nabla\nu$].

[25] Notice that family Σ_{∇} is not closed under arbitrary supersets, but this extension $\mathcal{K}_\Sigma \supseteq \Sigma_{\nabla}$ adds no definable subset: property $\Sigma[\varphi|v] \in \Sigma_{\nabla}$ iff $\Sigma[\varphi|v] \in \mathcal{K}_\Sigma$ follows from the schemata [$\rightarrow \nabla$] and [$\nabla\nu$].

[26] The inductive step for the new quantifier ∇, namely $\mathcal{H}^{\mathcal{K}_\Sigma} \models \nabla v\varphi$ iff $\nabla v\varphi \in \Sigma$, follows from the crucial property $\Sigma[\varphi|v] \in \Sigma_{\nabla}$ iff $\Sigma[\varphi|v] \in \mathcal{K}_\Sigma$ of the complex $\mathcal{K}_\Sigma$. The inductive steps for the propositional connectives as well as for the classical quantifiers $\forall$

We thus have the desired result: a Löwenheim-Skolem Theorem.

Theorem. Löwenheim-Skolem Theorem (for filter logic) Each filter-consistent set of sentences of $L^{\nabla}(\rho)$ has a filter model with cardinality at most $|L^{\nabla}(\rho)|$.

We now indicate how to adapt these ideas to our other logics.

- For up-closed logic, we use the same construction.[27]
- For basic and lattice logics, we take $\mathcal{K}_\Sigma := \Sigma_\nabla$.[28]
- For ultrafilter logic, we extend family Σ_∇ to an ultrafilter.[29]

4 Metamathematics of 'generally'

Our logics for 'generally' extend classical first-order logic. We have sound and complete deductive systems for these logics. As usual, such a result transfers the finitary character of derivability to the compactness of the corresponding semantic consequence. Thus, our extensions of classical first-order logic by generalised quantifiers have compactness.

We shall now examine some other metamathematical properties of these extensions of classical first-order logic by generalised quantifiers. We will take a closer look at these extensions comparing them to classical first-order logic.

4.1 Behaviour of quantifiers

We will first examine the behaviour of quantifiers in our logics for 'generally'. We wish to compare them to classical first-order logic, pointing out similarities and contrasts.

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and $\exists$ are as in Henkin's proof.

[27] Here, family Σ_∇ will be proper and we take $\mathcal{K}_\Sigma$ to be its closure under supersets. The crucial property $\Sigma[\varphi|v] \in \Sigma_\nabla$ iff $\Sigma[\varphi|v] \in \mathcal{K}_\Sigma$ follows from the schemata $[\rightarrow\nabla]$ and $[\nabla\nu]$.

[28] Family Σ_∇ already gives an appropriate complex.

[29] As family Σ_∇ has the finite intersection property, it can be extended to a proper ultrafilter $\mathcal{K}_\Sigma$. The property $\Sigma[\varphi|v] \in \Sigma_\nabla$ iff $\Sigma[\varphi|v] \in \mathcal{K}_\Sigma$ now follows from the schema $[\neg\nabla]$.

In our extensions of classical first-order logic, the behaviour of the classical quantifiers $\forall$ and $\exists$ remain the same, but what about the new quantifier ∇? We know that ∇ is intermediate between $\forall$ and $\exists$, in terms of behaviour[30], and we feel intuitively that it is closer to the universal quantifier[31].

Oppositions of quantifiers

We will now compare leading classical and generalised quantifiers.

First, we do not have instantiation for ∇.[32]

We now wish to examine some opposition relations between classical and generalised quantifiers.[33]

Consider the classical square of oppositions, involving affirmative and negative, universal and particular assertions, as well as the relations of contrary, subcontrary and contradictory.[34] We wish to consider analogue connections involving also generalised quantifiers.

First, we have to make room for ∇, placing it in between $\forall$ and $\exists$. This transforms the usual square of oppositions into a hexagon (see figure 1).

This hexagon of oppositions has interesting interpretations in terms of corroboration and refutation: generalised sentences are harder to corroborate than universal ones and harder to refute than existential ones.[35]

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[30] The common schemata $[\forall\nabla]$ and $[\nabla\exists]$ give $\vdash^{B} \forall v\varphi \to \nabla v\varphi$ and $\vdash^{B} \nabla v\varphi \to \exists v\varphi$. The converse implications are not valid ($\not\models^{\mathcal{M}} \nabla v\varphi \to \forall v\varphi$ and $\not\models^{\mathcal{M}} \exists v\varphi \to \nabla v\varphi$): either one would trivialise the new generalised quantifier, collapsing ∇ to $\forall$ or to $\exists$.

[31] One may feel the generalised quantifier ∇ to be closer to $\forall$ because of the intuitive interpretation "all, but negligibly few exceptions" for 'many', 'most', etc. One can define a dual generalised quantifier for 'rarely', closer to $\exists$.

[32] Indeed, $\nabla v\varphi$ does not yield $\varphi[v/t]$ (neither is the converse inference correct: $\varphi[v/t]$ does not yield $\nabla v\varphi$).

[33] Some square-of-opposition relations among 'few', 'many', and 'most' have been analysed [Pet'79].

[34] Contrary assertions cannot be both true, subcontrary assertions cannot be both false, and contradictory assertions cannot be both true nor false. The classical square of oppositions is as follows (where the diagonally opposed assertions are contradictory)

$$\begin{tikzcd}
 & \text{Affirmative} & & \text{Negative} \\
\text{Universal} & \forall v\varphi \arrow[rr, leftrightarrow, "contrar."] \arrow[d, Rightarrow] & & \forall v\neg\varphi \arrow[d, Rightarrow] \\
\text{Particular} & \exists v\varphi \arrow[rr, leftrightarrow, "subcontrar."] & & \exists v\neg\varphi
\end{tikzcd}$$

[35] Thus, generalised sentences fail to present a clear asymmetry between corroboration and refutation, of importance to some views of Popper (cf. [Pop'34], [Pop'75]).

Let us now take a closer look at the above hexagon of oppositions. We still have some further generalised assertions to place, namely those corresponding to $\neg\nabla v\varphi$ and $\neg\nabla v\neg\varphi$.[36] We can locate them by relying on equivalences concerning the behaviour of the classical quantifiers $\forall$ and $\exists$ under negation.[37] This transforms the above hexagon of oppositions into an octagon (see figure 2).[38]

This octagon displays oppositions holding in basic logic. In stronger logics for 'generally' one has some more information. For instance, we have some more oppositions in the octagon for filter logic (we have as contraries

$$\begin{tikzcd}
 & \text{Affirmative} & & \text{Negative} \\
\text{Universal} & \forall v\varphi \arrow[rr, leftrightarrow, "\mathit{contrar.}"] \arrow[d, Rightarrow] & & \forall v\neg\varphi \arrow[d, Rightarrow] \\
\text{Generalised} & \nabla v\varphi \arrow[d, Rightarrow] & & \nabla v\neg\varphi \arrow[d, Rightarrow] \\
\text{Particular} & \exists v\varphi \arrow[rr, leftrightarrow, "\mathit{subcontrar.}"] & & \exists v\neg\varphi
\end{tikzcd}$$

Table 1: Hexagon of oppositions in basic logic

$$\begin{tikzcd}
 & \text{Affirmative} & \text{Negative} & \\
\neg\exists v\neg\varphi \arrow[r, leftrightarrow] \arrow[dd, Rightarrow] & \forall v\varphi \arrow[r, leftrightarrow, "\mathit{contrar.}"] \arrow[d, Rightarrow] & \forall v\neg\varphi \arrow[r, leftrightarrow] \arrow[dd, Rightarrow] & \neg\exists v\varphi \arrow[d, Rightarrow] \\
 & \nabla v\varphi \arrow[rr, leftrightarrow, "\mathit{contradict.}"] \arrow[dd, Rightarrow] & & \neg\nabla v\varphi \arrow[dd, Rightarrow] \\
\neg\nabla v\neg\varphi \arrow[rr, leftrightarrow, "\mathit{contradict.}"] \arrow[d, Rightarrow] & & \nabla v\neg\varphi \arrow[d, Rightarrow] & \\
\neg\forall v\neg\varphi \arrow[r, leftrightarrow] & \exists v\varphi \arrow[r, leftrightarrow, "\mathit{subcontrar.}"] & \exists v\neg\varphi \arrow[r, leftrightarrow] & \neg\forall v\varphi
\end{tikzcd}$$

Table 2: Octagon of oppositions in basic logic

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[36] Indeed, the unary modalities ∇ and $\neg$ generate the four modalities with ∇: ∇, $\nabla\neg$, $\neg\nabla$ and $\neg\nabla\neg$.

[37] We have $\vdash \neg\exists v\varphi \leftrightarrow \forall v\neg\varphi$ and $\vdash \neg\forall v\varphi \leftrightarrow \exists v\neg\varphi$.

[38] The schemata $[\nabla\exists]$ and $[\forall\nabla]$ give $\vdash^B \neg\exists v\varphi \rightarrow \neg\nabla v\varphi$ and $\vdash^B \neg\nabla v\varphi \rightarrow \neg\forall v\varphi$.

$\neg\nabla v\neg\varphi$ and $\nabla v\varphi$ and as sub-contraries $\nabla v\neg\varphi$ and $\neg\nabla v\varphi$).[39] This octagon reduces back to a hexagon in the case of ultrafilter logic (because of the equivalences $\vdash^U \neg\nabla v\varphi \leftrightarrow \nabla v\neg\varphi$ and $\vdash^U \neg\nabla v\neg\varphi \leftrightarrow \nabla v\varphi$).[40]

Classical and generalised quantifiers

We now wish to examine how adjacent classical and generalised quantifiers interact. Having compared leading classical and generalised quantifiers, we now wish to examine some interactions among them.

We first consider some transfer principles where the behaviour of the new generalised quantifier is similar to that of the classical ones. In classical first-order logic we have the transfer principle $\vdash \exists u\forall z\varphi \to \forall z\exists u\varphi$. Since ∇ is between $\forall$ and $\exists$, one might expect some similar transfer principles for ∇.[41] Indeed, we can see that we have the transfer principles for ∇ in up-closed logic:

- $\vdash^C \nabla u\forall z\varphi \to \forall z\nabla u\varphi$
- $\vdash^C \exists u\nabla z\varphi \to \nabla z\exists u\varphi$.

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[39] The octagon for filter logic is follows.

$$\begin{tikzcd}
 & \text{Affirmative} & \text{Negative} & \\
\neg\exists v\neg\varphi \arrow[r, leftrightarrow] \arrow[d, Rightarrow] & \forall v\varphi \arrow[r, leftrightarrow, "\mathit{contrar.}"] \arrow[d, Rightarrow] & \forall v\neg\varphi \arrow[r, leftrightarrow] \arrow[d, Rightarrow] & \neg\exists v\varphi \arrow[d, Rightarrow] \\
\neg\nabla v\neg\varphi \arrow[r, leftrightarrow, "\mathit{contrar.}"] \arrow[d, Rightarrow] & \nabla v\varphi \arrow[r, leftrightarrow, "\mathit{contradict.}"] \arrow[d, Rightarrow] & \nabla v\neg\varphi \arrow[r, leftrightarrow, "\mathit{subcontrar.}"] \arrow[d, Rightarrow] & \neg\nabla v\varphi \arrow[d, Rightarrow] \\
\neg\forall v\neg\varphi \arrow[r, leftrightarrow] & \exists v\varphi \arrow[r, leftrightarrow, "\mathit{subcontrar.}"] & \exists v\neg\varphi \arrow[r, leftrightarrow] & \neg\forall v\varphi
\end{tikzcd}$$

In filter logic, formulae $\neg\nabla v\neg\varphi$ and $\nabla v\varphi$ are contraries because of the schemata $[\nabla\wedge]$ and $[\nabla\exists]$.

[40] The hexagon for ultrafilter logic is as follows.

$$\begin{tikzcd}
 & \text{Affirmative} & \text{Negative} & \\
\neg\exists v\neg\varphi \arrow[r, leftrightarrow] \arrow[d, Rightarrow] & \forall v\varphi \arrow[r, leftrightarrow, "\mathit{contrar.}"] \arrow[d, Rightarrow] & \forall v\neg\varphi \arrow[r, leftrightarrow] \arrow[d, Rightarrow] & \neg\exists v\varphi \arrow[d, Rightarrow] \\
\neg\nabla v\neg\varphi \arrow[r, leftrightarrow] \arrow[d, Rightarrow] & \nabla v\varphi \arrow[r, leftrightarrow, "\mathit{contradict.}"] \arrow[d, Rightarrow] & \nabla v\neg\varphi \arrow[r, leftrightarrow] \arrow[d, Rightarrow] & \neg\nabla v\varphi \arrow[d, Rightarrow] \\
\neg\forall v\neg\varphi \arrow[r, leftrightarrow] & \exists v\varphi \arrow[r, leftrightarrow, "\mathit{subcontrar.}"] & \exists v\neg\varphi \arrow[r, leftrightarrow] & \neg\forall v\varphi
\end{tikzcd}$$

In ultrafilter logic, we have the equivalence $\vdash^U \neg\nabla v\varphi \leftrightarrow \nabla v\neg\varphi$ (because of $[\neg\nabla]$, $[\nabla\wedge]$ and $[\nabla\exists]$).

[41] In basic logic, $\forall u\forall z\varphi$ yields $\forall u\nabla z\varphi$, $\nabla u\forall z\varphi$ and $\nabla u\nabla z\varphi$ (by $[\forall\nabla]$), and $\nabla u\nabla z\varphi$ yields $\exists u\nabla z\varphi$, $\nabla u\exists z\varphi$ and $\exists u\exists z\varphi$ (by $[\nabla\exists]$).

These transfer principles are easily interpreted and derived.[42] The converse transfers fail, as we will see shortly.

We now consider a case where the behaviour of the new generalised quantifier contrasts with that of the classical ones. In first-order logic, the classical quantifiers commute.[43] In contrast, iterated generalised quantifiers do not commute: $\nabla u \nabla z \varphi$ does not yield $\nabla z \nabla u \varphi$.

A single example can serve to establish the three failures just mentioned: the naturals with order $<$ and 'generally' meaning cofinite. Expand the structure $\mathcal{N} = \langle \mathbb{N}, < \rangle$ by a Fréchet ultrafilter $\mathcal{U}$: having no finite subset. We then see that structure $\mathcal{N}^{\mathcal{U}}$ satisfies none of

- $\forall u \nabla z\, u < z \to \nabla z \forall u\, u < z$,
- $\nabla u \exists z\, u < z \to \exists z \nabla u\, u < z$,
- $\nabla u \nabla z\, u < z \to \nabla z \nabla u\, u < z$.[44]

4.2 Deductive power

We shall now examine the deductive power of our logics for 'generally'. We will show that we have conservative extensions of classical first-order logic and that we can reduce some simple cases of consequences to classical conditions.

Concerning deductive powers, our extensions of classical first-order logic have increasing deductive powers.[45]

Conservative extension

We will first show that the addition of the new generalised quantifiers produces conservative extensions of classical first-order logic.

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[42] The behaviour of ∇ is reminiscent of that of $\exists$ in the first transfer (over $\forall$), and of that of $\forall$ in the second transfer (over $\exists$). These transfer principles follow from schema $[\to \nabla]$. If some u is related to several z's (via φ), then several z's are related to some u's: $\models^C \exists u \nabla z \varphi \to \nabla z \exists u \varphi$ (as $\mathcal{A}^{\mathcal{K}}[\varphi(\underline{a}, b|z)] \subseteq \mathcal{A}^{\mathcal{K}}[\exists u \varphi(\underline{a}|z)]$). The dual transfer $\vdash^C \nabla u \forall z \varphi \to \forall z \nabla u \varphi$ follows from schema $[\to \nabla]$ as $\vdash \forall u(\forall z \varphi \to \varphi)$.

[43] We have $\vdash \forall u \forall z \varphi \leftrightarrow \forall z \forall u \varphi$ and $\vdash \exists u \exists z \varphi \leftrightarrow \exists z \exists u \varphi$.

[44] Indeed, we see that $\mathcal{N}^{\mathcal{U}}$ satisfies $\forall u \nabla z\, u < z$ (for every $m \in \mathbb{N}$: $\{n \in \mathbb{N} : m < n\}$ is cofinite), so also $\nabla u \nabla z\, u < z$ and $\nabla u \exists z\, u < z$; but $\mathcal{N}^{\mathcal{U}}$ fails to satisfy $\exists z \nabla u\, u < z$ (for no $n \in \mathbb{N}$: $\{m \in \mathbb{N} : m < n\}$ is cofinite), whence $\mathcal{N}^{\mathcal{U}} \not\models \nabla z \nabla u\, u < z$ and $\mathcal{N}^{\mathcal{U}} \not\models \nabla z \forall u\, u < z$. This shows that these formulae are not valid in any modulated logic with intermediate module $\underline{\mathcal{M}}$: $\underline{\mathcal{B}} \subseteq \underline{\mathcal{M}} \subseteq \underline{\mathcal{U}}$.

[45] The increasing deductive powers can be seen by considering the schemata in our axiomatisations (cf. 4.3 in section 4).

It is easy to see that we have conservative extensions of classical first-order logic. [46]

Theorem. Conservative extension (of classical first-order logic) Consider a set $\Xi \subseteq \Phi_U$ of axioms for 'generally'. Given a set of sentences $\Delta \cup \tau \subseteq \mathrm{L}(\rho)$: $\Delta \vdash^{\Xi} \tau$ iff $\Delta \vdash \tau$.[47]

A pleasing consequence of having conservative extensions is that one can reuse classical first-order reasoning.

For instance, imagine that $P(d)$ and $M(d)$ stand for 'd is pleasant' and 'd is mild', respectively, and consider a classical first-order theory Δ (about days) where $\Delta \vdash \forall v[P(v) \leftrightarrow M(v)]$ ["pleasant days are mild days"]. We can then infer $\Delta \vdash^{B} \nabla v \neg P(v) \leftrightarrow \nabla v \neg M(v)$ ["days are generally unpleasant iff they are generally not mild"].[48]

Inference of simple generalised formulae

We will now examine some simple cases of consequences that reduce to classical conditions. The first step beyond first-order adds a single initial ∇. We shall examine some cases of inference and refutation of such formulae with a single initial ∇. We already know that, for classical formulae (without ∇), our logics for 'generally' have the same deductive power as classical first-order logic, but what about formulae with ∇? We will show that, for some formulae (with a single initial ∇), we can reduce inference and refutation to classical first-order conditions.

A *positive generalised formula* is one of the form $\nabla v \varphi$, where φ has no ∇. A *negative generalised formula* is the negation of a positive generalised formula: of the form $\neg \nabla v \varphi$, where φ has no ∇. The *simple generalised formulae* consist of the positive and negative generalised formulae. We shall examine some cases of inference and refutation of such formulae with a single (initial) ∇.

We first show that the absence of generalised information reduces inference and refutation to classical first-order logic.

As an example, consider a consistent purely first-order theory Δ, with three axioms expressing "Mercury is not solid", "Mercury is not the only

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[46] for classical formulae (without ∇), our ∇-axioms add no extra deductive power.

[47] Every first-order structure $\mathcal{A}$ can be expanded to an ultrafilter structure $\mathcal{A}^{\mathcal{U}} = \langle \mathcal{A}, \mathcal{U} \rangle$ satisfying the same first-order sentences.

[48] This assertion follows from the syntactic schema $[\leftrightarrow \nabla]$.

metal" and "Every metal, other than mercury, is solid". In this case, we cannot decide whether "metals generally are solid", as we will see.

The next result gives conditions for inferring or refuting simple generalised formulae from a purely first-order theory.

Theorem. Simple generalised consequences of first-order theory Consider a set $\Xi \subseteq \Phi_U$ of axioms for 'generally'. Given a set of sentences $\Delta \subseteq \mathrm{L}(\rho)$ and a formula $\varphi \in \mathrm{L}(\rho)$, we have the following conditions.

I : $\Delta \vdash^{\Xi} \nabla v\varphi$ iff $\Delta \vdash \forall v\varphi$

R : $\Delta \vdash^{\Xi} \neg\nabla v\varphi$ iff $\Delta \vdash \neg\exists v\varphi$

Proof. For I: the universe is the only set in every complex gives ($\Rightarrow$) and $[\forall\nabla]$ yields ($\Leftarrow$). For R: each nonempty set is in some complex gives ($\Rightarrow$) and $[\nabla\exists]$ yields ($\Leftarrow$).[49]

This result explains the preceding example.[50]

We now examine the effect of adding a single simple generalised formula to a purely first-order theory. The first-order formulae that are consequences of such extensions have similar first-order characterisations.[51]

Proposition. First-order consequences of extension by simple generalised formula Consider a set $\Xi \subseteq \Phi_U$ of axioms for 'generally'. Given a set of sentences $\Delta \subseteq \mathrm{L}(\rho)$ and formulae ψ and θ of $\mathrm{L}(\rho)$, we have the following conditions.

$+$: $\Delta \cup \{\nabla v\psi\} \vdash^{\Xi} \theta$ iff $\Delta \cup \{\exists v\psi\} \vdash \theta$

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[49] For (I$\Rightarrow$): if $\Delta \nvdash \forall v\varphi$, then some model $\mathcal{M} \models \Delta$ can be expanded by an ultrafilter $\mathcal{U}$ with $\mathcal{M}[\varphi(\ s|v)] \notin \mathcal{U}$, and thus $\mathcal{M}^{\mathcal{U}} \not\models \nabla v\varphi[\ s]$. Similarly for (R$\Rightarrow$): if $\Delta \nvdash \neg\exists v\varphi$, then some model $\mathcal{M} \models \Delta$ can be expanded by an ultrafilter $\mathcal{U}$ with $\mathcal{M}[\varphi(\ s|v)] \in \mathcal{U}$, thus $\mathcal{M}^{\mathcal{U}} \not\models \neg\nabla v\varphi[\ s]$.

[50] In the preceding example, the axioms of Δ are $\neg S(h)$, $\exists v \neg v \approx h$ and $\forall v(\neg v \approx h \rightarrow S(v))$. Then, $\Delta \nvdash^{\Xi} \nabla v S(v)$ (since $\Delta \nvdash \forall v S(v)$) and $\Delta \nvdash^{\Xi} \neg\nabla v S(v)$ (since $\Delta \nvdash \neg\exists v\, S(v)$).

[51] To illustrate the next result, consider a purely first-order theory Δ about birds. Imagine that we wish to know what kind of support the belief "birds generally fly" (i. e. $\nabla v F(v)$) may provide to the first-order question θ: "birds have wings". If one accepts $\nabla v F(v)$, then one can conclude θ iff θ follows from Δ and the existential assertion $\exists v F(z)$: "some bird flies". If, on the other hand, one accepts the negation $\neg\nabla v F(v)$, then one can conclude θ iff θ follows from Δ and the existential assertion $\exists v \neg F(v)$: "some bird does not fly".

$- \ : \Delta \cup \{\neg \nabla v\psi\} \vdash^{\Xi} \theta$ iff $\Delta \cup \{\exists v \neg \psi\} \vdash \theta$

Proof. The conditions follow from the preceding result by contraposition.[52] The conditions seen so far apply to basic logic and its extensions. They

can be summarised and interpreted as follows. We already know that ∇ is between $\forall$ and $\exists$; now, a single initial ∇ will behave as either extreme: as $\forall$ (in the case of conclusion) or as $\exists$ (in the case of hypothesis). By examining more closely the expressive power of the generalised quantifier in 5.3, we will be able to see that such reduction of consequences to classical logic is restricted to simple generalised formulae, failing for other, more complex, formulae. We will now examine a more general case: sufficient and necessary conditions for inferring simple generalised formulae from the extension of a purely first-order theory by a single simple generalised formula.

Proposition. Simple generalised consequences of simple generalised extension Given a set $\Xi \subseteq \Phi_U$ of axioms for 'generally', consider a set of sentences Δ of $\mathrm{L}(\rho)$ and positive generalised formulae $\nabla v\psi$ and $\nabla v\theta$ of $\mathrm{L}^{\nabla}(\rho)$.

$++ \ : \Delta \vdash \exists v\psi \rightarrow \forall v\theta \Rightarrow \Delta \cup \{\nabla v\psi\} \vdash^{\Xi} \nabla v\theta \Rightarrow \Delta \vdash \forall v\psi \rightarrow \exists v\theta$

$+- \ : \Delta \vdash \exists v\psi \rightarrow \forall v \neg\theta \Rightarrow \Delta \cup \{\nabla v\psi\} \vdash^{\Xi} \neg\nabla v\theta \Rightarrow \Delta \vdash \forall v\psi \rightarrow \exists v\neg\theta$

$-+ \ : \Delta \vdash \exists v\neg\psi \rightarrow \forall v\theta \Rightarrow \Delta \cup \{\neg\nabla v\psi\} \vdash^{\Xi} \nabla v\theta \Rightarrow \Delta \vdash \forall v\neg\psi \rightarrow \exists v\theta$

$-- \ : \Delta \vdash \exists v\neg\psi \rightarrow \forall v\neg\theta \Rightarrow \Delta \cup \{\neg\nabla v\psi\} \vdash^{\Xi} \neg\nabla v\theta \Rightarrow \Delta \vdash \forall v\neg\psi \rightarrow \exists v\neg\theta$

Proof. Sufficiency follows from $[\forall\nabla]$ and $[\nabla\exists]$. For necessity: the universe is in every complex and the empty set is in no complex.[53]

The conditions considered this far apply to basic logic and its extensions. We will now examine necessary and sufficient conditions for inferring simple generalised formulae from the extension of a purely first-order theory by a single simple generalised formula. These conditions hold for specific extensions of basic logic.[54]

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[52] By contraposition: condition (R) yields $(+)$ and $(-)$ follows from (I).

[53] Sufficiency of $(++)$: if $\Delta \vdash \exists v\psi \rightarrow \forall v\theta$, then $\Delta \cup \{\nabla v\psi\} \vdash^{\Xi} \forall v\theta$ (by $[\nabla\exists]$), whence $\Delta \cup \{\nabla v\psi\} \vdash^{\Xi} \nabla v\theta$ (by $[\forall\nabla]$). Necessity of $(++)$: if $\Delta \nvdash \forall v\psi \rightarrow \exists v\theta$, then some model $\mathcal{M} \models \Delta$ can be expanded by an ultrafilter $\mathcal{U}$ so that $\mathcal{M}^{\mathcal{U}} \models \nabla v\psi[\, s]$ but $\mathcal{M}^{\mathcal{U}} \not\models \nabla v\theta[\, s]$, whence $\Delta \cup \{\nabla v\psi\} \nvdash^{\Xi} \nabla v\theta$. The other cases are similar.

[54] To illustrate the next result, consider purely first-order information Δ about workers in a plant. Assume that one observes that "workers generally are careless": $\nabla v C(v)$. One can then conclude (in up-closed logic) that "workers generally are accident prone" (i. e. $\nabla v A(v)$) iff Δ entails the universal assertion $\forall v[C(v) \rightarrow A(v)]$: "all careless workers are accident prone".

Theorem. Behaviour of simple generalised formulae (in specific logics) Given a set of sentences $\Delta \subseteq \mathrm{L}(\rho)$, consider positive generalised formulae $\nabla v\psi$ and $\nabla v\theta$ of $\mathrm{L}^{\nabla}(\rho)$. We then have the following conditions.[55]

$+I$: $\Delta \cup \{\nabla v\psi\} \vdash^{C} \nabla v\theta$ iff $\Delta \vdash \forall v(\psi \to \theta)$

$+R$: $\Delta \cup \{\nabla v\psi\} \vdash^{F} \neg\nabla v\theta$ iff $\Delta \vdash \neg\exists v(\psi \wedge \theta)$

$-I$: $\Delta \cup \{\neg\nabla v\psi\} \vdash^{U} \nabla v\theta$ iff $\Delta \vdash \forall v(\psi \vee \theta)$

$-R$: $\Delta \cup \{\neg\nabla v\psi\} \vdash^{C} \neg\nabla v\theta$ iff $\Delta \vdash \neg\exists v(\neg\psi \wedge \theta)$

Proof. In each case, specific schemata yield ($\Leftarrow$), for ($\Rightarrow$): if the first-order condition fails, some model of Δ can be expanded to an appropriate ultrafilter model falsifying the generalised inference.[56]

Let us now examine the case of extending a purely first-order theory by a set of simple generalised formulae. For this case, we also have necessary and sufficient conditions for inferring simple generalised formulae in specific logics for 'generally'.

Corollary. Behaviour of extensions by simple generalised formulae Given a set of sentences $\Delta \subseteq \mathrm{L}(\rho)$, consider a positive generalised formula $\nabla w\varphi$ of $\mathrm{L}^{\nabla}(\rho)$ and a set $\Gamma_{+} \subseteq \mathrm{L}^{\nabla}(\rho)$ of positive generalised formulae.

$\{+\}$ Then $\Delta\cup\Gamma_{+} \vdash^{F} \nabla w\varphi$ iff, for some finite subset $\{\nabla u_1\psi_1, \dots, \nabla u_m\psi_m\}$ of Γ_{+} and a new variable z,

$$\Delta \vdash \forall z[(\psi_1[u_1/z] \wedge \dots \wedge \psi_m[u_m/z]) \to \varphi[w/z]].$$

$\{+-\}$ Given also a set $\Gamma_{-} \subseteq \mathrm{L}^{\nabla}(\rho)$ of negative generalised formulae, we have $\Delta\cup\Gamma_{+}\cup\Gamma_{-} \vdash^{U} \nabla w\varphi$ iff, for some finite subsets $\{\nabla u_1\psi_1, \dots, \nabla u_m\psi_m\}$ of Γ_{+} and $\{\neg\nabla v_1\theta_1, \dots, \neg\nabla v_n\theta_n\}$ of Γ_{-}, and a new variable z,

$$\Delta \vdash \forall z[(\psi_1[u_1/z] \wedge \dots \wedge \psi_m[u_m/z] \wedge \neg\theta_1[v_1/z] \wedge \dots \wedge \neg\theta_n[v_n/z]) \to \varphi[w/z]].$$

Proof. Compactness gives the finite subsets and specific schemata reduce the finite case to the preceding result $(+I)$.[57]

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[55] In each case, the first-order condition is necessary in basic logic.

[56] For $(+I)$: schema $[\to \nabla]$ yields ($\Leftarrow$); for ($\Rightarrow$), if $\Delta \not\vdash \forall v(\psi \to \theta)$, then some model $\mathcal{M} \models \Delta$ can be expanded by an ultrafilter $\mathcal{U} \in \underset{\sim}{C}$ so that $\mathcal{M}^{\mathcal{U}} \models \nabla v\psi[\,s]$ but $\mathcal{M}^{\mathcal{U}} \not\models \nabla v\theta[\,s]$, whence $\Delta \cup \{\nabla v\psi\} \not\vdash^{C} \nabla v\theta$. The other cases are similar.

[57] For $\{+\}$: $\Delta \cup \{\nabla u_1\psi_1, \dots, \nabla u_m\psi_m\} \vdash^{F} \nabla w\varphi$ iff [by filter schemata], for a new variable z, $\Delta \cup \{\nabla z(\psi_1[u_1/z] \wedge \dots \wedge \psi_m[u_m/z])\} \vdash^{F} \nabla w\varphi[w/z]$ iff [by the preceding result $(+I)$] $\Delta \vdash \forall z[(\psi_1[u_1/z] \wedge \dots \wedge \psi_m[u_m/z]) \to \varphi[w/z]]$. The case of $\{+-\}$ is similar.

We also have conditions for refuting a simple generalised formula, [58]

4.3 Expressive power

We shall now consider the expressive power of our logics for 'generally'. We will examine some conditions for eliminating the new generalised quantifier from some simple formulae and then show that we have proper extensions of classical first-order logic.

One would expect our logics for 'generally' to be strictly more expressive than classical first-order logic.[59] We will see that interesting generalised formulae equivalent to purely first-order formulae are indeed quite rare.

Elimination of 'generally'

We will first examine some conditions for eliminating the new quantifier ∇ from simple generalised formulae.

As an example, consider a consistent purely first-order theory Δ about birds asserting that "Some birds fly" and "Some birds do not fly". In this case, we cannot express within Δ "Birds generally fly", as we will see.

The next result gives conditions for eliminating the single initial ∇ from a positive generalised formula within a purely first-order theory.

Proposition. Reduction of positive generalised formula to first-order Consider a set $\Xi \subseteq \Phi_U$ of axioms for 'generally'. Given a set of sentences $\Delta \subseteq \mathrm{L}(\rho)$ and a formula $\psi \in \mathrm{L}(\rho)$, there exists a formula $\theta \in \mathrm{L}(\rho)$ such that $\Delta \vdash^{\Xi} \nabla v\psi \leftrightarrow \theta$ iff $\Delta \vdash \exists v\psi \rightarrow \forall v\psi$.
Proof. The conditions follow from previous results: (I) and $(+)$ in 5.2.[60] Thus, one can eliminate the single initial ∇ from $\nabla v\psi$ only when formula ψ becomes trivialised: this explains the preceding example.[61]

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[58] $\Delta \cup \Gamma_+ \vdash^F \neg\nabla w\varphi$ iff, for a finite subset $\{\nabla u_1\psi_1, \ldots, \nabla u_m\psi_m\} \subseteq \Gamma_+$ and a new variable z, $\Delta \vdash \forall z[(\psi_1[u_1/z]\wedge\ldots\wedge\psi_m[u_m/z]) \rightarrow \neg\varphi[w/z]]$ and $\Delta\cup\Gamma_+\cup\Gamma_- \vdash^U \neg\nabla w\varphi$ iff, for finite subsets $\{\nabla u_1\psi_1, \ldots, \nabla u_m\psi_m\} \subseteq \Gamma_+$ and $\{\neg\nabla v_1\theta_1, \ldots, \neg\nabla v_n\theta_n\} \subseteq \Gamma_-$ and a new variable z, $\Delta \vdash \forall z[(\psi_1[u_1/z]\wedge\ldots\wedge\psi_m[u_m/z]\wedge\neg\theta_1[v_1/z]\wedge\ldots\wedge\neg\theta_n[v_n/z])\neg\varphi[w/z]]$.

[59] We know that satisfaction of a formula with the generalised quantifier ∇ depends on the complex, which is not the case for purely first-order formulae. So, it is to be expected that some formulae with ∇ will not be equivalent to formulae without ∇.

[60] Each result gives one direction of the desired equivalence.

[61] In the preceding example, as consequences of consistent Δ we have $\exists vF(v)$ and $\exists v\neg F(v)$, so $\Delta \nvdash \exists vF(v) \rightarrow \forall vF(v)$; thus the result shows that one cannot express $\nabla vF(v)$ without ∇.

Proper extension

We will now show the expressive power of our logics for 'generally' extends properly that of classical first-order logic.

As mentioned, one expects the expressive power of our logics for 'generally' to be strictly more than that of classical first-order logic. It remains to exhibit specific examples of formulae from which the new generalised quantifier cannot be eliminated.

We will exhibit a formula that cannot be expressed within classical first-order logic: the (single) ∇ of $\exists u \nabla zu \approx z$ cannot be eliminated.

Theorem. Non-eliminable ∇
Consider the sentence $\exists u \nabla zu \approx z$. Given a set $\Xi \subseteq \Phi_U$ of axioms for 'generally' and a set of sentences $\Delta \subseteq \mathrm{L}(\rho)$ with infinite models, there exists no sentence $\tau \in \mathrm{L}(\rho)$ such that $\Delta \vdash^{\Xi} \exists u \nabla zu \approx z \leftrightarrow \tau$.

Proof. This sentence expresses that the complex has a singleton, and an infinite universe has principal and non-principal ultrafilters.[62]

In particular, there is no sentence $\tau \in \mathrm{L}(\rho)$ such that $\emptyset \vdash^{\Xi} \exists u \nabla zu \approx z \leftrightarrow \tau$.

We mentioned (in 5.2) that the reduction of consequences to classical logic is restricted to simple generalised formulae. The above sentence provides examples where such reductions fail.

Let γ be the sentence $\exists u \nabla zu \approx z$. Consider the conditions in 5.2 for inferring a simple generalised formula from a purely first-order theory. In contrast to (*I*), we have no sentence τ of $\mathrm{L}(\rho)$, such that $\Delta \vdash^{\Xi} \gamma$ iff $\Delta \vdash \tau$, for every set $\Delta \subseteq \mathrm{L}(\rho)$ of sentences having infinite models (e. g. $\Delta := \emptyset$).[63]

5 Conclusion

Logics for 'generally' are intended to express assertions with some vague notions, such as 'generally', by means of new generalised quantifiers, and to reason about them. The primary motivation is logics for precise treatment

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[62] Sentence $\exists u \nabla zu \approx z$ expresses that the filter is generated by a singleton, and an infinite universe has both principal and non-principal ultrafilters [B+S'71].

[63] Such a purely first-order sentence τ would provide an elimination of ∇ from $\exists u \nabla zu \approx z$ within the first-order theory Δ. Also, in contrast to (+), given a sentence σ of $\mathrm{L}(\rho)$, we have no sentence τ of $\mathrm{L}(\rho)$, such that $\Delta \cup \{\gamma\} \vdash^{\Xi} \sigma$ iff $\Delta \cup \{\tau\} \vdash \sigma$, for every set of sentences $\Delta \subseteq \mathrm{L}(\rho)$ such that $\Delta \cup \neg\{\sigma\}$ has infinite models.

of some vague notions (such as 'generally', 'several', 'many', 'most', etc.), which appear often in ordinary language and in some branches of science, much as one has logics embodying some mathematical notions [B+F'85].

Here, we have reviewed some logics for 'generally' and examined some metamathematical issues about them: axiomatisations, behaviour of the new quantifier, as well as deductive and expressive powers.

We have seen that our logics for 'generally' are proper conservative extensions of classical first-order logic with sound and complete deductive systems. Also, our logics are proper extensions of classical first-order logic with compactness and Löwenheim-Skolem properties. This feature may confer to our logics for 'generally' some independent model-theoretic interest.[64]

We also have considered some other metamathematical properties of our logics for 'generally'. We have compared them to classical first-order logic, with focus on the behaviour of the new generalised quantifiers, pointing out similarities and contrasts. We have examined extensions of the classical square of oppositions covering the new generalised quantifiers and some transfer principles involving classical and generalised quantifiers.

We have considered inference of simple generalised formulae, examining simple cases of consequences that reduce to classical conditions. This has led to conditions for eliminating the new quantifier ∇ from simple generalised formulae. We have also established the expressive power of our logics for 'generally' extends properly that of classical first-order logic by exhibiting formulae that cannot be expressed within classical first-order logic: with non-eliminable ∇.

We thus have logics for reasoning precisely about some versions of 'generally'. These logics are conservative extensions of classical first-order logic, with which they share various properties. This family of logics is undergoing further investigation [V+C'01,V+V'01, V+V'01a, RHV'01, Vel'02, Vel'02a, Vel'02b, V+V'02, V+V'02a].[65] They appear to have some interesting connections with fuzzy logic as used in expert systems, natural language and empirical reasoning. Such connections suggest the possibility of other applications ([C+V, Vel'98, Vel'99, V+C'01]).

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[64] The apparent conflict with Lindström's results ([Lin'66], [Bar'77]) is explained because we are using a non-standard notion of model (due to the complexes).

[65] These developments include proof methods and sorted versions (to express relative 'generally', since relativisation fails to express the intended meaning, due to properties of ∇ and $\rightarrow$, [C+V'97, V+C'01])

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On Reasoning about 'Generally' and 'Rarely' with Filter-like Family of Sets[1]

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Abstract

We examine some issues in reasoning about 'generally' and 'rarely'. The primary motivation is a precise qualitative approach to assertions and arguments involving such vague notions, which occur often in ordinary language and in some branches of science. We focus mainly on the intended meanings of such assertions and analyse some basic intuitions and their underlying presuppositions. This leads to distinguishing various versions according to their behaviour, which can be explained by means of filter-like families of sets. Such families provide bases for precise qualitative reasoning about some vague notions.

Keywords. Vague notions, generally, rarely, precise reasoning, families of sets, filter, ideal, logics for vague notions, several, many, most,few

Outline

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1 Introduction

In this paper we discuss, trying to explain, some fundamental issues in the precise treatment of assertions involving 'generally' and 'rarely'. We hope to clarify the role played by families of subsets in this context.[3]

Some vague notions occur often in ordinary language and in some branches of science and it would be desirable to reason about assertions involving them in a precise manner. The overall aim is having logics for (some versions of) these vague notions.[4] We shall focus mainly on the intended meanings of such assertions. By analysing some basic intuitions and their underlying presuppositions, we are led to distinguishing various versions according to their behaviour, which can be explained by means of families of sets. Such families, in turn, provide bases for precise qualitative reasoning about assertions involving some vague notions like 'generally' and 'rarely'.

Assertions and arguments involving some vague notions, such as 'generally', 'rarely', 'several', 'few', 'many', 'most', etc., occur often, both in ordinary language and in some branches of science. For instance, one often encounters assertions such as "Many bodies expand when heated", "Most birds fly" and "Few metals are liquid under ordinary conditions".[5] The assertions "Whoever likes sports watches Sports-TV" and "Boys generally like sports" appear to lead to "Boys generally watch Sports-TV". Such

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[3] A preliminary version of part of this exposition was presented at the II Simpósio Internacional Principia, held at Florianópolis in August 2002 (cf. [Vel'02]).

[4] We would like to have logics for some vague notions, much as one has logics embodying some mathematical notions ([B+F'85], p. 3).

[5] Such notions may also be useful in reporting experimental set-ups and results. More elaborate expressions are also used: a physician may say that a patient's background indicates a certain propensity, making him or her prone to some ailments.

qualitative arguments involving these vague notions appear frequently in several walks of life.[6]

Ideas concerning these notions have appeared in the literature. Some traditional square-of-opposition relations among 'few', 'many', and 'most' have been analysed [Pet'79] and a quantifier for 'most' in the sense of majority has been suggested [Res'62, Sla'88]. Systems with various generalised quantifiers, for notions such as 'many', 'few', 'most', etc., have been considered to be appropriate to treat quantified sentences in natural language (cf. [B+C'81], [Gra'03]). These works are also related to the tradition of analysis and formalisation of language [Fre'79, Tar'36, Chu'56, Mon'74].

We wish to reason about assertions involving vague notions in a precise manner. Here one may feel a certain tension. On the one hand, one needs a clear understanding of 'generally' and 'rarely' for precise reasoning; on the other hand, these notions appear to be quite vague.[7]

Our approach here will be as follows. We will first examine some intuitions behind 'generally' and 'rarely', which will indicate that we actually have various distinct versions of these vague notions. We will then consider abstract versions of these intuitions aiming at a unified treatment, which will in turn suggest how one can handle (some versions of) these vague notions by means of families of subsets. We will finally indicate how these ideas can be used to provide bases for logical systems.

2 Some accounts for 'generally' and 'rarely'

Various possible interpretations seem to be associated with vague notions 'generally' and 'rarely'. We shall consider some reasonable ones and examine some intuitions underlying them.

Consider assertions of the form "objects generally have a given property" and "objects rarely have a given property". How is one to understand these assertions? What would be the possible grounds for accepting them? We shall now examine some answers to these questions stemming from possible accounts for versions of 'generally' and 'rarely'.

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[6] A medical doctor usually prescribes a treatment considering it appropriate to a typical patient with such symptoms.

[7] Arguments involving such notions have been considered to be "unruly" to logical methods (cf., e. g., [Tou'58], p. 149). Vagueness is a source of controversies (cf., e. g., [Fin'75], [Wri'75], [Eva'78], [Pea'81]).

2.1 Numerical accounts for 'generally'

Some accounts for 'generally' try to explain it in terms of relative frequency or size.

For instance, consider the assertion "Brazilians generally like soccer". A relative-frequency account for it may be "The Brazilians that like soccer form a 'likely' portion", with more than, say, 75 % of the population.

Now, consider the assertion "Viennese generally like music". A size-based account for it might be "The Viennese that like music form a 'sizeable' set", in the sense that their number is above, say, 1 million.[8]

These two accounts of 'generally' are quite similar.[9] These two accounts may be termed "metric", as try to reduce it to a measurable aspect, so to speak. They seek to explicate "people generally have a property φ" as "the people having φ form a 'likely' (or 'sizeable') set", i. e. a set having "high" relative frequency (or cardinality), where 'high' is understood as above a given threshold.

These two metric accounts, however, differ in one important aspect, related to invariance. This can be seen by considering the relation of having the same size. On the one hand, the size accounts - cardinality above a given threshold - clearly fail to distinguish sets with the same cardinality: they are all either above or below the threshold. We may say that we have a non-local notion. In contrast, sets with the same size may very well have distinct probabilities.[10] Thus, in a probabilistic account of 'generally', the family of 'likely' sets, is not necessarily invariant under having the same size. It may be said to correspond to a local notion.

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[8] Notice that this threshold may depend on the person. Other persons may be inclined to use other thresholds and accept that a set of Viennese is 'sizeable' when its size exceeds 1.2 million or, say, 0.8 million.

[9] A size-based account for "Brazilians generally like soccer" might be "The Brazilians that like soccer form a 'sizeable' set": their number is above, say, 80 million (Brazil has about 170 million inhabitants). Also, a relative-frequency account for "Viennese generally like music" may be "The Viennese that like music form a 'likely' portion", with more than, say, 70 % of the population. Here, we use 'Brazilian' and 'Viennese' as inhabitant of Brazil and of Vienna, respectively.

[10] For instance, consider the even and odd naturals with probability 1/2; these sets have the same cardinality as the whole set of naturals (which has probability 1). Indeed, any infinite universe V can be partitioned as the union of two sets X and Y, both with the same cardinality as V; so V, X and Y cannot have all the same probability, even though they have the same size. For a finite universe, it suffices to consider a non-uniform distribution.

2.2 Relaxed accounts for 'rarely' and 'generally'

The preceding accounts hinge on assigning a threshold, which may seem somewhat arbitrary. Even though they may suffice for some situations, such approaches do not appear to be appropriate for other cases, where they may fail to clarify the underlying intuitions.

We shall now examine some more relaxed accounts. For instance, consider the assertion "Natural numbers rarely divide sixty". One may interpret, and explain, it by regarding it as asserting that "the divisors of sixty form a 'small' set", where 'small' is understood as finite. Similarly, one would understand the assertion "Real numbers rarely are rational" as "the rational reals form a 'small' set", with 'small' now taken as (at most) denumerable.

This account of 'rarely' is still quantitative and resorts to a threshold, but it is more relaxed. It tries to explicate "objects rarely have property φ" as "the objects having φ form a 'small' set", under a given sense of 'small' (capturing some idea of "having 'very few' elements").

The intended meaning of "objects generally have property φ" can also be given by means of the set of exceptions, i. e. those objects failing to have property φ. One may understand "Eagles generally fly" as "The non-flying eagles form a 'small' set", which suggests paraphrasing it as "Eagles rarely fail to fly" or "Eagles rarely are non-flying (birds)".

To illustrate some features of this relaxed account in contrast to the metric accounts, consider the universe of natural numbers and imagine that one accepts the following assertions:

- θ: "Natural numbers rarely are below thirteen",
- δ: "Natural numbers rarely divide twelve".

In this case, one would probably accept also the assertions:

- γ: "Naturals rarely are below thirteen and even",
- η: "Naturals rarely are below thirteen or divide twelve".

The acceptance of the first two assertions, as well as inferring γ from them, might be explained by a metric account as above. This, however, does not seem to be the case with assertion η.[11]. The more relaxed account can explain this situation, as we shall have occasion to see in section 4.

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[11] For instance, considering the universe of Brazilians, those liking basketball and those liking volleyball may have relative frequency below the threshold (of, say, 70 %), but those liking basketball or volleyball may happen to exceed the threshold.

2.3 Qualitative accounts for 'generally' and 'rarely'

The accounts of 'generally' and 'rarely' mentioned so far may be termed "quantitative". Even though they may suffice for various cases, such accounts do not seem to cover some situations, where these notions appear to present a qualitative character.

As an example, consider the assertion "Real numbers generally are rational". How is one to understand this assertion? What would be the possible grounds for accepting it? The rationals do not seem to form a "likely", "sizeable" or "large" set of reals in a quantitative sense: there are too few of them.[12] Yet, there seems to be a sense in which one may accept that' 'Real numbers generally are rational". Indeed, one may say that "the rationals are 'almost everywhere' within the reals", since near any real one finds a rational. In this sense, the rationals may be said to be "ubiquitous" within the reals [Gra'99, C+G'00]. More precisely, in any open neighbourhood of a real one finds a rational, thus the rational reals form a dense set of reals.

This example illustrates a local qualitative notion of 'generally'. One explicates "objects generally have property φ" by saying that "the set of objects having property φ is a dense set" in a given topology.[13]

We can perhaps distinguish the earlier quantitative accounts from the more flexible qualitative accounts in terms of the properties stressed. They are of a topological nature in the latter, rather than metrical as in the former. We can also see that the earlier quantitative versions can be subsumed under the more flexible qualitative notions.

2.4 Abstract versions: 'important' and 'negligible'

We now consider some abstract versions of 'generally' and 'rarely'.

We have seen various distinct notions of 'generally' and 'rarely'. In the accounts examined, the intended meaning of "objects generally have a given property φ" and of "objects rarely have property φ" can be given in terms of sets of objects: of those objects having φ and of those objects failing to have φ, respectively.

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[12] Indeed, it is the assertion "Real numbers generally are irrational" - in the sense "Real numbers rarely are rational" - that appears to be more reasonable, as explained above (in 2.2)

[13] A *dense* subset of the universe, in a given topology, is one having the universe as its closure (or equivalently, one intersecting every non-empty open set) [Kel'55].

We would like to give a unified treatment covering these various distinct notions of 'generally' and 'rarely'. For this purpose, we shall prefer to employ more neutral names encompassing these notions: we will use 'important' in lieu of 'sizeable', 'likely' or 'large' (corresponding to 'generally'), and, accordingly 'negligible' for 'non-sizeable', 'unlikely' or 'small' (corresponding to 'rarely').

The previous terms are somewhat vague, the more so with the new ones. Nevertheless, they present some advantages. First, the reliance on a - somewhat arbitrary - threshold is less stringent. Also, they have a wider range of applications, stemming from the somewhat liberal interpretation of 'important' as carrying considerable weight or importance. For instance, when saying "Unimportant meetings are those attended only by junior staff", one seems to be considering sets including only junior staff members as 'unimportant'.[14]

Notice that these notions of 'important' and 'negligible' are relative to the situation or person, as suggested by the examples.[15]

We have seen some ways of understanding, and explaining, these vague notions. We may summarise these various accounts of 'generally' and 'rarely' as follows.

1. Numeric accounts by 'likely' (high relative-frequency), e. g., "Brazilians generally like soccer" as "More than 75 % of the Brazilians like soccer".

2. Numeric accounts by 'sizeable' (large size), e. g., "Viennese generally like music" as ' 'The Viennese liking music are more than 1 million".

3. Relaxed quantitative accounts by 'small' (cardinality), e. g., "Natural numbers rarely divide sixty" as "The divisors of sixty form a finite set".

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[14] Another example is "Important parties are those attended by the celebrities" (cf. 4.1 in section 4).

[15] Further examples illustrating these points are as follows [Vel'99]. First, consider two sets with the same size: one consisting of a horse and an ox, and another one consisting of a horse and a dog. These sets may be just as important to a conservationist. But, the former may be more important to a farmer, whereas the latter might be preferred by an English gentleman, keen on fox hunting. Now, consider two sets with distinct sizes: one consisting of thirty birds, and another one consisting of a couple of elephants. The Zoo director is likely to consider them equally important. But, an ornithologist might rank the former as more important, whereas a truck driver in charge of transporting them would probably give more attention to the latter. So, a smaller set may be more important than a larger set, or just as important.

4. Qualitative accounts by ‘ubiquitous’ (dense set), e. g., “Real numbers generally are rational” as “The rationals are almost everywhere within the reals”.
5. Abstract accounts by‘important’/‘negligible’, e. g., “Unimportant meetings are those attended only by junior staff”.

These accounts differ with respect to their reliance on a threshold as well as invariance (local or non-local notions). We may summarise these features in the following table.

Account	Reading	Threshold	Invariance
frequency	likely	+	N
size	sizeable	+	Y
relaxed	small	±	Y
qualitative	ubiquitous	−	N
abstract	important	−	N

3 Families for ‘generally’ and ‘rarely’

We will now examine how one can handle (some versions of) vague notions like ‘generally’ and ‘rarely’ by means of families of subsets.

Various possible interpretations can be associated to the vague notions of ‘generally’ and ‘rarely’. We would like to give a unified treatment for (some of) them. We might say that we really have a family of notions and we attempt to describe some of their common properties. Towards this goal, we shall try to explain these notions by relying on a relation comparing subsets of a given universe.

As a first candidate for such a comparison one might consider the relation $\simeq$ of “having about the same size”. It is tempting to consider that we have an equivalence relation. Indeed, reflexivity and symmetry seem reasonable; but, what about transitivity?[16] Actually, a notion such as “having about the same size” is not such a good starting point. This is so because one is naturally led to think that sets with the same size should have about

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[16] Concerning transitivity: are we prepared to accept that the extremes X_0 and X_n of a long chain $X_0 \simeq X_1 \simeq \ldots \simeq X_n$ still have about the same size? Even though adjacent sets may differ by a very small amount, the extremes may differ substantially. Transitivity of vague relations is connected to the so-called sorites paradoxes [Sai’89, Edw’72]: my age a second ago and now are practically the same, but I am definitely quite older than when I was born.

the same size. In other words, this is a non-local notion, whereas some of our notions are, in contrast, local ones.

In view of the preceding considerations, we shall use 'roughly less important than' for our comparison between subsets of a universe V, which we shall denote by $\sqsubseteq$.[17] We shall try to explicate our notions of 'important' and 'negligible' by relying on some reasonable properties of this relation $\sqsubseteq$ between subsets of a given universe. Also, instead of assuming at the outset that we have an equivalence relation, we shall put forward some more basic - and hopefully more palatable - postulates. (This enterprise is somewhat reminiscent of that of "reverse mathematics", with an important difference.[18])

3.1 Basic ideas

We now have three vague notions, namely the properties 'negligible' and 'important', as well as the binary relation 'roughly less important than'. We shall attempt to explain them by means of some properties of these notions, based mainly on common sense and ordinary understanding.

In the sequel, we will examine properties connecting the vague notions 'negligible', 'important' and 'roughly less important than'. Given a universe V, we consider the families $\mathcal{N}$, of negligible subsets, and $\mathcal{W}$, of important subsets, of universe V.[19] We shall postulate some reasonable properties connecting these families and the dominance relation $\sqsubseteq$ of 'roughly less important than'. The relative character of 'negligible' and 'important' is embodied in these families and in properties of the binary relation $\sqsubseteq$ of 'roughly less important than', which may vary according to the situation. They, however, can be expected to share some general properties, if they are to be appropriate for capturing reasonable notions of 'generally' (and 'rarely'), corresponding to 'sizeable', 'likely' or 'large' (and 'non-sizeable', 'unlikely' or 'small').

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[17] Previous presentations relied on a more symmetric relation ('almost as important as') for comparing subsets of the universe [Vel'99, Vel'02]. The present organisation is more modular and incorporates considerable improvements over the previous approaches (see also 3.5).

[18] "The fundamental question in reverse mathematics is to determine which set existence axioms are required to prove particular theorems of mathematics" ([Sol'99], p. 45). Here, instead of locating familiar axioms, we will be suggesting some new postulates, whence the need for justifying their acceptance on intuitive grounds.

[19] Notice that it is the properties 'negligible' and 'important' as well as the binary relation that are somewhat vague; the subsets of universe are usual (non-vague) sets (rather than, say, fuzzy sets).

We find convenient to divide our postulates into two groups, namely common postulates and specific postulates, depending on whether the properties they express may be expected to be fundamental to our notions or shared only by some special versions of them.

3.2 Common postulates

We shall first consider our basic postulates, expressing properties that may be expected to be common to our vague notions.

We shall be resorting to two kinds of arguments, namely intuitive arguments - based mainly on common sense and ordinary understanding - to try to justify the acceptance of the proposed postulates, as well as (simple) mathematical proofs, to derive - as consequences of our postulates - some properties (that seem to be intuitively expected).

A dictionary explanation for 'negligible' is: "Something that is negligible is so small or unimportant that is not worth considering or worrying about" [Col'87].[20] Also, one usually understands 'negligible' as "fit to be neglected or discarded" [Web'70].

Our idea of a set being 'roughly less important than' another set is "being practically within": except for a part that may be discarded, the former is within the latter. These explanations suggest that it appears reasonable to say that "a set is practically within another when the difference between the former and the latter is negligible".

We are thus led to formulate our first basic postulate, explicating dominance in terms of the family of negligible subsets.

1. *Explicate dominance in terms of negligible*
For subsets S and T of the universe V: S is dominated by T iff the difference $S - T$ is negligible.[21]
$[\sqsubseteq \mathcal{N}]$: $S \sqsubseteq T \Leftrightarrow (S - T) \in \mathcal{N}$

We can now see some immediate - and intuitively reasonable - consequences of this basic postulate $[\sqsubseteq \mathcal{N}]$.

First, complementation reverses dominance: if $S \subseteq V$ is "practically within" $T \subseteq V$, then $\overline{T} \subseteq V$ is "practically within" $\overline{S} \subseteq V$.[22]

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[20] Notice that this explanation already suggests a connection between 'negligible' and 'important'.

[21] The difference (or relative complement) of two sets consists of the elements in one set but not in the other: $S - T := \{x \in V : x \in S \& x \notin T\}$.

[22] The (absolute) complement of a subset of the universe V consists of the elements (of the universe) outside the set: $\overline{S} := \{x \in V : x \notin S\}$.

1.a *Behaviour of dominance under complementation*
For subsets S and T of V: S is dominated by T iff the complement of S dominates the complement of T.
($\sqsubseteq^c$): $S \sqsubseteq T \Leftrightarrow \overline{T} \sqsubseteq \overline{S}$[23]

Second, the negligible subsets can be characterised as those "practically within" the empty set.

1.b *Characterisation of negligible subsets*
A subset $S \subseteq V$ is negligible iff S is dominated by the empty set.
($\mathcal{N} \sqsubseteq \emptyset$): $S \in \mathcal{N} \Leftrightarrow S \sqsubseteq \emptyset$[24]

Our second postulate is suggested by the above consequence 1.b: ($\mathcal{N} \sqsubseteq \emptyset$), which characterises the negligible subsets as those dominated by the empty set: those "practically within" the empty set. In a dual manner, we would expect the important subsets to be those with the universe "practically within", i. e. those dominating the universe.

2. *Explicate important as dominating the universe*
A subset $T \subseteq V$ is important iff T dominates the universe V.
[$V \sqsubseteq \mathcal{W}$]: $T \in \mathcal{W} \Leftrightarrow V \sqsubseteq T$

Our intuitive ideas about the notions of 'negligible' and 'important' suggest that an important subset has negligible complement. This duality is equivalent to 2: [$V \sqsubseteq \mathcal{W}$] in view of our first postulate [$\sqsubseteq \mathcal{N}$].

2.a *Duality of negligible and important under complementation*
A subset $S \subseteq V$ is negligible iff its complement $\overline{S} \subseteq V$ is important.
($\mathcal{N}^c\mathcal{W}$): $S \in \mathcal{N} \Leftrightarrow \overline{S} \in \mathcal{W}$[25]

Our relation of dominance might be trivial in two respects: no dominance or dominance for any pair of subsets. This is not what one would expect and our next two postulates concern aspects of non-triviality of dominance.

We still do not know whether there is any dominance of subsets. Our intuitive ideas about dominance indicate that the empty set is (practically) within the universe. This is the content of our next postulate.

3. *Empty set dominated by universe*
The empty set $\emptyset$ is dominated by the universe V.
[$\emptyset V$]: $\emptyset \sqsubseteq V$

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[23] Consequence 1.a: ($\sqsubseteq^c$) follows from 1: [$\sqsubseteq \mathcal{N}$], since $\overline{T} - \overline{S} = S - T$.

[24] Consequence 1.b: ($\mathcal{N} \sqsubseteq \emptyset$) follows from 1: [$\sqsubseteq \mathcal{N}$], since $S - \emptyset = S$.

[25] Consequence 2.a: ($\mathcal{N}^c\mathcal{W}$) is equivalent to 2: [$V \sqsubseteq \mathcal{W}$], by 1.a: ($\sqsubseteq^c$) and 1.b: ($\mathcal{N} \sqsubseteq \emptyset$), since $S \in \mathcal{N}$ iff $S \sqsubseteq \emptyset$ iff $\overline{\emptyset} \sqsubseteq \overline{S}$ iff $\overline{S} \in \mathcal{W}$.

We can now see some immediate - and reasonable - consequences of our basic postulates so far.

Our intuitive ideas about the notion of 'negligible' suggest that the empty set is (most) negligible (and, dually, the universe is (most) important). Our basic postulates corroborate these ideas.

3.a *Empty set negligible*
The empty set $\emptyset$ is negligible.
$(\emptyset\mathcal{N})$: $\emptyset \in \mathcal{N}$[26]

Our intuitive ideas suggest that a subset of a set is (practically) within the set. This is the content of our next consequence.

3.b *Subset is dominated*
For subsets S and T of the universe V: if S is a subset of T then S is dominated by T.
$(\subseteq\sqsubseteq)$: $S \subseteq T \Rightarrow S \sqsubseteq T$[27]

We are also led to accept a set as (practically) within itself, as expressed by our next consequence.

3.c *Reflexivity of dominance*
Each subset $S \subseteq V$ dominates itself.
$(=\sqsubseteq)$: $S \sqsubseteq S$[28]

As our next consequence, we have the empty set and the universe as the extremes of dominance.

3.d *Extremes of dominance*
Each subset $S \subseteq V$ dominates the empty set $\emptyset$ and is dominated by the universe V.
$(\bot\top)$: $\emptyset \sqsubseteq S$ & $S \sqsubseteq V$[29]

For all we know so far, our relation of dominance might collapse many subsets. Our intuitive ideas about dominance suggest that the (nonempty) universe is not (practically) within the empty set. This is the content of our fourth postulate.

4. *Universe not dominated by empty set*
The universe V is not dominated by the empty set $\emptyset$.

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[26] Consequence 3.a: $(\emptyset\mathcal{N})$ is equivalent to 3: $[\emptyset V]$ under 1: $[\sqsubseteq \mathcal{N}]$, since $\emptyset - V = \emptyset$. Clearly, $V \in \mathcal{W}$ is equivalent to $\emptyset \in \mathcal{N}$ in view of the duality 2.a: $(\mathcal{N}^c\mathcal{W})$.

[27] Consequence 3.b: $(\subseteq\sqsubseteq)$ follows from 1: $[\sqsubseteq \mathcal{N}]$ and 3.a: $(\emptyset\mathcal{N})$, as $S - T = \emptyset$, when $S \subseteq T$.

[28] Consequence 3.c: $(=\sqsubseteq)$ follows immediately from 3.b: $(\subseteq\sqsubseteq)$ (as $S \subseteq S$).

[29] Consequence 3.d: $(\bot\top)$ follows immediately from 3.b: $(\subseteq\sqsubseteq)$ (as $\emptyset \subseteq S \subseteq V$).

$[V\emptyset]$: $V \not\sqsubseteq \emptyset$

Our intuitive ideas about the notion of 'negligible' suggest that the universe is not negligible (and, dually, the empty set is not important). Our basic postulates corroborate these ideas yielding non-triviality of our families: the existence of non-negligible (and non-important) subsets of the (nonempty) universe.

4.a *Universe not negligible*
The universe V is not negligible.
$(V\mathcal{N})$: $V \notin \mathcal{N}$[30]

Summarising, we have four basic postulates (cf. table 1).

1	$\sqsubseteq\mathcal{N}$	$S \sqsubseteq T \Leftrightarrow (S - T) \in \mathcal{N}$	characterise $\sqsubseteq$ via $\mathcal{N}$
2	$V \sqsubseteq \mathcal{W}$	$T \in \mathcal{W} \Leftrightarrow V \sqsubseteq T$	characterise $\mathcal{W}$ via $\sqsubseteq$
3	$\emptyset V$	$\emptyset \sqsubseteq V$	dominance
4	$V\emptyset$	$V \not\sqsubseteq \emptyset$	non-dominance

Table 1: Basic postulates for dominance and families

These four basic postulates have some reasonably acceptable consequences (cf. table 2).

1.a	$\sqsubseteq^c$	$S \sqsubseteq T \Leftrightarrow \overline{T} \sqsubseteq \overline{S}$	$\sqsubseteq$ under complement
1.b	$\mathcal{N} \sqsubseteq \emptyset$	$S \in \mathcal{N} \Leftrightarrow S \sqsubseteq \emptyset$	$\mathcal{W}$ as dominated by $\emptyset$
2.a	$\mathcal{N}^c\mathcal{W}$	$S \in \mathcal{N} \Leftrightarrow \overline{S} \in \mathcal{W}$	$\mathcal{N}$ as complements of $\mathcal{W}$
3.a	$\emptyset\mathcal{N}$	$\emptyset \in \mathcal{N}$	void set is negligible
3.b	$\subseteq\sqsubseteq$	$S \subseteq T \Rightarrow S \sqsubseteq T$	subset dominated
3.c	$=\sqsubseteq$	$S \sqsubseteq S$	dominance is reflexive
3.d	$\bot\top$	$\emptyset \sqsubseteq S \sqsubseteq V$	$\emptyset$ & V: extremes of $\sqsubseteq$
4.a	$V\mathcal{N}$	$V \notin \mathcal{N}$	universe not negligible

Table 2: Consequences of the basic postulates

Thus, these basic postulates, expressing reasonable assumptions about our three vague notions, lead to the some basic properties of the families $\mathcal{N}$, of negligible subsets, and $\mathcal{W}$, of important subsets, of a (nonempty) universe V. These families $\mathcal{N}$ and $\mathcal{W}$ are

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[30] Consequence 4.a: $(V\mathcal{N})$ is equivalent to 4: $[V\emptyset]$ under 1.b: $(\mathcal{N} \sqsubseteq \emptyset)$. Clearly, $\emptyset \notin \mathcal{W}$ is equivalent to $V \notin \mathcal{N}$ in view of the duality 2.a: $(\mathcal{N}^c\mathcal{W})$.

- dual ($S \in \mathcal{N}$ iff $\overline{S} \in \mathcal{W}$);
- proper ($\emptyset \in \mathcal{N}$ & $V \notin \mathcal{N}$; $\emptyset \notin \mathcal{W}$ & $V \in \mathcal{W}$).

3.3 Specific postulates

We shall now consider our specific postulates, expressing properties that may be expected to be shared only by some notions corresponding to generally' and 'rarely'. By duality, each property about a family has its dual version concerning the other family.

These specific postulates will be of a special nature. So, we will introduce them from a somewhat algebraic viewpoint. We shall present some more intuitive reasons for accepting them later on (in 3.5).

Our first specific postulate concerns the behaviour of dominance under union. If two subsets of the universe are practically within a common subset, it may be reasonable to expect that so is their union.

A. *Dominance under union*
If a subset $R \subseteq V$ dominates subsets P and Q of the universe V, then R also dominates their union $P \cup Q$.
$[\cup \sqsubseteq]$: $P \sqsubseteq R \& Q \sqsubseteq R \Rightarrow P \cup Q \sqsubseteq R$

A consequence of this specific postulate A: $[\cup \sqsubseteq]$ (actually an equivalent formulation for it) is the following behaviour of the family of negligible subsets under union.

A'. *Closure of negligible subsets under union*
The union $N' \cup N"$ of negligible subsets N' and $N"$ is also negligible.
$(\mathcal{N}\cup)$: $N' \in \mathcal{N} \& N" \in \mathcal{N} \Rightarrow N' \cup N" \in \mathcal{N}$[31]

Consequence A': $(\mathcal{N}\cup)$ is equivalent by duality 2.a: $(\mathcal{N}^c\mathcal{W})$ to the closure of the family $\mathcal{W}$ of important subsets under intersection.

Our second specific postulate concerns the behaviour of dominance under intersection. Given two dominance relations, one may find reasonable to expect that the intersection of the lower subsets is dominated by the union of the upper subsets.

B. *Dominance under intersection*
For subsets P, Q, S and T of the universe V: if P is dominated by S and Q is dominated by T, then the intersection $P \cap Q$ is dominated by the union $S \cup T$.

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[31] Consequence A': $(\mathcal{N}\cup)$ is equivalent to A: $[\cup \sqsubseteq]$ under 1: $[\sqsubseteq \mathcal{N}]$, in view of the equality $(P \cup Q) - R = (P - R) \cup (Q - R)$.

$[\cap \sqsubseteq]$: $P \sqsubseteq S \& Q \sqsubseteq T \Rightarrow P \cap Q \sqsubseteq S \cup T$

A consequence of this specific postulate $[\cap \sqsubseteq]$ (actually equivalent to it) is the following behaviour of the family of negligible subsets under intersection.

B'. *Closure of negligible subsets under intersection*
The intersection $N' \cap N"$ of negligible subsets N' and $N"$ is negligible.
$(\mathcal{N}\cap)$: $N' \in \mathcal{N} \& N" \in \mathcal{N} \Rightarrow N' \cap N" \in \mathcal{N}$[32]

Much as before, this consequence B': $(\mathcal{N}\cap)$ is equivalent by duality 2.a: $(\mathcal{N}^c\mathcal{W})$ to the closure of the family $\mathcal{W}$ of the important subsets under union.

Our third specific postulate concerns the behaviour of dominance under inclusion. It may be reasonable to expect that, when a set is practically within another, the same will happen to each subset of the former.

C. *Dominance under inclusion*
For subsets Q and R of the universe V: if set R dominates set Q, then R will also dominate every subset $P \subseteq Q$.
$[\subseteq\sqsubseteq]$: $P \subseteq Q \& Q \sqsubseteq R \Rightarrow P \sqsubseteq R$

As a consequence of this specific postulate C: $[\subseteq\sqsubseteq]$ (actually an equivalent formulation), we have the following behaviour of the family of negligible subsets under inclusion.

C'. *Closure of negligible subsets under subset*
Each subset of a negligible subset is also negligible.
$(\mathcal{N} \subseteq)$: $S \subseteq N \& N \in \mathcal{N} \Rightarrow S \in \mathcal{N}$[33]

We notice that postulate C: $[\subseteq\sqsubseteq]$ is stronger than postulate B: $[\subseteq\sqsubseteq]$ (as a family closed under subsets must be closed under intersection). Also, as before by duality 2.a: $(\mathcal{N}^c\mathcal{W})$, this consequence C': $(\mathcal{N} \subseteq)$ is equivalent to the closure of the family $\mathcal{W}$ of important subsets under supersets.

Our fourth specific postulate concerns viewing negligible and important as alternatives. One may be willing to accept that a subset of the universe must be negligible or important.

D. *Alternatives negligible or important*
If a subset $S \subseteq V$ is not negligible, then it must be important.
$[\mathcal{N}|\mathcal{W}]$: $S \notin \mathcal{N} \Rightarrow S \in \mathcal{W}$

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[32] Consequence B': $(\mathcal{N}\cap)$ is equivalent to B: $[\subseteq\sqsubseteq]$ under 1: 1: $[\sqsubseteq \mathcal{N}]$, as $\emptyset \cap \emptyset = \emptyset$ and $(P \cap Q) - (S \cup T) = (P - S) \cap (Q - T)$.

[33] Consequence C': $(\mathcal{N} \subseteq)$ is equivalent to C: $[\subseteq\sqsubseteq]$ under 1: $[\sqsubseteq \mathcal{N}]$, since $P - R \subseteq Q - R$, when $P \subseteq Q$.

A consequence of this specific postulate $[\mathcal{N}|\mathcal{W}]$ (actually equivalent to it) by duality 2.a: $(\mathcal{N}^c\mathcal{W})$) is the following behaviour of the family of negligible subsets under complementation.

D'. *Negligible subsets under complementation*
If a subset $S \subseteq V$ is not negligible, then its complement $\overline{S} \subseteq V$ is negligible.
$(\mathcal{N}^c)$: $S \notin \mathcal{N} \Rightarrow \overline{S} \in \mathcal{N}$

Much as before, this consequence D': $(\mathcal{N}^c)$ is equivalent to the family $\mathcal{W}$ of important subsets being prime: if a subset $T \subseteq V$ is not important, then its complement $\overline{T} \subseteq V$ is important.

Summarising, we have four specific postulates, with equivalent formulations (cf. table 3).

A	$\cup\sqsubseteq$	$P \sqsubseteq R \& Q \sqsubseteq R \Rightarrow P \cup Q \sqsubseteq R$	$\sqsubseteq$ under union $\cup$
A'	$\mathcal{N}\cup$	$N', N" \in \mathcal{N} \Rightarrow N' \cup N" \in \mathcal{N}$	$\mathcal{N}$ $\cup$-closed
B	$\cap\sqsubseteq$	$P \sqsubseteq S \& Q \sqsubseteq T \Rightarrow P \cap Q \sqsubseteq S \cup T$	$\sqsubseteq$ under intersection $\cap$
B'	$\mathcal{N}\cap$	$N', N" \in \mathcal{N} \Rightarrow N' \cap N" \in \mathcal{N}$	$\mathcal{N}$ $\cap$-closed
C	$\subseteq\sqsubseteq$	$P \subseteq Q \& Q \sqsubseteq R \Rightarrow P \sqsubseteq R$	$\sqsubseteq$ under inclusion $\subseteq$
C'	$\mathcal{N} \subseteq$	$S \subseteq N \in \mathcal{N} \Rightarrow S \in \mathcal{N}$	$\mathcal{N}$ down-closed
D	$\mathcal{N}\|\mathcal{W}$	$S \notin \mathcal{N} \Rightarrow S \in \mathcal{W}$	Set in $\mathcal{N}$ or in $\mathcal{W}$
D'	$\mathcal{N}^c$	$S \notin \mathcal{N} \Rightarrow \overline{S} \in \mathcal{N}$	S or $\overline{S}$ must be negligible

Table 3: Specific postulates for dominance and families

Each specific postulate on its own may appear somewhat reasonable, but some combinations of them can lead to consequences that are perhaps less palatable. For instance, postulates A: $[\cup \sqsubseteq]$ and C: $[\subseteq\sqsubseteq]$ imply the transitivity of dominance: $P \sqsubseteq Q \& Q \sqsubseteq R \Rightarrow P \sqsubseteq R$.[34] Also, our postulates can be used to characterise some interesting classes of families of subsets, as we will see in the sequel.

3.4 Postulates and families

We will now examine how our postulates can be used to characterise some classes of families of subsets, corresponding to interesting versions of 'generally' and 'rarely'.

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[34] Postulates A: $[\cup \sqsubseteq]$ and C: $[\subseteq\sqsubseteq]$, under 1: $[\sqsubseteq \mathcal{N}]$, yield the transitivity of dominance $\sqsubseteq$, because of the inclusion $P - R \subseteq (P - Q) \cup (Q - R)$.

We will consider the following classes of families of (negligible) subsets (corresponding to versions of 'rarely').

- Lattices: closed under union and intersection.
- Down-closed: closed under subsets.
- Ideals: closed under union and subsets.
- Prime ideals: maximal proper ideals.

The dual families of (important) subsets (corresponding to versions of 'generally') form the following classes.

- Lattices: closed under intersection and union.
- Up-closed: closed under supersets.
- Filters: closed under intersection and supersets.
- Ultrafilters: maximal proper filters.

These families have many applications in Logic and in areas of Mathematics. In the sequel, we shall see how to use them for reasoning about (some versions of) 'generally' and 'rarely'.[35]

We now wish to see how our postulates can be used to characterise these classes of families of subsets, corresponding to versions of 'generally' and 'rarely'.

We have already seen (in 3.2) that, by our four basic postulates, the families $\mathcal{N}$ (of negligible subsets) and $\mathcal{W}$ (of important subsets) of a (nonempty) universe V are proper and as dual.[36].

We also know that our specific postulates correspond to closure properties of the families $\mathcal{N}$ and $\mathcal{W}$ (cf. table 3 in 3.3). Thus, our specific postulates lead to families of subsets as above.

We now wish to see the converse: how such families of subsets lead to models of our basic and specific postulates. Given such a family, we can

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[35] Some examples are as follows. The sets having more than, say, 70 % of the elements form an up-closed family (corresponding to a notion of 'several'). Both the finite unions of intervals of the reals and the cofinite open subsets of an infinite topological space form lattices (corresponding to notions of 'many'). The subsets including a given nonempty set as well as the cofinite subsets of an infinite universe form filters (corresponding to notions of 'most'). The subsets having a given element of an universe form an ultrafilter.

[36] We have $\emptyset \in \mathcal{N}$ & $V \notin \mathcal{N}$, $\emptyset \notin \mathcal{W}$ & $V \in \mathcal{W}$, and $S \in \mathcal{N}$ iff $\overline{S} \in \mathcal{W}$

construct a structure that will satisfy our four basic postulates, as well as some specific postulates.

This construction can be done as follows. Consider a family $\mathcal{W} \subseteq \wp(V)$ of (important) subsets of a nonempty universe V, and define

- family $\mathcal{N}$ of (negligible) subsets by $\mathcal{N} := \{S \subseteq V : \overline{S} \in \mathcal{W}\}$;
- relation $\sqsubseteq$ of (dominance) by $S \sqsubseteq T \Leftrightarrow (S - T) \in \mathcal{N}$.

We now have a structure $\mathcal{M} := \langle V, \sqsubseteq, \mathcal{N}, \mathcal{W} \rangle$. This structure $\mathcal{M}$ will satisfy our four basic postulates whenever family $\mathcal{W} \subseteq \wp(V)$ is proper: $\emptyset \notin \mathcal{W}$ and $V \in \mathcal{W}$.[37]

We can also see that each specific postulate corresponds to a closure property of the family $\mathcal{W} \subseteq \wp(V)$.[38]

Thus, each one of the above families of subsets corresponds to a set of our specific postulates. This is summarised in the following table.[39]

Postulates	$\Leftrightarrow$	Family $\mathcal{N}$	Family $\mathcal{W}$
A, B		lattice	lattice
C		downward	upward
A, (B,) C		ideal	filter
A, (B,) C, D		prime ideal	ultrafilter

We thus see that each one of the above families $\mathcal{N}$, of negligible subsets, and $\mathcal{W}$, of important subsets, gives rise to a model of our four basic postulates and corresponding specific postulates.

In this sense, we can say that our postulates characterise these classes of families of subsets (as well as some other interesting classes).

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[37] We can see that structure $\mathcal{M}$ satisfies postulate 1: $[\sqsubseteq \mathcal{N}]$, by the definition of $\sqsubseteq$. Now, by the definition of $\mathcal{N}$, $\mathcal{M}$ satisfies 2: $[V \sqsubseteq \mathcal{W}]$ (by the definition of $\sqsubseteq$), 3.a: $(\emptyset\mathcal{N})$ (whence 3: $[\emptyset V]$), as $V \in \mathcal{W}$, and 4.a: $(V\mathcal{N})$ (whence 4: $[V\emptyset]$), as $\emptyset \notin \mathcal{W}$.

[38] Indeed, we can see that we have

$$\begin{array}{lcr}
\mathcal{M} \models A & \Leftrightarrow & \mathcal{W}\ \cap\text{-closed} \\
\mathcal{M} \models B & \Leftrightarrow & \mathcal{W}\ \cup\text{-closed} \\
\mathcal{M} \models C & \Leftrightarrow & \mathcal{W}\ \text{up-closed} \\
\mathcal{M} \models D & \Leftrightarrow & \mathcal{W}\ \text{prime}
\end{array}$$

This shows a correspondence between specific postulates and closure properties.

[39] As noted before, postulate C implies postulate B.

3.5 Specific postulates revisited

We have introduced our specific postulates from a somewhat algebraic viewpoint in 3.3. We now wish to reconsider them: we will present some more intuitive reasons for accepting these specific postulates.

For our present purpose, it is convenient to consider another vague notion, namely a binary relation $\approx$ of similarity: 'about as important as'. Our intuitive idea of two sets being "about as important" is that they are "practically the same": except for a part that may be discarded, the sets are equal.

We are thus led to imagine that this notion of similarity can be explained in terms of the family of negligible subsets, much as before. We thus consider the following definition.

Define similarity in terms of negligible
Subsets S and T of the universe V are similar iff both differences $S - T$ and $T - S$ are negligible.
$\approx \mathcal{N}$: $S \approx T \Leftrightarrow (S - T) \in \mathcal{N} \& (T - S) \in \mathcal{N}$

We can now see some simple, and intuitively reasonable, consequences of this definition $\approx \mathcal{N}$ and our basic postulate 1: $[\sqsubseteq \mathcal{N}]$.[40]

Characterise similarity in terms of dominance
Subsets S and T of the universe V are similar iff they dominate each other.
$(\approx\sqsubseteq)$: $S \approx T \Leftrightarrow S \sqsubseteq T \& S \sqsubseteq T$

Invariance of similarity under complementation
Subsets S and T of the universe V are similar iff their complements are similar.
$(\approx^c)$: $S \approx T \Leftrightarrow \overline{S} \approx \overline{T}$

We will now examine some connections among our notions that may be considered somewhat reasonable.

A basic connection concerns the behaviour of the negligible subsets under similarity. One may expect that subset of the universe that is "practically the same" as a negligible one is negligible as well.

0. *Behaviour of negligible subsets under similarity*
A subset $S \subseteq V$ similar to a negligible subset $N \subseteq V$ is also negligible.
$< \mathcal{N} \approx>$: $S \approx N \& N \in \mathcal{N} \Rightarrow S \in \mathcal{N}$

Our next connection concerns the behaviour of similarity under union with negligible sets. One may find reasonable to accept that the addition

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[40] We can also characterise the sets that are both negligible and important as those similar to their own complements: $S \in \mathcal{N} \cap \mathcal{W} \Leftrightarrow S \approx \overline{S}$.

of a negligible set should have negligible impact, leaving a set practically the same as before.

I. *Behaviour of similarity under union with negligible set*
The union of subset $S \subseteq V$ with a negligible subset $N \subseteq V$ is similar to S.
$< +\mathcal{N} >$: $N \in \mathcal{N} \Rightarrow S \cup N \approx S$

This connection I: $< +\mathcal{N} >$ (in the presence of the basic connection 0: $< \mathcal{N} \approx >$) yields consequence A': $(\mathcal{N}\cup)$. Thus, these somewhat reasonable connections provide intuitive support for specific postulate A: $[\cup \sqsubseteq]$.

Connections 0: $< \mathcal{N} \approx >$ and I: $< +\mathcal{N} >$ yield consequence A': $(\mathcal{N}\cup)$.
$< \mathcal{N} \approx > \& < +\mathcal{N} > \vdash (\mathcal{N}\cup)$[41]

A connection analogous to the preceding one deals with the behaviour of similarity under removal of a negligible set. As before, one may be willing to accept that the removal of a negligible set should have negligible impact, leaving a set about as important as before.

II. *Behaviour of similarity under removal of negligible set*
The result $S - N$ of removing a negligible subset $N \subseteq V$ from subset $S \subseteq V$ is similar to S.
$< -\mathcal{N} >$: $N \in \mathcal{N} \Rightarrow S - N \approx S$

This connection II: $< -\mathcal{N} >$ (in the presence of the basic connection 0: $< \mathcal{N} \approx >$) yields consequence B': $(\mathcal{N}\cap)$. So, these somewhat reasonable connections provide intuitive justification for specific postulate B: $[\subseteq \sqsubseteq]$.

Connections 0: $< \mathcal{N} \approx >$ and II: $< -\mathcal{N} >$ yield consequence B': $(\mathcal{N}\cap)$.
$< \mathcal{N} \approx > \& < -\mathcal{N} > \vdash (\mathcal{N}\cap)$[42]

Another connection analogous to the preceding ones deals with the behaviour of the negligible subsets under dominance $\sqsubseteq$. Much as before, one may find reasonable to consider as negligible a subset of the universe that is practically within a negligible one.

III. *Behaviour of negligible subsets under dominance*
A subset $S \subseteq V$ dominated by a negligible subset $N \subseteq V$ is negligible.
$< \mathcal{N} \sqsubseteq >$: $S \sqsubseteq N \& N \in \mathcal{N} \Rightarrow S \in \mathcal{N}$

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[41] Indeed, from $N" \in \mathcal{N}$, connection I: $< +\mathcal{N} >$ yields $N' \cup N" \approx N'$, whence, with $N' \in \mathcal{N}$, connection 0: $< \mathcal{N} \approx >$ yields $N' \cup N" \in \mathcal{N}$. We thus have consequence A': $(\mathcal{N}\cup)$.

[42] Connections 0: $< \mathcal{N} \approx >$ and II: $< -\mathcal{N} >$ yield consequence B': $(\mathcal{N}\cap)$, in view of the equality $N' - (N' - N") = N' \cap N"$. Indeed, from $N" \in \mathcal{N}$, II: $< -\mathcal{N} >$ yields $N' - N" \approx N'$, whence, with $N' \in \mathcal{N}$, 0: $< \mathcal{N} \approx >$ yields $N' - (N' - N") \in \mathcal{N}$. So we have B': $(\mathcal{N}\cap)$.

This connection III: $< \mathcal{N} \sqsubseteq>$ (in the presence of the common postulates) yields consequence C': $(\mathcal{N} \subseteq)$. Thus, this somewhat reasonable connection lends some intuitive justification for specific postulate C: $[\subseteq\sqsubseteq]$.

Connection III: $< \mathcal{N} \sqsubseteq>$ (under consequence 3.b: $(\subseteq\sqsubseteq)$) yields consequence C': $(\mathcal{N} \subseteq)$.

$(\subseteq\sqsubseteq)\& < \mathcal{N} \sqsubseteq> \vdash (\mathcal{N} \subseteq)$

We now come to our final connection, which is probably the least intuitively acceptable one (and with more profound impact). The underlying idea is that the universe is so important (i. e. carries so much weight) that any attempt to cover it by finitely many subsets must employ an important subset (one carrying considerable weight, or equivalently, almost as important as the entire universe).[43]

IV. *Finite cover of universe and important subsets*

A finite cover of universe V must have an important set.

$< V\mathcal{W} >$: $V = T_1 \cup \ldots \cup T_n \Rightarrow \exists k : T_k \in \mathcal{W}$[44]

This connection IV: $< V\mathcal{W} >$ yields consequence D': $(\mathcal{N}^c)$. Thus, this connection provides some intuitive support for specific postulate D: $[\mathcal{N}|\mathcal{W}]$.

Connection IV: $< V\mathcal{W} >$ yields consequence D': $(\mathcal{N}^c)$.

$< \mathcal{N} \approx> \& < -\mathcal{N} > \vdash (\mathcal{N}\cap)$[45]

Summarising, we have seen some reasons (of varying intuitive appeal) providing some support for accepting our specific postulates, introduced from a somewhat algebraic viewpoint in 3.3.

4 Reasoning about ‘generally’ with families

The preceding ideas can be employed to provide bases for precise reasoning with assertions involving (some versions of) the vague notions ‘generally’ and ‘rarely’. In the sequel, we shall first illustrate how these ideas, giving some precise meanings to (versions of) of ‘generally’, also serve to reason and then indicate how one can set up logical systems on these bases.

The intended meaning of ‘generally’ (and ‘rarely’), at least in some cases, can be given by means of families of ‘important’ (and ‘negligible’)

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[43] Over an infinite universe, one may regard the finite subsets as not carrying considerable weight. Another example where this connection holds is provided by considering as carrying considerable weight the subsets with elephants.

[44] An equivalent formulation of connection IV: $< V\mathcal{W} >$ (by the duality 2.a: $(\mathcal{N}^c\mathcal{W})$) is: an empty finite intersection must have a negligible set.

[45] Case $n = 2$ of connection IV: $< V\mathcal{W} >$ yields consequence D': $(\mathcal{N}^c)$, since $V = S\cup\overline{S}$.

sets. Considering a given property φ, one can understand "objects generally have property φ" as "the objects having φ form an important set", in the sense of belonging to a given family of important subsets of the universe of discourse.

The preceding section shows that the families of important subsets corresponding to (some) notions of 'generally' can be characterised by postulates. Now, these postulates provide bases for analysing, and reasoning about, situations involving assertions with 'generally'.

4.1 Reasoning with families for 'generally'

We will now illustrate how the postulates characterising (some versions of) 'generally' can be used in analysing, and reasoning about, situations involving assertions with 'generally'.

We shall first examine a simple example. Consider the universe of Brazilians and imagine that one accepts the two assertions:

- φ: "Brazilians generally shave their faces";
- λ: "Brazilians generally shave their legs".

In this case, one would probably accept also the assertion

- μ: "Brazilians generally shave their faces or sport a moustache".

This, however, does not seem to be the case with the assertion

- ν: "Brazilians generally shave their faces and shave their legs".

The reason for accepting the assertion μ should be clear (see also below). Assertion ν can be seen not acceptable by considering males and females.[46]

For convenience, we will employ 'several' for the sense of 'generally' in this example. Thus, the explanation can be seen to hinge on the following ideas:

- if F has several elements F is a subset of M and $F \subseteq M$, then M also has several elements;

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[46] The "Brazilians that shave their faces" are generally males, whereas the "Brazilians that shave their legs" are generally females. So, the "Brazilians that shave their faces and shave their legs" form a rather small fraction of the population.

- even though both F and L have several elements, their intersection $F \cap L$ may fail to have several elements.

So, the situation in this example can be explained by considering the family $\mathcal{W}$ of important sets (corresponding 'generally') to be closed under supersets, but not under intersection.[47]

For another example, consider the universe of American males[48]. Imagine that one accepts the following three assertions:

- β: "American males generally like beer";
- σ: "American males generally like sports";
- ε: "American males generally are Democrats or Republicans".

In this case, one would probably accept also the two assertions:

- α: "American males generally like alcoholic beverages";
- τ: "American males generally like beer and sports".

Acceptance of assertion α should be clear and an explanation for accepting assertion τ can be given by means of the exceptions (see below). On the other hand, even though one accepts the assertion ϵ , neither one of the two assertions "American males generally are Democrats" and "American males generally are Republicans" seems to be equally acceptable.

For convenience, let us use 'most' for the sense of 'generally' in this example. Thus, the situation can be explained as follows:

- if B has most element and $B \subseteq A$, then A will have most elements as well;
- if both B and S have most elements, their complements $\overline{B}$ and $\overline{S}$ are small and so will be their union $\overline{B} \cup \overline{S}$ small, thus the intersection will have most elements;
- the union $D \cup R$ may have most elements, without either D or R having most elements.

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[47] Thus, family $\mathcal{W}$ is up-closed, but not a filter. An up-closed family $\mathcal{W}$ with both F and L also has a superset M of F, but not necessarily $F \cap L$.

[48] This example is similar to that of natural numbers in 2.2.

Thus, we can account for the situation in this example by considering the family $\mathcal{W}$ of important sets (corresponding 'generally') to be to be closed under supersets and under intersection, but not prime.[49]

The detection of the appropriate notion of 'generally', and of the nature of the corresponding family of important sets, will hinge on non-logical information depending on the situation. To illustrate this issue, imagine that a socialite, eager to attend interesting parties, receives pieces of advice as follows:

1. "Important parties are those attended by the celebrities";
2. "Important parties are those attended by Madonna".

The former advisor considers a set of guests as important when it includes the celebrities, whereas the latter advisor understands as important sets of guests those where Madonna is. In both interpretations, the family $\mathcal{W}$ of important sets is a filter, which is an ultrafilter in the Madonna interpretation, but not necessarily so in the celebrities interpretation.[50]

4.2 Logics for 'generally'

We shall now briefly indicate how one can set up logical systems for expressing and reasoning about assertions involving (some versions of) 'generally' on the basis of their characteristic postulates. The goal is having logics for some vague notions, much as we have "logics embodying mathematical concepts" [B+F'85].

Our logics for 'generally' add to classical first-order logic a (non-standard) generalised quantifier ∇, with intended interpretation "forming an important set of objects of the universe of discourse".

The syntax of our logics is obtained by extending the usual first-order syntax by the new quantifier. We extend the usual first-order syntax by adding the new quantifier ∇ together with a new (variable-binding) formation rule giving generalised formulas, of the form $\nabla z\varphi$.[51]

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[49] Thus, family $\mathcal{W}$ is a filter, but not an ultrafilter. A filter $\mathcal{W}$ with B, S and $D \cup R$ also has a superset A of B and $B \cap S$ in $\mathcal{W}$, but not necessarily D or R.

[50] In both cases, the family $\mathcal{W}$ is a principal filter: it consists of the sets including a generator. It is an ultrafilter when the generating set has a single element.

[51] With this new quantifier we can express assertions, such as "Birds generally fly" and "Metals generally are solid", as well as properties like "people are generally taller than x".

The semantics for our logics is obtained by extending the usual first-order definition of satisfaction to the new quantifier. For this purpose, we resort to complex structures, a complex structure $\mathcal{M}^{\mathcal{K}}$ being the expansion of a first-order structure $\mathcal{M}$ by a family $\mathcal{K}$ (of important) subsets of its universe. We then extend the usual Tarskian definition of satisfaction to generalised formulas, so as to capture the above interpretation: a generalised formula $\nabla z \varphi$ is satisfied iff the extension of φ belongs to the given complex $\mathcal{K}$. [52]

So, the propositional connectives as well as the classical quantifiers $\forall$ and $\exists$ will keep their familiar interpretations.[53]

We can set up deductive systems for our logics by adding to a calculus for classical first-order logic some schemata: basic and specific schemata. The basic schemata code fundamental properties common to proper complexes.[54]

The specific schemata code closure properties characterising complexes: up-closed families, lattices, filters, or ultrafilters.[55]

These systems provide sound and complete deductive calculi for reasoning about assertions involving 'generally'.[56] Our logics for 'generally' are (proper) conservative extensions of classical first-order logic[57], with which

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[52] More precisely, given a complex structure $\mathcal{M}^{\mathcal{K}} = \langle \mathcal{M}, \mathcal{K} \rangle$, for a formula $\nabla z \varphi(\underline{u}, z)$, we define $\mathcal{M}^{\mathcal{K}} \models \nabla z \varphi(\underline{u}, v)[\underline{a}]$ iff $\{b \in \mathsf{M} : \mathcal{M}^{\mathcal{K}} \models \varphi(\underline{u}, z)[\underline{a}, b]\}$ is in $\mathcal{K}$.

[53] For a purely first-order formula $\theta(\underline{u})$ (without ∇), $\mathcal{M}^{\mathcal{K}} \models \theta(\underline{u})[\underline{a}]$ iff $\mathcal{M} \models \theta(\underline{u})[\underline{a}]$.

[54] The four basic schemata are of two kinds.

$$\begin{array}{ll} \nabla z \varphi \rightarrow \exists z \varphi & [\emptyset \notin \mathcal{K}] \\ \forall z \varphi \rightarrow \nabla z \varphi & [M \in \mathcal{K}] \end{array}$$

These two basic schemata code proper complexes.

$$\begin{array}{lr} \nabla z \varphi(z) \leftrightarrow \nabla v \varphi(v), \text{ for a new } v & [\text{alphabetic variant}] \\ \forall z(\psi \leftrightarrow \theta) \rightarrow (\nabla z \psi \leftrightarrow \nabla z \theta) & [\text{extensionality}] \end{array}$$

These two basic schemata code invariance under syntax.

[55] The specific schemata are as follows

$$\begin{array}{lr} \forall z(\psi \rightarrow \theta) \rightarrow (\nabla z \psi \rightarrow \nabla z \theta) & [\text{up-closed}] \\ (\nabla z \psi \wedge \nabla z \theta) \rightarrow \nabla z(\psi \wedge \theta) & [\cap\text{-closed}] \\ (\nabla z \psi \wedge \nabla z \theta) \rightarrow \nabla z(\psi \vee \theta) & [\cup\text{-closed}] \\ \neg \nabla z \varphi \rightarrow \nabla z \neg \varphi & [\text{prime}] \end{array}$$

These schemata code properties of specific complexes.

[56] A sentence τ is derivable from a set Γ iff τ holds in every complex model of Γ (in each case).

[57] For classical formulas (without ∇), our ∇-axioms add no extra deductive power.

they share various metamathematical properties, such as compactness and Löwenheim-Skolem properties.[58]

5 Conclusion

We have examined some fundamental issues in the precise treatment of assertions involving 'generally' and 'rarely', trying to explain them and to clarify the role played by families of subsets in this context.

Assertions and arguments involving vague qualitative notions, such as 'rarely', 'generally', 'most', 'many', etc., occur often both in ordinary language and in some branches of science. This provides one of the motivations for undertaking such analyses.

We have examined some meanings for 'generally' and 'rarely'. The analysis of some basic intuitions and their underlying presuppositions has led to distinguishing various versions according to their behaviour. These various versions - corresponding to notions such as 'several' (or 'many') and 'most' - can be rendered precise by resorting to families of subsets. The properties of these families can be used for reasoning about assertions with (some versions of) 'generally' and 'rarely'.

By introducing generalised quantifiers over such families, one can obtain logical systems, which provide rigorous bases for qualitative reasoning about vague notions of 'generally'. These logics are conservative extensions of classical first-order logic, with which they share various properties. These systems are undergoing further investigation [V+C'01, V+V'01, RHV'01, V+V'02].[59] They appear to have some interesting connections with fuzzy logic [Zad'75] as used in expert systems [Tur'84], natural language [B+C'81, Mon'74] and empirical reasoning[60]. Such connections suggest the possibility of other applications [C+V'77, Vel'98].

These extensions are proper, because sentences, such as $\exists u \nabla z u \doteq z$, cannot be expressed without ∇ [V+C'01, Vel'99].

[58] The apparent conflict with Lindström's results ([Lin'66], [Bar'77]) is explained because we are using a non-standard notion of model (due to the complexes). This feature may confer to our logics for 'generally' some independent model-theoretic interest.

[59] These developments include proof methods and sorted versions (to express relative 'generally', since relativisation fails to express the intended meaning, due to properties of ∇ and $\rightarrow$ [C+V'97, V+C'01]).

[60] Some questions motivating the introduction of sorts to express relative 'generally' appear to be connected to the so-called paradoxes of confirmation ([Hem'45], [Sai'89]).

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Non Truth-Functional Many-Valuedness

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Abstract

Many-valued logics are standardly defined by logical matrices. They are truth-functional. In this paper non truth-functional many-valued semantics are presented, in a philosophical and mathematical perspective.

MSC Primary: 03B50, Secondary: 03B22 03B53 03B20 03B45

Contents

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Introduction

The aim of this paper is to present a new tool for the study of logics, the concept of non truth-functional many-valued semantics. [2]

We begin, in a first part, by a general discussion about many-valued logic [3] in order to show what is exactly the philosophical and mathematical meaning of this concept, in which position it stands in the logical space.

In a second part we recall the definition of logical matrix and the standard definition of many-valued logic based on this notion. It is important to have these definitions here in order to properly understand the distinction between truth-functional and non truth-functional many-valued semantics, and these definitions are also worthy to fix the terminology which can be misleading and ambiguous.

Notions discussed informally in the first part are given a precise meaning here and in the third part where we properly define the notion of non truth-functional many-valued semantics and give some examples of such semantics.

1 Logical matrices = many-valued logic?

1.1 Logical matrices do not reduce to many-valued logic

Many-valued logics are ones of the most famous non-classical logics. They appeared independently in the work of different people at the end of the XIXth / beginning of the XXth century, mainly C.S.Peirce, E.Post and

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[2] Non truth-functional many-valued semantics were introduced for the first time some years ago in one of my papers about non classical logics [3]. But in this paper they just appear as a side notion.

In some sense the present paper is a sequel of two other papers ([6], [18]), although it is self-contained. The paper [18] arose from a discussion about G.Malinowski's book on many-valued logics [25]. This little book is a good presentation of standard many-valued logic.

[3] We will alternatively use the singular "many-valued logic" or the plural "many-valued logics", depending on what we want to emphasize. The singular expression emphasizes many-valued logics as a whole.

J.Łukasiewicz. The work of Łukasiewicz was without doubt the most influential in the development of many-valued logics. One of the reasons is that Łukasiewicz's work promoted the concept of logical matrix, central concept for the construction of many-valued logics, implicit in the works of Peirce and Post. The concept of logical matrix was later on systematically used for the development of many-valued logic.

Moreover the notion of logical matrix has been used not only to generate many-valued logics. For example Bernays [2] used many-valued matrices at the metalogical level to prove the independence of sets of axioms for classical propositional logic. It has also been used at the metamathematical level by people like Kleene [23], Bochvar [13] or Girard [20]. It was the idea of Tarski that the concept of logical matrix could be used as a basic tool for the general theory of zero-order logics.[4] And in fact this concept really became fundamental in the so-called "Polish logic".[5]

So logical matrices do not reduce to many-valued logic. But what about the converse: does many-valued logic reduce to logical matrices? To generate many-valued logics do we need logical matrices?

1.2 Logical matrices do not violate the principle of bivalence

What is the definition of many-valued logic? One can define a many-valued logic as a logic which violates the principle of bivalence. But what it the principle of bivalence? It can be stated as follows:

Principle of Bivalence *A proposition is true or false: it cannot be true and false, or neither true nor false.*

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[4] In a footnote to the reedition of "Investigations into the sentential calculus" by Łukasiewicz and Tarski in Tarski's volume *Logic, semantics, metamathematics*, Tarski recalls that "the construction of many-valued systems of logic described here, are entirely due to Łukasiewicz and should not be referred to Łukasiewicz and Tarski." ([34], p.38) But later on when the concept of matrix is introduced, he adds the following footnote: "The view of matrix formation as a general method of constructing systems is due to Tarski" ([34], p.40).

[5] About the general theory of zero-order logics, Polish logic and this terminology see [10].

The interpretation of this principle is not necessarily obvious.[6] One can for example seriously doubt that the standard many-valued logics challenge this principle. The reason is the following: the principle of bivalence is present in many-valued matrices through the distinction between designated and undesignated values, as stressed by G.Malinowski: "The matrix method inspired by truth-tables embodies a distinct shadow of two-valuedness in the division of the matrix universe into two subsets of designated and undesignated elements." ([25], (p.72)). This very distinction is crucial for the definitions of logical truth and logical consequence.

When we have, for example, a many-valued matrix, with three values 0, 1/2, and 1, it is therefore misleading to call the value 0 true, the value 1 false and the value 1/2, indeterminate, or true-false, etc. The designated values must be considered as corresponding to truth, and the undesignated values as corresponding to falsity, because logical truth is defined as "designated for every valuation". The same many-valued matrix can generate totally different logics according to the choice of designated/undesignated values. For example Łukasiewicz took only 1 as designated, but in the constructions of paraconsistent logics people have taken 1/2 and 1 as designated (see [1], [19], [29], [30], [36], [27]). To consider 1/2 as designated but not calling it true can lead to erroneous interpretations of the logic generated by the matrix (see [12]).

1.3 Suszko's distinction between algebraic values and logical values

The concept of logical consequence seems essentially bivalent (and also the concept of logical truth which is a particular case of it). If one has a consequence relation $\vdash$ it can be interpreted semantically: $T \vdash a$ means every model of T is a model of a, and $T \nvdash a$ means there is a model of T which is not a model of a. So to be a model or not to be a model, that is the question. No matter how "truth in a model" is defined (using several designated values, interpretations, accessibility relations, etc.), what is important is that at the end we have the dichotomy true in a model / false in a model.

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[6] Very often the principle of bivalence is confused with the principle of excluded middle and sometimes with the principle of contradiction. This confusion has been discussed in [18] and [11] and will not be treated here. We have tried here to give a formulation of this principle which avoids confusions.

In fact if we have a consequence relation, it is always possible to find a bivalent semantics for it, just by taking as models, characteristic functions of some theories. As Suszko puts it: "In short, every logic is (logically) two-valued" ([33], p.378). Suszko indeed provided a bivalent semantics for Łukasiewicz's three many-valued logic $L3$ (see [32]). This may sound paradoxical since $L3$ is called a three many-valued logic because it cannot be defined with a two-valued matrix. But Suszko's semantics is not a matricial semantics. The values 0 and 1 in this semantics are not the domain of an algebra, they are not *algebraic values* but *logical values*, following Suszko's terminology.

In the case of propositional classical logic, algebraic values and logical values coincide in some sense because the characteristic functions of the maximal theories can be viewed at the same time as homomorphisms from the set of formulas into the Boolean algebra on $\{0, 1\}$. But in most cases this does not happen. So one must make clear the distinction between logical values and algebraic values. In the case of $L3$, we have two semantics, one with *two logical* values and a semantics with *three algebraic* values. But there are logics which cannot be characterized by (finite) matrices and therefore have no semantics with algebraic values. This is the case for example of the paraconsistent logic $C1$ of da Costa (cf. [14], [4]). For this logic, a semantics with two logical values was provided [15]. Later on da Costa and his pupils developed a general theory of logic based on such kind of semantics under the name "theory of valuation" (cf. [16], [17]).

For Suszko, "any multiplication of logical values is a mad idea and, in fact, Łukasiewicz did not actualize it" ([33], p.378). But what is exactly a logical value? Does the dichotomy algebraic value / logical value admit no third possibility? The aim of this paper is to present many-valued semantics where the values are not algebraic values in Suszko's sense, they are not elements of the algebra of a logical matrix. But we will not be mad enough to call these values logical values since, because as in the matrix case, we will make a distinction between designated and undesignated values, in order to define logical truth and logical consequence.

One can wonder why introducing such kind of many-valued semantics, since every logic is two-valued. Łukasiewicz's logic $L3$ has a bivalent semantics, so why multiplying the values and introducing a three-valued semantics? The reason is that this three-valued semantics gives a totally different look at $L3$ and is a very useful technical tool to prove theorems of $L3$ and metatheorems about $L3$.[7] Non truth-functional (i.e. non matri-

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[7] And vice-versa. The non truth-functional bivalent semantics for $L3$ was introduced

cial) many-valued semantics were introduced (cf. [3]) for the study of the paraconsistent logic $C1$ which doesn't have truth-functional semantics. As we have already said, this logic has a bivalent semantics. But the use of a many-valued semantics can give a better intuition of $C1$ and simplifies the proof, in the same way as the three-valued matricial semantics does for $L3$, even if this many-valued semantics is not truth-functional.

Non truth-functional many-valued semantics are a useful tool for the study of logics. As in the case of matricial semantics, the additional values are philosophically ambiguous, and in some sense they preserve the principle of bivalence through the dichotomy designated/undesignated values. But as in the case of matrix semantics it is also possible to use these non truth-functional semantics to generate logics which are many-valued in a deeper sense.[8]

2 The standard definition of many-valued logic

2.1 Logical matrices

A *logical matrix* $\mathcal{M}$ is a structure $\langle \mathcal{A}; \mathbb{D} \rangle$ where $\mathcal{A}$ is an abstract algebra $\langle \mathbb{A}; fun \rangle$ and $\mathbb{D}$ is a subset of the domain of the algebra $\mathbb{A}$.

fun is a finite sequence of finitary functions (i.e. functions of finite arity) defined on $\mathbb{A}$, called *truth-functions*. The *type* of the algebra is the specification of the length of this sequence and the arity of each truth-function. Elements of $\mathbb{A}$ are called *values*, an element of $\mathbb{A}$ is called a *designated value* if it is also a member of $\mathbb{D}$, *undesignated value* otherwise. Given a cardinal κ, a κ*-valued matrix* is a matrix where the domain of values is of cardinality κ.

A typical example of logical matrix is the 2-valued matrix of classical propositional logic: $\langle \langle \{0, 1\}; \neg, \vee, \wedge, \rightarrow \rangle; \{1\} \rangle$. It is important to note that here, the sign "$\rightarrow$", for example, represents a truth-function and not a

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by Suszko rather as an "exercise de style", and it didn't seem to have further utility. However this semantics was used later on to provided a sequent-calculus for $L3$ (see [9]) using the close connection between bivalent semantics and sequent calculus.

[8] G.Malinowski has used many-valued matrices to define consequence relations which are different than the usual one, which are in some sense more many-valued (see [26]). It is possible to use non truth-functional many-valued semantics in a similar way.

connective. Generally people use the same name, as we did, for truth-function and for connectives. This is a useful device but it can be sometimes misleading (see [10]). This 2-valued matrix is many-valued in the sense that "2 is many". But according to the standard convention a many-valued matrix is a matrix of cardinality superior or equal to three.

2.2 Logics

There are many ways to define what is a logical structure, we consider here three basic types of logical structures: $\mathcal{L}1= \langle\mathcal{F}; \vdash_1\rangle$, $\mathcal{L}2= \langle\mathcal{F}; \vdash_2\rangle$, $\mathcal{L}3= \langle\mathcal{F}; \vdash_3\rangle$.

For all these structures, $\mathcal{F}$ is an *absolutely free algebra* of type $\langle\mathbb{F}; con\rangle$. An element a of the domain $\mathbb{F}$ is called a *formula*. con is a sequence of functions called *connectives* which generate $\mathbb{F}$ from a subset $\mathbb{P}$ of $\mathbb{F}$. An element p of $\mathbb{P}$ is called an *atomic formula*. A set of formulas T is called a *theory*.

- $\vdash_1$ is a subset of $\mathbb{F}$, elements of $\vdash_1$ are called *tautologies*.
- $\vdash_2$ is a binary relation between theories and formulas. It is called a *consequence relation*.
- $\vdash_3$ is a binary relation between theories and theories. It is called a *multiple-consequence relation*.

Hereafter we will use the word logic as a generic term for these three kinds of structure.

2.3 Many-valued logics generated by logical matrices

Logical matrices are used to generate logics. With a logical matrix, one can generate a logic of type 1, 2 or 3 by a uniform method.

Given a matrix $\mathcal{M} = \langle\mathcal{A}; \mathbb{D}\rangle$, we consider the absolutely free algebra $\mathcal{F}$ of the same type as $\mathcal{A}$ and the set $\mathcal{HOM}$ of homomorphisms between $\mathcal{F}$ and $\mathcal{A}$.

An element of $\mathcal{HOM}$ will be called a *morpho-valuation*. A function from the set $\mathbb{P}$ of generators of $\mathcal{F}$ to the domain $\mathbb{A}$ of the algebra $\mathcal{A}$ is called an *atomic morpho-valuation.* Due to the fundamental property of absolutely free algebras, any atomic morpho-valuation has a unique extension which is a morpho-valuation. Thus, it is the same to consider morpho-valuations or atomic morpho-valuations, since there is a one-to-one correspondence between them.

Using the notion of morpho-valuations, we now define sets and relations on the domain of $\mathcal{F}$, which lead to the three basic types of logical structure. For any formula a and theories T, U:

- $\vdash_1 a$ iff for every morpho-valuation μ, $\mu(a)$ is a designated value.
- $T \vdash_2 a$ iff for every morpho-valuation μ, if $\mu(b)$ is a designated value for every element b of T, then $\mu(a)$ is a designated value.
- $T \vdash_3 U$ iff for every morpho-valuation μ, if $\mu(b)$ is a designated value for every element b of T, then there is an element c of U, such that $\mu(c)$ is a designated value.

Given a logic $\mathcal{L}$, one says that a matrix $\mathcal{M}$ characterized $\mathcal{L}$, or that $\mathcal{M}$ is a *characteristic matrix* for $\mathcal{L}$, iff $\mathcal{L}$ is the logic generated by $\mathcal{M}$.

According to the standard definition of many-valued logic, a logic is not a κ-valued logic if it is, or can be, generated by a κ-valued matrix. If this would be the definition, classical propositional logic would be a 242-valued logic since in fact it is not difficult to see that it can be generated by matrices of any cardinality superior to one.

A logic $\mathcal{L}$ is said to be κ-valued iff κ is the smallest cardinal such that there exists a κ-valued matrix which characterizes $\mathcal{L}$. A typical example of many-valued logic is Łukasiewicz's three-valued logic, which is generated by a three-valued, and cannot be characterized by a two-valued matrix. Concerning the cardinality of "many", the same convention applies here as in the case of matrices: the two-valued classical logic is not called a many-valued logic, a many-valued logic is a logic which is at least 3-valued.

3 Non truth-functional many-valuedness

3.1 Truth-functional logics

What is a truth-functional logic? Classical propositional logic is truth-functional, but what about intuitionistic logic? The various modal logics? etc.

One could just say that a truth-functional logic is a logic that can be characterized by a logical matrix. However this definition seems too weak, since following it, quite every logic would be truth-functional: according to a famous theorem, whose original idea is due to Lindenbaum, any logic of type 1 which is *structural*, i.e. close under substitutions, can be characterized by a matrix. And this theorem can be generalized in some sense to logic of types 2 and 3 (see [37]).

A reasonable definition runs as follows: a *truth-functional logic* is a logic that can be characterized by a finite matrix. In this sense, intuitionistic, standard modal logics ($S5$, $S4$, K, etc..), the paraconsistent logic $C1$, Jaśkowski's discussive logic and many other logics are not truth-functional.[9]

3.2 Non truth-functional semantics

Following our definition of truth-functional logic, we can say that a *truth-functional semantics* is a finite matrix. According to this definition a non truth-functional semantics is any semantics which is not a finite matrix. This definition is quite vague if we don't specify what is a semantics.

We can give a very general definition of a semantics for a logic: a *semantics* is a structure $\langle \mathbb{R}; mod \rangle$ where $\mathbb{R}$ is a set of objects called *representations*, and *mod* a function from the set of formulas to the power set of $\mathbb{R}$, which associates to each formula a the set $mod(a)$ of representations in which a is true.

For example in the case of $L3$, representations are homomorphisms from $\mathcal{F}$ to the algebra of the matrix and the function *mod* is defined by: $\mu \in mod(a)$ iff $\mu(a)$ is designated. In the case of modal logics, representations

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[9] One can generate logics with logical matrices in different other ways than the one explained in the preceding section. For example, Gödel has shown that intuitionistic logic cannot be characterized by a finite matrix [21], but Jaśkowski has shown that it can be defined by a set of finite matrices [22].

are frames, and the function *mod* is defined by: $\mu \in mod(a)$ iff a is true in every possible worlds of the frame μ. [10]

Let us now consider a very simple example of non truth-functional semantics, a bivalent one. We consider an algebra of formulas $\mathcal{F}$ built only with two connectives, $\neg$ and $\rightarrow$, and we define a set of functions $\mathbb{B}$ from $\mathbb{F}$ into $\{0, 1\}$ as follows: $\beta \in \mathbb{B}$ iff

- $\beta(a \rightarrow b) = 0$ iff $\beta(a) = 1$ and $\beta(b) = 0$
- if $\beta(a) = 1$ then $\beta(\neg a) = 0$.

The connective $\neg$ is defined by just "half" of the condition for classical negation. The logic generated by this semantics (taking 1 as designated) is called $K/2$ and has been studied in [8]. In this paper it has been shown in particular that classical logic is translatable in $K/2$. This logic is paracomplete in the sense that a formula and its negation can both be false. If we introduce a disjunction in a natural way, the formula $\neg(a \vee \neg a)$ is not a tautology.

The set of bivaluations $\mathbb{B}$ is not a set of homomorphisms and in particular cannot be generated from atomic bivaluations, i.e. functions from $\mathbb{P}$ to $\{0, 1\}$. The behavior of bivaluations for negation can be illustrated by the following table: [11]

p	$\neg p$	$\neg\neg p$	$\neg\neg\neg p$
0	0	0	0
0	0	0	1
0	0	1	0
0	1	0	0
0	1	0	1
1	0	0	0
1	0	0	1
1	0	1	0

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[10] More about this general definition of semantics can be found in [5], [7].

[11] It would be misleading to call such a table, a "truth-table", the similarity is rather visual than conceptual, since this table does not describe a truth-function. Anyway this kind of table can be used as a decidability method. When we say that this table is an "illustration", this means precisely this: given any bivaluation of $\mathbb{B}$, its restriction to the set of formulas which appear in the first line of the table, coincide with one of the other lines of the table; and any of these lines can be extended to a bivaluation of $\mathbb{B}$. This kind of tables were presented for the first-time in [15].

TABLE 1

We can interpret this non truth-functional semantics saying that the value of $\neg a$ is "not determined" by the value of a: if the value of a formula a is 1, the value of $\neg a$ must be 0, but if its value is 0, the value of $\neg a$ can be 0 or 1.

However in this semantics, the behavior of the implication is deterministic in the sense that if we know the values of the two direct subformulas of a conditional, we know the value of this conditional. The behavior of bivaluations for implication can be described by the following usual table:

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

TABLE 2

3.3 Examples of non-truth functional many-valued semantics

We will explain what is a non truth-functional many-valued semantics, generalizing the preceding example of non truth-functional bivalent semantics, to a three-valued non truth-functional semantics. In a three-valued truth-functional semantics, the value of the negation of a formula is determined by the value of this formula. For example in the case of the three-valued matrix of Łukasiewicz, if the value of a formula is 1/2, the value of its negation is 1/2. Now in a three-valued non truth-functional semantics, given a value for a formula, the value of its negation is not determined.

Let us consider a three-valued non truth-functional semantics with three values $\{\%, 0, 1\}$, where only 1 is considered as designated, defined by the following conditions: $\tau \in \mathbb{T}$ iff

- $\tau(a \to b)$ is undesignated iff $\tau(a) = 1$ and $\tau(b)$ is undesignated
- $\tau(a) = 0$ iff $\tau(\neg a) = 1$.

The behavior of threevaluations for negation can be described as follows:

p	$\neg p$	$\neg\neg p$
%	%	%
%	%	0
%	0	1
0	1	%
0	1	0
1	%	%
1	%	0
1	0	1

TABLE 3

The logic defined by this three-valued non truth-functional semantics is in fact the same as $K/2$. This can be explained as follows: in the case of the bivalent non truth-functional semantics, given p and $\neg p$, there are three possibilities than can be described by the following table:

p	$\neg p$
0	0
0	1
1	0

TABLE 4

Now in the three-valued non truth-functional semantics, these three possibilities are described by the three-values, as illustrated by the following table:

p
%
0
1

TABLE 5

The reader can check that the TABLE 3 is a reduction, in this spirit, of TABLE 1. This kind of reduction can be systematized by the following definition: given a bivaluation β of the bivalent non truth-functional semantics for $K/2$ we define a threevaluation τ_β as follows:

$\tau_\beta(a) = \%$ iff $\beta(a) = 0$ and $\beta(\neg a) = 0$
$\tau_\beta(a) = 0$ iff $\beta(a) = 0$ and $\beta(\neg a) = 1$
$\tau_\beta(a) = 1$ iff $\beta(a) = 1$ and $\beta(\neg a) = 0$

It is easy to see that with this method we get a one-to-one correspondence between $\mathbb{B}$ and $\mathbb{T}$ such that: $\beta(a)$ is designated iff $\tau_\beta(a)$ is designated. This proves that the logic generated by $\mathbb{B}$ and $\mathbb{T}$ are the same, namely $K/2$.

We can say that in the three-valued non truth-functional semantics for $K/2$, some information about the value of $\neg p$ is already given by the value of p. However this does not mean that the value of $\neg p$ is "determined" by the value of p. Therefore one may have some doubts about the usefulness of this semantics. The number of values has been increased and we still have indetermination, moreover the implication which was truth-functional is now getting non truth-functional, since the TABLE 2 must be replaced by the following one:

p	q	$p \to q$
%	%	1
%	0	1
%	1	1
0	%	1
0	0	1
0	1	1
1	%	%
1	%	0
1	0	%
1	0	0
1	1	1

TABLE 6

When the value of p is 1 and the value of q is undesignated, % or 0, then the value of $p \to q$ is not determined because it can be % or 0.

Let us see now a more convincing example of non truth-functional many-valued semantics. Imagine that we modify the definition of $\mathbb{B}$ adding the following condition:

- if $\beta(a) = 1$ and $\beta(b) = 0$ and $\beta(b) \neq \beta(\neg b)$ then $\beta(\neg(a \rightarrow b)) = 1$

This condition can be described by the following table:

p	q	$\neg q$	$p \rightarrow q$	$\neg(p \rightarrow q)$
0	0	0	1	0
0	0	1	1	0
0	1	0	1	0
1	0	0	0	0
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0

TABLE 7

In this table, we need to introduce not only direct subformulas but also negations of direct subformulas of the conditional.

Now we can "translate" this semantics in a three-valued non-truth functional semantics, adding to the two conditions which define $\mathbb{T}$, the "translation" of the above condition:

- if $\tau(a) = 1$ and $\tau(b) = 0$ then $\tau(a \rightarrow b) = 0$

which can be described by the following table:

p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$
%	%	1	0
%	0	1	0
%	1	1	0
0	%	1	0
0	0	1	0
0	1	1	0
1	%	%	%
1	%	0	1
1	0	0	1
1	1	0	1

TABLE 8

The subformula $\neg q$ does not appear neither in the above condition nor in the corresponding table. The three-valued non truth-functional semantics has the *subformula property* but not the bivalent non truth-functional semantics. That is basically what we have gained.

4 Conclusion

If this paper we have presented the concept of many-valued non-truth functional semantics, in particular comparing it to many-valued truth-functional semantics and bivalent non-truth functional semantics. We have tried to show the usefulness of this concept through an example of a three-valued non-truth functional semantics. But of course there are other examples. One can develop four-valued non-truth functional semantics, etc.

These non truth-functional many-valued semantics basically keep a bivalence feature through the distinction between designated and undesignated values, but this is also the case of the matricial truth-functional many-valued semantics, so it cannot be considered as an argument against them, unless it is also considered as an argument against standard many-valued logics.

From the point of view of truth-functional many-valuedness, a many-valued logic is a logic that cannot be characterized by a two-valued matrix. If we want to generalize this definition to non truth-functional many-valuedness, we face a problem since any logic can be characterized by a two-valued non truth-functional semantics.

Anyway it seems that the standard concept of many-valued logic is quite confuse. In fact if we do not limit the matricial definition of many-valued logic to logics that can be characterized by finite matrices, any logic is many-valued (due to Lindenbaum's theorem), except classical logic, which is not by convention, considering that 2 is not enough to be "many". It seems to us that the standard concept of many-valued logic should be withdrawn: logics which can be characterized by finite matrices should simply be called truth-functional, with the addition of the smallest cardinality for which they can be characterized by a matrix.

On the other hand the expression "many-valued semantics" should be kept but its meaning should be extended in order to include not only finite or infinite matrices, but also non truth-functional many-valued semantics.

These many-valued semantics are useful tools for the study of logics defined as sets of tautologies or consequence relations, but can also be used in a more radical way to generate logics which challenge the principle of bivalence in a deeper sense, and which truly deserve the name “many-valued logics”.

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J.-Y.Beziau, A.Costa-Leite and A.Facchini (eds)
Aspects of Universal Logic, 219-230

Towards a General Theory of the Combination of Logics

Alexandre Costa-Leite [1]

Abstract

The purpose of this article is to present some general concepts and problems related to the combination of logics.[2]

A brief history of the combination of logics is presented with the aim of finding unity in the methods for combining logics. Some general notions are analized (for instance, methods entail methods) - as well as some problems: the paradox of the combination of logics and the collapsing problem. Despite the existence of different methods for combining logics, is it be possible to come up with a universal and general approach able to unify the subject? I propose the *powerful method problem* and argue that a positive answer to this question can also be seen as an initial clue towards a general theory on the combination of logics.

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[1] Work supported by a grant of the Swiss National Science Foundation.

[2] I decided to write this article while I was reading the tutorial given by C. Caleiro in the *First World Congress and School on Universal Logic.* Caleiro says the following: "We adopt a methodological abstract viewpoint that is concerned with general universal mechanisms for combining logics. Rather than focusing on the specific details of the combination of particular logics, we aim at rigorously defining a logic combination mechanism at the adequate level of abstraction and then establishing meaningful transference results that may be used in many situations. The typical questions to be asked and answered are: (i) When does it make sense to combine two given logics and what is the result? (ii) If two logics with property P are combined does the resulting logic inherit the property P?"

1 Introduction

Questions about the nature of logical systems and logics are controversial and there is no global agreement related to the following problems: (A) What is a logical system? (B) What is a logic? Despite the plurality of answers, it is possible to *assume* that a logic is a pair composed by a set of propositions, indeed an algebra of propositions, and a consequence relation (i.e. logical consequence) on this set [3]. Not only are there many definitions of logic but also there is not just only one logic. The twentieth century has shown that logics are like some kind of computer virus, in the sense that each day a new one appears. To understand all these varieties of logics, it was urgent to find a way to unify this multiplicity. In this sense, *universal logic* was created in order to study general properties of logical systems (see [1]). Universal logic is a general theory of logical systems motivated basically by the proliferation of the logical systems available in the logical land. Universal logicians try to develop abstract tools which can be used to understand a plurality of logics from an abstract viewpoint instead of investigating a particular logic.[4]

In the same way that there are different logics, there are also many methods to combine logics: temporalization, synchronization, fusion, product, fibring etc. There is a vast literature about each one of these methods. They are useful in the sense that they allow us to better understand complex statements in natural languages as well as helping us understand problems related to technical fields like computer science (see [14]).

Indeed, combining logics is not an easy task, assuming that there are many definitions of the logical systems, as well as of different logics, and that there are many different presentations of the same logic: sequent calculus, natural deduction, tableaux, different semantics, etc. For this reason, it is urgent to develop a general theory on the combination of logics, able to answer, at least, one basic question: What is the definition of the combination of logics?

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[3] A consequence relation without restriction means that it does not need to respect the Tarskian axioms for logical consequence

[4] It seems to me that universal logic is a kind of platonism, given that it accepts that logics, despite of their different manifestations, have something in common, a kind of essence. It is the same idea but now applied to logics.

Such a theory must also be able to answer the following questions: How to unify all these methods for combining logics? What general properties are inherent in all of them? Is there a general theory about methods for combining logics? This article gives a clue about how to answer these questions applying a universal approach to the combination of logics.

2 A brief history of the combination of logics

There are many different perspectives on combination. One may want to combine two given logics, combining all their operators or one may want add a particular feature to a given logic, for example: temporalization - adding the concept of time to a given logic - fuzzification - adding a fuzzy character to a logic - paraconsistentization of logics - given a logic, how to obtain the paraconsistent counterpart of this logic. There are also different kinds of decomposition of a given logic into fragments such as for example possible-translation semantics [5] and [13].

At the beginning, combination of logics appeared in the environment of modal logic and, therefore, many methods were specially created to model on the one hand combination of Kripke Structures, on the other hand combination of axiomatic systems, although nowadays they are applied to a great variety of model-theoretical and proof-theoretical notions. The ideal would be to look for a universal conception of logical structure and thus to define an abstract method for combining logics independently of any particular conceptions of a logic.

The simplest method introduced to combine modal logics was fusion (see [19]). Semantically speaking, fusion consists in putting together without interferences two Kripke semantics, that is to say putting side by side the accessibility relations. Fusion of Hilbert proof systems also consists just of putting together rules and axioms. The basic idea underlying fusion is the combination of the languages of the two logics, done in a natural way, the rest follows straightforwardly. Fusion generally preserves soundness, completeness, finite model property etc.

Another method, more complicated, is product. It was first used by Shehtman to introduce a two-dimensional modal logic. Product defines the notion of dimension in logics.[5] Fusions and products have been very useful in modal logics [6] and in issues related to the modelling of philosophical concepts. A difficult and interesting task is to determine the proof-theoretical counterpart of a product of models (see [17]) [7], i.e. which method for combining proof-theoretical systems preserves completeness for product of semantics.

Even fibring, the most famous method, was developed in the context of modal logics. According to Dov Gabbay, who proposed fibring, this mechanism allows us to associate to each possible world a model using a fibring function. Fibring is a method used to combine logics while evaluating a formula which has an operator that can not be recognized in a particular language. Fibring is considered the most powerful method of combining logics because it allows, in one of its variations, interactions between languages. Gabbay uses an idea of fibring which is based on the idea of a fibring function, and on complex models with a fibring function - however, nowadays there are different definitions of fibring which are strongly related to different conceptions of logic (See for example the works of the Portuguese school and the importance of selecting the right level of abstraction or the working universe of logics.) To determine if metalogical notions as soundness and completeness are preserved by fibring depends strongly on a given conception of logic and fibring (see [12]).

3 What is combined?

A logician who combines logics performs a task similar to that of chemists, but instead of atoms and molecules, the logicians deal with languages, models and logics. The chemist has tools to realize the process of combining substances and afterwards he/she separates them. And the logicians also

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[5] Not just modal logics, but logics in general.

[6] A nice presentation of multi-dimensional modal logics, and of how it is obtained by products, is presented in [20]. Another more recent book about fusion, products and problems related to these constructions is [16]

[7] For many cases it is known.

have their own methods and techniques for combining logics and decomposing logics.

When logicians decide to combine logics they have to select the correct level of abstraction related to the nature of logics. It does not make sense to combine logics without stating, first, what is *assumed* as a logic.

To explain the problem let me mention the works developed by the Portuguese school, especially [12]. The simplest type of logic considered is called a *consequence system*, which is a pair composed of a set and a consequence operator obeying Tarski's axiom. However, this kind of consequence system does not constitute the right level of abstraction to combine logics, because the structure of the formulas is not defined, but it can be useful to introduce the combination of logics in a high level of abstraction. The next natural step is to determine the structure of the formulas and to work with a structural consequence operator in the sense of Łoś and Suszko. The Portuguese School go further by defining the notion of *deductive systems*, which is a structural consequence operator together with a set of inference rules [12]. Then they define the fibring of deductive systems. On the other hand they also introduce the concept of *interpretation systems* in order to combine semantics. An *interpretation system* is a structural consequence operator together with a set of interpretation structures, i.e. models [12]. They then define fibring of interpretation systems. To define what a logical system is, they make use of both notions. From [12] we learn that: logical systems are obtained by putting together deductive systems and interpretation systems in order to create a nice environment to talk about soundness and completeness. Therefore, logical systems are a good level of abstraction to begin to define particular combination methods between logics. Almost all results obtained in the combination of logics depend strongly on particular conceptions of the logical systems and of the methods for combining logics.[8]

In the same way that a theory of truth should be able to answer the question about what truth is and what criteria are used to determine when a proposition is true, a general theory on the combination of logics should be able to answer the question about what combination is and what the

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[8] I mentioned the article [12] just as an example, but these techniques appear in all the works of the Portuguese school, see for example Caleiro's PhD thesis [4] .

procedure is for combining logics[9]

There are many different methods of combination: fusion, product, fibring, synchronization, etc, each one applying to some particular proof systems, semantical structures, logical structures incorporating or not proof or semantical features. However, up to now there is no general clear definition of what combination of logics is, independent of circumstances.
Gabbay in [18] says that the combination of two logics is the least conservative extension defined on the combined language, but as Caleiro noted that two logics may not have a common conservative extension. Beziau gave a simple example: the combination of a logic with only a classical negation and a logic with only an intutionistic implication. In the combined logic, the intuitionistic implication becomes classical, therefore we don't have a conservative extension.

4 General questions about combination of logics

Instead of investigating particular conceptions of logics and particular methods for combining logics, we can inquire what are the general problems related to the combination of logics.

4.1 Preservation of properties

The first, and one of the most important and popular problems in the combination of logics, is the question about the *preservation of properties* or the *transfer theorems*. Preservation of properties is very normal in mathematical theories. It is also called invariance results. Just as an example, in the case of modal logics, the question is to know if modal satisfaction is preserved if we apply some operations to our models (bisimulations etc) [3]. This question can be stated as follows: given two logics each having

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[9] Carnielli and Coniglio have tried to give a categorial definition of combining logics [6].

the property P, which methods of combination of these two logics preserve this property, i.e. produce a combined logic which is P? This property can be truth-functionality, completeness, the finite model property, etc.[10]

To give an answer to the question above we should have information about the nature of the logics considered and about the nature of the method used in the combination. This problem - let us call it the *preserving properties problem* - is a good example of a universal question related to the combination of logics. As it is known, the process of combining logics can be realized in two different directions. The first one is the combination, literally, of logics. The second one is the decomposition of logics. The same general problem above applies also in the last case. Given a complex logic, if it is decomposed, how goes the preservation of its properties for its fragments?

4.2 Categorial Representation

Instead of investigating particular objects and particular transformations between these objects, the pure categorist studies general constructions with categories. Logical systems can be viewed as objects of categories. In this sense it is possible to construct categories of deductive systems, of interpretation systems and of logic systems [12].

An interesting problem that arises in questions about the combination of logics is the categorial representation of a given method. For example, in [10] it is proved that fibring is a kind of universal construction in a particular category. The second general problem related to the combination of logics is the *categorial representation problem* and can be stated as follows: *Given a method for combining logics, does this method have a universal construction in a category?*

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[10] Concerning the preservation of properties (soundness, completeness etc) related to particular methods I recommend [9, 10, 17, 18, 19].

4.3 Methods entail Methods

The third problem related to our universal approach to the combination of logics is the *methods entail methods problem.* This problem is a clue in the direction of finding a powerful and universal method for combining logics. When one selects a particular method for combining logics, would there be another method that could be generated by this method? To illustrate this problem, note that in [9] the authors show a plurality of relations between fibring, synchronization and parameterization and show how we can deduce one from another. This is a clear example of the *methods entail methods problem*, which can be stated as: Given a method for combining logics, does this method imply other methods?

5 Paradoxes related to the combination of logics

Although the fact that combining logics allows us to get more powerful logics, there are some problems related to the basic concepts in the subject.

5.1 The collapsing paradox

Gabbay has pointed out (see [17]) that the fibration of two logics leads to collapse. For example the fibring of classical implication and intuitionistic implication logic leads to collapses into classical implication. Logicians are developing many variations of fibring trying to solve this problem (see for instance [15] and [11]).

5.2 The copulation paradox

Beziau has pointed out a interesting problem arising with combination of logics [2]. It deals with combination of truth-tables. If one puts together in a natural way the standard semantics of conjunction and the standard semantics of disjunction, one gets a logic in which distributivity holds between conjunction and disjunction, so the combined logic produced by this combination of semantics is not the least conservative extension of the logic of conjunction and the logic of disjunction. Beziau calls this phenomenon by the suggestive name "copulation paradox", because the conjunction and the disjunction are interacting and producing a new property.

6 Conclusion

A very important task in the combination of logics is to find the right level of abstraction for logical structures. Logicians usually prefer to speak about logical structures where it is possible to express syntactical and semantical properties of the logics like, for instance, soundness and completeness [11]. After deciding about the best way to express what a logic is, it is then possible to define operations between these logics as, for example, fibring. In this sense, it is possible to define many varieties of fibring: fibring of deductive systems, interpretation systems, logical systems presentation and so on [12]. Methods for combining logics are tools which can be used by the universal logician to find again a unity in logic.

The *powerful method problem* is the intersection of the three general problems presented in this paper:

1) Is it possible to find a universal method able to show that most of all known methods are particular cases?
2) Is it possible to give a categorial characterization of this method?
3) What are the properties preserved by this method?

A positive answer to the above questions would probably be a paradise for those who are working in the combination of logics. Many people think that fibring would be a solution to this problem, but there are some problems with fibring such as the collapsing problem. This problem leads many logic combinators to propose new kinds of fibring as for example [11] and [15]. Despite these proposals, we do not have any guarantee that the powerful method problem is already solved.

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[11] For example, Caleiro's PhD thesis

In order to combine logics it should first be clear what a logic is. However, there is not a unique answer to this question. The most that one can do is suppose that a logic is something in particular and see what follows from the supposition. It is reasonable enough to depart from a conception of logic which permits us to speak about the two sides of a logical system: its syntax and semantics. Afterwards, we have to be able to determine exactly what are the properties of an abstract method for combining logics. Using a method for combining logics, we can enrich our languages and consequently our logics, being able thus to better understand formal languages and their applications.

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Introduction of Implication and Generalization in Axiomatic Calculi

Arthur Buchsbaum and Jean-Yves Beziau[1]

Abstract

Introduction of implication and generalization rules have a close relationship, for which there is a key idea for clarifying how they are connected: *varying objects.* Varying objects trace how generalization rules are used along a demonstration in an axiomatic calculus. Some ways for introducing implication and for generalization are presented here, taking into account some basic properties that calculi can have.

1 Introduction

The rules for introducing material implication share the following form:
- If $\Gamma \cup \{\alpha\} \vdash \beta$, then, under certain conditions, $\Gamma \vdash \alpha \to \beta$.

In the most simplest case there are no other additional conditions for concluding that $\Gamma \vdash \alpha \to \beta$. It occurs for *closed* and *partial strong calculi,* in which no of their rules involve any generalization, and all of them preserve in a way the consequents of implications. Closed calculi can also simulate generalization rules in some way if they also have non primitive but *admissible rules* dealing with some generalization forms.

More complex cases, related to *open calculi,* require a kind of tracing of how generalization rules (such as introduction of universal quantifier, introduction of necessity, and so on) must be used for deducing that $\Gamma \cup \{\alpha\} \vdash \beta$,

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in order to conclude that $\Gamma \vdash \alpha \rightarrow \beta$.

It can be observed, by a careful examination of the classical logic books, that two distinct choices for the introduction of implication and generalization rules have been made standard:

1st) The rule for introducing implication has no restrictions, but there are constraints for introducing the universal quantifier and other operators of this kind. A calculus adopting this strategy is called *closed* in our context. It is more often used in calculi presented in natural deduction and sequent calculus style. Examples of closed calculi may be found in [1], [13], [6], [7] and [11]. However, this closed option may be very cumbersome when used to calculi presented in axiomatic style having varying objects other than variables, such as in modal logics.

2nd) The introduction of implication is done with restrictions, but the introduction of the universal quantifier and other analogous operators is unconditional. This strategy is more often adopted for axiomatic formulations. These calculi are called *open*. Examples may be found in [8], [9] and [12].

Some well known formulations of introduction of implication rule, in the context of open calculi, presented in axiomatic style, that can be found in the literature, present some undesirable features, such as:

- explicit use of the concept of demonstration, instead of an idea of a higher level dealing with syntactic consequence;
- lack of an adequate tracing to accompany the use of varying objects in rules of generalization, making difficult in many situations to know when introduction of implication is allowed.

Below we will give two examples of formulations for introduction of implication, commonly found in the literature, that suffer from the abovementioned ills:

- "For the predicate calculus (or the full number-theoretic formal system), if $\Gamma, A \vdash B$ with the free variables held constant for the last assumption formula A, then $\Gamma \vdash A \rightarrow B$." According to [8], pg. 97.
- "Assume that $\Gamma, A \vdash B$, where, in the deduction, no application of Gen to a wf which depends upon A has as its quantified variable a free variable of A. Then $\Gamma \vdash A \rightarrow B$." According to [9], pg. 63.

In [8], chapter 5, pages 94–106, there are some ideas which were the main basis from which the present paper evolved, specially the idea of free variables being held constant with respect to the premises along a demonstration in an axiomatic calculus. This idea was extended and systematized by us to a more general concept, named *varying objects.* We have developed a study about them in an abstract way, for dealing with a broad spectrum of calculi. This paper is a new version and expansion of two former ones, [4] and [3].

Varying objects are a kind of extension of the concept of variables. They can trace what kind of applications of generalization rules are used along a demonstration. There are two ways of tracing, name here *dependence* and *supporting.* Both only consider applications of generalization rules having a hypothesis depending in the considered demonstration on some premise in it. The first tracing method, dependence, also considers if the varying objects used in applications of generalization rules are free in some hypothesis, whereas the second tracing method, supporting, does not take it into account.

Besides the usual binary syntactic consequence relation between a collection of formulas Γ and a formula α, defined by an axiomatic calculus $\mathbf{C}$, noted here by $\Gamma \vdash_{\mathbf{C}} \alpha$, the two tracing methods define two new syntactic consequence relations, named here *dependence consequence* and *supporting consequence*, which are ternary relations, in the sense that they relate a collection of formulas Γ, a formula α, and a collection of varying objects $\mathcal{V}$. They are noted respectively by $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$ and $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$.

These two new consequence relations share many common properties, but there are some properties exclusive to each one of them.

Under certain conditions, dependence consequence and supporting consequence can be partially (in *partial stable calculi*) or completely (in *stable calculi*) equivalent. Partial stable calculi have a weak formulation for generalization rules applied to supporting consequence, whereas stable calculi have a strong formulation for them.

Partial strong calculi have a weak formulation for introduction of implication with respect to supporting consequence, whereas *strong calculi* have a strong formulation for it.

The open calculi having the strongest formulations for introduction of implication and generalization rules are the *strong stable* ones.

The concepts here presented were already applied by the first author in the formulations and generalized proofs of metatheorems for some non classical calculi in [5], [10] and [2].

2 Basic Concepts

In this section we define some basic ideas related to axiomatic calculi, such as schemas, inference rules, axioms, applications and demonstrations, from which it is specified the basic consequence relation of an axiomatic calculi.

2.1 Notation. From now on, unless stated otherwise, we adopt the following conventions for the following letters, with or without primes and/or subscripts:

- $\mathbf{L}$ is a formal language;
- $\alpha, \beta, \gamma, \delta$ are formulas of $\mathbf{L}$;
- Γ, ϑ, ζ are collections of formulas of $\mathbf{L}$.

2.2 Definition. A collection of formulas of $\mathbf{L}$ is also said a *schema* (*in* $\mathbf{L}$). An *inference rule* (*in* $\mathbf{L}$) is a collection of n-tuples of formulas (of $\mathbf{L}$), for some $n \geq 2$. A *postulate* (*in* $\mathbf{L}$) is a schema (in $\mathbf{L}$) or an inference rule (in $\mathbf{L}$). An (*axiomatic*) *calculus* (*in* $\mathbf{L}$) is a pair $\langle \mathbf{L}, \mathbb{P} \rangle$, whereon $\mathbb{P}$ is a collection of postulates in $\mathbf{L}$. If $\mathbf{C} = \langle \mathbf{L}, \mathbb{P} \rangle$ is a calculus, then $\mathbf{L}$ is said to be *its language*, and $\mathbb{P}$ is said to be *its basis*. A schema belonging to the basis of a calculus is said to be *a schema of it*. A rule belonging to the basis of a calculus is said to be *a rule of it*. A schema or a rule of a calculus are also said to be *a postulate of it*.

2.3 Notation. From now on, unless stated otherwise, $\mathbf{C} = \langle \mathbf{L}, \mathbb{P} \rangle$ is an axiomatic calculus.

2.4 Definition. An *application* (*of an inference rule*) (*in* $\mathbf{L}$) is an element of an inference rule (in $\mathbf{L}$).

2.5 Notation. If $\langle \alpha_1, \ldots, \alpha_n, \beta \rangle$ is an application, we also note it by $\dfrac{\alpha_1, \ldots, \alpha_n}{\beta}$.

2.6 Definition. If $\dfrac{\alpha_1, \ldots, \alpha_n}{\beta}$ is an application, $\alpha_1, \ldots, \alpha_n$ are said to be *their hypotheses*, and β is said to be *its conclusion* or *consequence*. We also say that β is a *conclusion* or *consequence over* $\alpha_1, \ldots, \alpha_n$ *by* this application.

2.7 Notation. When there is no possibility of confusion, we note a given schema simply by writing down a generic element of it. The same is done for rules – we note a given rule simply by writing down a generic element of it.

2.8 Examples. The schema $\{(\alpha\wedge\beta)\to\beta \mid \alpha, \beta \text{ are formulas in } \mathbf{L}\}$ is noted simply by $(\alpha\wedge\beta)\to\beta$. At the same way, we note the rule $\left\{\dfrac{\alpha, \alpha\to\beta}{\beta} \mid \alpha, \beta \text{ are formulas in } \mathbf{L}\right\}$ by $\dfrac{\alpha, \alpha\to\beta}{\beta}$.

2.9 Definition. The *domain of an inference rule* is the collection of all tuples $\langle\alpha_1, \ldots, \alpha_n\rangle$ such that there exists β for which $\dfrac{\alpha_1, \ldots, \alpha_n}{\beta}$ is an application of this rule.

2.10 Definition. An inference rule is said to be *unary* if each application of it has only one hypothesis.

2.11 Definition. A formula belonging to a schema of $\mathbf{C}$ is said to be an *axiom of* $\mathbf{C}$. An application belonging to a rule of $\mathbf{C}$ is said to be an *application of (a rule of)* $\mathbf{C}$.

2.12 Definition. A *demonstration in* $\mathbf{C}$ *of* α *from* Γ is a finite non empty sequence $\mathcal{D}$ of formulas of $\mathbf{L}$ such that α is the last formula of $\mathcal{D}$ and, for each formula β of $\mathcal{D}$, at least one of the following conditions is satisfied:

- β is an axiom of $\mathbf{C}$;
- $\beta\in\Gamma$ (in this case we also say that β *is justified in* $\mathcal{D}$ *as being a premise*);
- there is an application $\dfrac{\beta_1, \ldots, \beta_n}{\beta}$ of $\mathbf{C}$ such that each β_i, for any $i \in \{1, \ldots, n\}$, precedes β in $\mathcal{D}$.

If there is a demonstration in $\mathbf{C}$ of α from Γ, we also note it by $\Gamma \vdash_{\mathbf{C}} \alpha$, and we say that α is a *consequence from* Γ *in* $\mathbf{C}$, or that α is a *theorem of* Γ *in* $\mathbf{C}$. We also note "$\emptyset \vdash_{\mathbf{C}} \alpha$" by "$\vdash_{\mathbf{C}} \alpha$". If $\vdash_{\mathbf{C}} \alpha$, α is also said to be a *thesis of* $\mathbf{C}$.

2.13 Theorem. A formula α is a consequence from Γ in $\mathbf{C}$ if, and only if, at least one of the following conditions is fulfilled:

- α is an axiom of $\mathbf{C}$;
- $\alpha\in\Gamma$;
- there is an application $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ of a rule of $\mathbf{C}$ such that $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_n$.

2.14 Theorem. The following properties are valid for the relation "$\vdash_{\mathbf{C}}$":

(i) if α is an axiom of $\mathbf{C}$, then $\vdash_{\mathbf{C}} \alpha$;

(ii) if $\alpha\in\Gamma$, then $\Gamma \vdash_{\mathbf{C}} \alpha$;

(iii) if $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ is an application of $\mathbf{C}$, then $\{\alpha_1, \ldots, \alpha_n\} \vdash_{\mathbf{C}} \alpha$;

(iv) if $\Gamma \vdash_{\mathbf{C}} \alpha$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash_{\mathbf{C}} \alpha$;
(v) if $\Gamma \vdash_{\mathbf{C}} \alpha_1, \ldots, \Gamma \vdash_{\mathbf{C}} \alpha_n$ and $\{\alpha_1, \ldots, \alpha_n\} \vdash_{\mathbf{C}} \beta$, then $\Gamma \vdash_{\mathbf{C}} \beta$;
(vi) if $\Gamma \vdash_{\mathbf{C}} \alpha$, then there is $\Gamma' \subseteqq \Gamma$ such that Γ' is finite and $\Gamma' \vdash_{\mathbf{C}} \alpha$.

Next we provide two simple syntactic ideas which are used in some examples given in this paper.

2.15 Definition. In a language in which "$\forall$" is a quantifier, a *universal generalization of* α is a formula of the form $\forall x_1 \ldots \forall x_n \, \alpha$ $(n \geq 0)$.

2.16 Definition. In a language in which "$\Box$" is an unary connective, we say the $\Box$ *is free in* a given formula α if there is a subformula of α out of the scope of $\Box$ in it.

3 Variation, Dependence and Supporting

In this section the idea of inference rule and its applications is expanded by attaching to each application a set of varying objects. From this new departure two new consequence relations are defined. Their basic properties and the interrelationship between them and with the basic consequence relation are presented.

3.1 Notation. From now on, unless stated otherwise, we adopt the following conventions for the following letters, with or without primes and/or subscripts:

- $\alpha, \beta, \gamma, \delta$ are formulas in $\mathbf{C}$;
- Γ, ϑ, ζ are collections of formulas in $\mathbf{C}$.

Below we extend the concept of application of a rule of inference by attaching to it a collection of things named *varying objects*.

3.2 Definition. For each application of a rule of inference, we attach to it a collection whose elements are named its *varying objects*. A rule whose applications do not have varying objects is said to be a *constant rule*; otherwise we say that it is a *varying rule*. We say that $\mathbf{o}$ is a *varying object in* $\mathbf{C}$ if there is an application of a rule in $\mathbf{C}$ such that $\mathbf{o}$ is a varying object of this application. For each calculus $\mathbf{C}$, it is specified when a varying object $\mathbf{o}$ is *free in* a given formula α. The following additional conditions are to be fulfilled:

- the number of varying objects of each application of a rule in $\mathbf{C}$ is finite;

- each varying object of an application of a rule is not free in the consequence of this application.

3.3 Examples. In practice, we find the following varying objects:

- variables used in universal quantification: "x" is the varying object of the application $\frac{\alpha}{\forall x\, \alpha}$ of the rule of universal generalization, which occurs in many quantificational logics;
- the hidden variable used for introducing connectives associated with modalities such as necessity; such variable can be indicated by the sign itself introduced by the rule: $\Box$ is the varying object of the rule $\frac{\alpha}{\Box\alpha}$, which is present in many modal logics.

3.4 Notation. From now on, unless stated otherwise, we adopt the following conventions for the following letters, with or without primes and/or subscripts:

- $\mathbf{o}$ is a varying object in $\mathbf{C}$;
- $\mathcal{V}, \mathcal{W}$ are collections of varying objects in $\mathbf{C}$.

3.5 Definition. A calculus is said to be *closed* if all their rules are constant, otherwise it is said to be *open*.

3.6 Definition. A given rule $\mathbf{r}$ is said to be *admissible in* a closed calculus $\mathbf{C}$ if it satisfies the following conditions:

- $\mathbf{r}$ is unary;
- the domain of $\mathbf{r}$ is the collection of all formulas of $\mathbf{C}$;
- $\mathbf{r}$ is a varying rule;
- $\mathbf{r}$ is not an inference rule of $\mathbf{C}$;
- if α is an axiom of $\mathbf{C}$ and $\frac{\alpha}{\alpha'}$ is an application of $\mathbf{r}$, then $\vdash_{\mathbf{C}} \alpha'$;
- if $\frac{\alpha}{\alpha'}$ is an application of $\mathbf{r}$ such that no varying object of it belongs to α, then $\alpha \vdash_{\mathbf{C}} \alpha'$;
- if $\frac{\alpha_1, \ldots, \alpha_n}{\alpha}$ is an application of a rule of $\mathbf{C}$ and $\alpha'_1, \ldots, \alpha'_n, \alpha'$ are respectively consequences of applications of $\mathbf{r}$ over $\alpha_1, \ldots, \alpha_n, \alpha$, using the same collection of varying objects, then $\alpha'_1, \ldots, \alpha'_n \vdash_{\mathbf{C}} \alpha'$.

3.7 Example. Let $\mathbf{C}$ be a calculus whose axioms have the following forms, including all their universal generalizations:

- $\alpha \to \forall x\, \alpha$, whereon x is not free in α;
- $\forall x\, (\alpha \to \beta) \to (\forall x\, \alpha \to \forall x\, \beta)$.

The only rule of **C** is $\dfrac{\alpha, \alpha \to \beta}{\beta}$, which is a constant rule.
We have that **C** is a closed calculus such that the varying rule $\dfrac{\alpha}{\forall x\, \alpha}$, whereon the varying object of each application is the quantified variable, is admissible in **C**.

3.8 Theorem.

$$\bullet\ \text{If} \begin{cases} * \ \mathbf{C} \text{ is a closed calculus,} \\ * \ \Gamma \vdash_{\mathbf{C}} \alpha, \\ * \ \dfrac{\alpha}{\alpha'} \text{ is an application of an admissible rule of } \mathbf{C}\text{, such that each of its} \\ \quad \text{varying objects are not free in } \Gamma, \end{cases}$$

then $\Gamma \vdash_{\mathbf{C}} \alpha'$.

Proof:
If α is an axiom, then, by definition 3.6, $\vdash_{\mathbf{C}} \alpha'$, therefore $\Gamma \vdash_{\mathbf{C}} \alpha'$.
If $\alpha \in \Gamma$, then no varying object of the application $\dfrac{\alpha}{\alpha'}$ is free in α, so, by definition 3.6, $\alpha \vdash_{\mathbf{C}} \alpha'$, therefore $\Gamma \vdash_{\mathbf{C}} \alpha'$.

If there is an application $\dfrac{\beta_1, \ldots, \beta_n}{\alpha}$ of **C** such that $\Gamma \vdash_{\mathbf{C}} \beta_1, \ldots, \Gamma \vdash_{\mathbf{C}} \beta_n$, and $\beta'_1, \ldots, \beta'_n$ are respectively consequences of $\beta_1, \ldots, \beta_n$ by the same rule in which α' is a consequence of α, using the same collection of varying objects, we have, by induction hypothesis, that $\Gamma \vdash_{\mathbf{C}} \beta'_1, \ldots, \Gamma \vdash_{\mathbf{C}} \beta'_n$. By definition 3.6, $\beta'_1, \ldots, \beta'_n \vdash_{\mathbf{C}} \alpha'$, therefore $\Gamma \vdash_{\mathbf{C}} \alpha'$. ☐

3.9 Definition. Let $\mathcal{D} = \alpha_1, \ldots, \alpha_n$ be a demonstration in **C**. We say that *α_i is relevant to α_j in $\mathcal{D}$* ($i, j \in \{1, \ldots, n\}$) if one of the following conditions is fulfilled:

- $i = j$ and α_j is justified in $\mathcal{D}$ as a premise;
- α_j is justified in $\mathcal{D}$ as a consequence of an application $\dfrac{\beta_1, \ldots, \beta_p}{\alpha_j}$ of a rule in **C** and there exists a hypothesis β_k ($k \in \{1, \ldots, p\}$) of this application such that α_i is relevant to β_k in $\mathcal{D}$.

3.10 Definition. We say that a demonstration $\mathcal{D}$ in **C** *depends on* a collection $\mathcal{V}$ of varying objects if $\mathcal{V}$ contains the collection of varying objects $\mathbf{o}$ of applications of rules in $\mathcal{D}$ having a hypothesis in which $\mathbf{o}$ is free such that there is a formula, justified as a premise in $\mathcal{D}$, whereon $\mathbf{o}$ is free too, relevant to this hypothesis in $\mathcal{D}$. If there is a demonstration in **C** of α from Γ such that it depends on $\mathcal{V}$, we say that α depends on $\mathcal{V}$ from Γ in **C**, and we note this by $\Gamma \overset{\mathcal{V}}{\vdash}_{\mathbf{C}} \alpha$. If $\mathcal{V} = \{\mathbf{o}_1, \ldots, \mathbf{o}_n\}$ and $n \geq 1$, we also note this by $\Gamma \overset{\mathbf{o}_1, \ldots, \mathbf{o}_n}{\vdash}_{\mathbf{C}} \alpha$. If $\mathcal{V} = \emptyset$, we say that $\mathcal{D}$ is an *unvarying* *demonstration in* $\mathbf{C}$. If α depends on $\emptyset$ from Γ in $\mathbf{C}$, we say that it is an *unvarying consequence of* Γ *in* $\mathbf{C}$.

3.11 Theorem. A formula α depends on $\mathcal{V}$ from Γ in $\mathbf{C}$ if, and only if, at least one of the following conditions is fulfilled:

- α is an axiom of $\mathbf{C}$;
- $\alpha \in \Gamma$;
- there is an application $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ of a rule in $\mathbf{C}$ such that $\Gamma \vdash_{\mathbf{C}}^{\mathcal{V}} \alpha_1, \ldots, \Gamma \vdash_{\mathbf{C}}^{\mathcal{V}} \alpha_n$ and, for every varying object $\mathbf{o}$ of this application such that $\mathbf{o} \notin \mathcal{V}$ and for every α_i $(1 \leq i \leq n)$, if $\mathbf{o}$ is free in α_i, then there is $\Gamma' \subseteq \Gamma$, such that $\mathbf{o}$ is not free in Γ' and $\Gamma' \vdash_{\mathbf{C}}^{\mathcal{V}} \alpha_i$.

If $\mathcal{V} = \emptyset$, we can replace the third clause above by the following condition:

- there exists an application $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ of a rule in $\mathbf{C}$, such that $\Gamma \vdash_{\mathbf{C}}^{\emptyset} \alpha_1, \ldots, \Gamma \vdash_{\mathbf{C}}^{\emptyset} \alpha_n$ and, for every varying object $\mathbf{o}$ of this application and for every α_i $(1 \leqslant i \leqslant n)$, if $\mathbf{o}$ is free in α_i, then there exists $\Gamma' \subseteq \Gamma$ such that $\mathbf{o}$ is not free in Γ' and $\Gamma' \vdash_{\mathbf{C}}^{\emptyset} \alpha_i$.

3.12 Examples. Let $\mathbf{C}$ be a calculus without schemas whose inference rules are the following:

- $\dfrac{\alpha}{\forall x \alpha}$, whereon the varying object of each application is the quantified variable;
- $\dfrac{\alpha}{\Box \alpha}$, whereon $\Box$ is the varying object of all applications.

The following propositions provide examples of dependence consequence:

- $p(x, y) \vdash_{\mathbf{C}}^{x,y} \forall x \forall y \forall z\, p(x, y)$;
- $p(x, y, z) \vdash_{\mathbf{C}}^{x,y,z} \forall x \forall y \forall z\, p(x, y, z)$;
- $p(x) \vdash_{\mathbf{C}}^{x,\Box} \Box \forall x\, p(x)$;
- $\Box p(x) \vdash_{\mathbf{C}}^{x} \Box \forall x\, \Box p(x)$;
- $\Box p(x, y) \vdash_{\mathbf{C}}^{x,y} \Box \forall x \forall y\, \Box p(x, y)$.

3.13 Theorem. The following properties are valid for the relation "$\vdash_{\mathbf{C}}^{\mathcal{V}}$":

(i) if there is a demonstration $\mathcal{D}$ in $\mathbf{C}$ of α from Γ whose collection of varying objects of applications of rules of $\mathbf{C}$ in $\mathcal{D}$ is $\mathcal{V}$, then $\Gamma \vdash_{\mathbf{C}}^{\mathcal{V}} \alpha$;

(ii) if $\Gamma \vdash_{\mathbf{C}}^{\mathcal{V}} \alpha$, then $\Gamma \vdash_{\mathbf{C}} \alpha$;

(iii) if $\Gamma \vdash_{\mathbf{C}} \alpha$, then there is a collection $\mathcal{V}$ of varying objects such that $\Gamma \vdash_{\mathbf{C}}^{\mathcal{V}} \alpha$;

(iv) $\vdash_{\mathbf{C}} \alpha$ iff $\vdash_{\mathbf{C}}^{\emptyset} \alpha$;

(v) if **C** is closed, then $\Gamma \vdash_{\mathbf{C}} \alpha$ iff $\Gamma \vdash^{\emptyset}_{\mathbf{C}} \alpha$;

(vi) if α is an axiom of **C**, then $\vdash^{\emptyset}_{\mathbf{C}} \alpha$;

(vii) if $\alpha \in \Gamma$, then $\Gamma \vdash^{\emptyset}_{\mathbf{C}} \alpha$;

(viii) if $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ is an application of **C** whose collection of varying objects is $\mathcal{V}$, then $\{\alpha_1, \ldots, \alpha_n\} \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$;

(ix) if $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ is an application of **C** such that $\mathcal{W}$ is the collection of all varying objects $\mathbf{o}$ of this application in which $\mathbf{o}$ is free in some of their hypotheses, then $\{\alpha_1, \ldots, \alpha_n\} \vdash^{\mathcal{W}}_{\mathbf{C}} \alpha$;

(x) if $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$ and $\mathcal{V} \subseteqq \mathcal{V}'$, then $\Gamma \vdash^{\mathcal{V}'}_{\mathbf{C}} \alpha$;

(xi) if $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$ and $\Gamma \subseteqq \Gamma'$, then $\Gamma' \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$;

(xii) if $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$, then there is $\mathcal{V}' \subseteqq \mathcal{V}$ such that $\mathcal{V}'$ is finite and $\Gamma \vdash^{\mathcal{V}'}_{\mathbf{C}} \alpha$;

(xiii) if $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$, then there is $\Gamma' \subseteqq \Gamma$ such that Γ' is finite and $\Gamma' \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$;

(xiv) if $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$ and, for each $\mathbf{o} \in \mathcal{W}$, $\mathbf{o}$ is not free in Γ, then $\Gamma \vdash^{\mathcal{V}-\mathcal{W}}_{\mathbf{C}} \alpha$;

(xv) if
- $*$ $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$,
- $*$ for each $\mathbf{o} \in \mathcal{W}$, there exists $\Gamma' \subseteq \Gamma$ such that $\mathbf{o}$ is not free in Γ' an
  $\Gamma' \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$,

then $\Gamma \vdash^{\mathcal{V}-\mathcal{W}}_{\mathbf{C}} \alpha$.

3.14 Example. The following assertions are not valid for the relation "$\vdash^{\mathcal{V}}$".

- if $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_n, \{\alpha_1, \ldots, \alpha_n\} \vdash^{\mathcal{V}}_{\mathbf{C}} \beta$, then $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \beta$;
- if
  - $*$ $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_p$,
  - $*$ $\{\alpha_1, \ldots, \alpha_p\} \vdash^{\mathbf{o}_1, \ldots, \mathbf{o}_n}_{\mathbf{C}} \beta$,
  - $*$ for all $i \in \{1, \ldots, n\}$ and for all $j \in \{1, \ldots, p\}$, if $\mathbf{o}_i \notin \mathcal{V}$ and $\mathbf{o}_i$ is free i
    α_j, then there exists $\Gamma' \subseteq \Gamma$ such that $\mathbf{o}_i$ is not free in Γ' and $\Gamma' \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_j$,

  then $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \beta$.

Proof:
Let **C** be a calculus whose schemas are "$\alpha \to \alpha \vee \beta$" and "$\forall x\, \alpha \to \alpha(x|t)$", and whose rules of inference are

$$\frac{\alpha,\ \alpha \to \beta}{\beta} \text{ and } \frac{\alpha}{\forall x\, \alpha},$$

such that the first is a constant rule and the second is a varying rule in which the varying object of each application is the corresponding quantified variable.

We have that $\begin{cases} \{\forall y\, q(y,z),\ q(y,z) \to r(y)\} \vdash^{\emptyset}_{\mathbf{C}} r(y) \\ r(y) \vdash^{\emptyset}_{\mathbf{C}} \forall z\, (r(y) \vee s(z)) \end{cases}$, however it is not true that $\{\forall y\, q(y,z),\ q(y,z) \to r(y)\} \vdash^{\emptyset}_{\mathbf{C}} \forall z\, (r(y) \vee s(z))$, from which we have a counterexample for the first proposition.

Likewise, we have that $\begin{cases} \{\forall y\, q(y,z),\ \forall y\, q(y,z) \to r(y)\} \vdash^{\emptyset}_{\mathbf{C}} r(y) \\ r(y) \vdash^{y}_{\mathbf{C}} \forall y \forall z\, (r(y) \vee s(z)) \end{cases}$, nevertheless it is not true that $\{\forall y\, q(y,z),\ q(y,z) \to r(y)\} \vdash^{\emptyset}_{\mathbf{C}} \forall y \forall z\, (r(y) \vee s(z))$, from which we have a counterexample for the second proposition. □

3.15 Definition. We say that a demonstration $\mathcal{D}$ in $\mathbf{C}$ *is supported by* a collection $\mathcal{V}$ of varying objects if $\mathcal{V}$ contains the collection of varying objects of applications of rules in $\mathcal{D}$ such that, for each conclusion of such applications, there exists a premise relevant to it in $\mathcal{D}$. If there exists a demonstration in $\mathbf{C}$ of α from Γ such that $\mathcal{D}$ is supported by $\mathcal{V}$, we say that α *is supported by* $\mathcal{V}$ *from* Γ *in* $\mathbf{C}$, and we note this by $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$. If $\mathcal{V} = \{\mathbf{o}_1, \ldots, \mathbf{o}_n\}$ and $n \geq 1$, we also note $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$ by $\Gamma \Vdash^{\mathbf{o}_1, \ldots, \mathbf{o}_n}_{\mathbf{C}} \alpha$. If $\mathcal{V} = \emptyset$, we say that $\mathcal{D}$ is a *stable demonstration in* $\mathbf{C}$. If α is supported by $\emptyset$ from Γ in $\mathbf{C}$, we say that α is a *stable consequence of* Γ *in* $\mathbf{C}$.

3.16 Theorem. A formula α is supported by $\mathcal{V}$ from Γ in $\mathbf{C}$ if, and only if, at least one of the following clauses is fulfilled:

- α is an axiom of $\mathbf{C}$;
- $\alpha \in \Gamma$;
- there exists an application $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ of a rule in $\mathbf{C}$ such that $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_n$ and, if there is a varying object $\mathbf{o}$ of this application such that $\mathbf{o} \notin \mathcal{V}$, then $\vdash_{\mathbf{C}} \alpha_1, \ldots, \vdash_{\mathbf{C}} \alpha_n$.

3.17 Examples. Let $\mathbf{C}$ be the calculus defined in 3.12. The following propositions provide examples of supporting consequence:

- $p(x,y) \Vdash^{x,y,z}_{\mathbf{C}} \forall x \forall y \forall z\, p(x,y)$;
- $p(x,y,z) \Vdash^{x,y,z}_{\mathbf{C}} \forall x \forall y \forall z\, p(x,y,z)$;
- $p(x) \Vdash^{x,\Box}_{\mathbf{C}} \Box \forall x\, p(x)$;
- $\Box p(x) \Vdash^{x,\Box}_{\mathbf{C}} \Box \forall x\, \Box p(x)$;
- $\Box p(x,y) \Vdash^{x,y,\Box}_{\mathbf{C}} \Box \forall x \forall y\, \Box p(x,y)$.

3.18 Theorem. The following properties are valid for the relation "$\Vdash^{\mathcal{V}}_{\mathbf{C}}$":

(i) if there exists a demonstration $\mathcal{D}$ in $\mathbf{C}$ of α from Γ whose collection of varying objects of applications of rules of $\mathbf{C}$ in $\mathcal{D}$ is $\mathcal{V}$, then $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$;

(ii) if $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$, then $\Gamma \vdash_{\mathbf{C}} \alpha$;

(iii) if $\Gamma \vdash_{\mathbf{C}} \alpha$, then there is a collection $\mathcal{V}$ of varying objects such that $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$;

(iv) $\vdash_{\mathbf{C}} \alpha$ iff $\Vdash^{\emptyset}_{\mathbf{C}} \alpha$;

(v) if $\mathbf{C}$ is closed, then $\Gamma \vdash_{\mathbf{C}} \alpha$ iff $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha$;

(vi) if α is an axiom of $\mathbf{C}$, then $\Vdash^{\emptyset}_{\mathbf{C}} \alpha$;

(vii) if $\alpha \in \Gamma$, then $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha$;

(viii) if $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ is an application of $\mathbf{C}$ whose collection of varying objects is $\mathcal{V}$, then $\{\alpha_1, \ldots, \alpha_n\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$;

(ix) if $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$ and $\mathcal{V} \subseteq \mathcal{V}'$, then $\Gamma \Vdash^{\mathcal{V}'}_{\mathbf{C}} \alpha$;

(x) if $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$;

(xi) if $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$, then there exists $\mathcal{V}' \subseteq \mathcal{V}$ such that $\mathcal{V}'$ is finite and $\Gamma \Vdash^{\mathcal{V}'}_{\mathbf{C}} \alpha$;

(xii) if $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$, then there exists $\Gamma' \subseteq \Gamma$ such that Γ' is finite and $\Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$;

(xiii) if $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_n, \{\alpha_1, \ldots, \alpha_n\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$, then $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$.

3.19 Example. The following assertions are not valid for the relation "$\Vdash^{\mathcal{V}}_{\mathbf{C}}$":

- if $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$ and, for each $\mathbf{o} \in \mathcal{W}$, $\mathbf{o}$ is not free in Γ, then $\Gamma \Vdash^{\mathcal{V}-\mathcal{W}}_{\mathbf{C}} \alpha$;
- if
  - $*$ $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$,
  - $*$ for each $\mathbf{o} \in \mathcal{W}$, there exists $\Gamma' \subseteq \Gamma$ such that $\mathbf{o}$ is not free in Γ' an $\Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$,

  then $\Gamma \Vdash^{\mathcal{V}-\mathcal{W}}_{\mathbf{C}} \alpha$.
- if
  - $*$ $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_p$,
  - $*$ $\{\alpha_1, \ldots, \alpha_p\} \Vdash^{\mathbf{o}_1, \ldots, \mathbf{o}_n}_{\mathbf{C}} \beta$,
  - $*$ for all $i \in \{1, \ldots, n\}$ and for all $j \in \{1, \ldots, p\}$, if $\mathbf{o}_i \notin$ and $\mathbf{o}_i$ is free in α_j, then there exists $\Gamma' \subseteq \Gamma$ such that $\mathbf{o}_i$ is not fre in Γ' and $\Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_j$,

  then $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$.

Proof:
Let **C** be a calculus with no schemas, and whose rules of inference are

$$\frac{\alpha \wedge \beta}{\alpha}, \ \frac{\forall x\, \alpha}{\alpha(x|t)} \text{ and } \frac{\alpha}{\forall x\, \alpha},$$

such that the first two ones are constant rules and the third one is a varying rule in which the varying object of each application is the quantified variable.
We have that $p(x) \Vdash^{y}_{\mathbf{C}} \forall y\, p(y)$, but it doesn't imply that $p(x) \Vdash^{\emptyset}_{\mathbf{C}} \forall y\, p(y)$, so we have a counterexample for the first two propositions.

Likewise, we have that $\begin{cases} \forall x\, p(x) \wedge p(y) \Vdash^{\emptyset}_{\mathbf{C}} \forall x\, p(x) \\ \forall x\, p(x) \Vdash^{z}_{\mathbf{C}} \forall z\, p(z) \end{cases}$, however it is not true that $\forall x\, p(x) \wedge p(y) \Vdash^{\emptyset}_{\mathbf{C}} \forall z\, p(z)$, therefore we have a counterexample for the third proposition. ☐

3.20 Theorem. The following proposition describes a way of expansion for the relation "$\vdash^{\mathcal{V}}_{\mathbf{C}}$" in a generic calculus.

- If $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_n, \{\alpha_1, \ldots, \alpha_n\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$, then $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \beta$.

3.21 Theorem. If $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$, then $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$.
Proof:
If α is an axiom of **C** or $\alpha \in \Gamma$, there is nothing to prove.
Let us suppose then that there is an application $\frac{\alpha_1, \ldots, \alpha_n}{\alpha}$ of a rule of **C** fulfilling the conditions of theorem 3.16. By induction hypothesis, we have that $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_n$. Given a varying object $\mathbf{o}$ of this application such that $\mathbf{o} \notin \mathcal{V}$, we have $\vdash_{\mathbf{C}} \alpha_1, \ldots, \vdash_{\mathbf{C}} \alpha_n$, and hence $\vdash_{\mathbf{C}} \alpha$, which is, according to propositions 4, 10 and 11 of theorem 3.13, a sufficient condition for concluding that $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$. ☐

3.22 Example. Consider again **C** the calculus defined in 3.12. We have that $p(x,y) \vdash^{x,y}_{\mathbf{C}} \forall x \forall y \forall z\, p(x,y)$, but it does not imply that $p(x,y) \Vdash^{x,y}_{\mathbf{C}} \forall x \forall y \forall z\, p(x,y)$; we have only that $p(x,y) \Vdash^{x,y,z}_{\mathbf{C}} \forall x \forall y \forall z\, p(x,y)$, so $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$ does not always imply that $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$.

4 Special Axiomatic Calculi

In this section some conditions are presented by which dependence and supporting consequences can be partially or completely equivalent, and by which generalization rules and introduction of implication can work in a weaker or in a stronger way.

4.1 Definition. A calculus **C** is said to be *partial stable* if the following conditions are valid:

- each varying rule of **C** is unary, its domain is the collection of all formulas in **C**, and each of its applications has exactly one varying object;
- for each application $\frac{\alpha'}{\alpha}$ of a varying rule in **C**, if its varying object is not free in α', then $\alpha' \Vdash^{\emptyset}_{\mathbf{C}} \alpha$;
- for each application $\frac{\alpha_1,\ldots,\alpha_n}{\alpha}$ of a constant rule in **C**, if $\alpha'_1,\ldots,\alpha'_n,\alpha'$ are respectively conclusions of applications of a varying rule over $\alpha_1,\ldots,\alpha_n,\alpha$, using the same varying object, then $\alpha'_1,\ldots,\alpha'_n \Vdash^{\emptyset}_{\mathbf{C}} \alpha'$.

4.2 Example. Let **C** be a calculus whose schemas are the following:

- $\alpha \rightarrow \forall x\, \alpha$, whereon x is not free in α;
- $\alpha \rightarrow \Box\alpha$, whereon $\Box$ is not free in α;
- $\forall x\,(\alpha \rightarrow \beta) \rightarrow (\forall x\, \alpha \rightarrow \forall x\, \beta)$;
- $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$.

The rules of **C** are the following:

- $\frac{\alpha,\, \alpha \rightarrow \beta}{\beta}$, which is a constant rule;
- $\frac{\alpha}{\forall x\, \alpha}$, whereon the varying object of each application is the quantified variable;
- $\frac{\alpha}{\Box\alpha}$, whereon $\Box$ is the varying object of each application.

We have that **C** is partial stable.

4.3 Theorem.

- If $\left\{\begin{array}{l} * \ \mathbf{C} \text{ is partial stable,} \\ * \ \Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha, \\ * \ \frac{\alpha}{\alpha'} \text{ is an application of a varying rule in } \mathbf{C} \text{ such that its varying object i} \\ \quad \text{not free in } \Gamma, \end{array}\right.$

  then $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha'$.

Proof: It is similar to the proof of theorem 4.16. ☐

4.4 Theorem. If **C** is partial stable, then $\Gamma \vdash^{\emptyset}_{\mathbf{C}} \alpha$ iff $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha$.

Proof: It is similar to the proof of theorem 4.17. ☐

4.5 Theorem. If $\mathbf{C}$ is partial stable, then "$\Vdash^{\emptyset}_{\mathbf{C}}$" has the following additional property:

- if
  - $*$ $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha_p$,
  - $*$ $\{\alpha_1, \ldots, \alpha_p\} \Vdash^{\mathbf{o_1,\ldots,o_n}}_{\mathbf{C}} \beta$,
  - $*$ for every $i \in \{1, \ldots, n\}$ and for every $j \in \{1, \ldots, p\}$, if $\mathbf{o_i}$ is free in α_j, then there exists $\Gamma' \subseteq \Gamma$ such that $\mathbf{o_i}$ is not free in Γ' and $\Gamma' \Vdash^{\emptyset}_{\mathbf{C}} \alpha_j$,

  then $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \beta$.

Proof: It is similar to the proof of theorem 4.18. □

4.6 Corollary. If $\mathbf{C}$ is partial stable, then the following additional properties are valid for the relation "$\vdash^{\emptyset}_{\mathbf{C}}$":

- $\Gamma \vdash^{\emptyset}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \vdash^{\emptyset}_{\mathbf{C}} \alpha_p, \{\alpha_1, \ldots, \alpha_p\} \vdash^{\emptyset}_{\mathbf{C}} \beta$, then $\Gamma \vdash^{\emptyset}_{\mathbf{C}} \beta$;
- if
  - $*$ $\Gamma \vdash^{\emptyset}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \vdash^{\emptyset}_{\mathbf{C}} \alpha_p$,
  - $*$ $\{\alpha_1, \ldots, \alpha_p\} \Vdash^{\mathbf{o_1,\ldots,o_n}}_{\mathbf{C}} \beta$,
  - $*$ for every $i \in \{1, \ldots, n\}$ and for every $j \in \{1, \ldots, p\}$, if $\mathbf{o_i}$ is free in α_j, then there exists $\Gamma' \subseteq \Gamma$ such that $\mathbf{o_i}$ is not free in Γ' and $\Gamma' \vdash^{\emptyset}_{\mathbf{C}} \alpha_j$,

  then $\Gamma \vdash^{\emptyset}_{\mathbf{C}} \beta$.

Proof: It suffices to use theorems 4.5 and 4.4, together with the proposition 13 of theorem 3.18. □

4.7 Definition. A calculus $\mathbf{C}$ is said to be *partial strong* if the following clauses are satisfied:

- $\vdash_{\mathbf{C}} \alpha \to \alpha$;
- $\beta \Vdash^{\emptyset}_{\mathbf{C}} \alpha \to \beta$;
- $\alpha, \alpha \to \beta \Vdash^{\emptyset}_{\mathbf{C}} \beta$;
- for each application $\dfrac{\beta_1, \ldots, \beta_n}{\beta}$ of a constant rule in $\mathbf{C}$,
  $\{\alpha \to \beta_1, \ldots, \alpha \to \beta_n\} \Vdash^{\emptyset}_{\mathbf{C}} \alpha \to \beta$.

4.8 Example. Let $\mathbf{C}$ be a calculus whose schemas are the following:

- $\alpha \to (\beta \to \alpha)$;
- $(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$.

The only inference rule of $\mathbf{C}$ is $\dfrac{\alpha, \alpha \to \beta}{\beta}$, which is a constant rule. We have that $\mathbf{C}$ is partial strong.

4.9 Example. Let $\mathbf{C}'$ be a calculus obtained from the calculus $\mathbf{C}$ of the preceding example by adding to it two new rules:

- $\dfrac{\alpha}{\forall x\, \alpha}$, whereon the varying object of each application is the quantified variable;
- $\dfrac{\alpha}{\Box\alpha}$, whereon $\Box$ is the varying object of each application.

We have that $\mathbf{C}'$ is also partial strong.

4.10 Theorem. The following propositions are equivalent:

- $\mathbf{C}$ is partial strong;
- for any Γ, α and β, $\Gamma \cup \{\alpha\} \Vdash^{\emptyset}_{\mathbf{C}} \beta$ iff $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha \rightarrow \beta$.

Proof: It is similar to the proof of theorem 4.23. ☐

4.11 Corollary. If $\mathbf{C}$ is closed, then the following propositions are equivalent:

- $\mathbf{C}$ is partial strong;
- for any Γ, α and β, $\Gamma \cup \{\alpha\} \vdash_{\mathbf{C}} \beta$ iff $\Gamma \vdash_{\mathbf{C}} \alpha \rightarrow \beta$.

Proof: It suffices to use theorem 4.10 and proposition 5 of theorem 3.18. ☐

4.12 Scholium. If the first, second and fourth clauses of definition 4.7 are valid for $\mathbf{C}$, then $\Gamma \cup \{\alpha\} \Vdash^{\emptyset}_{\mathbf{C}} \beta$ implies that $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha \rightarrow \beta$.

4.13 Corollary. If $\mathbf{C}$ is partial stable, then the following propositions are equivalent:

- $\mathbf{C}$ is partial strong;
- $\Gamma \cup \{\alpha\} \vdash^{\emptyset}_{\mathbf{C}} \beta$ iff $\Gamma \vdash^{\emptyset}_{\mathbf{C}} \alpha \rightarrow \beta$.

Proof: It suffices to use theorems 4.4 and 4.10. ☐

4.14 Definition. A partial stable calculus $\mathbf{C}$ is said to be *stable* if it has the following additional property:

- for each application $\dfrac{\beta}{\alpha}$ of a varying rule in $\mathbf{C}$, whereon $\mathbf{o}$ is its varying object, if β' and α' are respectively conclusions of applications of a varying rule in $\mathbf{C}$ over β and over α using a same varying object distinct from $\mathbf{o}$, then $\beta' \Vdash^{\mathbf{o}}_{\mathbf{C}} \alpha'$.

4.15 Example. The calculus defined in example 4.2 is partial stable, but it is not stable. If we add to it the schemas "$\forall x\, \alpha \rightarrow \alpha$" and "$\Box\alpha \rightarrow \alpha$", then we obtain a stable calculus.

4.16 Theorem.

- If $\left\{\begin{array}{l} * \ \mathbf{C} \text{ is stable,} \\ * \ \Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha, \\ * \ \frac{\alpha}{\alpha'} \text{ is an application of a varying rule in } \mathbf{C} \text{ such that its varying object} \\ \quad \text{is not free in } \Gamma, \end{array}\right.$

  then $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha'$.

Proof:
Let $\frac{\alpha}{\alpha'}$ be an application of a varying rule in $\mathbf{C}$, whose varying object, denoted by $\mathbf{o}'$ from now on, is not free in Γ.
If α is an axiom of $\mathbf{C}$, then $\vdash_{\mathbf{C}} \alpha$, so $\vdash_{\mathbf{C}} \alpha'$, therefore $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha'$.
If $\alpha \in \Gamma$, then $\mathbf{o}'$ is not free in α, so, as $\mathbf{C}$ is stable, $\alpha \Vdash^{\emptyset}_{\mathbf{C}} \alpha'$, therefore $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha'$.
If there is an application of a constant rule $\frac{\alpha_1, \ldots, \alpha_n}{\alpha}$ in $\mathbf{C}$ such that $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_n$, we have, by induction hypothesis, that $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha'_1, \ldots, \Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha'_n$, whereon $\alpha'_1, \ldots, \alpha'_n$ are respectively consequences over $\alpha_1, \ldots, \alpha_n$ by applications of the same rule from which $\frac{\alpha}{\alpha'}$ is an application, using the same varying object $\mathbf{o}'$. As $\mathbf{C}$ is stable, it follows that $\alpha'_1, \ldots, \alpha'_n \Vdash^{\emptyset}_{\mathbf{C}} \alpha'$, hence $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha'$.
Let us suppose now that there exists an application $\frac{\beta}{\alpha}$ of a varying rule in $\mathbf{C}$, whose varying object is $\mathbf{o}$, such that $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$. Consider β' a consequence of β by an application of the same rule in which α' is consequence of α, using the same varying object $\mathbf{o}'$. By induction hypothesis, $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta'$. If $\mathbf{o} \in \mathcal{V}$ and $\mathbf{o} = \mathbf{o}'$, then $\mathbf{o}' \in \mathcal{V}$, hence, from the hypothesis $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$, we have that $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha'$. If $\mathbf{o} \in \mathcal{V}$ and $\mathbf{o} \neq \mathbf{o}'$, then, as $\mathbf{C}$ is stable, $\beta' \Vdash^{\mathbf{o}}_{\mathbf{C}} \alpha'$, therefore $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha'$. If $\mathbf{o} \notin \mathcal{V}$, then $\vdash_{\mathbf{C}} \alpha$, thence $\vdash_{\mathbf{C}} \alpha'$, therefore $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha'$. □

4.17 Theorem. If $\mathbf{C}$ is stable, then $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$ iff $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$.

Proof:
By theorem 3.21, we have that $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$ implies $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$, so it remains to prove the converse.
Let us suppose that $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$.
Let $\mathcal{D}$ be a demonstration of α from Γ depending on $\mathcal{V}$, β the first occurrence of a formula in $\mathcal{D}$ justified as a consequence of an application of a varying

rule $\dfrac{\beta'}{\beta}$ such that its varying object does not belong to $\mathcal{V}$ and some premise is relevant to β' in $\mathcal{D}$. Let $\mathbf{o}$ be the varying object of this application.
If $\mathbf{o}$ is not free in β', then, as $\mathbf{C}$ is stable, we have that $\beta' \Vdash^{\emptyset}_{\mathbf{C}} \beta$, hence, as the considered occurrence of β' precedes β in $\mathcal{D}$, we have that $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta'$, and therefore, by transitivity of "$\Vdash^{\mathcal{V}}_{\mathbf{C}}$", $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$.
If $\mathbf{o}$ is free in β', then, as $\mathbf{o} \notin \mathcal{V}$, there exists $\Gamma' \subseteq \Gamma$ such that $\mathbf{o}$ is not free in Γ' and $\Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta'$, hence, as $\mathbf{C}$ is stable and in accordance with theorem 4.16, $\Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$, therefore $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$.
In any case, there is a demonstration $\mathcal{D}_\beta$ in $\mathbf{C}$ of β from Γ supported by $\mathcal{V}$. Replacing the considered occurrence of β in $\mathcal{D}$ by $\mathcal{D}_\beta$, we obtain, given $\mathcal{D}$, a demonstration in $\mathbf{C}$ of α from Γ, in which the number of applications of varying rules, whose varying objects do not belong to $\mathcal{V}$ and whose hypotheses have premises relevant to them in the new demonstration, has decreased one unit. Repeating the same process a finite number of times, we obtain a demonstration in $\mathbf{C}$ of α from Γ supported by $\mathcal{V}$, or rather, $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$. ☐

4.18 Theorem. If $\mathbf{C}$ is stable, then "$\Vdash^{\mathcal{V}}_{\mathbf{C}}$" has the following additional property:

- if $\left\{\begin{array}{l} * \ \Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_p, \\ * \ \{\alpha_1, \ldots, \alpha_p\} \Vdash^{\mathbf{o_1},\ldots,\mathbf{o_n}}_{\mathbf{C}} \beta, \\ * \ \text{for every } i \in \{1, \ldots, n\} \text{ and for every } j \in \{1, \ldots, p\}, \text{ if } \mathbf{o_i} \notin \\ \quad \text{and } \mathbf{o_i} \text{ is free in } \alpha_j, \text{ then there exists } \Gamma' \subseteq \Gamma \text{ such that } \mathbf{o_i} \text{ is not fr} \\ \quad \text{in } \Gamma' \text{ and } \Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_j, \end{array}\right.$

  then $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$.

Proof:
Let $\mathcal{D}_1, \ldots, \mathcal{D}_p$ be respectively demonstrations in $\mathbf{C}$ of $\alpha_1, \ldots, \alpha_p$ from Γ supported by $\mathcal{V}$, and let $\mathcal{E}$ be a demonstration in $\mathbf{C}$ of β from $\{\alpha_1, \ldots, \alpha_p\}$ supported by $\{\mathbf{o_1}, \ldots, \mathbf{o_n}\}$. Concatenating $\mathcal{D}_1, \ldots, \mathcal{D}_p, \mathcal{E}$, we obtain a demonstration $\mathcal{D}$ of β in $\mathbf{C}$ from Γ.
Let γ be the first occurrence of a formula in $\mathcal{D}$ justified as a consequence of an application $\dfrac{\gamma'}{\gamma}$ of a varying rule, such that its varying object does not belong to $\mathcal{V}$ and some element of Γ is relevant to γ' in $\mathcal{D}$. As $\mathcal{D}_1, \ldots, \mathcal{D}_p$ are demonstrations supported by $\mathcal{V}$, we have that the considered occurrence of

γ' appears in $\mathcal{E}$, hence, considering $\mathbf{o}$ the varying object of the application, we get that $\mathbf{o} \in \{\mathbf{o}_1, \ldots, \mathbf{o}_n\}$.
Let ϑ and ζ be defined by

$$\vartheta = \{\alpha_j \mid j \in \{1, \ldots, p\} \text{ and } \mathbf{o} \text{ is free in } \alpha_j\},$$
$$\zeta = \{\alpha_j \mid j \in \{1, \ldots, p\} \text{ and } \mathbf{o} \text{ is not free in } \alpha_j\}.$$

It is easy to verify that there exists a finite Γ', such that $\Gamma' \subseteq \Gamma$, $\mathbf{o}$ is not free in Γ' and, for every $\delta \in \vartheta$, $\Gamma' \Vdash_{\mathbf{C}}^{\mathcal{V}} \delta$. Therefore, by the construction of ζ, $\Gamma' \cup \zeta \Vdash_{\mathbf{C}}^{\mathcal{V}} \alpha_1, \ldots, \Gamma' \cup \zeta \Vdash_{\mathbf{C}}^{\mathcal{V}} \alpha_p$, and $\mathbf{o}$ is not free in $\Gamma' \cup \zeta$.
As the considered occurrence of γ' precedes γ in $\mathcal{D}$, we have that $\{\alpha_1, \ldots, \alpha_p\} \Vdash_{\mathbf{C}}^{\mathcal{V}} \gamma'$, and hence, by transitivity of "$\Vdash_{\mathbf{C}}^{\mathcal{V}}$", we get $\Gamma' \cup \zeta \Vdash_{\mathbf{C}}^{\mathcal{V}} \gamma'$, and therefore, by theorem 4.16, $\Gamma' \cup \zeta \Vdash_{\mathbf{C}}^{\mathcal{V}} \gamma$.
For every $\delta \in \Gamma' \cup \zeta$, we have that $\Gamma \Vdash_{\mathbf{C}}^{\mathcal{V}} \delta$, and hence, once again due to transitivity of "$\Vdash_{\mathbf{C}}^{\mathcal{V}}$", $\Gamma \Vdash_{\mathbf{C}}^{\mathcal{V}} \gamma$. Or rather, there exists a demonstration $\mathcal{D}_\gamma$ in $\mathbf{C}$ of γ from Γ supported by $\mathcal{V}$. Replacing the considered occurrence of γ in $\mathcal{D}$ by $\mathcal{D}_\gamma$, we have a new demonstration $\mathcal{D}'$ in $\mathbf{C}$ of β from Γ, in which the number of applications of varying rules, whose varying objects do not belong to $\mathcal{V}$ and each hypothesis has some premise relevant to it in $\mathcal{D}'$, has decreased one unit. Repeating the same process a finite number of times, we obtain a demonstration in $\mathbf{C}$ of β from Γ supported by $\mathcal{V}$, or rather, $\Gamma \Vdash_{\mathbf{C}}^{\mathcal{V}} \beta$. □

4.19 Corollary. If $\mathbf{C}$ is stable, then the following additional properties are valid for the relation "$\vdash_{\mathbf{C}}^{\mathcal{V}}$":

- if $\Gamma \vdash_{\mathbf{C}}^{\mathcal{V}} \alpha_1, \ldots, \Gamma \vdash_{\mathbf{C}}^{\mathcal{V}} \alpha_p, \{\alpha_1, \ldots, \alpha_p\} \vdash_{\mathbf{C}}^{\mathcal{V}} \beta$, then $\Gamma \vdash_{\mathbf{C}}^{\mathcal{V}} \beta$;
- if
  - $\Gamma \vdash_{\mathbf{C}}^{\mathcal{V}} \alpha_1, \ldots, \Gamma \vdash_{\mathbf{C}}^{\mathcal{V}} \alpha_p$,
  - $\{\alpha_1, \ldots, \alpha_p\} \vdash_{\mathbf{C}}^{\mathbf{o}_1, \ldots, \mathbf{o}_n} \beta$,
  - for every $i \in \{1, \ldots, n\}$ and for every $j \in \{1, \ldots, p\}$, if $\mathbf{o}_i \notin \mathcal{V}$ and $\mathbf{o}_i$ is free in α_j, then exists $\Gamma' \subseteq \Gamma$ such that $\mathbf{o}_i$ is not free in Γ' and $\Gamma' \vdash_{\mathbf{C}}^{\mathcal{V}} \alpha_j$,

  then $\Gamma \vdash_{\mathbf{C}}^{\mathcal{V}} \beta$.

Proof: It suffices to use theorems 4.18 and 4.17, together with the proposition 13 of theorem 3.18. □

4.20 Definition. A partial strong calculus $\mathbf{C}$ is said to be *strong* if it has the following additional property:

- for each application $\dfrac{\beta_1,\ldots,\beta_n}{\beta}$ of a varying rule of **C** whose collection of varying objects is $\mathcal{V}$, if no element of $\mathcal{V}$ is free in α, then $\{\alpha\rightarrow\beta_1,\ldots,\alpha\rightarrow\beta_n\}\Vdash^{\mathcal{V}}_{\mathbf{C}}\alpha\rightarrow\beta$.

4.21 Example. The calculus defined in example 4.8 is also strong.

4.22 Example. The calculus defined in example 4.9 is partial strong, but it is not strong. Consider **C** a new calculus obtained from it by adding two new schemas:

- $\forall x(\alpha\rightarrow\beta)\rightarrow(\alpha\rightarrow\forall x\,\beta)$, whereon x is not free in α;
- $\Box(\alpha\rightarrow\beta)\rightarrow(\alpha\rightarrow\Box\beta)$, whereon $\Box$ is not free in α.

We have that **C** is strong.

4.23 Theorem. The following propositions are equivalent:
(i) **C** is strong;
(ii) for any Γ, α, β and $\mathcal{V}$, such that each $\mathbf{o}\in\mathcal{V}$ is not free in α, $\Gamma\cup\{\alpha\}\Vdash^{\mathcal{V}}_{\mathbf{C}}\beta$ iff $\Gamma\Vdash^{\mathcal{V}}_{\mathbf{C}}\alpha\rightarrow\beta$.

(i) implies (ii):
Let us suppose that **C** is a strong calculus and that each $\mathbf{o}\in\mathcal{V}$ is not free in α.
If $\Gamma\Vdash^{\mathcal{V}}_{\mathbf{C}}\alpha\rightarrow\beta$, then, due to clause (iii) of definition 4.7, $\Gamma\cup\{\alpha\}\Vdash^{\mathcal{V}}_{\mathbf{C}}\beta$.
Consider now that $\Gamma\cup\{\alpha\}\Vdash^{\mathcal{V}}_{\mathbf{C}}\beta$.
If β is an axiom of **C**, then $\vdash_{\mathbf{C}}\beta$, hence, according to clause (ii) of definition 4.7, $\Gamma\vdash_{\mathbf{C}}\alpha\rightarrow\beta$, therefore $\Gamma\Vdash^{\mathcal{V}}_{\mathbf{C}}\alpha\rightarrow\beta$.
If $\beta\in\Gamma$, then $\Gamma\Vdash^{\mathcal{V}}_{\mathbf{C}}\beta$, hence, according to clause (ii) of definition 4.7, $\Gamma\Vdash^{\mathcal{V}}_{\mathbf{C}}\alpha\rightarrow\beta$.
If $\beta=\alpha$, then, according to clause (i) of definition 4.7, $\Gamma\vdash_{\mathbf{C}}\alpha\rightarrow\alpha$, therefore $\Gamma\Vdash^{\mathcal{V}}_{\mathbf{C}}\alpha\rightarrow\beta$.
If there is an application $\dfrac{\beta_1,\ldots,\beta_n}{\beta}$ of a rule of **C** such that $\Gamma\cup\{\alpha\}\Vdash^{\mathcal{V}}_{\mathbf{C}}\beta_1,\ldots,\Gamma\cup\{\alpha\}\Vdash^{\mathcal{V}}_{\mathbf{C}}\beta_n$, we have, by induction hypothesis, $\Gamma\Vdash^{\mathcal{V}}_{\mathbf{C}}\alpha\rightarrow\beta_1,\ldots,\Gamma\Vdash^{\mathcal{V}}_{\mathbf{C}}\alpha\rightarrow\beta_n$. If there is a varying object of this application that does not belong to $\mathcal{V}$, then, according to theorem 3.16, $\vdash_{\mathbf{C}}\beta$, hence, once again by clause (ii) of definition 4.7, $\vdash_{\mathbf{C}}\alpha\rightarrow\beta$, therefore $\Vdash^{\mathcal{V}}_{\mathbf{C}}\alpha\rightarrow\beta$. If every varying object of this application belongs to $\mathcal{V}$, then, as **C** is strong, $\{\alpha\rightarrow\beta_1,\ldots,\alpha\rightarrow\beta_n\}\Vdash^{\mathcal{V}}_{\mathbf{C}}\alpha\rightarrow\beta$, therefore $\Gamma\Vdash^{\mathcal{V}}_{\mathbf{C}}\alpha\rightarrow\beta$. □

(ii) implies (i):
Let us suppose that for any Γ, α, β and $\mathcal{V}$ such that each $\mathbf{o} \in \mathcal{V}$ is not free in α, $\Gamma \cup \{\alpha\} \Vdash_{\mathbf{C}}^{\mathcal{V}} \beta$ iff $\Gamma \Vdash_{\mathbf{C}}^{\mathcal{V}} \alpha \to \beta$.
As $\alpha \vdash_{\mathbf{C}} \alpha$, we have that $\vdash_{\mathbf{C}} \alpha \to \alpha$.
As $\{\beta, \alpha\} \Vdash_{\mathbf{C}}^{\emptyset} \beta$, we get $\beta \Vdash_{\mathbf{C}}^{\emptyset} \alpha \to \beta$.
As $\alpha \to \beta \Vdash_{\mathbf{C}}^{\emptyset} \alpha \to \beta$, we have that $\{\alpha, \alpha \to \beta\} \Vdash_{\mathbf{C}}^{\emptyset} \beta$.
Finally, let $\dfrac{\beta_1, \ldots, \beta_n}{\beta}$ be an application of a rule of $\mathbf{C}$ whose collection of varying objects is $\mathcal{V}$, and α a formula in $\mathbf{C}$ where no element of $\mathcal{V}$ is free. We have that

$$\{\alpha \to \beta_1, \ldots, \alpha \to \beta_n, \alpha\} \Vdash_{\mathbf{C}}^{\emptyset} \beta_1, \ldots, \{\alpha \to \beta_1, \ldots, \alpha \to \beta_n, \alpha\} \Vdash_{\mathbf{C}}^{\emptyset} \beta_n,$$

hence, as $\{\beta_1, \ldots, \beta_n\} \Vdash_{\mathbf{C}}^{\mathcal{V}} \beta$, we have that $\{\alpha \to \beta_1, \ldots, \alpha \to \beta_n, \alpha\} \Vdash_{\mathbf{C}}^{\mathcal{V}} \beta$, therefore $\{\alpha \to \beta_1, \ldots, \alpha \to \beta_n\} \Vdash_{\mathbf{C}}^{\mathcal{V}} \alpha \to \beta$. ☐

4.24 Scholium. If the first, second and fourth clauses of definition 4.7, together with the only clause of definition 4.20, are valid for $\mathbf{C}$, then the following proposition is true:

- if $\begin{cases} \text{each } \mathbf{o} \in \mathcal{V} \text{ is not free in } \alpha, \\ \Gamma \cup \{\alpha\} \Vdash_{\mathbf{C}}^{\mathcal{V}} \beta, \end{cases}$ then $\Gamma \Vdash_{\mathbf{C}}^{\mathcal{V}} \alpha \to \beta$.

4.25 Theorem. If $\mathbf{C}$ is stable, then the following propositions are equivalent:

- $\mathbf{C}$ is a strong calculus;
- for any Γ, α, β and $\mathcal{V}$, such that each $\mathbf{o} \in \mathcal{V}$ is not free in α, $\Gamma \cup \{\alpha\} \vdash_{\mathbf{C}}^{\mathcal{V}} \beta$ iff $\Gamma \vdash_{\mathbf{C}}^{\mathcal{V}} \alpha \to \beta$.

Proof: It suffices to use theorems 4.23 and 4.17. ☐

5 Conclusion

We have presented general formulations for generalization rules and for introduction of implication, valid for a large family of axiomatic calculi, from closed to open ones.

Closed calculi have the simplest formulation for introduction of implication, and generalization rules can be simulated through admissible rules. For these calculi, both these procedures can be performed by using only the basic consequence relation.

The same does not happen with respect to open calculi. In them, for managing the interrelationship between introduction of implication and generalization rules, the basic consequence relation is not sufficient for tracing how varying objects are used along a demonstration, so it is necessary to annotate them for all applications of inference rules, taking into account two adequate consequence relations, which can be partially or completely equivalent, depending on the particular calculus. In practice, at the worst case, it is necessary to work in a simultaneous way with two consequence relations for tracing varying objects, the dependence and supporting ones.

These results are very important for modelling new calculi that should have properties related to weaker or stronger forms of generalization and introduction of implication. Some of them were essential for obtaining an abstract completeness proof for a broad group of calculi with respect to their semantics, in [2], pgs. 72-88. A future paper will present a concise exposition of this proof.

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