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Extrinsic Estimates of Eigenvalues for the Laplacian with Wentzel Boundary Condition

Thèse

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À la mémoire de mon cher et regretté papa

Résumé

Ces travaux de recherche portent sur l'étude des valeurs propres de l'opérateur de Laplace avec les conditions au bord de Wentzel, données par $\beta\Delta_\Gamma u + \partial_{\mathbf{n}} u = \lambda u$ pour $\beta \geqslant 0$, ainsi que celles du Laplacien à densité, en fonction de la géométrie du domaine dans lequel ces problèmes sont posés.

Le problème de Wentzel sur des domaines euclidiens est bien connu et même dans le cadre plus général des variétés riemanniennes. Lorsque le coefficient $\beta = 0$, la condition au bord se réduit à la condition de Steklov $\partial_{\mathbf{n}} u = \lambda u$.

Pour estimer les valeurs propres de Wentzel-Laplace en fonction de la géométrie, nous établirons des bornes supérieures, asymptotiquement optimales au sens de la loi de Weyl.

Les premières estimations obtenues ici s'expriment en termes d'indice d'intersection du bord dans le cas euclidien, et de quantités géométriques extrinsèques dans le cadre riemannien. Pour un domaine compact Ω, à bord non vide et suffisamment régulier Γ, dans une variété riemannienne complète de dimension n, ces quantités interviennent dans une constante $\overline{B}(\Omega,\beta)$ telle que, pour tout $k \in \mathbb{N}$, les valeurs propres satisfont :

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant \left(\zeta_n\left[\kappa\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right] + 1\right)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} + \overline{B}(\Omega,\beta),$$

où ζ_n est une constante dépendant uniquement de la dimension, et $\kappa \in \mathbb{R}_{\geqslant 0}$ est la plus petite constante telle que la courbure de Ricci est minorée par $-(n-1)\kappa^2$.

Si la variété M a une courbure de Ricci non négative, alors :

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant C(\Omega,\beta) + B(\Omega)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{1}{n-1}} + A(\Omega)\beta\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}},$$

où $A(\Omega)$, $B(\Omega)$ et $C(\Omega,\beta)$ sont des constantes dépendant de la géométrie de Ω, avec $A(\Omega)$ et $B(\Omega)$ indépendantes de β.

Un autre résultat important que nous établissons montre que le rapport isopérimétrique permet de contrôler l'ensemble du spectre du problème de Wentzel-Laplace dans diverses variétés. Ces estimations sont particulièrement intéressantes dans la mesure où le rapport isopérimétrique d'un domaine est moins affecté par certaines « petites » déformations que des quantités géométriques telles que la courbure ou l'indice d'intersection du bord.

Les méthodes de preuve consistent à construites des décompositions bien appropriées de l'espace métrique mesuré associé au domaine, en fonction de la concentration volumique de la métrique riemannienne. Des domaines disjoints sont alors utilisés comme supports de fonctions tests bien choisies, permettant de borner les valeurs propres en utilisant la caractérisation du min-max. Ces résultats sont des conséquences d'inégalités plus précises — cependant un peu techniques — démontrées au fil de ce texte.

Concernant le Laplacien é densité, on considère une variété riemannienne compacte, connexe (M, g), avec d_M la forme volume associée à g, puis on s'intéresse à l'opérateur

$$L_h \cdot := e^{-h(\alpha-1)} (\Delta \cdot + \alpha\, g(\nabla h, \nabla \cdot)),$$

défini pour $\alpha \geqslant 0$ et $h \in C^2(M)$, sur M muni de la mesure pondérée $\mathrm{d}m := e^{-h}\mathrm{d}_M$. La question que nous explorons ici est celle de l'existence de bornes supérieures pour les valeurs propres de L_h dans $(M, \mathrm{d}m)$. Nous montrons que ces valeurs propres ne sont pas bornées lorsque $\alpha > 1$. Pour la démonstration, une suite de densités est construite, de sorte que leurs premières valeurs propres non nulles deviennent relativement grandes. Ce résultat est motivé et donne une réponse à une conjecture par des travaux antérieurs pour les valeurs de α entre 0 et 1.

Mots-clés : conditions de Wentzel, problème de Steklov, opérateur de Laplace-Beltrami, Laplaciens à densité, bornes sur les valeurs propres, variétés riemanniennes, géométrie spectrale, inégalités isopérimétriques, méthodes min-max, espaces métriques mesurés.

Abstract

This thesis investigates the problem of bounding the eigenvalues of the Laplace operator with Wentzel boundary condition $\beta\Delta_\Gamma u + \partial_{\mathbf{n}} u = \lambda u$ for $\beta \geqslant 0$ and weighted Laplacian, depending on the geometry of the domain in which these problems are defined.

The Wentzel problem is thoroughly studied for Euclidean domains and in the more general setting of Riemannian manifolds. When $\beta = 0$, the boundary condition reduces to Steklov condition $\partial_{\mathbf{n}} u = \lambda u$.

To estimate the Wentzel-Laplace eigenvalues in terms of the geometry, upper bounds that are asymptotically optimal according to the Weyl law, are established.

The first main estimates are in terms of the intersection index of the boundary for Euclidean domains and extrinsic geometric quantities for general Riemannian manifolds. For a compact domain Ω, with a non-empty and regular enough boundary Γ, in an n-dimensional complete Riemannian manifold, geometric quantities are involved in a constant $\overline{B}(\Omega,\beta)$ so that the eigenvalues satisfy for all $k \in \mathbb{N}$ the inequality $\lambda_{W,k}^{\beta}(\Omega) \leqslant \left(\zeta_n\left[\kappa\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right]+1\right)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} + \overline{B}(\Omega,\beta)$, where ζ_n is a dimensional constant and $\kappa \in \mathbb{R}_{\geqslant 0}$ is the smallest constant such that the Ricci curvature is bounded from below by $-(n-1)\kappa^2$. If M has non-negative Ricci curvature, then $\lambda_{W,k}^{\beta}(\Omega) \leqslant C(\Omega,\beta) + B(\Omega)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{1}{n-1}} + A(\Omega)\beta\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}$, where $A(\Omega)$, $B(\Omega)$, $C(\Omega,\beta)$ are constants depending on the geometry of Ω with $A(\Omega)$ and $B(\Omega)$ free from β.

Another major result we establish is that the isoperimetric ratio allows to control the entire spectrum of the Wentzel-Laplace problem in various manifolds.These estimates are particularly interesting in that the isoperimetric ratio of a domain is less affected by some «small» deformations than geometric quantities as curvatures or intersection index of the boundary.

The methods of proof consist in elaborating appropriate metric measure space decompositions for the domain, according to the volume concentration of the Riemannian metric. Then, disjoint regions are used to support well-chosen test functions that are conducive to bound the eigenvalues applying their min-max characterisation. These results are consequences of more precise, but a bit abstruse, inequalities that are proved in the text.

For the weighted Laplacian, we consider a compact, connected Riemannian manifold (M,g) and denote by d_M the volume element from g. We are interested in the operator $L_h\cdot := e^{-h(\alpha-1)}\Big(\Delta\cdot + \alpha g(\nabla h,\nabla\cdot)\Big)$, where $\alpha \geqslant 0$ and $h \in C^2(M)$ are given, on M equipped with the weighted volume form $\mathrm{d}m := e^{-h}\mathrm{d}_M$. The question that we explore is the existence of upper bounds for the eigenvalues of L_h in $(M,\mathrm{d}m)$. We prove that the eigenvalues are unbounded when $\alpha > 1$. For that purpose, a sequence of densities, such that their corresponding first non-zero eigenvalues become as large as desired, is constructed. This result is motivated by previous works for the other values of α (between zero and one) and was conjectured.

Keywords: Wentzel boundary conditions, Steklov problem, Laplace-Beltrami operator, weighted Laplacians, eigenvalue bounds, Riemannian manifolds, spectral geometry, isoperimetric inequalities, min-max methods, metric measure spaces.

Statement of originality

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

Aïssatou Mossèle Ndiaye
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1 Introduction (fr)

« La définition d'un bon problème mathématique, ce sont les mathématiques qu'il génère plutôt que le problème lui-même. » [Traduction libre]

– Andrew Wiles

De manière perceptible, la taille et la forme d'un objet affectent certaines de ses propriétés physiques. Par exemple, la percussion d'une énorme timbale produit une hauteur tonale plus basse (grave) que celles qui peuvent être jouées sur un tambour plus petit. Ce lien de causalité direct entre la taille et les ondes acoustiques dépend également de la forme et de la prévalence d'autres propriétés physiques comme la densité de la tête de tambour. En outre, cela est entièrement contrôlé par une condition préalable importante : la tension autour du cerceau au bord de la peau du tambour. C'est à ce point qu'extraire des informations sur les propriétés physiques d'un objet, telles que les tonalités produites par un instrument, à partir de sa géométrie et avec une précision mathématique, est subtile.

1.1 Motivation de l'étude des valeurs propres du laplacien

Des phénomènes physiques aussi divers que les vibrations d'une membrane (tambour), d'une corde, d'une masse d'air dans une salle de concert, le rayonnement d'un corps en équilibre thermique, sont régis par l'équation de Helmholtz :

$$\Delta u = \lambda u. \tag{1.1.1}$$

Il s'agit d'une équations aux dérivées partielles linéaire comportant l'opérateur de Laplace Δ (λ est une valeur propre, et u est une fonction propre) et correspondant à une forme indépendante du temps de l'équation d'onde. D'un point de vue physique, le laplacien sur une variété riemannienne compacte est un opérateur linéaire fondamental qui permet de décrire de nombreux phénomènes de propagation. Ceci motive la théorie spectrale du laplacien sur les variétés riemanniennes, dans le contexte de la géométrie différentielle.

Sur un domaine euclidien $\Omega \subset \mathbb{R}^n$, le laplacien Δ sur Ω est l'opérateur agissant sur les fonctions de Ω, donnée par la somme des dérivées secondes (supposées existantes) par rapport à chaque variable de position :

$$\Delta u := -\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

La généralisation naturelle sur une variété riemannienne (M, g), formellement nommée l'opérateur de Laplace-Beltrami, est la définition locale :

$$\Delta_g u := -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x_i}\left(g^{ij}\sqrt{\det g}\frac{\partial u}{\partial x_j}\right).$$

Indépendamment de tout système de coordonnées, l'opérateur de Laplace-Beltrami sur une variété riemannienne est défini de manière équivalente comme les applications successives, des opérateurs gradient et divergence :

$$\Delta_g := -\mathrm{div}_g \nabla_g$$

où div_g et ∇_g sont respectivement les opérateurs divergence et gradient associés à la métrique g.

Notez que nous adoptons la convention des géomètres qui est de placer un signe moins dans la définition et comme g est généralement compris, la référence à la métrique est omise en supprimant l'indice lorsque la métrique riemannienne est contextuellement claire. Le spectre de l'opérateur Δ est constitué de toutes les valeurs réelles pour lesquelles il existe une fonction propre correspondante telle que l'équation de Helmholtz (1.1.1) est satisfaite.

1.1.1 Quelques exemples

Voici des applications bien connues en physique où l'opérateur de Laplace joue le rôle clé :

Propagation des ondes. Soit Ω un domaine (ouvert, connexe) euclidien à bord. Si nous pensons à Ω comme une tête de tambour, le bord $\partial\Omega$ est attachée à un fût. Pour étudier la vibration générée lorsque qu'on frappe le tambour, nous résolvons l'équation d'onde :

$$\left(\Delta + \frac{\partial^2}{\partial t^2}\right)u(x,t) = 0. \tag{1.1.2}$$

Pour tenir compte du fait que le pourtour de la membrane est fixé, il est additionnellement imposé que la solution satisfasse la condition au bord :

$$u(x,t) = 0 \quad \text{pour tout point } x \in \partial\Omega. \tag{1.1.3}$$

La spécification d'une valeur imposée à la solution sur le bord du domaine est faite à travers la condition au bord de Dirichlet (1.1.3). L'interprétation physique de la condition au bord de Dirichlet peut être une membrane fixe sur le bord ou une température ambiante fixée à la limite. Autrement, une condition au bord d'ordinaire est un modèle mathématique efficace pour «approximer» convenablement une réalité physique. Il existe également d'autres conditions bien connues telles que la condition au bord de Neumann

$$\frac{\partial u}{\partial \mathbf{n}}(x,t) = 0 \quad \text{pour tout point } x \in \partial\Omega, \tag{1.1.4}$$

qui fixe une valeur prescrite pour la dérivée normale de la solution au bord du domaine. Il est autant possible d'avoir un mélange de conditions de Dirichlet et de Neumann sur des parties disjointes du bord pour donner ce que l'on appelle communément des conditions mixtes, ou une combinaison des deux donnant la condition de Robin :

$$\frac{\partial u}{\partial \mathbf{n}}(x,t) + \alpha u(x,t) = 0, \text{ pour tout point } x \in \partial\Omega, \quad \alpha > 0 \text{ est fixé.} \tag{1.1.5}$$

Un grand nombre de conditions au bord est possible, en fonction de la nature de l'équation à laquelle elles sont rattachées et du problème formulé.

Dans le même ordre d'idées que pour le tambour, le mouvement de la surface d'un fluide est décrit par l'équation :

$$\left(\Delta + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)u(x,t) = 0,$$

où c est la vitesse du son dans le fluide.

Diffusion de la chaleur. Pour connaître l'évolution dans le temps de la répartition de la température dans un solide $\Omega \subset \mathbb{R}^n$, il faut résoudre l'équation de la chaleur :

$$\left(\Delta + \frac{1}{c}\frac{\partial}{\partial t}\right)u(x,t) = 0$$

où c est la conductivité du matériau et $u(x,t)$ est la température au point $x \in \Omega$ au temps t. Lorsque la température d'équilibre est atteinte, après une période de temps suffisamment longue, les pertes et les gains de chaleur sont parfaitement équilibrés, alors la température est indépendante de la variable de temps t. Elle est exprimée sous forme de fonction harmonique :

$$\Delta u = 0.$$

Champ de gravité. Un champ gravitationnel newtonien $\mathbf{g}$ dû à l'influence d'un objet de densité de masse ρ_m, dérive d'un potentiel gravitationnel φ :

$$\mathbf{g} = -\nabla\varphi,$$

satisfaisant l'équation de Poisson :

$$\mathrm{div}\mathbf{g} = \Delta\varphi = 4\pi G\rho_m,$$

où G est la constante gravitationnelle.

Équation de Schrödinger pour les particules quantiques. Afin d'étudier le mouvement d'une particule quantique à l'intérieur d'un domaine Ω, lorsqu'aucune force externe n'est appliquée, nous devons résoudre l'équation :

$$\frac{\hbar^2}{2m}\Delta u(x,t) = i\hbar\frac{\partial}{\partial t}u(x,t),$$

où $i = \sqrt{-1}$, m est la masse de la particule, $\hbar$ est la constante de Planck réduite (ou constante de Dirac) et $u(x,t)$ est la fonction d'onde représentant la particule.

Comme décrit dans les exemples précédents, les fonctions propres de l'opérateur de Laplace sont matérialisées, en acoustique par des modes de vibration, en mécanique quantique par l'état physique d'un électron, etc. Les modes de vibration d'une fine membrane (une tête de tambour) à bord fixe sont donnés par les fonctions propres du laplacien de Dirichlet. Les fréquences propres sont les racines carrées des valeurs propres.

1.1.2 Géométrie du laplacien

La possibilité d'associer à n'importe quelle variété riemannienne compacte connexe M, avec ou sans bord, des opérateurs auto-adjoints comme l'opérateur de Laplace-Beltrami (nous l'appelons brièvement le laplacien), donne des objets fondamentaux : le spectre de l'opérateur. À l'avenant, on parle de problème fermé dans le cas des variétés compactes sans bord, à l'instar de la sphère. Une condition au bord, telle que la condition de Dirichlet ou celle de Neumann vues précédemment, étant nécessairement imposée en supplément dans le cas des variétés à bord, le spectre de Δ consiste en une suite infinie de valeurs propres réelles :

$$\mathrm{Spec}(\Delta) = \{0 \leqslant \lambda_1(M) < \lambda_2(M) \leqslant \lambda_3(M) \leqslant \cdots \leqslant \lambda_k(M) \leqslant \cdots\}.$$

Le lecteur peut se référer à [1, Thm. 14.6] et [4, Thm. $A.I.4$ $(S.I)$].

Le laplacien étant invariant par isométrie, chaque élément de la suite $\mathrm{Spec}(\Delta)$ est un invariant riemannien. Cette invariance pose la question de savoir si réciproquement le spectre (les fréquences) détermine (à isométrie près) la métrique riemannienne (Kac [42]).

La géométrie spectrale étudie les relations entre les caractéristiques géométriques de la variété et les propriétés du spectre du laplacien. Il est décomposée en deux domaines, respectivement appelés problèmes direct et inverse :

- Calculer ou rechercher des propriétés des valeurs propres à partir de la géométrie de la variété,
- Déterminer certains aspects de la géométrie de la variété, en supposant que nous connaissons toutes les valeurs propres.

Nous nous intéresserons, dans cette thèse, à trouver des informations sur les valeurs propres du laplacien en termes de données géométriques (problème direct). Il s'avère qu'en général, il n'est pas possible de déterminer explicitement les valeurs propres. Un problème théorique crucial consiste à trouver des inégalités pour les approcher.

1.2 Bornes supérieures pour les valeurs propres du laplacien

Un calcul explicite du spectre du laplacien d'une variété riemannienne est, au sens large du terme, quasiment impossible. Les valeurs propres sont difficiles à calculer et elles ne sont connues que dans quelques cas. À défaut de pouvoir expliciter le spectre plusieurs études porteront sur son estimation approximative relativement à la géométrie avec une attention particulière pour la première valeur propre non nulle. Pour donner quelques exemples, une corde vibrante de longueur $L > 0$ est décrite par l'équation d'onde unidimensionnelle suivante :

$$c^2 \frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial t^2},$$

où c est une constante et $\varphi(x,t)$ représente le déplacement transversal de la corde à la position x et au temps t. La fonction spatiale u résultant de la séparation des variables $\varphi(x,t) = u(x)\sin(\sqrt{\lambda}ct)$ est une fonction propre de l'opérateur dérivée seconde :

$$-\frac{\partial^2 u}{\partial x^2} = \lambda u.$$

La fréquence ou hauteur tonale de la corde correspond à la racine carrée de la valeur propre λ. Si les conditions aux limites imposées aux extrémités $x = 0$ et $x = L$ sont «extrémités fixes», c'est la condition de Dirichlet $u = 0$ à $x = 0$ et $x = L$, alors on a une infinité de valeurs propres données explicitement par :

$$\lambda_k = \left(\frac{k\pi}{L}\right)^2, \quad k = 1, 2, 3, \ldots$$

avec les fonctions propres correspondantes :

$$u_k(x) = \sin\left(\frac{k\pi x}{L}\right).$$

Pour une membrane (bidimensionnelle) ou une peau de tambour, vibrant transversalement dans la troisième direction, en opérant une séparation de variables dans l'équation (1.1.2), on obtient :

$$\Delta u = \lambda u, \text{ avec la condition de Dirichlet } u = 0 \text{ sur le bord } \partial\Omega,$$

où Ω décrit la position du tambour au repos avec un déplacement nul tout autour du bord. Tout comme dans le cas unidimensionnel, les valeurs propres λ_k sont positives et croissantes à l'infini. Malheureusement, les valeurs propres ne peuvent être calculées que dans quelques cas particuliers, par exemple : pour les disques (en utilisant les coordonnées polaires et les fonctions de Bessel), pour les rectangles (en utilisant les coordonnées rectangulaires et les fonctions sinus [48, § 52]), les triangles équilatéraux (en utilisant les coordonnées barycentriques et les fonctions trigonométriques [48, § 57]). Pour plus de détails et les formules résultantes, voir [49, Appendix A] et les références y figurant.

Pour en venir à présent à l'étude des bornes géométriques des valeurs propres, il est peut-être intéressant de souligner que l'inégalité de Faber-Krahn, dès 1925, donnait un minimum pour la première valeur propre de Dirichlet sur un domaine Ω du plan, en termes d'aire :

$$\frac{\pi j_{0,1}^2}{\text{Aire}(\Omega)} = \lambda_1(\mathcal{D})\frac{\text{Aire}(\mathcal{D})}{\text{Aire}(\Omega)} \leqslant \lambda_1(\Omega), \qquad \text{(Faber [24] et Krahn [47])}$$

où $j_{0,1}$ désigne le premier zéro de la fonction de Bessel d'ordre zéro, et $\mathcal{D}$ est un disque (voir [36] pour plus de détails sur les valeurs propres du disque). Il s'agit d'une inégalité isopérimétrique, donnant un sens précis et une preuve à une conjecture faite par Rayleigh :

"If the area of a membrane be given, there must evidently be some form of boundary for which the pitch (of the principal tone) is the gravest possible, and this form can be no other than the circle."

– Rayleigh, Theory of Sound [65, §210]

À noter également qu'avant Faber et Krahn, Courant [17] avait prouvé le résultat plus faible qui dit que parmi toutes les membranes de même périmètre L, la circulaire donne la valeur propre la plus petite, i.e

$$\frac{4\pi^2 j_0^2}{L^2} \leqslant \lambda_1(\Omega),$$

avec égalité si et seulement si la membrane est circulaire.

Borner la première valeur propre (non triviale) devenant dès lors un sujet de recherche de grand intérêt, de nombreux résultats sont apparus dans la littérature. Les contributions les plus intéressantes sont résumées dans [62]. Toutefois, selon toute vraisemblance, Hersch dans [37] a été le premier à obtenir une borne supérieure pour la première valeur propre non nulle du problème fermé en termes du volume. Il a développé dans une direction plus géométrique une approche de Szegö [69] pour prouver que :

$$\lambda_2(M) \leqslant \frac{8\pi}{\text{Vol}_g(M)}, \qquad (\lambda_1(M) = 0) \tag{1.2.1}$$

pour toute variété $M = (\mathbb{S}^2, g)$ donnée par la sphère $\mathbb{S}^2$ munie d'une métrique arbitraire g. En d'autres termes, parmi toutes les métriques riemanniennes sur une sphère d'aire fixe, la première valeur propre de l'opérateur de Laplace est maximale pour la métrique ronde standard qui satisfait

$$\lambda_2(\mathbb{S}^2)\text{Vol}(\mathbb{S}^2) = 8\pi.$$

Ce résultat de Hersch a été un incitant majeur pour toute une direction dans l'étude de l'existence de métriques extrémales pour les surfaces ainsi que des propriétés des valeurs propres d'ordre supérieur du laplacien.

Indépendamment, Reilly [66, Cor. 1] et Chavel [9, Thm. 1] ont complété l'inégalité de Hersch (1.2.1) pour tout domaine euclidien de dimension n, $\Omega \subset \mathbb{R}^n$ borné par une hypersurface compacte lisse Γ :

$$\lambda_2(\Gamma) \leqslant \left(\frac{\text{Vol}(\Gamma)}{\text{Vol}(\Omega)}\right)^2 \frac{n-1}{n^2}, \tag{1.2.2}$$

avec égalité si et seulement si Γ est la sphère $\mathbb{S}^{n-1}$ et $\mathrm{Vol}(\Gamma)$ désigne le volume en dimension $(n-1)$ de M. En dimension $n = 3$, en appliquant l'inégalité isopérimétrique classique, on peut remarquer que l'inégalité de Hersch (1.2.1) est plus forte que (1.2.2).

Dans cette thèse, nous nous intéresserons principalement à établir des inégalités du même type que Reilly-Chavel (que l'on retrouve également dans la littérature sous le nom d'inégalités de type Reilly et ainsi nommées en référence à (1.2.2)), pour les valeurs propres du laplacien avec condition au bord de Wentzel. Il s'agit d'un problème assez récent, introduit dans [72], qui n'a été étudié de manière significative qu'au cours des deux dernières décennies ; une description complète et détaillée est donnée au Chapitre 2.

1.3 Bornes uniformes et méthodes métriques

Rappelons qu'un grand nombre d'auteurs (Li et Yau [52], Heintze [34], El Soufi et Ilias [23], Veeravalli [71], Grosjean [32], Giménez, Miquel et Orengo [27], Wang et Xia [73], etc.) ont prouvé différentes versions de l'inégalité de Reilly-Chavel (1.2.2). En fait, cette borne est une version partielle de l'inégalité établie par Reilly qui est une borne supérieure pour la première valeur propre non nulle pour toute sous-variété fermée de l'espace euclidien en termes de courbure moyenne totale, ceci généralisant un résultat antérieur de Bleecker et Weiner [6]. Heintze [34], étend par la suite le résultat de Reilly aux hypersurfaces de variétés compactes et de variétés de Hadamard. Puis, El Soufi et Ilias [23] ont obtenu de meilleures bornes supérieures pour la première valeur propre non nulle de laplacien sur les hypersurfaces fermées de l'espace hyperbolique, en termes de courbure moyenne.

La plupart de ces estimations ont été obtenues en appliquant, comme noté par Colbois [10], des "méthodes de type barycentrique" et en utilisant des fonctions coordonnées comme fonctions test.

Une attention plus récente s'est concentrée sur les valeurs propres plus élevées du spectre. Bien que les méthodes barycentriques semblent efficaces avec les recherches antérieures pour borner la première valeur propre du laplacien, elles semblent difficiles à généraliser pour des valeurs propres d'ordre supérieur. Une nouvelle approche est donc nécessaire. Les estimations pour le problème de Wentzel, présentées dans cette thèse, ont été motivées par les résultats antérieurs sur le spectre du laplacien sur des variétés riemanniennes fermées basées sur une approche de géométrie métrique.

Soit M une variété connexe fermée (compacte sans bord), de dimension $n \geqslant 2$. Soit Δ, le laplacien agissant sur l'espace $C^\infty(M)$ des fonctions lisses à valeur réelle sur M. Nous considérons l'équation des valeurs propres

$$\Delta u = \lambda u.$$

Les valeurs propres λ sont réelles et la compacité de la variété permet de montrer qu'elles sont discrètes, la fonction constante étant une fonction propre associée à la première valeur propre $\lambda_1(M) = 0$:

$$0 = \lambda_1(M) < \lambda_2(M) \leqslant \lambda_3(M) \leqslant \cdots \lambda_k(M) \leqslant \cdots,$$

chaque valeur propre étant répétée autant de fois que sa multiplicité. On considère alors la valeur propre d'ordre k comme une fonction définie par

$$M \longrightarrow \lambda_k(M)$$

ou éventuellement une version normalisée avec les mêmes propriétés variationnelles (voir (2.1.1)). On étudie la corrélation entre les valeurs propres $\lambda_k(M)$ et les quantités géométriques extrinsèques qui dérivent de la seconde forme fondamentale de M immergée isométriquement dans l'espace euclidien, ou un espace ambiant plus général.

Korevaar [45] a conçu l'idée d'une nouvelle méthode utilisant une décomposition métrique de la variété pour prouver qu'il existe une constante universelle $C > 0$ telle que, pour toute surface compacte orientable M de genre γ, nous avons :

$$\lambda_k(M)\mathrm{Vol}(M) \leqslant C(\gamma+1)k, \quad \forall\, k \geqslant 2.$$

Ceci généralise aux valeurs propres d'ordre supérieur un résultat de Yang et Yau [77] :

$$\lambda_2(M)\mathrm{Vol}(M) \leqslant 8\pi(\gamma+1).$$

Dans la preuve du résultat de Korevaar, la surface M est décomposée en anneaux et régions disjoints selon la concentration volumique de la métrique. Cette décomposition annulaire de l'espace métrique [45, section 2] est ensuite utilisée pour constituer les supports de fonctions test propices à l'établissement des bornes pour les valeurs propres.

Un développement ultérieur de cette méthode par Grigor' yan, Netrusov et Yau [31, section 3], simplifiant la construction de Korevaar, a permis d'obtenir des bornes supérieures pour les valeurs propres via des capacités.

En appliquant la méthode de Grigor' yan et al. [31], Colbois, El Soufi et Girouard [14, Thm. 1.3] ont pu montrer que les valeurs propres de Steklov, tel que définies dans la section 2.2, d'un domaine dans une variété riemannienne complète sont bornées supérieurement en termes du rapport isopérimétrique.

Définition 1.3.1. Pour un domaine Ω à bord Γ, d'une variété riemannienne complète (M,g) de dimension n, le rapport isopérimétrique est défini par :

$$I(\Omega) := \frac{\mathrm{Vol}(\Gamma)}{\mathrm{Vol}(\Omega)^{\frac{n-1}{n}}}.$$

Au numérateur Vol représente le volume riemannien induite par la métrique g en dimension $(n-1)$.

Le théorème suivant montre qu'avec des hypothèses géométriques sur la variété, on peut contrôler les valeurs propres normalisées de Steklov en termes du rapport isopérimétrique du domaine.

Théorème 1.3.1 (Colbois et al. [14]). Soit M une variété riemannienne complète de dimension n avec une courbure de Ricci non négative. Soit Ω un domaine de M et Γ

son bord. Notons $\{\lambda_{S,k}\}_{k=1}^{\infty}$ le spectre du problème de Steklov :

$$\begin{cases} \Delta u = 0, & \text{dans } \Omega, \\ \partial_{\mathbf{n}} u = \lambda_S u, & \text{sur } \Gamma. \end{cases}$$

Alors, pour tout $k \geqslant 2$, on a :

$$\lambda_{S,k}(\Omega)\mathrm{Vol}(\Gamma)^{\frac{1}{n-1}} \leqslant \frac{\alpha_n}{I(\Omega)^{1-\frac{1}{n-1}}} k^{\frac{2}{n}}, \tag{1.3.1}$$

où α_n est une constante dépendant uniquement de la dimension n.

Les mêmes auteurs ont également obtenu des bornes, pour les valeurs propres du laplacien sur une hypersurface délimitant un domaine dans une variété riemannienne, en fonction du rapport isopérimétrique du domaine. Ils ont prouvé une borne supérieure uniforme pour ces valeurs propres, en utilisant la construction de Colbois et Maerten [12] qui ont élaboré une décomposition d'espace métrique mesuré complet pouvant être adaptée à diverses situations. C'est un outil principal dans [15] pour obtenir les bornes supérieures pour les valeurs propres du laplacien sur une hypersurface délimitant un domaine :

Théorème 1.3.2 (Colbois et al. [15, Thm. 1.2]). Soit (M,g) une variété riemannienne complète de dimension $n \geqslant 2$ et dont la courbure de Ricci est minorée par $-(n-1)\kappa^2$, $\kappa \in \mathbb{R}$. Soit $\Omega \subset M$ un domaine à bord lisse Γ et soit $(\eta_k)_{k\in\mathbb{N}}$ l'ensemble des valeurs propres du laplacien sur Γ. Pour tout $k \geqslant 2$, on a :

$$\eta_k(\Gamma) \leqslant \alpha_n \frac{I(\Omega)}{I_0(\Omega)}\kappa^2 + \beta_n \left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}} \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}, \tag{1.3.2}$$

où α_n et β_n sont des constantes dépendant de la dimension et

$$I_0(\Omega) := \inf\{I(U) \ : U \text{ ouvert inclus dans } \Omega\}.$$

En particulier, si Ω est un domaine de l'espace euclidien $M = \mathbb{R}^n$, on a $\kappa = 0$ et $I_0(\Omega) = I_0(\mathbb{R}^n) = n\omega_n^{\frac{1}{n}}$ où ω_n désigne le volume de la boule unitaire euclidienne de dimension n et donc :

$$\eta_k(\Gamma)\mathrm{Vol}(\Gamma)^{\frac{2}{n-1}} \leqslant c_n I(\Omega)^{1+\frac{2}{n-1}} k^{\frac{2}{n-1}},$$

où c_n est une constante qui ne dépend que de la dimension.

Remarque 1.3.1.
- Les méthodes métriques utilisées dans les théorèmes 1.3.1 et 1.3.2 sont différentes ; la construction de Colbois et Maerten [12] pour les valeurs propres du laplacien sur le bord et celle de Grigor'yan et al. [31] pour les valeurs propres de Steklov.
- Pour les valeurs propres de Steklov (1.3.1), comme pour celles de l'hypersurface que constitue le bord (1.3.2), le rapport isopérimétrique intervient dans les bornes, mais pour autant de manières très différentes. Dans (1.3.1), on voit que si

$n \geqslant 2$, les domaines avec de grands rapports isopérimétriques correspondent aux plus petites valeurs propres $\tilde{\lambda}_{S,k}(\Omega) = \lambda_{S,k}(\Omega)\mathrm{Vol}(\Gamma)^{\frac{1}{n-1}}$.

- Il faut remarquer que la puissance de k qui apparaît à droite de l'inégalité (1.3.2) est optimale, selon la loi de Weyl :

$$\eta_k(\Gamma) = C_n^2 k^{\frac{2}{n-1}} + o(k^{\frac{2}{n-1}}), \quad k \to \infty,$$

où $C_n = \frac{2\pi}{(\omega_{n-1}\mathrm{Vol}(\Gamma))^{\frac{1}{n-1}}}$.

Les valeurs propres de Steklov, quant à elles, se comportent selon la formule asymptotique suivante :

$$\lambda_{S,k} = C_n k^{\frac{1}{n-1}} + o(k^{\frac{1}{n-1}}), \quad k \to \infty.$$

Quant à savoir si (1.3.1) devrait apparaître avec $k^{\frac{1}{n-1}}$ plutôt que $k^{\frac{2}{n}}$, les auteurs soulignent effectivement que c'est impossible en dimension $n \geqslant 2$, à moins que le rapport isopérimétrique ne soit par contrainte majoré par $\frac{\alpha_n}{c_n}$. Néanmoins, une borne supérieure, ne dépendant pas du rapport isopérimétrique, avec $k^{\frac{1}{n-1}}$ reste possible.

1.4 Problème de Wentzel

Soit $n \geqslant 2$ et (M, g) une variété riemannienne complète de dimension n. Soit $\Omega \subset M$ un domaine borné, à bord lisse Γ. On note Δ et Δ_Γ les opérateurs de Laplace agissant sur des fonctions sur M et Γ, respectivement. Soit $\beta \in \mathbb{R}$ une constante, on considère le problème suivant :

$$\begin{cases} \Delta u = 0 & \text{dans } \Omega, \\ \beta\Delta_\Gamma u + \partial_{\mathbf{n}} u = \lambda u & \text{sur } \Gamma. \end{cases} \qquad \text{(Problème de Wentzel)} \qquad (1.4.1)$$

Dans ce qui suit, nous supposerons que β, que nous appelons le paramètre de bord, est non négatif. Dans ce cas, les valeurs propres de Wentzel forment une suite infinie de valeurs positives :

$$0 = \lambda_{W,1}^\beta < \lambda_{W,2}^\beta \leqslant \lambda_{W,3}^\beta \leqslant \cdots \leqslant \lambda_{W,k}^\beta \leqslant \cdots \nearrow +\infty. \qquad (1.4.2)$$

Nous adoptons la convention que chaque valeur propre est répétée selon sa multiplicité. La condition aux limites dans (1.4.1), nommée condition au bord de Wentzel, a été initialement introduite dans [72], afin de trouver les conditions aux limites les plus générales pour lesquelles l'opérateur associé génère un semigroupe markovien. Elle est souvent considérée sous une forme plus générale (cf. [25, eqn (1.2)], [26, eqn (2.32)]), et/ou parfois, elle accompagne l'équation de chaleur comme dans [44, eqn (1.3)], voir aussi [25]. Une bonne discussion sur les motivations et l'interprétation physique des conditions aux limites de Wentzel peut être trouvée dans [29].

Lorsque $\beta = 0$, le problème aux valeurs propres (1.4.1) se réduit au problème de

Steklov. Il s'agit, en quelque sorte, d'une combinaison entre le problème de Steklov et celui des valeurs propres du laplacien sur le bord (i.e. $\Delta_\Gamma u_k = \eta_k u_k$, $k \in \mathbb{N}$), puisque nous avons (Voir la section 2.3 et [18, §2.1] si $k = 2$.) :

$$\lim_{\beta\to\infty} \frac{\lambda^{\beta}_{W,k}(\Omega)}{\beta} = \eta_k(\Gamma), \quad \forall\, k \in \mathbb{N}. \tag{1.4.3}$$

Par conséquent, la question se pose de savoir si les méthodes métriques évoquées précédemment pourraient être étendues au problème de Wentzel. Concernant les deux premiers points de la remarque 1.3.1, quel type de méthode de décomposition métrique est approprié ? Comment la constante isopérimétrique apparaîtrait-t-elle dans l'inégalité ?

Ce n'est que récemment que le problème du laplacien avec la condition aux limites de Wentzel a été étudié de manière significative. Il a reçu beaucoup d'attention au cours des dix dernières années. À ce jour, plusieurs études ([18, 22, 75], etc.) se sont concentrées sur l'estimation des valeurs propres :

- Dambrine, Kateb et Lamboley [18, Cor. 1.2] ont obtenu une première borne supérieure pour la première valeur propre non triviale $\lambda^{\beta}_{W,2}$, en termes de quantités géométriques. Soit Ω un domaine de $\mathbb{R}^n$, à bord Γ. Soit $\wedge(\Omega)$ le rayon spectral de la matrice symétrique semi-définie positive

$$P(\Omega) \overset{\text{def}}{=} \left(\int_\Gamma \delta_{ij} - \mathbf{n}_i \mathbf{n}_j \mathrm{d}_\Gamma \right)_{i,j=1,\dots,n},$$

  où $\mathbf{n}$ est le vecteur normal extérieur à Γ. Alors, on a l'inégalité suivante :

$$\lambda^{\beta}_{W,2}(\Omega) \leqslant \frac{\mathrm{Vol}(\Omega) + \beta \wedge (\Omega)}{\omega_n^{-\frac{1}{n}} \mathrm{Vol}(\Omega)^{\frac{n+1}{n}} \left[1 + c_n \left(\frac{\mathrm{Vol}(\Omega \Delta B)}{\mathrm{Vol}(B)} \right)^2 \right]}, \quad c_n := \frac{(\sqrt[n]{2} - 1)(n+1)}{4n}. \tag{1.4.4}$$

  Ici, B est la boule ayant le même volume que Ω et avec le même centre de masse que Γ. La différence $\Omega \Delta B$ de Ω et B l'ensemble des éléments appartenant à Ω ou à B exclusivement. On a égalité si Ω est la boule.

- C'est dans ce même sens que Xia et Wang [75, Thm. 1.2], ont prouvé la borne suivante pour la même valeur propre :

$$\lambda^{\beta}_{W,2}(\Omega) \leqslant \frac{n\mathrm{Vol}(\Omega) + \beta(n-1)\mathrm{Vol}(\Gamma)}{n\mathrm{Vol}(\Omega)(\mathrm{Vol}(\Omega)\omega_n^{-1})^{\frac{1}{n}}}. \tag{1.4.5}$$

  Ils prouvent également une borne inférieure pour $\lambda^{\beta}_{W,2}(\Omega)$, dans le cadre d'une variété riemannienne, lorsque la courbure de Ricci et les courbures principales du bord sont bornées :

Soit M une variété riemannienne compacte connexe de dimension n à courbure de Ricci non négative et à bord ∂M. On suppose que les courbures principales de ∂M sont minorées par une constante positive c.

La première valeur propre de Wentzel non nulle de M satisfait l'inégalité :

$$\left[1+(n-1)c\beta+\sqrt{(n-1)^2c^2\beta^2+2(n-1)c\beta}\right]\frac{c}{2}<\lambda^{\beta}_{W,2}(M) \quad [75, (1.13)].$$

- Pour aller plus loin, Du, Wang et Xia [22] donne la borne supérieure suivante pour les n premières valeurs propres (n étant la dimension), lorsque Ω est une sous-variété immergée dans l'espace euclidien $\mathbb{R}^N$, $N \geqslant n$, équipé de la métrique euclidienne canonique. Si H désigne le champ de vecteurs courbure moyenne de Γ dans $\mathbb{R}^N$, alors on a :

$$\frac{1}{n-1}\Sigma_{j=1}^{n}(\lambda^{\beta}_{W,j})^{\frac{1}{2}} \leqslant \frac{\sqrt{[n\mathrm{Vol}(\Omega)+(n-1)\beta\mathrm{Vol}(\Gamma)]\int_\Gamma |H|^2 \mathrm{d}_\Gamma}}{\mathrm{Vol}(\Gamma)}. \tag{1.4.6}$$

  Quand $N=n$, c'est-à-dire que Ω est un domaine de $\mathbb{R}^N$, alors l'égalité est vraie dans (1.4.6) si et seulement si Ω est une boule.

Toutes les études citées ci-dessus fournissent des estimations précises des toutes premières valeurs propres du spectre en fonction des propriétés géométriques du domaine sur lequel le problème est défini. Cependant, il reste plusieurs aspects du spectre de Wentzel, sur lesquels relativement peu est connu. Cette thèse fournit un support supplémentaire significatif avec des bornes uniformes pour toutes les valeurs propres du problème (1.4.1).

1.4.1 Contribution

Soit (M,g) une variété riemannienne complète de dimension n et $\Omega \subset M$ un domaine à bord lisse Γ. Nous étudions les valeurs propres du laplacien avec la condition au bord de Wentzel

$$\beta\Delta_\Gamma u+\partial_{\mathbf{n}}u=\lambda u \qquad \beta\in\mathbb{R}_{\geqslant 0},$$

sur les domaines euclidiens et plus généralement sur les domaines dans les variétés riemanniennes. Parmi les données géométriques particulièrement intéressantes, on a le volume, l'aire du bord Γ et/ou le rapport isopérimétrique du domaine, le diamètre et les courbures.

Pour les domaines euclidiens de dimension $n \geqslant 2$, on obtient une borne supérieure qui, en plus de la dimension, du volume et de l'aire du bord, dépend de l'indice d'intersection noté $i(\Gamma)$, c'est-à-dire le nombre maximal de points d'intersection du bord avec les droites transversales. Pour tout $k \geqslant 2$, on a :

$$\lambda^{\beta}_{W,k}(\Omega)\leqslant\left[\zeta_n i(\Gamma)^{\frac{2}{n-1}}\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right)+1\right]\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}+B(\Omega,\beta). \qquad \text{(Cor. 3.0.5)}$$

La constante ζ_n dépend uniquement de la dimension et $B(\Omega,\beta)$ dépend de β, n, $\mathrm{Vol}(\Omega)$ et $\mathrm{Vol}(\Gamma)$.

Cette borne est optimale par rapport à la loi de Weyl (voir (2.4.1)) pour $\beta > 0$. En outre, cette estimation dans le cas euclidien, se généralise dans une certaine mesure, aux variétés riemanniennes à courbure de Ricci minorée :

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant \left(\zeta_n \left[\mathcal{K}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right] + 1\right)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} + \overline{B}(\Omega,\beta), \qquad \text{(Cor. 3.0.2)}$$

où la constante ζ_n dépend, comme ci-dessus, de la dimension n, $\mathcal{K} := \max\{1, K_-\}$ et $K_- \in \mathbb{R}_{\geqslant 0}$ est la plus petite constante telle que la courbure de Ricci de la variété soit minorée par $-(n-1)K_-^2$. Contrairement à $B(\Omega,\beta)$ dans le cas euclidien, $\overline{B}(\Omega,\beta)$ dépend de quantités géométriques moins explicites incluant la seconde forme fondamentale, le rayon d'injectivité normal, la courbure de Ricci du bord Γ. Cette constante implique tacitement des quantités géométriques ce qui a pour effet de durcir les hypothèses inhérentes au résultat. Par conséquent, l'optimalité de ces hypothèses ainsi que la comparaison avec des hypothèses alternatives s'avèrent intéressantes à étudier. Ces résultats seront significativement améliorés dans le théorème 3.3.6, voir (3.3.10).

Toutefois, avec les estimations ci-après, dans l'esprit de Reilly-Chavel (1.2.2), on prouve que l'intégralité du spectre peut être contrôlée par le rapport isopérimétrique du domaine :

Pour $n \geqslant 3$ et $\Omega \subset \mathbb{R}^n$ un domaine euclidien de dimension n, alors pour tout $k \geqslant 2$, nous avons :

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant C(\Omega,\beta)\left[\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{n-2} + \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}\right], \qquad \text{(Cor. 4.0.2)}$$

où $C(\Omega,\beta) = \zeta_n I(\Omega)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right] + 1$, ζ_n ne dépend que de n.

Ce qui rend cette borne particulièrement intéressante est la faible sensibilité à certaines petites déformations du rapport isopérimétrique par rapport à la courbure ou à l'indice d'intersection. Notamment en comparaison avec les résultats du Chapitre 3, où l'hypothèse porte sur l'existence de $\tilde{C}$ tel que défini en (3.0.2), une hypothèse basée sur le rapport isopérimétrique reste plus faible. En effet, l'existence de $\tilde{C}$ implique que le bord ne peut être concentré à aucun endroit, tandis qu'une hypothèse sur $I(\Omega)$ contrôle seulement qu'il ne soit pas globalement concentré.

De même, si $\Omega \subset M$ est un domaine à bord lisse Γ, dans une variété riemannienne compacte de dimension $n \geqslant 3$ dont la courbure de Ricci est minorée par $-(n-1)K_-^2$ avec $K_- \in \mathbb{R}_{>0}$, on prouve que les valeurs propres de Wentzel satisfont l'inégalité suivante :

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant A(\Omega,\beta) + B(\Omega,\beta)\left(\frac{k}{\mathrm{Vol}(\tilde{\Gamma})}\right)^{\frac{2}{n-1}}, \quad \forall\, k \geqslant 2, \qquad \text{(Thm. 4.0.4)}$$

où

$$A(\Omega,\beta)=\mathcal{K}^2\zeta(n)\left\{1+\left(\mathcal{K}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}+\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\mathcal{K}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right]\right\},$$
$$B(\Omega,\beta)=\zeta(n)\left\{1+\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\mathcal{K}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right]\right\},\quad \mathcal{K}:=\max\{1,K_-\}.$$

La constante $\zeta(n)$ ne dépend que de la dimension.

La dernière partie de cette thèse s'intéressera au spectre du laplacien associé à une densité.

1.5 Valeurs propres du laplacien avec densité

Soit (M,g) une variété riemannienne compacte connexe de dimension $n\geqslant 2$ à bord de classe C^1 ou sans bord. Soit $h\in C^2(M)$ et ρ la fonction positive définie par $\rho:=e^{-h}$. On note d_M, Δ et ∇ respectivement, la forme volume riemannienne, le laplacien et l'opérateur de gradient sur (M,g). Pour simplifier, on note également d_M la forme volume de la métrique induite sur le bord de M noté ∂M. Rappelons ici que nous définissons le laplacien avec un signe négatif.

Le laplacien de Witten (également appelé laplacien pondéré ou de Bakry-Emery) par rapport au volume pondéré $\rho\mathrm{d}_M$ est défini par

$$\Delta\cdot+g(\nabla h,\nabla\cdot).$$

On désigne par $\{\lambda_k(\rho,\rho)\}_{k\geqslant 1}$ son spectre avec la condition au bord de Neumann si le bord est non vide. Soit S_k l'ensemble de tous les sous-espaces vectoriels de dimension k de $H^1(M)$, le spectre est constitué d'une suite croissante de valeurs propres caractérisées par :

$$\lambda_k(\rho,\rho)=\inf_{V\in S_k}\sup_{u\in V\setminus\{0\}}\frac{\int_M|\nabla u|^2\rho\mathrm{d}_M}{\int_M u^2\rho\mathrm{d}_M},$$

pour tout $k\geqslant 1$.

Ces dernières années, le laplacien de Witten a reçu beaucoup d'attention de la part de nombreux auteurs (voir [21], [20], [76], [56], [54], [41], [35], [55], [53] et les références qui y figurent), en particulier le sujet de recherche classique de l'estimation des valeurs propres.

Lorsque h est une constante, le laplacien de Witten est exactement le laplacien. Un autre spectre a une caractérisation similaire à celle du laplacien de Witten : le spectre du laplacien associé à la métrique conforme $\rho^{\frac{2}{n}}g$. Il est naturel de désigner ce spectre par $\{\lambda_k(\rho,\rho^{\frac{n-2}{n}})\}_{k\geqslant 1}$, puisque les valeurs propres ont pour caractérisation variationnelle

$$\lambda_k(\rho,\rho^{\frac{n-2}{n}})=\inf_{V\in S_k}\sup_{u\in V\setminus\{0\}}\frac{\int_M|\nabla u|^2\rho^{\frac{n-2}{n}}\mathrm{d}_M}{\int_M u^2\rho\mathrm{d}_M}.$$

Dans le présent travail, nous nous intéressons au problème général avec l'énergie de

Dirichlet pondérée par ρ^α et par rapport au produit L^2 pondéré par ρ, où $\alpha \geqslant 0$ est une constante fixée. Ces valeurs propres sont celles de l'opérateur L_h défini par :

$$L_\rho \cdot = L_h \cdot := -\rho^{-1} \operatorname{div}(\rho^\alpha \nabla \cdot) = e^{-h(\alpha-1)}(\Delta \cdot + \alpha g(\nabla h, \nabla \cdot))$$

sur M équipé du volume pondéré d$m := \rho \mathrm{d}_M$. On considère la condition au bord de Neumann (1.1.4) si $\partial M \neq \emptyset$. Le spectre est constitué d'une suite de valeurs propres croissante à l'infinie :

$$\operatorname{Spec}(L_h) = \{0 = \lambda_1(\rho, \rho^\alpha) < \lambda_2(\rho, \rho^\alpha) \leqslant \lambda_3(\rho, \rho^\alpha) \leqslant \ldots \leqslant \lambda_k(\rho, \rho^\alpha) \leqslant \ldots\},$$

qui, pour tout $k \geqslant 1$, sont données par :

$$\lambda_k(\rho, \rho^\alpha) = \inf_{V \in S_k} \sup_{u \in V \setminus \{0\}} \frac{\int_M |\nabla u|^2 \rho^\alpha \mathrm{d}_M}{\int_M u^2 \rho \mathrm{d}_M}.$$

Ici, S_k désigne l'ensemble de tous les sous-espaces vectoriels de dimension k de $H^1(M)$. Les cas particuliers où $\alpha = 1$ et $\alpha = \frac{n-2}{n}$ correspondent aux problèmes mentionnés ci-dessus.

L'objectif est d'étudier l'effet de la densité, au regard des propriétés des valeurs propres $\lambda_k(\rho, \rho^\alpha)$ parmi des densités ρ avec une masse totale fixée. Un problème plus général où la fonctionnelle énergétique de Dirichlet est pondérée par une fonction positive σ, non nécessairement liée à ρ est présentée par Colbois et El-Soufi dans [11].

Dans l'article susmentionné, Colbois et El-Soufi présentent une borne supérieure pour le cas où $\alpha = 0$ [11, Cor. 4.1] :

Soit $\kappa > 0$ tel que $\operatorname{Ric}(M) \geqslant -(n-1)\kappa$, on a pour toute densité ρ telle que $⨍_M \rho \mathrm{d}_M = 1$

$$\lambda_k(\rho, 1) \operatorname{Vol}(M)^{\frac{2}{n}} \leqslant C_n k^{\frac{2}{n}} + C'_n \kappa, \tag{1.5.1}$$

où C_n et C'_n ne dépendent que de la dimension n.
Les valeurs propres $\lambda_k(\rho, 1)$ décrivent, en dimension 2, les vibrations d'une membrane non homogène et en dimensions supérieures, l'interprétation est une densité de masse ρ éventuellement non constante sur M.

La borne, lorsque $\alpha = 0$, comme constaté dans (1.5.1) est indépendante de ρ, tandis que, dans [16, Thm. 5.2], Colbois, El Soufi et Savo prouvent que dans le cas où $\alpha = 1$, la densité ρ variant mais satisfaisant $⨍_M \rho \mathrm{d}_M = 1$, il n'y a pas une telle borne pour toutes variétés. En effet, ils montrent que, sur une variété de révolution compacte, on peut avoir $\lambda_2(\rho, \rho)$ aussi grande que souhaité.

Par ailleurs, dans l'article [46], Kouzayha et Pétiard donnent une borne supérieure pour $\lambda_k(\rho, \rho^\alpha)$, quand $\alpha \in \left(0, \frac{n-2}{n}\right]$ et prouvent qu'il n'y en a pas pour $\lambda_2(\rho, \rho^\alpha)$ quand α passe sur l'intervalle $\left(\frac{n-2}{n}, 1\right)$.

Dans ce travail, nous traitons les cas restants, c'est-à-dire lorsque $\alpha > 1$. On prouve, comme conjecturé dans [46, Rem. 3], qu'il n'y a pas de borne supérieure pour $\lambda_2(\rho, \rho^\alpha)$ lorsque $\alpha > 1$, dans la classe des variétés M à bord convexe et courbure positive.

1.6 Plan

La structure générale de cette thèse prend la forme de quatre chapitres, divisés en deux parties imaginaires. Suite à cette introduction, nous avons :

> Partie *I* : Problème de valeurs propres de Wentzel, composée des chapitres 2, 3 et 4.
>
> Partie *II* : Le laplacien à densité, composée du chapitre 5.

Le chapitre 2 donne assez brièvement un aperçu des premières étapes caractérisant le spectre du laplacien de Wentzel - la base du cadre théorique. Des bornes uniformes dépendant de quantités géométriques et asymptotiquement optimales sont ensuite présentées au chapitre 3. Le chapitre 4 fournit ensuite un autre type de bornes, essentiellement en termes du rapport isopérimétrique avec des hypothèses claires sur les propriétés géométriques de la variété. La contribution de ce travail sur le spectre du laplacien à densité est ensuite présenté au chapitre 5. Les résultats présentés dans ce dernier chapitre ont été publiés dans [61].

1 Introduction

"The definition of a good mathematical problem is the mathematics it generates rather than the problem itself."
– Andrew Wiles

Perceptibly, either or both the size and the shape of a solid object effect some of its physical properties. For example, when one strikes a huge timpani, it makes a lower pitch than those that can be played on a smaller drum. This direct causal link between the size and sound waves also depends on the shape of the drumhead and on the prevalence of other physical properties as the density. Besides, this is fully controlled by an important prerequisite: the tension of the counter-hoop on the drumhead. At that point, extracting informations with mathematical precision, on physical properties as the sounds produced by an instrument, from its geometry is subtle.

1.1 Motivating the study of the Laplace operator and its eigenvalues

Physical phenomena as diverse as the vibrations of a membrane (drum), a string, a mass of air in a concert hall, the heat radiation from a body in thermal equilibrium are governed by the Helmholtz wave equation:

$$\Delta u = \lambda u. \tag{1.1.1}$$

It is a linear partial differential equation involving the Laplace operator Δ (λ is the eigenvalue, and u is the eigenfunction) and corresponds to a time-independent form of the wave equation. From a physical point a view, the Laplacian on a compact Riemannian manifold is a fundamental linear operator which describes numerous propagation phenomena. This motivates the spectral theory of the Laplacian on Riemannian manifolds within the context of differential geometry.

On a connected Euclidean domain $\Omega \subset \mathbb{R}^n$, the Laplacian Δ is the operator acting on functions in $C^\infty(\Omega)$ given by the sum after two times differentiations with respect to each position variable:

$$\Delta u := -\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

The natural generalisation on a Riemannian manifold (M,g) is the Laplace-Beltrami operator given by the local definition:

$$\Delta u := -\frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i}\left(g^{ij} \sqrt{\det g} \frac{\partial u}{\partial x_j}\right).$$

Independently of any coordinate system, the Laplace-Beltrami operator is equivalently defined as:

$$\Delta_g := -\mathrm{div}_g \nabla_g$$

where div_g and ∇_g are the divergence and the gradient operators, respectively.

Notice that we adopt the geometers' convention of placing a minus sign in the definition of the Laplace operator and we shall keep this convention throughout the text. Since g is usually understood, the dependency of the metric is suppressed by dropping the subscript, when the Riemannian metric is clear from context. The spectrum of the operator Δ consists of all real values for which there is a corresponding eigenfunction such that the Helmholtz equation 1.1.1 is satisfied.

1.1.1 Some examples

Here are well-known applications in physics where the Laplace operator plays a key role:

Wave propagation. Let Ω be a Euclidean bounded domain (non-empty, open and connected). If we think of Ω as a drum head, the boundary $\partial\Omega$ is attached to the rim of the drum. To study the vibration generated when the drum is beaten, we solve the wave equation:

$$\left(\Delta + \frac{\partial^2}{\partial t^2}\right) u(x,t) = 0. \tag{1.1.2}$$

We should also ask the solution to satisfy:

$$u(x,t) = 0 \quad \text{for all point } x \in \partial\Omega, \tag{1.1.3}$$

to take into account that the border of the membrane is fixed.

The specification of a value imposed on the solution on the boundary of the domain is made through the Dirichlet boundary condition (1.1.3). The physical interpretation of the Dirichlet boundary condition can be a fixed membrane on the edge or an ambient temperature fixed at the limit. Otherwise, a boundary condition is usually an efficient mathematical model to «approximate» a physical reality in a convenient way.

There are other well known conditions such as the Neumann boundary condition

$$\frac{\partial u}{\partial \mathbf{n}}(x,t) = 0 \quad \text{for all point } x \in \partial\Omega, \tag{1.1.4}$$

which sets a prescribed value for the normal derivative of the solution at the boundary of the domain. We can also have a mixture of Dirichlet and Neumann conditions on disjoint parts of the boundary to give what is commonly called mixed conditions, or a combination of both giving Robin boundary condition:

$$\frac{\partial u}{\partial \mathbf{n}}(x,t) + \alpha u(x,t) = 0 \text{ for all point } x \in \partial\Omega, \quad \alpha > 0 \text{ is fixed.} \tag{1.1.5}$$

There is a large number of possible boundary conditions, depending on the nature of the equation to which they are attached and on the formulated problem.

In the same line, the motion of the surface of the fluid is described by the Wave equation:

$$\left(\Delta + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)u(x,t) = 0,$$

where c is the speed of sound in the fluid.

Heat diffusion. To described the time evolution of temperature distribution in a solid $\Omega \subset \mathbb{R}^n$, one should solve the heat equation:

$$\left(\Delta + \frac{1}{c}\frac{\partial}{\partial t}\right)u(x,t) = 0$$

where c is the conductivity of the material and $u(x,t)$ is the temperature at the point $x \in \Omega$ at time t. If we assume that the steady state temperature distribution is reached after a long enough period of time, losses and gains of heat are perfectly balanced, then the temperature is independent of the time variable t. It is expressed as a harmonic function:

$$\Delta u = 0.$$

Gravity field. A Newtonian gravitational field $\mathbf{g}$ due to an attracting object of mass density ρ_m can be expressed in terms of a scalar potential φ:

$$\mathbf{g} = -\nabla\varphi\ .$$

Then gravitational potential function φ satisfies the Poisson equation:

$$\operatorname{div}\mathbf{g} = \Delta\varphi = 4\pi G\rho_m,$$

where G is the Newton's gravitational constant.

Schrödinger wave equation for Quantum particles. To study the motion of a quantum particle inside a domain Ω, when no external force is applied, we need to solve the quantum wave equation:

$$\frac{\hbar^2}{2m}\Delta u(x,t) = i\hbar\frac{\partial}{\partial t}u(x,t),$$

where $i = \sqrt{-1}$, m is the mass of the particle, $\hbar$ is the reduced Planck's constant and $u(x,t)$ is the wave function representing the particle.

As described on the previous examples, Laplace eigenfunctions are materialized, in acoustics by vibration modes, in quantum wave guides by the physical state of an electron, etc. Vibration modes of a thin membrane (a drum) with a fixed boundary are given by Dirichlet Laplace eigenfunctions. The natural frequencies correspond to the square roots of the eigenvalues.

1.1.2 Geometric structure and Laplace spectrum

The possibility to associate to any compact connected Riemannian manifold, with or without boundary, self-adjoint operators such as the Laplace-Beltrami (We briefly call it the Laplacian.), gives fundamental objects: the spectrum of the operator. The case of compact Riemannian manifold without boundary (i.e. a closed manifold - for example the sphere) is referred to as the closed eigenvalue problem. Additional conditions such as Dirichlet or Neumann conditions being added to the differential equation in the case that the boundary is non-empty, the spectrum of Δ consists of an infinite sequence of real eigenvalues:

$$\mathrm{Spec}(\Delta) = \{0 = \lambda_1(M) < \lambda_2(M) \leqslant \lambda_3(M) \leqslant \cdots \leqslant \lambda_k(M) \leqslant \cdots\}.$$

The reader should refer to [1, Thm. 14.6] and [4, Thm. $A.I.4$ $(S.I)$] for additional information

The Laplace operator being invariant under isometric deformations of the manifold, each element of this sequence is a geometric invariant. This analytic invariance raises the question of whether reciprocally the spectrum (frequencies) determines the metric up to isometry (Kac [42]).

Spectral geometry studies the relationships between geometric characteristics of the manifold and properties of the spectrum of the Laplacian. It is broken down into two areas, respectively known as the direct and inverse problems:

- Computing or finding properties on the eigenvalues from the geometric structure of the underlying manifold,
- Determine the geometry of the manifold, assuming that we know all the eigenvalues.

We will be interested, in this thesis, in finding informations about the eigenvalues of the Laplacian in terms of geometric data. It turns out that, it is usually not possible to determine explicitly the eigenvalues. A crucial theoretical problem consists in finding inequalities to approximate them.

1.2 Upper bounds for eigenvalues of the Laplacian - Starting and cornerstone of the theory

An explicit calculation of the Laplace spectrum of a Riemannian manifold is, in the broadest terms, impossible. The eigenvalues are hard to compute and they are only known in a few cases.

In the absence of being able to determine explicitly the spectrum, several studies will focus on its approximation with respect to the geometry and especially on the first non-zero eigenvalue.

To give some examples, a vibrating string of length L is described by the wave equation in one dimension:

$$c^2\frac{\partial^2\varphi}{\partial x^2} = \frac{\partial^2\varphi}{\partial t^2},$$

where $\varphi(x,t)$ represents the transverse displacement of the string at position x and at time t. The spatial function u resulting from separation of variables

$$\varphi(x,t) = u(x)\sin(\sqrt{\lambda}ct)$$

is an eigenfunction of the second derivative operator:

$$-\frac{\partial^2 u}{\partial x^2} = \lambda u.$$

The frequency or «pitch» of the string corresponds to the square root of the eigenvalue λ. If the imposed boundary conditions at the endpoints $x = 0$ and $x = L$ are «fixed endpoints», that is the Dirichlet condition $u = 0$ at $x = 0$ and $x = L$, then one has infinitely many eigenvalues given explicitly by:

$$\lambda_k = \left(\frac{k\pi}{L}\right)^2, \quad k = 1, 2, 3, \ldots$$

with corresponding eigenfunctions

$$u_k = \sin\left(\frac{k\pi x}{L}\right).$$

For a (two dimensional) membrane or drumhead, vibrating transversely in the third direction, by separating variables in the wave equation (1.1.2), we get the eigenvalue problem:

$$\Delta u = \lambda u, \text{ with Dirichlet condition } u = 0 \text{ on the boundary } \partial\Omega,$$

where Ω describes the rest shape of the drum which is fixed at zero displacement around the edge. Just like in one dimension, the eigenvalues λ_k are positive, and increase to infinity. Unfortunately, only in a few special cases can the eigenvalues be computed, for instance: for disks (using polar coordinates and Bessel functions), for rectangles (using rectangular coordinates and sine functions [48, § 52]), equilateral triangles (using barycentric coordinates and trigonometric functions [48, § 57]). For more details and resulting formulas, see [49] and references therein.

Turning now to the study of geometric bounds for the eigenvalues, it may be interesting to recall that the so-called Faber-Krahn inequality, as early as 1925, gave a minimum for the first Dirichlet eigenvalue on a domain Ω of the plane, in terms of area:

$$\frac{\pi j_{0,1}^2}{\text{Area}(\Omega)} = \lambda_1(\mathcal{D})\frac{\text{Area}(\mathcal{D})}{\text{Area}(\Omega)} \leqslant \lambda_1(\Omega), \qquad \text{(Faber [24] and Krahn [47])}$$

where $j_{0,1}$ denotes the first zero of the Bessel function of order zero, and $\mathcal{D}$ is a disk. This is an isoperimetric inequality, giving a precise meaning and a proof to a conjecture made by Lord Rayleigh:

"If the area of a membrane be given, there must evidently be some form of boundary for which the pitch (of the principal tone) is the gravest possible, and this form can be no other

than the circle."

– Rayleigh, Theory of Sound [65, §210]

It should also be noted that, before Faber and Krahn, Courant [17] proved the weaker result that among all membranes of the same perimeter L, the circular one yields the least lowest eigenvalue, i.e.

$$\frac{4\pi^2 j_0^2}{L^2} \leqslant \lambda_1(\Omega),$$

with equality if and only if the membrane is circular.

Bounding the first (non trivial) eigenvalue thus becoming a subject of broad research interest, many results have appeared in the literature. Most interesting contributions are summarized in [62]. Hersch in [37] was, most likely, the first to obtain an upper bound for the first non-zero closed eigenvalue in terms of area. He developed in a more geometric direction an approach of Szegö [69] to prove that:

$$\lambda_2(M) \leqslant \frac{8\pi}{\mathrm{Vol}(M)}, \tag{1.2.1}$$

for any manifold $M = (\mathbb{S}^2, g)$ given by the sphere $\mathbb{S}^2$ equipped with an arbitrary metric g. In other words, among all Riemannian metrics on a sphere of fixed area, the first eigenvalue of the Laplacian is maximal for the standard round metric which satisfies

$$\lambda_2(\mathbb{S}^2)\mathrm{Vol}(\mathbb{S}^2) = 8\pi.$$

This result of Hersch have been a strong incentive for a whole direction in the study of extremal metrics on surfaces and properties of higher order Laplace eigenvalues.

Independently, Reilly [66, Cor. 1] and Chavel [9, Thm. 1] complemented Hersch inequality (1.2.1) for an n-dimensional euclidean domain $\Omega \subset \mathbb{R}^n$ bounded by a smooth compact hypersurface Γ:

$$\lambda_2(\Gamma) \leqslant \left(\frac{\mathrm{Vol}(\Gamma)}{\mathrm{Vol}(\Omega)}\right)^2 \frac{n-1}{n^2}, \tag{1.2.2}$$

where equality holds if and only if Γ is the standard sphere $\mathbb{S}^{n-1}$ and $\mathrm{Vol}(M)$ denotes the $(n-1)$-volume of M. In dimension $n = 3$, applying the classical isoperimetric inequality, it can be noticed that Hersch's inequality (1.2.1) is stronger than (1.2.2).

In this thesis, we will be interested principally in establishing Reilly-Chavel type inequalities (also found in the literature as Reilly type inequalities and so named in reference to the approximation in (1.2.2)), for higher order eigenvalues of the Laplacian with Wentzel boundary condition. This is a rather recent boundary condition, introduced in [72], that has only been significantly investigated over the last two decades; see Chapter 2 for a full description.

1.3 Uniform bounds and metric methods

So far, great number of authors (Li and Yau [52], Heintze [34], El Soufi and Ilias [23], Veeravalli [71], Grosjean [32], Giménez et al. [27], Wang and Xia [73] etc.) have proved versions of the inequality of Reilly and Chavel in (1.2.2). In fact, this bound is a partial case of the inequality established by Reilly which is an upper bounds for the first non trivial eigenvalue of any closed submanifold of the Euclidean space in terms of the total mean curvature. This extends an earlier result of Bleecker and Weiner [6]. Heintze [34], extended Reilly's result to hypersurfaces of compact manifolds and Hadamard manifolds. El Soufi and Ilias [23] obtained the best upper bounds for the first non zero Laplace eigenvalue on closed hypersurfaces of the hyperbolic space, in terms of the total mean curvature.

Most of these estimates have been obtained by applying, as noticed by Colbois [10], "barycentric type methods" and the use of coordinate functions as test functions.

More recent attention has focused on the higher eigenvalues. Although barycentric methods appear efficient with prior research to bound the first non trivial eigenvalue of the Laplacian, they appear difficult to generalize for higher order eigenvalues. A new approach is therefore needed. The main estimates for the Wentzel eigenvalue problem, we proved in this thesis were motivated by prior estimates on the spectrum of the Laplacian on closed Riemannian manifolds based on metric geometry approach.

Let M be a closed (compact without boundary) connected manifold, of dimension $n \geqslant 2$. Let Δ denotes the Laplace operator acting on the space $C^\infty(M)$ of all smooth real valued functions on M. We consider the eigenvalue equation

$$\Delta u = \lambda u.$$

The eigenvalues λ are real and the compactness of the manifold allows to show that they are discrete, the constant function being eigenfunction associated to the first eigenvalue $\lambda_1(M) = 0$:

$$0 = \lambda_1(M) < \lambda_2(M) \leqslant \lambda_3(M) \leqslant \cdots \lambda_k(M) \leqslant \cdots,$$

each eigenvalue being repeated according to its multiplicity. One then consider the k-th eigenvalue as a functional given by

$$M \longrightarrow \lambda_k(M)$$

or possibly a normalised version with same variational properties (see (2.1.1)). The correlation between the eigenvalues $\lambda_k(M)$ and extrinsic geometric quantities constructed from the second fundamental form of M immersed isometrically into Euclidean space, or a more general ambient space, is investigated.

Korevaar in [45] conceived the idea of a new method using metric space decomposition to prove that there exists a universal constant $C > 0$ such that for any compact orientable surface M of genus γ, we have:

$$\lambda_k(M)\mathrm{Vol}(M) \leqslant C(\gamma+1)k, \quad \forall\, k \geqslant 2.$$

This extends to higher order eigenvalues the result of Yang and Yau:

$$\lambda_1(M)\mathrm{Vol}(M) \leqslant 8\pi(\gamma+1), \quad \forall\, k \geqslant 2.$$

In the proof of Korevaar's result, the surface M is decomposed into disjoint annuli and regions according to the volume concentration of the metric. This metric space annular decomposition is then used to support test functions, conducive to bounds for the eigenvalues.

Further development of this method by Grigor′yan, Netrusov, and Yau [31], providing a significantly simpler method of constructing capacitors to support test functions, allowed to obtain upper bounds for eigenvalues via capacities.

Applying the method of Grigor′yan et al. [31], Colbois, El Soufi, and Girouard [14, Thm. 1.3] has been able to show that the Steklov eigenvalues of a bounded domain in a complete Riemannian manifold are bounded above in terms of the isoperimetric ratio of the domain.

Definition 1.3.1. For a domain Ω of an n-dimensional complete Riemannian manifold (M,g), with boundary Γ, the isoperimetric ratio is defined by:

$$I(\Omega) := \frac{\mathrm{Vol}(\Gamma)}{\mathrm{Vol}(\Omega)^{\frac{n-1}{n}}}.$$

In the numerator Vol stands for the $(n-1)$-Riemannian volume induced by the metric g.

The following theorem shows that under a geometric assumptions, one can control the (normalized) Steklov eigenvalues in terms of the isoperimetric ratio.

Theorem 1.3.1 (Colbois et al. [14]). *Let M be a complete Riemannian manifold of dimension n with non-negative Ricci curvature. Let Ω be a domain of M and Γ its boundary. Denote by $\{\lambda_{S,k}\}_{k=1}^{\infty}$ the spectrum of the Steklov eigenvalue problem:*

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \partial_{\mathbf{n}} u = \lambda^S u, & \text{on } \Gamma. \end{cases}$$

Then, for every $k \geqslant 2$, one has:

$$\lambda_{S,k}(\Omega)\mathrm{Vol}(\Gamma)^{\frac{1}{n-1}} \leqslant \frac{\alpha_n}{I(\Omega)^{1-\frac{1}{n-1}}} k^{\frac{2}{n}}, \tag{1.3.1}$$

where α_n is a constant depending only on the dimension n.

The same authors also obtained bounds, for the eigenvalues of the Laplacian on a hypersurface bounding a domain in some ambient Riemannian manifold, depending on the isoperimetric ratio of the domain. They proved an uniform upper bound for these eigenvalues, using the construction in [12] by Colbois and Maerten who gave an elaborated decomposition of complete metric measure spaces which can be adapted to various situations. This is a main tool in [15, Thm. 1.2] to prove upper bounds for the eigenvalues of the Laplacian on a hypersurface bounding a domain:

Theorem 1.3.2 (Colbois et al. [15]). *Let (M,g) be a complete Riemannian manifold of dimension $n \geqslant 2$ with Ricci curvature bounded below by $-(n-1)\kappa^2$, $\kappa \in \mathbb{R}$.*

Let $\Omega \subset M$ a bounded domain with smooth boundary Γ and $(\eta_k)_{k\in\mathbb{N}}$ the eigenvalues of the laplacian Δ_Γ on Γ. For every $k \geqslant 2$, one has:

$$\eta_k(\Gamma) \leqslant \alpha_n \frac{I(\Omega)}{I_0(\Omega)}\kappa^2 + \beta_n \left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}} \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}, \tag{1.3.2}$$

where β_n, α_n are dimensional constants and

$$I_0(\Omega) := \inf\{I(U) : U \text{ open set in } \Omega\}.$$

In the particular case where $M = \mathbb{R}^n$ is the Euclidean space, one has $\kappa = 0$, $I_0(\Omega) = I_0(\mathbb{R}^n) = n\omega_n^{\frac{1}{n}}$, where ω_n denotes the volume of the unit Euclidean n-ball, and

$$\eta_k(\Gamma)\mathrm{Vol}(\Gamma)^{\frac{2}{n-1}} \leqslant c_n I(\Omega)^{1+\frac{2}{n-1}} k^{\frac{2}{n-1}},$$

where c_n is a dimensional constant.

Remark 1.3.1.

- The metric methods used in Theorems 1.3.1 and 1.3.2 are different; Colbois-Maerten [12] construction for the eigenvalues of the Laplacian on the boundary and that of Grigor′ yan-Netrusov-Yau [31] for the eigenvalues of Steklov of the domain.

- The isoperimetric constant intervenes in the two bounds in (1.3.1) and (1.3.2), but in different ways. In (1.3.1), we see that if $n \geqslant 2$, domains with large isoperimetric ratios correspond to the smallest Steklov eigenvalues $\tilde{\lambda}_{S,k}(\Omega) = \lambda_{S,k}(\Omega)\mathrm{Vol}(\Gamma)^{\frac{1}{n-1}}$.

- It should be noticed that the power of k appearing in the right-hand side of the estimate in (1.3.2) is optimal, according to Weyl's law:

  $$\eta_k(\Gamma) = C_n^2 k^{\frac{2}{n-1}} + \mathrm{o}(k^{\frac{2}{n-1}}), \quad k \to \infty,$$

  where $C_n = \frac{2\pi}{(\omega_{n-1}\mathrm{Vol}(\Gamma)^{\frac{1}{n-1}}}$.
  The Steklov eigenvalues, in turn, behave according to the following asymptotic formula:

  $$\lambda_{S,k} = C_n k^{\frac{1}{n-1}} + \mathrm{o}(k^{\frac{1}{n-1}}), \quad k \to \infty.$$

  Regarding whether (1.3.1) should hold with $k^{\frac{1}{n-1}}$ instead of $k^{\frac{2}{n}}$, the authors actually pointed out that it is impossible in dimension $n \geqslant 2$, unless the isoperimetric ratio is restricted to be less than $\frac{\alpha_n}{c_n}$. However, an upper bound which does not involve the isoperimetric ratio, with $k^{\frac{1}{n-1}}$, is still possible.

1.4 Wentzel eigenvalue problem

Let $n \geqslant 2$ and (M,g) be an n-dimensional complete Riemannian manifold. Let $\Omega \subset M$ be a compact bounded domain with smooth boundary Γ. We denote by Δ and Δ_Γ the Laplace operators acting on functions on M and Γ, respectively. Given an arbitrary constant $\beta \in \mathbb{R}$, consider the following eigenvalue problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \beta \Delta_\Gamma u + \partial_{\mathbf{n}} u = \lambda u & \text{on } \Gamma. \end{cases} \qquad \text{(Wentzel Problem)} \tag{1.4.1}$$

In what follows, we will assume that β, which we refer to as the boundary parameter, is non-negative. In this case, the Wentzel eigenvalues form a discrete infinite sequence $\{\lambda_{W,k}\}_{k=1}^{\infty}$ such that:

$$0 = \lambda_{W,1}^{\beta} < \lambda_{W,2}^{\beta} \leqslant \lambda_{W,3}^{\beta} \leqslant \cdots \leqslant \lambda_{W,k}^{\beta} \leqslant \cdots \nearrow \infty. \tag{1.4.2}$$

We adopt the convention that each eigenvalue is repeated according to its multiplicity.

The boundary condition in (1.4.1), which we call Wentzel boundary condition, was initially introduced by Ventcel' [72], in order to find the most general boundary conditions for which the associated operator generates a Markovian semigroup. It is often considered in a more general form (cf. [25, (1.2)],[26, (2.32)]), and/or sometimes it subordinates the heat equation as in [44, (1.3)], see also [25]. A good discussion on motivations and the physical interpretation of Wentzel boundary conditions can be found in [29].

When $\beta = 0$, the eigenvalue problem (1.4.1) reduced to the Steklov eigenvalue problem. It is somehow a mixture between the Steklov problem and the closed problem on the boundary (i.e. $\Delta_\Gamma u_k = \eta_k u_k$, $k \in \mathbb{N}$), since one has (See section 2.3 and [18, §2.1] for $k = 2$.) :

$$\lim_{\beta \to \infty} \frac{\lambda_{W,k}^{\beta}(\Omega)}{\beta} = \eta_k(\Gamma), \quad \forall\, k \in \mathbb{N}. \tag{1.4.3}$$

For the proof when $k = 1$, we refer to [18, §2.1], see Section 2.3. Therefore, the question is raised if the metric methods reported above could be extended to the Wentzel problem. Regarding Remark 1.3.1, which kind of metric decomposition method is appropriate and effective? How will the isoperimetric ratio appears in the inequality?

Only recently has the eigenvalue problem of the Laplacian with Wentzel boundary condition been significantly investigated. It has received much attention in the last ten years. To date, several studies ([18, 22, 75], etc.) focused on bounding the eigenvalues:

- Dambrine, Kateb and Lamboley in [18, Cor. 1.2] obtained a first upper bound for the first non-trivial eigenvalue $\lambda_{W,2}^{\beta}$ in terms of geometric quantities. Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary Γ. Let $\wedge(\Omega)$ denote the spectral radius of the symmetric and positive semi-definite matrix

$$P(\Omega) \stackrel{\text{def}}{=} \left(\int_\Gamma \delta_{ij} - \mathbf{n}_i \mathbf{n}_j \mathrm{d}_\Gamma \right)_{i,j=1,\dots,n},$$

where **n** is the outer normal vector field to Γ. The following inequality holds:

$$\lambda^{\beta}_{W,2}(\Omega) \leqslant \frac{\mathrm{Vol}(\Omega) + \beta \wedge (\Omega)}{\omega_n^{-\frac{1}{n}} \mathrm{Vol}(\Omega)^{\frac{n+1}{n}} \left[1 + c_n \left(\frac{\mathrm{Vol}(\Omega)\Delta B}{\mathrm{Vol}(B)}\right)^2\right]}, \quad c_n := \frac{(\sqrt[n]{2}-1)(n+1)}{4n}. \tag{1.4.4}$$

Here, B is the ball having the same volume as Ω and with the same center of mass than Γ and ω_n denotes the volume of the n-dimensional Euclidean unit ball. The symmetric difference $\Omega\Delta B$ is the union of the complement of Ω with respect to B and the complement of B with respect to Ω. Equality holds in (1.4.4) if Ω is a ball.

- In the same vein, Wang and Xia in [75, Thm. 1.2], proved the following bound for the same eigenvalue:

$$\lambda^{\beta}_{W,2}(\Omega) \leqslant \frac{n\mathrm{Vol}(\Omega) + \beta(n-1)\mathrm{Vol}(\Gamma)}{n\mathrm{Vol}(\Omega)(\mathrm{Vol}(\Omega)\omega_n^{-1})^{\frac{1}{n}}}. \tag{1.4.5}$$

They also prove a lower bound for $\lambda^{\beta}_{W,2}$, in the context of Riemannian manifold with bounded Ricci curvature and principle curvatures on the boundary Γ :

Let M be an n-dimensional compact connected Riemannian manifold with with smooth boundary ∂M and non-negative Ricci curvature. Assume that the principal curvatures of ∂M are bounded below by a positive constant c. The first non-zero Wentzel eigenvalue of M satisfies:

$$\left[1 + (n-1)c\beta + \sqrt{(n-1)^2c^2\beta^2 + 2(n-1)c\beta}\right]\frac{c}{2} < \lambda^{\beta}_{W,2}(M) \quad [75, (1.13)].$$

- Going further, Du et al. [22] provided the following upper bound for the first n eigenvalues (n being the dimension), when Ω is an immersed submanifold in an Euclidean space $\mathbb{R}^N$ equipped with the canonical Euclidean metric. Let H denote the mean curvature vector field of Γ in $\mathbb{R}^N$, then one has:

$$\frac{1}{n-1}\Sigma_{j=1}^{n}(\lambda^{\beta}_{W,j})^{\frac{1}{2}} \leqslant \frac{\sqrt{[n\mathrm{Vol}(\Omega) + (n-1)\beta\mathrm{Vol}(\Gamma)]\int_\Gamma |H|^2 \mathrm{d}_\Gamma}}{\mathrm{Vol}(\Gamma)}. \tag{1.4.6}$$

When $N = n$, that is Ω is a bounded domain of $\mathbb{R}^N$, then equality holds in (1.4.6) if and only if Ω is a ball.

All of the studies reviewed here provide sharp estimates for the very first eigenvalues in the spectrum according to geometric properties of the domain on which the problem is defined. However, there remain several aspects of the Wentzel-Laplace spectrum, about which relatively little is known. This thesis provides significant additional support with uniform bounds for all the eigenvalues of the problem (1.4.1).

1.4.1 Contribution

Let $n \geqslant 2$, (M, g) an n-dimensional complete Riemannian manifold and Ω a bounded domain in M with smooth boundary Γ. We study the eigenvalues of the Laplacian with Wentzel boundary condition

$$\beta\Delta_\Gamma u + \partial_{\mathbf{n}} u = \lambda u \qquad \beta \in \mathbb{R}_{\geqslant 0},$$

on Euclidean domains and more generally on Riemannian manifolds. Among the geometric data of particular interest are the volume, the area of the boundary and/or the isoperimetric ratio of the domain, the diameter and curvatures.

For the Euclidean domains of dimension $n \geqslant 2$, we obtain an upper bound which, in addition to the dimension, the volume and the area of the boundary, depends on the intersection index denoted by $i(\Gamma)$, i.e. the maximal number of intersection points of the boundary with transversal lines. For any $k \geqslant 2$, we have:

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant \left[\zeta_n i(\Gamma)^{\frac{2}{n-1}}\beta + 1\right]\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} + B(\Omega,\beta). \qquad \text{(Cor. 3.0.5)}$$

The constant ζ_n depends only on the dimension and $B(\Omega,\beta)$ depends on β, n, $\mathrm{Vol}(\Omega)$ and $\mathrm{Vol}(\Gamma)$.

This bound is asymptotically optimal in the sense of the Weyl law when $\beta > 0$ (See (2.4.1).). Moreover, this estimate in the Euclidean case, to a certain extent, generalizes to domains in Riemannian manifolds with bounded Ricci curvature:

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant \left(\zeta_n\left[\mathcal{K}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right] + 1\right)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} + \overline{B}(\Omega,\beta), \qquad \text{(Cor. 3.0.2)}$$

where the constant ζ_n depends, as above, on the dimension n, $\mathcal{K} := \max\{1, K_-\}$ and $K_- \in \mathbb{R}_{\geqslant 0}$ is the smallest constant such that the Ricci curvature of the manifold is bounded from below by $-(n-1)K_-^2$. Unlike $B(\Omega,\beta)$ in the Euclidean case, $\overline{B}(\Omega,\beta)$ depends on less explicit geometric quantities including the second fundamental form, the normal injectivity radius and the Ricci curvature of Γ. This constant tacitly involves geometric quantities. This has the effect of hardening the assumptions inherent in the result. The optimality of these hypotheses as well as the comparison with alternative hypotheses are therefore interesting to study. These results will be significantly improved in Theorem 3.3.6, see (3.3.10).

Nonetheless, with the following inequalities in the spirit of Reilly-Chavel (1.2.2), we prove that the whole spectrum can be controlled in terms of the isoperimetric ratio of the domain:

Let $n \geqslant 3$ and $\Omega \subset \mathbb{R}^n$ an n-dimensional Euclidean domain, for all $k \geqslant 2$, one has

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant C(\Omega,\beta)\left[\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{n-2} + \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}\right], \qquad \text{(Cor. 4.0.2)}$$

where $C(\Omega,\beta) = \zeta_n I(\Omega)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right] + 1$ and ζ_n depends only on n.

What makes this bound particularly interesting is the low sensitivity to some small deformations of the isoperimetric ratio when compared with the curvature or the intersection index. In particular, in comparison with the results in Chapter 3, where the hypothesis concerns the existence of $\tilde{C}$ (See (3.0.2).), an assumption based on the isoperimetric ratio is weaker. Indeed, the existence of $\tilde{C}$ implies that the boundary cannot be concentrated anywhere, while an assumption on $I(\Omega)$ only make sure that it is not globally concentrated.

Similarly, if $\Omega \subset M$ is a domain with smooth boundary Γ, in a compact Riemannian manifold of dimension $n \geqslant 3$ whose Ricci curvature is bounded from below by $-(n-1)K_-^2$ with $K_- \in \mathbb{R}_{>0}$, we prove that the Wentzel eigenvalues satisfy:

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant A(\Omega,\beta) + B(\Omega,\beta)\left(\frac{k}{\mathrm{Vol}(\tilde{\Gamma})}\right)^{\frac{2}{n-1}}, \quad \forall\, k \geqslant 2, \qquad \text{(Thm. 4.0.4)}$$

where

$$A(\Omega,\beta) = \mathcal{K}^2\zeta(n)\left\{1 + \left(\mathcal{K}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}} + \left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\mathcal{K}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]\right\},$$

$$B(\Omega,\beta) = \zeta(n)\left\{1 + \left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\mathcal{K}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]\right\}, \quad \mathcal{K} := \max\{1, K_-\}.$$

The constant $\zeta(n)$ depends only on the dimension.

The second part of this thesis is concerned with the spectrum of the Laplacian associated to a density.

1.5 Eigenvalues of the Laplacian with density

Let (M,g) be a compact connected n-dimensional Riemannian manifold, possibly with boundary of class C^1. Let $h \in C^2(M)$ and ρ be the positive function define by $\rho := e^{-h}$. Let d_M, Δ and ∇ denote respectively, the Riemannian volume measure, the Laplace and the gradient operator on (M,g). For simplicity, we also denote by d_M the volume element for the induced metric on ∂M.

The Witten Laplacian (also called drifting, weighted or Bakry-Emery Laplacian) with respect to the weighted volume measure $\rho\mathrm{d}_M$ is define by

$$\Delta \cdot + g(\nabla h, \nabla \cdot).$$

We designate by $\{\lambda_k(\rho,\rho)\}_{k\geqslant 1}$ its spectrum under Neumann conditions if the boundary is non-empty. Let S_k be the set of all k-dimensional vector subspaces of $H^1(M)$, the spectrum consists of a non-decreasing sequence of eigenvalues variationally defined for all $k \geqslant 1$ by

$$\lambda_k(\rho,\rho) = \inf_{V\in S_k}\sup_{u\in V\setminus\{0\}} \frac{\int_M |\nabla u|^2 \rho \mathrm{d}_M}{\int_M u^2 \rho \mathrm{d}_M}.$$

In recent years, the Witten Laplacian received much attention from many mathematicians (See [21], [20], [76], [56], [54], [41], [35], [55], [53] and the references therein.), in particularly the classical research topic of estimating eigenvalues.

When h is a constant, the Witten Laplacian is exactly the Laplacian. Another spectrum has a similar characterisation with the one of the Witten laplacian: the spectrum of the Laplacian associated with the metric $\rho^{\frac{2}{n}}g$, which is conformal to g. It is natural to denote its spectrum by $\{\lambda_k(\rho,\rho^{\frac{n-2}{n}})\}_{k\geqslant 1}$, since the eigenvalues are variationally characterised by

$$\lambda_k(\rho,\rho^{\frac{n-2}{n}}) = \inf_{V\in S_k}\sup_{u\in V\setminus\{0\}} \frac{\int_M |\nabla u|^2 \rho^{\frac{n-2}{n}} \mathrm{d}_M}{\int_M u^2 \rho \mathrm{d}_M}.$$

In the present work, we are interested in the expanded eigenvalue problem of the Dirichlet energy functional weighted by ρ^α, with the L^2 inner product weighted by ρ, where $\alpha \geqslant 0$ is a given constant. These eigenvalues are those of the operator $L_\rho \cdot = L_h \cdot := -\rho^{-1}\operatorname{div}(\rho^\alpha \nabla \cdot) = e^{-h(\alpha-1)}(\Delta \cdot + \alpha g(\nabla h, \nabla \cdot))$ on M endowed with the weighted volume form $\mathrm{d}m := \rho \mathrm{d}_M$. The spectrum consists of an unbounded increasing sequence of eigenvalues

$$\operatorname{Spec}(L_h) = \{0 = \lambda_1(\rho,\rho^\alpha) < \lambda_2(\rho,\rho^\alpha) \leqslant \lambda_3(\rho,\rho^\alpha) \leqslant \ldots \leqslant \lambda_k(\rho,\rho^\alpha) \leqslant \ldots\},$$

which, for all $k \geqslant 1$, are given by

$$\lambda_k(\rho,\rho^\alpha) = \inf_{V\in S_k}\sup_{u\in V\setminus\{0\}} \frac{\int_M |\nabla u|^2 \rho^\alpha \mathrm{d}_M}{\int_M u^2 \rho \mathrm{d}_M}.$$

As already mentioned, S_k is the set of all k−dimensional vector subspaces of $H^1(M)$. The particular cases where $\alpha = 1$ and $\alpha = \frac{n-2}{n}$ correspond to the problems mentioned above.

A main interest is to investigate the interplay between the geometry of (M,g) and the effect of the weights, looking at the behaviour of $\lambda_k(\rho,\rho^\alpha)$, among densities ρ of fixed total mass. The more general problem where the Dirichlet energy functional is weighted by a positive function σ, not necessarily related to ρ is presented by Colbois and El-Soufi in [11].

In the aforementioned paper, Colbois and El-Soufi exhibit an upper bound for the case where $\alpha = 0$ ([11, Cor. 4.1]):

Let $\kappa > 0$ such that $\operatorname{Ric}(M) \geqslant -(n-1)\kappa$, for every density ρ such that $⨍_M \rho \mathrm{d}_M = 1$, one has

$$\lambda_k(\rho,1)\operatorname{Vol}(M)^{\frac{2}{n}} \leqslant C_n k^{\frac{2}{n}} + C'_n\kappa, \tag{1.5.1}$$

where C_n and C'_n depend only on the dimension n. Whereas, in [16, Thm. 5.2], Colbois, El Soufi and Savo prove that, when $\alpha = 1$, there is no upper bound among all manifolds. Indeed, they show that, on a compact revolution manifold, with the density ρ varying but satisfying the condition $⨍_M \rho \mathrm{d}_M = 1$, one has $\lambda_2(\rho,\rho)$ as large as desired. In their work in [46], Kouzayha and Pétiard give an upper bound for $\lambda_k(\rho,\rho^\alpha)$, when $\alpha \in \left(0, \frac{n-2}{n}\right]$ and prove that there is none for $\lambda_2(\rho,\rho^\alpha)$ when α runs over the interval $\left(\frac{n-2}{n}, 1\right)$.

Our contribution to this line of research deals with the remaining cases, that is when $\alpha > 1$. We prove, as conjectured in [46, Rem 3], that there is no upper bound for

$\lambda_2(\rho, \rho^\alpha)$, in the class of manifolds M with convex boundary and positive curvature.

1.6 Thesis outline

The overall structure of this thesis takes the form of four chapters, divided into two imaginary parts. Following this introduction, we have:

Part I: Wentzel eigenvalue problem, consisting of chapters 2, 3 et 4.

Part II: Laplacian with density, consisting of chapter 5.

Chapter 2 gives a brief overview of the first stages characterising the spectrum of the Wentzel-Laplace eigenvalue problem – the backbone of the theoretical framework. Uniform bounds which are asymptotically sharp are then presented in Chapter 3. Chapter 4 then provides other type of bounds essentially in terms of the isoperimetric ratio with clear assumptions on the geometric properties of the underlying manifold. The core of the contribution of this work about boundedness of spectrum of the weighted Laplacian is then presented in Chapter 5. The new material in this chapter has been published in [61].

2 Wentzel-Laplace operator and functional framework

Let $n \geqslant 2$ and (M, g) be an n-dimensional Riemannian manifold. From now on, all Riemannian manifolds are assumed to be connected and geodesically complete. Let Ω be a bounded domain in M with smooth boundary Γ. Let Δ and Δ_Γ denote the Laplace operators acting on functions on Ω and on the hypersurface Γ, respectively. The gradient operators on Ω and Γ will be denoted by ∇ and ∇_Γ, respectively and the outer normal derivative on Γ by $\partial_{\mathbf{n}}$. We denote by d_M and d_Γ the (Riemannian) volume elements of M and Γ. Let $\beta \in \mathbb{R}_{\geqslant 0}$, we consider the Wentzel eigenvalue problem on Ω:

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega, \\ \beta \Delta_\Gamma u + \partial_{\mathbf{n}} u = \lambda u \quad \text{on } \Gamma. \end{cases} \tag{2.0.1}$$

A "classical" solution of (2.0.1) is a twice differentiable function $u \in C^2(\overline{\Omega})$ satisfying (2.0.1) with a corresponding eigenvalue $\lambda \in \mathbb{R}$. Multiplying by an arbitrary test function v and integrating by parts, we have that each solution (u, λ) of (2.0.1) satisfies the equality:

$$\beta \int_\Gamma g(\nabla_\Gamma u, \nabla_\Gamma v)\, \mathrm{d}_\Gamma + \int_\Omega g(\nabla u, \nabla v)\, \mathrm{d}_M = \lambda \int_\Gamma uv\, \mathrm{d}_\Gamma, \quad \text{for every test function v.} \tag{2.0.2}$$

Equation (2.0.1) can then be cast in a variational (or weak) form. For this, the function space suitable for the integral equation (2.0.2) is considered:

$$\text{If } \beta = 0,\ H(\Omega) := H^1(\Omega), \text{ otherwise,}$$

$$H(\Omega) := \{u \in H^1(\Omega); u_{|_\Gamma} \in H^1(\Gamma)\}.$$

The space $H(\Omega)$, endowed with the norm $\|u\|^2_{H(\Omega)} := \|u\|^2_{H^1(\Omega)} + \|u_{|_\Gamma}\|^2_{H^1(\Gamma)}$, is a Hilbert space (See for instance [57, §2, Rem. 2.2], see also [25, §2].

Define on $H(\Omega)$ the bilinear forms $a, b : H(\Omega) \times H(\Omega) \longrightarrow \mathbb{R}$ by:

$$a(u, v) := \int_\Omega g(\nabla u, \nabla v)\, \mathrm{d}_M + \beta \int_\Gamma g(\nabla_\Gamma u, \nabla_\Gamma v)\, \mathrm{d}_\Gamma,$$

$$b(u, v) := \int_\Gamma uv\, \mathrm{d}_\Gamma \quad \forall\ u, v \in H(\Omega). \tag{2.0.3}$$

The weak formulation of problem (2.0.1) is the following:

Find $(u, \lambda) \in H(\Omega) \times \mathbb{R}$ such that, $u \neq 0$ and $a(u, v) = \lambda b(u, v)$ for all $v \in H(\Omega)$.

The two bilinear forms are positive, since $\beta \geqslant 0$. In addition, a is continuous (i.e $\exists$ $C > 0$ such that, $\forall\ u, v \in H(\Omega)$, $|a(u, v)| \leqslant C\|u\|_{H(\Omega)}\|v\|_{H(\Omega)}$), and coercive (i.e. $\exists \alpha > 0$ such that $a(v, v) \geqslant \alpha \|v\|^2_{H(\Omega)}$, $\forall\ v \in H(\Omega)$). The verification of coercivity and continuity relies on a Poincaré type inequality ([51, Lemma 18.8]) and Cauchy-Schwarz inequality, respectively. The existence and uniqueness of a solution for a weak formulation with a coercive and continuous bilinear form is a consequence of the Lax-Milgram theorem ([50, Thm. 2.1.]), applied here in the Hilbert space $H(\Omega)$. Classical solutions are recovered using the classical regularity theory of weak solutions. Problem (2.0.1) has then a real discrete spectrum that consists of non-negative eigenvalues with finite multiplicities:

$$0 = \lambda^{\beta}_{W,1} < \lambda^{\beta}_{W,2} \leqslant \lambda^{\beta}_{W,3} \leqslant \cdots \leqslant \lambda^{\beta}_{W,k} \leqslant \cdots \nearrow \infty. \tag{2.0.4}$$

We adopt the convention that each eigenvalue is repeated according to its multiplicity. The Dirichlet-to-Neumann map $\mathrm{N_D}$, as defined in (2.2.2), allows to purely rewrite the eigenvalue problem (2.0.1) on the boundary Γ:

$$\beta \Delta_{\Gamma} u + \mathrm{N_D} u = \lambda u \quad \text{on } \Gamma. \tag{2.0.5}$$

This approach allows to split Wentzell type boundary conditions of the Laplace operator into a combination of two simpler operators, giving a simple way to describe the spectrum. It is known that $\mathrm{N_D}$ is a self-adjoint, positive pseudo-differential operator.

Following [8, §3], we let $\mathrm{Re} : L^2(\Gamma) \longrightarrow H(\Omega)$ denote the resolvent operator associated to (2.0.5) and $\mathrm{Tr} : H(\Omega) \longrightarrow H^1(\Gamma)$ the trace operator. The operator $\mathrm{Tr} \circ \mathrm{Re} : L^2(\Gamma) \longrightarrow H^1(\Gamma)$ is compact and injective. Therefore, the spectrum of the the inverse operator consists of an increasing and diverging sequence of non-negative eigenvalues $\{\lambda^{\beta}_{W,k}\}_{k=1}^{\infty}$ of finite multiplicity. The whole theory on analytic perturbations and operators with compact resolvent is detailed, for instance, in [43, Ch. 2].

2.1 Min-max characterization of Wentzel eigenvalues

For $\beta \geqslant 0$, the eigenvalue sequence $\{\lambda^{\beta}_{W,k}\}_{k=1}^{\infty}$ given in (2.0.4) is subject to the following min-max characterization (See [68, Thm. 1.2 and §3.3] and [26, Eq. (2.33)].). Let $\mathfrak{V}(k)$ denotes the set of all k-dimensional subspaces of $H(\Omega)$. For every $k \in \mathbb{N}$ and $\beta \geqslant 0$, the kth eigenvalue of the Wentzel-Laplace operator satisfies:

$$\lambda^{\beta}_{W,k} = \min_{V \in \mathfrak{V}(k)} \max_{u \in V \setminus \{0\}} R_{\beta}(u), \tag{2.1.1}$$

where $R_{\beta}(u)$, the Rayleigh quotient, is defined by:

$$R_{\beta}(u) := \frac{a(u,u)}{b(u,u)} = \frac{\int_{\Omega} |\nabla u|^2 \mathrm{d}_M + \beta \int_{\Gamma} |\nabla_{\Gamma} u|^2 \mathrm{d}_{\Gamma}}{\int_{\Gamma} u^2 \mathrm{d}_{\Gamma}}, \quad \text{for all } u \in H(\Omega)\setminus\{0\}. \tag{2.1.2}$$

The variational characterization provided in (2.1.1) is of great use in the study of bounds for the eigenvalues; this will be illustrated in the next chapters. When $\beta = 0$, the eigenvalue problem (2.0.1) corresponds to a first limit case, the well-known Steklov problem. There is a second limit case as $\beta \longrightarrow +\infty$ that is the closed eigenvalue problem on the boundary.

2.2 Steklov problem

Consider the map $\wedge : H^{\frac{1}{2}}(\Gamma) \longrightarrow H^1(\Omega)$, related to the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u_{|_\Gamma} = f & \text{on } \Gamma, \end{cases} \tag{2.2.1}$$

which associate to any function $f \in H^{\frac{1}{2}}(\Gamma)$ its harmonic extension, i.e. the unique function u in $H^1(\Omega)$ satisfying (2.2.1). See [70, Prop. 1.7, (1.40)] and [44, Appendix C2, Theorem C2.1], for more details. The notation $u_{|_\Gamma} = \mathrm{Tr}u \in H^{\frac{1}{2}}(\Gamma)$ stands for the trace of $u \in H^1(\Omega)$ at the boundary Γ. The Sobolev space of order $\frac{1}{2}$ on Γ is the trace space define by $H^{1/2}(\Gamma) := \mathrm{Tr}(H^1(\Omega)) := \{v \in L^2(\Gamma) \mid \exists u \in H^1(\Omega)\colon \mathrm{Tr}(u) = v\}$. This is also denoted simply by u, if no ambiguity can result.

Then, the Dirichlet-to-Neumann operator is defined by:

$$\begin{aligned} \mathrm{N_D} : & H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma) \\ & f \mapsto \partial_{\mathbf{n}}(\wedge f). \end{aligned} \tag{2.2.2}$$

The negative space $H^{-\frac{1}{2}}(\Gamma)$ is the dual space of $H^{\frac{1}{2}}(\Gamma)$. The eigenvalues for the Dirichlet-to-Neumann map N_D are those of the Steklov problem:

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \partial_{\mathbf{n}} u = \lambda^S u, & \text{on } \Gamma. \end{cases} \tag{2.2.3}$$

For a general introduction to Steklov problem, we refer to the survey paper [28] and the references therein. It is known that in a domain with Lipschitz boundary, the Steklov problem possesses an infinitely increasing sequence of eigenvalues $\{\lambda_{S,k}\}_{k=1}^{\infty}$, see [68] or [60, Eq. (9)]. They behave according to the following asymptotic formula:

$$\lambda_{S,k} = C_n \left(\frac{k}{\mathrm{Vol}(\Gamma)} \right)^{\frac{1}{n-1}} + \mathrm{o}(k^{\frac{1}{n-1}}), \quad k \to \infty, \tag{2.2.4}$$

where $C_n = \frac{2\pi}{\omega_{n-1}}$.

2.3 Eigenvalues of the Laplace operator on the boundary

We denote by $\{\eta_k\}_{k=1}^{\infty}$ the eigenvalues of Δ_Γ, the Laplacian on the boundary Γ. For $k \geqslant 1$, by Hörmander [38], one has

$$\eta_k = C_n^2 \left(\frac{k}{\mathrm{Vol}(\Gamma)} \right)^{\frac{2}{n-1}} + \mathrm{o}(k^{\frac{2}{n-1}}), \quad k \to \infty. \tag{2.3.1}$$

The eigenvalue η_k of Γ are variationally characterized as follows:

$$\eta_k = \min_{V\in\mho(k)} \max_{0\neq u\in V} \frac{\int_\Gamma |\nabla_\Gamma u|^2 \mathrm{d}_\Gamma}{\int_\Gamma u^2 \mathrm{d}_\Gamma}, \quad \forall\, k \geqslant 1, \tag{2.3.2}$$

where $\mho(k)$ is the set all k-dimensional linear subspaces of $H^1(\Gamma)$ and d_Γ is the Riemannian volume measure induced by the metric g on Γ.

Following from (2.1.1) and (2.3.2), we have the relation (1.4.3):

$$\lim_{\beta\to\infty} \frac{\lambda^\beta_{W,k}(\Omega)}{\beta} = \eta_k(\Gamma), \quad \forall\, k \in \mathbb{N}.$$

Proof.

- Let $u_1,\ldots,u_k$ be the Wentzel eigenfunctions associated respectively with the eigenvalues $\lambda^\beta_{W,1}$, …, $\lambda^\beta_{W,k}$.

  The eigenfunctions associated to $\lambda^\beta_{W,1}$ are constant functions, then for $k \geqslant 2$, one has:

$$\lambda^\beta_{W,k} = \min\left\{ \frac{\int_\Omega |\nabla u|^2 \mathrm{d}_M + \beta \int_\Gamma |\nabla_\Gamma u|^2 \mathrm{d}_\Gamma}{\int_\Gamma u^2 \mathrm{d}_\Gamma},\ v \in H(\Omega),\ \int_\Gamma v u_i \mathrm{d}_\Gamma = 0\ \forall\ i = 0,\ldots,k-1 \right\}. \tag{2.3.3}$$

  If $k = 2$, the minimum is taken over the functions orthogonal to the eigenfunctions associated to $\lambda^\beta_{W,1}$.

  Assume that $u_1,\ldots,u_k$ are normalized such that $\int_\Gamma u_i u_j \mathrm{d}_\Gamma = \delta_{i,j}$. We have then

$$\int_\Omega \nabla u_i \cdot \nabla u_j \mathrm{d}_M + \beta \int_\Gamma \nabla_\Gamma u_i \cdot \nabla_\Gamma u_j \mathrm{d}_\Gamma = \lambda^\beta_{W,i} \delta_{i,j} \text{ for all } i,j = 1,\ldots,k.$$

  Let $V \subset H^1(\Gamma)$ be the space generated by $u_{1|_\Gamma},\ldots,u_{k|_\Gamma}$. Any function $u \in V$ such that $\int_\Gamma u^2 \mathrm{d}_\Gamma = 1$ can be written as a linear combination $u = \sum_{i=1}^k c_i u_i|_\Gamma$, where $c = (c_1,\ldots,c_k) \in \mathbb{R}^k$ is such that $|c| = 1$. Using the variational formula (2.3.2) we obtain:

$$\eta_k \leqslant \max_{\substack{u\in V\setminus\{0\}\\ \int_\Gamma u^2 \mathrm{d}_\Gamma = 1}} \int_\Gamma |\nabla_\Gamma u|^2 \mathrm{d}_\Gamma = \max_{\substack{c=(c_1,\ldots,c_k)\\ |c|=1}} \int_\Gamma |\nabla_\Gamma \sum_{i=1}^k c_i u_i|^2 \mathrm{d}_\Gamma = \max_{\substack{c=(c_1,\ldots,c_k)\\ |c|=1}} \int_\Gamma |\sum_{i=1}^k c_i \nabla_\Gamma u_I|^2 \mathrm{d}_\Gamma.$$

$$\begin{aligned} \beta\eta_k &\leqslant \beta \max_{\substack{c=(c_1,\ldots,c_k)\\ |c|=1}} \int_\Gamma |\sum_{i=1}^k c_i \nabla_\Gamma u_I|^2 \mathrm{d}_\Gamma + \max_{\substack{c=(c_1,\ldots,c_k)\\ |c|=1}} \int_\Omega |\sum_{i=1}^k c_i \nabla u_I|^2 \mathrm{d}_M \\ &= \max_{\substack{c=(c_1,\ldots,c_k)\\ |c|=1}} \sum_{i,j=1}^k c_i c_j \left(\beta \int_\Gamma \nabla_\Gamma u_i \cdot \nabla_\Gamma u_j \mathrm{d}_\Gamma + \int_\Omega \nabla u_i \cdot \nabla u_j \mathrm{d}_M \right). \end{aligned}$$

Since $\int_\Omega \nabla u_i \cdot \nabla u_j \mathrm{d}_M + \beta \int_\Gamma \nabla_\Gamma u_i \cdot \nabla_\Gamma u_j \mathrm{d}_\Gamma = \lambda^\beta_{W,i}\delta_{i,j}$, we have

$$\beta\eta_k \leqslant \max_{\substack{c=(c_1,\ldots,c_k)\\|c|=1}} \sum_{i,j=1}^{k} c_i c_j \lambda^\beta_{W,i}\delta_{i,j} \leqslant \max_{\substack{c=(c_1,\ldots,c_k)\\|c|=1}} \sum_{i=1}^{k} c_i^2 \lambda^\beta_{W,k} = \lambda^\beta_{W,k}.$$

- As is well known, $L^2(\Gamma)$ has an orthonormal basis made of eigenfunctions of Δ_Γ (See, for instance, [2, Thm. 18].). Let $\{\eta_k\}_{k=1}^\infty$ be the sequence of eigenvalues of problem $\Delta_\Gamma u = \lambda u$ and let $\{\varphi_k\}_{k=1}^\infty$ denote the sequence of eigenfunctions associated with the eigenvalues η_k , normalized such that $\int_\Gamma \varphi_i\varphi_j \mathrm{d}_\Gamma = \delta_{i,j}$ for all $i,j \in \mathbb{N}$. Hence, $\{\varphi_k\}_{k=1}^\infty$ satisfies

$$\int_\Gamma \nabla_\Gamma\varphi_i \cdot \nabla_\Gamma\varphi_j \mathrm{d}_\Gamma = \eta_i\delta_{i,j} \quad \forall\ i,j \in \mathbb{N}.$$

Let k be fixed and for each $j = 1,\ldots,k$, ϕ_j be the harmonic extensions of φ_j, i.e.

$$\begin{cases} \Delta\phi_j = 0 & \text{in } \Omega, \\ \phi_j|_\Gamma = \varphi_j & \text{on } \Gamma. \end{cases}$$

We have $\varphi_j \in H^1(\Gamma)$, for all $j = 1,\ldots,k$ and a unique $\phi_i \in H^1(\Omega)$ solving (2.3), see [70, p. 360, (1.39)-(1.40)]. for every $i = 1,\ldots,k$, $\phi_i \in H(\Omega)$, let $V := \mathrm{span}\{\phi_i,\ i = 1,\ldots,k\}$ be the space generated by $\{\phi_j\}_{j=1}^k$.

Every function ϕ in V can be expressed as $\phi = \Sigma_{j=1}^k \alpha_j\phi_j$ and

$$\frac{\int_\Gamma |\nabla_\Gamma\phi|^2 \mathrm{d}_\Gamma}{\int_\Gamma \phi^2 \mathrm{d}_\Gamma} = \frac{\Sigma_{i,j=1}^k \alpha_i\alpha_i \int_\Gamma \nabla_\Gamma\phi_i \cdot \nabla_\Gamma\phi_j \mathrm{d}_\Gamma}{\Sigma_{i,j=1}^k \alpha_i\alpha_j \int_\Gamma \phi_i \cdot \phi_j \mathrm{d}_\Gamma} = \frac{\Sigma_{i=1}^k \alpha_i^2\eta_i}{\Sigma_{i=1}^k \alpha_i^2} \leqslant \eta_k.$$

Then, $\max\limits_{\phi\in V\setminus\{0\}} \frac{\int_\Gamma |\nabla_\Gamma\phi|^2 \mathrm{d}_\Gamma}{\int_\Gamma \phi^2 \mathrm{d}_\Gamma} \leqslant \eta_k$.

$$\frac{\lambda^\beta_{W,k}}{\beta} \leqslant \max_{\phi\in V\setminus\{0\}} \left(\frac{1}{\beta} \frac{\int_\Omega |\nabla\phi_i|^2 \mathrm{d}_M + \int_\Gamma |\nabla_\Gamma\phi_i|^2 \mathrm{d}_\Gamma}{\int_\Gamma \phi^2 \mathrm{d}_\Gamma} \right) \leqslant \max_{\phi\in V\setminus\{0\}} \frac{1}{\beta} \frac{\int_\Omega |\nabla\phi_i|^2 \mathrm{d}_M}{\int_\Gamma \phi^2 \mathrm{d}_\Gamma} + \eta_k.$$

Hence, one has $\eta_k \leqslant \frac{\lambda^\beta_{W,k}}{\beta} \leqslant \max\limits_{\phi\in V\setminus\{0\}} \frac{1}{\beta} \frac{\int_\Omega |\nabla\phi_i|^2 \mathrm{d}_M}{\int_\Gamma \phi^2 \mathrm{d}_\Gamma} + \eta_k$ for all $\beta > 0$. Since $\phi \neq 0$ and $\phi \in H^1(\Omega)$ the quotient $\frac{\int_\Omega |\nabla\phi_i|^2 \mathrm{d}_M}{\int_\Gamma \phi^2 \mathrm{d}_\Gamma}$ is always finite and this yields the limit equality

$$\lim_{\beta\to\infty} \frac{\lambda^\beta_{W,k}(\Omega)}{\beta} = \eta_k(\Gamma), \quad \forall\ k \in \mathbb{N}.$$

■

2.4 Asymptotic behaviour of Wentzel eigenvalues

A perspective, used for instance in Gal [26], is to think about the Wentzel eigenvalue problem as a perturbed version (unperturbed when $\beta = 0$) of the Steklov problem, see [26, (2.32)].

For $\beta > 0$, the Weyl asymptotic for $\lambda^{\beta}_{W,k}$ can be deduced from properties of perturbed forms , accordingly to (2.0.5), using (2.2.4) and the asymptotic behaviour of the spectrum of $\beta\Delta_\Gamma$ from (2.3.1). The Weyl law for eigenvalues of the problem (2.0.1) reads:

$$\lambda^{\beta}_{W,k} = \beta C_n^2 \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} + \mathrm{o}(k^{\frac{2}{n-1}}), \quad k \to \infty. \tag{2.4.1}$$

See [26, (2.37)], a detailed discussion about the spectral properties of the Wentzel Laplacian can be found in [26, §2].

3 Asymptotically Sharp Bounds

In this chapter, we prove asymptotically optimal upper bounds for the eigenvalues of the Wentzel–Laplace operator on Riemannian manifolds with Ricci curvature bounded below. These bounds depend highly on the geometry of the boundary in addition to the dimension and the volume of the manifold.

Theorem 3.0.1. *Let $n \geqslant 2$ and (M,g) be a complete n-dimensional Riemannian manifold with $\mathrm{Ric}(M,g) \geqslant -(n-1)K_-^2$, with $K_- \in \mathbb{R}_{\geqslant 0}$. Let $\Omega \subset M$ be a domain with smooth boundary Γ. Then for every $k \geqslant 2$, one has:*

$$\begin{aligned}\lambda_{W,k}^{\beta}(\Omega) \leqslant A(n,K_-)\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right]\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}\\ + B(M,\Omega)\left[\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}} + \frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right]\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}} + C(M,\Omega)\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right]\\ + \beta\left[A(n,K_-)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} + B(M,\Omega)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}} + C(M,\Omega)\right],\end{aligned} \tag{3.0.1}$$

where the constant $A(n,K_-)$ depends on the dimension n and K_-, $B(M,\Omega)$ and $C(M,\Omega)$ depend on the geometry of Ω and the ambient manifold M.

When $\beta = 0$, we immediately obtain an estimate for the Steklov eigenvalues, but which is less qualitative than (1.3.1).

As a corollary of this result, we have the following upper bound.

Corollary 3.0.2. *Under the assumptions of Theorem 3.0.1, one has:*

$$\lambda_{W,k}^{\beta}(\Omega) \leqslant \left(A(n)\left[\mathcal{K}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right] + 1\right)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} + C(M,\Omega,\beta),$$

where $\mathcal{K} := \max\{1, K_-\}$, the constant $A(n)$ depends only on the dimension n and $C(M,\Omega,\beta)$ depends on β, on the geometry of the domain $\Omega \subset M$.

Remark 3.0.1. It is important to notice that, in addition to the requirement of a bounded Ricci curvature, Theorem 3.0.1 requires further assumptions. Indeed, the proof of Theorem 3.0.1 uses Nardulli's lemma (Lem 3.2.2), where assumptions about the geometry of both Γ and the ambient Riemannian manifold M are made when describing the constants R_0 and C_0. The constants R_0 and C_0 depend, amongst others, on upper and lower bounds on the second fundamental form of Γ, on bound on Ricci curvature tensor of Γ and an upper bound on curvature tensor of ambient manifold.

Without thinking that one can allow much weaker assumptions, it would be interesting to compare them with alternative assumptions on the curvature of M and principal curvatures of Γ.

A general important assumption that might be envisaged is roughly the control of the "volume concentration" of the boundary Γ. This might be comprehended through the result given in Theorem 3.0.3 below. A sufficient assumption underlying this volume control is expressed in the following definition.

Definition 3.0.1. Let $n \geq 2$ and $\tilde{C}$ be a positive real number. We designate by $\mathcal{M}(n,\tilde{C})$ the class of all n-dimensional Riemannian manifolds Ω with boundary Γ such that, for all $x \in \Gamma$ and all radius $0 < r < 1$, we have:

$$\mathrm{Vol}(B(x,r) \cap \Gamma) \leqslant \tilde{C} r^{n-1}, \tag{3.0.2}$$

where $B(x,r)$ denotes the n-dimensional extrinsic metric ball of center x and radius $r > 0$.

This allows us to ceil the boundary volume absorbed by n-dimensional metric balls. The eigenvalues of manifolds in $\mathcal{M}(n,\tilde{C})$ are uniformly controlled. We establish the following very general result:

Theorem 3.0.3. *Let $n \geqslant 2$, $K_- \in \mathbb{R}_{\geqslant 0}$ and $\tilde{C}$ be a positive real number. Let (M,g) be a complete n-dimensional Riemannian manifold with $\mathrm{Ric}(M,g) \geqslant -(n-1)K_-^2$. Let $\Omega \subset M$ be a domain with smooth boundary Γ such that $\Omega \in \mathcal{M}(n,\tilde{C})$, that is (3.0.2) is fulfilled. Let $\mathcal{K} := \max\{1, K_-\}$, for every $k \geqslant 2$, one has:*

$$\lambda_{W,k}^{\beta}(\Omega) \leqslant A(n)\mathcal{K}^2\left[\mathcal{K}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]\left[\left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} + 1\right] + B(n)\mathcal{K}\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}}, \tag{3.0.3}$$

where the constants $A(n)$ and $B(n)$ depend only on the dimension of M.

When $\beta = 0$, we get:

$$\lambda_{S,k}(\Omega) \leqslant A(n)\mathcal{K}^2\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right]\left[\left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} + 1\right] + B(n)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}}. \tag{3.0.4}$$

The upper bound in (3.0.3) is more readily understood than (3.0.1) and may be developed further. It is improved when $K_- = 0$ in Theorem 3.3.6.
When β goes to infinity, one has

$$\lim_{\beta\to\infty} \frac{\lambda_{W,k}^{\beta}(\Omega)}{\beta} \leqslant A(n)\mathcal{K}^2\left[\left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} + 1\right]$$

and then from (1.4.3)), we have $\eta_k(\Gamma) \leqslant A(n)\left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} + A(n)\mathcal{K}^2$ which is optimal in view of (2.3.2) and which gives us the result in [13, Thm 1.3] for the Euclidean case. Hence, hypothesis on the volume concentration of the boundary, made via the existence of $\tilde{C}$, grants then asymptotically optimal upper bounds under reasonable assumptions.

On the form of the upper bound provided in (3.0.3), a geometric term has to appear with $\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}$. Indeed explicit counter examples have been provided in [64, Chapter 2] by Pétiard to show that, there is no possible Kröger type inequality for large k.

Compared to Theorem 3.0.1, Theorem 3.0.3 provides a bound that is easier to decipher. We will see in the next chapter, in which we mainly work with the isoperimetric ratio of the domain, that similar upper bounds can be proved with somewhat weaker assumptions.

Theorem 3.0.3 yields the following corollary.

Corollary 3.0.4. *Let the assumptions of Theorem 3.0.3 hold. For all $k \in \mathbb{N}$, one has:*

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant \left(A(n)\mathcal{K}^2\left[\mathcal{K}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right]\tilde{C}^{\frac{2}{n-1}}+1\right)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}+B(\Omega,\beta).$$

where the constant $A(n)$ depends only on the dimension and $B(\Omega,\beta)$ depends on β and on the geometry, precisely on n, K_-, $\mathrm{Vol}(\Omega)$ and $\mathrm{Vol}(\Gamma)$ (See (3.2.20) together with (3.2.19).).

Notice that, due to the considerable number of constants, whose exact values have no particular interest, the same letters are often used to denote different constants of the same kind.

Remark 3.0.2. An invariant that measures the concentration of the volume as in Definition 3.0.1, which is used by the authors in [13], is the intersection index.

If Ω is an Euclidean domain of $\mathbb{R}^n$ so that its boundary Γ is a compact hypersurface, the intersection index of Γ is defined as the supremum number of transversal intersections of real lines with Γ:

$$i(\Gamma) := \sup\{\sharp(\Gamma\cap\pi),\ \pi \text{ transversal line to } \Gamma\}.$$

From [13, Prop 2.1], for every $x \in \mathbb{R}^n$ and $r > 0$, one has

$$\mathrm{Vol}\left(\Gamma\cap B(x,r)\right) \leqslant \frac{i(\Gamma)}{2}\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)r^{n-1},$$

where $B(x,r)$ denotes the Euclidean ball of center x and radius r in $\mathbb{R}^n$ and $\mathbb{S}^{n-1}$ the denotes the unit sphere in $\mathbb{R}^n$. This gives the following corollary to Theorem 3.0.3.

Corollary 3.0.5. *Let $n \geqslant 2$ and $\Omega \subset \mathbb{R}^n$ be an Euclidean domain with boundary Γ, then for every $k \geqslant 2$, one has:*

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant \left[A_n i(\Gamma)^{\frac{2}{n-1}}\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right)+1\right]\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}+B(\Omega,\beta),$$

where the constant A_n depends only on the dimension n and $B(\Omega,\beta)$ is the same constant as in Corollary 3.0.4 with $K_- = 0$ and depends on the geometry (n, $\mathrm{Vol}(\Omega)$, $\mathrm{Vol}(\Gamma)$) and on β. If in addition, Γ is convex, we have $i(\Gamma) = 2$ and there is a dimensional constant A'_n such that:

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant (A'_n\beta+1)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}+B(\Omega,\beta).$$

In the next section, we present the most technical construction in this chapter. It will be used in the sequel to prove the main results.

3.1 Metric measure space decomposition

Much of what we do in this section carries over to the metric space setting and is inspired by [12], [15] and [33]. We adopt the notation (X,d,μ) to designate a metric measure space such that X is complete and locally compact with respect to the distance d and μ is a Borel measure supported in a bounded Borelian subset $\overline{Y} \subset X$, such that $\mu(\overline{Y} \setminus Y) = 0$ and $\mu(Y) \in (0,\infty)$. For every $x \in X$ and $r > 0$, $B(x,r) := \{y \in X : d(x,y) < r\}$ designates the metric open ball. We denominate capacitor any couple (A,B) of subsets such that $\emptyset \neq A \subset B \subset X$. Two capacitors (A_1,B_1) and (A_2,B_2) are disjoint if $B_1 \cap B_2 = \emptyset$. A family of capacitors is a finite set of capacitors in X that are pairwise disjoint.

Definition 3.1.1 (Covering property). Let (X,d) be a complete, locally compact metric space. We denote by $\operatorname{diam}(X)$ the diameter, defined as the maximal distance between any two points of X. Let $\varepsilon > 1$, $\rho > 0$ and an non-decreasing function $N : (0,\rho] \longrightarrow \mathbb{N}_{\geqslant 2}$. We say that (X,d) satisfies the $(N,\varepsilon;\rho)$-covering property if each ball of radius r such that $0 < r \leqslant \rho$ can be covered by $N(r)$ balls of radius $\frac{r}{\varepsilon}$. In order to simplify notation, we will write N_r instead of $N(r)$. We shall omit the symbol ε in the notation if and only if $\varepsilon = 2$. In the same way, we drop the term ρ from the notation if and only if $\rho \geqslant \operatorname{diam}(X)$, which we call a global covering property.

Definition 3.1.2. Let (X,d,μ) be as described above. We say that the measure μ is radially monotone in X if:

$$\lim_{r\to 0}\{\sup \mu(B(x,r)),\ x \in X\} = 0. \tag{3.1.1}$$

If (X,d,μ), as above, is radially monotone and satisfies the covering property, then $\lim_{r\to 0} N_r^2 \mu(B(x,r)) = 0$ for all $x \in X$.

Essential tools used in our later constructions are given by Lemma 3.3.2 which is an adapted version of the following result due to Colbois and Maerten.

Lemma 3.1.1 (Colbois and Maerten [12]). *Let (X,d,μ) be as above with μ radially monotone. Let $N : (0,\operatorname{diam}(X)] \longrightarrow \mathbb{N}_{\geqslant 2}$ non-decreasing. Assume that (X,d,μ) satisfies the $(N,4)$-covering property.*

Let $r > 0$ and $K \in \mathbb{N}$ such that for every $x \in X$, $\mu(B(x,r)) \leqslant \frac{\mu(X)}{4KN_r^2} =: \alpha$. Then there exists a family of K capacitors $\{(A_i,B_i)\}_{1\leqslant i\leqslant K}$ with the following properties for $1 \leqslant i,j \leqslant K$:

1. *$\mu(A_i) \geqslant 2N_r\alpha$,*
2. *$B_i = A_i^r := \{x \in X,\ d(x,A_i) < r\}$ is the r-neighbourhood of A_i and $d(B_i,B_j) > 2r$ whenever $i \neq j$.*

Definition 3.1.3. Let (X,d,μ) be as above. We use the acronym CM-capacitors for those capacitors obtained by applying Colbois-Maerten's construction (Lemma 3.1.1). Consistently, we call spherical capacitor any capacitor (A,B) such that A and B are both metric balls.

In the following lemma, we provide a useful procedure to construct a general family of (spherical or CM) capacitors. This is the main tool for the proof of Proposition 3.2.1.

Lemma 3.1.2. *Let (X,d,μ) be a metric measure space such that X is complete and locally compact with respect to the distance d and μ is a Borel measure supported in a bounded Borelian subset $\overline{Y} \subset X$, such that $\mu(\overline{Y}\setminus Y) = 0$ and $\mu(Y) \subset (0,\infty)$.*

Let $N : (0, \mathrm{diam}(X)] \longrightarrow \mathbb{N}_{\geqslant 2}$ such that (X,d,μ) satisfies the $(N,4)$-covering property. In parrticular, (X,d,μ) satisfies the $(N,4;1)$-covering property, with $N = N(1) \in \mathbb{N}_{\geqslant 2}$ constant. Let $0 < r_0 \leqslant \frac{1}{10}$ be fixed and $\alpha = \frac{\mu(X)}{4KN^2}$, then for every $K \in \mathbb{N}$, X satisfies at least one of the following properties:

1. *X contains a family of K spherical capacitors $\{(A_j,B_j)\}_{j=1}^K$ such that*
   - *$A_j = B(x_j,r_j)$ and $\mu(A_j) \geqslant \alpha$, with $x_j \in X$, $r_j \in (0,2r_0]$,*
   - *$B_j = B(x_j,2r_j)$.*
2. *X contains a family of K CM-capacitors $\{(A_j,A_j^{\tilde{r}_0})\}_{j=1}^K$ where $\tilde{r}_0 = \min\{r_0,\tau_1\}$ and*

$$\tau_1 := \sup\{r \in \mathbb{R}_{>0} : \mu(B(x,r)) \leqslant \alpha, \forall x \in X\}. \tag{3.1.2}$$

Lemma 3.1.2, applied on its own, is weaker than the construction in [33, Theorem 2.1] which is based partially on the Grigor′ yan et al. [31]'s method making use of annuli. Here in Lemma 3.1.2, in addition to the construction of Colbois and Maerten, we only use balls which are somewhat a restricted type of annuli with zero interior radius. The achieved results are penalised by the lack of knowledge concerning the volumes of CM-capacitors (See (3.2.10) and (3.2.14) in the proof of the proposition 3.2.1.), and especially when $\beta \searrow 0$ for Steklov eigenvalues.

Nevertheless, the advantage of the approach in the construction of Lemma 3.1.2 is its simplicity and by adapting slightly the CM-capacitors, we obtain good estimates (See Section 3.3.).

Proof. The proof consists of two steps. The first part is an iterative scheme presenting a method to construct the proof objects. And, if the assumptions to achieve the construction of the K spherical capacitors in the first step do not hold, then we shall be able to use Lemma 3.1.1, and solve the problem once in Step 2.

Step 1. To start, define

$$X_1 := X, \quad \mu_1(A) := \mu(A),$$

for every measurable set $A \subset X$ and

$$\tau_1 = \sup\{r \in \mathbb{R}_{>0} : \mu_1(B(x,r)) \leqslant \alpha, \forall\, x \in X_1\}.$$

We have the two possible cases:

<u>Case $\tau_1 > r_0$:</u> In this case, $\mu(B(x,r_0)) \leqslant \alpha$ for every $x \in X$. We set $\tilde{r}_0 := r_0$ and move to Step 2.

Case $\tau_1 \leqslant r_0$: Set $r_1 := \frac{3}{2}\tau_1$, there exists $x_1 \in X_1$ such that $\mu_1(B(x_1, r_1)) > \alpha$. We define

$$A_1 := B(x_1, r_1), \quad B_1 := B(x_1, 2r_1), \quad C_1 := B(x_1, 4r_1).$$

We have then $\mu(A_1) > \alpha$. There are two important observations for the inductive step. First, C_1 can be covered by N^2 balls of radius $\frac{r_1}{4}$ (Since $4r_1 = 6\tau_1 \leqslant 6r_0 \leqslant \frac{6}{10} < 1$, the ball C_1 can be covered by N balls of radius r_1 and each of those balls can be covered by N balls of radius $\frac{r_1}{4}$.). Second, $\mu(B(x, \frac{r_1}{4})) \leqslant \alpha$ for all $x \in X$, since $\frac{r_1}{4} < \tau_1$. Hence $\mu_1(C_1) \leqslant N^2\alpha$ and

$$\mu(X \backslash C_1) \geqslant \mu(X) - \mu(C_1) \geqslant \mu(X)\left(1 - \frac{1}{4K}\right) > \frac{\mu(X)}{2}.$$

First iteration. We define

$$X_2 := X \backslash C_1, \quad \mu_2(A) := \mu(A \cap X_2),$$

for every measurable set $A \subset X$ and

$$\tau_2 := \sup\{r \in \mathbb{R}_{>0} : \mu_2(B(x, r)) \leqslant \alpha, \forall\, x \in X_2\}.$$

We have:

Case $\tau_2 > \tau_1$: In this case we set $\tilde{r}_0 := \tau_1$ and move to Step 2.

Case $\tau_2 \leqslant \tau_1$: Set $r_2 := \frac{3}{2}\tau_2$. Then, there exists $x_2 \in X_2$ such that $\mu_2(B(x_2, r_2)) > \alpha$. We define

$$A_2 := B(x_2, r_2), \quad B_2 := B(x_2, 2r_2),$$
$$C_2 := C_1 \cup B(x_2, 4r_1).$$

We have $\mu(A_2) = \mu(A_2 \cap X) \geqslant \mu(A_2 \cap X_2) = \mu_2(A_2) > \alpha$.
In addition, $r_2 < r_1$ hence $B_1 \cap B_2 = \emptyset$.

Similarly, $B(x_2, 4r_1)$ can be covered by N^2 balls of radius $\frac{r_1}{4}$ and since $\mu(B(x, \frac{r_1}{4})) \leqslant \alpha$ for all $x \in X$, $\mu(B(x_2, 4r_1)) \leqslant N^2\alpha$. Notice that $\mu(C_2) \leqslant \mu(C_1) + \mu(B(x_2, 4r_1)) \leqslant 2N^2\alpha$. Hence, one has

$$\mu(X \backslash C_2) = \mu(X) - \mu(C_2) \geqslant \mu(X)\left(1 - \frac{1}{2K}\right) > \frac{\mu(X)}{2}.$$

Iteration j, with $1 < j \leqslant K$. Suppose that we have constructed $j-1$ capacitors

$$\{(A_1, B_1), \ldots, (A_{j-1}, B_{j-1})\}$$

satisfying for $1 \leqslant i \neq l \leqslant j-1$

$$\begin{cases} \mu(A_i) > \alpha \\ B_i \cap B_l = \emptyset \\ A_i = B(x_i, r_i), B = B(x_i, 2r_i), \text{ where } r_i \leqslant r_1 \\ C_{j-1} = \bigcup_{i=1}^{j-1} B(x_i, 4r_1) \quad \text{and} \quad \mu(C_{j-1}) \leqslant (j-1)N^2\alpha. \end{cases}$$

Define $X_j := X \backslash C_{j-1}$, then

$$\begin{aligned} \mu(X_j) &= \mu(X \backslash C_{j-1}) = \mu(X) - \mu(C_{j-1}) \\ &\geqslant \mu(X)\left(1 - \frac{j-1}{4K}\right) > \frac{\mu(X)}{2} > 0. \end{aligned}$$

Then, define the measure $\mu_j(A) := \mu(A \cap X_j)$, for all measurable set $A \subset X$ and

$$\tau_j := \sup\{r \in \mathbb{R}_{>0} : \mu_j(B(x,r)) \leqslant \alpha, \forall\, x \in X_j\}.$$

Case $\tau_j > \tau_1$: We move to Step 2.

Case $\tau_j \leqslant \tau_1$: Set $r_j := \frac{3}{2}\tau_j$. Then, there exists $x_j \in X_j$ such that $\mu_j(B(x_j, r_j)) > \alpha$. We define

$$A_j := B(x_j, r_j), \quad B_j := B(x_j, 2r_j),$$
$$C_j := C_{j-1} \cup B(x_j, 4r_1).$$

One has $\mu(A_j) \geqslant \mu(A_j \cap X_j) = \mu_j(A_j) > \alpha$.
Since $4r_1 < 1$, $B(x_j, 4r_1)$ can be covered by N^2 balls of radius $\frac{r_1}{4}$, hence

$$\mu(B(x_j, 4r_1)) \leqslant N^2\alpha.$$

We have

$$\mu(C_j) \leqslant \mu(C_{j-1}) + \mu(B(x_j, 4r_1)) \leqslant jN^2\alpha \leqslant KN^2\alpha.$$

$$\mu(X \backslash C_j) \geqslant \mu(X)\left(1 - \frac{j}{4K}\right) \geqslant \mu(X)\left(1 - \frac{1}{4}\right) > \frac{\mu(X)}{2}.$$

Since for $i = 1, \ldots, j$ we have $r_i < r_1$, $B_j \cap B_i = \emptyset$ for all $1 \leqslant i \leqslant j-1$.

Step 2. Let $\tilde{r}_0 := \min\{r_0, \tau_1\}$, applying Lemma 3.1.1 to (X, d, μ) with the radius $\tilde{r}_0$, we obtain a family of K CM-capacitors $\{(A_j, A_j^{\tilde{r}_0})\}_{j=1}^K$. This concludes the proof.

∎

3.2 Proofs

In this section we prove the main results in this chapter. We start with the following proposition which will be useful to prove Theorem 3.0.3.

Proposition 3.2.1. *Let $n \geqslant 2$ and $\tilde{C}$ be a positive real number. Let (M,g) be a complete connected n-dimensional Riemannian manifold with $\mathrm{Ric}(M,g) \geqslant -(n-1)K_-^2$, with $K_- \in \mathbb{R}_{\geqslant 0}$. Let $\Omega \subset M$ be a domain with smooth boundary Γ such that $\Omega \in \mathcal{M}(n,\tilde{C})$. Then for every $k \geqslant 2$, one has:*

$$\lambda_{W,k}^{\beta}(\Omega) \leqslant A(n,\beta,K_-,\Omega)\left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} + B(n,K_-,\Omega)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}} + C(n,\beta,K_-,\Omega),$$

where

$$A(n,\beta,K_-,\Omega) = A(n,K_-)\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]$$
$$B(n,K_-,\Omega) = B(n,K_-)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}$$
$$C(n,\beta,K_-,\Omega) = C(n,K_-)\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right],$$

$A(n,K_-)$, $B(n,K_-)$ and $C(n,K_-)$ are in terms of $c_n e^{K_-}$, where c_n is a constant depending only on the dimension n, see (3.2.9) *and* (3.2.13).

Proof. Consider the metric measure space (M,d,μ), where d is the distance from the metric g and μ is the Borel measure with support Γ defined for each Borelian A of M by

$$\mu(A) := \int_{A\cap\Gamma} \mathrm{d}_\Gamma.$$

This measures the area of the part of the hypersurface Γ lying inside the subset A.

We will start by showing that the metric space (M,d,μ) satisfies the assumptions of Lemma 3.1.2. Then, according to the nature of the capacitors obtained after applying Lemma 3.1.2, we define a disjointly supported family of test functions and bound their Rayleigh quotient. This allows us to conclude the proof using the variational characterisation of $\lambda_{W,k}^{\beta}(\Omega)$.

Being a topological manifold, M is locally compact and the radial monotonicity is fulfilled. Thanks to the Hopf-Rinow theorem (M,d) is a complete metric space. The measure μ is supported in Γ, we have clearly $\mu(\overline{\Gamma}\backslash\Gamma) = \mu(\emptyset) = 0$ and $\mu(\Gamma) = \mathrm{Vol}(\Gamma) \in (0,\infty)$.

To show that the metric space (M,d) satisfies the covering property, we take $x \in M$, $0 < r \leqslant \mathrm{diam}(M)$ and $\{B(x_i,\frac{r}{8})\}_{i\in I}$ a maximal family of disjoint balls with center $x_i \in B(x,r)$ for all $i \in I$. The maximality assumption is in the following sense: for any $y \in B(x,r)$, the ball $B(y,\frac{r}{8})$ overlaps with at least one ball in $\{B(x_i,\frac{r}{8})\}_{i\in I}$. So, the family $\{B(x_i,\frac{r}{4})\}_{i\in I}$ covers $B(x,r)$ and if we prove that the cardinality of I is bounded by a constant N, the assertion follows.

Notice that the following chain of inclusions holds for arbitrary $i \in I$:

$$\bigcup_{j\in I} B(x_j,\frac{r}{8}) \subset B(x,r+\frac{r}{8}) \subset B(x,2r) \subset B(x_i,4r). \qquad (3.2.1)$$

One has $\sum_{i\in I}\inf_{i\in I}\mathrm{Vol}\Big(B\Big(x_i,\frac{r}{8}\Big)\Big)\leqslant\sum_{i\in I}\mathrm{Vol}(B(x_i,\frac{r}{8}))$,

$$\mathrm{card}(I)\inf_{i\in I}\mathrm{Vol}\Big(B\Big(x_i,\frac{r}{8}\Big)\Big)\leqslant\sum_{i\in I}\mathrm{Vol}(B(x_i,\frac{r}{8})),$$

where card(I) denotes the cardinality of I. Since the balls $B(x_i,\frac{r}{8})$ are pairwise disjoint, we have $\sum_{i\in I}\mathrm{Vol}(B(x_i,\frac{r}{8}))=\mathrm{Vol}(\bigcup_{i\in I}B(x_i,\frac{r}{8}))$ and then

$$\mathrm{card}(I)\inf_{i\in I}\mathrm{Vol}\Big(B\Big(x_i,\frac{r}{8}\Big)\Big)\leqslant\mathrm{Vol}(\bigcup_{i\in I}B(x_i,\frac{r}{8}))\leqslant\mathrm{Vol}(B(x,2r)). \tag{3.2.2}$$

Thanks to the volume comparison Theorem (Bishop 1964, Gromov 1980), the volume ratio $r\longrightarrow\frac{\mathrm{Vol}(B(x_{i_0},r))}{\nu(n,-K_-^2,r)}$ is a non increasing function whose limit is 1 as $r\searrow 0$. Here $\nu(n,K_-^2,r)$ denotes the volume of a ball of radius r in the constant curvature model space $M^n_{K_-^2}$. Therefore, one has

$$\frac{\mathrm{Vol}(B(x_i,4r))}{\nu(n,-K_-^2,4r)}\leqslant\frac{\mathrm{Vol}(B(x_i,\frac{r}{8}))}{\nu(n,-K_-^2,\frac{r}{8})},\quad\forall\ i\in I.$$

We have then, for every $i\in I$,

$$\frac{\mathrm{Vol}\Big(B\Big(x_i,4r\Big)\Big)}{\mathrm{Vol}\Big(B\Big(x_i,\frac{r}{8}\Big)\Big)}\leqslant\frac{\nu(n,-K_-^2,4r)}{\nu(n,-K_-^2,\frac{r}{8})}=\begin{cases}\frac{\int_0^{4r}\sinh^{n-1}(K_-t)\mathrm{d}t}{\int_0^{\frac{r}{8}}\sinh^{n-1}(K_-t)\mathrm{d}t} & \text{if } K_->0\\ \frac{\int_0^{4r}t^{n-1}\mathrm{d}t}{\int_0^{\frac{r}{8}}t^{n-1}\mathrm{d}t}=2^{5n} & \text{if } K_-=0\end{cases}$$

$$\frac{\int_0^{4r}\sinh^{n-1}(K_-t)\mathrm{d}t}{\int_0^{\frac{r}{8}}\sinh^{n-1}(K_-t)\mathrm{d}t}\leqslant\frac{\int_0^{4r}\Big[(K_-t)e^{K_-t}\Big]^{n-1}\mathrm{d}t}{\int_0^{\frac{r}{8}}(K_-t)^{n-1}\mathrm{d}t}\leqslant\frac{e^{4r(n-1)K_-}\int_0^{4r}t^{n-1}\mathrm{d}t}{\int_0^{\frac{r}{8}}t^{n-1}\mathrm{d}t}$$
$$=2^{5n}e^{4r(n-1)K_-}$$

Thus, setting $N_r:=\lceil 2^{5n}e^{4r(n-1)K_-}\rceil$ and applying the last inclusion in (3.2.1), we get:

$$\frac{\mathrm{Vol}\Big(B\Big(x,2r\Big)\Big)}{\mathrm{Vol}\Big(B\Big(x_i,\frac{r}{8}\Big)\Big)}<N_r,\quad\forall\ i\in I.$$

In other words, $\frac{\mathrm{Vol}\big(B\big(x,2r\big)\big)}{N_r}$ is a minorant of $\Big\{\mathrm{Vol}\Big(B\Big(x_i,\frac{r}{8}\Big)\Big),\ i\in I\Big\}$ and therefore

$$\frac{\mathrm{Vol}\Big(B\Big(x,2r\Big)\Big)}{N_r}\leqslant\inf_{i\in I}\mathrm{Vol}\Big(B\Big(x_i,\frac{r}{8}\Big)\Big).$$

Combining this with (3.2.2), we get:

$$\operatorname{card}(I)\frac{\operatorname{Vol}\Big(B\big(x,2r\big)\Big)}{N_r} \leqslant \operatorname{card}(I)\inf_{i\in I}\operatorname{Vol}\Big(B\Big(x_i,\frac{r}{8}\Big)\Big) \leqslant \operatorname{Vol}(B(x,2r)).$$

We conclude that $\operatorname{card}(I) \leqslant N_r$ and (M,d) satisfies the $(N_r,4)$-covering property as required.

- Setting $N := 2^{5n}e^{4(n-1)K_-}$ constant, (M,d) satisfies the $(N,4;1)$-covering property.
- When $K_- = 0$, we have actually proved that the $(2^{5n},4)$-covering property is satisfied. In particular, if M is the Euclidean space $\mathbb{R}^n$, each ball of radius $r>0$ can be covered by 32^n balls of radius $\frac{r}{4}$.

Now, set $K := 4k$ and $r_0 := \frac{1}{10}$ and $\tilde{r}_0 = \min\{r_0,\tau_1\}$. Applying Lemma 3.1.2, there exists in M either a family $\mathfrak{B} = \{(A_j,B_j)\}_{j=1}^{4k}$ of spherical capacitors such that

- $A_j = B(x_j,r_j),\ x_j \in X,\ r_j \in (0,2\tilde{r}_0],\ \mu(A_j) \geqslant \alpha = \frac{\operatorname{Vol}(\Gamma)}{16kN^2},$
- $B_j = B(x_j,2r_j),$

or a family $\mathfrak{C} = \{(A_j,A_j^{\tilde{r}_0})\}_{j=1}^{4k}$ of $4k$ CM-capacitors such that $\mu(A_j) \geqslant 2N\alpha$.

First case $M \supset \mathfrak{B}$. For each $1 \leqslant j \leqslant 4k$, we consider the function f_j supported in $B_j = B(x_j,2r_j) \in \mathfrak{B}$ and defined by

$$f_j(x) := \begin{cases} \min\{1,2-\frac{d(x_j,x)}{r_j}\} & \forall x \in B_j,\\ 0 & \forall x \in M\backslash B_j. \end{cases} \tag{3.2.3}$$

We have $R_\beta(f_j) \leqslant \frac{\int_{\Omega\cap B_j}|\nabla f_j|^2 \mathrm{d}_M + \beta\int_{\Gamma\cap B_j}|\nabla f_j|^2\mathrm{d}_\Gamma}{\int_{\Gamma\cap A_j} {f_i}^2\mathrm{d}_\Gamma}$.

i) Since for every $x \in A_j$, $f_j(x) = 1$, one has

$$\int_{\Gamma\cap A_j} {f_j}^2 \mathrm{d}_\Gamma \geqslant \int_{\Gamma\cap A_j}\mathrm{d}_\Gamma \geqslant \mu(A_j) \geqslant \frac{\operatorname{Vol}(\Gamma)}{16N^2k}.$$

ii) Set for $x \in M$, $d_j(x) := d(x_j,x)$, then

$$|\nabla f_j| \leqslant \left|\nabla(2-\frac{d_j(x)}{r_j})\right| = \left|\frac{1}{r_j}\nabla(d_j(x))\right| \leqslant \frac{1}{r_j}.$$

By Hölder's inequality, we have

$$\int_{\Omega\cap B_j}|\nabla f_j|^2 \mathrm{d}_M \leqslant \left(\int_{\Omega\cap B_j}|\nabla f_j|^n \mathrm{d}_M\right)^{\frac{2}{n}}\left(\int_{\Omega\cap B_j}\mathrm{d}_M\right)^{1-\frac{2}{n}}$$
$$\leqslant \left(\frac{1}{r_j^n}\int_{\Omega\cap B_j}1\mathrm{d}_M\right)^{\frac{2}{n}}\left(\int_{\Omega\cap B_j}\mathrm{d}_M\right)^{1-\frac{2}{n}} \leqslant \left(\frac{1}{r_j^n}\mathrm{Vol}(B_j)\right)^{\frac{2}{n}}\left(\mathrm{Vol}(\Omega\cap B_j)\right)^{1-\frac{2}{n}}.$$

However, one has

$$\mathrm{Vol}(B_j)\leqslant \nu(n,-K_-^2,r_j)\leqslant \frac{2^n}{n}r_j^n e^{2(n-1)r_jK_-}$$
$$\leqslant \frac{2^n}{n}r_j^n e^{2(n-1)K_-} =: c(n,K_-)r_j^n. \tag{3.2.4}$$

In addition, the B_j's are pairwise disjoint then $\sum_{j=1}^{4k}\mathrm{Vol}(\Omega\cap B_j)\leqslant \mathrm{Vol}(\Omega)$. We deduce that at least $2k$ of B_j's satisfy

$$\mathrm{Vol}(\Omega\cap B_j)\leqslant \frac{\mathrm{Vol}(\Omega)}{k}. \tag{3.2.5}$$

Up to re-ordering, we assume that for the first $2k$ of the B_j's we have (3.2.5). Hence, $\int_{\Omega\cap B_j}|\nabla f_j|^2 \leqslant c(n,K_-)^{\frac{2}{n}}\left(\frac{\mathrm{Vol}(\Omega)}{k}\right)^{1-\frac{2}{n}}, \quad \forall\ 1\leqslant j\leqslant 2k.$

iii) Notice that $2r_j\leqslant 4r_0<1$. We have

$$\mathrm{Vol}(\Gamma\cap B(x_j,2r_j))\leqslant \tilde{C}(2r_j)^{n-1}, \tag{3.2.6}$$

$$\int_{\Gamma\cap B_j}|\nabla f_j|^2\mathrm{d}_M \leqslant \left(\int_{\Gamma\cap B_j}|\nabla f_j|^n\mathrm{d}_M\right)^{\frac{2}{n-1}}\left(\int_{\Gamma\cap B_j}\mathrm{d}_M\right)^{1-\frac{2}{n-1}}$$
$$\leqslant \left(\frac{1}{r_j^{n-1}}\int_{\Gamma\cap B_j}\mathrm{d}_M\right)^{\frac{2}{n-1}}\left(\int_{\Gamma\cap B_j}\mathrm{d}_M\right)^{1-\frac{2}{n-1}}$$
$$\leqslant \left(\frac{1}{r_j^{n-1}}\mathrm{Vol}(\Gamma\cap B(x_j,2r_j))\right)^{\frac{2}{n-1}}\left(\mathrm{Vol}(\Gamma\cap B_j)\right)^{1-\frac{2}{n-1}}$$
$$\leqslant \left(2^{n-1}\tilde{C}\right)^{\frac{2}{n-1}}\left(\mathrm{Vol}(\Gamma\cap B_j)\right)^{1-\frac{2}{n-1}}.$$

In addition, again the B_j's are pairwise disjoint then $\sum_{j=1}^{2k}\mathrm{Vol}(\Gamma\cap B_j) < \sum_{j=1}^{4k}\mathrm{Vol}(\Gamma\cap B_j)\leqslant \mathrm{Vol}(\Gamma)$. Hence at least k elements of $\{B_j,\ j=1,\ldots,2k\}$ satisfy

$$\mathrm{Vol}(\Gamma\cap B_j)\leqslant \frac{\mathrm{Vol}(\Gamma)}{k}. \tag{3.2.7}$$

Without loss of generality, we make the simplifying assumption that for the first k elements of $\{B_j,\ j=1,\dots,2k\}$, inequality (3.2.7) holds. Hence,

$$\int_{\Gamma\cap B_j}|\nabla f_j|^2 \mathrm{d}_M \leqslant 4\tilde{C}^{\frac{2}{n-1}}\left(\frac{\mathrm{Vol}(\Gamma)}{k}\right)^{1-\frac{2}{n-1}}.$$

Combining i), ii), iii), one has

$$\begin{aligned} R_\beta(f_j) &\leqslant \frac{16N^2k}{\mathrm{Vol}(\Gamma)}\left[c(n,K_-)^{\frac{2}{n}}\left(\frac{\mathrm{Vol}(\Omega)}{k}\right)^{1-\frac{2}{n}} + 4\beta\tilde{C}^{\frac{2}{n-1}}\left(\frac{\mathrm{Vol}(\Gamma)}{k}\right)^{1-\frac{2}{n-1}}\right] \\ &\leqslant B(n,K_-)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}}\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}} + A(n,K_-)\beta\left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}. \end{aligned} \tag{3.2.8}$$

Here $A(n,K_-) := 2^{16n}e^{8(n-1)K_-}$ and $B(n,K_-) := 2^{14n}e^{8(n-1)K_-}c(n,K_-)^{\frac{2}{n}}$, (3.2.9)

with $c(n,K_-)$ defined in (3.2.4).

Second case $M \supset \mathfrak{C}$**.** For each $1 \leqslant j \leqslant 4k$, we consider the function φ_j supported on $A_j^{\tilde{r}_0}$ defined by

$$\varphi_j(x) := \begin{cases} 1 - \frac{d(A_j,x)}{\tilde{r}_0} & \forall x \in A_j^{\tilde{r}_0}, \\ 0 & \forall x \in M\backslash A_j^{\tilde{r}_0}. \end{cases}$$

We have $R_\beta(\varphi_j) \leqslant \frac{\int_{\Omega\cap A_j^{\tilde{r}_0}}|\nabla\varphi_j|^2\mathrm{d}_M + \beta\int_{\Gamma\cap A_j^{\tilde{r}_0}}|\nabla\varphi_j|^2\mathrm{d}_\Gamma}{\int_{\Gamma\cap A_j}{\varphi_i}^2\mathrm{d}_\Gamma}$.

With the same argument as above, we can assume that for the first k of the $A_j^{\tilde{r}_0}$'s, we have:

$$\mathrm{Vol}(\Omega\cap A_j^{\tilde{r}_0}) \leqslant \frac{\mathrm{Vol}(\Omega)}{k} \quad \text{and} \quad \mathrm{Vol}(\Gamma\cap A_j^{\tilde{r}_0}) \leqslant \frac{\mathrm{Vol}(\Gamma)}{k}. \tag{3.2.10}$$

Then, one has:

$$\int_{\Gamma\cap A_j}{\varphi_j}^2\mathrm{d}_\Gamma \geqslant \int_{\Gamma\cap A_j}\mathrm{d}_\Gamma \geqslant \mu(A_j) \geqslant \frac{\mathrm{Vol}(\Gamma)}{8Nk},$$

since for every $x \in A_j$, $\varphi_j(x) = 1$ and

$$\begin{aligned} R_\beta(\varphi_j) &\leqslant \frac{2Nk}{\mathrm{Vol}(\Gamma)}\left[\frac{1}{\tilde{r}_0^2}\frac{\mathrm{Vol}(\Omega)}{k} + \beta\frac{1}{\tilde{r}_0^2}\frac{\mathrm{Vol}(\Gamma)}{k}\right] \\ &= \frac{2N}{\tilde{r}_0^2}\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]. \end{aligned} \tag{3.2.11}$$

- Hence, if $\tilde{r}_0 = r_0$, then

$$R_\beta(\varphi_j) \leqslant C(n,K_-)\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right], \tag{3.2.12}$$

$$\text{where} \quad C(n,K_-) := \frac{2^{5n+3}e^{4(n-1)K_-}}{r_0^2} = 5^2 \cdot 2^{5(n+1)}e^{4(n-1)K_-}. \tag{3.2.13}$$

- Otherwise, $\tilde{r}_0 = \tau_1$ and there exists $x \in M$ such that $\mu(B(x,2\tilde{r}_0)) > \alpha$. Take $y \in B(x,2\tilde{r}_0) \cap \Gamma$, since $B(y,4\tilde{r}_0) \supset B(x,2\tilde{r}_0)$, we have $\tilde{C}(4\tilde{r}_0)^{n-1} \geqslant \mu(B(y,4\tilde{r}_0)) > \alpha$. Consequently, $\frac{1}{\tilde{r}_0^2} \leqslant 4\left(\frac{16\tilde{C}kN^2}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}$ and we have

$$R_\beta(\varphi_j) \leqslant A'(n,K_-)\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]\left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}, \tag{3.2.14}$$

with $A'(n,K_-)$ depending on n and K_-.

In both cases $R_\beta(\varphi_j)$ is bounded from above by the sum of the right-hand sides in (3.2.8), (3.2.12) and (3.2.14). Without loss of generality, one can assume that $A(n,K_-) \geqslant A'(n,K_-)$. One concludes the argument by applying the min-max characterization of $\lambda^\beta_{W,k}(\Omega)$. ∎

Proof of Theorem 3.0.3. From Proposition 3.2.1, for every $k \geqslant 2$, we have

$$\begin{aligned}\lambda^\beta_{W,k}(\Omega) \leqslant{}& A(n,K_-)\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]\left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} \\ &+ B(n,K_-)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}} \\ &+ C(n,K_-)\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right],\end{aligned} \tag{3.2.15}$$

where the constants $A(n,K_-)$, $B(n,K_-)$ and $C(n,K_-)$ are in the form $c(n)e^{K_-}$, $c(n)$ being a term involving n and K_- free.

- If $K_- \leqslant 1$, then $A(n,K_-)$, $B(n,K_-)$ and $C(n,K_-)$ can be replace by constants depending only on n:

$$\begin{aligned}\lambda^\beta_{W,k}(\Omega) \leqslant{}& A(n)\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]\left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} \\ &+ B(n)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}} \\ &+ C(n)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right).\end{aligned}$$

- Otherwise, we assume that $\mathrm{Ric}(M,g) \geqslant -(n-1)K_-^2 g$ with $K_- > 1$. Then the Ricci curvature $\mathrm{Ric}(M,\tilde{g})$ of the rescaled metric $\tilde{g} := K_-^2 g$ is bounded by $-(n-1)\tilde{g}$. We mark with a tilde quantities associated with the metric $\tilde{g}$, while those unmarked with such will be still associated with the metric g. Then we have from (3.0.2),

$$\mathrm{Vol}_{\tilde g}(B(x,r)\cap\Gamma) = K_-^{n-1}\mathrm{Vol}_g(B(x,r)\cap\Gamma) \leqslant K_-^{n-1}\tilde C r^{n-1},$$

for all $x\in\Gamma$ and all radius $0<r<1$ and

$$\tilde\lambda^{\beta}_{W,k}(\Omega) \leqslant A(n)\left[\frac{\mathrm{Vol}_{\tilde g}(\Omega)}{\mathrm{Vol}_{\tilde g}(\Gamma)}+\beta\right]\left(\frac{K_-^{n-1}\tilde C k}{\mathrm{Vol}_{\tilde g}(\Gamma)}\right)^{\frac{2}{n-1}} + B(n)\left(\frac{\mathrm{Vol}_{\tilde g}(\Omega)}{\mathrm{Vol}_{\tilde g}(\Gamma)}\right)^{1-\frac{2}{n}}\left(\frac{k}{\mathrm{Vol}_{\tilde g}(\Gamma)}\right)^{\frac{2}{n}} + C(n)\left(\frac{\mathrm{Vol}_{\tilde g}(\Omega)}{\mathrm{Vol}_{\tilde g}(\Gamma)}+\beta\right),$$

In addition, since $K_->1$, for all $u\in H(\Omega)\backslash\{0\}$ we have

$$\tilde R_\beta(u) = \frac{K_-\int_\Omega |\nabla u|^2 \mathrm{d}_M + \beta\int_\Gamma |\nabla_\Gamma u|^2\mathrm{d}_\Gamma}{K_-^2\int_\Gamma u^2\mathrm{d}_\Gamma} \geqslant \frac{1}{K_-^2}R_\beta(u). \tag{3.2.16}$$

Indeed,

$$\tilde R_\beta(u) = \frac{\int_\Omega \tilde g(\tilde\nabla u,\tilde\nabla u)\tilde{\mathrm{d}}_M + \beta\int_\Gamma \tilde g(\tilde\nabla_\Gamma u,\tilde\nabla_\Gamma u)\tilde{\mathrm{d}}_\Gamma}{\int_\Gamma u^2\tilde{\mathrm{d}}_\Gamma}.$$

where $\tilde g = K_-^2 g$. The gradient of u with respect to the metric $\tilde g$, denoted by $\tilde\nabla u$ is defined as the vector field satisfying $\tilde g(\tilde\nabla u, X) = du(X)$ for all vector field X. The differential $df\ :\ TM \longrightarrow \mathbb{R}$ which measures the change in the function, is given in local coordinates by $df = \partial_i(f)dx^i$. Defined in this way,$\tilde\nabla u$ satisfies for all vector field X

$$\tilde g(\tilde\nabla u, X) = K_-^2 g(\tilde\nabla u, X) = g(K_-^2\tilde\nabla u, X)$$

Since the gradient ∇u is the unique vector-field such that, for all vector field X, $g(\nabla u, X) = du(X)$, we have:

$$K_-^2\tilde\nabla u = \nabla u.$$

We know that the volume form $\tilde{\mathrm{d}}_M = \sqrt{\det\ \tilde g}\ dx^1\wedge\cdots\wedge dx^n = K_-^n\sqrt{\det\ g}\ dx^1\wedge\cdots\wedge dx^n = K_-^n\mathrm{d}_M$. Similarly, we have $\tilde\nabla_\Gamma u = K_-^{-2}\nabla_\Gamma u$ and $\tilde{\mathrm{d}}_\Gamma = K_-^{n-1}\mathrm{d}_\Gamma$. Hence,

$$\begin{aligned}\tilde R_\beta(u) &= \frac{\int_\Omega \tilde g(\tilde\nabla u,\tilde\nabla u)\tilde{\mathrm{d}}_M + \beta\int_\Gamma \tilde g(\tilde\nabla_\Gamma u,\tilde\nabla_\Gamma u)\tilde{\mathrm{d}}_\Gamma}{\int_\Gamma u^2\tilde{\mathrm{d}}_\Gamma}\\ &= \frac{\int_\Omega K_-^2 g(K_-^{-2}\nabla u, K_-^{-2}\nabla u)K_-^n\mathrm{d}_M + \beta\int_\Gamma K_-^2 g(K_-^{-2}\nabla_\Gamma u, K_-^{-2}\nabla_\Gamma u)K_-^{n-1}\mathrm{d}_\Gamma}{\int_\Gamma u^2 K_-^{n-1}\mathrm{d}_\Gamma}.\end{aligned}$$

Every orthonormal basis of a k-dimensional subspaces $V\in\mathfrak{V}(k)$ of $H(\Omega)$ remains orthogonal with the metric $\tilde{g}$, then using the variational characterisation, we have

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant K_-^2 \tilde{\lambda}^{\beta}_{W,k}(\Omega) \leqslant A(n) K_-^2 \left[\frac{\mathrm{Vol}_{\tilde{g}}(\Omega)}{\mathrm{Vol}_{\tilde{g}}(\Gamma)} + \beta\right] \left(\frac{K_-^{n-1}\tilde{C}k}{\mathrm{Vol}_{\tilde{g}}(\Gamma)}\right)^{\frac{2}{n-1}}$$
$$+ B(n) K_-^2 \left(\frac{\mathrm{Vol}_{\tilde{g}}(\Omega)}{\mathrm{Vol}_{\tilde{g}}(\Gamma)}\right)^{1-\frac{2}{n}} \left(\frac{k}{\mathrm{Vol}_{\tilde{g}}(\Gamma)}\right)^{\frac{2}{n}} + C(n) K_-^2 \left(\frac{\mathrm{Vol}_{\tilde{g}}(\Omega)}{\mathrm{Vol}_{\tilde{g}}(\Gamma)} + \beta\right).$$

However $\mathrm{Vol}_{\tilde{g}}(\Omega) = K_-^n \mathrm{Vol}_g(\Omega)$ and $\mathrm{Vol}_{\tilde{g}}(\Gamma) = K_-^{n-1}\mathrm{Vol}_g(\Gamma)$, thus

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant A(n) \left[K_- \frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right] \left(\frac{K_-^{n-1}\tilde{C}k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}$$
$$+ B(n) K_- \left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}} \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}} + C(n) K_-^2 \left(K_- \frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right) \tag{3.2.17}$$

In each case,

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant A(n) \mathcal{K}^2 \left[\mathcal{K} \frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right] \left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}$$
$$+ B(n) \mathcal{K} \left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}} \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}} + C(n) \mathcal{K}^2 \left(\mathcal{K} \frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right). \tag{3.2.18}$$

Without loss of generality one can assume that $C(n) \leqslant A(n)$. Then

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant A(n) \mathcal{K}^2 \left[\mathcal{K} \frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right] \left[\left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} + 1\right] + B(n) \mathcal{K} \left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}} \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}}.$$

In the particular case where $\beta = 0$, in (3.2.16) we have $\tilde{R}_\beta(u) \geqslant \frac{1}{K_-} R_\beta(u)$. and (3.2.17) becomes

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant A(n) \mathcal{K}^2 \left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right] \left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} + B(n) \left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}} \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}} + C(n) K_-^2 \frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}$$
$$\leqslant A(n) \mathcal{K}^2 \left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right] \left[\left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} + 1\right] + B(n) \left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}} \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}}.$$

■

Proof of Corollary 3.0.4. We rewrite the second term in the right hand side of (3.2.18) that we refer as T_2:

$$T_2 = \frac{\overline{B}}{k^{\frac{2}{n(n-1)}}} \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}},$$

where $\overline{B} := B(n,\mathcal{K})\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}} \mathrm{Vol}(\Gamma)^{\frac{2}{n(n-1)}}$ and $B(n,\mathcal{K}) = B(n)\mathcal{K}$.

- If $k \leqslant \overline{B}^{\frac{n(n-1)}{2}}$ then T_2 is bounded from above by

$$\frac{\overline{B}^{1+\frac{2}{n}}}{\mathrm{Vol}(\Gamma)^{\frac{2}{n-1}}} = B(n,\mathcal{K})^{1+\frac{2}{n}} \frac{\mathrm{Vol}(\Omega)^{1-\frac{4}{n^2}}}{\mathrm{Vol}(\Gamma)^{1-\frac{2}{n(n-1)}+\frac{2}{n+1}-\frac{4}{n^2(n-1)}-\frac{4}{n^2}}} =: \overline{\overline{B}}, \tag{3.2.19}$$

  which is a geometric constant free from k.

- Otherwise, we have $\frac{\overline{B}}{k^{\frac{2}{n(n-1)}}} < 1$ and then

$$T_2 < \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}.$$

In each case, we have

$$T_2 \leqslant \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} + \overline{\overline{B}}.$$

Hence replacing in (3.2.18) and setting

$$B(\Omega,\beta) := \overline{\overline{B}} + C(n,\mathcal{K})\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right), \tag{3.2.20}$$

we get

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant \left(A(n)\mathcal{K}^2\left[\mathcal{K}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]\tilde{C}^{\frac{2}{n-1}} + 1\right)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} + B(\Omega,\beta).$$

■

Proof of Corollary 3.0.5. The result follows immediately from Corollary 3.0.4, setting $K_- = 0$ and $\tilde{C} = \frac{i(\Gamma)}{2}\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)$. ■

The following lemma, from [59, Lemma 3.2] gives a volume estimate result which will be very useful to achieve our estimate in Theorem 3.0.1.

Lemma 3.2.2 (Nardulli [59]). *Let (M,g) be a Riemannian manifold of dimension $n \geqslant 2$ and $\Gamma \subset M$ a smooth hypersurface. Then there exist two constants $R_0 > 0$ and $C_0 > 0$ such that for every $x \in M$ at distance $d < R_0$ from Γ, one has*

$$\mathrm{Vol}_g(\Gamma \cap B(x,R)) \leqslant (1 + 2RC_0)\omega_{n-1}(2R)^{n-1}, \quad \forall\, R \in [d, R_0), \tag{3.2.21}$$

where ω_{n-1} is the volume of the unit ball of $\mathbb{R}^{n-1}$. Here R_0 is a constant depending on the geometry of the ambient Riemannian manifold M, in particular the injectivity radius, on a bound on the second fundamental form of Γ and on the normal injectivity radius of Γ. The constant C_0 depends on an upper bound on curvature tensor of ambient manifold, on a lower bound on Ricci curvature tensor of Γ and an upper bound on the second fundamental form of Γ .

Remark 3.2.1. In the statement of [59, Lemma 3.2] the inequality holds for every $x \in M$ with $\text{dist}(\Gamma, x) < R_0$ and $R < R_0$. One can consider $R \in [\text{dist}(\Gamma, x), R)$ since in the case $\text{dist}(\Gamma, x) > R$, the intersection $\Gamma \cap B(x, r)$ is empty and the inequality is trivial. Assuming that $\text{dist}(\Gamma, x) \leqslant R < R_0$, we have in the right hand side $\text{dist}(\Gamma, x) + R \leqslant 2R$ which leads to our statement.

The proof in [59] reduces the problem to an application of Bishop-Gromov inequality estimating the volume of an intrinsic ball of Γ. This is done by using comparison theorems for distortion of the normal exponential map based on a submanifold, to compare the extrinsic and intrinsic distance functions on Γ.

Proof of Theorem 3.0.1. Following the same lines as the proof of Proposition 3.2.1, and setting $r_0 := \frac{1}{10}\min\{1, R_0\}$ (R_0 as in Lemma 3.2.2), we notice that in the case of spherical capacitors, one has

$$d(x_j, \Gamma) \leqslant 2r_j \leqslant 4r_0 < R_0.$$

Applying Lemma 3.2.2 with $R = 2r_j$, we have

$$\text{Vol}(\Gamma \cap B(x_j, 2r_j)) \leqslant (1 + 4r_j C_0)\omega_{n-1}(4r_j)^{n-1}.$$

Either $1 \geqslant 4r_j C_0$ and then one has

$$\begin{aligned}
\int_{\Gamma\cap B_j} |\nabla f_j|^2 \mathrm{d}_M &\leqslant \left(\int_{\Gamma\cap B_j} |\nabla f_j|^{n-1} \mathrm{d}_M\right)^{\frac{2}{n-1}} \left(\int_{\Gamma\cap B_j} \mathrm{d}_M\right)^{1-\frac{2}{n-1}} \\
&\leqslant \left(\frac{1}{r_j^{n-1}} \int_{\Gamma\cap B_j} \mathrm{d}_M\right)^{\frac{2}{n-1}} \left(\int_{\Gamma\cap B_j} \mathrm{d}_M\right)^{1-\frac{2}{n-1}} \\
&\leqslant \left(\frac{1}{r_j^{n-1}} \text{Vol}(\Gamma \cap B(x_j, 2r_j))\right)^{\frac{2}{n-1}} \left(\text{Vol}(\Gamma \cap B_j)\right)^{1-\frac{2}{n-1}} \\
&\leqslant \left(2^{2n-1}\omega_{n-1}\right)^{\frac{2}{n-1}} \left(\text{Vol}(\Gamma \cap B_j)\right)^{1-\frac{2}{n-1}}.
\end{aligned}$$

Or, $1 \leqslant 4r_j C_0$ and then

$$\begin{aligned}\int_{\Gamma\cap B_j} |\nabla f_j|^2 \mathrm{d}_M &\leqslant \left(\int_{\Gamma\cap B_j} |\nabla f_j|^n \mathrm{d}_M\right)^{\frac{2}{n}} \left(\int_{\Gamma\cap B_j} \mathrm{d}_M\right)^{1-\frac{2}{n}}\\ &\leqslant \left(\frac{1}{r_j^n}\int_{\Gamma\cap B_j} 1\,\mathrm{d}_M\right)^{\frac{2}{n}} \left(\int_{\Gamma\cap B_j} \mathrm{d}_M\right)^{1-\frac{2}{n}}\\ &\leqslant \left(\frac{1}{r_j^n}\mathrm{Vol}(\Gamma\cap B(x_j,2r_j))\right)^{\frac{2}{n}} \left(\mathrm{Vol}(\Gamma\cap B_j)\right)^{1-\frac{2}{n}}\\ &\leqslant \left(2^{3+2(n-1)}C_0\omega_{n-1}\right)^{\frac{2}{n}} \left(\mathrm{Vol}(\Gamma\cap B_j)\right)^{1-\frac{2}{n}}.\end{aligned}$$

In each case, one has

$$\begin{aligned}\int_{\Gamma\cap B_j} |\nabla f_j|^2 \mathrm{d}_M \leqslant& \left(2^{2n-1}\omega_{n-1}\right)^{\frac{2}{n-1}} \left(\mathrm{Vol}(\Gamma\cap B_j)\right)^{1-\frac{2}{n-1}}\\ &+ \left(2^{3+2(n-1)}C_0\omega_{n-1}\right)^{\frac{2}{n}} \left(\mathrm{Vol}(\Gamma\cap B_j)\right)^{1-\frac{2}{n}}.\end{aligned}$$

Hence,

$$\int_{\Gamma\cap B_j} |\nabla f_j|^2 \mathrm{d}_M \leqslant \left(2^{2n-1}\omega_{n-1}\right)^{\frac{2}{n-1}} \left(\frac{\mathrm{Vol}(\Gamma)}{k}\right)^{1-\frac{2}{n-1}} + \left(2^{3+2(n-1)}C_0\omega_{n-1}\right)^{\frac{2}{n}} \left(\frac{\mathrm{Vol}(\Gamma)}{k}\right)^{1-\frac{2}{n}}.$$

We get:

$$\begin{aligned}R_\beta(f_j) \leqslant \frac{16N^2k}{\mathrm{Vol}(\Gamma)}\Big[& c(n,K_-)^{\frac{2}{n}}\left(\frac{\mathrm{Vol}(\Omega)}{k}\right)^{1-\frac{2}{n}}\\ &+\beta\left(2^{2n-1}\omega_{n-1}\right)^{\frac{2}{n-1}}\left(\frac{\mathrm{Vol}(\Gamma)}{k}\right)^{1-\frac{2}{n-1}} + \beta\left(2^{3+2(n-1)}C_0\omega_{n-1}\right)^{\frac{2}{n}}\left(\frac{\mathrm{Vol}(\Gamma)}{k}\right)^{1-\frac{2}{n}}\Big].\end{aligned}$$

Finally,

$$R_\beta(f_j) \leqslant A(n,K_-)\beta\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} + B(n,K_-,C_0)\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right]\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}}, \tag{3.2.22}$$

where

$$\begin{aligned}A(n,K_-) &:= 2^{28n}\omega_{n-1}^{\frac{2}{n-1}}e^{8(n-1)K_-}\\ (n,K_-,C_0) &:= 2^{24n}(C_0\omega_{n-1})^{\frac{2}{n}}e^{8(n-1)K_-} + 2^{24n}e^{12(n-1)K_-}.\end{aligned}$$

In the Second case $M \supset \mathfrak{C}$, we have CM-capacitors. Taking the same test functions

φ_j, one has

$$R_\beta(\varphi_j) \leqslant \frac{8Nk}{\mathrm{Vol}(\Gamma)}\left[\frac{1}{\tilde{r}_0^2}\frac{\mathrm{Vol}(\Omega)}{k} + \beta\frac{1}{\tilde{r}_0^2}\frac{\mathrm{Vol}(\Gamma)}{k}\right] = \frac{2N}{\tilde{r}_0^2}\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]. \tag{3.2.23}$$

Hence, if $\tilde{r}_0 = r_0$, then

$$R_\beta(\varphi_j) = C(n,K_-,R_0)\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right],$$

where $C(n,K_-,R_0) := \frac{2^{10n}e^{4(n-1)K_-}}{r_0^2}$ and $r_0 := \frac{1}{10}\min\{1,R_0\}$.

Otherwise, $\tilde{r}_0 = \tau_1 < r_0$ and there exist $x \in X$ such that $\mu(B(x,2\tau_1)) > \alpha = \frac{\mathrm{Vol}(\Gamma)}{16N^2k}$. Using Lemma 3.2.2, we have

$$(1 + 4\tilde{r}_0C_0)\,\omega_{n-1}(4\tilde{r}_0)^{n-1} > \alpha.$$

Either $4\tau_1C_0 < 1$ then $\frac{1}{\tilde{r}_0^2} \leqslant a'(n)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}$ where $a'(n)$ depends only on n, or $4\tau_1C_0 \geqslant 1$ and $\frac{1}{\tilde{r}_0^2} \leqslant b'(n,,C_0)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}}$ where $b'(n,C_0)$ depends on n and C_0. Replacing in (3.2.23), these partial results are combined by a global upper bound after summation

$$\begin{aligned} R_\beta(\varphi_j) \leqslant A'(n,K_-)&\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} \\ &+ B'(n,K_-,C_0)\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}} \\ &+ C(n,K_-,R_0)\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right], \end{aligned} \tag{3.2.24}$$

where $A'(n,K_-)$, $B'(n,K_-,C_0)$ and $C(n,K_-,R_0)$ depend on the respective terms in parentheses.

We can assume that $A(n,K_-) \geqslant A'(n,K_-)$ and $B(n,K_-,C_0) \geqslant B'(n,K_-,C_0)$. Hence, from (3.2.22) and (3.2.24), we have in both cases, applying the min-max characterization of $\lambda^\beta_{W,k}(\Omega)$:

$$\begin{aligned} \lambda^\beta_{W,k}(\Omega) \leqslant A(n,K_-)&\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} \\ &+ B(n,K_-,C_0)\left[\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}} + \frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}} \\ &+ C(n,K_-,R_0)\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right], \end{aligned} \tag{3.2.25}$$

where the constants $B(n,K_-,C_0)$ and $C(n,K_-,R_0)$ depend on n and geometric quantities K_-, C_0 and R_0 respectively, where C_0 and R_0 are constants that are implied by Narduli's

Lemma. The important thing to note is that we know what these constants depend on and the index k is not included in this list of dependencies. We set $B(M,\Omega) := B(n,K_-,C_0)$ and $C(M,\Omega) := C(n,K_-,C_0)$. ∎

Proof of Corollary 3.0.2. With the same arguments as in the proof of Theorem 3.0.3, we have from (3.2.25):

$$\begin{aligned}\lambda^{\beta}_{W,k}(\Omega) \leqslant A(n)\left[K\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right]\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} \\ +\overline{B}(n,C_0)K\left[\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}+\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right]\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}} \\ +\overline{C}(n,R_0)K^2\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right]. \quad (3.2.26)\end{aligned}$$

The remaining of the proof is similar to the proof of Corollary 3.0.4.

∎

3.3 Sharp upper bound for Riemannian manifolds with non-negative Ricci curvature

In the previous section, the applied construction of Colbois and Maerten utilizes unions of small balls in the underlying manifold to form the capacitors. A slight adjustment of this construction gives adapted capacitors for an easy use of tube volume estimates helping us to sharpen the upper bounds for the Wentzel eigenvalues proved in the previous section.

3.3.1 Adapted CM-capacitors

Let (X,d,μ) be a metric measure space such that X is complete and locally compact with respect to the distance d and μ is a Borel measure supported in a bounded Borelian subset $\overline{Y} \subsetneq X$, such that $\mu(\overline{Y}\backslash Y) = 0$ and $\mu(Y) \in (0,\infty)$. Let $N : (0,\mathrm{diam}(X)] \ni r \longrightarrow N_r \in \mathbb{N}_{\geqslant 2}$ such that (X,d,μ) satisfies the $(N,4)$-covering property. In parrticular, (X,d,μ) satisfies the $(N,4;1)$-covering property (Definition 3.1.1), with $N \in \mathbb{N}_{\geqslant 2}$ constant.

Lemma 3.3.1. *Let $\boldsymbol{\alpha} \in]0,\mu(X)[$ and $0 < r < \frac{1}{4}$ be fixed. Assume that for every point x in X, $\mu(B(x,r)) \leqslant \boldsymbol{\alpha}$ i.e*

$$r \in \{s \in \mathbb{R}_{>0} : \mu(B(x,s)) \leqslant \boldsymbol{\alpha}, \forall\, x \in X\} \cap (0,\frac{1}{4})$$

Then, there exists a finite union U of balls of radius r centred in X such that the subset $A := U \cap Y$ satisfies the following inequalities:

$$\boldsymbol{\alpha} < \mu(A) \leqslant \mu(A^{4r}) \leqslant 2N\boldsymbol{\alpha}, \quad (3.3.1)$$

where $A^{4r} := \{x \in X : d(A,x) < 4r\}$ denotes the $4r$-neighbourhood of A in X.

Proof. Let Ψ be the positive non-decreasing function defined by:

$$\Psi : \mathbb{N} \ni m \longrightarrow \max\left\{\mu\left(\bigcup_{i=1}^{m} B(x_i, r)\right), x_i \in X \ \forall\ 1 \leqslant i \leqslant m\right\}. \tag{3.3.2}$$

The application Ψ is well defined since, for each $m \in \mathbb{N}$, at one or more sets of m points, the function $\Psi_m : X^m \ni (x_1, \ldots, x_m) \longrightarrow \mu(\bigcup_{i=1}^m B(x_i, r))$ achieves its maximum value in X^m.

Since $\mu(B(x,r)) \leqslant \boldsymbol{\alpha}$ for every $x \in X$, we have $\Psi(1) \leqslant \boldsymbol{\alpha}$. In addition, there necessarily exists a radius $R > 0$ large enough such that for some $x \in X$ one has

$$\mu(B(x,R)) = \mu(X) > \boldsymbol{\alpha}.$$

Noticing that one can cover $B(x,R)$ by a finite union of balls of radius r, we have $\Psi(m_R) > \boldsymbol{\alpha}$ for some finite number $m_R \in \mathbb{N}$. In sum, there is $m_R \in \mathbb{N}$ such that $\Psi(1) \leqslant \boldsymbol{\alpha} < \Psi(m_R)$ with Ψ a discrete non-decreasing function in $\mathbb{N} \supset \{1, \ldots, m_R\}$. Therefore, there exists a unique integer $m_{(\boldsymbol{\alpha},r)} \in \{2, \ldots, m_R\}$ satisfying

$$\Psi(m_{(\boldsymbol{\alpha},r)} - 1) \leqslant \boldsymbol{\alpha} < \Psi(m_{(\boldsymbol{\alpha},r)}). \tag{3.3.3}$$

Let $(\overline{x}_i)_{1 \leqslant i \leqslant m_{(\boldsymbol{\alpha},r)}} \in X^{m_{(\boldsymbol{\alpha},r)}}$ be a maximal argument of the function $\Psi_{m_{(\boldsymbol{\alpha},r)}}$. That is

$$\mu\left(\bigcup_{i=1}^{m_{(\boldsymbol{\alpha},r)}} B(\overline{x}_i, r)\right) = \max_{(x_i)_{i=1}^{m_{(\boldsymbol{\alpha},r)}} \in X^{m_{(\boldsymbol{\alpha},r)}}} \Psi_{m_{(\boldsymbol{\alpha},r)}}(x_1, \ldots, x_{m_{(\boldsymbol{\alpha},r)}}) = \Psi(m_{(\boldsymbol{\alpha},r)}).$$

We set

$$U := \bigcup_{i=1}^{m_{(\boldsymbol{\alpha},r)}} B(\overline{x}_i, r), \qquad V := \bigcup_{i=1}^{m_{(\boldsymbol{\alpha},r)}} B(\overline{x}_i, 4r) \qquad \text{and}$$

$$A = U \cap Y.$$

Since μ is supported in $\overline{Y}$ and $\mu(\overline{Y} \backslash Y) = 0$, one easily see that

$$\mu(A) = \mu(U) = \Psi(m_{(\boldsymbol{\alpha},r)}) > \boldsymbol{\alpha}.$$

Now, since $A \subset U$, we have $A^{4r} \subset \{x \in X : d(U,x) < 4r\} \subset V$ by construction. Notice that V can be covered by $m_{(\boldsymbol{\alpha},r)}$ times N balls of radius r and $m_{(\boldsymbol{\alpha},r)} \leqslant 2(m_{(\boldsymbol{\alpha},r)} - 1)$. Then, one can cover V with $2N(m_{(\boldsymbol{\alpha},r)} - 1)$ balls of radius r. Every $(m_{(\boldsymbol{\alpha},r)} - 1)$-union of balls of radius r measures at most $\Psi(m_{(\boldsymbol{\alpha},r)} - 1)$. Hence $\mu(A^{4r}) \leqslant \mu(V) \leqslant 2N\Psi(m_{(\boldsymbol{\alpha},r)} - 1) \leqslant 2N\boldsymbol{\alpha}$. ■

Lemma 3.3.2. *Let the assumptions of Lemma 3.3.1 hold with $0 < r < \frac{1}{4}$ and $\boldsymbol{\alpha} = \frac{\mu(X)}{2NK}$ for some integer $K \in \mathbb{N}$. Then, there exist K measurable subsets $\{A_j\}_{j=1}^K$ of Y, such that for all $j \in \{1, \ldots, K\}$,*

(a) $\boldsymbol{\alpha} < \mu(A_j) \leqslant \mu(A_j^{4r}) \leqslant 2N\boldsymbol{\alpha}$,

(b) $\operatorname{dist}(x, Y) \leqslant 2r,\ \forall\ x \in A_j^r$,

(c) $A_i^{2r} \cap A_j^{2r} = \emptyset$ *for every* $i \neq j \in \{1, \dots, K\}$.

Remark 3.3.1. Lemma 3.1.2 still holds with the family of CM(adapted)-capacitors given by Lemma 3.3.2. The benefit of this approach is that, the proof of Proposition 3.2.1 hence yields a new $\mathrm{Vol}(A_j^r \cap \Omega)$ upper bound. This system of capacitors is chosen because it is more feasible to bound their volumes by estimating the volume of a well chosen tube about the boundary Γ.

Lemma 3.3.3. *Let* (X, d, μ) *be as we described above and* $N \in \mathbb{N}$ *such that* (X, d, μ) *satisfies the* $(N, 4; 1)$*-covering property. Let* $0 < r_0 \leqslant \frac{1}{10}$ *be fixed, then for every* $K \in \mathbb{N}$*, at least one of the following cases holds:*

1. *X contains a family of K spherical capacitors* $\{(A_j, B_j)\}_{j=1}^K$ *such that*
   - $A_j = B(x_j, r_j)$ *and* $\mu(A_j) \geqslant \alpha$*, with* $x_j \in X$*,* $r_j \in (0, 2r_0]$*,*
   - $B_j = B(x_j, 2r_j)$*.*
2. *X contains a family of K CM(adapted)-capacitors* $\{(A_j, A_j^{\tilde{r}_0})\}_{j=1}^K$ *such that*
   - $\mathrm{dist}(x, Y) \leqslant 2\tilde{r}_0$ *for all* $x \in A_j^{\tilde{r}_0}$ *and* $j \in \{1, \dots, K\}$*, where*

$$\tilde{r}_0 = \min\{r_0, \tau_1\} \quad \text{and} \quad \tau_1 := \sup\{r \in \mathbb{R}_{>0} : \mu(B(x, r)) \leqslant \alpha,\ \forall\, x \in X\}. \tag{3.3.4}$$

The proof is exactly the same provided for Lemma 3.1.2, Lemma 3.3.2 is applied in state of Lemma 3.1.1.

Proof of Lemma 3.3.2. We shall construct recursively k subsets $A_1, \dots, A_K$ of Y, such that for all j in $\{1, \dots, K\}$, one has:

(i) $\boldsymbol{\alpha} < \mu(A_j) \leqslant \mu(A_j^{4r_j}) \leqslant 2N\boldsymbol{\alpha}$,

(ii) $X_j = X \bigcap \left(\bigcup_{i=1}^{j-1} A_i^{4r}\right)^c = X_{j-1} \cap (A_{j-1}^{4r})^c$. (By convention we set $X_0 = X$ and $A_0^{3r} = \emptyset$.)

(iii) $A_j^{2r} \subset X \bigcap \left(\bigcup_{i=1}^{j-1} A_i^{2r}\right)^c$.

So, (b) is satisfied since $A_j \subset Y$ and (c) follows from (iii).

First, we claim that,

$$\mu(X) - (K-1)2N\boldsymbol{\alpha} > \boldsymbol{\alpha}.$$

Indeed, one has $\boldsymbol{\alpha} = \frac{\mu(X)}{2NK} < \frac{\mu(X)}{K} = \mu(X) - (K-1)\frac{\mu(X)}{K} = \mu(X) - (K-1)2N\boldsymbol{\alpha}$. This will be useful in each step of the iteration, since from (i) and (iii), one has then

$$\mu(X_j) \geqslant \mu(X) - (j-1)2N\boldsymbol{\alpha} > \boldsymbol{\alpha}, \quad \forall 1 \leqslant j \leqslant K.$$

$j = 1$ Consider the a metric measure space $(X_1, d, \mu) := (X, d, \mu)$, applying Lemma 3.3.1, one has $\emptyset \neq A_1 \subset Y_1 := Y$ such that:

$$\boldsymbol{\alpha} < \mu(A_1) \leqslant \mu(A_1^{4r}) \leqslant 2N)\boldsymbol{\alpha}.$$

$j=2$ Set $Y_2 = Y_1 \backslash (A_1^{4r})$ and $X_2 = X_1 \backslash (A_1^{4r})$.

$$\mu(X_2) = \mu(X_1 \cap (A_1^{4r})^c) \geqslant \mu(X_1) - \mu(A_1^{4r}) \geqslant \mu(X) - 2N\boldsymbol{\alpha} > \boldsymbol{\alpha}.$$

We consider the a metric measure space $(X_2, d, \mu_{|X_2})$. The restricted measure $\mu_{|X_2}$ is supported in Y_2 and $\mu_{|X_2}(X_2) = \mu(X_2) > \boldsymbol{\alpha}$. Then, from Lemma 3.3.1, one has $\emptyset \neq A_2 \subset Y_2$ such that:

$$\boldsymbol{\alpha} < \mu(A_2) \leqslant \mu(A_2^{4r}) \leqslant 2N_r\boldsymbol{\alpha}$$

and

$$A_2^{2r} \subset X \bigcap \left(\bigcup_{i=1}^{j-1} A_2^{2r} \right)^c.$$

$2 < j \leqslant K$ Assume that we have all ready constructed $A_1, \ldots, A_{j-1}$ satisfying induction hypotheses. We set $Y_j = Y_{j-1} \cap (A_{j-1}^{4r})^c$ and $X_j = X_{j-1} \cap (A_{j-1}^{4r})^c$. Then

$$\mu(X_j) = \mu \left(X \bigcap (\bigcup_{i=1}^{j-1} A_i^{4r})^c \right) \geqslant \mu(X) - 2(j-1)N\boldsymbol{\alpha} > \boldsymbol{\alpha}.$$

Applying Lemma 3.3.1 to the metric measures space $(X_j, d, \mu_{|X_j})$, We get $A_j \subset Y_j$ satisfying:

$$\boldsymbol{\alpha} < \mu(A_j) \leqslant \mu(A_j^{4r}) \leqslant 2N\boldsymbol{\alpha},$$

and

$$A_j^{2r} \subset X \bigcap \left(\bigcup_{i=1}^{j-1} A_i^{2r} \right)^c.$$

■

Having discussed how we construct the capacitors, the next section gives an estimation of the volume of a tube about a hypersurface in a Riemannian manifold.

3.3.2 Tubes about hypersurfaces

In this section we recall some facts about estimation of the volume of a tube about a submanifold of a Riemannian manifold, especially with non negative Ricci curvature. We use them in the next one to deduce the result we aim to prove.

Studying how the volume of a tubular neighbourhood of a submanifold depends on metric invariants of that submanifold has generated considerable interest because of its application in certain statistical problems.

Hermann Weyl [74], in 1939, has found a formula for the volume of a tube of (small) radius r around a submanifold P of dimension q in the n dimensional Euclidean space $\mathbb{R}^n$:

$$V_P^{\mathbb{R}^n} = \frac{(\pi r^2)^{\frac{(n-q)}{2}}}{(\frac{1}{2}n-q)!} \sum_{i}^{\lceil \frac{q}{2} \rceil} \frac{k_{2i}(P) r^{2i}}{(n-q+2)(n-q+4)\cdots((n-q+2i)}, \tag{3.3.5}$$

where the first coefficient $k_2(P)$ is just the volume of P and the coefficients $k_{2i}(P)$, $i > 1$ are functions. The most striking aspect of this formula is the polynomial behaviour of the volume function in terms of the radius r. This responded to a question posed by Harold Hotelling [39] who had proved that, in the n-dimensional Euclidean space, the volume of a tube of small radius about a curve depends only on the radius and the length of the curve. For details, please refer to [30].

Let M be a Riemannian manifold and P be a topologically embedded submanifold in M.

Definition 3.3.1. Formally, a tube $T(P,r)$ of radius $r \geqslant 0$ about P is a set

$$T(P,r) = \{x \in M \mid \text{ there exists a geodesic } \epsilon \text{ of length } L(\epsilon) \leqslant r \text{ from } x \text{ meeting } P \text{ orthogonally } \}. \quad (3.3.6)$$

It is important to stress that if P has a boundary, then

$$T(P,r) \neq \{x \in M \mid \operatorname{dist}(x,P) \leqslant r\}. \quad (3.3.7)$$

Indeed, the set on the right-hand side has additional end caps.

This definition of a tube makes sense for any topologically embedded submanifold of any Riemannian manifold. The set $T(P,r)$ is measurable, therefore it has volume which we denote by $V_P^M(r)$.

Let $\nu := \{(p,v) \mid p \in P \text{ and } v \in P_p^{\perp}\}$. denotes the normal bundle of P in M. Here $P_p^{\perp}$ is the orthogonal complement of P_p in M_p. Indeed, ν is a sub-bundle of the restriction to P of the tangent bundle of M. The exponential map of ν (the normal exponential map of P) is given for every $(p,v) \in \nu$ by

$$\exp_\nu : \nu \ni (p,v) \longrightarrow \exp_\nu = \exp_p(v) \in M,$$

where $\exp_p$ denotes the exponential map of M at p.

It is known that $\exp_\nu$ is defined and non singular at least in a small neighbourhood of the zero section of ν. Then one defines focal and cut focal points of the submanifold P as follows:

Definition 3.3.2. Focal points of P are every points m in the manifold M such that the exponential map $\exp_\nu$ of the normal bundle ν of P is singular somewhere on $\exp_\nu^{-1}(m)$.

The function e_f, defined by

$$e_f : \{(p,v) \mid p \in P, v \in P_p^{\perp}, \|v\| = 1\} \longrightarrow \mathbb{R}$$
$$e_f(p,v) := \inf\{t > 0 \mid \operatorname{kernel}(((\exp_\nu)_*)_{(p,tv)}) \neq 0\},$$

gives then, for every $(p,v) \in \nu$, the distance from p to its first focal point along the geodesic $t \longrightarrow \exp_\nu(p,tv)$. Equivalently, $e_f(p,v)$ is the distance from p to its nearest focal point in the direction v.

Definition 3.3.3. Let m be a point on a geodesic ξ meeting P orthogonally. The point m is a cut-focal point along the geodesic ξ, if the distance from m to P is no longer minimized along ξ after m.

That is to say, m is the first point on ξ with at least one other geodesic that meets P orthogonally and minimizing the distance from m to P.

Let e_c denotes the function defined by

$$e_c : \{(p,v) \mid p \in P, v \in P_p^{\perp}, \|v\| = 1\} \longrightarrow \mathbb{R}$$
$$e_c(p,v) := \sup\{t > 0 \mid \mathrm{dist}(\exp_\nu(p,tv),P) = t\}.$$

Then $e_c : (p,v)$, when finite, gives the distance from p to its cut-focal point in the direction v.

Remark 3.3.2. For all $p \in P$ and $v \in P_p^{\perp}$, the following inequality about $e_c(p,u)$ and $e_f(p,u)$ is well known.

$$e_c(p,u) \leqslant e_f(p,u). \tag{3.3.8}$$

Let O_P denotes the largest neighbourhood of the zero section of ν for which $\exp_\nu : O_P \longrightarrow \exp_\nu(O_p)$ is a diffeomorphism:

$$O_P = \{(p,tv) \in \nu \mid \|u\| = 1 \text{ and } 0 \leqslant t < e_c(p,u)\}.$$

The boundary of $\exp_\nu(O_P)$ is the set of cut-focal points.

If a tube satisfies $T(P,r) \subseteq \exp_\nu(O_P)$, then $\exp_\nu$ maps a tube of radius r about the zero section in ν diffeomorphically onto $T(P,r)$. Otherwise, $\exp_\nu$ only maps a subset of $\{(p,v)\nu v \mid \|v\| \leqslant r\}$ diffeomorphically onto $T(P,r)$. We therefore need to define a minimal focal distance:

Definition 3.3.4. The minimal focal distance of P in M is

$$\mathrm{minfoc}(P) := \inf\{e_c(p,v) \mid (p,u) \in \nu, \|v\| = 1\}.$$

In the case that P is a point, the minimal focal distance coincides with the injectivity radius. If P is a hypersurface bounding a domain Ω in M then the minimal focal distance of P, can be related to the rolling radius of Ω. For instance, if P is the boundary of a strictly convex domain in Euclidean space, then the minimal focal distance is equal to the rolling radius $\mathrm{roll}(\Omega)$ (Blaschke's rolling theorem). In [40], are specified few other conditions that imply $\mathrm{roll}(\Omega) = \mathrm{minfoc}(P)$, see also references therein.

Lemma 3.3.4. *Assuming that the manifold M is complete, for all*

$$0 < r \leqslant \mathrm{minfoc}(P),$$

a tube of radius r about the zero section in ν, is mapped diffeomorphically onto $T(P,r)$ by $\exp_\nu$.

The formula for hypersurfaces that we will use is special case of formula derived for general submanifolds. Consider a complete n-dimensional Riemannian manifold M and an embedded hypersurface P with compact closure. Let I denotes the second fundamental form of the immersion of P in M.

The mean curvature vector field H of P is defined, for any local orthonormal frame $\{E_l, \ldots, E_{n-1}\}$ on P, by:

$$\overline{H} = \sum_{i=1}^{n-1} I_{E_i} E_i.$$

The mean curvature in the direction $v \in P_p^\perp$ is then given by $(\overline{H}(p), v)$. Considering an orientable hypersurface, there are two unit normal vectors $\pm\mathbf{n} \in P_p^\perp$. Then one fixes $\mathbf{n}$ and $H := (\overline{H}(p), \mathbf{n})$ is the mean curvature.

We have the following tube volume estimate involving Ricci Curvature. In fact, lower bounds on Ricci curvature yield upper bounds on tube volumes:

Lemma 3.3.5. *Let M be a complete n-dimensional Riemannian manifold with non-negative Ricci curvature. Let P be an embedded hypersurface with compact closure in M. Then for all*

$$0 < r \leqslant \operatorname{minfoc}(P),$$

one has:

$$V_P^M(r) \leqslant \frac{2r}{n} \sum_{i=0}^{\lceil \frac{n-1}{2} \rceil} \binom{n}{2i+1} \left(\frac{r}{n-1}\right)^{2i} \int_p H^{2i} dP. \tag{3.3.9}$$

A proof of Lemma 3.3.5 can be found in [30, Corollary 10.24].

3.3.3 Application

We use estimation of the volume of a tube about a submanifold of a Riemannian manifold to improve the sharpness of our results proved in Section 3.2, especially when the Ricci curvature is positive. Proposition 3.2.1 is refined as follows.

Theorem 3.3.6. *Let (M, g) be a complete n-dimensional Riemannian manifold with non-negative Ricci curvature. Let $\tilde{C} \geqslant 1$ and $\Omega \subset M$ be a domain with a smooth boundary Γ such that, for all $x \in \Gamma$ and every $0 < r < 1$, one has:*

$$\operatorname{Vol}(B(x,r) \cap \Gamma) \leqslant \tilde{C} r^{n-1}.$$

Let $\overline{h} \in \mathbb{R}$, such that the minimal focal distance of Γ satisfies:

$$\operatorname{minfoc}(\Gamma) \geqslant \overline{h} > 0$$

and that the mean curvature on the boundary Γ satisfies $\kappa_- \leqslant H \leqslant \kappa_+$, $\kappa_-, \kappa_+ \in \mathbb{R}$. Let $\overline{\kappa}_+ := \max\{1, |\kappa_-|, |\kappa_+|\}$, for every $k \in \mathbb{N}$ one has:

$$\lambda_{W,k}^{\beta}(\Omega) \leqslant a_n \overline{\kappa}_+^{2n} \left[\left(\frac{\tilde{C}k}{\operatorname{Vol}(\Gamma)} \right)^{\frac{1}{n-1}} + \frac{1}{\left(\min\{1, \overline{h}\}\right)^2} \right] + \beta b_n \left[\left(\frac{\tilde{C}k}{\operatorname{Vol}(\Gamma)} \right)^{\frac{2}{n-1}} + \frac{1}{\left(\min\{1, \overline{h}\}\right)^2} \right], \tag{3.3.10}$$

where a_n and b_n depend only on the dimension n.

One important consequence of the above result is what happens in the limiting cases :

Corollary 3.3.7. *Under the assumptions of Theorem, for every $k \in \mathbb{N}$ we have:*

1.

$$\lambda_{S,k}\Omega) \leqslant a_n \overline{\kappa}_+^{2n} \left[\left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)} \right)^{\frac{1}{n-1}} + \frac{1}{\left(\min\{1, \overline{h}\}\right)^2} \right]. \tag{3.3.11}$$

2.

$$\eta_k(\Gamma) = \lim_{\beta\to\infty} \frac{\lambda^{\beta}_{W,k}(\Omega)}{\beta} = b_n \left[\left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)} \right)^{\frac{2}{n-1}} + \frac{1}{\left(\min\{1, \overline{h}\}\right)^2} \right], \quad \forall\, k \in \mathbb{N}. \tag{3.3.12}$$

Proof. We set $K := 4k$, $r_0 := \frac{1}{10}\min\{1,\overline{h}\}$ and $\tilde{r}_0 := \min\{r_0, \tau_1\}$ and apply Lemma 3.3.3 the metric measure space (M, d, μ), where d is the distance from the metric g and μ is the Borel measure with support Γ defined for each Borelian A of M by

$$\mu(A) := \int_{A\cap\Gamma} \mathrm{d}_\Gamma.$$

We choose the same test functions as for the proof of Proposition 3.2.1.

In the first case, we have a family of $4k$ spherical capacitors $\{(A_j, B_j)\}_{j=1}^{4k}$ with $A_j = B(x_j, r_j)$ and $B_j = B(x_j, 2r_j)$, with $x_j \in M$, $r_j \in (0, 2r_0]$ for every $j \in \{1,\dots,4k\}$.

Since $\mu(A_j) \geqslant \alpha > 0$, we have $A_j \cap \Gamma \neq \emptyset$ and $B_j \subset B(y_j, 4r_j)$ with $y_j \in \Gamma$. Therefore, $\bigcup_{j=1}^{4k} B_j \subseteq T(\Gamma, 4r_j) \subseteq T(\Gamma, 8r_0) \subset T(\Gamma, \overline{h})$. Since $B_i \cap B_j = \emptyset$ for every $i \neq j\{1,\dots,4k\}$, half of the B_j's satisfy:

$$\mathrm{Vol}(\Gamma \cap B_j) \leqslant \frac{\mathrm{Vol}(\Gamma)}{k}. \tag{3.3.13}$$

In addition, since $\bigcup_{j=1}^{4k} B_j \subseteq T(\Gamma, 4r_j) \subset T(\Gamma, \overline{h})$ and they are disjoint, at least for k of those $2k$ B_j's satisfying (3.3.13) we have:

$$\mathrm{Vol}(\Omega \cap B_j) \leqslant \frac{V_\Gamma^M(4r_j)}{k} \leqslant \frac{8r_j}{nk} \overline{\kappa}_+^{2n} \mathrm{Vol}(\Gamma) \left[\sum_{i=0}^{\lceil \frac{n-1}{2} \rceil} \binom{n}{2i+1} \left(\frac{1}{n-1} \right)^{2i} \right].$$

Set $C(n) := \sum_{i=0}^{\lceil \frac{n-1}{2} \rceil} \binom{n}{2i+1} \left(\frac{1}{n-1} \right)^{2i}$, we have

$$\begin{aligned} \int_{\Omega\cap B_j} |\nabla f_j|^2 \mathrm{d}_M &\leqslant \left(\int_{\Omega\cap B_j} |\nabla f_j|^{2(n-1)} \mathrm{d}_M \right)^{\frac{1}{n-1}} \left(\int_{\Omega\cap B_j} \mathrm{d}_M \right)^{1-\frac{1}{n-1}} \\ &\leqslant \left(\frac{1}{r_j^{2(n-1)}} \int_{\Omega\cap B_j} \mathrm{d}_M \right)^{\frac{1}{n-1}} \left(\int_{\Omega\cap B_j} \mathrm{d}_M \right)^{1-\frac{1}{n-1}}. \end{aligned}$$

$$\text{Then, } \int_{\Omega\cap B_j} |\nabla f_j|^2 \mathrm{d}_M \leqslant \left(\frac{1}{r_j^{2(n-1)}}\mathrm{Vol}(B_j)\right)^{\frac{1}{n-1}} \Big(\mathrm{Vol}(\Omega\cap B_j)\Big)^{1-\frac{1}{n-1}}$$
$$\leqslant \left(\frac{1}{r_j^{2(n-1)}}\omega_n 2^n r_j^n\right)^{\frac{1}{n-1}} \left(\frac{8r_j}{nk}\overline{\kappa}_+^{2n}\mathrm{Vol}(\Gamma)\left[\sum_{i=0}^{\lceil\frac{n-1}{2}\rceil}\binom{n}{2i+1}\left(\frac{1}{n-1}\right)^{2i}\right]\right)^{1-\frac{1}{n-1}}$$
$$\leqslant (\omega_n 2^n)^{\frac{1}{n-1}}\left(\frac{8}{n}C(n)\overline{\kappa}_+^{2n}\frac{\mathrm{Vol}(\Gamma)}{k}\right)^{1-\frac{1}{n-1}}.$$

It follows that

$$R_\beta(f_j) \leqslant \frac{16N^2k}{\mathrm{Vol}(\Gamma)}\left[c(n)^{\frac{2}{n}}\overline{\kappa}_+^{\frac{2n(n-2)}{n-1}}\left(\frac{\mathrm{Vol}(\Gamma)}{k}\right)^{1-\frac{1}{n-1}} + 4\beta\tilde{C}^{\frac{2}{n-1}}\left(\frac{\mathrm{Vol}(\Gamma)}{k}\right)^{1-\frac{2}{n-1}}\right]$$
$$\leqslant A(n)\left[\overline{\kappa}_+^{2n}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{1}{n-1}} + \beta\left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}\right]$$
$$\leqslant A(n)\left[\overline{\kappa}_+^{2n}\left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{1}{n-1}} + \beta\left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}\right], \tag{3.3.14}$$

where $c(n)$ and $A(n)$ is a constant depending only on the dimension n.

In the second case, for each $1 \leqslant j \leqslant 4k$, we still consider the function φ_j supported on $A_j^{\tilde{r}_0}$ defined by

$$\varphi_j(x) := \begin{cases} 1 - \frac{d(A_j,x)}{\tilde{r}_0} & \forall x \in A_j^{\tilde{r}_0}, \\ 0 & \forall x \in M\backslash A_j^{\tilde{r}_0}, \end{cases}$$

We have $R_\beta(\varphi_j) \leqslant \dfrac{\int_{\Omega\cap A_j^{\tilde{r}_0}} |\nabla\varphi_j|^2 \mathrm{d}_M + \beta\int_{\Gamma\cap A_j^{\tilde{r}_0}} |\nabla\varphi_j|^2 \mathrm{d}_\Gamma}{\int_{\Gamma\cap A_j} {\varphi_i}^2 \mathrm{d}_\Gamma}$ and for the first $2k$ of the $A_j^{\tilde{r}_0}$'s, we can assume that

$$\mathrm{Vol}(\Gamma\cap A_j^{\tilde{r}_0}) \leqslant \frac{\mathrm{Vol}(\Gamma)}{k}. \tag{3.3.15}$$

Since $A_j^{\tilde{r}_0} \subset T(\Gamma, \tilde{r}_0)$ for every $i \in \{1,\ldots,2k\}$ and they are one-to-one disjoint, after rearrangement, for the first k of the $A_j^{\tilde{r}_0}$'s, in addition to 3.3.15 we have:

$$\mathrm{Vol}(\Omega\cap A_j^{\tilde{r}_0}) \leqslant \frac{V_\Gamma^M(\tilde{r}_0)}{k}.$$

Applying Lemma 3.3.5, we have

$$\mathrm{Vol}(\Omega\cap A_j^{\tilde{r}_0}) \leqslant \frac{2\tilde{r}_0}{nk}\overline{\kappa}_+^{2n}\mathrm{Vol}(\Gamma)\left[\sum_{i=0}^{\lceil\frac{n-1}{2}\rceil}\binom{n}{2i+1}\left(\frac{1}{n-1}\right)^{2i}\right].$$

Since for every $x \in A_j$, $\varphi_j(x) = 1$, one has:

$$\int_{\Gamma\cap A_j} {\varphi_j}^2 \mathrm{d}_\Gamma \geqslant \int_{\Gamma\cap A_j} \mathrm{d}_\Gamma \geqslant \mu(A_j) \geqslant \frac{\mathrm{Vol}(\Gamma)}{8Nk},$$

and

$$R_\beta(\varphi_j) \leqslant \frac{8Nk}{\mathrm{Vol}(\Gamma)}\left[c(n)\frac{1}{\tilde{r}_0^2}\frac{\tilde{r}_0\overline{\kappa}_+^{2n}\mathrm{Vol}(\Gamma)}{k} + \beta\frac{1}{\tilde{r}_0^2}\frac{\mathrm{Vol}(\Gamma)}{k}\right] = 8N\left[\frac{c(n)\overline{\kappa}_+^{2n}}{\tilde{r}_0} + \beta\frac{1}{\tilde{r}_0^2}\right],$$

$c(n)$ being a dimensional constant.

Hence, if $\tilde{r}_0 = r_0$, then

$$R_\beta(\varphi_j) \leqslant \frac{C(n)}{\left(\min\{1,\ \overline{h}\}\right)^2}\left[\overline{\kappa}_+^{2n} + \beta\right], \quad \text{where } C(n) := 2^{5n+10}c(n). \tag{3.3.16}$$

Otherwise, $\tilde{r}_0 = \tau_1$ and there exists $x \in M$ such that $\mu(B(x, 2\tilde{r}_0)) > \alpha$. Take $y \in B(x, 2\tilde{r}_0) \cap \Gamma$, since $B(y, 4\tilde{r}_0) \supset B(x, 2\tilde{r}_0)$, we have $\tilde{C}(4\tilde{r}_0)^{n-1} \geqslant \mu(B(y, 4\tilde{r}_0)) > \alpha$. Consequently, $1 \leqslant \frac{1}{\tilde{r}_0} \leqslant 2\left(\frac{16\tilde{C}kN^2}{\mathrm{Vol}(\Gamma)}\right)^{\frac{1}{n-1}}$ and we have

$$R_\beta(\varphi_j) \leqslant C(n)\left[\overline{\kappa}_+^{2n}\left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{1}{n-1}} + \beta\left(\frac{\tilde{C}k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}\right], \tag{3.3.17}$$

with $C(n)$ depending on n. Without loss of generality, one can assume that $A(n) \geqslant C(n)$. Hence from (3.3.14), (3.3.16) and (3.3.17), we conclude the argument by applying the min-max characterization of $\lambda_{W,k}^{\beta}(\Omega)$. ■

4 Isoperimetric bounds

In this chapter, are presented asymptotically optimal estimates, according to the Weyl law, through bounds that are given in terms of the isoperimetric ratio of the domain. These results show in particular that the isoperimetric ratio allows to control the entire spectrum of the Wentzel-Laplace operator in various ambient spaces.

Assumption based on the isoperimetric ratio is less demanding than the hypothesis concerning the existence of $\tilde{C}$ in the previous chapter formulated in (3.0.2). Here, we only requires that the boundary is not concentrated everywhere.

Our first result provides an upper bound in the case of Euclidean domains. We respectively denote by ω_n and $\rho_{n-1} = n\omega_n$ the volumes of the unit ball and the unit sphere in the n-dimensional Euclidean space.

Theorem 4.0.1. *Let $n \geqslant 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded euclidean domain with smooth boundary Γ. Then, for every $k \geqslant 2$, one has*

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant \zeta_1(n)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}} + \zeta_2(n) I(\Omega)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}, \quad (4.0.1)$$

where $\zeta_1(n) := 2^{10(n+1)}\omega_n^{\frac{2}{n}}$ and $\zeta_2(n) := \frac{2^{10(n+3)}}{n}\omega_n^{\frac{1}{n}}$.

With regard to the inequality dimension $n \geqslant 3$, note that the assumption we made is necessary only to ensure that the inequality (4.1.23) holds, applying (4.1.22). We do not know if an inequality similar to (4.0.1) is valid in the two-dimensional case.

The results in this chapter, for the limit cases, are to be compared with (1.3.1) and (1.3.2).

Corollary 4.0.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded euclidean domain of dimension $n \geqslant 3$ with smooth boundary Γ. Then, for every $k \geqslant 2$, we have*

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant C_1(\Omega,\beta) + C_2(\Omega,\beta)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}. \quad (4.0.2)$$

Here $C_1(\Omega,\beta)$ and $C_2(\Omega,\beta)$ are geometric constants given by:

$$C_2(\Omega,\beta) = \zeta_2(n) I(\Omega)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right] + 1$$

$$C_1(\Omega,\beta) = \zeta_1^n(n)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{n-2} C_2(\Omega,\beta).$$

The constants $\zeta_1(n)$ and $\zeta_2(n)$ are the same as in Theorem 4.0.1.

For bounded domains in Riemannian manifold with Ricci curvature bounded from below, we have an isoperimetric upper bound, which also depends on the infimum isoperimetric ratio that we define as follows:

Definition 4.0.1. Let (M,g) be a complete Riemannian manifold of dimension $n \geqslant 2$ and Ω a bounded domain in M. The infimum isoperimetric ratio of Ω is the quantity $I_0(\Omega) := \inf\{I(U) : U$ open set in $\Omega\}$; we recall that $I(U) := \frac{\mathrm{Vol}(\partial U)}{\mathrm{Vol}(U)^{\frac{n-1}{n}}}$ where ∂U denotes the boundary of U. In particular, if Ω is an Euclidean domain, one has $I_0(\Omega) = I_0(\mathbb{R}^n) = n\omega_n^{\frac{1}{n}}$.

Theorem 4.0.3. *Let (M,g) be a complete Riemannian manifold of dimension $n \geqslant 3$ with non-negative Ricci curvature. If $\Omega \subset M$ is a bounded domain with smooth boundary Γ, then for every $k \in \mathbb{N}$ the following inequality holds:*

$$\lambda_{W,k}^{\beta}(\Omega) \leqslant \zeta_1(n)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}} + \zeta_2(n)\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}, \quad (4.0.3)$$

where $\zeta_1(n) := 2^{10(n+1)}\omega_n^{\frac{2}{n}}$ and $\zeta_2(n) := 2^{5(n+5)}\rho_{n-1}^{\frac{2}{n-1}}$.

Theorem 4.0.4. *Let (M,g) be a complete Riemannian manifold of dimension $n \geqslant 3$ with Ricci curvature bounded from below by $-(n-1)\kappa^2$, $\kappa \in \mathbb{R}_{>0}$. Let $\Omega \subset M$ be a bounded domain with smooth boundary Γ. Then for every $k \geqslant 2$, we have:*

$$\lambda_{W,k}^{\beta}(\Omega) \leqslant A(\Omega,\beta) + B(\Omega,\beta)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}, \quad (4.0.4)$$

The constants are given by:

$$A(\Omega,\beta) = \overline{\kappa}^2\zeta(n)\left\{1 + \left(\overline{\kappa}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}} + \left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\overline{\kappa}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]\right\},$$

$$B(\Omega,\beta) = \zeta(n)\left\{1 + \left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\overline{\kappa}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]\right\},$$

$\zeta(n)$ being a constant depending only on the dimension n and $\overline{\kappa} := \max\{\kappa, 1\}$.

Theorems 4.0.3 and 4.0.4 emanate from a technical result (Theorem 4.1.5) that we prove in Section 4.1.

This leads to the following corollary.

Corollary 4.0.5. *Let (M,g) be a smooth Cartan-Hadamard manifold of dimension $n \geqslant 3$ with Ricci curvature bounded from below by $-(n-1)\kappa^2$, $\kappa \in \mathbb{R}_{>0}$ and $\Omega \subset M$ a bounded* *domain with smooth boundary* Γ. *Then for every* $k \geqslant 2$, *we have*

$$\lambda_{W,k}^{\beta}(\Omega) \leqslant A(\Omega,\beta) + B(\Omega,\beta)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}, \tag{4.0.5}$$

with

$$A(\Omega,\beta) = \overline{\kappa}^2\zeta(n)\left\{1 + \left(\overline{\kappa}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}} + I(\Omega)^{1+\frac{2}{n-1}}\left[\overline{\kappa}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]\right\},$$
$$B(\Omega,\beta) = \zeta(n)\left\{1 + I(\Omega)^{1+\frac{2}{n-1}}\left[\overline{\kappa}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]\right\},$$

$\zeta(n)$ *being a constant depending on the dimension* n *and* $\overline{\kappa} := \max\{\kappa, 1\}$.

4.1 General inequality

In this section, we establish some needed technical results and the major result in this chapter used to prove our main theorems. Let $n \geqslant 2$ and (M,g) be a complete n-dimensional Riemannian manifold. Let $\Omega \subset M$ a bounded domain with smooth boundary Γ. Let $r \in \mathbb{R}_{>0}$, we denote by $B(x,r) = \{p \in M, d(x,p) < r\}$ the metric ball of radius r centred at $x \in M$, where d is the Riemannian distance associated to the metric g. We assume Γ satisfies the following hypothesis:

(H_0): There exists a radius $r_-(\Gamma) > 0$ and a constant $C \in \mathbb{N}_{>1}$ such that for all $x \in \Gamma$ and $r < r_-(\Gamma)$, one has

$$\mathrm{Vol}(B(x,r)) < C\omega_n r^n \quad \text{and} \quad \mathrm{Vol}(\partial B(x,r)) < C\rho_{n-1} r^{n-1}. \tag{4.1.1}$$

Here $\partial B(x,r)$ denotes the geodesic sphere of radius r centred at x.

Lemma 4.1.1. *Let* (M,g), Ω *and* Γ *be as above. For every* $K \in \mathbb{N}$, *let* r_K *be an associated "maximal" radius defined by*

$$r_K := \left(\frac{\mathrm{Vol}(\Omega)}{2}\right)^{\frac{1}{n}}\left(\frac{I_0(\Omega)}{KC\rho_{n-1}}\right)^{\frac{1}{n-1}}. \tag{4.1.2}$$

Let $\{x_j\}_{j=1}^K$ *be an arbitrary set of points in* Γ. *Then for every* $r > 0$ *satisfying both* $r < \frac{1}{2}r_-(\Gamma)$ *and* $r \leqslant \frac{1}{2}r_K$, *one has*

$$\mathrm{Vol}\left(\Gamma \setminus \bigcup_{j=1}^{K} B(x_j, 2r)\right) > 0. \tag{4.1.3}$$

Proof. We denote by Ω_0 (respectively Γ_0) the subset $\Omega \setminus \bigcup_{j=1}^K B(x_j,2r)$ (respectively $\Gamma \setminus \bigcup_{j=1}^K B(x_j,2r)$). One can think of $\overline{\Omega}_0$ as a holed cheese.

It is straightforward to check that $\overline{\Omega}_0$ is non-empty. Indeed, we have,

$$\mathrm{Vol}(\Omega_0) = \mathrm{Vol}(\Omega\backslash\bigcup_{j=1}^{K} B(x_j, 2r)) \geqslant \mathrm{Vol}(\Omega) - \mathrm{Vol}\left(\bigcup_{j=1}^{K} B(x_j, 2r)\right) \geqslant \mathrm{Vol}(\Omega) - \sum_{j=1}^{K} \mathrm{Vol}\Big(B(x_j, 2r)\Big).$$

$$\begin{aligned}\mathrm{Vol}\Big(B(x_j, 2r)\Big) < C\omega_n r_K^n = C\omega_n \left(\frac{\mathrm{Vol}(\Omega)}{2}\right)\left(\frac{I_0(\Omega)}{KC\rho_{n-1}}\right)^{\frac{n}{n-1}} \\ \leqslant C\omega_n \left(\frac{\mathrm{Vol}(\Omega)}{2}\right)\left(\frac{I_0(\mathbb{R}^n)}{KC\rho_{n-1}}\right)^{\frac{n}{n-1}} = C\left(\frac{\mathrm{Vol}(\Omega)}{2}\right)\frac{1}{(KC)^{\frac{n}{n-1}}}.\end{aligned}$$

$$\mathrm{Vol}(\Omega_0) \geqslant \mathrm{Vol}(\Omega) - \frac{KC}{(KC\rho_{n-1})^{\frac{n}{n-1}}}\left(\frac{\mathrm{Vol}(\Omega)}{2}\right) > \left(\frac{\mathrm{Vol}(\Omega)}{2}\right). \tag{4.1.4}$$

Since the boundary of Ω_0, that we denote by $\partial\Omega_0$, is contained in the union $\Gamma_0 \bigcup\Big(\bigcup_{j=1}^{K} \mathrm{Vol}(\partial B(x_j, 2r))\Big)$, one has:

$$\begin{aligned}\mathrm{Vol}(\Gamma_0) &\geqslant \mathrm{Vol}(\partial\Omega_0) - \sum_{j=1}^{K} \mathrm{Vol}(\partial B(x_j, 2r)) \\ &= I(\Omega_0)\mathrm{Vol}(\Omega_0)^{\frac{n-1}{n}} - \sum_{j=1}^{K} \mathrm{Vol}(\partial B(x_j, 2r)).\end{aligned}$$

Then, since $2r < r_-(\Gamma)$, one has

$$\mathrm{Vol}(\Gamma_0) > I(\Omega_0)\mathrm{Vol}(\Omega_0)^{\frac{n-1}{n}} - KC\rho_{n-1}(2r)^{n-1}. \tag{4.1.5}$$

Noticing that $I_0(\Omega) \leqslant I_0(\mathbb{R}^n)$, we have then

$$\begin{aligned}I(\Omega_0)\mathrm{Vol}(\Omega_0)^{\frac{n-1}{n}} &> I(\Omega_0)[\mathrm{Vol}(\Omega) - KC\omega_n(2r)^n]^{\frac{n-1}{n}} \\ &\geqslant [I_0(\Omega)^{\frac{n}{n-1}}\mathrm{Vol}(\Omega) - KC\rho_{n-1}^{\frac{n}{n-1}}(2r)^n]^{\frac{n-1}{n}}.\end{aligned}$$

Replacing in (4.1.5), this leads to the following inequality:

$$\mathrm{Vol}(\Gamma_0) > [I_0(\Omega)^{\frac{n}{n-1}}\mathrm{Vol}(\Omega) - KC\rho_n^{\frac{n}{n-1}}(2r)^n]^{\frac{n-1}{n}} - KC\rho_{n-1}(2r)^{n-1}. \tag{4.1.6}$$

The right hand side is non-negative if

$$\left(\frac{I_0(\Omega)}{KC\rho_{n-1}}\right)^{\frac{n}{n-1}}\frac{\mathrm{Vol}(\Omega)}{(KC)^{-\frac{1}{n-1}} + 1} \geqslant (2r)^n. \tag{4.1.7}$$

We Notice that $\frac{1}{(KC)^{-\frac{1}{n-1}}+1} \geqslant \frac{1}{2}$. Inequality (4.1.7) is then satisfied whenever

$$r \leqslant \frac{1}{2}\left(\frac{I_0(\Omega)}{KC\rho_{n-1}}\right)^{\frac{1}{n-1}}\left(\frac{\mathrm{Vol}\Omega)}{2}\right)^{\frac{1}{n}} = \frac{1}{2}r_K. \tag{4.1.8}$$

Hence (4.1.8) implies that $\mathrm{Vol}(\Gamma_0) > 0$. ■

Lemma 4.1.2. *Let the assumptions of Lemma 4.1.1 be fulfilled. We define*

$$K_0 := \left\lfloor \frac{I_0(\Omega)}{C\rho_{n-1}r_-(\Gamma)^{n-1}}\left(\frac{\mathrm{Vol}(\Omega)}{2}\right)^{\frac{n-1}{n}}\right\rfloor + 1, \tag{4.1.9}$$

where $\lfloor\ \rfloor$ *denotes the floor function, so that* $r_K < r_-(\Gamma)$ *if* $K \geqslant K_0$. *Let* $\{x_j\}_{j=1}^K$ *be an arbitrary set of points in* Γ. *Then, for every* $K \geqslant K_0$ *and* $0 < r \leqslant \frac{1}{16}r_k$, *we have*

$$\mathrm{Vol}\left(\Gamma\backslash\bigcup_{j=1}^K B(x_j,2r)\right) > \left(\frac{r}{r_K}\right)^{\frac{n-1}{n}} I_0(\Omega)\mathrm{Vol}(\Omega)^{\frac{n-1}{n}}. \tag{4.1.10}$$

Proof. From (4.1.6) in the proof of Lemma 4.1.1, one has

$$\mathrm{Vol}(\Gamma_0) > [I_0(\Omega)^{\frac{n}{n-1}}\mathrm{Vol}(\Omega) - KC\rho_n^{\frac{n}{n-1}}(2r)^n]^{\frac{n-1}{n}} - KC\rho_{n-1}(2r)^{n-1}.$$

Setting $\alpha := \frac{r_K}{r}$ (we notice that $\alpha \geqslant 2^4$ since $r \leqslant \frac{1}{2^4}r_K$), we have

$$\begin{aligned}(2r)^n = \left(\frac{2}{\alpha}r_K\right)^n &= \left(\frac{2}{\alpha}\right)^n \frac{\mathrm{Vol}(\Omega)}{2}\left(\frac{I_0(\Omega)}{KC\rho_{n-1}}\right)^{\frac{n}{n-1}} \\ &\leqslant \frac{1}{KC\rho_{n-1}^{\frac{n}{n-1}}}\left(\frac{2^{\frac{n-1}{n}}}{\alpha}\right)^n I_0(\Omega)^{\frac{n}{n-1}}\mathrm{Vol}(\Omega),\end{aligned} \tag{4.1.11}$$

where we have used that $KC \geqslant 1$. On the other hand,

$$\begin{aligned}(2r)^{n-1} = \left(\frac{2}{\alpha}r_K\right)^{n-1} &= \left(\frac{2}{\alpha}\right)^{n-1}\left(\frac{\mathrm{Vol}(\Omega)}{2}\right)^{\frac{n-1}{n}}\frac{I_0(\Omega)}{KC\rho_{n-1}} \\ &\leqslant \frac{1}{KC\rho_{n-1}}\left(\frac{2^{\frac{n-1}{n}}}{\alpha}\right)^{n-1} I_0(\Omega)^{\frac{n}{n-1}}\mathrm{Vol}(\Omega).\end{aligned} \tag{4.1.12}$$

From inequalities (4.1.11) and (4.1.12), we get

$$\mathrm{Vol}\left(\Gamma\backslash\bigcup_{j=1}^K B(x_j,2r)\right) > \left[\left(1-\left(\frac{2^{\frac{n-1}{n}}}{\alpha}\right)^n\right)^{\frac{n-1}{n}} - \left(\frac{2^{\frac{n-1}{n}}}{\alpha}\right)^{n-1}\right] I_0(\Omega)\mathrm{Vol}(\Omega)^{\frac{n-1}{n}}.$$

We notice that, since $\alpha > 2$,

$$\begin{aligned}\left(1-\left(\frac{2^{\frac{n-1}{n}}}{\alpha}\right)^{n}\right)^{\frac{n-1}{n}}-\left(\frac{2^{\frac{n-1}{n}}}{\alpha}\right)^{n-1} &\geqslant \left(1-\alpha^{-1}\right)^{\frac{n-1}{n}}-\alpha^{-\frac{n-1}{n}}\\ &= \alpha^{-\frac{n-1}{n}}\left[(\alpha-1)^{\frac{n-1}{n}}-1\right]\\ &\geqslant \alpha^{-\frac{n-1}{n}}\left[15^{\frac{n-1}{n}}-1\right].\end{aligned}$$

It follows that

$$\mathrm{Vol}\left(\Gamma\setminus\bigcup_{j=1}^{K}B(x_j,2r)\right) > \alpha^{-\frac{n-1}{n}}I_0(\Omega)\mathrm{Vol}(\Omega)^{\frac{n-1}{n}},$$

since $15^{\frac{n-1}{n}} \geqslant 2$ for every $n \geqslant 2$. ■

Let (M,g), Ω and Γ be as described above and $r \in \mathbb{R}_{>0}$. The external covering number $N_r^{ext}(\Gamma)$ of Γ in M with respect to r is defined as the fewest number of points $x_1,\dots,x_N \in M$ such that the balls $B(x_1,r),\dots,B(x_N,r)$ cover Γ. Lemmas 4.1.1 and 4.1.2 imply the following principal lemma.

Lemma 4.1.3. *Let $n \geqslant 2$ and (M,g) be an n-dimensional Riemannian manifold. Let $\Omega \subset M$ a bounded domain with smooth boundary Γ. Then for every $K \geqslant K_0$ and $0 < r \leqslant \frac{1}{2}r_K$,*

i. $K < N_r^{ext}(\Gamma)$.

ii. If in addition $r \leqslant \frac{1}{16}r_K$ then for every arbitrary set of points $\{x_j\}_{j=1}^{K}$ in M, one has

$$\mathrm{Vol}(\Gamma\setminus\bigcup_{j=1}^{K}B(x_j,r)) > \left(\frac{r}{r_K}\right)^{\frac{n-1}{n}} I_0(\Omega)\mathrm{Vol}(\Omega)^{\frac{n-1}{n}}. \qquad (4.1.13)$$

Proof. Suppose $N_r^{ext}(\Gamma) \leqslant K$ and let $\{B(x_j,r)\}_{j=1}^{N_r^{ext}(\Gamma)}$ be a minimal covering of Γ. By the minimality assumption, every $B(x_j,r)$ intersects Γ. For $j \in \{1,\dots,N_r^{ext}(\Gamma)\}$, let $x_j' \in B(x_j,r)\cap\Gamma$, one has

$$B(x_j,r) \subset B(x_j',2r), \quad \text{for every } i \in \{1,\dots,N_r^{ext}(\Gamma)\}.$$

This implies

$$\mathrm{Vol}\left(\Gamma\setminus\bigcup_{j=1}^{N_r^{ext}(\Gamma)}B(x_j',2r)\right) \leqslant \mathrm{Vol}\left(\Gamma\setminus\bigcup_{j=1}^{N_r^{ext}(\Gamma)}B(x_j,r)\right).$$

We complete the family $\{B(x_j',2r)\}_{j=1}^{N_r^{ext}(\Gamma)}$ to $\{B(x_j',2r)\}_{j=1}^{K}$ by setting $x_j' := x_1'$ for $N_r^{ext}(\Gamma) <$

$j \leqslant K$. Then, applying Lemma 4.1.1, we have

$$\begin{aligned}\mathrm{Vol}(\Gamma \backslash \bigcup_{j=1}^{N_r^{ext}(\Gamma)} B(x_j, r)) &\geqslant \mathrm{Vol}(\Gamma \backslash \bigcup_{j=1}^{N_r^{ext}(\Gamma)} B(x'_j, 2r)) \\ &= \mathrm{Vol}(\Gamma \backslash \bigcup_{j=1}^{K} B(x'_j, 2r)) > 0.\end{aligned}$$

Hence, it is contradictory to $\Gamma \subset \bigcup_{j=1}^{N_r^{ext}(\Gamma)} B(x_j, r))$. To prove (ii.), we notice that if $B(x_j, r) \cap \Gamma \neq \emptyset$ then $B(x_j, r) \subset B(x'_j, 2r)$ with $x' \in \Gamma$. The inequality follows applying Lemma 4.1.2. ■

Again Lemma 3.1.1 provides the final ingredient to prove the most technical results in this chapter presented in Theorems 4.1.4 and 4.1.5.

Theorem 4.1.4. *Let (M, g) be a complete Riemannian manifold of dimension $n \geqslant 2$. Let $\Omega \subset M$ be a bounded domain whose boundary Γ is a smooth hypersurface satisfying (H_0). We assume that M, with respect to the distance associated to the metric g satisfies the $(N, 4)$-covering property for some discrete positive and non decreasing function N.*
Then, for every integer $k \geqslant \frac{1}{4}K_0$ (K_0 is the same as in (4.1.9)), one has

$$\begin{aligned}\lambda_{W,k}^{\beta}(\Omega) \leqslant C_1 \left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}} \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}} \\ + C_2 \left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}} \left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right] \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}, \quad (4.1.14)\end{aligned}$$

where $C_1 = 2^8 (C\omega_n)^{\frac{2}{n}} N_r^2$, $C_2 = 2^{21} (C\rho_{n-1})^{\frac{2}{n-1}} N_r$ and $r = \frac{1}{64} r_{(4k)}$, r_{4k} is as defined in (4.1.2).

Proof. We consider the metric measure space (M, d, μ), where d is the distance from the metric g and μ is the Borel measure with support Γ defined for each Borelian A of M by

$$\mu(A) := \int_{A \cap \Gamma} \mathrm{d}_\Gamma.$$

Fix $K = 4k$ and $r := \frac{1}{64} r_{(4k)} = \frac{1}{64} \left(\frac{\mathrm{Vol}(\Omega)}{2}\right)^{\frac{1}{n}} \left(\frac{I_0(\Omega)}{4kC\rho_{n-1}}\right)^{\frac{1}{n-1}}$. We choose, in M, a family of points $\{x_j\}_{j=1}^{K}$ satisfying

$$\begin{cases} B(x_j, 2r) \cap B(x_i, 2r) = \emptyset & \text{for all } 1 \leqslant i \neq j \leqslant K, \\ \mu(B(x_1, r)) \geqslant \mu(B(x_2, r)) \geqslant \ldots \geqslant \mu(B(x_K, r)) \geqslant \mu(B(x, r)), \end{cases} \quad (4.1.15)$$

for all $x \in M_0 := M \backslash \bigcup_{j=1}^{K} B(x_j, 4r)$. This can be done inductively, selecting the point x_1 such that

$$\mu(B(x_1, r)) = \sup\{\mu(B(x, r)),\ x \in M\},$$

and the points x_j, for $j = 2,\dots,K$, such that

$$\mu(B(x_j,r)) = \sup\{\mu(B(x,r)),\ x \in M\setminus \bigcup_{i=1}^{j-1} B(x_i,4r)\}.$$

There are two possible cases:

Assuming that $\mu(B(x_K,r)) \leqslant \frac{\mu(M)}{4KN_r^2}$. We consider the metric measure space (M_0,d,μ_0) where μ_0 is defined by

$$\mu_0(A) := \int_{A\cap\Gamma_0} \mathrm{d}_\Gamma, \quad \Gamma_0 := \Gamma\setminus \bigcup_{i=1}^{K} B(x_i,4r).$$

for every Borelian A in M. Since $4r = \frac{1}{16} r_K$, it follows from Lemma 4.1.3, that

$$\mu_0(M_0) = \mathrm{Vol}(\Gamma_0) > \frac{1}{16^{\frac{n-1}{n}}} I_0(\Omega)\mathrm{Vol}(\Omega)^{\frac{n-1}{n}}.$$

From (4.1.15) one has $\mu_0(B(x,r)) \leqslant \mu(B(x,r)) \leqslant \frac{\mu(M)}{4KN_r^2}$ for every $x \in M_0$. Applying Lemma 3.1.1, we have a family of K capacitors $\{(A_i,B_i)\}_{1\leqslant i\leqslant K}$ with the following properties for $1 \leqslant i,j \leqslant K$:

1. $\mu_0(A_i) \geqslant \frac{\mu_0(M_0)}{2N_rK}$,
2. $B_i = A_i^r = \{x \in X,\ d(x,A_i) < r\}$ is the r-neighbourhood of A_i and $d(B_i,B_j) > 2r$ whenever $i \neq j$.

We notice that $\mu_0(M_0) = \mathrm{Vol}(\Gamma_0)$.
For each $1 \leqslant j \leqslant K$, we consider the function φ_j supported in A_j^r defined by

$$\varphi_j(x) := \begin{cases} 1 - \frac{d(A_j,x)}{r} & \forall\ x \in A_j^r, \\ 0 & \forall\ x \in M\setminus A_j^r. \end{cases}$$

We can assume that (3.2.10) holds for $j = 1,\dots,k$. Since $\int_{\Gamma\cap A_j} \varphi_j{}^2 \mathrm{d}_\Gamma \geqslant \int_{\Gamma_0\cap A_j} \mathrm{d}_\Gamma = \mu_0(A_j) \geqslant \frac{\mathrm{Vol}(\Gamma_0)}{8N_rk}$, we have

$$\begin{aligned} R_\beta(\varphi_j) &\leqslant \frac{\int_{\Omega\cap A_j^r} |\nabla\varphi_j|^2 \mathrm{d}_M + \beta \int_{\Gamma\cap A_j^r} |\nabla\varphi_j|^2 \mathrm{d}_\Gamma}{\int_{\Gamma\cap A_j} \varphi_j{}^2 \mathrm{d}_\Gamma} \\ &\leqslant \frac{8N_rk}{\mathrm{Vol}(\Gamma_0)}\left[\frac{1}{r^2}\frac{\mathrm{Vol}(\Omega)}{k} + \beta\frac{1}{r^2}\frac{\mathrm{Vol}(\Gamma)}{k}\right] \\ &= \frac{8N_r}{r^2\mathrm{Vol}(\Gamma_0)}[\mathrm{Vol}(\Omega) + \beta\mathrm{Vol}(\Gamma)]. \end{aligned}$$

However,

$$\frac{1}{r^2} = \left(\frac{2^6}{r_{(4k)}}\right)^2 = 2^{12}\left(\frac{2}{\mathrm{Vol}(\Omega)}\right)^{\frac{2}{n}}\left(\frac{4kC\rho_{n-1}}{I_0(\Omega)}\right)^{\frac{2}{n-1}}$$

and

$$\mathrm{Vol}(\Gamma_0) > \frac{1}{16^{\frac{n-1}{n}}} I_0(\Omega)\mathrm{Vol}(\Omega)^{\frac{n-1}{n}}.$$

Thus,

$$\frac{1}{r^2\mathrm{Vol}(\Gamma_0)} \leqslant 2^{19}\frac{(kC\rho_{n-1})^{\frac{2}{n-1}}}{\mathrm{Vol}(\Omega)^{1+\frac{1}{n}} I_0(\Omega)^{1+\frac{2}{n-1}}}.$$

We get

$$\begin{aligned} R_\beta(\varphi_j) &\leqslant 2^{21}N_r\frac{(kC\rho_{n-1})^{\frac{2}{n-1}}}{\mathrm{Vol}(\Omega)^{1+\frac{1}{n}} I_0(\Omega)^{1+\frac{2}{n-1}}}\left[\mathrm{Vol}(\Omega)+\beta\mathrm{Vol}(\Gamma)\right] \\ &\leqslant 2^{21}N_r(C\rho_{n-1})^{\frac{2}{n-1}}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}\frac{\mathrm{Vol}(\Gamma)^{1+\frac{2}{n-1}}}{\mathrm{Vol}(\Omega)^{1+\frac{1}{n}} I_0(\Omega)^{1+\frac{2}{n-1}}}\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right] \\ &= 2^{21}N_r(C\rho_{n-1})^{\frac{2}{n-1}}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\Gamma)^{\frac{1}{n-1}}}{I(\Omega)^{1+\frac{1}{n-1}}}+\beta\right]. \end{aligned}$$

Hence,

$$\begin{aligned} \lambda^\beta_{W,k}(\Omega) &\leqslant \max_{1\leqslant j\leqslant k} R_\beta(\varphi_j) \\ &\leqslant 2^{21}N_r(C\rho_{n-1})^{\frac{2}{n-1}}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\Gamma)^{\frac{1}{n-1}}}{I(\Omega)^{1+\frac{1}{n-1}}}+\beta\right]. \end{aligned} \tag{4.1.16}$$

Assuming that $\mu(B(x_K,r)) > \frac{\mu(X)}{4KN_r^2}$. From (4.1.15) one has $\mu(B(x_j,r)) \geqslant \frac{\mu(X)}{4KN_r^2}$ for every $1 \leqslant j \leqslant K$. We consider the same test functions as in (3.2.3). For $1 \leqslant j \leqslant 4k$,the function f_j supported in $B_j := B(x_j, 2r)$ and defined by

$$f_j(x) := \begin{cases} \min\{1, 2-\frac{d(x_j,x)}{r}\} & \forall\, x \in B_j, \\ 0 & \forall\, x \in M\backslash B_j. \end{cases}$$

Set $A_j := B(x_j, r)$, then

$$\int_{\Gamma\cap A_j} {f_j}^2 \mathrm{d}_\Gamma \geqslant \int_{\Gamma\cap A_j}\mathrm{d}_\Gamma \geqslant \mu(A_j) \geqslant \frac{\mathrm{Vol}(\Gamma)}{16N_r^2 k}.$$

Set for $x \in M$, $d_j(x) := \mathrm{dist}(x_j, x)$, then

$$|\nabla f_j| \leqslant |\nabla(2-\frac{d_j(x)}{r})| = |\frac{1}{r}\nabla(d_j(x))| \leqslant \frac{1}{r}.$$

By Hölder's inequality, we have

$$\begin{aligned}\int_{\Omega\cap B_j}|\nabla f_j|^2\mathrm{d}_M &\leqslant \left(\int_{\Omega\cap B_j}|\nabla f_j|^n\mathrm{d}_M\right)^{\frac{2}{n}}\left(\int_{\Omega\cap B_j}\mathrm{d}_M\right)^{1-\frac{2}{n}}\\ &\leqslant \left(\frac{1}{r^n}\mathrm{Vol}(B_j)\right)^{\frac{2}{n}}\left(\mathrm{Vol}(\Omega\cap B_j)\right)^{1-\frac{2}{n}}.\end{aligned}$$

Notice that $B_j\cap\Gamma\supset A_j\cap\Gamma\neq\emptyset$. Let $x_j'\in B_j\cap\Gamma$, one has $B_j\subset B(x_j',4r)$. Since $4r\leqslant r_K<r_-(\Gamma)$,

$$\mathrm{Vol}(B_j)\leqslant \mathrm{Vol}(B(x_j',4r))<C\omega_n(4r)^n.$$

Hence,

$$\int_{\Omega\cap B_j}|\nabla f_j|^2\leqslant (C\omega_n4^n)^{\frac{2}{n}}\left(\frac{\mathrm{Vol}(\Omega)}{k}\right)^{1-\frac{2}{n}},\quad \forall\ 1\leqslant j\leqslant k.$$

Replacing in the Rayleigh quotient, we get:

$$\begin{aligned}R_\beta(f_j)&\leqslant \frac{\int_{\Omega\cap B_j}|\nabla f_j|^2\mathrm{d}_M+\beta\int_{\Gamma\cap B_j}|\nabla f_j|^2\mathrm{d}_\Gamma}{\int_{\Gamma\cap A_j}{f_i}^2\mathrm{d}_\Gamma}\\ &\leqslant \frac{16N_r^2k}{\mathrm{Vol}(\Gamma)}\left[(C\omega_n4^n)^{\frac{2}{n}}\left(\frac{\mathrm{Vol}(\Omega)}{k}\right)^{1-\frac{2}{n}}+\beta\frac{1}{r^2}\frac{\mathrm{Vol}(\Gamma)}{k}\right]\\ &\leqslant 2^8N_r^2(C\omega_n)^{\frac{2}{n}}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}}\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\\ &\quad+\beta 2^{10}N_r^2(C\rho_{n-1})^{\frac{2}{n-1}}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{\frac{2}{n-1}}.\end{aligned}$$

Since $\frac{I(\Omega)}{I_0(\Omega)}\geqslant 1$, regarding the right hand side of (4.1.16), we have

$$\begin{aligned}R_\beta(f_j)\leqslant& 2^8N_r^2(C\omega_n)^{\frac{2}{n}}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}}\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\\ &+\beta 2^{21}N_r^2(C\rho_{n-1})^{\frac{2}{n-1}}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}.\end{aligned}$$

Then, in this case

$$\begin{aligned}\lambda^{\beta}_{W,k}(\Omega) \leqslant &\max_{1\leqslant j\leqslant k} R_{\beta}(\varphi_j)\\ \leqslant &2^8 N_r^2 (C\omega_n)^{\frac{2}{n}} \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}} \left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\\ &+ \beta 2^{21} N_r^2 (C\rho_{n-1})^{\frac{2}{n-1}} \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} \left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}. \qquad (4.1.17)\end{aligned}$$

From (4.1.16) and (4.1.17), in both possible cases we have

$$\begin{aligned}\lambda^{\beta}_{W,k}(\Omega) \leqslant &C_1 \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}} \left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\\ &+ C_2 \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} \left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}} \left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right], \qquad (4.1.18)\end{aligned}$$

where $C_1 := 2^8 N_r^2 (C\omega_n)^{\frac{2}{n}}$ and $C_2 := 2^{21} N_r (C\rho_{n-1})^{\frac{2}{n-1}}$. This ends the proof. ■

When $n \geqslant 3$, Theorem 4.1.4 can be extended to cover all eigenvalues as follows:

Theorem 4.1.5. *Let (M,g) be a complete Riemannian manifold of dimension $n \geqslant 3$ and let $\Omega \subset M$ be a bounded domain whose boundary Γ is a smooth hypersurface satisfying the hypothesis (H_0). We assume that M, with respect to the distance associated to the metric g satisfies the $(N,4)$-covering property for some discrete positive function N.*
Then, for every $k \geqslant 2$, one has

$$\begin{aligned}\lambda^{\beta}_{W,k}(\Omega) \leqslant C(\Omega,\beta) + C_1 &\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}} \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}}\\ &+ C_2 \left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}} \left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right] \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}, \qquad (4.1.19)\end{aligned}$$

where the constants C_1 and C_2 are the same as in Theorem 4.1.4 and

$$\begin{aligned}C(\Omega,\beta) := &\frac{C_1}{(C\rho_{n-1} r_-(\Gamma)^{n-1})^{\frac{2}{n}}} \left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\\ &+ \frac{C_2}{(C\rho_{n-1} r_-(\Gamma)^{n-1})^{\frac{2}{n-1}}} \left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}} \left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]. \qquad (4.1.20)\end{aligned}$$

Proof. For $1 \leqslant k < K_0$, one has

$$\begin{aligned}\lambda^{\beta}_{W,k}(\Omega) \leqslant& \lambda^{\beta}_{W,K_0}(\Omega)\\ \leqslant& C_1\left(\frac{K_0}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}}\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\\ &+C_2\left(\frac{K_0}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\Gamma)^{\frac{1}{n-1}}}{I(\Omega)^{1+\frac{1}{n-1}}}+\beta\right]. \end{aligned}\tag{4.1.21}$$

However, $K_0 \leqslant \frac{I_0(\Omega)}{C\rho_{n-1}r_-(\Gamma)^{n-1}}\left(\frac{\mathrm{Vol}(\Omega)}{2}\right)^{\frac{n-1}{n}}+1$, using the triangle inequality, we obviously have for every $p \in \mathbb{N}_{\geqslant 2}$:

$$\begin{aligned}K_0^{\frac{2}{p}} &\leqslant \left(\frac{I_0(\Omega)}{C\rho_{n-1}r_-(\Gamma)^{n-1}}\left(\frac{\mathrm{Vol}(\Omega)}{2}\right)^{\frac{n-1}{n}}\right)^{\frac{2}{p}}+1\\ &\leqslant \left(\frac{\mathrm{Vol}(\Gamma)}{2^{\frac{n-1}{n}}C\rho_{n-1}r_-(\Gamma)^{n-1}}\frac{I_0(\Omega)}{I(\Omega)}\right)^{\frac{2}{p}}+1\\ &\leqslant \left(\frac{\mathrm{Vol}(\Gamma)}{C\rho_{n-1}r_-(\Gamma)^{n-1}}\right)^{\frac{2}{p}}+k^{\frac{2}{p}}. \end{aligned}\tag{4.1.22}$$

We set $C_3 := \left(\frac{1}{C\rho_{n-1}r_-(\Gamma)^{n-1}}\right)^{\frac{2}{p}}$, replacing in (4.1.21), we get

$$\begin{aligned}\lambda^{\beta}_{W,k}(\Omega) \leqslant& C_1\left\{C_3+\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}}\right\}\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\\ &+C_2\left\{C_3+\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}\right\}\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right]. \end{aligned}\tag{4.1.23}$$

To obtain the first (resp. second) summand at the right-hand side of (4.1.23), we applied (4.1.22) with $p = n$ (resp. $p = n-1$).

Rearranging terms in above inequality, we have

$$\begin{aligned}\lambda^{\beta}_{W,k}(\Omega) \leqslant C(\Omega,\beta) &+ C_1\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}}\\ &+C_2\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right]\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}, \end{aligned}\tag{4.1.24}$$

where

$$C(\Omega,\beta) := \frac{C_1}{(C\rho_{n-1}r_-(\Gamma)^{n-1})^{\frac{2}{n}}}\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}} + \frac{C_2}{(C\rho_{n-1}r_-(\Gamma)^{n-1})^{\frac{2}{n-1}}}\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right]. \quad (4.1.25)$$

The result follows applying Theorem 4.1.4 when $k \geqslant K_0$. ∎

4.2 Proofs

Proof of Theorem 4.0.1. We have in the Euclidean case:

$$r_-(\Gamma) = +\infty, \quad C = 2, \quad I_0(\Omega) = I_0(\mathbb{R}^n) = n\omega_n^{\frac{1}{n}}, \quad N = 32^n.$$

Applying Theorem 4.1.5, we get for every $k \geqslant 2$

$$\lambda_{W,k}^{\beta}(\Omega) \leqslant \zeta_1(n)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}} + \dot{\zeta}_2(n)\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right]\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}, \quad (4.2.1)$$

where $\zeta_1(n) := 2^{10(n+1)}\omega_n^{\frac{2}{n}}$ and $\dot{\zeta}_2(n) := 2^{10(n+3)}\rho_{n-1}^{\frac{2}{n-1}}$. The result follows replacing $I_0(\Omega)$ by $n\omega^{\frac{1}{n}}$. ∎

Proof of Corollary 4.0.2. From Theorem 4.0.1, one has

$$\begin{aligned}\lambda_{W,k}^{\beta}(\Omega) &\leqslant \zeta_1(n)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}} \\ &\quad + \zeta_2(n)I(\Omega)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right]\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} \\ &= \left\{\zeta_1(n)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\left(\frac{\mathrm{Vol}(\Gamma)}{k}\right)^{\frac{2}{n(n-1)}} \right. \\ &\quad \left. + \zeta_2(n)I(\Omega)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right]\right\}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}.\end{aligned} \quad (4.2.2)$$

1. If $\zeta_1(n)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\left(\frac{\mathrm{Vol}(\Gamma)}{k}\right)^{\frac{2}{n(n-1)}} < 1$, then

$$\lambda_{W,k}^{\beta}(\Omega) < C_2(\Omega,\beta)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}.$$

2. Otherwise, $\zeta_1(n)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\left(\frac{\mathrm{Vol}(\Gamma)}{k}\right)^{\frac{2}{n(n-1)}} \geqslant 1$. That is,

$$\frac{k}{\mathrm{Vol}(\Gamma)} \leqslant \left[\zeta_1(n)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{\frac{n-2}{n}}\right]^{\frac{n(n-1)}{2}},$$

$$\begin{cases}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}} \leqslant \zeta_1^{n-1}(n)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{\frac{(n-2)(n-1)}{n}} \\ \left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} \leqslant \zeta_1^{n}(n)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{n-2}.\end{cases} \tag{4.2.3}$$

Replacing in (4.2.2), we get

$$\begin{aligned}\lambda_{W,k}^{\beta}(\Omega) \leqslant & \zeta_1^n(n)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{n-2} \\ & + \zeta_1^n(n)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{n-2}\zeta_2(n)I(\Omega)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right] \\ & \leqslant \zeta_1^n(n)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{n-2}\left\{1+\zeta_2(n)I(\Omega)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right]\right\} \\ & = C_1(\Omega,\beta)\end{aligned}$$

In both cases, one has $\lambda_{W,k}^{\beta}(\Omega) \leqslant C_1(\Omega,\beta) + C_2(\Omega,\beta)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}$. ■

Proof of Theorem 4.0.4. Let $r > 0$, we denote by $\nu(n,-\kappa^2,r)$ (respectively $\nu_\partial(n,-\kappa^2,r)$) the volume of a ball (respectively a sphere) of radius r in the constant curvature model space $M^n_{-\kappa^2}$. As a consequence of the relative Bishop-Gromov volume comparison theorem, we have the following volume and area comparisons, for every $r > 0$ and $x \in M$:

$$\mathrm{Vol}(B(x,r)) \leqslant \nu(n,-\kappa^2,r) \quad \text{and} \quad \mathrm{Vol}(\partial B(x,r)) \leqslant \nu_\partial(n,-\kappa^2,r).$$

The sphere of radius r in the model space $M^n_{-\kappa^2}$ has area

$$\nu_\partial(n,-\kappa^2,r) = \rho_{n-1} sn_{-\kappa^2}(r)^{n-1}$$

and the ball of radius r has volume

$$\nu(n,-\kappa^2,r) = \rho_{n-1}\int_0^r sn_{-\kappa}(t)^{n-1}\mathrm{d}t,$$

where $sn_\varkappa : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$sn_\varkappa(t) = \begin{cases}\frac{1}{\sqrt{\varkappa}}\sin(\sqrt{\varkappa}t) & \text{if } \varkappa > 0 \\ t & \text{if } \varkappa = 0 \\ \frac{1}{\sqrt{-\varkappa}}\sinh(\sqrt{-\varkappa}t) & \text{if } \varkappa < 0.\end{cases}$$

$$\begin{aligned}\rho_{n-1}\int_0^r sn_{-\kappa^2}(r)^{n-1}\mathrm{d}t &= \rho_{n-1}\int_0^r \Big((\frac{1}{\kappa}\sinh(\kappa t)\Big)^{n-1}\mathrm{d}t\\ &\leqslant \rho_{n-1}\int_0^r \Big[te^{\kappa t}\Big]^{n-1}\mathrm{d}t\\ &\leqslant \rho_{n-1}e^{r(n-1)\kappa}\int_0^r t^{n-1}\mathrm{d}t\\ &\leqslant \omega_n r^n e^{r(n-1)\kappa}\end{aligned}$$

and

$$\rho_{n-1}sn_{-\kappa^2}(r)^{n-1}\leqslant e^{r(n-1)\kappa}\rho_{n-1}r^{n-1}.$$

Hence, for every $0<r<1$ and $x\in M$, we have

$$\mathrm{Vol}(B(x,r))<C\omega_n r^n \quad \text{and} \quad \mathrm{Vol}(\partial B(x,r))<C\rho_{n-1}r^{n-1}, \tag{4.2.4}$$

with $C:=e^{n\kappa}$.

On the other hand, for every $0<r<1$ and $x\in M$, $B(x,r)$ can be covered by $N:=2^{5n}e^{4r(n-1)\kappa t}<2^{5n}e^{4(n-1)\kappa}$ balls of radius $\frac{r}{4}$.

Then applying Theorem 4.1.5 with

$$r_-(\Gamma)=1, \quad C=e^{n\kappa}, \quad N=2^{5n}e^{4(n-1)\kappa},$$

we get, for every $k\geqslant 2$,

$$\begin{aligned}\lambda^{\beta}_{W,k}(\Omega)\leqslant e^{c_0(n)\kappa}\Bigg\{&C(\Omega,\beta)\\ &+c_1(n)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n}}\\ &+c_2(n)\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right]\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}\Bigg\},\end{aligned}$$

$$\text{where } C(\Omega,\beta):=c_1'(n)\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}}+c_2'(n)\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right]$$

and the constants $c_0(n)$, $c_1(n)$, $c_2(n)$, $c_1'(n)$ and $c_2'(n)$ depend only on n.

Following the same arguments as the proof of Corollary 4.0.2, we have for every $k\geqslant 2$, one has

$$\lambda^{\beta}_{W,k}(\Omega)\leqslant e^{c_0(n)\kappa}\left\{\overline{C}_1(\Omega,\beta)+\overline{C}_2(\Omega,\beta)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}\right\},$$

where $\overline{C}_2(\Omega,\beta)=1+c_2(n)\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}+\beta\right]$ and $\overline{C}_1(\Omega,\beta)=C(\Omega,\beta)+c_1^n(n)\overline{C}_2(\Omega,\beta)$.

- If $\kappa \leqslant 1$, then

$$\lambda^{\beta}_{W,k}(\Omega) \leqslant e^{c_0(n)}\left\{\overline{C}_1(\Omega,\beta) + \overline{C}_2(\Omega,\beta)\left(\frac{k}{\mathrm{Vol}(\Gamma)}\right)^{\frac{2}{n-1}}\right\},$$

which implies (4.0.4).

- Otherwise, we assume that $\mathrm{Ric}(M,g) \geqslant -(n-1)\kappa^2 g$ with $\kappa > 1$. Then the Ricci curvature $\mathrm{Ric}(M,\tilde{g})$ of the rescaled metric $\tilde{g} := \kappa^2 g$ is bounded by $-(n-1)\tilde{g}$. We mark with a tilde quantities associated with the metric $\tilde{g}$, while those unmarked with such will be still associated with the metric g. Then we have

$$\lambda^{\beta}_{W,k}(\tilde{\Omega}) \leqslant e^{c_0(n)}\left\{\overline{C}_1(\tilde{\Omega},\beta) + \overline{C}_2(\tilde{\Omega},\beta)\left(\frac{k}{\mathrm{Vol}(\tilde{\Gamma})}\right)^{\frac{2}{n-1}}\right\}. \tag{4.2.5}$$

However $\mathrm{Vol}(\tilde{\Omega}) = \mathrm{Vol}_{\tilde{g}}(\Omega) = \kappa^n \mathrm{Vol}(\Omega)$ and $\mathrm{Vol}(\tilde{\Gamma}) = \kappa^{n-1}\mathrm{Vol}(\Gamma)$. Thus,

$$\begin{aligned}\overline{C}_2(\tilde{\Omega},\beta) &= 1 + c_2(n)\left(\frac{I(\tilde{\Omega})}{I_0(\tilde{\Omega})}\right)^{1+\frac{2}{n-1}}\left[\frac{\mathrm{Vol}(\tilde{\Omega})}{\mathrm{Vol}(\tilde{\Gamma})} + \beta\right] \\ &= 1 + c_2(n)\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\kappa\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]. \end{aligned} \tag{4.2.6}$$

Likewise, since

$$C(\tilde{\Omega},\beta) = c_1'(n)\left(\kappa\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}} + c_2'(n)\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\kappa\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right],$$

we have

$$\begin{aligned}\overline{C}_1(\tilde{\Omega},\beta) &= C(\tilde{\Omega},\beta) + c_1^n(n)\overline{C}_2(\tilde{\Omega},\beta) \\ &= c_1^n(n) + c_1'(n)\left(\kappa\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}} \\ &\quad + \left(c_1^n(n)c_2(n) + c_2'(n)\right)\left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\kappa\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]. \end{aligned} \tag{4.2.7}$$

We set $\overline{\zeta}(n) := \max\{1, c_2(n), c_1^n(n), c_1'(n), c_1^n(n)c_2(n) + c_2'(n)\}$ so that

$$\overline{C}_2(\tilde{\Omega},\beta) \leqslant \overline{\zeta}(n)\left\{1 + \left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\kappa\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]\right\} \text{ and}$$

$$\overline{C}_1(\tilde{\Omega},\beta) \leqslant \overline{\zeta}(n)\left\{1 + \left(\kappa\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)}\right)^{1-\frac{2}{n}} + \left(\frac{I(\Omega)}{I_0(\Omega)}\right)^{1+\frac{2}{n-1}}\left[\kappa\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta\right]\right\}.$$

In addition, since $\kappa > 1$, for all $u \in H(\Omega)\backslash\{0\}$ we have

$$\tilde{R}_\beta(u) = \frac{\kappa \int_\Omega |\nabla u|^2 \mathrm{d}_M + \beta \int_\Gamma |\nabla_\Gamma u|^2 \mathrm{d}_\Gamma}{\kappa^2 \int_\Gamma u^2 \mathrm{d}_\Gamma} \geqslant \frac{1}{\kappa^2} R_\beta(u).$$

Every orthonormal basis of a k-dimensional subspaces $V \in \mathfrak{V}(k)$ of $H(\Omega)$ remains orthogonal with the metric $\tilde{g}$, then using the variation characterisation with (4.2.5), (4.2.6) and (4.2.7), we have

$$\begin{aligned}
\lambda^\beta_{W,k}(\Omega) &\leqslant \kappa^2 \lambda^\beta_{W,k}(\tilde{\Omega}) \leqslant \kappa^2 e^{c_0(n)} \left\{ \overline{C}_1(\Omega,\beta) + \overline{C}_2(\tilde{\Omega},\beta) \left(\frac{k}{\mathrm{Vol}(\tilde{\Gamma})} \right)^{\frac{2}{n-1}} \right\} \\
&\leqslant \kappa^2 \zeta(n) \left\{ 1 + \left(\kappa \frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} \right)^{1-\frac{2}{n}} + \left(\frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n-1}} \left[\kappa \frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta \right] \right\} \\
&\quad + \zeta(n) \left\{ 1 + \left(\frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n-1}} \left[\kappa \frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Gamma)} + \beta \right] \right\} \left(\frac{k}{\mathrm{Vol}(\Gamma)} \right)^{\frac{2}{n-1}}, \qquad (4.2.8)
\end{aligned}$$

where $\zeta(n) := e^{c_0(n)} \overline{\zeta}(n)$ is a dimensional constant.

■

Proof of Theorem 4.0.3. We have for every $r > 0$, $sn_0(r) = r$, then for every $r > 0$ and $x \in M$, one has

$$\mathrm{Vol}(B(x,r)) \leqslant \nu(n,0,r) = \rho_{n-1} \int_0^r t^{n-1} \mathrm{d}t = \omega_n r^n$$

and

$$\mathrm{Vol}(\partial B(x,r)) \leqslant \nu_\partial(n,0,r) = \rho_{n-1}(r)^{n-1}, \quad \forall\, x \in M.$$

On the other hand, for every $r > 0$ and $x \in M$, $B(x,r)$ can be covered by $N := 32^n$ balls of radius $\frac{r}{4}$.

Then the ressult follows from Theorem 4.1.5 with $r_-(\Gamma) = +\infty$, $C = 2$ and $N = 32^n$. ■

5 Eigenvalues of the Laplacian with density

Let M denote a complete, connected Riemannian manifold of dimension $n \in \mathbb{N}$. We assume that M has a smooth and connected boundary ∂M. Denote by g and d_M respectively, the Riemannian metric on M and the associated volume element. Let Δ be the Laplace operator on M equipped with the weighted volume form $\mathrm{d}m := e^{-h}\mathrm{d}_M$. We consider the operator

$$L_h \cdot := e^{-h(\alpha-1)}\Big(\Delta \cdot + \alpha g(\nabla h, \nabla \cdot)\Big),$$

where $\alpha > 1$ and $h \in C^2(M)$ are given. The result proved in this chapter states about the existence of upper bounds for the eigenvalues of the weighted Laplacian L_h with the Neumann boundary condition if the boundary is non-empty.

On ∂M, let ∇_∂ the tangential gradient, Δ_∂ the Laplace operator on ∂M and $\partial_{\mathbf{n}}$ the derivative with respect to the outward pointing unit normal vector $\mathbf{n}$ to M. The second fundamental form on ∂M is defined by $I(X,Y) := g(\nabla_X \mathbf{n}, Y)$ for any vector fields X and Y. Recall that the boundary is called convex, if the second fundamental form of ∂M with respect to $\mathbf{n}$ is non-negative, $I_p \geqslant 0$, for all $p \in \partial M$. As above, we denote by $H := \mathrm{Tr} I$ the mean curvature of ∂M and by $\sec(M)$ (resp. $\mathrm{Ric}(M)$) the sectional curvature (resp. the Ricci curvature) on M.

We write $\sec(M) \geqslant \kappa$ (resp. $\mathrm{Ric}(M) \geq \kappa$) standing for $\sec_p(P) \geqslant \kappa$ for every point $p \in M$ and every 2-plane $P \subset T_pM$ (resp. $\mathrm{Ric}_p(v) \geqslant \kappa$ for every $p \in M$ and every $v \in T_pM$). In particular, if $\kappa \in \mathbb{R}$ and $\sec(M) \geqslant \frac{\kappa}{n-1}$ then $\mathrm{Ric}(M) \geqslant \kappa$. We recall that, by the classical Myers's theorem ([58]), if M satisfies $\sec(M) \geqslant \kappa > 0$ then M is compact and $\mathrm{diam}(M,g) \leqslant \frac{\pi}{\sqrt{\kappa}} = \mathrm{diam} S^n_K$, S^n_κ denotes the n-dimensional space form of constant sectional curvature κ.

Theorem 5.0.1. *Let $\alpha > 1$ be a given real constant. Let (M,g) be a compact connected $n-$dimensional Riemannian manifold whose sectional curvature satisfies $\sec(M) \geqslant \frac{\kappa}{n-1}$, for some positive constant κ. If M has convex boundary, then there exists a sequence of densities $\{\rho_j\}_{j\geqslant 2}$ and $j_0 \in \mathbb{N}$, such that*

$$\lambda_2(\rho_j, \rho_j^\alpha)\left(\frac{\mathrm{Vol}(M)}{\int_M \rho_j \mathrm{d}_M}\right)^{\alpha-1} \geqslant 2\kappa j, \quad \forall j \geqslant j_0.$$

Here, $\mathrm{Vol}(M)$ *denotes the volume of M.*

This inequality provides a lower bound that grows linearly to infinity in j as $j \to \infty$, showing that with respect to these densities, $\lambda_2(\rho, \rho^\alpha)$ becomes as large as desired. Unfortunately, I do not know any other way to prove it, than the following long and painful computation.

Our aim is to show that, there exists a family of densities $\rho_j = e^{-h_j}$, $j \in \mathbb{N}$, such that their corresponding first non-zero eigenvalues become as large as desired. For this, we use the extended Reilly formula presented in Theorem 5.0.2, to provide a lower bound that grows linearly to infinity in j, as $j \to \infty$.

Theorem 5.0.2. *(Adapted Reilly type Formula) Consider M equipped with the weighted volume form* $\mathrm{d}m = \mathrm{e}^{-\mathrm{h}}\,\mathrm{d}_M$ *for some* $h \in C^2(M)$*. Then, for every* $u \in C^\infty(M)$*, we have:*

$$\begin{aligned}\int_M \mathrm{e}^{\mathrm{h}(\alpha-1)}|\mathrm{L_h}\,u|^2 - \mathrm{e}^{-\mathrm{h}(\alpha-1)}|\mathrm{Hess}\,u|^2\mathrm{d}m \\ = \int_M \mathrm{e}^{-\mathrm{h}(\alpha-1)}(\mathrm{Ric} + \alpha\mathrm{Hess}\,h)(\nabla u, \nabla u)\,\mathrm{d}m \\ + \int_{\partial M} \mathrm{e}^{-\mathrm{h}(\alpha-1)}\partial_{\mathbf{n}}u\Big(H\partial_n u - \alpha g(\nabla h, \nabla u) - \Delta_\partial u\Big)\mathrm{d}m \\ + \int_{\partial M} \mathrm{e}^{-\mathrm{h}(\alpha-1)}[I(\nabla_\partial u, \nabla_\partial u) - g(\nabla_\partial u, \nabla_\partial \partial_{\mathbf{n}} u)]\,\mathrm{d}m. \quad (5.0.1)\end{aligned}$$

In the next section, we prove these two theorems.

5.1 Proofs

5.1.1 Proof of Theorem 5.0.1

Let (M,g) be a compact connected n-dimensional Riemannian manifold with sectional curvature satisfying $\sec(M) \geqslant \frac{\kappa}{n-1} > 0$ and convex boundary ∂M. Let $h \in C^2(M)$ and assume that λ is the first non-zero eigenvalue of L_h. Let $u \neq 0$ be an eigenfunction with corresponding eigenvalue λ, i.e. u satisfies $\mathrm{L_h}\,u = \lambda u$.

Lemma 5.1.1. *If* $\mathrm{Ric} + \alpha\mathrm{Hess}\,h \geqslant \left(\alpha^2\frac{|\nabla h|^2}{nz} + A\right)g$*, for some $A > 0$ and $z > 0$, holds when evaluated at* $(\nabla u, \nabla u)$*, then:*

$$A\lambda \int_M u^2\mathrm{d}m \leqslant \frac{\lambda^2}{n(z+1)} \int_M u^2\left(\mathrm{e}^{\mathrm{h}(\alpha-1)}n(z+1) - \mathrm{e}^{-\mathrm{h}(\alpha-1)}\right)\mathrm{d}m. \quad (5.1.1)$$

Proof. With the Neumann boundary condition, we have

$$\lambda \int_M u^2\,\mathrm{d}m = \int_M \mathrm{e}^{-\mathrm{h}(\alpha-1)}|\nabla u|^2\,\mathrm{d}m. \quad (5.1.2)$$

Indeed,

$$\begin{aligned}\lambda \int_M u^2\,\mathrm{d}m = \int_M \mathrm{e}^{-\mathrm{h}(\alpha-1)}\mathrm{L_h}\,u \cdot u\,\mathrm{d}m \\ = \int_M \mathrm{e}^{-\mathrm{h}(\alpha-1)}\Delta u.u\,\mathrm{d}m + \int_M \mathrm{e}^{-\mathrm{h}(\alpha-1)}\alpha g(\nabla h, \nabla u)\,u\,\mathrm{d}m \\ = \int_M \nabla u \cdot \nabla(\mathrm{e}^{-\mathrm{h}\alpha}u)\,\mathrm{d}_M + \int_M \mathrm{e}^{-\mathrm{h}(\alpha-1)}\alpha g(\nabla h, \nabla u)\,u\,\mathrm{d}m \\ = \int_M \mathrm{e}^{-\mathrm{h}(\alpha-1)}|\nabla u|^2\,\mathrm{d}m - \int_M \mathrm{e}^{-\mathrm{h}(\alpha-1)}\alpha g(\nabla h, \nabla u)\,u\,\mathrm{d}m + \int_M \mathrm{e}^{-\mathrm{h}(\alpha-1)}\alpha g(\nabla h, \nabla u)\,u\,\mathrm{d}m.\end{aligned}$$

Equality (5.0.1) becomes

$$\int_M e^{h(\alpha-1)}|L_h u|^2 - e^{-h(\alpha-1)}|\text{Hess } u|^2 \, dm$$
$$= \int_M e^{-h(\alpha-1)}(\text{Ric} + \alpha \text{Hess } h)(\nabla u, \nabla u)\, dm$$
$$+ \int_{\partial M} e^{-h(\alpha-1)} I(\nabla_\partial u, \nabla_\partial u)\, dm.$$

Since ∂M is convex, then $I(\nabla_\partial u, \nabla_\partial u) \geqslant 0$ and one has

$$\begin{aligned}\int_M e^{h(\alpha-1)}|L_h u|^2 - e^{-h(\alpha-1)}|\text{Hess } u|^2 &\geqslant \int_M e^{-h(\alpha-1)}(\text{Ric} + \alpha \text{Hess } h)(\nabla u, \nabla u)\, dm \\ &\geqslant \alpha^2 \int_M e^{-h(\alpha-1)}|\nabla u|^2 \frac{|\nabla h|^2}{nz}\, dm + A \int_M e^{-h(\alpha-1)}|\nabla u|^2\, dm \\ &= \alpha^2 \int_M e^{-h(\alpha-1)}|\nabla u|^2 \frac{|\nabla h|^2}{nz}\, dm + A\lambda \int_M u^2\, dm. \end{aligned} \tag{5.1.3}$$

The last equality following from (5.1.2). Notice that the same inequality also holds if ∂M is empty. On the other hand, $|\text{Hess } u|^2 \geqslant \frac{|\Delta u|^2}{n}$ (See [3, p. 409].), and

$$\begin{aligned}\int_M e^{h(\alpha-1)}|L_h u|^2 - e^{-h(\alpha-1)}|\text{Hess } u|^2\, dm &\leqslant \int_M e^{h(\alpha-1)} \lambda^2 u^2 - \frac{1}{n} e^{-h(\alpha-1)}|\Delta u|^2\, dm \\ &= \int_M e^{h(\alpha-1)} \lambda^2 u^2 - \frac{1}{n} e^{-h(\alpha-1)}(\lambda u - \alpha g(\nabla h, \nabla u))^2\, dm \\ &\leqslant \int_M e^{h(\alpha-1)} \lambda^2 u^2 - \frac{1}{n} e^{-h(\alpha-1)}\left(\frac{\lambda^2 u^2}{z+1} - \alpha^2 \frac{|g(\nabla h, \nabla u)|^2}{z}\right) dm \\ &= \lambda^2 \int_M u^2 \frac{e^{h(\alpha-1)} n(z+1) - e^{-h(\alpha-1)}}{n(z+1)}\, dm + \alpha^2 \int_M e^{-h(\alpha-1)} \frac{|g(\nabla h, \nabla u)|^2}{nz}\, dm. \end{aligned} \tag{5.1.4}$$

In the second to last inequality we have used Young's inequality. Indeed, given any $\epsilon > 0$,

$$\lambda u \alpha g(\nabla h, \nabla u) \leqslant \frac{\lambda^2 u^2}{2\epsilon} + \frac{\epsilon}{2}\alpha^2 |g(\nabla h, \nabla u)|^2,$$

since $\left(\frac{\lambda u}{\sqrt{2\epsilon}} - \sqrt{\frac{\epsilon}{2}}\alpha g(\nabla h, \nabla u)\right)^2$ is non-negative. Adding the expression $-\frac{1}{2}\left(\lambda^2 u^2 + \alpha^2 |g(\nabla h, \nabla u)|^2\right)$ to both sides of this inequality, we get

$$-\Big(\lambda u - \alpha g(\nabla h, \nabla u)\Big)^2 \leqslant -\left[\lambda^2 u^2 \left(1 - \frac{1}{\epsilon}\right) + \alpha^2 |g(\nabla h, \nabla u)|^2 (1-\epsilon)\right].$$

Then choosing $\epsilon := \frac{z+1}{z}$, one has

$$-\Big(\lambda u - \alpha g(\nabla h, \nabla u)\Big)^2 \leqslant -\left[\frac{\lambda^2 u^2}{z+1} - \frac{\alpha^2 |g(\nabla h, \nabla u)|^2}{z}\right].$$

Now, combining (5.1.3) and (5.1.4), we have

$$A\lambda \int_M u^2 \mathrm{d}m \leqslant \lambda^2 \int_M u^2 \frac{\mathrm{e}^{\mathrm{h}(\alpha-1)}\, n(z+1) - \mathrm{e}^{-\mathrm{h}(\alpha-1)}}{n(z+1)}\, \mathrm{d}m.$$

■

Now, we consider $\tilde{\lambda} := \lambda \left(\frac{\mathrm{Vol}(M)}{\int_M e^{-h} \mathrm{d}_M} \right)^{\alpha-1}$ which is invariant under rescaling of the density. Indeed, for any non-zero scalar a,

$$\frac{\int_M |\nabla u|^2 (ae^{-h})^\alpha \mathrm{d}_M}{\int_M u^2 (ae^{-h}) \mathrm{d}_M} \cdot \left(\frac{\mathrm{Vol}(M)}{\int_M (ae^{-h}) \mathrm{d}_M} \right)^{\alpha-1} = \frac{\int_M |\nabla u|^2 e^{-h\alpha} \mathrm{d}_M}{\int_M u^2 e^{-h} \mathrm{d}_M} \left(\frac{\mathrm{Vol}(M)}{\int_M e^{-h} \mathrm{d}_M} \right)^{\alpha-1}.$$

Replacing λ by $\tilde{\lambda} \left(\frac{\int_M e^{-h} \mathrm{d}_M}{\mathrm{Vol}(M)} \right)^{\alpha-1}$ in (5.1.1) and under the assumptions of Lemma 5.1.1, we get the following inequality:

$$A\tilde{\lambda} \int_M u^2 \, \mathrm{d}m \leqslant \tilde{\lambda}^2 \int_M u^2 \left(\frac{\int_M e^{-h} \mathrm{d}_M}{\mathrm{Vol}(M)} \right)^{\alpha-1} \left(\frac{\mathrm{e}^{\mathrm{h}(\alpha-1)}\, n(z+1) - \mathrm{e}^{-\mathrm{h}(\alpha-1)}}{n(z+1)} \right) \mathrm{d}m. \tag{5.1.5}$$

Let $r : M \ni x \longrightarrow d(x_0, x) \in \mathbb{R}_{\geqslant 0}$, where $x_0 \in M$ is a fixed point, be the distance function from x_0. Recall that, except at x_0 itself and its cut locus, r is smooth.

The cut locus is the union of the set of conjugate points to x_0 (the conjugate locus) and the set of cut points, i.e., those points at which there are at least two minimizing geodesics. Refer to [19, Chapter 13, Propostion 2.2], [67, Proposition 4,8]. The conjugate points consist of the critical values of the exponential map $\exp_{x_0}$ and Sard's theorem implies that the conjugate locus has measure zero since $\exp_{x_0}$ is smooth. If there are at least two distinct minimal geodesics going from x_0 to $p \in M$, then r is not differentable at p. The distance function beying Lipschitz, thanks to Radamacher's differentiation theorem the set of cut points is a set of points where r is not differentiable and is therefore also of measure zero.

Thus the cut locus is of measure zero and

$$|\nabla r| = 1, \quad \mathrm{Hess}\, r = I, \quad \Delta r = H, \qquad \text{a.e. on } M.$$

Here I is the second fundamental form with respect to the inward normal and H the mean curvature, of the level hypersurfaces of $|\,|$ (i.e. the geodesic spheres $\partial B(x_0, r)$).

Let $j \geqslant 2$, $z \in \mathbb{R}_{>0}$, $\alpha > 1$. We define

$$c_0 := \sqrt{n(z+1)e^{\alpha-1}\left(e^{\alpha-1} - \frac{1}{j} \right)}, \quad C_j := -\frac{1}{\alpha-1}\log(c_0) \quad \text{and}$$

$$h_j(x) := e^{-\frac{|x|^2}{j}} + C_j.$$

The following properties hold.

Lemma 5.1.2. *The function h_j satisfies the following inequalities almost a.e.:*

(i) $\left(\frac{\int_M e^{-h_j}\mathrm{d}_M}{\mathrm{Vol}(M)}\right)^{\alpha-1} \leqslant c_0$,

(ii) $\frac{e^{h_j(\alpha-1)} n(z+1) - e^{-h_j(\alpha-1)}}{n(z+1)} \leqslant \frac{1}{jc_0}$,

(iii) $|\nabla h_j|^2 g - \alpha \mathrm{Hess}\, h_j \leqslant \frac{2\alpha}{j} g$ *at every* $(X,X) \in TM \times TM$.

Proof. (i) $h_j(x) > C_j$ implies that $\int_M e^{-h_j(x)}\mathrm{d}_M \leqslant \int_M c_0^{\frac{1}{\alpha-1}}\mathrm{d}_M = c_0^{\frac{1}{\alpha-1}}\mathrm{Vol}(M)$.

(ii) Let us set $b := n(z+1)$ and $u := e^{h_j(\alpha-1)}$.
We want to prove that $\frac{(u^2b-1)jc_0 - bu}{ubjc_0} \leqslant 0$.
Notice that $u > 0$, $bjc_0 > 0$ and $\frac{(u^2b-1)jc_0 - bu}{u} = \frac{(u-u_1)(u-u_2)}{u}$, where $u_1 := \frac{b - \sqrt{b^2+4bj^2c_0^2}}{2bjc_0} < 0$ and $u_2 := \frac{b + \sqrt{b^2+4bj^2c_0^2}}{2bjc_0} > 0$.
Moreover, $0 < e^{h_j(\alpha-1)} \leqslant u_2$.
Indeed, $e^{h_j(\alpha-1)} = e^{(\alpha-1)(e^{-\frac{|x|^2}{j}} + C_j)}$, so the first inequality is immediate.
For the second inequality, we have

$$\sqrt{\frac{4}{b} + \frac{1}{j^2c_0^2}} = \frac{1}{c_0}\left(\sqrt{\frac{4c_0^2}{b} + \frac{1}{j^2}}\right) = \frac{1}{c_0}\sqrt{\left(2\,\mathrm{e}^{\alpha-1} - \frac{1}{j}\right)^2} = \frac{1}{c_0}\left(2\,\mathrm{e}^{\alpha-1} - \frac{1}{j}\right).$$

Hence, $\log(\frac{1}{c_0}) = \log\left[\frac{1}{2}\left(\sqrt{\frac{4}{b} + \frac{1}{j^2c_0^2}} + \frac{1}{jc_0}\right)\right] - (\alpha - 1)$ and

$$C_j = \frac{1}{(\alpha-1)}\log\left[\frac{1}{2}\left(\sqrt{\frac{4}{b} + \frac{1}{j^2c_0^2}} + \frac{1}{jc_0}\right)\right] - 1$$

$$h_j(x) \leqslant 1 + C_j \leqslant \frac{1}{\alpha - 1}\log\left[\frac{1}{2}\left(\sqrt{\frac{4}{b} + \frac{1}{j^2c_0^2}} + \frac{1}{jc_0}\right)\right].$$

Hence, $e^{h_j(\alpha-1)} \leqslant \frac{1}{2}\left(\sqrt{\frac{4}{b} + \frac{1}{j^2c_0^2}} + \frac{1}{jc_0}\right) = u_2$.

(iii) Notice that r is radial and we have

$$h_j(x) := e^{-\frac{r(x)^2}{j}} + C_j, \qquad \nabla h_j = -\frac{2r}{j} e^{-\frac{r^2}{j}} \nabla r$$

and Hess $h_j(x) = e^{-\frac{r^2}{j}}\left[\left(-\frac{2}{j}+(\frac{2}{j}r)^2\right)\mathrm{d}r^2 - \frac{2}{j}r\text{Hess } r\right]$. Hence,

$$\begin{aligned}
|\nabla h_j(r)|^2 g - \alpha\text{Hess } h_j(r) &\leqslant (\frac{2r}{j})^2 e^{-\frac{2r^2}{j}} g - \alpha e^{-\frac{r^2}{j}}\left[\left(-\frac{2}{j}+(\frac{2}{j}r)^2\right)\mathrm{d}r^2 - \frac{2}{j}r\text{Hess } r\right]\\
&= (\frac{2r}{j})^2 e^{-\frac{2r^2}{j}} g - \alpha(\frac{2}{j}r)^2 e^{-\frac{r^2}{j}}\mathrm{d}r^2 + \alpha\frac{2}{j}e^{-\frac{r^2}{j}}\left[\mathrm{d}r^2 + r\text{Hess } r\right]\\
&\leqslant (\frac{2r}{j})^2 e^{-\frac{r^2}{j}}\left(e^{-\frac{r^2}{j}} - \alpha\right) g + \alpha\frac{2}{j}e^{-\frac{r^2}{j}}\left[\mathrm{d}r^2 + r\text{Hess } r\right]\\
&\leqslant \frac{2\alpha}{j}e^{-\frac{r^2}{j}}\left[\mathrm{d}r^2 + r\text{Hess } r\right], \text{ since } e^{-\frac{r^2}{j}} \leqslant 1 < \alpha\\
&\leqslant \frac{2\alpha}{j}\left[\text{Hess } (\frac{1}{2}r^2)\right].
\end{aligned}$$

Since $\sec(M) \geqslant 0$, from the Hessian Comparison (see for instance [63, Lemma 57]), one has Hess $(\frac{1}{2}r^2) \leqslant g$,.

■

Proof of Theorem 5.0.1. We set $z = \frac{\alpha^2}{n}$, $A := \frac{\kappa}{2}$ and $j_0 := \lceil\frac{4\alpha}{\kappa}\rceil$. Then from Lemma 5.1.2 (iii), we have

$$\text{Ric} + \alpha\text{Hess } h_j \geqslant \kappa + \alpha^2\frac{|\nabla h_j|^2}{nz} - \frac{2\alpha}{j} \geqslant \alpha^2\frac{|\nabla h_j|^2}{nz} + A, \quad \forall\, j \geqslant j_0.$$

Combining inequality (5.1.5), Lemma 5.1.2 (i) and (ii), we finally get

$$A\tilde{\lambda}\int_M u^2 \mathrm{d}m \leqslant \tilde{\lambda}^2\int_M u^2 c_0\frac{1}{jc_0}\,\mathrm{d}m.$$

Hence, for every $j \geqslant j_0$, one has $Aj \leqslant \tilde{\lambda}$. ■

5.1.2 Proof of Theorem 5.0.2

To prove Theorem 5.0.2, one needs the following adapted Bochner formula deduced from the standard one for smooth functions (See e.g.[3, Thm. 346].):

$$\frac{1}{2}\Delta(|\nabla u|^2) = -|\text{Hess } u|^2 + g(\nabla u, \nabla\Delta u) - \text{Ric}(\nabla u, \nabla u). \quad \text{(Bochner [7, (9)])}$$

Lemma 5.1.3. *Let u be a smooth function on (M, g). Then,*

$$\frac{1}{2}\mathrm{L_h}|\nabla u|^2 = -\mathrm{e}^{\mathrm{-h}(\alpha-1)}\Big(|\text{Hess } u|^2 + (\text{Ric} + \alpha\text{Hess } h)(\nabla u, \nabla u)\Big) + g\Big(\nabla u, \nabla\mathrm{L_h}\, u + (\alpha-1)\mathrm{L_h}\, u\nabla h\Big). \quad (5.1.6)$$

Proof.

$$\begin{aligned}\frac{1}{2}\,\mathrm{L_h}\,|\nabla u|^2 &= \frac{1}{2}\,\mathrm{e}^{\text{-h}(\alpha\text{-}1)}\Big(\Delta\,|\nabla u|^2 + \alpha g(\nabla h, \nabla|\nabla u|^2)\Big)\\ &= \mathrm{e}^{\text{-h}(\alpha\text{-}1)}\Big(-|\text{Hess } u|^2 + g(\nabla u, \nabla\,\Delta\,u) - \mathrm{Ric}(\nabla u, \nabla u)\Big)\\ &\quad + \frac{1}{2}\alpha\,\mathrm{e}^{\text{-h}(\alpha\text{-}1)}\,g(\nabla h, \nabla|\nabla u|^2)\\ &= -\,\mathrm{e}^{\text{-h}(\alpha\text{-}1)}\Big(|\text{Hess } u|^2 + \mathrm{Ric}(\nabla u, \nabla u)\Big) + \mathrm{e}^{\text{-h}(\alpha\text{-}1)}\,g(\nabla u, \nabla\,\Delta\,u)\\ &\quad - \alpha\,\mathrm{e}^{\text{-h}(\alpha\text{-}1)}\,\text{Hess } h(\nabla u, \nabla u) + \alpha\,\mathrm{e}^{\text{-h}(\alpha\text{-}1)}\,D_{\nabla u}g(\nabla h, \nabla u).\end{aligned}$$

For the last line, we have used $\frac{1}{2}g(\nabla h, \nabla|\nabla u|^2) = g(\nabla h, \nabla_{\nabla u}\nabla u)$ $= D_{\nabla u}g(\nabla h, \nabla u) - g(\nabla_{\nabla u}\nabla h, \nabla u) = D_{\nabla u}g(\nabla h, \nabla u) - \text{Hess } h(\nabla u, \nabla u)$. Moreover,

$$\begin{aligned}g\,(\nabla(\mathrm{L_h}\,u), \nabla u) &= D_{\nabla u}(\mathrm{L_h}\,u)\\ &= -(\alpha-1)D_{\nabla u}(h)\,\mathrm{L_h}\,u + \mathrm{e}^{\text{-h}(\alpha\text{-}1)}\,D_{\nabla u}\,(\Delta\,u) + \alpha\,\mathrm{e}^{\text{-h}(\alpha\text{-}1)}\,D_{\nabla u}g(\nabla h, \nabla u)\\ &= -(\alpha-1)g(\nabla h, \nabla u)\,\mathrm{L_h}\,u + \mathrm{e}^{\text{-h}(\alpha\text{-}1)}\,g\,(\nabla\,\Delta\,u, \nabla u) + \alpha\,\mathrm{e}^{\text{-h}(\alpha\text{-}1)}\,D_{\nabla u}g(\nabla h, \nabla u).\end{aligned}$$

Finally,

$$\begin{aligned}\frac{1}{2}\,\mathrm{L_h}\,|\nabla u|^2 = -\,\mathrm{e}^{\text{-h}(\alpha\text{-}1)}\Big(|\text{Hess } u|^2 + (\mathrm{Ric} + \alpha\text{Hess } h)(\nabla u, \nabla u)\Big)\\ + g\Big(\nabla u, \nabla\,\mathrm{L_h}\,u + (\alpha-1)\,\mathrm{L_h}\,u\nabla h\Big).\end{aligned}$$

■

Proof of Theorem 5.0.2. We shall integrate equality (5.1.6). On the left-hand side, we have

$$\begin{aligned}\frac{1}{2}\int_M L_h|\nabla u|^2\mathrm{d}m &= \frac{1}{2}\int_M \mathrm{e}^{\text{-h}(\alpha\text{-}1)}\Big(\Delta\,|\nabla u|^2 + \alpha g\Big(\nabla h, \nabla|\nabla u|^2\Big)\Big)\mathrm{d}m\\ &= \frac{1}{2}\int_M g\Big(\nabla(|\nabla u|^2), \nabla(\mathrm{e}^{-\alpha h})\Big)\mathrm{d}_M - \frac{1}{2}\int_{\partial M}\partial_{\mathbf{n}}(|\nabla u|^2)\,\mathrm{e}^{-\alpha h}\,\mathrm{d}_M\\ &\quad + \frac{1}{2}\alpha\int_M \mathrm{e}^{\text{-h}(\alpha\text{-}1)}\,g\Big(\nabla h, \nabla|\nabla u|^2\Big)\,\mathrm{d}m\\ &= -\int_{\partial M}\mathrm{e}^{\text{-h}(\alpha\text{-}1)}\,g\,(\partial_{\mathbf{n}}(\nabla u), \nabla u)\,\mathrm{d}m.\end{aligned}$$

The second term on the right-hand side gives

$$\int_M g\Big(\nabla u, \nabla \mathrm{L_h}\, u + (\alpha-1)\,\mathrm{L_h}\, u \nabla h\Big)\mathrm{d}m$$
$$= \int_M g\left(\nabla u, \mathrm{e}^{\text{-h}}\nabla L_h u\right)\mathrm{d}_M + (\alpha-1)\int_M g\Big(\nabla u, L_h u \nabla h\Big)\mathrm{d}m$$
$$= \int_M g\left(\nabla u, \nabla (L_h u\, \mathrm{e}^{\text{-h}})\right)\mathrm{d}_M + \int_M g(\nabla h, \nabla u) L_h u\,\mathrm{d}m + (\alpha-1)\int_M g(\nabla h, \nabla u) L_h u\,\mathrm{d}m$$
$$= \int_M \Delta\, u L_h u\,\mathrm{d}m + \alpha \int_M g\,(\nabla h, \nabla u)\, L_h u\,\mathrm{d}m + \int_{\partial M} \partial_{\mathbf{n}} u L_h u\,\mathrm{d}m$$
$$= \int_M \mathrm{e}^{\mathrm{h}(\alpha\text{-}1)}\,|L_h u|^2 \mathrm{d}m + \int_{\partial M} \partial_{\mathbf{n}} u L_h u\,\mathrm{d}m.$$

Then, replacing in (5.1.6), one has

$$-\int_{\partial M} \mathrm{e}^{\text{-h}(\alpha\text{-}1)} g\Big(\partial_{\mathbf{n}}(\nabla u), \nabla u\Big)\mathrm{d}m = -\int_M \mathrm{e}^{\text{-h}(\alpha\text{-}1)}\,|\mathrm{Hess}\ u|^2\ \mathrm{d}m$$
$$-\int_M \mathrm{e}^{\text{-h}(\alpha\text{-}1)}(\mathrm{Ric} + \alpha \mathrm{Hess}\ h)\,(\nabla u, \nabla u)\,\mathrm{d}m$$
$$+\int_M \mathrm{e}^{\mathrm{h}(\alpha\text{-}1)}\,|L_h u|^2 \mathrm{d}m + \int_{\partial M} \partial_{\mathbf{n}} u L_h u\,\mathrm{d}m,$$

$$\int_M \mathrm{e}^{\mathrm{h}(\alpha\text{-}1)}\,|L_h u|^2 - \mathrm{e}^{\text{-h}(\alpha\text{-}1)}\,|\mathrm{Hess}\ u|^2 \mathrm{d}m = \int_M \mathrm{e}^{\text{-h}(\alpha\text{-}1)}(\mathrm{Ric} + \alpha \mathrm{Hess}\ h)\,(\nabla u, \nabla u)\,\mathrm{d}m$$
$$-\int_{\partial M} g(\partial_{\mathbf{n}}(\nabla u), \nabla u)\,\mathrm{e}^{\text{-h}(\alpha\text{-}1)} + \partial_{\mathbf{n}} u L_h u\,\mathrm{d}m. \quad (5.1.7)$$

Now, it remains to estimate $\left[g(\partial_{\mathbf{n}}(\nabla u), \nabla u)\,\mathrm{e}^{\text{-h}(\alpha\text{-}1)} + \partial_{\mathbf{n}} u L_h u\right]$ which is equal to

$$\mathrm{e}^{\text{-h}(\alpha\text{-}1)}\,[g(\partial_{\mathbf{n}}(\nabla u), \nabla u) + \partial_{\mathbf{n}} u\, \Delta\, u + \alpha\, g(\nabla h, \nabla u)\partial_{\mathbf{n}} u].$$

We notice that on ∂M on has:

$$\Delta\, u = -H\partial_{\mathbf{n}} u + \Delta_\partial\, u - \partial_{\mathbf{n}}^2 u, \quad (5.1.8)$$

(See e.g [5, (3)].). We recall that our sign convention for the operators Δ and Δ_∂ is the opposite of that in [5]. Moreover, one has (See for instance [5, eq. (5)].):

$$g(\partial_{\mathbf{n}}(\nabla u), \nabla u) = (-\Delta\, u + \Delta_\partial\, u - H\partial_{\mathbf{n}} u)\partial_{\mathbf{n}} u - I(\nabla_\partial u, \nabla_\partial u) + g(\nabla_\partial u, \nabla_\partial \partial_{\mathbf{n}} u)$$
$$= \partial_{\mathbf{n}} u\ \partial_{\mathbf{n}}^2 u - I(\nabla_\partial u, \nabla_\partial u) + g(\nabla_\partial u, \nabla_\partial \partial_{\mathbf{n}} u). \quad (5.1.9)$$

We then combine equalities (5.1.8) and (5.1.9) to derive an expression for the second integral in the right-hand side of (5.1.7):

$$
\begin{aligned}
g(\partial_{\mathbf{n}}(\nabla u), \nabla u)\, e^{-h(\alpha-1)} + \partial_{\mathbf{n}} u L_h u
&= e^{-h(\alpha-1)}\left[-I(\nabla_\partial u, \nabla_\partial u) + g(\nabla_\partial u, \nabla_\partial \partial_{\mathbf{n}} u)\right] \\
&\quad + e^{-h(\alpha-1)}\, \partial_{\mathbf{n}} u\, (-H\partial_{\mathbf{n}} u + \alpha g(\nabla h, \nabla u) + \Delta_\partial u). \quad (5.1.10)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int_M e^{h(\alpha-1)} |L_h u|^2 - e^{-h(\alpha-1)} |\mathrm{Hess}\ u|^2 \mathrm{d}m &= \int_M e^{-h(\alpha-1)} (\mathrm{Ric} + \alpha \mathrm{Hess}\ h)(\nabla u, \nabla u)\, \mathrm{d}m \\
&\quad + \int_{\partial M} e^{-h(\alpha-1)}\, \partial_{\mathbf{n}} u \Big(H\partial_{\mathbf{n}} u - \alpha g(\nabla h, \nabla u) - \Delta_\partial u \Big)\, \mathrm{d}m \\
&\quad + \int_{\partial M} e^{-h(\alpha-1)} \left[I(\nabla_\partial, \nabla_\partial u) - g(\nabla_\partial u, \nabla_\partial \partial_{\mathbf{n}} u) \right]\, \mathrm{d}m.
\end{aligned}
$$

■

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