

## LOGICAL GROUNDING AND FIRST-DEGREE ENTAILMENTS

Fabrice CORREIA  
University of Neuchâtel

### *Summary*

I give a characterisation of three relations of logical grounding in sequent format, which I use to highlight some connections between logical grounding and first-degree entailments understood à la Anderson and Belnap.

The view that there is a distinction among inferential connections between those which are explanatory and those which are not has found proponents throughout the history of philosophy—the most well-known being perhaps Aristotle (*Posterior Analytics*, I, 2 and 13) and Bernard Bolzano (*Theory of Science*, esp. II, §198 and IV, §525). In my “Logical Grounds” (Correia 2014) I developed a theory of *logical grounding* which is based on this distinction. For present purposes, grounding may be taken to be a many-one relation between propositions, and the notion may be explicated by saying that some given propositions ground another given proposition when the former propositions’ being true *makes it the case that* the latter proposition is true. A central tenet of “Logical Grounds” is that grounding comes in various kinds, and that one of these kinds is distinctively logical. The paper gives a precise characterisation of logical grounding and shows that the notion can be fruitfully used in certain areas of logical inquiry.

One of the main results put forward in “Logical Grounds” is that various well-known consequence relations—in particular, the relation of classical logical consequence defined on propositional or first-order languages—can be fully characterised in terms of logical grounding. One such relation, associated with the so-called first-degree entailments of Anderson and Belnap (1962, 1963), is actually more closely tied to logical grounding than the other ones. The aim of the present paper is to further the study of these ties. In the first section, I present the theory of logical grounding for propositional languages more or less as it is formulated in “Logical Grounds”. In the second section, I characterise the consequence relation

of interest to us, still relative to propositional languages, and in the final section I present and discuss some important connections between that relation and logical grounding.

### 1. *Logical grounding*

The theory of logical grounding to be formulated here supposes given a standard propositional language with  $\wedge$ ,  $\vee$  and  $\neg$  as primitive connectives. Following common usage, the formulas of the language which are either atoms or negated atoms are called *literals*, and I use  $\phi$ ,  $\psi$ , etc. for formulas of the language and  $\Delta$ ,  $\Gamma$ , etc. for sets thereof. Standard definitions and notational conventions will be used throughout the paper.

The theory assumes the following basic rules of inference:

$$\begin{array}{lll}
 (\wedge 1) \quad \frac{\phi \quad \psi}{\phi \wedge \psi} & (\wedge 2) \quad \frac{\neg \phi}{\neg(\phi \wedge \psi)} & (\wedge 3) \quad \frac{\neg \psi}{\neg(\phi \wedge \psi)} \\
 (\vee 1) \quad \frac{\neg \phi \quad \neg \psi}{\neg(\phi \vee \psi)} & (\vee 2) \quad \frac{\phi}{\phi \vee \psi} & (\vee 3) \quad \frac{\psi}{\phi \vee \psi} \\
 (\neg) \quad \frac{\phi}{\neg \neg \phi}
 \end{array}$$

Each rule, read from top to bottom, is supposed to encode a link of logical grounding:  $(\wedge 1)$  states that for any  $\phi$  and  $\psi$ ,  $\phi$  and  $\psi$  together ground  $\phi \wedge \psi$ ,  $(\neg)$  states that for any  $\phi$ ,  $\phi$  grounds  $\neg \neg \phi$ , and so on.

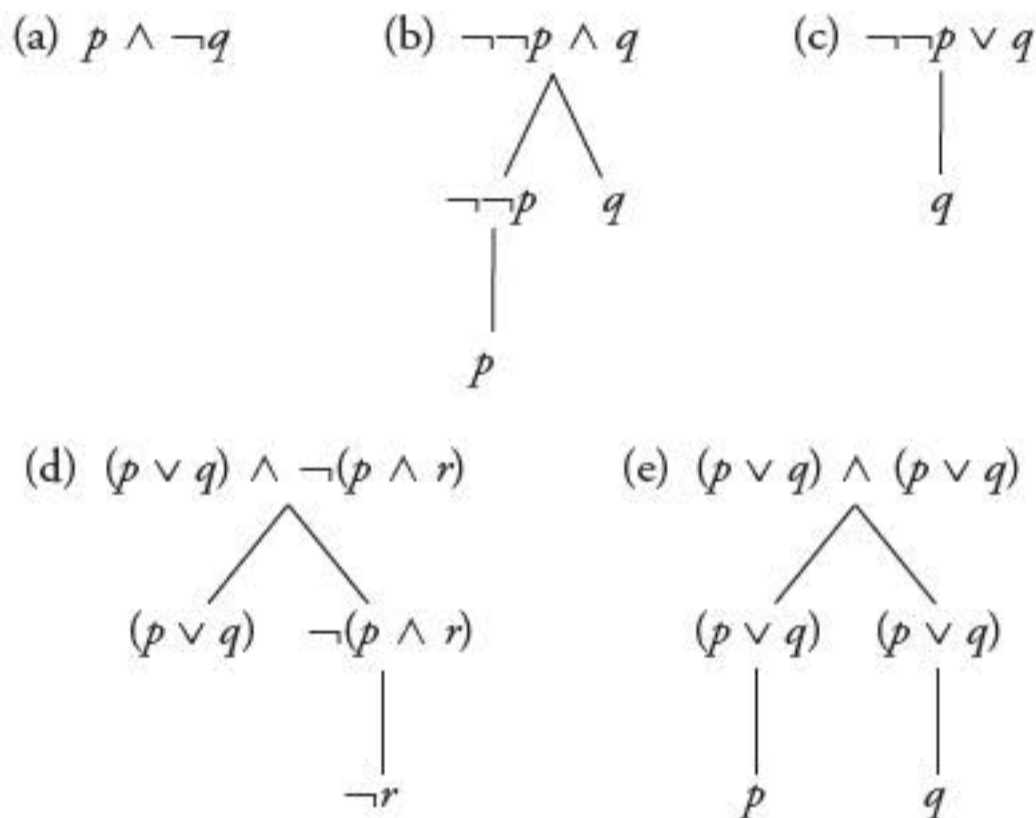
The theory also assumes that these rules are sufficient, in the sense that all the connections of logical grounding between formulas of our language can be described in terms of these rules. There are actually various relations that can be defined in terms of the basic rules, which correspond to various concepts of logical grounding.

Let a *grounding tree* be a rooted tree  $T$  which satisfies the following conditions:

1. Each of  $T$ 's nodes is occupied by a formula;
2. No parent node in  $T$  is occupied by a literal;
3. Given a parent node  $N$  in  $T$ :

- If  $N$  is occupied by  $\phi \wedge \psi$ ,  $N$  has two children, one occupied by  $\phi$  and the other one by  $\psi$
- If  $N$  is occupied by  $\phi \vee \psi$ ,  $N$  has one child, occupied by  $\phi$  or by  $\psi$
- If  $N$  is occupied by  $\neg(\phi \wedge \psi)$ ,  $N$  has one child, occupied by  $\neg\phi$  or by  $\neg\psi$
- If  $N$  is occupied by  $\neg(\phi \vee \psi)$ ,  $N$  has two children, one occupied by  $\neg\phi$  and the other one by  $\neg\psi$
- If  $N$  is occupied by  $\neg\neg\phi$ ,  $N$  has one child, occupied by  $\phi$ .

The following are examples of grounding trees:



A grounding tree *for* a formula is a grounding tree whose root node is occupied by the formula itself, and a grounding tree for a formula is said to be *from* a set of formulas  $\Delta$  iff  $\Delta$  is the set of all the formulas which occupy leaves on the grounding tree. Thus, (a) above is a grounding tree for  $p \wedge \neg q$  from  $\{p \wedge \neg q\}$ , (b) a grounding tree for  $\neg\neg p \wedge q$  from  $\{p, q\}$ , etc. A grounding tree is said to be *degenerate* iff it consists of just one node.

Three concepts of logical grounding may then naturally be defined:<sup>1</sup>

1. Here as in "Logical Grounds", I treat logical grounding as a relation between sets of formulas and formulas, and I treat it as a non-factive relation (' $\Delta$  logically grounds  $\phi$ ' does not entail ' $\phi$  and all the members of  $\Delta$  are true'). Some might demur on both counts. Yet my treatment

**Definition 1.1.** For  $\Delta$  a set of formulas and  $\phi$  a formula:

- $\Delta$  STRICTLY GROUNDS\*  $\phi$ —in symbols:  $\Delta \triangleright^* \phi$ —*iff<sub>def</sub>* there is a non-degenerate grounding tree for  $\phi$  from  $\Delta$ .
- $\Delta$  STRICTLY GROUNDS  $\phi$ —in symbols:  $\Delta \triangleright \phi$ —*iff<sub>def</sub>* there is a covering of  $\Delta$  (i.e. a family of sets whose union is  $\Delta$ ) such that for each  $\Delta'$  in this covering,  $\Delta' \triangleright^* \phi$ .
- $\Delta$  WEAKLY GROUNDS  $\phi$ —in symbols:  $\Delta \trianglerighteq \phi$ —*iff<sub>def</sub>* for some  $\Delta' \subseteq \Delta$ , there is a grounding tree for  $\phi$  from  $\Delta'$ .

One can readily verify that for every set of formulas  $\Delta$  and every formula  $\phi$ :

- $\Delta \trianglerighteq \phi$  iff either  $\phi \in \Delta$ , or for some  $\Delta' \subseteq \Delta$ ,  $\Delta' \triangleright^* \phi$
- $\Delta \triangleright \phi$  iff either  $\phi \in \Delta$ , or for some  $\Delta' \subseteq \Delta$ ,  $\Delta' \triangleright \phi$ .

Thus, the weak relation is definable in terms of either of the strict relations. One can also verify that  $\triangleright^*$  is strictly stronger than  $\triangleright$ , which in turn is strictly stronger than  $\trianglerighteq$ . Further important properties of these three relations are listed below:

Properties of STRICT GROUNDS\*:

1. If  $\Delta \triangleright^* \phi$ , then  $\Delta \neq \emptyset$  and is finite
2. If  $\Delta \triangleright^* \phi$ , then  $\phi$  is not a literal
3. If  $\Delta \triangleright^* \phi$ , then  $\text{Complexity}(\phi) > \text{Complexity}(\psi)$  for any  $\psi \in \Delta$
4. Not:  $\Delta, \phi \triangleright^* \phi$  *Generalised Irreflexivity*
5. If  $\Delta \triangleright^* \psi$  and  $\psi, \Delta' \triangleright^* \phi$  and  $\psi \notin \Delta'$ , then  $\Delta, \Delta' \triangleright^* \phi$  *Restricted Cut*

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4. Not:  $\Delta, \phi \triangleright \phi$  *Generalised Irreflexivity*
5. If  $\Delta \triangleright \psi$  and  $\psi, \Delta' \triangleright \phi$ , then  $\Delta, \Delta' \triangleright \phi$  *Cut*
6. If  $\Delta \triangleright \phi$  and  $\Delta' \triangleright \phi$ , then  $\Delta, \Delta' \triangleright \phi$  *Amalgamation*

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of logical grounding as a relation between sets of formulas and formulas is inessential and could easily be abandoned in favour of other treatments. On factivity, see “Logical Grounds”, 35f.

Properties of WEAK GROUNDING:

1. If  $\Delta \supseteq \phi$ , then  $\Delta \neq \emptyset$
2. If  $\Delta \supseteq \phi$ , and  $\phi \notin \Delta$ , then  $\phi$  is not a literal
3.  $\phi \supseteq \phi$  *Reflexivity*
4. If  $\Delta \supseteq \psi$  and  $\psi, \Delta' \supseteq \phi$ , then  $\Delta, \Delta' \supseteq \phi$  *Cut*
6. If  $\Delta \supseteq \phi$ , then  $\Delta, \Delta' \supseteq \phi$  *Weakening*

The strict relations satisfy neither Reflexivity nor Weakening, and for that reason they are presumably closer to our intuitive conception of logical grounding than the weak relation. The weak relation is nevertheless theoretically very useful, as we will see below. Whether one of the strict relations is closer than the other one to our intuitive conception of logical grounding I do not know.

Since the basic rules of inference for grounding are all classically valid, the weaker relation is stronger than classical logical consequence: for every set of formulas  $\Delta$  and every formula  $\phi$ , if  $\Delta \supseteq \phi$ , then  $\phi$  is a classical consequence of  $\Delta$ . It should actually also be clear that weak grounding is stronger than certain consequence relations which are strictly stronger than classical consequence. One important such relation is the relevant consequence relation associated with Anderson and Belnap's (1962, 1963) first-degree entailments. The connections between weak grounding and that notion of consequence—FDE-consequence, for short—run actually very deep.

## 2. FDE

Let me here characterise FDE-consequence in a general form, and present a proof system which captures the relation so defined. (On the basics of FDE and the related logics mentioned at the end of this section, see e.g. (Priest 2008).)

Let a *valuation* be a distribution of truth-values (T and F) over the atoms of our language. Neither gaps nor gluts are excluded: an atom may be assigned no truth-value at all, or both T and F. Truth ( $\models$ ) and falsity ( $\models$ ) for formulas relative to a valuation  $v$  are then defined recursively as follows:

- For  $\phi$  atomic:  $v \models \phi$  iff  $p$  is assigned T by  $v$
- For  $\phi$  atomic:  $v \models \phi$  iff  $p$  is assigned F by  $v$

- $v \models \phi \wedge \psi$  iff  $v \models \phi$  and  $v \models \psi$
- $v \models \phi \vee \psi$  iff  $v \models \phi$  or  $v \models \psi$
- $v \models \neg\phi$  iff  $v \not\models \phi$
- $v \models \phi$  iff  $v \models \neg\neg\phi$

FDE-consequence is most frequently characterised as a relation between two formulas, and one way of doing it makes use of valuations as just defined and runs as follows:  $\psi \models_{\text{FDE}} \phi$  iff for every valuation  $v$ , if  $v \models \psi$ , then  $v \models \phi$ . I generalise a bit and define it as a relation between two sets of formulas:

**Definition 2.1.** For all sets of formulas  $\Delta$  and  $\Gamma$ :  $\Delta \models_{\text{FDE}} \Gamma$  iff<sub>def</sub> for every valuation  $v$ , if ( $v \models \psi$  for all  $\psi \in \Delta$ ), then ( $v \models \phi$  for some  $\phi \in \Gamma$ ).

FDE-consequence so defined can be proof-theoretically characterised in an elegant way by means of a multiple-conclusion sequent calculus defined by the following axioms and rules:<sup>2</sup>

### System FDE

*Introduction axioms:*

- i1.  $\phi, \psi \vdash \phi \wedge \psi$
- i2.  $\phi \vdash \phi \vee \psi$
- i3.  $\psi \vdash \psi \vee \phi$
- i4.  $\neg\phi, \neg\psi \vdash \neg(\phi \vee \psi)$
- i5.  $\neg\phi \vdash \neg(\phi \wedge \psi)$
- i6.  $\neg\psi \vdash \neg(\phi \wedge \psi)$
- i7.  $\phi \vdash \neg\neg\phi$

*Elimination axioms:*

- e1.  $\phi \wedge \psi \vdash \phi$
- e2.  $\phi \wedge \psi \vdash \psi$
- e3.  $\phi \vee \psi \vdash \phi, \psi$
- e4.  $\neg(\phi \vee \psi) \vdash \neg\phi$
- e5.  $\neg(\phi \vee \psi) \vdash \neg\psi$
- e6.  $\neg(\phi \wedge \psi) \vdash \neg\phi, \neg\psi$
- e7.  $\neg\neg\phi \vdash \phi$

*Structural rules:*

Cut:

$$\frac{\Delta \vdash \Gamma, \phi \quad \phi, \Delta' \vdash \Gamma'}{\Delta, \Delta' \vdash \Gamma, \Gamma'}$$

2. The axioms could, of course, be replaced by rules with zero premisses.

Weakening:

$$\frac{\Delta \vdash \Gamma}{\Delta, \Delta' \vdash \Gamma, \Gamma'}$$

(Notice that the only axioms with a multiple conclusion are the elimination axioms e3 and e6.) The sequents provable in this calculus are said to be *FDE-provable*, and I will write  $\Delta \Vdash_{\text{FDE}} \Gamma$  to say that the sequent  $\Delta \vdash \Gamma$  is FDE-provable. (The same type of notation will be used for other systems below.)

As previously announced:

**Theorem 2.2.** *For all  $\Delta$  and  $\Gamma$ :  $\Delta \Vdash_{\text{FDE}} \Gamma$  iff  $\Delta \models_{\text{FDE}} \Gamma$ .*

*Proof.* The proof of soundness is straightforward. For completeness, suppose that  $\Delta \Vdash_{\text{FDE}} \Gamma$ . Enumerate the formulas:  $\phi_0, \phi_1, \dots$ , and define a series of sets of formulas  $(S_n)_{n \in \mathbb{N}}$  as follows:

- $S_0 = \Delta$ ;
- $S_{n+1} = S_n \cup \{\phi_n\}$  if  $S_n, \phi_n \Vdash_{\text{FDE}} \Gamma$ , and  $S_n$  otherwise.

Define  $\Delta^+$  as  $\bigcup_{n \in \mathbb{N}} S_n$ . Notice that by construction,  $\Delta^+ \cap \Gamma = \emptyset$ . One can establish that for all formulas  $\phi$  and  $\psi$ :

1.  $\phi \wedge \psi \in \Delta^+$  iff both  $\phi \in \Delta^+$  and  $\psi \in \Delta^+$ ;
2.  $\neg(\phi \wedge \psi) \in \Delta^+$  iff  $\neg\phi \in \Delta^+$  or  $\neg\psi \in \Delta^+$ ;
3.  $\phi \vee \psi \in \Delta^+$  iff  $\phi \in \Delta^+$  or  $\psi \in \Delta^+$ ;
4.  $\neg(\phi \vee \psi) \in \Delta^+$  iff both  $\neg\phi \in \Delta^+$  and  $\neg\psi \in \Delta^+$ ;
5.  $\neg\neg\phi \in \Delta^+$  iff  $\phi \in \Delta^+$ .

Define valuation  $v$  by stipulating that for every atom  $\phi$ ,  $v$  assigns T to  $\phi$  iff  $\phi \in \Delta^+$  and  $v$  assigns F to  $\phi$  iff  $\neg\phi \in \Delta^+$ . Using points 1–5 above, one can prove by induction on the length of the formulas that for all formulas  $\phi$ :  $v \models \phi$  iff  $\phi \in \Delta^+$ , and  $v \models \neg\phi$  iff  $\neg\phi \in \Delta^+$ . Since  $\Delta \subseteq \Delta^+$ , it follows that  $v \models \psi$  for all  $\psi \in \Delta$ . On the other hand, since  $\Delta^+ \cap \Gamma = \emptyset$ , it also follows that there is no formula  $\phi \in \Gamma$  such that  $v \models \phi$ . Consequently,  $\Delta \not\models_{\text{FDE}} \Gamma$ .

As an aside, notice that if we alter definition 2.1 by imposing certain conditions on valuations in the definiens, we obtain characterisations of other well-known consequence relations: if one excludes gaps, one gets LP-consequence; if one excludes gluts, one gets K3-consequence; and if one excludes both, one gets classical consequence. In order to obtain an adequate calculus for LP-consequence, it suffices to add the axiom  $\vdash \phi, \neg\phi$  to the calculus for FDE-consequence presented above; for K3 consequence it suffices to add  $\phi, \neg\phi \vdash$  instead; and for classical consequence, it suffices to add both.

### 3. Connections

Let a *situation* be a set of literals. In “Logical Grounds”, I worked with a many-one notion of FDE-consequence and I established the following connection between that notion and weak grounding:

- Given any set of formulas  $\Delta$  and any formula  $\phi$ :  $\Delta \vDash_{\text{FDE}} \phi$  iff for every situation  $\Lambda$ , if  $(\Lambda \supseteq \psi$  for all  $\psi \in \Delta)$ , then  $\Lambda \supseteq \phi$ .

A more general result concerning many-many FDE-consequence can be established in much the same way:

**Theorem 3.1.** *For all  $\Delta$  and  $\Gamma$ :  $\Delta \vDash_{\text{FDE}} \Gamma$  iff for every situation  $\Lambda$ , if  $(\Lambda \supseteq \psi$  for all  $\psi \in \Delta)$ , then  $(\Lambda \supseteq \phi$  for some  $\phi \in \Gamma)$ .*

This shows that FDE-consequence can be defined in terms of WEAK GROUNDING, and hence ultimately in terms of either of its strict counterparts. (The same holds of LP-consequence, K3-consequence and classical consequence.)

An almost immediate upshot of this result, which is not stated in “Logical Grounds”, is that:

**Theorem 3.2.** *For every situation  $\Delta$  and every formula  $\phi$ :  $\Delta \supseteq \phi$  iff  $\Delta \vDash_{\text{FDE}} \phi$ .*

This shows that WEAK GROUNDING can be defined in terms of FDE-consequence when restricted to a relation between sets of literals and arbitrary formulas.

These theorems state deep links between logical grounding as characterised above and FDE-consequence. But one can go significantly fur-

ther thanks to the sequent calculus for FDE introduced in the previous section.

Notice that the seven introduction axioms of that calculus correspond, in an obvious sense, to the seven basic rules for grounding, while none of the elimination axioms corresponds, in the same sense, to a acceptable links of ground. Also, remember that WEAK GROUNDING obeys both a principle of Cut and a principle of Weakening. Would dropping the elimination axioms from the sequent calculus for FDE give us an adequate characterisation of WEAK GROUNDING?

Not quite, for two reasons: first, weak grounding is not many-many, and second, weak grounding is reflexive, and while  $\phi \vdash \phi$  is FDE-provable, its proofs all make use of elimination axioms. Yet consider the system  $FDE_0$ , defined from system FDE by dropping the elimination axioms and adding an axiom for reflexivity:

### System $FDE_0$

*Introduction axioms:*

- i1.  $\phi, \psi \vdash \phi \wedge \psi$
- i2.  $\phi \vdash \phi \vee \psi$
- i3.  $\psi \vdash \psi \vee \phi$
- i4.  $\neg\phi, \neg\psi \vdash \neg(\phi \vee \psi)$
- i5.  $\neg\phi \vdash \neg(\phi \wedge \psi)$
- i6.  $\neg\psi \vdash \neg(\phi \wedge \psi)$
- i7.  $\phi \vdash \neg\neg\phi$

*Structural axiom:*

Reflexivity:

$$\phi \vdash \phi$$

*Structural rules:*

Cut:

$$\frac{\Delta \vdash \Gamma, \psi \quad \psi, \Delta' \vdash \Gamma'}{\Delta, \Delta' \vdash \Gamma, \Gamma'}$$

Weakening:

$$\frac{\Delta \vdash \Gamma}{\Delta, \Delta' \vdash \Gamma, \Gamma'}$$

Then  $\models_{\text{FDE}_0}$  is essentially many-one, i.e. for all sets of formulas  $\Delta$  and  $\Gamma$ ,  $\Delta \models_{\text{FDE}_0} \Gamma$  iff  $\Delta \models_{\text{FDE}_0} \phi$  for some  $\phi \in \Gamma$  (the right-to-left direction is immediate thanks to Weakening, and the left-to-right direction can easily be proved by induction on the length of the derivations). And it can be shown that:

**Theorem 3.3.** *For all  $\Delta$  and  $\phi$ :  $\Delta \triangleright \phi$  iff  $\Delta \Vdash_{\text{FDE}_0} \phi$ .*

*Proof.* For the left-to-right direction, by induction on the height of the grounding trees, and for the right-to-left direction, by induction on the length of the derivations.

This being established, it is clear that a more direct characterisation of WEAK GROUNDING is provided by the following calculus for many-one sequents, which differs from the previous calculus only in its structural rules:

### System $\text{FDE}_1$ (for WEAK GROUNDING)

*Introduction axioms:*

- i1.  $\phi, \psi \vdash \phi \wedge \psi$
- i2.  $\phi \vdash \phi \vee \psi$
- i3.  $\psi \vdash \psi \vee \phi$
- i4.  $\neg\phi, \neg\psi \vdash \neg(\phi \vee \psi)$
- i5.  $\neg\phi \vdash \neg(\phi \wedge \psi)$
- i6.  $\neg\psi \vdash \neg(\phi \wedge \psi)$
- i7.  $\phi \vdash \neg\neg\phi$

*Structural axiom:*

Reflexivity:

$$\phi \vdash \phi$$

*Structural rules:*

Cut:

$$\frac{\Delta \vdash \psi \quad \psi, \Delta' \vdash \phi}{\Delta, \Delta' \vdash \phi}$$

Weakening:

$$\frac{\Delta \vdash \phi}{\Delta, \Delta' \vdash \phi}$$

**Theorem 3.4.** For all  $\Delta$  and  $\phi$ :  $\Delta \triangleright \phi$  iff  $\Delta \Vdash_{FDE_1} \phi$ .

*Proof.* Same strategy as for the previous theorem.

The strict relations can also be characterised in a similar way. The case of STRICT GROUNDING is straightforward:

### System FDE<sub>2</sub> (for STRICT GROUNDING)

*Introduction axioms:*

- i1.  $\phi, \psi \vdash \phi \wedge \psi$
- i2.  $\phi \vdash \phi \vee \psi$
- i3.  $\psi \vdash \psi \vee \phi$
- i4.  $\neg\phi, \neg\psi \vdash \neg(\phi \vee \psi)$
- i5.  $\neg\phi \vdash \neg(\phi \wedge \psi)$
- i6.  $\neg\psi \vdash \neg(\phi \wedge \psi)$
- i7.  $\phi \vdash \neg\neg\phi$

*Structural rules:*

Cut:

$$\frac{\Delta \vdash \psi \quad \psi, \Delta' \vdash \phi}{\Delta, \Delta' \vdash \phi}$$

Amalgamation:

$$\frac{\Delta \vdash \phi \quad \Delta' \vdash \phi}{\Delta, \Delta' \vdash \phi}$$

**Theorem 3.5.** For all  $\Delta$  and  $\phi$ :  $\Delta \triangleright \phi$  iff  $\Delta \Vdash_{FDE_2} \phi$ .

*Proof.* Same strategy again.

The case of STRICT GROUNDING\* is more complicated and leads to a calculus that is less elegant:

### System FDE<sub>3</sub> (for STRICT GROUNDING\*)

*Introduction axioms:*

- i1.  $\phi, \psi \vdash \phi \wedge \psi$
- i2.  $\phi \vdash \phi \vee \psi$
- i3.  $\psi \vdash \psi \vee \phi$
- i4.  $\neg\phi, \neg\psi \vdash \neg(\phi \vee \psi)$
- i5.  $\neg\phi \vdash \neg(\phi \wedge \psi)$
- i6.  $\neg\psi \vdash \neg(\phi \wedge \psi)$
- i7.  $\phi \vdash \neg\neg\phi$

*Introduction rules:*

For conjunction:

$$\frac{\Delta \vdash \phi \quad \Delta' \vdash \psi}{\Delta, \Delta' \vdash \phi \wedge \psi}$$

$$\frac{\Delta \vdash \phi}{\Delta, \phi \vdash \phi \wedge \phi}$$

For disjunction:

$$\frac{\Delta \vdash \neg\phi \quad \Delta' \vdash \neg\psi}{\Delta, \Delta' \vdash \neg(\phi \vee \psi)}$$

$$\frac{\Delta \vdash \neg\phi}{\Delta, \neg\phi \vdash \neg(\phi \vee \phi)}$$

*Structural rules:*

Restricted Cut ( $\psi \notin \Delta'$ ):

$$\frac{\Delta \vdash \psi \quad \psi, \Delta' \vdash \phi}{\Delta, \Delta' \vdash \phi}$$

**Theorem 3.6.** *For all  $\Delta$  and  $\phi$ :  $\Delta \triangleright^* \phi$  iff  $\Delta \Vdash_{FDE_3} \phi$ .*

*Proof.* Same strategy again.

Theorems 3.1 to 3.6 constitute substantial elements of our understanding of the relationships between logical grounding and first-degree entailments. I surmise that they are far from telling the whole story, and in particular I would not be surprised if the three relations of logical grounding involved in these theorems could be neatly characterised in terms of FDE-consequence.<sup>3</sup>

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3. Material from which this paper stemmed has been presented at the *Kit Fine Conference* (Varano Borghi, Italy, July-August 2013), the workshop *Groundedness in Semantics and Beyond* (Oslo, August 2013) and the workshop *Proofs That and Proofs Why* (IHPST, Paris, November 2013). I am grateful to the audiences of these events for helpful comments and discussions. This work was carried out while I was in charge of the Swiss National Science Foundation projects ‘Grounding—Metaphysics, Science, and Logic’ (Neuchâtel, CRSII1-147685) and ‘The Nature of Existence: Neglected Questions at the Foundations of Ontology’ (Neuchâtel, 100012-150289), and a member of the Spanish Ministry of Economy and Competitiveness ‘The Makings of Truth: Nature, Extent, and Applications of Truthmaking’ (Barcelona, FFI2012-35026).

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