IMPRIMATUR POUR LA THESE

Normalizers of maximal tori and classifying spaces of compact Lie groups

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FACULTE DES SCIENCES

La Faculté des sciences de l'Université de Neuchâtel sur le rapport des membres du jury,

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autorise l'impression de la présente thèse.

Neuchâtel, le 7 septembre 2000

Le doyen:

J.-P. Derendinger

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THÈSE

présentée à la Faculté des sciences
pour obtenir le grade de docteur ès sciences par

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A ceux qui s'en sont allés

A ceux qui sont venus

A Marie

Remerciements

Je remercie tout d'abord mon directeur de thèse, le Professeur Ueli Suter, pour son soutien mathématique et moral constant et décisif tout au long de l'élaboration de ce travail.

Je remercie mon ami, collègue et camarade de bureau Michel Matthey d'avoir partagé de nombreux moments de ces années d'étude et de recherche avec moi. Sa relecture très minutieuse d'une grande partie de ce travail, ainsi que ses nombreuses suggestions et commentaires pertinents, m'ont permis d'en améliorer le contenu.

Je remercie Akimou Osse pour ses nombreux encouragements. Il est à l'origine de beaucoup des questions qui sont traitées dans ce travail et le chapitre 5 n'existerait pas sans sa collaboration fructueuse.

Je remercie les Professeurs François Sigrist, Dominique Arlettaz et Mamoru Mimura, de m'avoir fait l'honneur de faire partie du Jury, parfois au prix d'un très long voyage, et de leur lecture attentive de ce travail.

Je remercie Catherine, et nos enfants Numa et Ulysse, de leur présence et de leur patience.

Je remercie Alain Jeanneret et Nicolas Louvet.

Je remercie mes parents, ainsi que tous mes proches et amis.

Je remercie mes Professeurs, mes collègues assistants, ainsi que tous les collaborateurs de l'Institut de Mathématiques de l'Université de Neuchâtel.

Je remercie le Fonds National Suisse de la Recherche Scientifique de son soutien financier par la Requête no. 2000-056816.99 durant trois années. Je remercie également la Japan Society for the Promotion of Science de me permettre de poursuivre mon travail grâce à l'octroi d'une bourse post-doctorale.

Je remercie tous ceux qui ne se sont pas encore retrouvés et qui ont participé d'une manière ou d'une autre à l'élaboration de ce travail.

Je remercie la Vie de tout ce qu'elle m'accorde.

Contents

Introduction

The origin of the notion of Lie group goes back to the work of the mathematician Marius Sophus Lie, in the second half of the 19th century. Since then, the notion has been thoroughly studied and the dual nature - algebraic and geometric - of Lie groups make them fundamental tools in many areas of mathematics and physics, especially for the study of the notion of symmetry. Following Hilbert's idea of getting rid of the hypotheses of differentiability in Lie theory, a homotopy-theoretic version of Hilbert's fifth problem developed in the late '60s, the first precise statement being due to Rector [57]. In this context, the relevant object is the classifying space of the Lie group, i.e. the orbit space of a free action of the group on a contractible space. Moreover, a fundamental structure theorem, based on the work of Iwasawa, shows that the homotopy-theoretic study of "reasonable" Lie groups (i.e. Lie groups with a finite number of connected components), restricts to the homotopy-theoretic study of *compact* Lie groups. Now, when focusing on *connected* compact Lie groups, the normalizer of a maximal torus has proved to be a key invariant. One of the purpose of the present work is to investigate whether this is also true in the nonconnected case; the main actors will thus be normalizers of maximal tori in *nonconnected* compact Lie groups. Notice that, since the Lie algebra associated to a Lie group only gives information on the connected component of the identity, Lie groups have often been studied under the hypothesis of connectedness. However, nonconnected compact Lie groups arise naturally in mathematics: the orthogonal group $O(n)$ of rigid motions that fix the origin of the euclidean space $\mathbb{R}^n$, or more generally, the isometry group of a compact Riemannian manifold are examples of such groups. Moreover, some physicists have recently revived the idea that *nonconnected* compact Lie groups might be the relevant mathematical objects in certain gauge theories (see for instance [41]).

To a compact Lie group G, there is a naturally associated group extension given by $G_o \hookrightarrow G \twoheadrightarrow \Gamma$, where G_o denotes the connected component of the identity of G and $\Gamma = G/G_o$ denotes the finite group of connected components. For a fixed maximal torus T in $G_o \subset G$, we will consider two normalizers: the normalizer of T in G, denoted by $N = N_G(T)$, and its subgroup $N_o = N_{G_o}(T)$ obtained by intersecting N with G_o, i.e. the classical normalizer studied in the theory of connected compact Lie groups. For these latter groups, a well-known and remarkable theorem of Curtis, Wiederhold and Williams [16] shows that the normalizer of a maximal torus characterizes the group. Even more: the knowledge of this normalizer allows to reconstruct the corresponding connected compact Lie group. The precise statement is as follows:

> *Two connected compact Lie groups are isomorphic if and only if the normalizers of their maximal tori are isomorphic. Moreover, a connected compact Lie group can be reconstructed from the normalizer of one of its maximal tori.*

The original proof only covered the semisimple case, however Osse later generalized the result to all connected compact Lie groups [54, 55]. This theorem has inspired most parts of the present work, but before explaining in which way, we introduce a problem, apparently not directly related to it. In the beginning of the '90s, Mislin asked the following question motivated by the study of Lie groups from the homotopy point of view [26]:

> *Let G_o and H_o be two connected compact Lie groups. If the classifying spaces BG_o and BH_o are homotopy equivalent, does this imply that G_o and H_o are isomorphic as Lie groups?*

The answer is affirmative and was first given by Osse in [53]. In [50], Notbohm even shows that the hypothesis of connectedness is superfluous. However, concerning the nonconnected case, Theorem 1.2 in [50] is crucial and depends on an important affirmation to which we have an objection. Indeed, example 2 on page 23 of the present work shows that the description in [50] of the set of isomorphism classes of compact Lie groups does not hold in general. One of the origins of this work was thus to give another proof of the affirmative answer to Mislin's question for *nonconnected* compact Lie groups. The theorem of Curtis-Wiederhold-Williams now comes back on stage: the main idea in our approach is trying to find a generalization of this theorem in the nonconnected case, and then apply it to give a new proof. Examples, which in particular show that the hypothesis of connectedness cannot simply be left out in the original theorem of Curtis-Wiederhold-Williams, lead us to conjecture a "generalized Curtis-Wiederhold-Williams theorem".

> **Conjecture.** *Two compact Lie groups G and G' are isomorphic if and only if their normalizers (N, N_o) and (N', N_o') are isomorphic as group pairs.*

Concerning the original theorem of Curtis-Wiederhold-Williams, we also note that it does not answer the following natural question, formulated by Osse:

> *What is the relation between the automorphisms of the normalizer of a maximal torus and those of the corresponding connected compact Lie group?*

A precise answer turns out to be needed in our approach to solving Mislin's question.

Still related to the homotopy-theoretic version of Hilbert's fifth problem, it should also be mentioned that the notions of "maximal torus", "normalizer" and "automorphism" are well defined in the context of p-compact groups [23]. The conjecture and the second question above have translations in this setting. In particular, a major problem in the theory of finite loop spaces is completing the classification of p-compact groups and the normalizer of a maximal torus has already proved to be the distinguishing invariant in many cases [47].

We end up this introduction with a description of the contents of the chapters. Chapter 1 is a collection of definitions and results on compact Lie groups and normalizers of maximal tori that will be needed in this work. Most of these results are classical and cited without any proof.

Chapter 2 gives a classification *up to isomorphism* of compact Lie groups with nonabelian connected component. This classification might belong to the category of results *"well-known to the experts"* (be cautious, this is not a category in the mathematical sense). However, we could not find any reference for it in the literature. Even the rather exhaustive work of de Siebenthal [18] in the '50s does not give the answer we need for the sequel. In more details, after reviewing the classical relation between group extensions with nonabelian kernel and low dimensional cohomology groups, we apply it to the particular case of compact Lie groups. We then define an action of the product of the outer automorphism group of G_o by the automorphism group of Γ, $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$, on the set $\mathcal{E}$ of classes of equivalent extensions of the form $G_o \hookrightarrow E \twoheadrightarrow \Gamma$. Finally, we use centralizers of principal subgroups of rank one for passing from "*up to equivalence of extensions*" to "*up to isomorphism of Lie groups*". More precisely, we get

> **Theorem.** *Two compact Lie groups are isomorphic if and only if the two corresponding cohomology classes in $\mathcal{E}$ are in the same orbit under the action of* $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$.

We also give examples of the classification and take a look at the importance of semidirect products in compact Lie groups with a revisited "sandwich" theorem.

In chapter 3, we study the automorphism group of N_o, i.e. of the normalizer in the connected case. We start with an overview of the notion of root diagram. We then give a description, based on classical cohomology of groups, of the automorphisms that induce the identity on the maximal torus T. Next, we present a decomposition of the outer automorphism group of N_o as a semidirect product, which answers the second question above and can be interpreted in the following terms: *"the automorphisms of N_o are those of G_o together with automorphisms that induce the identity on the torus, which are non-extendable to G_o."* Precisely, we show the following:

Theorem. *Let $W_o = N_o/T$ denote the Weyl group of G_o. The outer automorphism group of N_o canonically decomposes as*

$$\mathrm{Out}(N_o) \cong H^1(W_o;T) \rtimes \mathrm{Out}(G_o)\,,$$

where the W_o-module T is endowed with the natural action induced by the extension $T \hookrightarrow N_o \twoheadrightarrow W_o$.

In the last section of this chapter, we introduce some notions related to the Tits presentation of N_o that are then used for the explicit calculation of the outer automorphism group of some normalizers.

Chapter 4 begins with the proof of a reconstruction process: starting from the group pair (N, N_o), i.e. from the inclusion $N_o \hookrightarrow N$, corresponding to the normalizers of a maximal torus in some compact Lie group G, we can cite the isomorphism class of G. The rest of the chapter is dedicated to showing that the "generalized Curtis-Wiederhold-Williams theorem" is true in special cases, by proving that the mentioned reconstruction process is invariant under isomorphisms of pairs of normalizers. In particular, we get:

Theorem. *The conjecture is true in the following cases:*

1. *the isomorphism of pairs of normalizers induces an automorphism of N_o that is the restriction of some automorphism of G_o;*
2. *the connected compact Lie group G_o corresponding to N_o is of adjoint type;*
3. *the connected compact Lie group G_o corresponding to N_o is simple.*

We also show that the general case can be reduced to a question about semidirect products.

The purpose of chapter 5 is to present our approach for proving that two compact Lie groups are isomorphic if and only if their classifying spaces are homotopy equivalent, taking the connected case for granted. It starts with some recollections on classifying spaces and fibrations. Then some known results on mapping spaces involving classifying spaces are gathered. Section 5.3 contains a key result showing that the maps from BG to the classifying space $B\bar{G}$ of the compact Lie group $\bar{G}$ corresponding to the quotient of G by the center of G_o are essentially unique. It also contains the following reconstruction result: knowing its classifying space BG, one can reconstruct the compact Lie group G up to isomorphism. Then, using point (1) of the above theorem of chapter 4, and a proposition due to Zabrodsky, we prove in section 5.5 that this reconstruction process is invariant under homotopy equivalence for all compact Lie groups. We therefore get:

> **Theorem.** *Two (not necessarily connected) compact Lie groups are isomorphic if and only if their classifying spaces are homotopy equivalent.*

On the way to proving this last theorem, we also get the following result for the normalizer N_o.

> **Theorem.** *The map*
>
> $$\beta_{N_o} : \mathrm{Out}(N_o) \longrightarrow \mathrm{Aut}(BN_o)\,,\ [\psi] \longmapsto [B\psi]$$
>
> *is an isomorphism of groups. In particular*
>
> $$\mathrm{Aut}(BN_o) \cong H^1(W_o;T) \rtimes \mathrm{Out}(G_o)\,.$$

In other words, the group of homotopy classes of self-homotopy equivalences of BN_o and the outer automorphism group of N_o are isomorphic. This extends to the class of normalizers N_o a result for connected compact Lie groups of Jackowski, McClure and Oliver [37]. The fact that β_{N_o} is an isomorphism in this last theorem is a particular case of an unpublished theorem by Møller. However, we believe that the precise description of $\mathrm{Aut}(BN_o)$ is new.

Finally, the appendix contains some material and calculations related to the third chapter.

Chapter 1

Compact Lie groups and normalizers of maximal tori

Starting with a minimal background on compact Lie groups, this chapter introduces some properties of normalizers of maximal tori, both in the connected and nonconnected cases.

1.1 Background on Lie groups

A *(real) Lie group* G is a group G equipped with the structure of a smooth manifold such that both maps

$$G \times G \longrightarrow G\,,\ (g,h) \longmapsto gh$$

and

$$G \longrightarrow G\,,\ g \longmapsto g^{-1}$$

are smooth. A *homomorphism* $\varphi : G \to H$ of Lie groups is a homomorphism of groups that is also a smooth map. A connected Lie group will be called *simple* if it is not abelian and has no proper closed normal subgroup of strictly positive dimension.

Examples 1.1

1. Any finite or countable group equipped with the discrete topology and the structure of a 0-dimensional manifold is a Lie group.

2. The *unit circle* $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ is a connected compact abelian Lie group.

3. The *unitary group* $U(n) = \{A \in GL_n(\mathbb{C}) : {}^t\bar{A}\cdot A = \mathbf{1}\}$ is a connected compact Lie group; note that $U(1) = S^1$.

4. The *general linear group* $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det A \neq 0\}$ is a Lie group; it is non-compact (for $n \geqslant 2$) and has two connected components.

5. The *direct product* of Lie groups is the direct product of the corresponding abstract groups endowed with the smooth structure of the direct product of smooth manifolds. One can show that any connected compact abelian Lie group is a *torus*, i.e. a group isomorphic to the *n-dimensional torus* $\mathbb{T}^n$ defined as the direct product of n copies of the circle group S^1.

In the sequel we will only consider the compact case and G will always denote a compact Lie group. To G, there is a naturally associated group extension given by $G_o \hookrightarrow G \twoheadrightarrow \Gamma$, where G_o denotes the connected component of the neutral element of G and $\Gamma = G/G_o = \pi_o(G)$ denotes the finite group of connected components. We give some examples of *nonconnected* compact Lie groups.

Examples 1.2

1. Any group extension
$$\mathbb{T}^n \hookrightarrow E \twoheadrightarrow F,$$
with F a finite group.

2. The *orthogonal group* $\{A \in GL_n(\mathbb{R}) : {}^tA\cdot A = \mathbf{1}\}$, with the associated group extension
$$SO(n) = O(n)_o \hookrightarrow O(n) \overset{\det}{\twoheadrightarrow} \mathbb{Z}/2.$$
In fact $O(n) = SO(n) \rtimes \langle \mathrm{diag}(-1,1,\dots,1)\rangle \cong SO(n) \rtimes \mathbb{Z}/2$.

3. The orthogonal group has a double cover called $\mathrm{Pin}(n)$. It is a nonconnected compact Lie group, and for $n \geqslant 2$ we have
$$\mathrm{Spin}(n) = \mathrm{Pin}(n)_o \hookrightarrow \mathrm{Pin}(n) \twoheadrightarrow \mathbb{Z}/2.$$
For $n \geqslant 3$ the group $\mathrm{Spin}(n)$ is the universal cover of $SO(n)$.

4. There is a famous "triality phenomenon" in Spin(8) expressed by the fact that its outer automorphism group is isomorphic to the symmetric group on three letters, i.e. $\mathrm{Out}(\mathrm{Spin}(8)) \cong \Sigma_3$. In the exceptional Lie group F_4 this phenomenon is embodied by a nonconnected compact Lie group. Indeed F_4 contains Spin(8) as a subgroup of maximal rank and its normalizer $N_{F_4}(\mathrm{Spin}(8))$

fits in the extension

$$\mathrm{Spin}(8) \hookrightarrow N_{\mathrm{F}_4}(\mathrm{Spin}(8)) \twoheadrightarrow \Sigma_3\,,$$

each element in Out(Spin(8)) being represented by an inner automorphism c_n, witb $n \in N_{\mathrm{F}_4}(\mathrm{Spin}(8))$ (see [3, Chap. 16] and [10, Exercice 6, p. 113]).

From works of Killing, E. Cartan and Weyl the precise structure of connected compact Lie groups is known.

Theorem 1.3

(i) If G_o is a connected compact Lie group, then

$$G_o \cong (S \times H)/K\,,$$

where S is a torus, H a 1-connected compact Lie group and K a finite central subgroup of $S \times H$.

(ii) Any 1-connected compact Lie group is a direct product of simple 1-connected compact Lie groups.

(iii) Let H be a 1-connected simple compact Lie group. Then H is isomorphic to one of the following 1-connected simple compact Lie groups: $A_n = \mathrm{SU}(n+1)$, $B_n = \mathrm{Spin}(2n+1)$, $C_n = \mathrm{Sp}(n)$, $D_n = \mathrm{Spin}(2n)$, G_2, F_4, E_6, E_7, E_8, with $n = 1, 2, \ldots$ (and $n \geqslant 3$ for the case D_n).

A connected compact Lie group G_o will be said *semisimple* if the torus S in the previous theorem is trivial; it is equivalent to saying that the center $Z_o = Z(G_o)$ of G_o is finite. The unitary group $\mathrm{U}(n)$ is a connected compact Lie group that is *not* semisimple. The centerless quotient $\bar{G}_o = G_o/Z_o$ is called the *adjoint group* of G_o. It is isomorphic to the image of G_o in the adjoint representation, which is the action of G_o on its Lie algebra LG_o induced by conjugation, i.e. $\mathrm{Ad} : G_o \to \mathrm{GL}(LG_o)$, $g \mapsto (c_g)_*$. We will also say that a centerless connected compact Lie group is of *adjoint type*. For more details on compact Lie groups see, for instance, the books of Adams [1], Bourbaki [10], Bröcker-tom Dieck [14], Mimura-Toda [42].

1.2 The normalizer of a maximal torus in the connected case

A closed subgroup $T \subset G_o$ is called a *maximal torus* if it is a torus and if it is strictly contained in no other torus. It is not difficult to check that any non-trivial connected compact Lie group has at least one maximal torus. The following fundamental theorem is a first illustration of the crucial role played by maximal tori in connected compact Lie groups.

Theorem 1.4 *Let G_o be a connected compact Lie group.*

(i) Any element in G_o is contained in a maximal torus.

(ii) Any two maximal tori T and T' of G_o are conjugate, i.e. there exists an element $g \in G_o$ such that $c_g(T) = gTg^{-1} = T'$. In particular, the dimension of a maximal torus is an invariant of G_o, called the rank of G_o.

(iii) The conjugates of a maximal torus T cover the group, i.e.

$$G_o = \bigcup_{g \in G_o} gTg^{-1} .$$

A proof of tbis theorem can be found in [14, Chap. IV].

We now fix a maximal torus T in G_o. One of the most important object in this work will be the normalizer N_o of T in G_o, defined by

$$N_o = N_{G_o}(T) = \{n \in G_o : c_n(T) = nTn^{-1} = T\} .$$

Tbe *Weyl group* of G_o is the finite quotient $W_o = N_o/T$. Tbe map

$$W_o \times T \to T , \ (w = nT, t) \mapsto w \cdot t = ntn^{-1}$$

defines a faithful action of W_o on T. Note that if G_o is oonabelian, N_o is a *nonconnected* compact Lie group, with the associated extensioa

$$T = (N_o)_o \hookrightarrow N_o \overset{\pi}{\twoheadrightarrow} W_o .$$

The notation N_o may seem clumsy, but it will be justified when we consider the case of a nonconnected compact Lie group G.

Examples 1.5

1. In $\mathrm{U}(n)$ tbe *standard maximal torus* $T(\mathrm{U}(n))$ consists of the subgroup of diagonal matrices. The Weyl group $W_o(\mathrm{U}(n))$ is tbe symmetric group Σ_n and its action on $T(\mathrm{U}(n))$ is given by permutation of the elements on the diagoual, i.e. for all $\sigma \in W_o(\mathrm{U}(n)) = \Sigma_n$, $t = \mathrm{diag}(z_1, \dots, z_n) \in T(\mathrm{U}(n))$:

$$\sigma \cdot t = \sigma \cdot \mathrm{diag}(z_1, \dots, z_n) = \mathrm{diag}(z_{\sigma^{-1}(1)}, \dots, z_{\sigma^{-1}(n)}) .$$

   The same bolds for $\mathrm{SU}(n) \subset \mathrm{U}(n)$, except that the standard maximal torus is given by $T(\mathrm{SU}(n)) = T(\mathrm{U}(n)) \cap \det^{-1}(1) \cong \mathbb{T}^{n-1}$.

2. In the particular case of the group of quaternions of unit norm $\mathbb{S}^3 \cong \mathrm{SU}(2)$, the previous example becomes

$$T(\mathbb{S}^3) = \{\cos\varphi + i\sin\varphi\,,\ \varphi \in \mathbb{R}\} = \mathbb{S}^1$$

and

$$N_o(\mathbb{S}^3) = \mathbb{S}^1 \amalg j\mathbb{S}^1 \cong \mathrm{Pin}(2)\,.$$

We will use the same notation for $N_o(\mathrm{SU}(2))$, with the abuses

$$\{\mathrm{diag}(z, z^{-1}) : |z| = 1\} = \mathbb{S}^1$$

and

$$j \longmapsto j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SU}(2)\,.$$

under Cayley's isomorphism.

3. In $\mathrm{SO}(3)$, the standard maximal torus is given by

$$T(\mathrm{SO}(3)) = \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} : \varphi \in \mathbb{R} \right\}.$$

Let $d = \mathrm{diag}(-1, 1, -1) \in \mathrm{SO}(3)$; then for the normalizer we have

$$N_o(\mathrm{SO}(3)) = T(\mathrm{SO}(3)) \amalg d{\cdot}T(\mathrm{SO}(3)) \cong \mathrm{O}(2)\,.$$

We now recall the precise statement of the theorem of Curtis, Wiederhold and Williams.

Theorem 1.6 (Curtis-Wiederhold-Williams; Osse) *Two connected compact Lie groups are isomorphic if and only if the normalizers of their maximal tori are isomorphic. Moreover, a connected compact Lie group can be reconstructed from the normalizer of one of its maximal tori.*

The original proof of the theorem for semisimple connected compact Lie groups can be found in the paper by Curtis-Wiederhold-Williams [16]; Osse later observed that the concept of root diagrams (see chapter 3) allows to extend the original proof to all connected compact Lie groups, and also proposed a new proof [55]. An independent proof due to Notbohm is presented in [50].

There is another remarkable result that we will need: it is a theorem due to Tits saying that, in the connected case, the normalizer has a presentation where the generating set is a finite set added to the toral part. Before stating it we need to recall some facts and notations. Denote by LT the Lie algebra of the maximal torus T of G_o. Let $R = R(G_o, T) \subset LT^*$ be the root system of G_o associated to T and $R^\vee \subset LT$ the corresponding coroot system. The Weyl group W_o acts on LT and is generated by the reflections s_α for $\alpha \in R$, explicitly given by

$$s_\alpha : LT \longmapsto LT\,,\ X \longmapsto s_\alpha(X) = X - \alpha(X)\alpha^\vee\,.$$

Once a W_o-invariant inner product $(.,.)$ has been fixed the formula becomes

$$s_\alpha(X) = X - 2\frac{(X\,,\alpha^\vee)}{(\alpha^\vee\,,\alpha^\vee)}\cdot\alpha^\vee\,.$$

For $w = s_\alpha \in W_o$, $t \in T$, we will also use the notation $w\cdot t = s_\alpha(t)$. Define $T_\alpha = \exp(\mathbb{R}\alpha^\vee) = \{t \in T : s_\alpha(t) = t^{-1}\}$; this is a closed subgroup of T isomorphic to $\mathbb{S}^1$. Also define $h_\alpha = \exp(\frac{\alpha^\vee}{2})$. Recall that for each root $\alpha \in R$ there exists a homomorphism of Lie groups $\nu_\alpha : \mathrm{SU}(2) \to G_o$, unique up to conjugation by an element of $T(\mathrm{SU}(2)) = \mathbb{S}^1$, and satisfying, among other properties,

- $\nu_\alpha(\mathbb{S}^1) = T_\alpha$
- $\nu_\alpha(j\mathbb{S}^1) \subset \pi^{-1}(s_\alpha)\ (\subset N_o \setminus T)$
- $\nu_\alpha(-1) = h_\alpha$

(see [10, §4, N° 5]). Fix a basis B of R, and denote by $\ell_{\alpha\beta}$ the order of the element $s_\alpha s_\beta$ in W_o.

Theorem 1.7 (Tits) *For all roots $\alpha \in B$ let us choose an element q_α in $\nu_\alpha(j\mathbb{S}^1)$. Then for all α, $\beta \in B$, $\alpha \neq \beta$, and for all $t \in T$, the following relations hold:*

$$\begin{array}{lrcl} (1) & q_\alpha^2 & = & h_\alpha \\ (2) & \underbrace{q_\alpha q_\beta q_\alpha \cdots}_{l_{\alpha\beta}\ factors} & = & \underbrace{q_\beta q_\alpha q_\beta \cdots}_{l_{\alpha\beta}\ factors} \\ (3) & q_\alpha t q_\alpha^{-1} & = & s_\alpha(t)\,. \end{array}$$

Moreover the group N_o is generated by the set $T \cup \{q_\alpha\}_{\alpha\in B}$ and is defined by relations (1), (2) and (3), added to the fact that T is a subgroup of N_o. More precisely, any relation between elements of N_o is a consequence of (1), (2), (3) and of relations between elements of T.

Tits proved this theorem for split reductive semisimple algebraic groups [67, 68], however his proof can be translated for compact Lie groups (see the remark in the proof of Lemma 3 in [16], or Chapter 3 in [40]).

We end up this section with a result relating the center $Z(G_o)$ of G_o to the center $Z(N_o)$ of N_o, which shows that they coincide in most cases. This result is due to Matthey and its proof is not published [19].

Theorem 1.8 (Matthey)

(i) The center of N_o is the set of invariants of the W_o-module T, i.e.

$$Z(N_o) = T^{W_o}.$$

(ii) The quotient $Z(N_o)/Z(G_o)$ is an elementary abelian 2-group; more precisely

$$Z(N_o)/Z(G_o) \cong (\mathbb{Z}/2)^k,$$

where k is the number of direct factors of type $\mathrm{SO}(2\ell+1)$ *in G_o.*

For closely related results, consult the preprint of Dwyer and Wilkerson [24].

1.3 The normalizer of a maximal torus and related subgroups in the nonconnected case

In this section we introduce and study some of the properties of an other fundamental object of this work: the normalizer $N = N_G(T)$ of a maximal torus $T \subset G_o$ in the nonconnected compact Lie group G. We also define some related subgroups. Aiming at a generalization of the theorem of Curtis, Wiederhold and Williams to the nonconnected case we will suppose for the rest of this work that G_o is *nonabelian.*

Let B still denote a basis of the root system R associated to the fixed maximal torus T. The *principal diagonal* of G_o with respect to B is the 1-dimensional subspace

$$D(B) = \{X \in LT : \alpha(X) = \beta(X), \forall\, \alpha, \beta \in B\} \cap \mathcal{H} \subset LT,$$

where $\mathcal{H}$ is the maximal semisimple ideal of the Lie algebra LG_o. Then, taking the exponential, $\Delta = \Delta(B) = \exp(D(B))$ is a closed subgroup of T isomorphic to $\mathbb{S}^1$.

Definition 1.9 (de Siebenthal) A *principal subgroup* of G_o (associated to T) is a connected closed subgroup H satisfying

- H is not contaioed in any proper connected closed subgroup of maximal rank,
- there exists a basis B of R such that $\Delta(B) \subset H$.

In particular, a principal subgroup verifies $(H, T') = G_o$ for any maximal torus T'. The remarkable work of de Siehenthal shows that any compact Lie group possesses a priocipal subgroup of rank 1, thus isomorphic to $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$, and that two principal subgroups are conjugate [17, Chap. IV]. Let H_T be such a priocipal subgroup of rank 1, associated to a fixed maximal torus T. We will denote by $Z = Z_G(H_T)$ its ceotralizer in G. Before stating the main theorem of this section, we consider one more subgroup defined as the centralizer $Q = Z_G(\Delta)$ in G of the subgroup corresponding to a principal diagonal.

Theorem 1.10 *For any compact Lie group G there exists a commutative diagram*

$$\begin{array}{ccccc}
Z_o & \hookrightarrow & Z & \twoheadrightarrow & \Gamma \\
\downarrow & & \downarrow & & \| \\
T & \hookrightarrow & Q & \twoheadrightarrow & \Gamma \\
\downarrow & & \downarrow & & \| \\
N_o & \hookrightarrow & N & \twoheadrightarrow & \Gamma \\
\downarrow & & \downarrow & & \| \\
G_o & \hookrightarrow & G & \twoheadrightarrow & \Gamma
\end{array}$$

where each row is a group extension, and where Z_o is the center of G_o.

Proof. The centralizer of H_T in G_o is equal to the center Z_o of G_o (in fact, by a theorem of Borel and de Siebenthal [8], this property characterizes the closed subgroups of G_o that are not contaioed in any proper connected closed subgroup of maximal rank [10, Ex. 15, p. 116]). As Z intersects every component of G [17, Théorème 4, pp. 253-254], we get an extension $Z_o \hookrightarrow Z = Z_G(H_T) \twoheadrightarrow \Gamma$. The other statements are deduced from the fact that $\Delta \subset T$ contains a regular element, i.e. an element that is contained in exactly one maximal torus, T in the present case. (Consult [31] or [40] for more details.) □

Remarks 1.11

1. We say that the subgroup Q is an *extended maximal torus* in G. These subgroups share some important properties with maximal tori: they are all conjugate, and fixing one of them, its conjugates by the elements of G_o cover the whole group G. They appear in the literature under various disguises (see for instance Oliver [52, Section 1.] and Segal [61, §4.]), as explained in [31].

2. The *Weyl group* of G is defined to be the quotient $W = N/T$. Associated to it, there are two group extensions, namely

$$T \hookrightarrow N \twoheadrightarrow W$$

and

$$W_o \hookrightarrow W \twoheadrightarrow \Gamma .$$

An extended maximal torus furnishes a splitting of the second extension (since $\Gamma = Q/T \subset N/T = W$). This shows that $W \cong W_o \rtimes \Gamma$.

The following theorem is another application of principal subgroups of rank 1. It will be very useful when applying the theory of group extensions to the case of compact Lie groups in next chapter. For the proof, we refer to [18, Théorème, pp. 46-47] (for another approach consult [10, §4.10.]).

Theorem 1.12 (de Siebenthal) *Let G_o be a connected compact Lie group and $H_T \subset G_o$ a principal subgroup of rank* 1. *Then the extension*

$$\mathrm{Inn}(G_o) \overset{i}{\hookrightarrow} \mathrm{Aut}(G_o) \overset{\pi}{\twoheadrightarrow} \mathrm{Out}(G_o)$$

is split, i.e.

$$\mathrm{Aut}(G_o) \cong \mathrm{Inn}(G_o) \rtimes \mathrm{Out}(G_o) .$$

A possible splitting is given by $s : \mathrm{Out}(G_o) \to \mathrm{Aut}(G_o)$, where $s(\alpha)$ is the unique automorphism in $\pi^{-1}(\alpha)$ fixing H_T pointwise.

The fact that the extension associated to $\mathrm{Aut}(G_o)$ is split was known before the work of de Siebenthal, at least in the semisimple case, and appeared in a paper of Dynkin [25].

Chapter 2

Cohomological classification of compact Lie groups

This chapter presents a cohomological classification of *nonconnected* compact Lie groups *up to isomorphism*, including some examples. It ends up with some related semidirect decompositions stated in a "sandwich" theorem.

2.1 Extension associated to a compact Lie group and cohomology

The relation between group extensions with nonabelian kernel and low dimensional cohomology groups is classical and well-known since the work of Eilenberg and MacLane [27]. As it is a basic tool in our approach to the classification of compact Lie groups, we start this section with a brief overview of the general case before focusing on the special case of compact Lie groups. For a thorough treatment of the general case, we recommend the textbooks by MacLane [39], Robinson [59], or Adem-Milgram [4]; a more concise approach can be found in Kirillov's book [38], and a sketch, which we roughly follow, in Brown's [15]. An extension E of the group Q by the group K (some authors say of K by Q), denoted by $K \hookrightarrow E \twoheadrightarrow Q$, induces an "outer action", i.e. a homomorphism $\varphi : Q \to \mathrm{Out}(K)$. Conversely being given a pair of groups K and Q and a homomorphism $\varphi : Q \to \mathrm{Out}(K)$ it is natural to try to classify all the possible extensions $K \hookrightarrow E \twoheadrightarrow Q$ that induce φ . We will soon see that this can be done *up to equivalence of extensions*. But before, recall that two extensions $K \hookrightarrow E \twoheadrightarrow Q$ and $K \hookrightarrow E' \twoheadrightarrow Q$ are said to be *equivalent* if there exists a homomorphism $\rho : E \to E'$ with the commutative diagram

$$\begin{array}{ccccc} K & \hookrightarrow & E & \twoheadrightarrow & \Gamma \\ \| & & \downarrow{\scriptstyle\rho} & & \| \\ K & \hookrightarrow & E' & \twoheadrightarrow & \Gamma \end{array}$$

It is then not difficult to show that ρ is an isomorphism and that the induced "outer action" homomorphisms are equal. The fundamental theorem of group extension theory shows that two cohomology groups completely describe the set $\mathcal{E}(K, Q, \varphi)$ of equivalence classes of extensions inducing φ. Let $A = Z(K)$ be the center of K; then φ induces, by restriction, a homomorphism $\bar{\varphi} : Q \to \mathrm{Aut}(A)$.

Theorem 2.1 (Eilenberg-MacLane)

(i) The set $\mathcal{E}(K, Q, \varphi)$ admits a free and transitive action by the abelian group $H^2_{\bar{\varphi}}(Q; A)$. Hence either $\mathcal{E}(K, Q, \varphi) = \varnothing$ or else there is a (non-canonical) bijection $\mathcal{E}(K, Q, \varphi) \approx H^2_{\bar{\varphi}}(Q; A)$, which depends on the choice of a particular element of $\mathcal{E}(K, Q, \varphi)$.

(ii) A homomorphism $\varphi : Q \to \mathrm{Out}(K)$ gives rise to an "obstruction" in $H^3_{\bar{\varphi}}(Q; A)$, which vanishes if and only if $\mathcal{E}(K, Q, \varphi) \neq \varnothing$.

A proof of this theorem can be found in MacLane's book [39]. As an immediate consequence one gets the following simple special case:

Corollary 2.2 *If K has a trivial center then there is, up to equivalence, exactly one extension of Q by K corresponding to a given homomorphism $Q \to \mathrm{Out}(K)$.*

Remarks 2.3

1. An even simpler situation arises when both the center and the outer automorphism group of K are trivial (such groups are called *complete*). This is *exactly* the case for which any extension with K as normal subgroup is a direct product. As a first application to compact Lie groups, we observe that $\mathrm{SO}(2\ell + 1)$ is complete, and thus for any extension $\mathrm{SO}(2\ell + 1) \hookrightarrow E \twoheadrightarrow Q$ we have

$$E \cong \mathrm{SO}(2\ell + 1) \times Q\,.$$

2. The cohomology of groups gives a satisfactory solution to the problem of group extensions. However, it does *not* classify them *up to isomorphism* as the classical example $Q = K = \mathbf{Z}/3$ shows. In this case $H^2(\mathbf{Z}/3; \mathbf{Z}/3) \cong \mathbf{Z}/3$ but there are only two non-isomorphic groups with 9 elements. In fact, there are two non-equivalent extensions corresponding to $E = \mathbf{Z}/9$.

We now want to apply the classical theory of group extensions to the case of compact Lie groups. We fix a nonabelian connected compact Lie group G_o, a finite group Γ, and a homomorphism $\varphi : \Gamma \to \mathrm{Out}(G_o)$. We write Z_o for the center of G_o. Choosing a principal subgroup $H_T \subset G_o$ and fixing s as in theorem 1.12, we get the commutative diagram

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\varphi} & \mathrm{Out}(G_o) & \xrightarrow{s} & \mathrm{Aut}(G_o) \\ & \searrow^{\bar\varphi} & \downarrow r & & \downarrow r \\ & & \mathrm{Aut}\,(Z_o) & = & \mathrm{Aut}\,(Z_o) \end{array}$$

In the sequel, we will use the notation $\sigma_\gamma = (s \circ \varphi)(\gamma)$, for $\gamma \in \Gamma$. In the particular case of compact Lie groups, we get:

Proposition 2.4

(i) The set of equivalence classes of extensions $\mathcal{E}(\Gamma, G_o, \varphi)$ is in bijection with the cohomology group $H^2_{\bar\varphi}(\Gamma; Z_o)$.

(ii) For all $u \in H^2_{\bar\varphi}(\Gamma; Z_o)$ the corresponding extension $G_o \hookrightarrow G \twoheadrightarrow \Gamma$ carries a natural structure of Lie group.

Proof. By theorem 2.1, it suffices to see that $\mathcal{E}(\Gamma, G_o, \varphi) \neq \varnothing$ to prove (i). But this follows from theorem 1.12: the semidirect product $G = G_o \rtimes_{s\circ\varphi} \Gamma$ exists for any φ. The second statement is easily deduced from classical Lie group theory. □

The bijection in this last proposition is not canonical. As stated in theorem 2.1, it depends on the choice of a particular element in $\mathcal{E}(\Gamma, G_o, \varphi)$. On the other hand, there is a canonical bijection between $\mathcal{E}(\Gamma, Z_o, \bar\varphi)$ and $H^2_{\bar\varphi}(\Gamma; Z_o)$, thus a bijection $\Lambda : \mathcal{E}(\Gamma, Z_o, \bar\varphi) \to \mathcal{E}(\Gamma, G_o, \varphi)$ still depending on the previous choice. Let us describe this bijection by first recalling that $H^2_{\bar\varphi}(\Gamma; Z_o) = Z^2_{\bar\varphi}(\Gamma; Z_o)/B^2_{\bar\varphi}(\Gamma; Z_o)$, where, keeping the multiplicative notation in Z_o, the cocycles are functions $h : \Gamma \times \Gamma \to Z_o$ satisfying

- $h(\gamma_1, e) = h(e, \gamma_2) = \mathrm{e}$ (normalization)
- $(\delta h)(\gamma_1, \gamma_2, \gamma_3) = \sigma_{\gamma_1}(h(\gamma_2, \gamma_3)) \cdot h(\gamma_1\gamma_2, \gamma_3)^{-1} \cdot h(\gamma_1, \gamma_2\gamma_3) \cdot h(\gamma_1, \gamma_2)^{-1} = e$

for all $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$; the coboundaries satisfy the extra condition that there exists a function $k : \Gamma \to Z_o$, with $k(e) = e$, and such that

$$h(\gamma_1, \gamma_2) = (\delta k)(\gamma_1, \gamma_2) = \sigma_{\gamma_1}(k(\gamma_2)) \cdot k(\gamma_1\gamma_2)^{-1} \cdot k(\gamma_1)$$

for all $\gamma_1, \gamma_2 \in \Gamma$. Let us choose the semidirect product $G_o \rtimes \Gamma$ associated to the section s as the extension corresponding to the neutral element in $H^2_{\bar\varphi}(\Gamma; Z_o)$. Then for an element $u = [h] \in H^2_{\bar\varphi}(\Gamma; Z_o)$, the corresponding class of extensions is given by $[G_o \hookrightarrow G_h \twoheadrightarrow \Gamma]$, where G_h is the *set* $G_o \times \Gamma$ equipped with the multiplication

$$(g,\gamma) *_h (g',\gamma') = \big(g\cdot\sigma_\gamma(g')\cdot h(\gamma,\gamma'), \gamma\cdot\gamma'\big)$$

(see [39], Chapter IV, §4 and §8). We will also denote by $G_o \hookrightarrow G_u \twoheadrightarrow \Gamma$ any representative of the class of extensions corresponding to $u \in H^2_{\bar\varphi}(\Gamma; Z_o)$. We next give a canonical description of the inverse of Λ, i.e. a description that does not depend on the choice of a particular element in $\mathcal{E}(\Gamma, G_o, \varphi)$.

Lemma 2.5 *For any principal subgroup of rank one H in G_o, the map*

$$\begin{array}{rccc} \Theta: & \mathcal{E}(\Gamma, G_o, \varphi) & \longrightarrow & \mathcal{E}(\Gamma, Z_o, \bar\varphi) \\ & [G_o \hookrightarrow G \twoheadrightarrow \Gamma] & \longrightarrow & [Z_o \hookrightarrow Z_G(H) \twoheadrightarrow \Gamma] \end{array}$$

is the inverse of Λ (and does not depend on the choice of H). In particular it is a bijection.

Proof. As centralizers of principal subgroups of rank 1 are preserved by isomorphisms of G, Θ does not depend on the choice of a representative in $[G_o \hookrightarrow G \twoheadrightarrow \Gamma]$. Let $u = [Z_o \hookrightarrow E_h \twoheadrightarrow \Gamma] = [h] \in H^2_{\bar\varphi}(\Gamma, Z_o)$. Then we have the commutative diagram

$$\begin{array}{ccccc} Z_o & \hookrightarrow & E_h & \twoheadrightarrow & \Gamma \\ \downarrow & & \downarrow & & \| \\ G_o & \hookrightarrow & G = G_h & \twoheadrightarrow & \Gamma \end{array}$$

where E_h is $Z_o \times \Gamma$ as a set. Let us show that $E_h = Z_{G_h}(H_T)$, H_T being the principal subgroup of rank 1 of G_o corresponding to the fixed section s. By proposition 1.10 it is enough to check that E_h is contained in $Z_{G_h}(H_T)$. Let $(z,\gamma) \in E_h$ and $(x,e) \in H_T \leqslant G_o \leqslant G_h$. We calculate

$$\begin{aligned} (z,\gamma) *_h (x,e) &= (z\cdot\sigma_\gamma(x)\cdot h(\gamma,e),\gamma) \\ &= (z\cdot x, \gamma) \end{aligned}$$

and

$$\begin{aligned} (x,e) *_h (z,\gamma) &= (x\cdot\sigma_e(z)\cdot h(e,\gamma),\gamma) \\ &= (x\cdot z, \gamma) \\ &= (z\cdot x, \gamma), \end{aligned}$$

by normalization, and because the restriction of σ_γ to H_T is the identity by the choice of the section s.

Now, as the principal subgroups of rank 1 are all conjugate by an element of G_o (see [17], Théorème, pp. 46-47), so are their centralizers. Therefore the extensions $Z_o \hookrightarrow Z_G(H) \twoheadrightarrow \Gamma$, for H running through the family of principal subgroups of rank 1, all belong to the same class. Thus we have just proven that Θ is well defined and satisfies $\Theta \circ \Lambda = id_{\mathcal{E}(\Gamma, Z_o, \bar{\varphi})}$. As Λ is bijective, this shows that $\Theta = \Lambda^{-1}$. □

We summarize the situation exposed in this section.

Theorem 2.6 *Suppose given G_o, a homomorphism $\varphi : \Gamma \to \mathrm{Out}(G_o)$ and an extension $Z_o \hookrightarrow Z \twoheadrightarrow \Gamma$ for which the homomorphism $\Gamma \to \mathrm{Aut}(Z_o)$ coincides with $\bar{\varphi}$. Then, up to equivalence of extensions, there exists a unique compact Lie group G fitting into the commutative diagram*

$$\begin{array}{ccccc} Z_o & \hookrightarrow & Z & \twoheadrightarrow & \Gamma \\ \downarrow & & \downarrow & & \| \\ G_o & \hookrightarrow & G & \twoheadrightarrow & \Gamma \\ \downarrow & & \downarrow & & \downarrow \varphi \\ \mathrm{Inn}(G_o) & \hookrightarrow & \mathrm{Aut}(G_o) & \twoheadrightarrow & \mathrm{Out}(G_o) \end{array}$$

where the rows are group extensions. Moreover the given data allow to construct an extension $G_o \hookrightarrow G \twoheadrightarrow \Gamma$, in which the subgroup Z is the centralizer of a principal subgroup of rank 1.

Conversely, one can recover the class of the extension $Z_o \hookrightarrow Z \twoheadrightarrow \Gamma$ in the extension $G_o \hookrightarrow G \twoheadrightarrow \Gamma$ by taking the centralizer of any principal subgroup of rank 1.

2.2 Classification of compact Lie groups up to isomorphism

All the possible classes of extensions of Γ by G_o correspond to the disjoint union

$$\mathcal{E} = \mathcal{E}(\Gamma, G_o) \approx \coprod_{\varphi \in Hom(\Gamma, Out(G_o))} H^2_{\bar{\varphi}}(\Gamma; Z_o).$$

We now define an action of $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$ on this set. Let $u = [Z_o \overset{\mu}{\hookrightarrow} Z \overset{\nu}{\twoheadrightarrow} \Gamma] \in H^2_{\bar{\varphi}}(\Gamma; Z_o)$. For $\alpha \in \mathrm{Out}(G_o)$, choose $\tilde{\alpha} \in \mathrm{Aut}(G_o)$ such that $\pi(\tilde{\alpha}) = \alpha$ and denote by $\bar{\alpha} \in \mathrm{Aut}(Z_o)$ the restricted automorphism.

Lemma 2.7

(i) The map $\mathrm{Out}(G_o) \times \mathcal{E} \to \mathcal{E}$, $(\alpha, u) \mapsto u \cdot \alpha = \alpha^*(u) = [Z_o \overset{\mu \circ \bar{\alpha}}{\hookrightarrow} Z \overset{\nu}{\twoheadrightarrow} \Gamma]$ *defines a right action. The image $\alpha^*(u)$ corresponds to the extension $G_o \overset{i \circ \tilde{\alpha}}{\hookrightarrow} G \overset{p}{\twoheadrightarrow} \Gamma$, and*

belongs to $H^2_\psi(\Gamma; Z_o) \subset \mathcal{E}$, *where* $\psi = c_{\alpha^{-1}} \circ \varphi$, *with* $c_{\alpha^{-1}}$ *denoting conjugation by* α^{-1}.

(ii) *The map* $\mathrm{Aut}(\Gamma) \times \mathcal{E} \to \mathcal{E}$, $(\beta, u) \mapsto \beta \cdot u = \beta_*(u) = [Z_o \overset{\mu}{\hookrightarrow} Z \overset{\beta\circ\nu}{\twoheadrightarrow} \Gamma]$ *defines a left action. The image* $\beta_*(u)$ *corresponds to the extension* $G_o \overset{i}{\hookrightarrow} G \overset{\beta\circ p}{\twoheadrightarrow} \Gamma$, *and belongs to* $H^2_\theta(\Gamma; Z_o) \subset \mathcal{E}$, *where* $\theta = \varphi \circ \beta^{-1}$.

Proof. As the proofs of the two parts of the lemma are very similar, we only treat the first one. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
Z_o & \xrightarrow{\tilde{\alpha}} & Z_o & \overset{\mu}{\hookrightarrow} & Z = Z_G(H_T) & \overset{\nu}{\twoheadrightarrow} & \Gamma \\
\downarrow & & \downarrow & & \downarrow & & \| \\
G_o & \xrightarrow{\tilde{\alpha}} & G_o & \overset{i}{\hookrightarrow} & G & \overset{p}{\twoheadrightarrow} & \Gamma \\
\downarrow & & & & \downarrow c & & \downarrow \psi \\
\mathrm{Inn}(G_o) & & \hookrightarrow & & \mathrm{Aut}(G_o) & \overset{\pi}{\twoheadrightarrow} & \mathrm{Out}(G_o)
\end{array}$$

The principal subgroups of rank one are preserved by isomorphisms. As $\tilde{\alpha}^{-1}(H_T)$ is clearly centralized by any element in Z, the statement about which extension corresponds to $\alpha^*(u)$ follows from theorem 2.6. At the same time, this shows that the map is well defined. It is then straightforward to check that it is a right action. For the resulting homomorphism, we choose a set theoretic section $v : \Gamma \to G$ of $p : G \twoheadrightarrow \Gamma$, and compute for $\gamma \in \Gamma$:

$$\begin{aligned}
\psi(\gamma) &= \pi\Big((i \circ \tilde{\alpha})^{-1} \circ c_{v(\gamma)} \circ (i \circ \tilde{\alpha})\Big) \\
&= \pi\Big(\tilde{\alpha}^{-1} \circ (i^{-1} \circ c_{v(\gamma)} \circ i) \circ \tilde{\alpha})\Big) \\
&= \pi(\tilde{\alpha})^{-1} \circ \pi(i^{-1} \circ c_{v(\gamma)} \circ i) \circ \pi(\tilde{\alpha})) \\
&= \alpha^{-1} \circ \varphi(\gamma) \circ \alpha \\
&= (c_{\alpha^{-1}} \circ \varphi)(\gamma).
\end{aligned}$$

□

Clearly for $u \in \mathcal{E}$ and a corresponding representative $G_o \hookrightarrow G_u \twoheadrightarrow \Gamma$, we have $G_u \cong G_{\alpha^*(u)} \cong G_{\beta_*(u)}$ for all $\alpha \in \mathrm{Out}(G_o)$, $\beta \in \mathrm{Aut}(\Gamma)$. Moreover, it is clear that the two actions commute and so we get a left action of $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$ on $\mathcal{E}$. Elements in the same orbit represent isomorphic groups; the main result in this chapter tells that the converse is true.

Theorem 2.8 *Two compact Lie groups* G_{u_1} *and* G_{u_2} *are isomorphic if and only if the corresponding cohomology classes* $u_1 \in H^2_{\varphi_1}(\Gamma; Z_o) \subset \mathcal{E}$ *and* $u_2 \in H^2_{\varphi_2}(\Gamma; Z_o) \subset \mathcal{E}$ *are in the same orbit under the action of* $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$.

Proof. Let $\rho : G_{u_1} \to G_{u_2}$ be an isomorphism of compact Lie groups. As the connected component of the identity is preserved by an isomorphism, this gives rise to the commutative diagram

$$\begin{array}{ccccc} G_o & \hookrightarrow & G_{u_1} & \twoheadrightarrow & \Gamma \\ \cong\downarrow\tilde{\rho} & & \cong\downarrow\rho & & \cong\downarrow\beta \\ G_o & \hookrightarrow & G_{u_2} & \twoheadrightarrow & \Gamma \end{array}$$

Let us define $\alpha = \pi(\tilde{\rho}) \in \mathrm{Out}(G_o)$ and $\bar{\alpha} = \rho|_{Z_o}$. As the centralizers of principal subgroups of rank one are preserved by isomorphisms, and by theorem 2.6, this induces a new commutative diagram that we write as follows:

$$\begin{array}{ccccc} Z_o & \xrightarrow{\mu_1 \circ \bar{\alpha}^{-1}} & Z_{u_1} & \xrightarrow{\beta \circ \nu_1} & \Gamma \\ \| & & \cong\downarrow\bar{\rho} = \rho|_{Z_{u_1}} & & \| \\ Z_o & \xrightarrow[\mu_2]{} & Z_{u_2} & \xrightarrow[\nu_2]{} & \Gamma \end{array}$$

Thus, by Lemma 2.7, we have $u_2 = (\alpha^{-1})^* \beta_*(u_1)$ and so u_1 and u_2 are in the same orbit. □

2.3 Examples of the classification

1. We take $G_o = \mathrm{SU}(2)$ and $\Gamma = \mathbb{Z}/2$. As $\mathrm{Out}(G_o)$ is trivial and $Z_o \cong \mathbb{Z}/2$, we have

$$\mathcal{E}(\mathbb{Z}/2, \mathrm{SU}(2)) \approx \coprod_{\varphi \in \mathrm{Hom}(\mathbb{Z}/2, 0)} H^2_{\varphi}(\mathbb{Z}/2; \mathbb{Z}/2) = H^2(\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2 .$$

   The group $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$ being trivial, these two elements correspond to two non-isomorphic compact Lie groups. The first one is clearly $G_{u_o} = \mathrm{SU}(2) \times \mathbb{Z}/2$. Let us give a description of the second one. Conjugating a matrix in $\mathrm{SU}(2)$ by j amounts to taking the complex conjugate of each entry in the matrix, i.e. $c_j : \mathrm{SU}(2) \to \mathrm{SU}(2)$, $g \mapsto c_j(g) = \bar{g}$. Let us denote by $G_{u_1} = \mathrm{SU}(2) \rtimes_j \mathbb{Z}/2$ the semidirect product where the generator t of $\mathbb{Z}/2$ acts as c_j on $\mathrm{SU}(2)$. As the center of G_{u_1} is given by $\langle (j, t) \rangle \cong \mathbb{Z}/4$, G_{u_o} and G_{u_1} are non-isomorphic. Therefore G_{u_1} is the second compact Lie group that we were looking for.

2. We now take $G_o = \mathrm{SU}(2) \times \mathrm{SU}(2) \cong \mathrm{Spin}(4)$ and keep $\Gamma = \mathbb{Z}/2$. This time the outer automorphism group is given by $\mathrm{Out}(G_o) = \langle [\tau] \rangle$, where τ is the automorphism that exchanges the two factors, i.e.

$$\tau : \mathrm{SU}(2) \times \mathrm{SU}(2) \longrightarrow \mathrm{SU}(2) \times \mathrm{SU}(2), \ (g, h) \longmapsto (h, g)$$

and $Z_o = \mathbf{Z}/2 \times \mathbf{Z}/2$. We thus have

$$\begin{aligned}\mathcal{E}(\mathbf{Z}/2, \mathrm{SU}(2)\times\mathrm{SU}(2)) &\approx \coprod_{\varphi\in\mathrm{Hom}(\mathbf{Z}/2,\mathbf{Z}/2)} H^2_{\varphi}(\mathbf{Z}/2;\mathbf{Z}/2\times\mathbf{Z}/2) \\ &= H^2(\mathbf{Z}/2;\mathbf{Z}/2\times\mathbf{Z}/2)) \amalg H^2_{id}(\mathbf{Z}/2;\mathbf{Z}/2\times\mathbf{Z}/2)) \\ &\approx (\mathbf{Z}/2\times\mathbf{Z}/2)\amalg\{0\}.\end{aligned}$$

One then verifies that as extensions of the center, i.e. for centralizers of a principal subgroup of rank 1, these five non-equivalent extensions are in fact represented by only three non-isomorphic groups, namely

$$\begin{aligned}Z_{u_o} &\cong \mathbf{Z}/2\times\mathbf{Z}/2\times\mathbf{Z}/2\\ Z_{u_1} &\cong \mathbf{Z}/2\times\mathbf{Z}/4\\ Z_{u_2} &\cong \mathbf{Z}/2\times\mathbf{Z}/4\\ Z_{u_3} &\cong \mathbf{Z}/2\times\mathbf{Z}/4\end{aligned}$$

for the elements of $H^2(\mathbf{Z}/2;\mathbf{Z}/2\times\mathbf{Z}/2))$, and

$$Z_{v_o} \cong (\mathbf{Z}/2\times\mathbf{Z}/2)\rtimes\mathbf{Z}/2 \cong D_8$$

(the group of symmetries of the square) for the element of $H^2_{id}(\mathbf{Z}/2;\mathbf{Z}/2\times\mathbf{Z}/2)$. The group $\mathbf{Z}/2\times\mathbf{Z}/4$ yields three non-equivalent extensions, because among its three elements of order 2, only one is divisible by 2 (the element $(0,2)$ in additive notation). Therefore this element must be characteristic and changing the non-trivial element of $\mathbf{Z}/2\times\mathbf{Z}/2$ that is mapped to it gives three extensions that must clearly be oon-equivalent. At the level of Lie groups the five non-equivalent extensions are represented by

$$\begin{aligned}G_{u_o} &= \mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathbf{Z}/2\\ G_{u_1} &= (\mathrm{SU}(2)\times\mathrm{SU}(2))\rtimes_{j\times id}\mathbf{Z}/2\\ G_{u_2} &= (\mathrm{SU}(2)\times\mathrm{SU}(2))\rtimes_{id\times j}\mathbf{Z}/2\\ G_{u_3} &= (\mathrm{SU}(2)\times\mathrm{SU}(2))\rtimes_{j\times j}\mathbf{Z}/2\\ G_{v_o} &= (\mathrm{SU}(2)\times\mathrm{SU}(2))\rtimes_{\tau}\mathbf{Z}/2\end{aligned}$$

(one checks that $(-\mathbf{1},\mathbf{1},e)$ corresponds to the characteristic element of order 2 in Z_{u_1} whereas it is $(\mathbf{1},-\mathbf{1},e)$ in Z_{u_2}, and therefore G_{u_1} and G_{u_2} are certainly not equivalent). Finally, the group $\mathrm{Out}(G_o)\times\mathrm{Aut}(\Gamma)\cong\mathbf{Z}/2\times 0$ acts on this set of equivalent extensions and it is clear that the only non-trivial orbit is $\{G_{u_1}, G_{u_2}\}$, and that there are 4 non-isomorphic extension of $\mathbf{Z}/2$ by the Lie group $\mathrm{SU}(2)\times\mathrm{SU}(2)\cong\mathrm{Spin}(4)$.

Remarks 2.9

1. As we have chosen Γ to be cyclic, the calculations of cohomology groups in these examples can be done by hand using the classical explicit resolution (see for instance [29, §2.1, pp. 5-6]).

2. It is clear, from what has been done in section 2.2, that the elements in $H^2_{\varphi}(\Gamma; Z_o)$ and in $H^2_{c_{\alpha^{-1}}\circ\varphi}(\Gamma; Z_o)$ will be identified (at least) pairwise under the action of an element $[\alpha] \in \mathrm{Out}(G_o)$. The second example is intended to show that identifications can even occur inside a given cohomology group (i.e. without changing the "outer" action of Γ on G_o).

2.4 A "sandwich" theorem

In this section we propose a structure theorem for compact Lie groups using centralizers of principal subgroups of rank 1 similar to the "Sandwich Theorem for compact Lie groups" that appears in the book by Hofmann and Morris [33, Corollary 6.75, p. 272].

Theorem 2.10 *For any compact Lie group there are two surjective homomorphisms*

$$\begin{array}{ccccc} G_s = G_o \rtimes Z & \xrightarrow{\pi_1} & G & \xrightarrow{\pi_2} & \bar{G} \cong \bar{G}_o \rtimes \Gamma \\ (g_o, z) & \longmapsto & g_o \cdot z & & \\ & & g & \longmapsto & [g] = gZ_o \end{array}$$

where the centralizer Z of a fixed principal subgroup of rank 1 acts on G_o by conjugation, and where π_2 is the canonical projection corresponding to the normal subgroup Z_o of G.

For the kernels, we have $\ker \pi_1 \cong Z_o$ *and* $\ker \pi_2 = Z_o$*; in particular*

$$G \cong (G_o \rtimes Z)/Z_o \quad \textit{and} \quad G/Z_o \cong \bar{G}_o \rtimes \Gamma\,.$$

Proof. The assertion about π_2 are clear.

The map π_1 is well-defined and surjective (because Z intersects every component of G). A straightforward computation shows that it is a homomorphism. Finally we have

$$\begin{aligned} \ker \pi_1 &= \{(g,z) \in G_o \rtimes Z : g \cdot z = e\} \\ &= \{(g, g^{-1}) \in G_o \rtimes Z\} \\ &= \{(g, g^{-1}) \in Z_o \times Z_o\} \\ &\cong Z_o\,. \end{aligned}$$

because $G_o \cap Z = Z_o$. □

Remarks 2.11

1. The origin of the name of this theorem is clear: any compact Lie group is "sandwiched" in between two semidirect products closely related to it.

2. The component of the identity of G_s is equal to G_o if and only if G_o is semisimple.

3. Our version of the "sandwich" theorem has the advantage of being more explicit than the one in [33] (the result therein is an existence theorem). Its drawback is the fact that Z is finite if and only if G_o is semisimple.

4. Given a homomorphism $\varphi : \Gamma \to \mathrm{Out}(G_o)$ and an extension $Z_o \overset{\mu}{\hookrightarrow} E \overset{\nu}{\twoheadrightarrow} \Gamma$ for which the action coincide with the restriction $\bar{\varphi}$, there is a more direct way than the cohomological one to recover the corresponding compact Lie group G, i.e. the one that fits into the commutative diagram

$$\begin{array}{ccccc} Z_o & \hookrightarrow & E & \twoheadrightarrow & \Gamma \\ \downarrow & & \downarrow & & \| \\ G_o & \hookrightarrow & G & \twoheadrightarrow & \Gamma \end{array}$$

Let us define the composition $\bar{\sigma} : E \overset{\nu}{\twoheadrightarrow} \Gamma \overset{\varphi}{\to} \mathrm{Out}(G_o) \overset{s}{\to} \mathrm{Aut}(G_o)$, where s is as in theorem 1.12. Then following Bourbaki [9, Lemme 7, pp. 210-211], we have

$$G = (G_o \times_{\bar{\sigma}} E)/\Delta Z_o,$$

where ΔZ_o is the image of the injection $z_o \mapsto (z_o^{-1}, \mu(z_o))$. Taking $E = Z$, this gives another proof of our version of the "sandwich" theorem.

2.5 Splitting of the extension associated to a compact Lie group

In this last section we make a few observations on the splitting of the extension associated to a compact Lie group G. Notice that for all the examples of compact Lie groups (with nonabelian component of the identity) we have considered so far, the associated extension was split. In fact this situation occurs in many particular cases. Using Cartan subgroups (in the sense of Segal [61], i.e. those Adams called "SS subgroups" in honour of Segal and de Siebenthal [2]), de Siebenthal showed the following theorem [18, Théorème, p 74]:

Theorem 2.12 (de Siebenthal) *Let G be a compact Lie group with G_o simply connected, or of adjoint type (i.e. Z_o is trivial). If $\Gamma = \pi_o(G)$ is cyclic then G is a semidirect product, i.e. $G \cong G_o \rtimes \Gamma$.*

As an application of the "sandwich" theorem of the previous section, we will construct a compact Lie group G with an extension that is *not* split, i.e. a compact Lie group G that cannot be decomposed as a semidirect product of the form $G_o \rtimes \Gamma$.

We start relating the splitting of the extension associated to G to that of the extension associated to the normalizer of a maximal torus N in G.

Proposition 2.13 *If the group of components Γ of G is nilpotent, then the extension $G_o \hookrightarrow G \twoheadrightarrow \Gamma$ is split if and only if the extension $N_o \hookrightarrow N \twoheadrightarrow \Gamma$ is split.*

Proof. The "if" part is clear. Conversely, let $s : \Gamma \to G$ be a section. By a result in Bourbaki [10, Corollaire 4, p.49], any nilpotent subgroup of a compact Lie group is contained in the normalizer of some maximal torus. Therefore, if needed after conjugation by an element in G_o, we have $s(\Gamma) \subset N$, and we can conclude that the extension associated to N is split. □

For an extended maximal torus Q we have the following situation.

Proposition 2.14 *If the group of components Γ of G is cyclic, then the extension $G_o \hookrightarrow G \twoheadrightarrow \Gamma$ is split if and only if the extension $T \hookrightarrow Q \twoheadrightarrow \Gamma$ is split.*

Proof. The proposition readily follows from the fact that the conjugates of Q cover G. □

Example 2.15 Consider the dihedral group $D_8 = \langle r, s \mid r^4 = s^2 = e\,,\ srs^{-1} = r^{-1} \rangle$. Then the quotient

$$G = (\mathrm{SU}(2) \times D_8)/\Delta\mathbb{Z}/2\,,$$

where $\Delta\mathbb{Z}/2$ denotes the central subgroup generated by $(-\mathbf{I}, r^2)$, is a compact Lie group with $G_o \cong \mathrm{SU}(2)$ and $\Gamma \cong V$, the 4-group of Klein.
Claim: The associated extension

$$\mathrm{SU}(2) \hookrightarrow G \twoheadrightarrow V$$

is not split. Let us check this claim. Let $\mathbb{S}^1$ denote the standard maximal torus in $\mathrm{SU}(2)$, and N_o, respectively N, the normalizer of $\mathbb{S}^1$ in G_o, respectively G. We have

$$\begin{aligned} N = \ & \{[t,e] : t \in \mathbb{S}^1\} \ \amalg \ \{[jt,e] : t \in \mathbb{S}^1\} \ \amalg \\ & \{[t,r] : t \in \mathbb{S}^1\} \ \amalg \ \{[jt,r] : t \in \mathbb{S}^1\} \ \amalg \\ & \{[t,s] : t \in \mathbb{S}^1\} \ \amalg \ \{[jt,s] : t \in \mathbb{S}^1\} \ \amalg \\ & \{[t,rs] : t \in \mathbb{S}^1\} \ \amalg \ \{[jt,rs] : t \in \mathbb{S}^1\}\,. \end{aligned}$$

By contradiction, suppose that the extension associated to G is split. As V is abelian, thus nilpotent, we deduce, by proposition 2.13, that the extension associated to N is also split. We want to show that this is not possible by enumerating the elements of order 2 in N. In the component corresponding to r we calculate

- $$\begin{aligned} [t,r]^2 &= [t^2,r^2] = \Delta\mathbb{Z}/2 \\ &\Longleftrightarrow (t^2,r^2) = (-\mathbb{1},r^2) \\ &\Longleftrightarrow t = \pm i\,, \end{aligned}$$

- $$\begin{aligned} [jt,r]^2 &= [(jt)^2,r^2] \\ &= [-\mathbb{1},r^2] \\ &= \Delta\mathbb{Z}/2 \end{aligned}$$

  (every element in this sub-component is of order 2).

In a similar way we calculate, for $n = 0\,,1$, that in the component corresponding to $r^n s$ the element $[t,r^n s]$ is of order 2 if and only if $t = -\mathbb{1}$ and that the sub-component $\{\,[jt,r^n s] : t \in \mathbb{S}^1\,\}$ does not contain any element of order 2. Two of the three non-trivial elements in $\Gamma = V$ must thus be mapped by the section to $[-\mathbb{1},s]$ and $[-\mathbb{1},rs]$. Therefore, the image of the third non-trivial element is

$$[-\mathbb{1},s]\cdot[-\mathbb{1},rs] = [\mathbb{1},srs] = [\mathbb{1},r^{-1}] = [-\mathbb{1},r]\,,$$

which is not of order 2. A contradiction that proves our claim.

Remarks 2.16

1. One can verify that this example is "minimal".

2. An obstruction to the splitting of the extension associated to the extended maximal torus Q can be found in a paper by Oliver; this obstruction involves the representation ring of G and its relation with the family of all p-toral subgroups of G [52, Corollary 3.11, p. 376]. In particular, on pp. 376-377, Oliver constructs a compact Lie group $G = \mathrm{SU}(2) \rtimes (\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/3)$ such that the extension corresponding to the extended maximal torus Q is *not* split.

Chapter 3

Automorphisms of the normalizer N_o

For connected compact Lie groups, the remarkable theorem of Curtis, Wiederhold and Williams shows that the normalizer of a maximal torus characterizes the group. However, it does not say anything on the relation between the endomorphisms of the normalizer and those of the group (the "functoriality" of the theorem). In this chapter, we study this relation for the special case of automorphisms. This problem was first proposed to us by Osse; we later realized that it was a crucial ingredient when trying to generalize the theorem of Curtis, Wiederhold and Williams, to nonconnected compact Lie groups (see chapter 3).

By a result of de Siebenthal [18, Proposition 2, p. 56] (see also Bourbaki [10, Théorème 1, p. 48]), every automorphism of a connected compact Lie group preserves some maximal torus, thus its normalizer, and yields, therefore, an automorphism of the normalizer by restriction. Conversely, can every automorphism of the normalizer be extended to an automorphism of the whole group? The following example shows that the answer is no!

Example 3.1 Take $G_o = \mathrm{SU}(2m)$ with $m \geqslant 2$. Recall that the normalizer N_o of a maximal torus T is given by an extension $T \hookrightarrow N_o \overset{\pi}{\twoheadrightarrow} \Sigma_{2m}$. Then the map

$$\psi : N_o \longrightarrow N_o\,,\ n \longmapsto (-\mathbf{1})^{\epsilon(\pi(n))}.n\,,$$

where $\epsilon : \Sigma_{2m} \twoheadrightarrow \mathbb{Z}/2 = \{0,1\}$ denotes the signature homomorphism, defines an automorphism of the normalizer. We investigate ψ for the case of the standard maximal torus consisting of the diagonal matrices of determinant 1. By contradiction, suppose that ψ can be extended to $\mathrm{SU}(2m)$. As it is trivial on the maximal torus, this extension is a conjugation by an element $t \in T$ by a result of Bourbaki [10, Proposition 9, p. 30] [10, Proposition 9, p. 30]. We now check what it implies on the maximal torus and on an element $n \in N_o$ that exchanges the first two elements on the diagonal, i.e.

$$n\cdot\mathrm{diag}(z_1, z_2, z_3, \ldots, z_{2m})\cdot n^{-1} = \mathrm{diag}(z_2, z_1, z_3, \ldots, z_{2m}).$$

We have $t\cdot n\cdot t^{-1} = \psi(n) = -n$, which is equivalent to $n^{-1}\cdot t\cdot n = n\cdot t\cdot n^{-1} = -t$. As $m \geqslant 2$, this is clearly in contradiction with how n is acting on T.

Once a maximal torus and its normalizer are fixed, the relation we want to study is naturally expressed in terms of outer automorphism groups. In this chapter, we will give a precise description of the *non-extendable* automorphisms of the normalizer and show that, together with the outer automorphisms of the whole group, they furnish a decomposition of the outer automorphism group of the normalizer as a semidirect product.

3.1 Root diagrams

The notion of root diagram generalizes the notion of root system and allows to treat the case of connected compact Lie group that are *not* semisimple. Root diagrams are introduced and studied in Bourbaki [10, §8-9], where it is shown that the classification of connected compact Lie groups can be done in this purely algebraic setting. In fact, knowing the root diagram is equivalent to knowing the Stiefel diagram together with the integral lattice in the Lie algebra of the maximal torus. In this section, we propose a slightly modified version of the definition of root diagrams that was discussed at a seminar on normalizers of maximal tori and compact Lie groups, held at Neuchâtel during the year 1997-1998 [19]. The participants in this seminar were U. Suter, A. Osse, M. Matthey and the author. The idea of this definition of root diagram is due to Osse; its advantage is twofold: it is more tractable than Bourbaki's definition, and it shows clearly that the notion of root diagram is a generalization of that of root system. However, we will see that the two definitions are equivalent once an invariant inner product is fixed. For a thorough study of root systems we refer to the classical text of Bourbaki [11].

Definition 3.2 A *(reduced) root diagram* is a triple $D = (V, M, \Phi)$ satisfying:

1. V is a euclidean space, M is a lattice in it, and Φ is a finite subset of $M \setminus \{0\}$;
2. for all $\alpha \in \Phi$ and for all $k \in \mathbb{Z}$, $k\alpha \in \Phi$ if and only if $k = \pm 1$;
3. for all $\alpha \in \Phi$, the map

$$s_\alpha : V \longrightarrow V\,,\ x \longmapsto x - \frac{2(x, \alpha)}{(\alpha, \alpha)}\alpha\,,$$

called *reflection of vector* α, leaves the set Φ globally invariant;

4. for all $\alpha \in \Phi$ and $x \in M$, $\dfrac{2(x\,,\alpha)}{(\alpha\,,\alpha)} \in \mathbb{Z}$.

The group of automorphisms of V generated by the reflections s_α is called the *Weyl group of* D and is denoted by $W(D)$. The inner product $(.\,,.)$ is obviously $W(D)$-invariant. Hence the set Φ is a reduced root system in the euclidean space $\mathbb{R}\Phi$, whose Weyl group is precisely $W(D)$.

The next lemma will link the definition presented here with Bourbaki's.

Lemma 3.3 *Let* $D = (V, M; \Phi)$ *be a root diagram. Consider the set*

$$M_o = \{x \in M : (x\,,\alpha) = 0,\ \forall \alpha \in \Phi\}.$$

Then M_o *is the subgroup* $M^{W(D)}$ *of* $W(D)$*-invariants of* M. *Moreover it is a direct summand in* M *and the set* $M_o \cup \Phi$ *spans the vector space* V.

Proof. It is clear that $M_o = M^{W(D)}$.

Since M_0 is a subgroup of the free abelian group M of rank n, it is itself a free abelian group of rank $r \leqslant n$. By the elementary divisors theorem, there exist bases $\{e_1, \dots, e_n\}$ of M and $\{\epsilon_1, \dots, \epsilon_r\}$ of M_o, and integers $d_1, \dots, d_r$ such that $\epsilon_j = d_j e_j$, for all $j = 1, \dots, r$. Let $j \in \{1, \dots, r\}$; by definition of M_o, we have $(\epsilon_j, \alpha) = 0$ for all $\alpha \in \Phi$, hence the same holds for e_j, and therefore $e_j \in M_o$ and $d_j = 1$. This proves that M_o is a direct summand in M.

Let us now show that $M_o \cup \Phi$ spans V. Let $x \in M$; for all $\alpha \in \Phi$, we have $s_\alpha(x) \equiv x \pmod{\mathbb{Z}\Phi}$. As $W = W(D)$ is generated by the s_α's and $\mathbb{Z}\Phi$ is globally W-invariant, it implies $wx \equiv x \pmod{\mathbb{Z}\Phi}$ for all $w \in W$. Define $y = \sum_{w\in W} wx \in M^W$; obviously, $y \equiv |W|\cdot x \pmod{\mathbb{Z}\Phi}$, and therefore $x = (1/|W|)\cdot y + z$ with $y \in M_o$ and $z \in \mathbb{Z}\Phi$. □

Remark 3.4 Since M_o is a direct summand in M, one can find a $\mathbb{Z}$-module N such that $M = M_o \oplus N$. The example of U(2) shows, however, that this decomposition is neither orthogonal nor compatible with the action of $W(D)$.

As a consequence of this lemma, our definition of the notion of root diagram is equivalent to Bourbaki's, once a $W(D)$-invariant inner product has been fixed (which is practically always the case).

The example that will interest us is the *(covariant) root diagram* of a connected compact Lie group G_o (with respect to the maximal torus T): it is given by the triple $D(G_o) = (LT, \Gamma(T), R^\vee)$, where $\Gamma(T)$ is the integral lattice in LT, and $R^\vee$

is the set of coroots of G_o (see page 12). The Weyl group $W(D(G_o))$ is canonically isomorphic to the Weyl group W_o of G_o. In this particular case, M_o is equal to the intersection $\Gamma(T) \cap \mathfrak{Z}(LG_o)$ of the integral lattice with the center of the Lie algebra, i.e. the non-semisimple part of the integral lattice.

Let us now define the notion of isomorphism of root diagrams.

Definition 3.5 Let $D = (V, M, \Phi)$ and $D' = (V', M', \Phi')$ be root diagrams. An *isomorphism of root diagrams* between D and D' is an isomorphism of vector spaces $f : V \longrightarrow V'$ (which is not necessarily an isometry) mapping M bijectively onto M', Φ bijectively onto Φ', and satisfying, for all $x \in M$ and all $\alpha \in \Phi$, the compatibility condition

$$\frac{2(x,\alpha)}{(\alpha,\alpha)} = \frac{2(f(x), f(\alpha))}{(f(\alpha), f(\alpha))}.$$

As an immediate consequence of this definition, the inner product on V is more or less irrelevant. More precisely, two $W(D)$-invariant inner products on V yield two isomorphic root diagrams.

The *automorphism group* of a root diagram will be denoted by $\mathrm{Aut}(D)$; it can be shown that $W(D)$ is a normal subgroup of $\mathrm{Aut}(D)$. We finally collect several results on the relationship between root diagrams and connected compact Lie groups.

Theorem 3.6 (Bourbaki)

(1) For any root diagram D there exists a connected compact Lie group G_o with $D(G_o)$ isomorphic to D.

(2) Two connected compact Lie groups G_o and G'_o are isomorphic if and only if their root diagrams $D(G_o)$ and $D(G'_o)$ are isomorphic.

(3) For any connected compact Lie group G_o, we have an isomorphism

$$\mathrm{Aut}(D(G_o))/W(D(G_o)) \cong \mathrm{Out}(G_o).$$

All these results are proved in Bourbaki [10, pp. 40-42].

3.2 Automorphisms of N_o inducing the identity on the maximal torus T

As T is the component of the identity of N_o, an automorphism $\psi \in \mathrm{Aut}(N_o)$ of the normalizer induces two automorphisms $\psi_{|T}$ and $\bar{\psi}$ fitting in the commutative diagram

$$\begin{array}{ccccc} T & \hookrightarrow & N_o & \xrightarrow{\pi} & W_o \\ \cong\downarrow\psi_{|T} & & \cong\downarrow\psi & & \cong\downarrow\bar{\psi} \\ T & \hookrightarrow & N_o & \xrightarrow{\pi} & W_o \end{array}$$

The automorphism ψ also induces a vector space automorphism $\psi_* : LT \to LT$ on the Lie algebra of the torus.

Before stating the first lemma in this section, let us recall that for a root $\alpha \in R$, its associated coroot $\alpha^\vee \in LT$ and reflection s_α, the Lie algebra of the torus decomposes as $LT = \mathbb{R}\alpha^\vee \oplus H_\alpha$, where H_α denotes the hyperplane fixed by s_α, i.e. the hyperplane orthogonal to $\alpha^\vee$.

Lemma 3.7 *The automorphism $\bar{\psi}$ permutes the reflections in W_o. Moreover for $s_\beta = \bar{\psi}(s_\alpha)$, one has*

(*i*) $s_\beta = \psi_* \circ s_\alpha \circ \psi_*^{-1}$;

(*ii*) $\psi_*(H_\alpha) = H_\beta$;

(*iii*) $\psi_*(\mathbb{R}\alpha^\vee) = \mathbb{R}\beta^\vee$.

Proof. Let $s_\alpha \in W_o$ be a reflection, and let $q_\alpha \in N_o$ be an element such that $\pi(q_\alpha) = s_\alpha$. Let us denote $w = \bar{\psi}(s_\alpha) = \pi(\psi(q_\alpha))$, and $H = \psi_*(H_\alpha)$. Let also $Y = \psi_*(X) \in H$, with $X \in H_\alpha$. The following calculation shows that w fixes H pointwise:

$$\begin{aligned} w\cdot Y &= (c_{\psi(q_\alpha)})_*(Y) \\ &= (\psi \circ c_{q_\alpha} \circ \psi^{-1})_*(Y) \\ &= \psi_* \circ (c_{q_\alpha})_* \circ \psi_*^{-1}(Y) \\ &= \psi_* \circ s_\alpha \circ \psi_*^{-1}(Y) \\ &= \psi_* \circ s_\alpha \circ \psi_*^{-1} \circ \psi_*(X) \\ &= \psi_*(X) \\ &= Y. \end{aligned}$$

Now the subgroup of W_o that fixes H pointwise has the form $\langle s_\beta \rangle = \{id, s_\beta\}$ for some reflection s_β [34, Theorem 1.12 (d), p.22, and Proposition 1.14, p.24]. Since w is an element of order 2, we can conclude that $w = s_\beta$. Moreover the previous calculation shows that (*i*) and (*ii*) are verified.

The last assertion follows from (i) and (ii): for all $X \in LT$ we have

$$\begin{aligned} s_\beta \circ \psi_*(X) &= \psi_* \circ s_\alpha(X) \\ \Longleftrightarrow \quad \psi_*(X) - 2\tfrac{(\psi_*(X),\beta^\vee)}{(\beta^\vee,\beta^\vee)}\cdot\beta^\vee &= \psi_*\Big(X - 2\tfrac{(X,\alpha^\vee)}{(\alpha^\vee,\alpha^\vee)}\cdot\alpha^\vee\Big) \\ &= \psi_*(X) - 2\tfrac{(X,\alpha^\vee)}{(\alpha^\vee,\alpha^\vee)}\cdot\psi_*(\alpha^\vee) \\ \Longleftrightarrow \quad \tfrac{(\psi_*(X),\beta^\vee)}{(\beta^\vee,\beta^\vee)}\cdot\beta^\vee &= \tfrac{(X,\alpha^\vee)}{(\alpha^\vee,\alpha^\vee)}\cdot\psi_*(\alpha^\vee)\,. \end{aligned}$$

Fixing an element $X \in LT\backslash H_\alpha$, we have $\psi_*(X) \in LT\backslash H_\beta$, and the last equality shows that $\psi_*(\alpha^\vee)$ is a (non-zero) multiple of $\beta^\vee$. □

Corollary 3.8 *If $\psi_{|T}$ is the identity of T, then $\bar\psi$ is the identity of W_o.*

Proof. Let us write $s_\beta = \bar\psi(s_\alpha)$. As $\psi_{|T} = id_T$, we also have $\psi_* = id_{LT}$. Therefore $H_\beta = \psi_*(H_\alpha) = H_\alpha$, and thus $s_\beta = s_\alpha$. □

We end up this section by recalling a relation between the automorphisms of an extension inducing the identity, on both the kernel and the quotient, and a group of cohomology. It will be used in our description of the outer automorphism group of the normalizer. Let $K \hookrightarrow E \twoheadrightarrow Q$ be a group extension. Let us denote the center of K by A, endowed with the natural action $\theta : Q \to \mathrm{Aut}(A)$ induced by the extension. Exploiting an exact sequence introduced by Wells [70], it can be shown, under some hypotheses (see below), that there is an injection of the cohomology group $H^1(Q;A)$ into the outer automorphism group of E [70], [58]. We will be interested in the case where K is abelian, i.e. $K = A$. Let also $\mathrm{Aut}(E,A,Q)$ be the subgroup consisting of elements ψ in $\mathrm{Aut}(E)$ that fix A pointwise and such that the induced automorphism $\bar\psi : Q \to Q$ is also the identity, i.e.

$$\mathrm{Aut}(E,A,Q) = \{\psi \in \mathrm{Aut}(E) : \psi_{|A} = id_A \text{ and } \bar\psi = id_Q\}\,.$$

The precise statement we will need is as follows.

Theorem 3.9 *Let $A \hookrightarrow E \twoheadrightarrow Q$ a group extension with A abelian.*

(1) There is an identification

$$H^1(Q;A) = \mathrm{Aut}(E,A,Q)/\,\mathrm{Inn}_E(A)\,,$$

where $\mathrm{Inn}_E(A) = \{\psi \in \mathrm{Aut}(E,A,Q) : \psi = c_a\,,\ \textit{with } a \in A\}$.

(2) *Moreover if A is a maximal abelian normal subgroup of E (this is the case if and only if the restriction of θ to the center of Q is an injection), then the induced application*

$$H^1(Q;A) \longrightarrow \mathrm{Out}(E)$$

is injective.

A proof of this theorem can be found in the book by Adem and Milgram [4, pp. 87-88].

3.3 Out(N_o) as a semidirect product

This is the central section of the chapter where we formulate and prove the theorem describing the precise semidirect decomposition of the outer automorphsim group of N_o.

Theorem 3.10 *Let G_o be a connected compact Lie group, T a maximal torus in G_o, and N_o the normalizer of T in G_o. Let $W_o = N_o/T$ be the Weyl group of G_o. Then the outer automorphism group of N_o canonically decomposes as*

$$\mathrm{Out}(N_o) \cong H^1(W_o;T) \rtimes \mathrm{Out}(G_o)\,,$$

where the W_o-module T is endowed with the natural action induced by the extension $T \hookrightarrow N_o \twoheadrightarrow W_o$.

Remark 3.11 Since T is a maximal abelian normal subgroup of N_o, theorem 3.9 says that the cohomology group $H^1(W_o;T)$ can be seen as the subgroup of $\mathrm{Out}(N_o)$ given by

$$H^1(W_o;T) = \mathrm{Aut}(N_o,T,W_o)/\,\mathrm{Inn}_{N_o}(T) \cong \mathrm{Aut}(N_o,T,W_o)/\big(T/Z(N_o)\big)\,.$$

In the sequel, we will always make this identification.

We will prove theorem 3.10 by showing that there exists a split short exact sequence

$$H^1(W_o;T) \hookrightarrow \mathrm{Out}(N_o) \twoheadrightarrow \mathrm{Out}(G_o)\,.$$

The first step will consist in showing that there is a canonical injection of the outer automorphism group $\mathrm{Out}(G_o)$ into $\mathrm{Out}(N_o)$. Then, in the most important step, we will construct a surjective homomorphism $\mathrm{Out}(N_o) \twoheadrightarrow \mathrm{Out}(G_o)$, for which the injection in the first step is a splitting. Finally, we will identify the kernel of this surjection with the cohomology group $H^1(W_o;T)$. We subdivide these three steps in as many propositions.

Proposition 3.12 *There is a canonical inclusion*

$$i : \mathrm{Out}(G_o) \hookrightarrow \mathrm{Out}(N_o).$$

of the outer automorphism group of G_o *into that of* N_o.

Proof. Let us consider $\mathrm{Aut}(G_o,T) = \{\varphi \in \mathrm{Aut}(G_o) : \varphi(T) = T\}$, the subgroup of automorphisms of G_o preserving T. As $\varphi(N_o) = N_{G_o}(\varphi(T)) = N_o$ for all $\varphi \in \mathrm{Aut}(G_o,T)$, we get a canonical homomorphism $\psi : \mathrm{Aut}(G_o,T) \to \mathrm{Aut}(N_o) \twoheadrightarrow \mathrm{Out}(N_o)$. On the other hand, we know, by Bourbaki [10, Proposition 18, p. 42], that

$$\mathrm{Aut}(G_o,T)/\Big(\mathrm{Aut}(G_o,T) \cap \mathrm{Inn}(G_o)\Big) \cong \mathrm{Out}(G_o),$$

where $\mathrm{Aut}(G_o,T) \cap \mathrm{Inn}(G_o) = \{\varphi \in \mathrm{Aut}(G_o) : \exists n \in N_o \text{ with } \varphi = c_n\}$. Thus we get an induced map

$$\begin{array}{ccc} \mathrm{Aut}(G_o,T) & \xrightarrow{\psi} & \mathrm{Out}(N_o) \\ \downarrow & \nearrow_{\bar{\psi} = i} & \\ \mathrm{Out}(G_o) & & \end{array}$$

We conclude by showing that i is injective. Let $\varphi \in \mathrm{Aut}(G_o,T)$; we have $[\varphi] \in \ker i$ if and only if $\varphi|_{N_o} \in \mathrm{Inn}(N_o)$. Therefore we find $n \in N_o$ such that $\varphi|_{N_o} = c_n$. So $c_n^{-1} \circ \varphi|_T = \mathrm{id}_T$ and thus, again by a result of Bourbaki [10, Proposition 9, p. 30], the automorphism $c_n^{-1} \circ \varphi$ is a conjugation by an element $t \in T$. Hence $[\varphi] = [c_{nt}]$ is trivial in $\mathrm{Aut}(G_o,T)/(\mathrm{Aut}(G_o,T) \cap \mathrm{Inn}(G_o)) \cong \mathrm{Out}(G_o)$ and i is injective. $\square$

Proposition 3.13 *There exists a surjective homomorphism*

$$p : \mathrm{Out}(N_o) \twoheadrightarrow \mathrm{Out}(G_o)$$

for which the injection i *is a splitting, i.e. such that* $p \circ i$ *is the identity of* $\mathrm{Out}(G_o)$.

At some point in the proof of this proposition, we will need a result on finite reflection groups that we introduce first.

Lemma 3.14 *Let* W *be a finite real reflection group and let* $W = W_1 \times W_2 \times \ldots \times W_n$ *be its decomposition into irreducible components. Let* φ *be an automorphism of* W *that preserves the set of reflections in* W *(i.e. for any reflection* s_α*, the element* $\varphi(s_\alpha)$ *is a reflection). Then* φ *preserves the decomposition of* W *into irreducible components, i.e. for all* i *we have* $\varphi(W_i) = W_j$*, for some* j.

Proof. By definition of an irreducible component, and by the fact that any irreducible component is generated by the reflections it contains, two reflections s_α and $s_{\alpha'}$ are in the same component if and only if there exists m reflections $s_{\alpha_1} = s_\alpha, \dots, s_{\alpha_m}$ such that s_{α_i} et $s_{\alpha_{i+1}}$ do not commute, for all $1 \leqslant i \leqslant m-1$, and such that $s_{\alpha'} = s_{\alpha_m} \dots s_{\alpha_2} s_{\alpha_1}$. As φ preserves reflections, the previous equivalence implies that $\varphi(s_\alpha)$ and $\varphi(s_{\alpha'})$ are in the same component if and only if s_α et $s_{\alpha'}$ are in the same component. □

Remark 3.15 One cannot drop the hypothesis on φ in this lemma: the Weyl group of SU(2) × SU(2) has two irreducible components; it is isomorphic to the Klein group whose automorphisms permute the three non-trivial elements, but it contains only two reflections.

Proof of proposition 3.13. Let $\psi \in \mathrm{Aut}(N_o)$; the main part of the proof consists in showing that the vector space automorphism $\psi_* : LT \to LT$ is an automorphism of the root diagram $D(G_o)$. As $(\psi_1 \circ \psi_2)_* = (\psi_1)_* \circ (\psi_2)_*$, it will yield, by theorem 3.6, a homomorphism $\bar{p} : \mathrm{Aut}(N_o) \to \mathrm{Out}(G_o)$. We finally check that $\bar{p}$ factorizes through a homomorphism $p : \mathrm{Out}(N_o) \to \mathrm{Out}(G_o)$ with the required properties.

Let us show that $\psi_* \in \mathrm{Aut}(D(G_o))$ in several steps.

Step 1: Let $X \in \Gamma(T)$; by naturality of the exponential, we have $\exp \circ \psi_*(X) = \psi \circ \exp(X) = e$, which implies $\psi_*(\Gamma(T)) \subset \Gamma(T)$. As ψ_* is invertible, we can conclude that ψ_* maps $\Gamma(T)$ bijectively onto itself.

Step 2: We derive some formulae that will be useful in the next step. In particular, we get that the the compatibility condition in definition 3.5 is satisfied. Let $\alpha^\vee \in R^\vee$; fix an element $X \in LT$ and decompose it as $X = Y + Z$, with $Y \in \mathbb{R}\alpha^\vee$ and $Z \in H_\alpha$. Explicitly, we have $Y = \frac{(X,\alpha^\vee)}{(\alpha^\vee,\alpha^\vee)} \cdot \alpha^\vee$, and thus

$$\psi_*(Y) = \frac{(X,\alpha^\vee)}{(\alpha^\vee,\alpha^\vee)} \cdot \psi_*(\alpha^\vee).$$

On the other hand, we have $\psi_*(X) = \psi_*(Y) + \psi_*(Z)$, with, again by lemma 3.7, $\psi_*(Y) \in \mathbb{R}\beta^\vee$ and $\psi_*(Z) \in H_\beta$. Therefore we have

$$\psi_*(Y) = \frac{(\psi_*(X),\psi_*(\alpha^\vee))}{(\psi_*(\alpha^\vee),\psi_*(\alpha^\vee))} \cdot \psi_*(\alpha^\vee).$$

Comparing these two expressions for $\psi_*(Y)$, we get that

$$\frac{(X,\alpha^\vee)}{(\alpha^\vee,\alpha^\vee)} = \frac{(\psi_*(X),\psi_*(\alpha^\vee))}{(\psi_*(\alpha^\vee),\psi_*(\alpha^\vee))}, \forall X \in LT, \forall \alpha^\vee \in R^\vee.$$

In particular, this equality shows that the compatibility condition in definition 3.5 is satisfied. Now setting $X = \beta^\vee \in R^\vee$, we get

$$\frac{\|\beta^\vee\|}{\|\alpha^\vee\|}\cdot\cos\theta = \frac{\|\psi_*(\beta^\vee)\|}{\|\psi_*(\alpha^\vee)\|}\cdot\cos\phi\,,$$

where $\theta = \measuredangle(\alpha^\vee, \beta^\vee)$ et $\phi = \measuredangle(\psi_*(\alpha^\vee), \psi_*(\beta^\vee))$. Symmetrically we also get

$$\frac{\|\alpha^\vee\|}{\|\beta^\vee\|}\cdot\cos\theta = \frac{\|\psi_*(\alpha^\vee)\|}{\|\psi_*(\beta^\vee)\|}\cdot\cos\phi\,.$$

For two non-orthogonal coroots $\alpha^\vee$ and $\beta^\vee$, these last two expressions finally yield

$$\frac{\|\alpha^\vee\|}{\|\beta^\vee\|} = \frac{\|\psi_*(\alpha^\vee)\|}{\|\psi_*(\beta^\vee)\|}\,. \qquad (*)$$

Step 3: This is the longer step of the proof in which we show that ψ_* maps the set of coroots $R^\vee$ bijectively onto itself. We first show that the troublesome cases are those where G_o has direct factors of type $\mathrm{SO}(2\ell+1)$. By lemma 3.7, ψ_* permutes the directions generated by the coroots. We have also checked in step 1 that ψ_* preserves $\Gamma(T)$. Moreover the set of $R^\vee$ is contained in the integer lattice $\Gamma(T)$, and it is almost always characterized by the fact that $\alpha^\vee$ is "minimal", i.e. $\alpha^\vee$ is the element of $\mathbb{R}_+\alpha^\vee \cap (\Gamma(T)\backslash\{0\})$ that is the nearest to the origin. The only exception is the following case [55, pp. 86-87]: R has an irreducible component of type B_ℓ ($l \geqslant 1$, $B_1 = A_1$) that corresponds in G_o, to a direct factor $\mathrm{SO}(2\ell+1)$, and α is a short root. This translates in $R^\vee$ by a dual irreducible component of type C_ℓ, with a long coroot $\alpha^\vee$; in this case $1/2\cdot\alpha^\vee$ is the "minimal" element (we will say that α is a *strechable* root). By "minimality" of coroots in all the other cases, it only remains to treat this exceptional case. We subdivide step 3.

• **Simple case:** We can suppose that G_o is isomorphic to $\mathrm{SO}(2\ell+1)$. The case $\ell = 1$ is straightforward: the integral lattice is isomorphic to $\mathbb{Z}$ and is preserved by ψ_*, thus the generator $1/2\cdot\alpha^\vee$ is mapped to $\pm 1/2\cdot\alpha^\vee$, and therefore ψ_* preserves $R^\vee = \{\pm\alpha^\vee\}$. If $\ell > 1$, there are two possible lengths for the coroots, and the factor between the norms of a long and a short coroot is $\sqrt{2}$. The only pathological case that could occur is the following: a long coroot $\alpha^\vee$ (i.e. one corresponding to a strechable root) maps to twice a short coroot. A short coroot could then map to either a short coroot or half a long coroot (because $\psi_*(\Gamma(T)) = \Gamma(T)$). Now $\alpha^\vee = \alpha_\ell^\vee$ is an element of some basis $\{\alpha_1^\vee, \dots, \alpha_{\ell-1}^\vee, \alpha_\ell^\vee\}$ of $R^\vee$, with $\alpha_\ell^\vee$ and $\alpha_{\ell-1}^\vee$ non-orthogonal. Let us write $2\cdot\beta_\ell^\vee = \psi_*(\alpha_\ell^\vee)$, for some short coroot $\beta_\ell^\vee$. We apply $(*)$ with $\alpha^\vee = \alpha_\ell^\vee$ and $\beta^\vee = \alpha_{\ell-1}^\vee$. Distinguishing the two possible cases for the length of the image of the short coroot $\alpha_{\ell-1}^\vee$, we get

$$\sqrt{2} = \frac{\|\alpha_\ell^\vee\|}{\|\alpha_{\ell-1}^\vee\|} = \frac{\|2\cdot\beta_\ell^\vee\|}{\|\psi_*(\alpha_{\ell-1}^\vee)\|} = \frac{2\|\alpha_{\ell-1}^\vee\|}{\|\psi_*(\alpha_{\ell-1}^\vee)\|} = \begin{cases} 2\frac{\|\alpha_{\ell-1}^\vee\|}{\|\alpha_{\ell-1}^\vee\|} & = 2 \\ 2\frac{\|\alpha_{\ell-1}^\vee\|}{1/2\|\alpha_\ell^\vee\|} & = 2\sqrt{2}\,, \end{cases}$$

which is manifestly impossible and ends up the simple case.

• **Semisimple case:** The only case to check is when G_o possesses at least one *direct factor isomorphic to* $\mathrm{SO}(2\ell+1)$. By the classification of connected compact Lie groups (theorem 1.3), we can suppose that $G_o = H \times G_s$, where $G_s = \mathrm{SO}(2\ell+1)$, and $H = (G_1 \times \ldots \times G_{s-1})/K$, with K a central subgroup of the universal cover $\widetilde{H} = G_1 \times \ldots \times G_{s-1}$ of H. This also implies the following corresponding decompositions: $T = (T_1 \times \ldots \times T_{s-1})/K \times T_s$, with $LT = LT_1 \oplus \ldots \oplus LT_{s-1} \oplus LT_s$, as well as $N_o = (N_1 \times \ldots \times N_{s-1})/K \times N_s$, and $W_o = W_1 \times \ldots \times W_{s-1} \times W_s$. By lemma 3.7, the automorphism $\bar{\psi} : W_o \to W_o$ permutes the reflections, and thus, by lemma 3.14, it permutes the irreducible components of W_o. If the factor W_s (corresponding to $\mathrm{SO}(2\ell+1)$) is preserved, we can conclude as in the simple case. If not, suppose, without loss of generality, that $\bar{\psi}(W_1) = W_s$. We therefore also have $\psi_*(LT_1) = LT_s$, and we have to check that the coroots of LT_1 map to those of LT_s. Let us consider the composite homomorphism

$$\kappa : G_1 \hookrightarrow G_1 \times \ldots \times G_{s-1} \twoheadrightarrow (G_1 \times \ldots \times G_{s-1})/K \hookrightarrow G_o\,,$$

which factorizes through the injection $\alpha : \bar{G}_1 = G_1/\ker\kappa \hookrightarrow G_o$, with $\ker\kappa$ central. By restriction, we also have $\alpha_{|\bar{N}_1} : \bar{N}_1 = N_1/\ker\kappa \hookrightarrow N_o$, and $\alpha_{|\bar{T}_1} : \bar{T}_1 = T_1/\ker\kappa \hookrightarrow T$. Let us then consider the composition

$$\psi_1 : \bar{N}_1 \overset{\alpha}{\hookrightarrow} N_o \overset{\psi}{\to} N_o \overset{p_s}{\twoheadrightarrow} N_s\,,$$

where p_s denotes the canonical projection.

Claim. ψ_1 is an isomorphism. First we show that ψ_1 is surjective. From the commutative diagram

$$\begin{array}{ccccccc} LT_1 & \xrightarrow[\cong]{\alpha_*} & LT_1 & \xrightarrow[\cong]{\psi_{*|LT_1}} & LT_s & \xrightarrow[\cong]{(p_s)_{*|LT_s}} & LT_s \\ \downarrow & & & & & & \downarrow \\ \bar{T}_1 & & & \xrightarrow{\qquad\psi_{1|\bar{T}_1}\qquad} & & & T_s \end{array}$$

we get that $\psi_{1|\bar{T}_1} : \bar{T}_1 \to T_s$ is surjective. On the other hand, we have

$$\bar{\psi} \circ \bar{\alpha} : \bar{W}_1 = \bar{N}_1/\bar{T}_1 \xrightarrow{\cong} W_1 \xrightarrow{\cong} W_s\,,$$

and we deduce that im ψ_1 meets every connected component of N_s. So im ψ_1 is a subgroup of N_s that contains T_s and that meets all components, therefore it is equal to N_s and we get the surjectivity of ψ_1. For the injectivity, we first observe that the situation we have just described at the level of Weyl groups implies that $\ker\psi_1 \subset \bar{T}_1$. Now $\psi \circ \alpha$ is clearly injective, and maps $\bar{T}_1$ bijectively onto T_s. As p_s maps T_s bijectively onto itself, we get that ψ_1 is injective. This proves our claim.

Then, by the theorem of Curtis-Wiederhold-Williams, $\bar{G}_1$ must be isomorphic to $SO(2\ell+1)$, and as $\bar{G}_1 \hookrightarrow G_o$, the situation reduces to the exchange of two isomorphic factors of type $SO(2\ell+1)$. We can conclude by the same arguments as in the simple case that the coroots of LT_1 map to those of LT_s. Doing the same for each factor $SO(2\ell+1)$, it follows that ψ_* preserves $R^\vee$ in the semisimple case.

• **General case:** It is deduced from the following classical decomposition of the Lie algebra of the torus:

$$LT = \mathfrak{Z}(LG_o) \oplus \bigoplus_{\alpha\in B} \mathbb{R}\alpha^\vee ,$$

where B denotes a basis of the root system. Moreover, we have $\bigoplus_{\alpha\in B}\mathbb{R}\alpha^\vee = \sum_{\alpha\in R}\mathbb{R}\alpha^\vee$. Therefore, by lemma 3.7, ψ_* preserves this decomposition of LT and, in this case as well, maps $R^\vee$ bijectively onto itself.

Steps 1,2 and 3 together show that ψ_* is an automorphism of $D(G_o)$.

Finally, it is clear that inner automorphisms of N_o act as elements of the Weyl group of $D(G_o)$. By theorem 3.6, this implies that the homomorphism $\bar{p}$ factorizes through $p : \mathrm{Out}(N_o) \to \mathrm{Out}(G_o)$. By construction, it is also clear that the composition $p \circ i : \mathrm{Out}(G_o) \hookrightarrow \mathrm{Out}(N_o) \to \mathrm{Out}(G_o)$ is the identity. This shows that p is split-surjective, with i as a splitting, and completes the proof. □

Proposition 3.16 *The kernel of p is*

$$\ker \mathrm{p} = H^1(W_o; T).$$

Proof. By definitions:

$$[\psi] \in \ker p \iff p[\psi] \in W(D)$$
$$\iff \Big(\psi_* : LT \to LT\Big) \in W_o .$$

So we find $\mathbf{n} \in N_o$ such that $\psi_* = (c_n)_*$. Clearly $[\psi] = [c_n^{-1} \circ \psi] \in \mathrm{Out}(N_o)$. Now $(c_n^{-1} \circ \psi)_* = id_{LT}$ and thus $c_n^{-1} \circ \psi_{|T} = id_T$. Therefore, by corollary 3.8, the induced automorphism on W_o is also the identity, and we can conclude by invoking theorem 3.9 (see remark 3.11). □

Proof of theorem 3.10. Propositions 3.12, 3.13 and 3.16 clearly constitute a proof. □

3.4 Tits systems and related notions

In chapter 1, we stated a theorem of Tits on a presentation of the normalizer N_o (see page 12). Related to it, we introduce some definitions that will be useful for explicit calculations of $H^1(W_o;T)$. First recall that for every $\alpha \in B$, we chose an element $q_\alpha \in \nu_\alpha(j\mathbb{S}^1) \subset \pi^{-1}(s_\alpha)$; also recall that $q_\alpha^2 = h_\alpha = \exp(\frac{\alpha^\vee}{2})$.

Definitions 3.17

1. We will call the finite subset $A = \{q_\alpha\}_{\alpha \in B}$ of $N_o \setminus T$ a *Tits system* and each coset $\nu_\alpha(j\mathbb{S}^1) = q_\alpha \cdot T_\alpha$ a *Tits circle*.

2. A *pseudo Tits system* is a finite subset $\tilde{A} = \{\tilde{q}_\alpha\}_{\alpha \in B}$ such that

   - $\tilde{q}_\alpha \in \pi^{-1}(s_\alpha)$ for all $\alpha \in B$
   - $\tilde{q}_\alpha^2 = h_\alpha$ for all $\alpha \in B$
   - $\tilde{q}_\beta \notin \nu_\beta(j\mathbb{S}^1)$ for some $\beta \in B$.

   The corresponding coset $\tilde{q}_\beta \nu_\beta(\mathbb{S}^1)$ will be called a *pseudo Tits circle.*

Before introducing a few properties of these objects, we recall that for any root $\alpha \in R$, the Lie algebra of the maximal torus decomposes as $LT = \mathbb{R}\alpha^\vee \oplus H_\alpha$, where $\mathbb{R}\alpha^\vee = \{X \in LT : s_\alpha(X) = -X\}$ and $H_\alpha = \{X \in LT : s_\alpha(X) = X\}$. Also recall that $T_\alpha = \nu_\alpha(\mathbb{S}^1) = \exp(\mathbb{R}\alpha^\vee) \cong \mathbb{S}^1$ and define $U_o^\alpha = \exp(H_\alpha)$. For the subgroup of fixed points $F^\alpha = \{t \in T : s_\alpha(t) = t\}$, it is clear that we have $U_o^\alpha \subseteq F^\alpha$. By the decomposition of the Lie algebra, we have $T = T_\alpha \cdot U_o^\alpha = T_\alpha \cdot F^\alpha$, with a generally non-trivial intersection $T_\alpha \cap F^\alpha$. (The notation U_o^α comes from the fact that this group is the connected component of an important closed subgroup U^α of T: to each $\alpha \in R$ corresponds a unique homomorphism $\rho_\alpha : LT \to \mathbb{S}^1$ such that $\alpha = (\rho_\alpha)_*$, and U^α is defined as $U^\alpha = \ker \rho_\alpha = \ker \rho_{-\alpha}$.)

Lemma 3.18 *Let $\alpha \in B$ and $t = ru \in T$, with $r \in T_\alpha$ and $u \in F^\alpha$. Then $t \cdot \nu_\alpha(j\mathbb{S}^1)$ is a pseudo Tits circle if and only if $u^2 = e$ and $u \in F^\alpha \setminus (T_\alpha \cap F^\alpha)$, i.e.*

$$u \in P_\alpha = (F^\alpha \cap S) \setminus (T_\alpha \cap F^\alpha \cap S)$$

where $S = \{t \in T : t^2 = e\}$ is the subgroup of elements of order 2 in T.

Proof. Let $q_\alpha \in \nu_\alpha(j\mathbb{S}^1)$. We have

$$(tq_\alpha)^2 = (ruq_\alpha)^2 = ruq_\alpha ruq_\alpha = u^2 q_\alpha^2 = u^2 h_\alpha,$$

which is equal to h_α if and only if $u^2 = e$. The lemma then follows from the definition of pseudo Tits systems. □

Corollary 3.19 *For a root $\alpha \in B$ the number of distinct pseudo Tits circles is finite and given by*

$$\frac{\#P_\alpha}{2}.$$

Proof. The elements $u, u' \in P_\alpha$ are on the same pseudo Tits circle if and only if there exists $r \in T_\alpha$ such that $ru' = u$. So $r = uu' \in T_\alpha \cap S$ and therefore $r = \nu_\alpha(\pm 1)$. □

Example 3.20 By first considering the case SU(2), we get that the matrices

$$q_{\alpha_1} = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & \mathbb{O} & \\ & & 1 & & & \\ & & & 1 & & \\ & \mathbb{O} & & & \ddots & \\ & & & & & 1 \end{pmatrix}, \quad q_{\alpha_2} = \begin{pmatrix} 1 & & & & & \\ & 0 & 1 & & \mathbb{O} & \\ & -1 & 0 & & & \\ & & & 1 & & \\ & \mathbb{O} & & & \ddots & \\ & & & & & 1 \end{pmatrix}, \ldots$$

$$q_{\alpha_{n-1}} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & \mathbb{O} & \\ & & 1 & & & \\ & & & 1 & & \\ & \mathbb{O} & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}$$

form a Tits system in $SU(n)$. The corresponding Tits circles are given by $q_{\alpha_j} T_{\alpha_j} = q_{\alpha_j} \cdot \{\operatorname{diag}(1, \ldots, 1, z_j = z, z_{j+1} = z^{-1}, 1, \ldots, 1) : |z| = 1\}$; more explicitly, for $j = 1$:

$$q_{\alpha_1} T_{\alpha_1} = \left\{ \begin{pmatrix} 0 & z & & & \\ -z^{-1} & 0 & & \mathbb{O} & \\ & & 1 & & \\ & \mathbb{O} & & \ddots & \\ & & & & 1 \end{pmatrix} : |z| = 1 \right\}.$$

It is easy to check that there is no pseudo Tits circle in SU(2). Then, for $n \geqslant 3$, considering the subgroup of elements of order 2 in $T(SU(n))$ and the action of the Weyl group, we calculate that for a given root α, the number of distinct pseudo Tits circles is

$$\frac{\#P_\alpha}{2} = 2^{n-3} - 1.$$

The "first" pseudo Tits circle thus lives in SU(4); for α_1 it is given by translating the corresponding Tits circle by the element $t_1 = \operatorname{diag}(1, 1, -1, -1)$, i.e.

$$t_1 q_{\alpha_1} T_{\alpha_1} = \left\{ \begin{pmatrix} 0 & z & 0 & 0 \\ -z^{-1} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} : |z| = 1 \right\}.$$

Lemma 3.21

(1) Let $A = \{q_\alpha\}_{\alpha\in B}$ and $A' = \{q'_\alpha\}_{\alpha\in B}$ be two Tits systems. Then A and A' are conjugate by an element of the maximal torus, i.e. there exists $t \in T$ such that $c_t(q_\alpha) = q'_\alpha$ for all $\alpha \in B$.

(2) Let $\tilde{A} = \{\tilde{q}_\alpha\}_{\alpha\in B}$ and $\tilde{A}' = \{\tilde{q}'_\alpha\}_{\alpha\in B}$ be two pseudo Tits systems such that $\tilde{q}_\alpha$ and $\tilde{q}'_\alpha$ are on the same Tits or pseudo Tits circle for all $\alpha \in B$. Then $\tilde{A}$ and $\tilde{A}'$ are conjugate by an element of the maximal torus.

Proof. By definition, (1) clearly implies (2). It suffices to check (1) in the semisimple case. Let us write $R = \{\alpha_1, \dots, \alpha_\ell\}$. By definition there exist elements $r_1 \in T_{\alpha_1}, \dots, r_\ell \in T_{\alpha_\ell}$ such that $q'_{\alpha_1} = r_1 q_{\alpha_1}, \dots, q'_{\alpha_\ell} = r_\ell q_{\alpha_\ell}$. We have to construct an element $t \in T$ such that

$$c_t(q_{\alpha_j}) = r_j q_{\alpha_j} \iff t q_{\alpha_j} t^{-1} = r_j q_{\alpha_j} \iff t q_{\alpha_j} t^{-1} q_{\alpha_j}^{-1} = r_j \qquad (*)$$

for all j. We show that there exists an element $X \in LT$ such that $t = \exp(X)$ has the right property. Fix elements $\lambda_j \in \mathbb{R}$ satisfying $\exp(\lambda_j \alpha_j^\vee) = r_j$. In LT the equality $(*)$ then becomes

$$\exp(X)\exp(s_{\alpha_j}(X))^{-1} = \exp(\lambda_j\alpha_j^\vee) \iff \exp(X - s_{\alpha_j}(X)) = \exp(\lambda_j\alpha_j^\vee).$$

As $s_{\alpha_j}(X) = X - 2\frac{(X, \alpha_j^\vee)}{(\alpha_j^\vee, \alpha_j^\vee)} \cdot \alpha_j^\vee$, this last expression becomes

$$\exp(2\frac{(X, \alpha_j^\vee)}{(\alpha_j^\vee, \alpha_j^\vee)} \cdot \alpha_j^\vee - \lambda_j\alpha_j^\vee) = e$$

for all j. As $R^\vee$ is a basis of LT, the ℓ scalars defined by

$$a_j = (X, \alpha_j^\vee) = \frac{\lambda_j(\alpha_j^\vee, \alpha_j^\vee)}{2}$$

determine the covariant coordinates of a vector $X \in LT$, and, by construction, $t = \exp(X)$ has the desired property. □

3.5 Some explicit calculations of $\mathrm{Out}(N_o)$

We want to explicitly calculate the outer automorphism group of the normalizer for some connected compact Lie groups. As the outer automorphism groups of connected compact Lie groups are well-known and classified, it suffices to calculate the subgroups corresponding to non-extendable automorphisms, i.e. by the previous section, the cohomology groups $H^1(W_o; T)$. To do so, we are going to use the Tits presentation of the normalizer. We start with a group-theoretic characterization of $H^1(W_o; T)$.

Lemma 3.22 *Every non-trivial element in $H^1(W_o;T)$ is of order 2. Moreover $H^1(W_o;T)$ is finite and, in particular,*

$$H^1(W_o;T) \cong \bigoplus_{\mathit{finite}} \mathbf{Z}/2\,,$$

i.e. $H^1(W_o;T)$ is an elementary abelian 2-group.

Proof. Considering $H^1(W_o;T)$ as a subgroup of $\mathrm{Out}(N_o)$, we know that for $[\psi] \in H^1(W_o;T)$, there exists a representative $\psi \in \mathrm{Aut}(N_o, T, W_o)$. Having fixed a Tits system $A = \{q_\alpha\}_{\alpha\in B}$, it is clear, from the first Tits' relation in theorem 1.7, that $\psi(q_\alpha)$ is on a Tits or pseudo Tits circle, for all $\alpha \in B$. By lemmas 3.18 and 3.21, there exist elements $t \in T$, and $t_\alpha \in F^\alpha \cap S$, such that the automorphism defined by $\tilde{\psi} = c_t \circ \psi$ satisfies $\tilde{\psi}(q_\alpha) = t_\alpha q_\alpha$, for all $\alpha \in B$. In particular $q_\alpha t_\alpha q_\alpha^{-1} = t_\alpha$ and $t_\alpha^2 = e$, and $[\tilde{\psi}] = [\psi] \in H^1(W_o;T)$. Now, as $\tilde{\psi} \in \mathrm{Aut}(N_o, T, W_o)$, we have

$$\tilde{\psi}^2(q_\alpha) = \tilde{\psi}(t_\alpha q_\alpha) = \tilde{\psi}(t_\alpha)\tilde{\psi}(q_\alpha) = t_\alpha^2 q_\alpha = q_\alpha$$

for all $\alpha \in B$, and we can conclude that $\tilde{\psi}^2$ is the identity. Notice that $[\psi] = [\tilde{\psi}] \in H^1(W_o;T)$ is completely determined by the set of parameters $\{t_\alpha\}_{\alpha\in B}$. Since $t_\alpha \in F^\alpha \cap S$ for each $\alpha \in B$, and since S is finite, the finiteness of $H^1(W_o;T)$ follows.

□

We proceed with explicit calculations; these will be needed in chapter 4, and, except for quotients of $\mathrm{SU}(n)$, they cover all connected simple compact Lie groups with non-trivial center and non-trivial group of outer automorphisms.

Proposition 3.23 *In each case let T denote the respective maximal torus. We calculate:*

(1) $H^1(W_o(\mathrm{SU}(2));T) = 0$

$$H^1(W_o(\mathrm{SU}(n));T) \cong \begin{cases} \mathbf{Z}/2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \qquad \text{for } n \geqslant 3$$

(2) $$H^1(W_o(\mathrm{Spin}(2n));T) \cong \begin{cases} \mathbf{Z}/2 \oplus \mathbf{Z}/2 & n \text{ even} \\ \mathbf{Z}/2 & n \text{ odd} \end{cases} \qquad \text{for } n \geqslant 2$$

(3) $H^1(W_o(\mathrm{SO}(2n));T) \cong \mathbf{Z}/2$ *for* $n \geqslant 3$

(4) $H^1(W_o(\mathrm{E}_6);T) = 0$.

Moreover, each non-trivial element in these cohomology groups has a representative $\psi \in \mathrm{Aut}(N_o, T, W_o)$ such that, for all q_α in a fixed Tits system $A = \{q_\alpha\}_{\alpha\in B}$, we have $\psi(q_\alpha) = z q_\alpha$ for some $z \in Z(N_o) \cap S$.

Proof. In each case we work with the standard maximal torus. We fix a basis $B = \{\alpha_1, \dots, \alpha_l\}$, an associated Tits system $A = \{q_j = q_{\alpha_j}\}_{\alpha_j \in B}$, and write $w_j = \pi(q_j) = s_{\alpha_j}$. The general strategy is to look for a representative $\psi \in \mathrm{Aut}(N_o, T, W_o)$ as in the proof of lemma 3.22. As two such representatives differ by a conjugation with an element in T, every equality in this proof will be meant up to such a conjugation. For t_j defined by $\psi(q_j) = t_j q_j$, we will always choose one of the two elements in $F^{\alpha_j} \cap S$, i.e. such that $w_j \cdot t_j = t_j$ and $t_j^2 = e$. A quick glance at the Dynkin diagram of the groups we are considering shows that, for $\alpha_i \neq \alpha_j$, the only possible values for $\ell_{ij} = \ell_{\alpha_i \alpha_j}$ are 2 and 3. From this observation, we derive two formulae that will be used throughout the proof. They are both directly deduced from the second Tits relation:

- for $\ell_{ij} = 2$ we have $q_i q_j = q_j q_i$ and, as ψ is an automorphism, we get

$$\begin{array}{rrcll}
 & \psi(q_i q_j) & = & \psi(q_j q_i) & \\
\Longleftrightarrow & \psi(q_i)\psi(q_j) & = & \psi(q_j)\psi(q_i) & \\
\Longleftrightarrow & t_i q_i t_j q_j & = & t_j q_j t_i q_i & \\
\Longleftrightarrow & t_i (w_i \cdot t_j) q_i q_j & = & t_j (w_j \cdot t_i) q_j q_i & \\
\Longleftrightarrow & t_i (w_i \cdot t_j) & = & t_j (w_j \cdot t_i) & \\
\Longleftrightarrow & t_i (w_j \cdot t_i) & = & t_j (w_i \cdot t_j) & \qquad (*)
\end{array}$$

- for $\ell_{ij} = 3$ we have $q_i q_j q_i = q_j q_i q_j$ and, similarly, we get

$$\begin{array}{rrcll}
 & \psi(q_i q_j q_i) & = & \psi(q_j q_i q_j) & \\
\Longleftrightarrow & t_i (w_j \cdot t_i)((w_i w_j) \cdot t_i) & = & t_j (w_i \cdot t_j)((w_j w_i) \cdot t_j) & \qquad (**)
\end{array}$$

The core of the proof is then a case-by-case checking.

(1): We will denote a diagonal element t in $S \subset T \subset \mathrm{SU}(n)$ by $t = (\epsilon_1, \dots, \epsilon_n) = (\epsilon_k)$, where $\epsilon_k = \pm 1$ for all k (and of course $\prod \epsilon_k = 1$). Recall that w_j exchanges the j-th and $(j+1)$-st coordinates on the diagonal. Example 3.20 shows that there is no pseudo Tits circle in $\mathrm{SU}(2)$ and $\mathrm{SU}(3)$, and therefore that $H^1(W_o(\mathrm{SU}(2)); T)$ and $H^1(W_o(\mathrm{SU}(3)); T)$ are trivial. We now treat $\mathrm{SU}(4)$; a quick analysis of the 8 elements in S shows that $t_j = \pm \mathbb{1}$ for $j = 1, 2, 3$. Then, as $\ell_{12} = 3 = \ell_{23}$, applying formula $(**)$ yields $t_1 = t_2 = t_3$. Therefore, there is only one non-trivial element in $H^1(W_o(\mathrm{SU}(4)); T)$, represented by the automorphism described in example 3.1. The cases $n = 2, 3$ constitute the base steps of an induction. Consider now the canonical inclusions

$$\begin{array}{ccccc}
S_{n-2} = S \cap \mathrm{SU}(n-2) & \hookrightarrow & S_{n-1} = S \cap \mathrm{SU}(n-1) & \hookrightarrow & S \\
\downarrow & & \downarrow & & \downarrow \\
T_{n-2} = T \cap \mathrm{SU}(n-2) & \hookrightarrow & T_{n-1} = T \cap \mathrm{SU}(n-1) & \hookrightarrow & T \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{SU}(n-2) & \hookrightarrow & \mathrm{SU}(n-1) & \hookrightarrow & \mathrm{SU}(n)
\end{array}$$

The subgroup S_{n-2} is of index 2 in S_{n-1}, itself of index 2 in S. Define then $u = (1, \dots, 1, -1, -1, 1)$ and $v = (1, \dots, 1, -1, -1) \in S$; we have the decompositions

$$\begin{aligned} S &= S_{n-1} \amalg vS_{n-1} \\ &= S_{n-2} \amalg uS_{n-2} \amalg vS_{n-2} \amalg uvS_{n-2}. \end{aligned}$$

We distinguish two cases, depending on whether the automorphism ψ preserves the normalizer of the maximal torus in $\mathrm{SU}(n-1)$ or not.

Case 1: $\psi_{|N_{\mathrm{SU}(n-1)}(T_{n-1})} \in \mathrm{Aut}(N_{\mathrm{SU}(n-1)}(T_{n-1}))$

By induction, only two sub-cases can occur.

- $\psi_{|N_{\mathrm{SU}(n-1)}(T_{n-1})} = id_{N_{\mathrm{SU}(n-1)}(T_{n-1})}$ and thus $t_j = \mathbb{1}$ for $j = 1, 2, \dots, n-2$. For $j = 1, 2, \dots, n-3$, we have $\ell_{j,n-1} = 2$ and therefore

$$\begin{aligned} & \mathbb{1} = t_j(w_{n-1} \cdot t_j) = t_{n-1}(w_j \cdot t_{n-1}) \\ \Longrightarrow \quad & t_{n-1} = w_j \cdot t_{n-1} \\ \Longrightarrow \quad & t_{n-1} = (\epsilon, \dots, \epsilon, \epsilon_{n-1}, \epsilon_{n-1}) = (\epsilon, \dots, \epsilon, 1, 1). \end{aligned}$$

  Similarly, $\ell_{n-2,n-1} = 3$ and formula $(**)$ gives on the last three coordinates

$$(1,1,1) = (\epsilon, 1, 1) \cdot (1, \epsilon, 1) \cdot (1, 1, \epsilon) = (\epsilon, \epsilon, \epsilon),$$

  and therefore $\psi = id$.

- In case n is odd, $\psi_{|N_{\mathrm{SU}(n-1)}(T_{n-1})}$ can also correspond to the non-trivial outer automorphism, i.e. for $j = 1, 2, \dots, n-2$, $\psi(q_j) = tq_j$ with $t = (-1, \dots, -1, 1)$. We exclude this case using $\ell_{1,n-1} = 2$; indeed, on the last two coordinates formula $(*)$ becomes

$$\begin{aligned} (-1,1) \cdot (1,-1) &= (\epsilon_{n-1}, \epsilon_{n-1}) \cdot (\epsilon_{n-1}, \epsilon_{n-1}) \\ \Longrightarrow \qquad (-1,-1) &= (1,1), \end{aligned}$$

  which is clearly impossible.

Case 2: $\psi_{|N_{\mathrm{SU}(n-1)}(T_{n-1})} \notin \mathrm{Aut}(N_{\mathrm{SU}(n-1)}(T_{n-1}))$.
Write $t_{n-3} = (\mu_k)$, $t_{n-2} = (\lambda_k)$ and $t_{n-1} = (\delta_k)$. By hypothesis there exists $j_o \in \{1, 2, \ldots, n-2\}$ such that $t_{j_o} \notin S_{n-1}$, i.e. $t_{j_o} = vs_{j_o}$ with $s_{j_o} \in S_{n-1}$. We claim that

$$t_j = vs_j\,, \quad \text{with } s_j \in S_{n-1} \text{ for all } j = 1, 2, \ldots, n-2\,.$$

First observe that $w_j \cdot v = v$ for all $j = 1, 2, \ldots, n-3$ and that the last coordinate of v is not exchanged by w_{n-2}. As $\ell_{j_o, j_o \pm 1} = 3$ we get, by applying (**) and inspecting the last coordinate on each side of the equality, that $t_{j_o \pm 1} = vs_{j_o \pm 1}$ with $s_{j_o \pm 1} \in S_{n-1}$. By induction, this demonstrates the claim.

Now we extract information from $\ell_{j,n-1} = 2$, for all $j = 1, 2, \ldots, n-3$. For instance, choosing $j = n-3$ formula (**) reads on the last four coordinates

$$\begin{array}{rrcl} & (\mu_{n-3}, \mu_{n-2}, \mu_{n-1}, -1) \cdot & & (\delta_{n-3}, \delta_{n-2}, \delta_{n-1}, \delta_n) \cdot \\ & (\mu_{n-3}, \mu_{n-2}, -1, \mu_{n-1}) & = & (\delta_{n-2}, \delta_{n-3}, \delta_{n-1}, \delta_n) \\ \Longrightarrow & (1, 1, -\mu_{n-1}, -\mu_{n-1}) & = & (\delta_{n-2}\delta_{n-3}, \delta_{n-2}\delta_{n-3}, 1, 1)\,. \end{array}$$

Repeating the argument for the other j's shows that the last two coordinates of t_j are $(-1, -1)$, $j = 1, 2, \ldots, n-3$, and that $t_{n-1} = (\delta, \ldots, \delta, \delta_{n-1}, \delta_n)$. By the decomposition of S in cosets associated to S_{n-2}, we deduce that, in fact,

$$t_j = vs_j\,, \quad \text{with } s_j \in S_{n-2} \text{ for all } j = 1, 2, \ldots n-3\,.$$

But, since v is invariant under w_j for $j = 1, 2, \ldots, n-3$, the fact that the elements $\psi(q_j) = vs_jq_j$ satisfy the three Tits relations implies that the elements s_jq_j have the same property. Therefore, setting $\tilde{\psi}(q_j) = s_jq_j$ for all $j = 1, 2, \ldots, n-3$, defines an automorphism $\tilde{\psi}$ of $N_{\mathrm{SU}(n-2)}(T_{n-2})$. Again by induction, two sub-cases can occur.

- $\tilde{\psi} = id_{N_{\mathrm{SU}(n-2)}(T_{n-2})}$ and thus $s_j = \mathbb{1}$, i.e. $t_j = v$ for $j = 1, 2, \ldots, n-3$. But $\ell_{1,n-2} = 2$ shows that this case is impossible by inspecting the last three coordinates.

- In case n is even, we can also have $s_j = (-1, \ldots, -1, 1, 1)$, i.e. $t_j = -\mathbb{1}$ for $j = 1, 2, \ldots, n-3$. Now $\ell_{j,n-2} = 2$ for $j = 1, 2, \ldots, n-4$ shows that $t_{n-2} = (\lambda, \ldots, \lambda, -1, -1, -1)$. We already know that $t_{n-1} = (\delta, \ldots, \delta, -1, -1)$. Using $\ell_{n-3,n-2} = 3$ and checking the last four coordinates, one sees that $t_{n-2} = -\mathbb{1}$; then $\ell_{n-2,n-1} = 3$, and the last three coordinates finally implies $t_{n-1} = -\mathbb{1}$. In case n is even, we therefore get one non-trivial outer automorphism, as described in example 3.1. This concludes the proof for $\mathrm{SU}(n)$.

The proofs of the three other cases are similar and relegated to the appendix. □

Remarks 3.24

1. Explicit calculations in adjoint groups show that there exist non-trivial elements in some cohomology groups $H^1(W_o;T)$ that are not of the form described at the end of the proposition.

2. Thanks to the isomorphism $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ we know that

$$H^1(W_o(\mathrm{SU}(2) \times \mathrm{SU}(2)); \mathbb{S}^1 \times \mathbb{S}^1) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2\,.$$

Explicitly, this cohomology group is generated by the classes of

$$\psi_\pm \text{ and } \psi_\mp \in \mathrm{Aut}\big(N_o(\mathrm{SU}(2)) \times N_o(\mathrm{SU}(2)), \mathbb{S}^1 \times \mathbb{S}^1, W_o(\mathrm{SU}(2) \times \mathrm{SU}(2))\big)\,,$$

entirely defined by

$$\begin{array}{rrcc} \psi_\pm : & N_o(\mathrm{SU}(2)) \times N_o(\mathrm{SU}(2)) & \longrightarrow & N_o(\mathrm{SU}(2)) \times N_o(\mathrm{SU}(2)) \\ & q_1 = (j, \mathbf{1}) & \longmapsto & (j, -\mathbf{1}) \\ & q_2 = (\mathbf{1}, j) & \longmapsto & (\mathbf{1}, -j) \end{array}$$

and

$$\begin{array}{rrcc} \psi_\mp : & N_o(\mathrm{SU}(2)) \times N_o(\mathrm{SU}(2)) & \longrightarrow & N_o(\mathrm{SU}(2)) \times N_o(\mathrm{SU}(2)) \\ & q_1 = (j, \mathbf{1}) & \longmapsto & (-j, \mathbf{1}) \\ & q_2 = (\mathbf{1}, j) & \longmapsto & (-\mathbf{1}, j) \end{array}$$

Recalling that $H^1(W_o(\mathrm{SU}(2)); \mathbb{S}^1) = 0$, this shows that passing from the simple to the semisimple case generates new outer automorphisms of the normalizer that are not just obtained from permuting isomorphic factors or built up from outer automorphisms of each factor. So far, we have not find a way to completely control and describe this phenomenon.

Chapter 4

A generalization of the theorem of Curtis-Wiederhold-Williams

4.1 Formulation of the generalization as a conjecture

Examples show that one cannot generalize the theorem of Curtis, Wiederhold and Williams (see page 11) by just dropping the hypothesis of connectedness: SO(3) and O(2) have isomorphic normalizers of maximal tori, even though they are not isomorphic. Other examples can be constructed if one finds a nonabelian connected compact Lie group G_o in which the normalizer of a maximal torus N_o is not a maximal subgroup. In this case the two nonisomorphic compact Lie groups, both with a nonabelian connected component of the identity, having isomorphic normalizers of maximal tori are G_o and any closed subgroup G such that $N_o \subsetneq G \subsetneq G_o$; for instance take $G = N_{F_4}(\mathrm{Spin}(8)) \subsetneq F_4 = G_o$ as mentioned in chapter 1. The exhaustive list of such examples in the simple case can be found in Bourbaki [10, Exercise 6, p. 113] (consult the paper of Borel and de Siebenthal [8] for a thorough study of closed subgroups of maximal rank). These examples lead us to the following natural question, which we formulate as a conjecture.

Conjecture *Two compact Lie groups G and G' are isomorphic if and only if the normalizers of their maximal tori (N, N_o) and (N', N'_o) are isomorphic as group pairs, i.e. if and only if there exists an isomorphism $\rho : N \to N'$ giving rise to the commutative diagram*

$$\begin{array}{ccccc} N_o & \hookrightarrow & N & \twoheadrightarrow & N/N_o \\ \cong\downarrow\tilde{\rho} & & \cong\downarrow\rho & & \cong\downarrow\bar{\rho} \\ N'_o & \hookrightarrow & N' & \twoheadrightarrow & N'/N'_o \end{array}$$

This chapter is dedicated to show how far we could go in solving this conjecture.

4.2 Reconstructing G from the inclusion $N_o \hookrightarrow N$

One way to prove the original Curtis-Wiederhold-Williams theorem is, given the normalizer N_o of a maximal torus, to first construct a connected compact Lie group with normalizer N_o and then check that starting from an isomorphic normalizer yields an isomorphic connected compact Lie group. Our general strategy for trying to solve the conjecture is the same; the method, however, is quite different from the original one in the connected case. We start by comparing the set of equivalence classes of extensions corresponding to G with that corresponding to N. Fix the normalizer N_o of a maximal torus T in a connected compact Lie group G_o, and a finite group Γ. We will denote by

$$\mathcal{E}_{Lie}(\Gamma, N_o)$$

the subset of the set of equivalence classes of extensions $\mathcal{E}(\Gamma, N_o)$ corresponding to classes having a representative element $N_o \hookrightarrow N \twoheadrightarrow \Gamma$ that is the normalizer of a maximal torus in some compact Lie group G. Clearly, the subset $\mathcal{E}_{Lie}(\Gamma, N_o)$ is the relevant one in our problem; we first aim at finding a way to describe it. Since the extension $N_o \hookrightarrow N \twoheadrightarrow \Gamma$ is a normalizer in G, the "outer action" homomorphism $\theta : \Gamma \to \mathrm{Out}(N_o)$ comes, by restriction, from an "outer action" on the connected component G_o of G, i.e. there is a commutative diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\ \theta\ } & \mathrm{Out}(N_o) \\ & \searrow^{\varphi} & \uparrow i \\ & & \mathrm{Out}(G_o) \end{array}$$

where $\mathrm{Out}(G_o) \hookrightarrow \mathrm{Out}(N_o)$ is the canonical injection of theorem 3.10. We now fix a pair of homomorphisms θ and φ as in this last diagram. Recall from proposition 2.4 that there is a bijection $H^2_\varphi(\Gamma; Z(G_o)) \approx \mathcal{E}(\Gamma, G_o, \varphi)$; similarly, we have $H^2_\theta(\Gamma; Z(N_o)) \approx \mathcal{E}(\Gamma, N_o, \theta)$, since taking the normalizer of a maximal torus in the semidirect product $G_o \rtimes_{s\circ\varphi} \Gamma$, where s denotes a section of $\mathrm{Aut}(G_o) \twoheadrightarrow \mathrm{Out}(G_o)$, shows that $\mathcal{E}(\Gamma, N_o, \theta)$ is not empty.

Remark 4.1 As a principal subgroup of rank 1 associated to T contains a regular element, every automorphism $\sigma = s(\alpha)$ in the image of the section s associated to this principal subgroup (see theorem 1.12) globally preserves T. Moreover σ fixes H_T pointwise, thus σ_* fixes the associated principal diagonal $D(B)$ in LT. From this we deduce that both Weyl chambers containing $D(B)$ are globally preserved. Therefore the induced automorphism σ^* on LT^* preserves the basis B and σ_* preserves the dual basis $B^\vee$. The subgroup of automorphisms having this property will be denoted by $\mathrm{Aut}(G_o, T, B^\vee)$. *Summarizing, the section s associated to a principal subgroup of* rank 1 yields, by restriction, a commutative diagram

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\theta} & \mathrm{Out}(N_o) & \xrightarrow{s} & \mathrm{Aut}(N_o, B^\vee) \\ & \searrow^{\varphi} & \uparrow & & \uparrow r \\ & & \mathrm{Out}(G_o) & \xrightarrow{s} & \mathrm{Aut}(G_o, T, B^\vee) \end{array}$$

In particular, there also exists a class in $\mathcal{E}(\Gamma, N_o, \theta)$ represented by a semidirect product, namely by $N_o \rtimes_{s\circ\theta} \Gamma$.

Before giving a description of $\mathcal{E}_{Lie}(\Gamma, N_o)$, we introduce more notations. Recall that the centers of G_o and N_o coincide except on direct factors of type $SO(2\ell+1)$ (theorem 1.8); decomposing the connected component as $G_o = G_o^1 \times G_o^2$ where all such direct factors are gathered in G_o^2, we also have corresponding decompositions for the torus and for its normalizer, i.e. $N_o = N_o^1 \times N_o^2$. Thus we get two extensions

$$Z(G_o) = Z(N_o^1) \overset{i_1}{\hookrightarrow} Z(N_o) = Z(N_o^1) \oplus Z(N_o^2) \overset{p_2}{\twoheadrightarrow} Z(N_o^2) \cong \mathbb{Z}/2 \oplus \ldots \oplus \mathbb{Z}/2$$

and

$$Z(N_o^2) \overset{i_2}{\hookrightarrow} Z(N_o) = Z(N_o^1) \oplus Z(N_o^2) \overset{p_1}{\twoheadrightarrow} Z(G_o) = Z(N_o^1).$$

As the "outer action" preserves these decompositions, we get, by restriction to the centers, a commutative diagram

$$\begin{array}{ccccccc} \Gamma & \xrightarrow{\theta} & \mathrm{Out}(N_o) & \xrightarrow{r} & \mathrm{Aut}(Z(G_o)) \times \mathrm{Aut}(Z(N_o^2)) & \hookrightarrow & \mathrm{Aut}(Z(N_o)) \\ & \searrow^{\varphi} & \uparrow & & \uparrow & & \\ & & \mathrm{Out}(G_o) & \xrightarrow{r} & \mathrm{Aut}(Z(G_o)) & & \end{array}$$

We define $\bar{\xi}$ by $\bar{\theta}(\gamma) = r(\theta(\gamma)) = (\bar{\varphi}(\gamma), \bar{\xi}(\gamma))$ ($\bar{\xi}$ is the restriction to the center of N_o^2 of the "outer action" on G_o^2). The fact that the "outer action" preserves the decomposition says that the two above extensions are short exact sequences of Γ-modules.

Lemma 4.2 *The maps induced in cohomology by the inclusions of centers yield the decomposition*

$$\begin{aligned} H^2_{\bar{\theta}}(\Gamma; Z(N_o)) &= i_{1*}\Big(H^2_{\bar{\varphi}}(\Gamma; Z(G_o))\Big) \oplus i_{2*}\Big(H^2_{\bar{\xi}}(\Gamma; Z(N_o^2))\Big) \\ &\cong H^2_{\bar{\varphi}}(\Gamma; Z(G_o)) \oplus H^2_{\bar{\xi}}(\Gamma; Z(N_o^2)). \end{aligned}$$

In particular, we have

$$\mathcal{E}_{Lie}(\Gamma, N_o, \theta) \approx \operatorname{im}(i_{1*}) \cong H^2_{\bar\varphi}(\Gamma; Z(G_o)).$$

Proof. The long exact coefficient sequence [15, Chap. III, Proposition 6.1] gives

$$\ldots \longrightarrow H^2_{\bar\varphi}(\Gamma; Z(G_o)) \xrightarrow{i_{1*}} H^2_{\bar\theta}(\Gamma; Z(N_o)) \xrightarrow{p_{2*}} H^2_{\bar\xi}(\Gamma; Z(N_o^2)) \longrightarrow \ldots$$

and

$$\ldots \longrightarrow H^2_{\bar\xi}(\Gamma; Z(N_o^2)) \xrightarrow{i_{2*}} H^2_{\bar\theta}(\Gamma; Z(N_o)) \xrightarrow{p_{1*}} H^2_{\bar\varphi}(\Gamma; Z(G_o)) \longrightarrow \ldots$$

By functoriality, we have $p_{1*} \circ i_{1*} = id_{H^2(\Gamma; Z(G_o))}$ and $p_{2*} \circ i_{2*} = id_{H^2(\Gamma; Z(N_o^2))}$. Therefore p_{1*} and p_{2*} are both surjective and i_{1*} and i_{2*} are corresponding splittings. This shows that the above two parts of long exact sequences are in fact split extensions and gives the decomposition announced in the lemma. Finally, by definition $\mathcal{E}_{Lie}(\Gamma, N_o, \theta)$ corresponds to $\operatorname{im}(i_{1*})$. □

Applying these observations to all "outer action" homomorphisms $\theta : \Gamma \to \operatorname{Out}(N_o)$ that factorize through $\operatorname{Out}(G_o)$ proves that we have the following identification:

Corollary 4.3

$$\mathcal{E}_{Lie}(\Gamma, N_o) \approx \coprod_{\varphi \in \operatorname{Hom}(\Gamma, \operatorname{Out}(G_o))} H^2_{\bar\varphi}(\Gamma; Z(G_o)) \approx \mathcal{E}(\Gamma, G_o).$$

The proof of lemma 4.2 shows that, starting from an extension class u corresponding to the normalizer of a maximal torus in a compact Lie group, it is possible to recover the extension class corresponding to the group: it is given by $p_{1*}(u) \in H^2_{\bar\varphi}(\Gamma; Z(G_o))$. We now state this as a proposition and give an explicit and "constructive" proof.

Proposition 4.4 *Given the group pair (N, N_o) corresponding to the normalizer of a maximal torus in some compact Lie group G, one can reconstruct G, up to equivalence of extensions.*

Proof. Starting from the inclusion $N_o \hookrightarrow N$, the theorem of Curtis-Wiederhold-Williams gives the connected component G_o of G and thus its center $Z_o = Z(G_o)$. So we get the inclusions $Z_o \subset T \subset N_o \subset G_o$.

The "outer action" homomorphism for G is easily recovered. Indeed, the given inclusion trivially yields the corresponding extension

$$N_o \hookrightarrow N \twoheadrightarrow \Gamma = N/N_o.$$

Thus we get the homomorphism $\theta : \Gamma \to \mathrm{Out}(N_o)$ that factorizes, by hypothesis, as in the commutative diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\theta} & \mathrm{Out}(N_o) \\ & \searrow^{\varphi} & \uparrow i \\ & & \mathrm{Out}(G_o) \end{array}$$

where φ is the homomorphism we were looking for.

We proceed with the central idea of the proof, which is to pass to the adjoint group of G_o, i.e. the connected compact Lie group $\bar{G}_o = G_o/Z_o$. Observe that Z_o is normal in the compact Lie group G that we try to reconstruct and therefore also in N. The extension corresponding to G gives rise, when passing to quotients, to an extension $\bar{G}_o \hookrightarrow \bar{G} = G/Z_o \twoheadrightarrow \Gamma$. Since Z_o is characteristic in G_o, there is a natural homomorphism $\mathrm{Out}(G_o) \to \mathrm{Out}(\bar{G}_o)$ which leads to the homomorphism $\bar{\varphi} : \Gamma \xrightarrow{\varphi} \mathrm{Out}(G_o) \to \mathrm{Out}(\bar{G}_o)$. As the center of $\bar{G}_o$ is trivial, by theorem 2.4 there exists, up to equivalence, a unique extension of Γ by $\bar{G}_o$ giving rise to $\bar{\varphi}$, namely $\bar{G} \cong \bar{G}_o \rtimes_{\bar{s}\circ\bar{\varphi}} \Gamma$, where $\bar{s}$ is a section of the canonical projection $\mathrm{Aut}(\bar{G}_o) \twoheadrightarrow \mathrm{Out}(\bar{G}_o)$, chosen as in theorem 1.12. Moreover, by taking the quotient of the given normalizer extension by Z_o one gets another extension $\bar{N}_o = N_o/Z_o \hookrightarrow \bar{N} = N/Z_o \twoheadrightarrow \Gamma$ that is split. To see it, let us consider the maximal torus $\bar{T} = T/Z_o$ in $\bar{G}_o$. We claim that $\bar{N}_o = N_{\bar{G}_o}(\bar{T})$ and $\bar{N} = N_{\bar{G}}(\bar{T})$. The proof is straightforward and we only show the first equality:

$$\begin{aligned} \bar{n} \in N_{\bar{G}_o}(\bar{T}) &\iff \begin{cases} \bar{n} = gZ_o\,,\ g \in G_o : \\ gZ_o \cdot tZ_o \cdot g^{-1}Z_o = gtg^{-1}Z_o \in \bar{T}\,,\ \forall t \in T \end{cases} \\ &\iff \bar{n} = gZ_o\,,\ g \in N_o \\ &\iff \bar{n} \in N_o/Z_o\,. \end{aligned}$$

Now by theorem 1.10 there is a commutative diagram

$$\begin{array}{ccccc} 1 & \hookrightarrow & Z & \xrightarrow{p} & \Gamma \\ \downarrow & & \downarrow & & \| \\ \bar{N}_o & \hookrightarrow & \bar{N} & \twoheadrightarrow & \Gamma \\ \downarrow & & \downarrow & & \| \\ \bar{G}_o & \hookrightarrow & \bar{G} & \twoheadrightarrow & \Gamma \end{array}$$

where each row is a group extension. Thus p is an isomorphism and $\nu = p^{-1}$ is the desired section.

The last part of the reconstruction of G consists in taking the pullback of the pair of morphisms (ν, π) :

$$\begin{array}{ccc}
Z_o & = & Z_o \\
\downarrow & & \downarrow \\
P & \hookrightarrow & N \\
\downarrow & & \downarrow \pi \\
\Gamma & \overset{\nu}{\hookrightarrow} & \bar{N}
\end{array}$$

The universal property of the pullback shows that, up to equivalence, this is the only extension fitting in the above commutative diagram. Moreover, as priacipal subgroups of rank 1 in a connected compact Lie group are all conjugate [18, Théorème 2, p. 252], it implies that, again up to equivaleoce, P does oot depeod on the priocipal subgroup chosen to get the section $\bar{s}$. Now suppose we know G, then it must fit into the commutative diagram

$$\begin{array}{ccccccc}
Z_o & = & Z_o & \hookrightarrow & N_o & \hookrightarrow & G_o \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P & \hookrightarrow & N & = & N & \hookrightarrow & G \\
\downarrow & & \downarrow \pi & & \downarrow & & \downarrow \\
\Gamma & \overset{\nu}{\hookrightarrow} & \bar{N} & \twoheadrightarrow & \Gamma & = & \Gamma
\end{array}$$

Since ν is a section we get a last commutative diagram

$$\begin{array}{ccccc}
Z_o & \hookrightarrow & P & \twoheadrightarrow & \Gamma \\
\downarrow & & \downarrow & & \| \\
G_o & \hookrightarrow & G & \twoheadrightarrow & \Gamma
\end{array}$$

So we have enough data to actually reconstruct a unique extension correspoodiag to G and conclude the proof (iavoke theorem 2.6 or apply the construction described io remark 2.11, 3). □

4.3 Problem of invariance under isomorphisms of group pairs

At this stage, one has to be careful not to conclude that the coajecture is solved; the fact that the reconstruction process is well behaved with respect to isomorphisms of the pair (N, N_o) has yet to be proved. Let $\rho : N_u \to N_v$ be an isomorphism of normalizers of maximal tori that fits in the commutative diagram

$$\begin{array}{ccccc}
N_o & \hookrightarrow & N_u & \twoheadrightarrow & \Gamma \\
\cong \downarrow \psi = \tilde{\rho} & & \cong \downarrow \rho & & \cong \downarrow \bar{\rho} \\
N_o & \hookrightarrow & N_v & \twoheadrightarrow & \Gamma
\end{array}$$

Let θ_u and θ_v be the corresponding "outer action" homomorphisms, both with image in $\mathrm{Out}(G_o) \overset{i}{\hookrightarrow} \mathrm{Out}(N_o)$. Let h_u and h_v be representative cocycles of the classes $u \in H^2_{\theta_u}(\Gamma; Z(N_o))$ and $v \in H^2_{\theta_v}(\Gamma; Z(N_o))$ corresponding to the extension associated to N_u and N_v. Let G_u and G_v denote the compact Lie groups corresponding to N_u and N_v. To solve the conjecture, we have to show that G_u and G_v are isomorphic. By theorem 2.8 in chapter 2, we can tell when two compact Lie groups are isomorphic from the knowledge of the corresponding cohomology classes. Now we also have, by corollary 4.3, a bijection $\mathcal{E}_{Lie}(\Gamma, N_o) \approx \mathcal{E}(\Gamma, G_o)$. Therefore $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$ acts on $\mathcal{E}_{Lie}(\Gamma, N_o)$ by carrying, through this bijection, the action described in lemma 2.7. Clearly, two extensions in the same orbit correspond to isomorphic normalizers. Showing that the converse is true would prove the "generalized Curtis-Wiederhold-Williams theorem". These considerations, together with the decomposition $\mathrm{Out}(N_o) \cong H^1(W_o; T) \rtimes \mathrm{Out}(G_o)$ and the description of $H^1(W_o; T)$ in lemma 3.22, show that the situation we consider can be reduced as follows:

Lemma 4.5 *The conjecture is true if it holds for all isomorphisms of the pairs (N_u, N_o) and (N_v, N_o) that fit in a commutative diagram*

$$\begin{array}{ccccccc}
T & \hookrightarrow & N_o & \hookrightarrow & N_u & \twoheadrightarrow & \Gamma \\
\| & & \cong\downarrow\psi & & \cong\downarrow\rho & & \| \\
T & \hookrightarrow & N_o & \hookrightarrow & N_v & \twoheadrightarrow & \Gamma
\end{array}$$

with $[\psi] \in H^1(W_o; T)$ and $\psi^2 = id_{N_o}$.

Remark 4.6 We give another interpretation of the problem of invariance under isomorphisms of group pairs. Both $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$ and $\mathrm{Out}(N_o) \times \mathrm{Aut}(\Gamma)$ act on $\mathcal{E}_{Lie}(\Gamma, N_o)$; to solve the conjecture, it suffices to show that there are no less orbits for the action of $\mathrm{Out}(N_o) \times \mathrm{Aut}(\Gamma)$ than for the action of $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$. Obviously, the "generalized Curtis-Wiederhold-Williams theorem" is false if there exist two non-isomorphic compact Lie groups having isomorphic pairs of normalizers. In this case, when the projection $[\psi] \in \mathrm{Out}(N_o)$ of the restricted automorphism ψ of N_o is decomposed according to $\mathrm{Out}(N_o) \cong H^1(W_o; T) \rtimes \mathrm{Out}(G_o)$, the component in $H^1(W_o; T)$ is necessarily non-trivial.

4.4 Extension of automorphisms

We want to study more carefully the isomorphisms ρ that appear in lemma 4.5 and derive some conditions under which the conjecture is true. Suppose for a while that we look at the problem from another viewpoint. Namely, suppose we are given the automorphism $\psi : N_o \to N_o$ and asked if it can be extended to an automorphism $\bar{\rho}$ as in the commutative diagram

$$\begin{array}{ccccc} N_o & \hookrightarrow & N_u & \twoheadrightarrow & \Gamma \\ \cong\downarrow\psi & & \cong\downarrow\exists?\bar{\rho} & & \| \\ N_o & \hookrightarrow & N_u & \twoheadrightarrow & \Gamma \end{array}$$

If $\bar{\rho}$ exists, then the corresponding compact Lie groups G_u and G_v are isomorphic. Indeed, in this case we have

$$\begin{array}{ccccc} N_o & \hookrightarrow & N_u & \twoheadrightarrow & \Gamma \\ \cong\downarrow\psi^{-1} & & \cong\downarrow\bar{\rho}^{-1} & & \| \\ N_o & \hookrightarrow & N_u & \twoheadrightarrow & \Gamma \\ \cong\downarrow\psi & & \cong\downarrow\rho & & \| \\ N_o & \hookrightarrow & N_v & \twoheadrightarrow & \Gamma \end{array}$$

and therefore $u = v$ and we can conclude that $G_u \cong G_v$. We have proved the following lemma:

Lemma 4.7 *If the automorphism ψ in lemma 4.5 can be extended to an automorphism of N_u then $G_u \cong G_v$, i.e. the conjecture is true in this case.*

The following easy observation will be useful.

Lemma 4.8 *Let $E = K \rtimes Q$ be a semidirect product; let $k \in K$ and $q \in Q$. If $c_k(q) = kqk^{-1}$ is an element of Q then $c_k(q) = q$.*

Proof. As $K \triangleleft E$, we have $c_k(q) = kqk^{-1} = kqk^{-1}q^{-1}q = k'q$, with $k' \in K$. But, by hypothesis, $k'q \in Q$ and therefore $k' \in K \cap Q$, which is reduced to the neutral element. □

Lemma 4.9 *In the reduced situation depicted in lemma 4.5, θ_u and θ_v are equal, and, for $\theta = \theta_u = \theta_v$, we have*

$$[\psi] \in H^1(W_o;T) \cap Z_{\mathrm{Out}(N_o)}(\theta(\Gamma)).$$

In particular, for this reduced situation, u and v are classes in the same cohomology group $H^2_\theta(\Gamma; Z(N_o))$.

Proof. As in lemma 2.7, a direct computation shows that $\theta_v(\gamma) = c_{[\psi]}(\theta_u(\gamma))$ for all $\gamma \in \Gamma$ (i.e. $\theta_v = c_{[\psi]} \circ \theta_u$). By hypothesis, $\theta_u(\gamma)$ and $\theta_v(\gamma)$ are both elements of $\mathrm{Out}(G_o)$, and $[\psi]$ is an element of $H^1(W_o;T)$. Recalling that $\mathrm{Out}(N_o) \cong H^1(W_o;T) \rtimes \mathrm{Out}(G_o)$, we deduce, by lemma 4.8, that $[\psi]$ centralizes $\theta_u(\Gamma)$ and therefore $\theta_u = \theta_v$. □

Proposition 4.10 *Suppose that the connected compact Lie group G_o corresponding to N_o is of adjoint type, i.e. Z_o is trivial. Then the conjecture is true.*

Proof. It suffices to show that this holds for the reduced situation of lemma 4.5. As Z_o is trivial, every cohomology group appearing in the description of $\mathcal{E}_{Lie}(\Gamma, N_o)$ in corollary 4.3 is trivial. Therefore, by lemma 4.9, $u = v$ and we can conclude that $G_u \cong G_v$. □

For a fixed section s as in theorem 1.12, recall that there is a commutative diagram

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\theta} & \mathrm{Out}(N_o) & \xrightarrow{s} & \mathrm{Aut}(N_o) \\ & \searrow^{\varphi} & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle r} \\ & & \mathrm{Out}(G_o) & \xrightarrow{s} & \mathrm{Aut}(G_o, T) \end{array}$$

We will write $\sigma = s \circ \theta$ and $\sigma_\gamma = \sigma(\gamma)$ for $\gamma \in \Gamma$. Recall from chapter 2, page 19, that the extension corresponding to N_u is equivalent to $[N_o \hookrightarrow N_{h_u} \twoheadrightarrow \Gamma]$, where N_{h_u} is the set $N_o \times \Gamma$ with multiplication

$$(n, \gamma) *_{h_u} (n', \gamma') = \left(n \cdot \sigma_\gamma(n') \cdot h_u(\gamma, \gamma'), \gamma \cdot \gamma'\right).$$

The same holds for N_v, replacing h_u by h_v.

We will denote by $\mathrm{Aut}_p(N_u, N_v)$ (where "p" stands for *pathological*) the set of isomorphisms of N_u to N_v that induce an automorphism ψ of N_o satisfying

(1) $\psi_{|T} = id_T$,

(2) $\psi^2 = id_{N_o}$,

(3) $[\psi] \in H^1(W_o; T) \cap Z_{\mathrm{Out}(N_o)}(\theta(\Gamma)) = H^1(W_o; T) \cap Z_{\mathrm{Out}(N_o)}(\varphi(\Gamma))$.

Definition 4.11 An automorphism ψ of N_o satisfying conditions (1) and (2) and such that $[\psi]$ is a non-trivial outer automorphism will be called *exotic*.

Lemma 4.12 *There is a bijection between* $\mathrm{Aut}_p(N_u, N_v)$ *and the set of normalized maps*

$$f : \Gamma \longrightarrow S_Z = \{t \in T : t^2 \in Z(N_o)\} \subset T \subset N_o$$

such that

(i) the map

$$\begin{array}{rcl} \Gamma \times \Gamma & \longrightarrow & Z(N_o) \\ (\gamma, \gamma') & \longmapsto & \sigma_\gamma(f(\gamma')) \cdot f(\gamma\gamma')^{-1} \cdot f(\gamma) \end{array}$$

is equal to the cocycle $h_u \cdot h_v^{-1}$, i.e. represents the cohomology class $u - v \in H^2_\theta(\Gamma; Z(N_o))$;

(ii) for all $\gamma \in \Gamma$ *we have*

$$\psi \circ \sigma_\gamma \circ \psi^{-1} = \psi \circ \sigma_\gamma \circ \psi = c_{f(\gamma)} \circ \sigma_\gamma .$$

Explicitly, the bijection is given by

$$f \longmapsto \Big((n,\gamma) \mapsto \big(\psi(n)f(\gamma),\gamma\big)\Big) \in \mathrm{Aut}_p(N_u, N_v) .$$

Proof. Let $\rho \in \mathrm{Aut}_p(N_u, N_v)$. We proceed in several steps to show that ρ is defined by a function f as stated.

Step 1: As ψ induces the identity on W_o, ρ can be written as $\rho(n,\gamma) = (\rho_\gamma(n),\gamma)$ for some maps $\rho_\gamma : N_o \to N_o$ defined for all $\gamma \in \Gamma$. Since ρ induces ψ on N_o, we have in particular $\rho_e = \psi$. Decomposing an element in N_u as $(n,\gamma) = (n,e) *_{h_u} (e,\gamma)$, we calculate

$$\begin{aligned}
\rho(n,\gamma) &= \rho\big((n,e) *_{h_u} (e,\gamma)\big) \\
&= \rho(n,e) *_{h_v} \rho(e,\gamma) \\
&= \big(\psi(n),e\big) *_{h_v} \big(\rho_\gamma(e),\gamma\big) \\
&= \big(\psi(n)\sigma_e(\rho_\gamma(e))h_v(e,\gamma),\gamma\big) \\
&= \big(\psi(n)\rho_\gamma(e),\gamma\big) .
\end{aligned}$$

Therefore, defining $f : \Gamma \to N_o$, $\gamma \mapsto f(\gamma) = \rho_\gamma(e)$, we get

$$\rho(n,\gamma) = \big(\psi(n)f(\gamma),\gamma\big)$$

with $f(e) = e$.

Step 2: We derive a formula which will show that f satisfies (i), and later (ii). On one hand, we have

$$\begin{aligned}
\rho\big((n,\gamma) *_{h_u} (n',\gamma')\big) &= \rho\big(n\sigma_\gamma(n')h_u(\gamma,\gamma'),\gamma\gamma'\big) \\
&= \big(\psi(n\sigma_\gamma(n')h_u(\gamma,\gamma'))f(\gamma\gamma'),\gamma\gamma'\big) \\
&= \big(\psi(n)\psi(\sigma_\gamma(n'))h_u(\gamma,\gamma')f(\gamma,\gamma'),\gamma\gamma'\big)
\end{aligned}$$

because ψ is the identity on the center $Z(N_o)$. On the other hand, we have

$$\begin{aligned}
\rho(n,\gamma) *_{h_v} \rho(n',\gamma') &= \big(\psi(n)f(\gamma),\gamma\big) *_{h_v} \big(\psi(n')f(\gamma'),\gamma'\big) \\
&= \big(\psi(n)f(\gamma)\sigma_\gamma(\psi(n')f(\gamma'))h_v(\gamma,\gamma'),\gamma\gamma'\big) .
\end{aligned}$$

These two expressions being equal, we get

$$\psi(\sigma_\gamma(n'))f(\gamma\gamma')h_u(\gamma,\gamma') = f(\gamma)\sigma_\gamma(\psi(n')f(\gamma'))h_v(\gamma,\gamma') \qquad (\sharp)$$

for all $n' \in N_o, \gamma, \gamma' \in \Gamma$. In particular, for $\gamma' = e$, it becomes

$$\begin{array}{rrcll} & \psi(\sigma_\gamma(n'))f(\gamma) & = & f(\gamma)\sigma_\gamma(\psi(n')) & \forall n' \in N_o, \gamma \in \Gamma \\ \Longleftrightarrow & (\psi \circ \sigma_\gamma)(n') & = & f(\gamma)(\sigma_\gamma \circ \psi)(n')f(\gamma)^{-1} & \forall n' \in N_o, \gamma \in \Gamma \\ \Longleftrightarrow & \psi \circ \sigma_\gamma & = & c_{f_\gamma} \circ \sigma_\gamma \circ \psi & \forall \gamma \in \Gamma \\ \Longleftrightarrow & \psi \circ \sigma_\gamma \circ \psi^{-1} & = & c_{f_\gamma} \circ \sigma_\gamma & \forall \gamma \in \Gamma, \end{array}$$

which is condition (ii).

Step 3: We check that $f(\Gamma) \subset T$. For all $\gamma \in \Gamma$ condition (ii) gives

$$\begin{array}{rrcll} & (\psi \circ \sigma_\gamma \circ \psi)(t) & = & (c_{f_\gamma} \circ \sigma_\gamma)(t) & \forall t \in T \\ \Longleftrightarrow & \sigma_\gamma(t) & = & c_{f_\gamma}(\sigma(t)) & \forall t \in T \\ \Longleftrightarrow & t & = & c_{f_\gamma}(t) & \forall t \in T \end{array}$$

because ψ is the identity on T and $\sigma_{|T} \in \mathrm{Aut}(T)$. Therefore $f(\gamma)$ is in the centralizer $Z_{N_o}(T)$ of T which is equal to T.

Step 4: For $n' = e$, formula $(\sharp)$ becomes

$$f(\gamma\gamma')h_u(\gamma, \gamma') = f(\gamma)\sigma_\gamma(f(\gamma'))h_v(\gamma, \gamma'),$$

which is equivalent to condition (i).

Step 5: Finally, we show that $f(\gamma)^2 \in Z(N_o)$ for all $\gamma \in \Gamma$. Let $\eta = \rho^{-1} \in \mathrm{Aut}_p(N_u, N_v)$; by the first steps we have $\eta(\mathrm{n}, \gamma) = (\psi(\mathrm{n})g(\gamma), \gamma)$ for some map $g : \Gamma \to N_o$. For all n and γ, we calculate

$$\begin{array}{rcl} (\mathrm{n}, \gamma) & = & (\rho^{-1} \circ \rho)(n, \gamma) \\ & = & \eta\big(\psi(n)f(\gamma), \gamma\big) \\ & = & \big(\psi(\psi(n)f(\gamma))g(\gamma), \gamma\big) \\ & = & \big(\psi^2(n)\psi(f(\gamma))g(\gamma), \gamma\big) \\ & = & \big(nf(\gamma)g(\gamma), \gamma\big). \end{array}$$

Therefore $g(\gamma) = f(\gamma)^{-1}$ for all $\gamma \in \Gamma$. Now g satisfies condition (ii), so we get

$$c_{f(\gamma)^{-1}} \circ \sigma_\gamma = c_{g(\gamma)} \circ \sigma_\gamma = \psi \circ \sigma_\gamma \circ \psi = c_{f(\gamma)} \circ \sigma_\gamma$$

for all γ. This implies that $c_{f(\gamma)^2} = id_{N_o}$ and ends up step 5.

Conversely, given a map $f : \Gamma \to S_Z$ satisfying (i) and (ii), a straightforward computation shows that

$$\Big((\mathrm{n}, \gamma) \mapsto \big(\psi(n)f(\gamma), \gamma\big)\Big) \in \mathrm{Aut}_p(N_u, N_v).$$

□

Remark 4.13 For $u = v$, this lemma is a particular case of a lemma in Wells' paper [70, p. 191].

Corollary 4.14 *There exists $\rho \in \mathrm{Aut}_p(N_u, N_v)$ if and only if there exists $\rho' \in \mathrm{Aut}_p(N_{u'}, N_{v'})$, where $u' = 0$ and $v' = v - u$ in $H^2_\theta(\Gamma; Z(N_o))$.*

Proof. Let $\rho \in \mathrm{Aut}_p(N_u, N_v)$ with induced automorphism ψ. The corresponding map f satisfies condition (ii) of lemma 4.12, which does not depend on the cocyles u and v in $H^2_\theta(\Gamma; Z(N_o))$. Now f also satisfies (i), which only depends on the difference $u - v$. Therefore, f defines an isomorphism $\rho' \in \mathrm{Aut}_p(N_{u'}, N_{v'})$, inducing the same automorphism ψ, for all pairs of cocycles $u', v' \in H^2_\theta(\Gamma; Z(N_o))$ such that $u' - v' = u - v$. The other direction is treated symmetrically and the corollary follows as a particular case. □

Corollary 4.15 *Let $\rho \in \mathrm{Aut}_p(N_u, N_v)$ and $\psi = \rho_{|N_o} \in \mathrm{Aut}(N_o)$. Suppose there exists an exotic automorphism $\eta \in [\psi]$ and a section s fitting in the commutative diagram*

$$\begin{array}{ccccc} \sigma : \Gamma & \xrightarrow{\theta} & \mathrm{Out}(N_o) & \xrightarrow{s} & \mathrm{Aut}(N_o) \\ & \searrow^{\varphi} & \updownarrow & & \uparrow r \\ & & \mathrm{Out}(G_o) & \xrightarrow{s} & \mathrm{Aut}(G_o, T) \end{array}$$

such that $\eta \in Z_{\mathrm{Aut}(N_o)}(\sigma(\Gamma))$. Then $u = v$, i.e. the extensions corresponding to N_u and N_v are equivalent. In particular the corresponding compact Lie groups G_u and G_v are isomorphic.

Proof. By hypothesis, η commutes with every σ_γ. Therefore condition (ii) in the lemma implies that the map $f : \Gamma \to S_Z$ corresponding to η has its image in $Z(N_o)$. In this case, (i) exactly says that $h_u \cdot h_v^{-1}$ is a coboundary (see page 19), and thus $u = v$. □

Remark 4.16 The section s in this corollary does not necessarily have to be associated to a principal subgroup of rank one.

4.5 Cases where the conjecture is true

From what has been done in the previous sections, we already know that the "generalized Curtis-Wiederhold-Williams theorem" is true for particular cases. We now summarize the situation, and go one step further by showing that the conjecture is true in the simple case.

Theorem 4.17 *In the following cases, the conjecture is true:*

(1) the automorphism ψ is the restriction of some automorphism of G_o, i.e.

$$[\psi] \in \mathrm{Out}(G_o) \stackrel{i}{\hookrightarrow} \mathrm{Out}(N_o)\,;$$

(2) the connected compact Lie group G_o corresponding to N_o is of adjoint type;

(3) the connected compact Lie group G_o corresponding to N_o is simple.

Clearly, point (1) in the theorem follows from lemma 4.5, and point (2) is proposition 4.10. The rest of this section is dedicated to prove point (3). The proof will be based on corollary 4.15 and on case-by-case checking using the classification of connected compact Lie groups and the calculations of $H^1(W_o;T)$ done in chapter 3. More precisely, we will consider pairs of automorphisms ψ,σ of the normalizer N_o, such that ψ is exotic and $[\sigma] \in \mathrm{Out}(G_o)$. By lemmas 4.5 and 4.9, cases for which $[\psi]\circ[\sigma]\circ[\psi]^{-1} \neq [\sigma]$ will not have to be considered; if, however, $[\psi]\circ[\sigma]\circ[\psi]^{-1} = [\sigma]$ we will show that well-chosen representatives commute in $\mathrm{Aut}(N_o)$ and that corollary 4.15 can always be invoked to conclude.

Before actually proving point (3) we reduce the cases to be checked. If $\mathrm{Out}(G_o)$ or $H^1(W_o;T)$ is trivial, corollary 4.15 clearly holds and point (3) follows. If G_o is centerless, then point (2) gives the conclusion. Therefore, from the classification of connected compact Lie groups and from proposition 3.23, point (3) is verified for $G_o = \mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8, \mathrm{SU}(2), \mathrm{SU}(2n+1), \mathrm{Spin}(2n+1), \mathrm{Sp}(n), \mathrm{sSpin}(4n)\ (n \geqslant 3)$, and all their quotients except for $\mathrm{SU}(2n+1)$. The remaining cases are: $\mathrm{SU}(2n)$ and possibly quotients of $\mathrm{SU}(n)$, $\mathrm{SO}(2n)$ and $\mathrm{Spin}(2n)$ for $n \geqslant 4$ (recall that the triality principle implies $\mathrm{sSpin}(8) \cong \mathrm{SO}(8)$).

We now provide several lemmas needed to establish (3) for the remaining cases.

Lemma 4.18 *Let $\sigma \in \mathrm{Aut}(N_o)$ such that $[\sigma] \in \mathrm{Out}(G_o)$. Then σ preserves the set of Tits circles.*

Proof. This is a direct consequence of the unicity, up to conjugation by an element of $\mathrm{S}^1 \subset \mathrm{SU}(2)$, of the homomorphisms $\nu_\alpha : \mathrm{SU}(2) \to G_o$. □

Let us fix a basis B of the root system associated to the maximal torus $T \subset G_o$. Denote by $B^\vee$ the dual basis and choose an associated Tits system (see page 41) $A = \{q_{\alpha^\vee} = q_\alpha\}_{\alpha\in B}$.

Lemma 4.19 *Let ψ be an exotic automorphism defined by $\psi(q_\alpha) = zq_\alpha$, for all $q_\alpha \in A$, for some $z \in Z(N_o) \cap S$. Suppose $\sigma \in \mathrm{Aut}(N_o)$ satisfies $\sigma(A) = A$ and $\sigma(z) = z$. Then ψ and σ commute, i.e. $\psi\circ\sigma\circ\psi = \psi\circ\sigma\circ\psi^{-1} = \sigma$.*

Proof. The automorphisms ψ and σ clearly commute on T. It then suffices to check that $\psi \circ \sigma \circ \psi = \sigma$ on A. But this follows from a straightforward computation. Indeed, let $q_\beta = \sigma(q_\alpha)$, then

$$\psi\circ\sigma\circ\psi(q_\alpha) = \psi(\sigma(\psi(q_\alpha))) = \psi(\sigma(zq_\alpha)) = \psi(zq_\beta) = \psi(z)\psi(q_\beta) = z^2 q_\beta = q_\beta = \sigma(q_\alpha).$$

□

Lemma 4.20 *Let $\sigma \in \mathrm{Aut}(N_o, B^\vee)$ such that $[\sigma] \in \mathrm{Out}(G_o)$. Then there exists $t \in T$ such that*

$$c_t \circ \sigma(q_{\alpha^\vee}) = q_{\sigma_*(\alpha^\vee)},$$

for all $\alpha^\vee \in B^\vee$.

Proof. By point (iii) in lemma 3.7, we know that $\sigma(q_{\alpha^\vee}) = n$, with $\pi(n) = s_{\sigma_*(\alpha^\vee)} \in W_o$. Now, by lemma 4.18, n is on the Tits circle in $\pi^{-1}(s_{\sigma_*(\alpha^\vee)})$. The conclusion then follows from the fact that Tits systems are conjugate by an element of T (see lemma 3.21). □

Corollary 4.21 *Let G_o be without factors of type $\mathrm{SO}(2\ell+1)$. If $\mathrm{Out}(G_o)$ is cyclic, then the section s in the following commutative diagram*

$$\begin{array}{ccc}
\mathrm{Out}(N_o) & \xrightarrow{s} & \mathrm{Aut}(N_o) \\
\uparrow & & \uparrow r \\
\mathrm{Out}(G_o) & \xrightarrow{s} & \mathrm{Aut}(G_o, T)
\end{array}$$

can be chosen such that every σ in the image of s globally fixes the Tits system A.

Proof. By remark 4.1, there exists a section s such that $B^\vee$ is preserved. Let σ be a generator of $s(\mathrm{Out}(G_o))$. By the lemma there exists $t \in T$ such that $\tilde{\sigma} = c_t \circ \sigma$ preserves A. To prove the corollary we have to show that the order of $\tilde{\sigma}$ is equal to the order of σ. Now if σ^m is the identity, we have $\tilde{\sigma}^m_{|T} = \sigma^m_{|T} = id_{|T}$. Moreover $\tilde{\sigma}^m(q_{\alpha^\vee}) = q_{\sigma^m_*(\alpha^\vee)} = q_{\alpha^\vee}$ for all $\alpha^\vee \in B^\vee$. Therefore $\tilde{\sigma}^m$ is the identity on N_o. By a result of Bourbaki [10, Proposition 9, p. 30], $\tilde{\sigma}^m$ must be a conjugation by an element in $Z(N_o) = Z(G_o)$. □

Recall that $\mathrm{Out}(\mathrm{Spin}(8)) = \langle [\sigma], [\tau] \rangle \cong \Sigma_3$, for some elements $[\sigma]$ of order 2 and $[\tau]$ of order 3. The next lemma will give a description of the triality on the normalizer of the standard maximal torus in $\mathrm{Spin}(8)$. In fact, a description of a representative τ of $[\tau]$ on a Tits system and on the center $Z(N_o(\mathrm{Spin}(8))) = Z(\mathrm{Spin}(8))$ will be good enough for later use. Recall that $T(\mathrm{Spin}(8))$ consists of the elements

$$(\cos x_1 + e_1e_2 \sin x_1)(\cos x_2 + e_3e_4 \sin x_2) \cdots (\cos x_4 + e_7e_8 \sin x_4)$$

in Spin(8). A basis of the Lie algebra $LT(\text{Spin}(8)) \cong \mathbb{R}^4$ is given, in the Clifford algebra $C\ell(\mathbb{R}^8)$, by $\{v_1 = e_1e_2, v_2 = e_3e_4, v_3 = e_5e_6, v_4 = e_7e_8\}$ and the exponential map is defined by $e^{e_{2j-1}e_{2j}t} = \cos t + e_{2j-1}e_{2j}\sin t$. One checks that a basis $B^\vee$ of the coroot system of $LT(\text{Spin}(8))$ is given in $\{v_j\}$ by

$$\begin{aligned} \alpha_1^\vee &= (-1,-1,0,0,) \\ \alpha_2^\vee &= (1,-1,0,0,) \\ \alpha_3^\vee &= (0,1,-1,0) \\ \alpha_4^\vee &= (0,0,1,-1). \end{aligned}$$

By corollary 4.21, there exists $\tau \in r \circ s([\tau]) \in \text{Aut}(N_o(\text{Spin}(8)), B^\vee)$ such that

$$\begin{aligned} &\alpha_1^\vee \xmapsto{\tau_*} \alpha_2^\vee \xmapsto{\tau_*} \alpha_4^\vee \xmapsto{\tau_*} \alpha_1^\vee \\ &\alpha_3^\vee \xmapsto{\tau_*} \alpha_3^\vee \end{aligned}$$

Let $\{q_1, q_2, q_3, q_4\}$ be the Tits system associated to $B^\vee$.

Lemma 4.22 *The action of τ on $\{q_1, q_2, q_3, q_4\}$ is given by*

$$\begin{aligned} &q_1 \xmapsto{\tau} q_2 \xmapsto{\tau} q_4 \xmapsto{\tau} q_1 \\ &q_3 \xmapsto{\tau} q_3 \end{aligned}$$

Moreover, for the non-trivial elements of the center of Spin(8), *we have*

$$-1 \xmapsto{\tau} a = e_1e_2e_3e_4e_5e_6e_7e_8 \xmapsto{\tau} -a = -e_1e_2e_3e_4e_5e_6e_7e_8 \xmapsto{\tau} -1.$$

Proof. The assertion about the Tits system follows from lemma 4.20. Now the action on the center can be computed using the explicit description of the automorphism τ_* of $LT(\text{Spin}(8))$. By linear algebra it is given, in the basis $\{v_j\}$, by the matrix

$$M = \frac{1}{2}\begin{pmatrix} -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}.$$

The element $w = e_1e_2\pi$ is mapped to -1 by the exponential map. We thus have

$$\tau(-1) = \tau \circ \exp(w) = \exp(\tau_*(w)).$$

Straightforward computations give $Mw = \frac{1}{2}(-\pi, \pi, \pi, -\pi)$ and $\tau(-1) = e^{Mw} = a = e_1e_2e_3e_4e_5e_6e_7e_8$. In particular, this implies that the orbit of -1 under the action of τ has three elements and give the conclusion. $\square$

For $\mathrm{SU}(n)$, $n \geqslant 3$, the theorem will follow from the next lemma. Let $K \subset Z(\mathrm{SU}(n))$ be any central subgroup and $Q\,\mathrm{SU}(n) = \phi(\mathrm{SU}(n)) = \mathrm{SU}(n)/K$ be the associated quotient group. It is easy to check that a representative of the only non-trivial outer automorphism $[\sigma] \in \mathrm{Out}(\mathrm{SU}(n)) \cong \mathbf{Z}/2$ of $\mathrm{SU}(n)$ is given by complex conjugation, i.e.

$$\sigma : \mathrm{SU}(n) \longrightarrow \mathrm{SU}(n)\,,\ g \longmapsto \bar{g}\,.$$

Clearly σ fixes pointwise both the Tits system A associated to the standard maximal torus $T(\mathrm{SU}(n))$ described in example 3.20 and the subgroup S of elements of order 2 in $T(\mathrm{SU}(n))$. As σ preserves any central subgroup, we also get that the only outer automorphism of $Q\,\mathrm{SU}(n)$ is given by

$$\sigma' : Q\,\mathrm{SU}(n) \longrightarrow Q\,\mathrm{SU}(n)\,,\ [g] \longmapsto [\bar{g}]\,.$$

Now we can take for the Tits system associated the standard maximal torus $QT = T(Q\,\mathrm{SU}(n)) = \phi(T(\mathrm{SU}(n)))$ the image $QA = \phi(A)$ of A under the quotient map. Let S_{QT} denote the subgroup of elements of order 2 in QT.

Lemma 4.23 *For any central subgroup K, the automorphism σ' fixes pointwise both the Tits system QA and the subgroup S_{QT}.*

Proof. The first assertion is clear because $QA = \phi(A)$ and σ fixes A pointwise. If $K \cap S_{QT} = \{\mathbb{I}\}$ the second assertion follows because $S_{QT} = \phi(S)$ in this case. Otherwise, $K \cap S_{QT} = \{\pm\mathbb{I}\}$. Let $\omega = e^{2\pi i/k}\cdot\mathbb{I}$ be a generating element of K and m the integer such that $mk = n$. The subgroup of elements of order 2 in $Q\,\mathrm{SU}(n)$ decomposes in two cosets $S_{QT} = \phi(S) \amalg [\zeta]\cdot\phi(S)$, where one can choose $\zeta = \mathrm{diag}(e^{2\pi i/2k}, e^{2\pi i/2k}, \ldots, e^{2\pi i/2k}, (-1)^m e^{2\pi i/2k}) \in \mathrm{SU}(n)$. To conclude it is enough to check that $\sigma'[\zeta] = [\zeta]$. Now $\sigma'[\zeta] = [\sigma(\zeta)] = [\zeta^{-1}]$, and we are finished because $\zeta^2 = \omega \in K$ shows that $[\zeta] = [\zeta^{-1}]$. □

Proof of theorem 4.17. It only remains to check point (3) for the cases we described previously in this section. Let $\psi, \sigma \in \mathrm{Aut}(N_o)$ such that ψ is exotic and $[\sigma] \in \mathrm{Out}(G_o)$.

Case 1: $Q\,\mathrm{SU}(n)$ for $n \geqslant 3$. We can choose the exotic automorphism ψ such that each $q_\alpha \in QA$ is mapped to $t_\alpha q_\alpha$ for some $t_\alpha \in S_{QT}$. Now, by lemma 4.23, the automorphism σ' is the identity on QA and on S_{QT}. A simple variation on lemma 4.19 shows that ψ and σ' commute and we can conclude this case by invoking corollary 4.15.

For the remaining cases, we know by inspection of the results of proposition 3.23, that the non-trivial outer automorphisms in $H^1(W_o; T)$ are in bijection with the central elements of order 2. Each one can be represented by an exotic automorphism ψ_z given by $\psi_z(q_\alpha) = zq_\alpha$, with $z \in Z(N_o) \cap S = Z(G_o) \cap S$. For each σ, we choose a section s as in corollary 4.21.

Case 2: $\mathrm{SO}(2n)$ for $n \geqslant 4$. Obviously the only non-trivial central element $z = -\mathbb{1}$ is invariant under any automorphism. This case then follows from lemmas 4.19 and 4.15. (Recall that $\mathrm{Out}(\mathrm{SO}(2n)) = \langle[\sigma]\rangle \cong \mathbb{Z}/2$; we could also treat this case explicitly by working with the representative $\sigma = c_E$, where $E = \mathrm{diag}(-1, 1, \ldots, 1) \in \mathrm{O}(2n)$).

Case 3: $\mathrm{Spin}(2n)$ for $n \geqslant 4$. Recall that $\mathrm{Out}(\mathrm{Spin}(2n)) = \langle[\sigma]\rangle \cong \mathbb{Z}/2$ for $n \geqslant 5$. We subdivide this case:

- ψ_{-1}: since σ preserves $z = -1$, we can conclude as in case 2 (except for $n = 4$).
- for n even we also have $\psi_{\pm a}$: we are going to show that these two automorphisms do not centralize $[\sigma]$ in $\mathrm{Out}(N_o)$. The automorphism σ permutes a and $-a$ (this is easy to check with $[c_{e_1}] = [\sigma]$, where $e_1 \in \mathrm{Pin}(2n)$). Let q_α be an element of A that is fixed by σ (two elements in A are permuted, the other ones being fixed). Then a straightforward computation gives $\psi_{\pm a} \circ \sigma \circ \psi_a^{-1}(q_\alpha) = -a^2 q_\alpha = -q_\alpha$. A quick glance at the Tits circle in the appendix shows that $-q_\alpha$ is not on a Tits circle. Therefore, by lemma 4.18, $[\psi_a \circ \sigma \circ \psi_a^{-1}]$ is not in $\mathrm{Out}(G_o)$.
- for $n = 8$, we must also consider the outer automorphism $[\tau] \in \mathrm{Out}(\mathrm{Spin}(8))$ of order 3. We choose the representative τ given in lemma 4.22. We now check the effect on the fixed element q_3 in A. Again we compute

  - $\psi_{-1} \circ \tau \circ \psi_{-1}^{-1}(q_3) \;=\; -aq_3$,
  - $\psi_a \circ \tau \circ \psi_a^{-1}(q_3) \;=\; -q_3$,
  - $\psi_{-a} \circ \tau \circ \psi_{-a}^{-1}(q_3) \;=\; -q_3$.

  None of these images is on the Tits circle (in fact, as explained in the appendix, the Tits circle associated to q_3 is $\{(\cos t + e_3e_4 \sin t)(\cos t - e_5e_6 \sin t) : t \in \mathbf{R}\}$ and the element of order 2 is $-e_3e_4e_5e_6$). Therefore if ψ is an exotic automorphism, the "outer action" homomorphism must necessarily be contained in $\langle[\sigma]\rangle \subset \mathrm{Out}(\mathrm{Spin}(8))$ and we can conclude as in case 2.

□

4.6 Reduction to semidirect products

We extract from sections 4.3 and 4.4 a reformulation of the critical situation for the conjecture in terms of semidirect products. Let $\pi : \mathrm{Aut}(G_o, T) \twoheadrightarrow \mathrm{Out}(G_o)$ denote the canonical projection.

Proposition 4.24 *The conjecture is true if and only if the following situation never holds: there exist two homomorphisms* $\sigma_1, \sigma_2 : \Gamma \to \mathrm{Aut}(G_o, T)$ *such that* $\varphi = \pi \circ \sigma_1 = \pi \circ \sigma_2$, *and an isomorphism of pairs of normalizers*

$$\rho : N_{[0]} = N_o \rtimes_{\sigma_1} \Gamma \longrightarrow N_v = N_o \rtimes_{\sigma_2} \Gamma$$

such that

- $v \neq [0] \in i_*(H^2(\Gamma, Z(G_o))) \subset H^2(\Gamma, Z(N_o))$,
- $[\psi = \rho_{|N_o}] \in H^1(W_o; T) \cap Z_{\mathrm{Out}(N_o)}(\varphi(\Gamma))$

Proof. By lemma 4.5 and corollary 4.14, if the conjecture does not hold there exists a situation as depicted in the statement of the proposition. Conversely, the hypothesis $v \neq [0]$ implies that $p_*(v) \neq p_*([0]) = [0] \in H^2(\Gamma, Z(G_o))$. So the corresponding extensions $Z_{[0]} = Z_o \rtimes_{\sigma_1} \Gamma$ and $Z_{p_*(v)}$ are not isomorphic. But they correspond to centralizers of principal subgroups of rank 1 in $G_{[0]}$ and G_v. Therefore these latter groups cannot be isomorphic even though their normalizers are isomorphic as pairs of groups. This concludes the proof. □

Remark 4.25 A different approach for trying to solve the conjecture is to find in the normalizer N subgroups that play the same role as centralizers of principal subgroups of rank 1 do in the compact Lie group G (these are difficult to describe as subgroups of N and are not exactly the appropriate ones if the centers of N_o and G_o do not coincide). In fact, we can show, by a careful comparison of the approaches of de Siebenthal and of Bourbaki for finding a section of $\mathrm{Aut}(G_o) \twoheadrightarrow \mathrm{Out}(G_o)$ ([18, Théorème, pp. 46-47] and [10, §4.10.]), that the subgroups we are looking for are normalizers of a fixed Tits system A in the extended maximal torus $Q \subset N$. To solve the conjecture, it would suffice to show that such subgroups are preserved by isomorphisms of pairs of normalizers. This approach is essentially a reformulation of what has been done in the present chapter, however having it in mind might be useful for solving the conjecture in all cases.

Chapter 5

Normalizers and classifying spaces

In the beginning of the '90s, Mislin asked the following question [26]:

> *Let G_o and H_o be two connected compact Lie groups. If the classifying spaces BG_o and BH_o are homotopy equivalent, does this imply that G_o and H_o are isomorphic as Lie groups?*

The answer is affirmative and was first given by Osse in [53]; there now exist several proofs in the literature [50, 45, 55, 72]. In [50], Notbohm even shows that the connectedness hypothesis is superfluous. The purpose of this chapter is to give another proof of this last statement, taking the result in the connected case for granted. The interest is twofold: our approach is constructive and it highlights the crucial role played by normalizers of maximal tori.

5.1 Classifying spaces and fibrations

To set up the background, we want to recall some material on classifying spaces and homotopy theory. All the definitions and results, especially on fibrations, that are used without mention can be found in either one of the textbooks by Whitehead [71], Spanier [62], or Switzer [65]. In the whole chapter, "space" and "map" always mean topological space and continuous map. We will call a map $p : E \to B$ a *fibration* (or *Hurewicz fibration*) if it has the homotopy lifting property with respect to all spaces, i.e. if the following commutative diagram can be completed as indicated

$$\begin{array}{ccc} X \times \{0\} & \longrightarrow & E \\ \big\downarrow & \nearrow & \big\downarrow p \\ X \times I & \longrightarrow & B \end{array}$$

for any space X. We will use the common notation $F \to E \xrightarrow{p} B$ for a fibration p with "fibre" $F = p^{-1}(b)$ over some designated point $b \in B$.

We first introduce the notion of principal bundle. Let G be a topological group; recall that an action of G on a space E is called free if $g{\cdot}x = x$ always implies $g = e$.

Definition 5.1 A *principal G-bundle* $\xi = (p : E \to B)$ consists of a free right G-action

$$E \times G \longrightarrow E\,,\ (x, g) \longmapsto x{\cdot}g$$

and a surjective map $p : E \to B$ such that

(*i*) $p(x{\cdot}g) = p(x)$ for all $x \in E$, $g \in G$;

(*ii*) for each $b \in B$, there exists an open neighbourhood V_b of b and an G-equivariant homeomorphism $\varphi : p^{-1}(V_b) \to V_b \times G$ such that

$$\begin{array}{ccc} p^{-1}(V_b) & \xrightarrow{\quad\varphi\quad} & V_b \times G \\ & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle pr_1} & \\ & V_b & \end{array}$$

is commutative.

More precisely, the G-equivariance of φ means

$$\varphi(x{\cdot}g) = \varphi(x){\cdot}g\,, \quad \text{for all } x \in p^{-1}(V_b)\,,\ g \in G\,,$$

with G acting trivially on V_b and by right translation on itself.

The space E is called the *total space*, B is called the *base space*, and φ a *trivialisation over* V_b. Let also $E_b = p^{-1}(b)$ be the *fibre over* the point $b \in B$.

Remarks 5.2

1. The map p induces a homeomorphism between the orbit space E/G and the base space B, because condition (*ii*) implies that p is an open mapping.

2. As the G-action on E is free, the trivialisation over V_b implies that for all $b \in B$ and all $x \in E_b$ the map

$$G \longrightarrow E_b\,,\ g \longmapsto x{\cdot}g$$

is a homeomorphism.

There is a natural notion of morphism associated to principal G-bundles:

Definition 5.3 Let $\xi = (p : E \to B)$ and $\xi' = (p' : E' \to B')$ be two principal G-bundles. A *morphism* between these two bundles is a pair of maps (u, f) such that

- the diagram

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{f} & B' \end{array}$$

  is commutative,

- the map u is G-equivariant, i.e. $u(x \cdot g) = u(x) \cdot g$, for all $x \in E$ and $g \in G$.

The two bundles are said to be *isomorphic* if u and f are homeomorphisms. A principal G-bundle is *trivial* if it is isomorphic to the *product principal G-bundle* $B \times G \to B$.

Many geometric problems involve the study of G-bundles over a fixed space X. Classifying spaces will now enter the scene naturally, because they are intimately related to the classification of principal G-bundles (hence their name). We will give a brief account of this in the quite general setting of *numerable* principal G-bundles. Recall that $\xi = (p : E \to B)$ is numerable if there exists a open cover $\{V_i\}_{i \in \mathbb{N}}$ of B such that $\xi_{|V_i}$ is trivial, and a partition of unity associated to it. For instance every principal G-bundle over a paracompact space, hence over a CW-complex, is numerable. More details can be found for instance in Dold [20], Husemoller [36], or Steenrod (in case the base space is a finite CW-complex) [63]. Recall that for a principal G-bundle $\xi = (p : E \to B)$, the pullback $f^*(\xi)$ of ξ along a map $f : X \to B$ is a principal G-bundle over X [36, Proposition 4.1, p. 44]. A first step towards translating the study of the set $\mathrm{Prin}_G(X)$ of isomorphism classes of numerable principal G-bundles over X into a homotopy problem is the following:

Theorem 5.4 *Let ξ be a numerable principal G-bundle over B. If $f, g : X \to B$ are homotopic, then the principal G-bundles $f^*(\xi)$ and $g^*(\xi)$ are isomorphic over X.*

Proof. See Theorem 9.9, p. 52 in Husemoller [36]. □

For two spaces X and Y, let $[X, Y]$ be the set of homotopy classes of maps of X to Y.

Definition 5.5 A principal G-bundle $\xi_G = (p_G : EG \to BG)$ is *universal* provided ξ_G is numerable and for all spaces X the map

$$[X, BG] \longrightarrow \mathrm{Princ}_G(X)\,,\ [f] \longmapsto [f^*(\xi_G)]$$

is a natural bijection, i.e. the functor $\mathrm{Princ}_G(-)$ is represented by the space BG.

The space BG is called a *classifying space* for the topological group G.

This definition immediately implies that if $\xi_G = (p_G : EG \to BG)$ and $\xi'_G = (p'_G : EG' \to BG')$ are universal for G, then there exists a homotopy equivalence $f : BG \to BG'$ with $\xi_G = f^*(\xi'_G)$. With a slight abuse of language, one can thus speak of *the* classifying space of a topological group. Its existence is given, for instance, by the *Milnor construction* [36, pp. 54-56]. We now collect several properties of classifying spaces.

Theorem 5.6

(i) There exists a functor ξ_- from the category of topological groups to that of universal principal bundles.

(ii) A numerable principal G-bundle is universal if and only if its total space EG is contractible.

(iii) If G and G' are topological groups, then $B(G \times G') \simeq BG \times BG'$.

(iv) Any inner automorphism c_g of G induces a map $Bc_g : BG \to BG$ that is homotopic to the identity.

(v) Let H be a closed subgroup of G. Then the inclusion $i : H \hookrightarrow G$ induces a fibration

$$G/H \longrightarrow BH \xrightarrow{Bi} BG\,.$$

(vi) A short exact sequence of topological groups

$$K \overset{i}{\hookrightarrow} G \overset{\pi}{\twoheadrightarrow} Q$$

with K closed in G, yields a fibration

$$BK \xrightarrow{Bi} BG \xrightarrow{B\pi} BQ\,.$$

(vii) If A is an abelian topological group, with a compactly generated topology (i.e. a set is closed if and only if it meets every compact set in a closed set), then there is a model for BA that is an abelian topological group.

(viii) Let G be a compact Lie group, Z a closed central subgroup of G and $\pi : G \to G/Z$ the quotient map. Then

$$B\pi : BG \longrightarrow B(G/Z)$$

is a principal BZ-bundle.

(ix) If G is a compact Lie group, then BG has the homotopy type of a CW-complex of finite type.

The explicit proofs are scattered in the literature: For (i) and (ii) see Dold [20, §7 and §8]. Proofs of (iii)-(v) and (vii) using the Milgram-Steenrod construction can be found either in Steenrod [64] or Adem-Milgram [4, Chap. II, §1]. For (vi) (and (v)), see Piccinini-Spreafico [56] or the exercises in tom Dieck [69, Chap. I, §8]. Point (viii) is a special case of Theorem 7.7 in [56]. Finally, the last point, as well as the specific and concise approach for compact Lie groups and bundles over CW-complexes, can be found in Mimura-Toda [42, Chap. II, §6].

Examples 5.7

1. For a discrete group Γ, the long exact homotopy sequence of the fibration $\Gamma \to E\Gamma \to B\Gamma$ shows that $B\Gamma$ is an *Eilenberg-MacLane space* $K(\Gamma, 1)$. In particular $B\mathbb{Z}/2 = K(\mathbb{Z}/2, 1) = \mathbb{R}P^\infty$, and $B\mathbb{Z} = K(\mathbb{Z}, 1) = \mathbb{S}^1$.

2. For the circle group $\mathbb{S}^1$, we have $B\mathbb{S}^1 = BB\mathbb{Z} = K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$.

3. The *Stiefel manifold* of (orthonormal) k-frames in $\mathbb{R}^n$ is the subspace

$$V_k(\mathbb{R}^n) = \{(v_1, \dots, v_k) : v_i \in S^{n-1},\ (v_i|v_j) = \delta_{ij}\} \subset (S^{n-1})^k .$$

The *Grassmann manifold* $G_k(\mathbb{R}^n)$ is the set of k-dimensional subspaces of $\mathbb{R}^n$ with the quotient topology defined by the surjection

$$V_k(\mathbb{R}^n) \longrightarrow G_k(\mathbb{R}^n),\ (v_1, \dots, v_k) \longmapsto \langle v_1, \dots, v_k\rangle .$$

The standard inclusions

$$\mathbb{R}^n \subset \mathbb{R}^{n+1} \subset \mathbb{R}^{n+2} \subset \dots$$

given by $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0)$ induce inclusions

$$V_k(\mathbb{R}^n) \subset V_k(\mathbb{R}^{n+1}) \subset V_k(\mathbb{R}^{n+2}) \subset \dots$$

and

$$G_k(\mathbb{R}^n) \subset G_k(\mathbb{R}^{n+1}) \subset G_k(\mathbb{R}^{n+2}) \subset \dots$$

We take $V_k(\mathbb{R}^\infty) = \bigcup_{n\geqslant k} V_k(\mathbb{R}^n) = \lim\limits_{n\to\infty} V_k(\mathbb{R}^n)$ and $G_k(\mathbb{R}^\infty) = \bigcup_{n\geqslant k} G_k(\mathbb{R}^n) = \lim\limits_{n\to\infty} G_k(\mathbb{R}^n)$, both with the *inductive topology* (a set is closed in the limit if and only if its intersection with every subspace in the sequence is closed). Then $\pi : V_k(\mathbb{R}^\infty) \to G_k(\mathbb{R}^\infty)$, where π is the obvious map, is a universal $\mathrm{O}(k)$-principal bundle and thus $B\,\mathrm{O}(k) = G_k(\mathbb{R}^\infty)$.

In a similar fashion, we have $B\,\mathrm{U}(k) = G_k(\mathbb{C}^\infty)$ and $B\,\mathrm{Sp}(k) = G_k(\mathbb{H}^\infty)$. For the special orthogonal, resp. special unitary, group one has $B\,\mathrm{SO}(k) = SG_k(\mathbb{R}^\infty)$, resp. $B\,\mathrm{SU}(k) = SG_k(\mathbb{C}^\infty)$, the space of oriented k-subspaces in $\mathbb{R}^\infty$, resp. $\mathbb{C}^\infty$ [36, p. 91].

4. By the theorem of Peter and Weyl, any compact Lie group G admits a faithful representation, i.e. there exists an injection $G \hookrightarrow \mathrm{U}(N)$, for some N. Therefore, one can take $EG = V_N(\mathbb{C}^\infty)$ and $BG = V_N(\mathbb{C}^\infty)/G$.

We end up this review on classifying spaces, by recalling that the study of Lie groups from the homotopy point of view can be restricted to the compact case. Indeed let G be a Lie group with a finite number of connected components. Then a fundamental structure theorem says that, up to conjugation, G has a unique compact subgroup K and that the homogeneous space G/K is contractible [32, Theorem XV.3.1]. Therefore by point (v) of theorem 5.6 there is a homotopy equivalence $BG \simeq BK$. From now on, we will only consider classifying spaces of *compact* Lie groups, and always choose a model that is a CW-complex of finite type.

Next, we collect a few definitions and results on fibrations.

Definition 5.8 If $p : E \to B$ is a fibration, then a homotopy $H : X \times I \to E$ is a *vertical homotopy* if $p \circ H$ is stationary (in other words, H only moves points around in their fibre).

Definition 5.9 Let $g : B \to B'$ be a map. A *fibre map (over g)* between two fibrations $p : E \to B$ and $p' : E' \to B'$ is a map $f : E \to E'$ such that

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{g} & B' \end{array}$$

is homotopy commutative.

In this last definition, using the homotopy lifting property for the fibration p', one can always replace f by a homotopic map $\bar{f}$ such that the diagram is (strictly) commutative.

The next lemma is well-known; one can find a formulation with weaker hypotheses in Dold [20, Theorem 6.3].

Lemma 5.10 *Let $F \to E \to B$ and $F' \to E' \to B'$ be fibrations in the category of CW-complexes. Let $f : E \to E'$ be a fibre map inducing a homotopy equivalence on the fibres and on the base spaces, i.e. inducing the following homotopy commutative diagram*

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ g\downarrow\simeq & & f\downarrow & & h\downarrow\simeq \\ F' & \longrightarrow & E' & \longrightarrow & B' \end{array}$$

Then f is a homotopy equivalence.

Proof. The result follows from the long exact homotopy sequence, the five lemma for groups, and the classical result of J.H.C. Withehead [71, Theorem 3.5, p. 220], saying that a weak homotopy equivalence between CW-complexes is a homotopy equivalence. □

Another classical fact is that the pullback is not well behaved in the homotopy category. However, if one of the maps is a fibration, then things become clearer. Given a diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

where p is a fibration and f is any map, one can take its (usual) pullback and get the commutative diagram

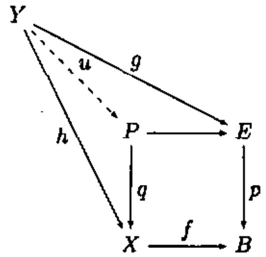
$$\begin{array}{ccc} P & \longrightarrow & E \\ q\downarrow & & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

where $q : P \to X$ is also a fibration, called the *induced fibration from p by f*. Then, in the homotopy category, the universal property of the pullback becomes

Lemma 5.11 *For any homotopy commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{g} & E \\ h\downarrow & & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

where p is a fibration, there exists, up to homotopy, a unique map $u : Y \to P$ such that



is homotopy commutative.

Proof. Applying the homotopy lifting property, one can replace g by a homotopic map g' such that the first diagram is strictly commutative, i.e. such that $p \circ g' = f \circ h$. Then the universal property of the (usual) pullback gives a unique map $u : Y \to P$ that makes the second diagram strictly commutative. Clearly, any map homotopic to u makes the diagram homotopy commutative. Conversely, applying the homotopy lifting property for $q : P \to X$, any map $u' : Y \to P$ such that the second diagram is homotopy commutative has to be homotopic to u. This gives the uniqueness part. □

In the sequel, we will use the two previous lemmas implicitly.

5.2 Recollection of results on mapping spaces

We recall, without proof, several theorems on mapping spaces involving classifying spaces that will be used in our approach. Let us start with some standard notations: For two spaces X and Y, let $\mathrm{map}(X,Y)$ be the space of maps from X to Y, with the usual compact-open topology; $\mathrm{map}(X,Y)_f$ will denote the path component of $f : X \to Y$ in $\mathrm{map}(X,Y)$. For two groups Γ_1 and Γ_2, let

$$\mathrm{Rep}(\Gamma_1, \Gamma_2) = \mathrm{Hom}(\Gamma_1, \Gamma_2)/\mathrm{Inn}(\Gamma_2)$$

denote the set of homomorphisms $\Gamma_1 \to \Gamma_2$ modulo inner automorphisms of Γ_2. Recall that a space is called *aspherical* if it is arcwise connected and if $\pi_i(X) = 0$ for all $i \geqslant 2$. The study of maps between classifying spaces goes back to the following classical result of Hurewicz [35, p. 219]:

Theorem 5.12 (Hurewicz) *For any pair of aspherical spaces X and Y, the map*

$$[X, Y] \longrightarrow \mathrm{Rep}(\pi_1(X), \pi_1(Y))$$

is a bijection.

A more general formulation and a proof can be found in Whitehead [71, Chap. V, §4, pp. 224-226]. As a particular case, this theorem applies to classifying spaces of discrete groups, where it says that all maps between such spaces are induced by homomorphisms (for compact Lie groups, it is a well-known fact that this is not true in general). The answer to the natural question of when a homomorphism of Lie groups induces a homotopy equivalence is given by

Theorem 5.13 (Evens, Minami) *Let $\varphi : G \to H$ be a homomorphism of compact Lie groups. Then φ is an isomorphism if and only if $B\varphi_* : H_*(BG;\mathbb{Z}) \to H_*(BH;\mathbb{Z})$ is an isomorphism.*

In particular, if $B\varphi$ is a homotopy equivalence, then φ is an isomorphism.

Evens mentions the finite group case in [28] and Minami gives a general proof of this result in [43].

For a space X let $\mathrm{Aut}(X)$ be the group of classes of homotopy equivalences of X, i.e. $\mathrm{Aut}(X)$ is the group of classes $[f] \in [X,X]$ such that there exists $g : X \to X$ with $g \circ f \sim id_X$ and $f \circ g \sim id_X$. Let also $X_p^\wedge$ denote the Bousfield-Kan p-completion of X [12, Chap. VI]. We will need [37, Proposition 2.7 and Corollary 3.7]:

Theorem 5.14 (Jackowski-McClure-Oliver) *Let G_o be a connected compact Lie group and let Z_o be its center.*

(1) The group $\mathrm{Aut}(BG_o)$ is isomorphic to the outer automorphism group $\mathrm{Out}(G_o)$ of G_o. More precisely, the map

$$\beta_{G_o} : \mathrm{Out}(G_o) \longrightarrow \mathrm{Aut}(BG_o),\ [\alpha] \longmapsto [B\alpha]$$

is a group isomorphism (see also Theorem 5.2 of [55]).

(2) The natural homomorphism $Z_o \times G_o \to G_o$ induces a homotopy equivalence $(BZ_o)_p^\wedge \xrightarrow{\simeq} \mathrm{map}((BG_o)_p^\wedge, (BG_o)_p^\wedge)_{id}$.

Recall that a *p-toral group* P is a compact Lie group whose component of the identity P_o is a torus and whose group of components P/P_o is a (finite) p-group. As in the case of discrete groups, the homotopy classes of maps from the classifying space of a p-toral group to that of a compact Lie group all come from homomorphisms [22, Theorem 1.1'],[72, Theorem A],[48, Theorem 5.1]:

Theorem 5.15 (Dwyer-Zabrodsky, Notbohm) *Let P be a p-toral group and G a compact Lie group. Then the obvious map*

$$B : \mathrm{Rep}(P, G) \longrightarrow [BP, BG]$$

is a bijection.

We will also need results from Thom's theory on the topology of function spaces [66], as revisited by Notbohm and Smith [51]. First, recall tbe following generalization of the notion of principal G-bundle: a fibration $p : E \to B$ is called *principal* if the fibre F is an H-space acting on E and the action satisfies $p(x \cdot f) = p(x)$, for all $x \in E$, $f \in F$. A classical theorem of Hurewicz shows that a principal G-bundle witb paracompact base is a fibration, hence a principal fibration [21, Cbap. XX, §3-4].

Theorem 5.16 (Thom) *Let X be a connected space.*

(1) Let A be an abelian group and n a positive integer. Then for any map $f : X \to K(A, n)$ there is a homotopy equivalence

$$\mathrm{map}(X, K(A, n))_f \simeq \prod_{i=1}^{n} K(H^{n-i}(X, A), i).$$

(2) Let $\pi : Y \to B$ be a principal fibration with structure group H, and $f : X \to Y$ a fixed map. Then the map

$$p : \mathrm{map}(X, Y)_f \longrightarrow \mathrm{map}(X, B)_{\pi \circ f},\ h \longmapsto \pi \circ h,$$

is a principal fibration with fibre a union of components of $\mathrm{map}(X, H)$.

Cf. theorems 1.1 and 1.2 in [51] for the proof of these results.

5.3 Reconstructing G from BG

We first state some facts involving standard covering space theory. The following lemma is a direct consequence of the homotopy lifting property.

Lemma 5.17 *Let $p : Y \to X$ be a covering projection, where X is a connected locally connected space and Y is connected. Suppose that there exist self-homotopy equivalences h and f such that*

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{h} & X \end{array}$$

is homotopy commutative. If h is homotopic to the identity id_X then f is homotopic to a covering transformation of p.

Remark 5.18 Recall that a *covering transformation* of p, or *automorphism* of p, is a homeomorphism $\phi : Y \to Y$ such that $p \circ \phi = p$. Under the hypotheses of the lemma, the group of automorphisms of p is isomorphic to $N_{\pi_1(X)}(p_*\pi_1(Y))/p_*\pi_1(Y)$ [62, Thm. 2, p. 85].

Let G be a compact Lie group; let $K \overset{i}{\hookrightarrow} G \overset{\pi}{\twoheadrightarrow} Q$ be an extension of Lie groups, with K closed in G and Q finite. Recall that this extension gives rise to a homomorphism $\varphi : Q \to \mathrm{Out}(K)$ which can be composed with the map $\beta_K : \mathrm{Out}(K) \to \mathrm{Aut}(BK)$ (as defined in theorem 5.14). On the other hand, the monodromy action for the corresponding fibration $BK \overset{Bi}{\to} BG \overset{B\pi}{\to} BQ$ produces a homomorphism $\Theta : \pi_1(BQ) \to \mathrm{Aut}(BK)$. A careful analysis of the situation shows the following lemma, together with its corollary.

Lemma 5.19 *There exists an isomorphism $\delta : \pi_1(BQ) \to Q$ such that the diagram*

$$\begin{array}{ccc} \pi_1(BQ) & \xrightarrow{\quad\delta\quad} & Q \\ & {\scriptstyle\Theta}\searrow \quad \swarrow{\scriptstyle\beta_K \circ \varphi} & \\ & \mathrm{Aut}(BK) & \end{array}$$

is commutative.

Corollary 5.20 *Let $H \overset{i}{\hookrightarrow} G$ be a closed subgroup of G, with G/H finite. Choose models for the classifying spaces such that $p = Bi : BH \to BG$ is a covering map (with fibre $F \approx G/H$). Then any covering transformation of p is homotopic to Bc_g for some $g \in N_G(H)$.*

As in chapter 4, we will now consider the extensions corresponding to a compact Lie group G and to the associated group $\bar{G} = G/Z_o$, where Z_o denotes the center of the component of the identity G_o of G. Abusing language, we will say that the (not necessarily centerless) group $\bar{G}$ is of "adjoint type". Recall the commutative diagram

$$\begin{array}{ccccc} Z_o & = & Z_o & & \\ \downarrow & & \downarrow & & \\ G_o & \hookrightarrow & G & \twoheadrightarrow & \Gamma \\ {\scriptstyle\pi|G_o}\downarrow & & \downarrow{\scriptstyle\pi} & & \| \\ \bar{G}_o & \hookrightarrow & \bar{G} & \twoheadrightarrow & \Gamma \end{array}$$

Applying the functor $B(-)$ to it yields a corresponding commutative diagram for the fibrations $BG_o \to BG \to B\Gamma$ and $B\bar{G}_o \to B\bar{G} \to B\Gamma$. The next theorem tells that $B\pi : BG \to B\bar{G}$ is essentially the unique fibre map under $g = B\pi_{|G_o}$ and over id_Γ.

Theorem 5.21 *Up to homotopy, there is a unique fibre map $f : BG \to B\bar{G}$ such that the diagram*

$$\begin{array}{ccccc} BG_o & \longrightarrow & BG & \longrightarrow & B\Gamma \\ {\scriptstyle g = B\pi_{|G_o}}\downarrow & & \downarrow{\scriptstyle f} & & \| \\ B\bar{G}_o & \longrightarrow & B\bar{G} & \longrightarrow & B\Gamma \end{array}$$

is homotopy commutative.

This theorem is one of the main ingredients in our approach for solving Mislin's question in the *nonconnected* case, as it appears both in the "reconstruction" and "invariance" parts. Before proving it, we recall a procedure, outlined by Møller in [46] and based on previous papers of Booth, Heath, and Piccinini [6, 7], that relates the problem of finding maps between the total spaces of two fibrations with the space of based sections of another related fibration. We keep notations as in [46]; let $p : U \to A$ and $q : V \to B$ be fibrations over connected and pointed base spaces. Let $g : p^{-1}(*) \to q^{-1}(*)$ be a map between the fibres and $h : (A, *) \to (B, *)$ be a map between the base spaces such that the pair (g, h) is compatible with the monodromy action of the fundamental group of the bases on the fibres, i.e. such that for all $\omega \in \pi_1(A, *)$ the diagram

$$\begin{array}{ccc} p^{-1}(*) & \xrightarrow{g} & q^{-1}(*) \\ {\scriptstyle \omega}\downarrow & & \downarrow{\scriptstyle \pi_1(h)(\omega)} \\ p^{-1}(*) & \xrightarrow{g} & q^{-1}(*) \end{array}$$

is homotopy commutative. The goal is to describe the space of fibre maps $f : U \to V$ under g and over h, i.e. of maps that fits in the following homotopy commutative diagram

$$\begin{array}{ccc} p^{-1}(*) & \xrightarrow{g} & q^{-1}(*) \\ \downarrow & & \downarrow \\ U & \overset{f}{\dashrightarrow} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{h} & B \end{array}$$

One defines the set

$$\mathrm{fibmap}(U, V)_g^h = \coprod_{a \in A} \mathrm{map}\left(p^{-1}(a), q^{-1}(h(a))\right)_{g_a}$$

where $g_a \in [p^{-1}(a), q^{-1}(h(a))]$ is the homotopy class making

$$\begin{array}{ccc} p^{-1}(*) & \xrightarrow{g} & q^{-1}(*) \\ \zeta\downarrow & & \downarrow h(\zeta) \\ p^{-1}(a) & \xrightarrow{g_a} & q^{-1}(h(a)) \end{array}$$

homotopy commutative for any path ζ from the base point $*$ to $a \in A$. By pulling-back q along h, we get the identification

$$\operatorname{fibmap}(U, V)_g^h = \operatorname{fibmap}(U, h^*(V))_g^{id_A}.$$

The set $\operatorname{fibmap}(U, V)_g^h$ carries a natural topology as defined in [7]. The forthcoming lemma (see [46, pp. 310-311]) is a special case of corollary 2 and proposition 6 in [7], or of theorem 4.1 in [5].

Lemma 5.22 *The space* $\operatorname{map}(U, V)_g^h$ *of maps of* U *into* V *under* g *and over* h *with the compact-open topology is homeomorphic to the space of based sections of the fibration*

$$\operatorname{map}(p^{-1}(*), q^{-1}(*))_g \longrightarrow \operatorname{fibmap}(U, V)_g^h \longrightarrow A,$$

where the projection onto A *is the natural one.*

The proof of theorem 5.21 will consist in showing that the space of based sections of this last fibration, in our case, is trivial. To do so, we will need the general obstruction theory in fibre spaces involving homology with local coefficients and bundles of groups presented in Whitehead's book [71, Chap. VI]. We will keep notations as in this book, and refer to it for a thorough presentation of the theory. The lemma we will need in this context is the following:

Lemma 5.23 *Let* $p : E \to B = K(\Gamma, 1)$ *be a fibration, with* Γ *a finite group. Suppose that the fibre* F *is connected and that* $\pi_n(F)$ *is a* $\mathbb{Q}$*-vector space for all* n *(in particular* F *is* 1*-simple, i.e.* $\pi_1(F)$ *is abelian). Then, up to vertical homotopy, there is a unique based section of* p.

Proof. This section problem corresponds to the diagram

$$\begin{array}{ccc} L = * & \xrightarrow{f} & E \\ \cap\downarrow & \exists ?s \nearrow & \downarrow p \\ K = B & \overset{\phi = id}{=\!=\!=} & B \end{array}$$

Following §6 of Chap. VI in [71], the primary obstructions to extending f lie in $H^{n+1}(K,*;\phi^*\pi_n(\mathcal{F})) = H^{n+1}(K(\Gamma,1);\pi_n(\mathcal{F}))$ and the primary differences of two liftings lie in $H^n(K,*;\phi^*\pi_n(\mathcal{F})) = H^n(K(\Gamma,1);\pi_n(\mathcal{F}))$ (where $\pi_n(\mathcal{F})$ denotes the local coefficient system associated to the fibration). However, as the base of the fibration is $K(\Gamma,1)$, these cohomology groups reduce to the usual cohomology groups of $K(\Gamma,1)$ with coefficients in the Γ-module $\pi_n(F)$ [71, Theorem 3.5*, p. 281], i.e.

$$H^{n+1}(K(\Gamma,1);\pi_n(\mathcal{F})) \cong H^{n+1}(K(\Gamma,1);\pi_n(F))$$

and

$$H^n(K(\Gamma,1);\pi_n(\mathcal{F})) \cong H^n(K(\Gamma,1);\pi_n(F)).$$

Now, as Γ is finite and $\pi_n(F)$ is a rational vector space for all n, we conclude, by a classical transfer argument [15, Corollary 10.2, p. 84], that all the cohomology groups of interest vanish. Existence and uniqueness of the section then follows from theorems 6.3 and 6.5, pp. 298-299, in [71]. □

Next we will study the component $\mathrm{map}(BG_o, BZ_o)_c$ of the constant map. As a last ingredient for proving theorem 5.21 we show that this mapping space is homotopy equivalent to BZ_o. This is already known if G_o is semisimple [49, Proof of Proposition 1.2], however we did not find any reference for the general case. In the proof, we will use the first part of theorem 5.16 on Thom's theory.

Lemma 5.24 *The evaluation map*

$$ev : \mathrm{map}(BG_o, BZ_o)_c \longrightarrow BZ_o\,,\ f \longmapsto f(*)$$

is a homotopy equivalence.

A homotopy inverse of ev is given by the map

$$\psi : BZ_o \longrightarrow \mathrm{map}(BG_o, BZ_o)_c\,,\ z \longmapsto (\psi(z) : x \mapsto z)\,.$$

Proof. We first recall that $Z_o \cong A \times S$, with A a finite abelian group and S a torus, and so we can take $BZ_o = BA \times BS = K(A,1) \times K(\pi_1(S),2)$ (where $\pi_1(S) \cong \mathbb{Z}^n$). Thus we have

$$\begin{aligned} \mathrm{map}(BG_o, BZ_o)_c &= \mathrm{map}(BG_o, BA \times BS)_c \\ &\cong \mathrm{map}(BG_o, BA)_{c_1} \times \mathrm{map}(BG_o, BS)_{c_2}\,, \end{aligned}$$

for two constant maps c_1 and c_2. We compute these two terms separately using theorem 5.16:

$$\begin{aligned} \mathrm{map}(BG_o, BA)_{c_1} &\simeq K(H^0(BG_o, A), 1) \\ &\simeq K(A,1) \\ &= BA\,, \end{aligned}$$

and

$$\begin{aligned}\mathrm{map}(BG_o, BS)_{c_2} &\simeq K(H^1(BG_o, \pi_1(S)), 1) \times K(H^0(BG_o, \pi_1(S)), 2)\\ &\simeq K(H^0(BG_o, \pi_1(S)), 2)\\ &\simeq K(\pi_1(S), 2)\\ &= BS,\end{aligned}$$

where the first factor vanishes because the first cohomology group of a 1-connected space is trivial for any coefficient (by applying the universal coefficient theorem [13, 7.2, p. 282]). Therefore we have $\mathrm{map}(BG_o, BZ_o)_c \simeq BZ_o$.

Now notice that $ev \circ \psi$ is the identity map of BZ_o. Thus at the level of homotopy groups, we get for all n

$$id = ev_* \circ \psi_* : \pi_n(BZ_o) \xrightarrow{\psi_*} \pi_n(\mathrm{map}(BG_o, BZ_o)_c) \xrightarrow{ev_*} \pi_n(BZ_o).$$

We deduce that ev_* is a surjective homomorphism between isomorphic groups for all n. But there are only two non-trivial cases: for $n = 1$, we have

$$\pi_1(\mathrm{map}(BG_o, BZ_o)_c) \cong \pi_1(BZ_o) \cong A,$$

a finite abelian group, and for $n = 2$,

$$\pi_2(\mathrm{map}(BG_o, BZ_o)_c) \cong \pi_2(BZ_o) \cong \mathbb{Z}^r.$$

In both cases the surjective homomorphism ev_* has to be an isomorphism. Therefore ev is a weak homotopy equivalence between CW-complexes and thus a homotopy equivalence. It is then straightforward that the right homotopy inverse ψ of ev is a homotopy inverse of ev. □

We are now ready to prove the theorem.

Proof of theorem 5.21. By lemma 5.22, it suffices to show that the fibration

$$\mathrm{map}(BG_o, B\bar{G}_o)_g \longrightarrow \mathrm{fibmap}(BG, B\bar{G})_g^{id} \longrightarrow B\Gamma$$

has, up to vertical homotopy, a unique based section. To do so, we will show that it fulfills the hypotheses of lemma 5.23. First, by (viii) in theorem 5.6, the fibration $BZ_o \to BG_o \xrightarrow{g} B\bar{G}_o$ is principal; applying the second part of theorem 5.16 to it and to the map $id : BG_o \to BG_o$, we get a principal fibration $\mathrm{map}(BG_o, BG_o)_{id} \to \mathrm{map}(BG_o, B\bar{G}_o)_g$. Following the argument in Notbohm and Smith [51, pp. 302-303] for this particular case, one checks that its fibre can be identified with the sections of the trivial principal BZ_o-bundle $BG_o \times BZ_o \to BG_o$ that are vertically homotopic to the trivial section. Therefore the fibre is reduced to $\mathrm{map}(BG_o, BZ_o)_c$, where c is the constant map. Let us now consider the diagram

$$\begin{array}{ccc} \mathrm{map}(BG_o, BZ_o)_c & \xrightarrow[\simeq]{ev} & BZ_o \\ \Big\downarrow i & \swarrow ad & \\ \mathrm{map}(BG_o, BG_o)_{id} & & \\ \Big\downarrow & & \\ \mathrm{map}(BG_o, B\bar{G}_o)_g & & \end{array}$$

where the diagonal map ad is as in theorem 5.14 (2), i.e. it is the adjoint of $B\alpha$, where $\alpha : G_o \times Z_o \to G_o$ is the natural homomorphism. As it corresponds to the BZ_o-principal action on BG_o, one checks that $ad = i \circ \psi$, where ψ is the homotopy inverse of ev, appearing in lemma 5.24. Therefore the above diagram is homotopy commutative. Thus we get a principal fibration

$$BZ_o \longrightarrow E = \mathrm{map}(BG_o, BG_o)_{id} \longrightarrow X = \mathrm{map}(BG_o, B\bar{G}_o)_g .$$

Now both BZ_o and $E = \mathrm{map}(BG_o, BG_o)_{id} = \{f : BG_o \xrightarrow{\simeq} BG_o \mid f \sim id\}$ are nilpotent, because so are connected H-spaces [30, p. 62]. By proposition 5.5 in [12, Chap. III, p. 84] we get that X is nilpotent too. We also deduce from the long exact sequence in homotopy, which gives a surjection $\pi_1(E) \twoheadrightarrow \pi_1(X)$, that $\pi_1(X)$ is abelian. These facts allow to invoke proposition 6.5. in [12, Chap VI, p. 187], which says that

$$(BZ_o)^\wedge_p \longrightarrow E^\wedge_p \longrightarrow X^\wedge_p$$

is a fibration. With the long exact sequence in homotopy and the second result in theorem 5.14, one shows that $\pi_n(X^\wedge_p) = 0$ for all n and all primes p. By proposition 5.1 in [12, Chap. VI, p. 183], these homotopy groups fit in a short exact sequence and their vanishing gives $Ext(\mathbb{Z}/p^\infty, \pi_n(X)) = 0$ and $Hom(\mathbb{Z}/p^\infty, \pi_n(X)) = 0$. But as $\pi_n(X)$ is abelian and thus nilpotent, $Ext(\mathbb{Z}/p^\infty, \pi_n(X)) = 0$ is equivalent to $\pi_n(X)$ is p-divisible [12, Lemma 3.6, Chap. VI, p. 176]. Using this and $Hom(\mathbb{Z}/p^\infty, \pi_n(X)) = 0$, one easily checks by contraposition that $\pi_n(X)$ has no p-torsion. These two last properties being true for all primes p, this is equivalent to saying that $\pi_n(X)$ is divisible and has no torsion, which is equivalent to saying that the abelian group $\pi_n(X)$ has a structure of $\mathbb{Q}$-vector space [60, Example 10.8, p. 320], for all n. The conclusion is then given by lemma 5.23. □

Remarks 5.25

1. In the previous proof, if G_o is semisimple it is easier to identify the fibre of

$$\mathrm{map}(BG_o, BG_o)_{id} \longrightarrow \mathrm{map}(BG_o, B\bar{G}_o)_g .$$

Indeed, in this case one has

$$[BG_o, BZ_o] = [BG_o, K(Z_o, 1)] \cong H^1(BG_o, Z_o) .$$

But as BG_o is 1-conoected $H^1(BG_o, Z_o) = 0$ (see the proof of lemma 5.24). Therefore $\mathrm{map}(BG_o, BZ_o)$ reduces to $\mathrm{map}(BG_o, BZ_o)_c$.

2. If the fibratioo

$$\mathrm{map}(BG_o, B\bar{G}_o)_g \longrightarrow \mathrm{fibmap}(BG, B\bar{G})_g^{id} \longrightarrow B\Gamma$$

appearing in the proof were *simple* (i.e. if the fundamental group of the base would act trivially on the homotopy groups of the fibre), it would allow to use a "lighter" version of obstruction theory. It is certainly iustructive to find an explicit example where the fibration is *not* simple.

We are ready to state and prove the main result of this section.

Theorem 5.26 *Given BG, the classifying space of a compact Lie group G, one can reconstruct G up to isomorphism.*

Proof. Starting from BG, one gets a model for BG_o as its universal cover $\widetilde{BG}$. Assuming the result in the connected case, we get the group G_o and hence the inclusions $Z_o \subset T \subset N_o \subset G_o$, as well as the adjoint group $\bar{G}_o$ associated to G_o. The group of compooents Γ of G is simply given by $\pi_1(BG)$. Supposing that we have a model for the fibratioo $BG_o \to BG \to B\Gamma$, lemma 5.19 gives a representative homomorphism $\varphi : \Gamma \to \mathrm{Out}(G_o)$. So we have eoough data to get the unique adjoint type group $\bar{G}$ corresponding to G, with all its relevant subgroups.

Now we want to recover the normalizer N of the maximal torus in G. Pulling-back the fibration $BZ_o \to BG \overset{B\pi}{\to} B\bar{G}$ along $B\bar{i}$, where $\bar{i}$ denotes the inclusioo $\bar{i} : \bar{N} \hookrightarrow \bar{G}$, furnishes a new fibration with the commutative diagram

$$\begin{array}{ccc} BZ_o & = & BZ_o \\ \downarrow & & \downarrow \\ X & \longrightarrow & BG \\ \downarrow & & \downarrow{\scriptstyle B\pi} \\ B\bar{N} & \xrightarrow{B\bar{i}} & B\bar{G} \end{array}$$

We claim that X is a model for BN. Indeed the universal property of the pullback yields a fibre map $v : BN \to X$ that fits in the homotopy commutative diagram

$$\begin{array}{ccccc} BZ_o & \xrightarrow{u} & BZ_o & = & BZ_o \\ \downarrow & & \downarrow & & \downarrow \\ BN & \xrightarrow{v} & X & \longrightarrow & BG \\ \downarrow & & \downarrow & & \downarrow \\ B\bar{N} & = & B\bar{N} & \longrightarrow & B\bar{G} \end{array}$$

The situation at the group level implies that the composition $id_{BZ_o} \circ u$ is the identity, and so u must be the identity. Therefore $v : BN \to X$ is a homotopy equivalence and we can set $BN = X$.

Invoking once more the universal property of the pullback gives a homotopically unique map $BN_o \xrightarrow{u} BN$ that is necessarily a model for Bi, where $i : N_o \hookrightarrow N$ denotes the inclusion. Applying the "constructive" part in the proof of proposition 2.3. in [50] to both fibrations in

$$\begin{array}{ccc} BT & = & BT \\ \downarrow & & \downarrow \\ BN_o & \longrightarrow & BN \\ \downarrow & & \downarrow \\ BW_o & \longrightarrow & BW \end{array}$$

gives

$$\begin{array}{ccc} T & = & T \\ \downarrow & & \downarrow \\ N_o & \overset{i}{\dashrightarrow} & N \\ \downarrow & & \downarrow \\ W_o & \hookrightarrow & W \end{array}$$

where the inclusion i is recovered by observing that N_o is in fact a pullback in this diagram (see exercise 1, p. 94 in [15]). We conclude by invoking proposition 4.4. □

Remark 5.27 In the above proof, we show that BN is obtained as a pullback. In fact it is not difficult to check that the same situation holds at the group level. However, one cannot immediately conclude that this implies the result at the classifying space level, as the functor $B(-)$ does not preserve pullbacks in general.

5.4 Automorphisms of BN_o

For giving an affirmative answer to Mislin's question in the nonconneted case, we will need some information on the group $\mathrm{Aut}(BN_o)$ of classes of self-homotopy equivalences of the classifying space of the normalizer N_o. We will in fact give a complete description of this group. The key ingredient for the last two sections is the following proposition due to Zabrodsky [72, Proposition 2.3]:

Proposition 5.28 (Zabrodsky) *Let $T^n \hookrightarrow L \twoheadrightarrow W$ and $T^m \hookrightarrow L' \twoheadrightarrow W'$ be finite extensions of tori. Then any map $f : BL \to BL'$ is homotopic to a map $B\rho$, for some homomorphism $\rho : L \to L'$.*

Lemma 5.29 *Let $[\psi] \in H^1(W_o;T) \subset \mathrm{Out}(N_o)$. Then $B\psi$ is homotopic to the identity id_{BN_o} if and only if $[\psi]$ is trivial in $H^1(W_o;T)$.*

Proof. The "if" part is clear. Conversely, suppose that $B\psi \sim id_{BN_o}$. By contradiction, suppose that $[\psi] \neq [id_{N_o}]$. By section 3.5 in chapter 3, there exists a Tits element q_α in N_o such that $\psi(q_\alpha)$ is not on a Tits circle. Let $N_\alpha \overset{i}{\hookrightarrow} N_o$ denote the 2-toral subgroup generated by T and q_α. Any conjugation by an element of N_o that preserves N_α represents an element different from $\psi_{|N_\alpha}$ in $\mathrm{Rep}(N_\alpha, N_\alpha)$, since such conjugations preserve Tits circles. We choose models for the classifying spaces such that the induced map $p = Bi$ is a covering projection; we get the homotopy commutative diagram

$$\begin{array}{ccc} BN_\alpha & \xrightarrow{f = B(\psi_{|N_\alpha})} & BN_\alpha \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ BN_o & \xrightarrow{h = B\psi} & BN_o \end{array}$$

By lemma 5.17, f is thus homotopic to an automorphism of p, which can be identified, by remark 5.18, with an element of $N_{W_o}(\langle s_\alpha \rangle)/\langle s_\alpha \rangle$, where $s_\alpha = \pi(q_\alpha) \in W_o$. Therefore, by corollary 5.20, $f = B(\psi_{|N_\alpha}) \sim Bc_n$, for some $n \in N_o$. By theorem 5.15 applied to $P = G = N_\alpha$, this contradicts the fact that $\psi_{|N_\alpha}$ and c_n are different in $\mathrm{Rep}(N_\alpha, N_\alpha)$. □

Theorem 5.30 *The map*

$$\beta_{N_o} : \mathrm{Out}(N_o) \longrightarrow \mathrm{Aut}(BN_o)\,,\ [\psi] \longmapsto [B\psi]$$

is an isomorphism of groups. In particular

$$\mathrm{Aut}(BN_o) \cong H^1(W_o;T) \rtimes \mathrm{Out}(G_o)\,.$$

Proof. By proposition 5.28 of Zabrodsky the map is surjective. Thus, it remains to check that $B\rho \sim B\rho'$ implies $[\rho] = [\rho'] \in \mathrm{Out}(N_o)$. By theorem 3.10, we can decompose the representative elements as $\rho = \psi \circ \sigma$ and $\rho' = \psi' \circ \sigma'$, with $[\psi], [\psi'] \in H^1(W_o;T)$ and $[\sigma], [\sigma'] \in \mathrm{Out}(G_o)$. Define $\eta = \psi'^{-1} \circ \psi$ and $\tau = \sigma' \circ \sigma^{-1}$; clearly $[\eta] \in H^1(W_o;T)$ and $[\tau] \in \mathrm{Out}(G_o)$. Now, as $B\rho \sim B\rho'$, we have $B\eta = B(\psi'^{-1} \circ \psi) \sim B(\sigma' \circ \sigma^{-1}) = B\tau$. We claim that both are homotopic to the identity id_{BN_o}. Indeed, take models such that $p = Bi : BT \to BN_o$ is the universal covering map. Then the diagram

$$\begin{array}{ccc} BT & \xrightarrow{B(\eta \circ \tau_{|T}^{-1})} & BT \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ BN_o & \xrightarrow{B(\eta \circ \tau^{-1}) \sim \mathrm{id}_{BN_o}} & BN_o \end{array}$$

is homotopy commutative. Lemma 5.17 implies that $B(\eta_{|T})$ are $B(\tau_{|T})$ homotopic modulo an automorphism of p, which are given by the action of the Weyl group. Thus $B(\tau_{|T})$ is homotopic to Bc_n for some $n \in N_o$. By theorem 5.15 applied to $P = G = T$, this is only possible if $\tau_{|T}$ is given by c_n and the same must hold for $\tau \in \mathrm{Aut}(N_o)$. Therefore $B\tau \sim B\eta$ are both homotopic to the identity as claimed. In particular, we also get that $[\tau]$ is trivial in $\mathrm{Out}(N_o)$. Moreover, by lemma 5.17, the same holds for $[\eta]$. Therefore $[\sigma] = [\sigma'] \in \mathrm{Out}(G_o) \subset \mathrm{Out}(N_o)$ and $[\psi] = [\psi'] \in H^1(W_o;T) \subset \mathrm{Out}(N_o)$ and we can conclude that $[\rho] = [\rho'] \in \mathrm{Out}(N_o)$. □

Remark 5.31 In an unpublished paper [44], Møller shows that the map β_{N_o} is in fact an isomorphism for any nonconnected compact Lie group. Also consult [47] for related results.

5.5 Invariance under homotopy equivalences

In this section, we want to present our proof that the answer to Mislin's question is affirmative for *nonconnected* compact Lie groups as well.

Theorem 5.32 *Let G and G' be two compact Lie groups. Then G and G' are isomorphic if and only if their classifying spaces BG and BG' are homotopy equivalent.*

As in the case of the "generalized Curtis-Wiederhold-Williams theorem" (see chapter 4), reconstructing the isomorphism class of G starting from BG is not enough to answer positively to Mislin's question. We must still show that the reconstruction process we proposed is invariant under homotopy equivalences. Before presenting the proof of theorem 5.32, we introduce two technical lemmas.

Lemma 5.33 *Let $f : BG \to BG'$ be a homotopy equivalence. Then f fits into a homotopy commutative diagram*

$$\begin{array}{ccc} BG_o & \xrightarrow{B\alpha} & BG_o \\ {\scriptstyle Bi}\downarrow & & \downarrow{\scriptstyle Bi'} \\ BG & \xrightarrow{f} & BG' \\ {\scriptstyle Bp}\downarrow & & \downarrow{\scriptstyle Bp'} \\ B\Gamma & \xrightarrow{B\beta} & B\Gamma \end{array}$$

with $\alpha \in \mathrm{Aut}(G_o, T)$ and $\beta \in \mathrm{Aut}(\Gamma)$. Moreover there exists an isomorphism of adjoint type groups $\eta : \bar{G} \to \bar{G}'$ such that the diagram

$$\begin{array}{ccc} BG_o & \xrightarrow{B\alpha} & B\bar{G}_o \\ {\scriptstyle Bi}\downarrow & & \downarrow{\scriptstyle B\bar{i}'} \\ BG & \xrightarrow{f} & BG' \\ {\scriptstyle B\pi}\downarrow & & \downarrow{\scriptstyle B\pi'} \\ B\bar{G} & \xrightarrow{B\eta} & B\bar{G}' \end{array}$$

is homotopy commutative.

Proof. As BG' is path connected, we can suppose that f is a pointed map [62, Chap. 7, Lemma 2, p. 380]. By the homotopy lifting property, f induces a homotopy equivalence on the universal covers $BG_o = \widetilde{BG}$ and $BG'_o = \widetilde{BG'}$. Therefore we can suppose $G_o = G'_o$, theorem 5.32 being true in the connected case [53]. Thus f induces a homotopy commutative diagram

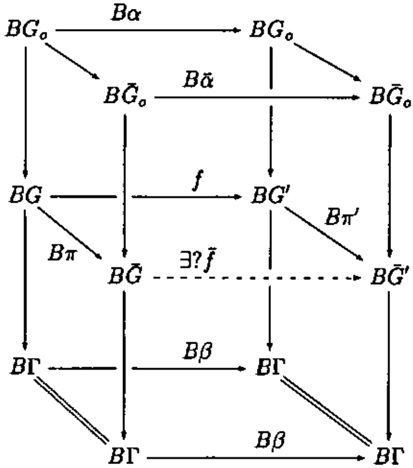
$$\begin{array}{ccc} BG_o & \xrightarrow{g} & BG_o \\ {\scriptstyle Bi}\downarrow & & \downarrow{\scriptstyle Bi'} \\ BG & \xrightarrow{f} & BG' \\ {\scriptstyle q = Bp}\downarrow & & \downarrow{\scriptstyle q' = Bp'} \\ B\Gamma & \overset{\exists ?h}{\dashrightarrow} & B\Gamma \end{array}$$

For the fundamental groups, it produces a commutative diagram

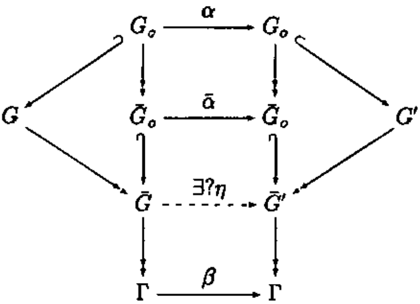
$$\begin{array}{ccc} \pi_1(BG) & \underset{\cong}{\xrightarrow{f_*}} & \pi_1(BG') \\ {\scriptstyle q_*}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle q'_*} \\ \pi_1(B\Gamma) & \underset{\cong}{\overset{q'_* \circ f_* \circ q_*^{-1}}{\dashrightarrow}} & \pi_1(B\Gamma) \\ \downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow \\ \Gamma & \underset{\cong}{\overset{\beta}{\dashrightarrow}} & \Gamma \end{array}$$

Defining $h = B\beta$ completes the first diagram and makes it homotopy commutative, because $B\Gamma$ is an aspherical space [71, Chap. V, §4]. Finally, by theorem 5.14, there exists $\alpha \in \mathrm{Aut}(G_o, T)$ such that $g \sim B\alpha$, and we get the first assertion.

For the second assertion, let us consider the homotopy commutative diagram



Let us construct maps at the level of groups:



Up to equivalence of extensions, $\bar{G}$ and $\bar{G}'$ are unique and given by semidirect products. The two actions are related as shown in the commutative diagram

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\theta} & \mathrm{Out}(\bar{G}_o) & \xrightarrow{s} & \mathrm{Aut}(\bar{G}_o) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle c_{[\bar{\alpha}]}} & & \downarrow{\scriptstyle c_{\bar{\alpha}}} \\ \Gamma & \longrightarrow & \mathrm{Out}(\bar{G}_o) & \longrightarrow & \mathrm{Aut}(\bar{G}_o) \end{array}$$

i.e. $\bar{G} = \bar{G}_o \rtimes_\sigma \Gamma$ and $\bar{G}' = \bar{G}_o \rtimes_{\sigma'} \Gamma$, with $\sigma = s \circ \theta$ and $\sigma' = c_{\bar{\alpha}} \circ s \circ \theta \circ \beta^{-1}$. One then checks that $\eta : \bar{G} \to \bar{G}'$, $(g_o, \gamma) \mapsto \eta(g_o, \gamma) = (\alpha(g_o), \beta(\gamma))$ is an isomorphism making the diagram at the group level commutative. By lemma 5.19, setting $\bar{f} = B\eta$ makes the "front face" of the first diagram homotopy commutative. As $B\eta$ is a homotopy equivalence, we get, by theorem 5.21, that $B\eta \circ B\pi$ is unique up to homotopy, and conclude that the whole diagram is homotopy commutative. □

Lemma 5.34 *Let $\rho : N \to N'$ be an isomorphism of normalizers. Suppose that $B\rho : BN \to BN'$ fits in the homotopy commutative diagram*

$$\begin{array}{ccc} BN_o & \xrightarrow{B\sigma} & BN_o \\ {\scriptstyle Bi}\downarrow & & \downarrow{\scriptstyle Bi'} \\ BN & \xrightarrow{B\rho} & BN' \\ {\scriptstyle Bp}\downarrow & & \downarrow{\scriptstyle Bp'} \\ B\Gamma & \xrightarrow{B\beta} & B\Gamma \end{array}$$

with $[\sigma] \in \mathrm{Out}(G_o)$. *Then, at the group level, there is a commutative diagram*

$$\begin{array}{ccc} N_o & \xrightarrow{\psi = \rho_{|N_o}} & N_o \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ N & \xrightarrow{\rho} & N' \end{array}$$

with $[\psi] = [\rho_{|N_o}] \in \mathrm{Out}(G_o)$.

Proof. The first diagram induces the homotopy commutative diagram

$$\begin{array}{ccc} BW_o & \xrightarrow{\bar{B}\sigma} & BW_o \\ {\scriptstyle Bi}\downarrow & & \downarrow{\scriptstyle Bi'} \\ BW & \xrightarrow{\bar{B}\rho} & BW' \\ {\scriptstyle Bp}\downarrow & & \downarrow{\scriptstyle Bp'} \\ B\Gamma & \xrightarrow{\bar{B}\beta} & B\Gamma \end{array}$$

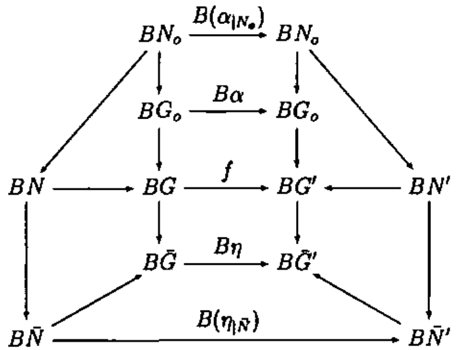
Applying the functor $\pi_1(-)$ yields, by 5.12, a commutative diagram at the group level

$$\begin{array}{ccc} W_o & \xrightarrow{\bar{\sigma}} & W_o \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ W & \xrightarrow{\bar{\rho}} & W' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ \Gamma & \xrightarrow{\beta} & \Gamma \end{array}$$

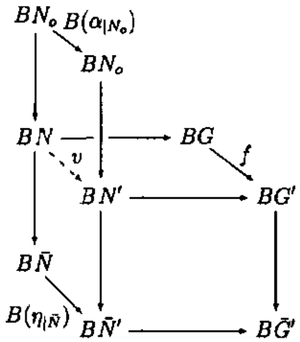
where the homomorphisms are given up to conjugation in the target groups. This implies that $\rho(N_o) = N_o$ and gives the second diagram in the lemma. To prove the last assertion, we reapply the functor $B(-)$ to it and get, by the homotopy lifting property and lemma 5.17, that $B(\sigma \circ \psi^{-1})$ is homotopic to a covering transformation of $Bi : BN_o \to BN$. By corollary 5.20, these are given by the action of Γ on BN_o, and thus are coming from $\mathrm{Out}(G_o)$ in $\mathrm{Out}(N_o)$. Therefore, by theorem 5.30, $[\psi] \in \mathrm{Out}(G_o)$ as claimed.

□

Proof of theorem 5.32. Let $f : BG \to BG'$ be a homotopy equivalence. Applying lemma 5.33, as well as the construction of theorem 5.26, yields the homotopy commutative diagram



with $\alpha \in \mathrm{Aut}(G_o)$ and $\eta \in \mathrm{Aut}(\bar{G}_o)$. From this diagram we extract



where the map $v : BN \to BN'$ exists and makes the diagram homotopy commutative by the universal property of the pullback. As a fibre map inducing the homotopy equivalence $B(\eta_{|\bar{N}})$ on the base spaces and the self-homotopy equivalence $B(\alpha_{|Z_o})$ on the fibre BZ_o, v is necessarily a homotopy equivalence. By proposition 5.28, there exists a homomorphism $\rho : N \to N'$ such that $v \sim B\rho$; by theorem 5.13, ρ has to be an isomorphism. It satisfies the hypothesis of lemma 5.34, and thus we get a commutative diagram at the group level as in this lemma. We conclude by invoking theorem 4.17. □

Appendix

Tits circles in SO($2n$) and Spin($2n$)

First recall tbat the standard maximal torus in SO($2n$) consists of the matrices

$$\begin{pmatrix} \cos 2\pi x_1 & -\sin 2\pi x_1 & & & \\ \sin 2\pi x_1 & \cos 2\pi x_1 & & & \\ & & \ddots & & \\ & & & \cos 2\pi x_n & -\sin 2\pi x_n \\ & & & \sin 2\pi x_n & \cos 2\pi x_n \end{pmatrix}$$

with $x_1, \ldots, x_n \in \mathbb{R}$. Choosing the root system described on page 9 of Adams' book [3], the action of tbe elements of an associated Tits system $A = \{q_1, \ldots, q_n\}$ is as follows: q_1 permutes x_1 and x_2 and changes both signs, q_2 permutes x_1 and x_2, q_3 permutes x_2 and x_3, ..., q_n permutes x_{n-1} and x_n. Calculations in the Lie algebra gives tbe coroots, from whicb one gets the circles $T_j = T_{\alpha_j}$ and the elements $h_j = h_{\alpha_j} = \exp(\frac{\alpha^\vee}{2})$ in T, and can deduce an associated Tits system. Writing

$$D(x) = \begin{pmatrix} \cos 2\pi x & -\sin 2\pi x \\ \sin 2\pi x & \cos 2\pi x \end{pmatrix} \in \mathrm{SO}(2),$$

and

$$E = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

one explicitly gets

$$\begin{aligned} T_1 &= \Big\{\mathrm{diag}\big(D(x), D(x), \mathbb{1}, \ldots, \mathbb{1}\big) : x \in \mathbb{R}\Big\} \\ T_2 &= \Big\{\mathrm{diag}\big(D(x), D(-x), \mathbb{1}, \ldots, \mathbb{1}\big) : x \in \mathbb{R}\Big\} \\ T_3 &= \Big\{\mathrm{diag}\big(\mathbb{1}, D(x), D(-x), \mathbb{1}, \ldots, \mathbb{1}\big) : x \in \mathbb{R}\Big\} \\ &\ \ \vdots \\ T_n &= \Big\{\mathrm{diag}\big(\mathbb{1}, \ldots, \mathbb{1}, D(x), D(-x)\big) : x \in \mathbb{R}\Big\} \end{aligned}$$

and

$$q_{\alpha_1} = \begin{pmatrix} 0 & E & & & & \\ -E & 0 & & & & \\ & & \mathbb{1} & & & \\ & & & \mathbb{1} & & \\ & & & & \ddots & \\ & & & & & \mathbb{1} \end{pmatrix}, \qquad q_{\alpha_2} = \begin{pmatrix} 0 & \mathbb{1} & & & & \\ -\mathbb{1} & 0 & & & & \\ & & \mathbb{1} & & & \\ & & & \mathbb{1} & & \\ & & & & \ddots & \\ & & & & & \mathbb{1} \end{pmatrix},$$

$$q_{\alpha_3} = \begin{pmatrix} \mathbb{1} & & & & \\ & 0 & \mathbb{1} & & \\ & -\mathbb{1} & 0 & & \\ & & & \mathbb{1} & \\ & & & & \ddots & \\ & & & & & \mathbb{1} \end{pmatrix}, \ \dots, \ q_n = \begin{pmatrix} \mathbb{1} & & & & \\ & \mathbb{1} & & & \\ & & \ddots & & \\ & & & \mathbb{1} & & \\ & & & & 0 & \mathbb{1} \\ & & & & -\mathbb{1} & 0 \end{pmatrix}.$$

Notice that for $j = 2, 3, \dots, n$, this corresponds, under the standard inclusion $\mathrm{SU}(n) \subset \mathrm{SO}(2n)$, to the situation depicted in example 3.20 (by uniqueness of the homomorphisms ν_{α_j}, it shows that the corresponding q_j are the right ones). The Tits circles are then simply given by q_jT_j.

For $\mathrm{Spin}(2n)$, it suffices to consider the covering homomorphism $\phi : \mathrm{Spin}(2n) \twoheadrightarrow \mathrm{SO}(2n)$ and to "lift" the situation just described. The standard maximal torus $T(\mathrm{Spin}(2n)) = \tilde{T} = \phi^{-1}(T)$ consists then of the elements

$$(\cos x_1 + e_1e_2 \sin x_1)(\cos x_2 + e_3e_4 \sin x_2) \cdots (\cos x_n + e_{2n-1}e_{2n} \sin x_n).$$

Defining $\tilde{T}_j = \phi^{-1}(T_j)$ and $\tilde{q}_j$ as one of the two elements in $\phi^{-1}(q_j)$ gives the Tits circles $\tilde{q}_j\tilde{T}_j$, for $j = 1, 2, \dots, n$. Recalling that for $v \in \mathbb{S}^{2n-1} \subset \mathbb{R}^{2n}$, the element $\phi(v) \in \mathrm{O}(2n)$ is the reflection in the hyperplane orthogonal to v, one explicitly gets

$$\begin{aligned}
\tilde{T}_1 &= \Big\{(\cos x + e_1e_2 \sin x)(\cos x + e_3e_4 \sin x) : x \in \mathbb{R}\Big\} \\
\tilde{T}_2 &= \Big\{(\cos x + e_1e_2 \sin x)(\cos x - e_3e_4 \sin x) : x \in \mathbb{R}\Big\} \\
\tilde{T}_3 &= \Big\{(\cos x + e_3e_4 \sin x)(\cos x - e_5e_6 \sin x) : x \in \mathbb{R}\Big\} \\
&\ \vdots \\
\tilde{T}_n &= \Big\{(\cos x + e_{2n-3}e_{2n-2} \sin x)(\cos x - e_{2n-1}e_{2n} \sin x) : x \in \mathbb{R}\Big\}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{q}_1 &= \tfrac{1}{2}e_1e_4(e_1 - e_3)(e_2 - e_4) \\
\tilde{q}_2 &= \tfrac{1}{2}e_3e_4(e_1 - e_3)(e_2 - e_4) \\
\tilde{q}_3 &= \tfrac{1}{2}e_5e_6(e_3 - e_5)(e_4 - e_6) \\
&\ \vdots \\
\tilde{q}_n &= \tfrac{1}{2}e_{2n-1}e_{2n}(e_{2n-3} - e_{2n-1})(e_{2n-2} - e_{2n}).
\end{aligned}$$

End of the proof of proposition 3.23

The labels (∗) and (∗∗) refer to the formulae on page 45.

(2): Recall that the subgroup S of elements of order 2 in $T \subset \mathrm{Spin}(2n)$ consists of ± 1 together with the words of the form

$$\pm \prod_j e_{2j-1}e_{2j}$$

of total length a multiple of 4 (as a convention, when we say "word" or "subword", we always mean reduced ones). We first consider $\mathrm{Spin}(4)$. In this case, the subgroup S of elements of order 2 in T is equal to the center $Z(N_o) = Z(G_o)$ and both t_1 and t_2 have two possible values:

$$\begin{array}{rcrcr} t_1 &=& 1 &\sim& e_1e_2e_3e_4 \\ t_1 &=& -1 &\sim& -e_1e_2e_3e_4 \\ t_2 &=& 1 &\sim& -e_1e_2e_3e_4 \\ t_2 &=& -1 &\sim& e_1e_2e_3e_4\,, \end{array}$$

where, for each value, we have indicated the two equivalent choices according to the description of the Tits circles in the previous section. As $\ell_{12} = 2$, the result follows for $\mathrm{Spin}(4)$ (see also remarks 3.24). The case $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$ has already been treated. We will also need the case $\mathrm{Spin}(8)$. An analysis of the 16 elements in S shows that

$$t_j \in \{\pm 1, \pm e_1e_2e_3e_4e_5e_6e_7e_8\} = Z(N_o) = Z(G_o)\,, \quad \text{for } j = 1,2,3,4\,.$$

Now $\ell_{j,3} = 3$ for $j = 1,2,4$ and (∗∗) yields

$$t_j = t_j^3 = t_3^3 = t_3\,, \quad \text{for } j = 1,2,4\,,$$

which shows that the result holds in this case as well. The cases $\mathrm{Spin}(6)$ and $\mathrm{Spin}(8)$ constitute the base steps of the induction. As for $\mathrm{SU}(n)$, we will consider the canonical inclusions

$$\begin{array}{ccccc} S_{n-2} = S \cap \mathrm{Spin}(2(n-2)) & \hookrightarrow & S_{n-1} = S \cap \mathrm{Spin}(2(n-1)) & \hookrightarrow & S \\ \downarrow && \downarrow && \downarrow \\ T_{n-2} = T \cap \mathrm{Spin}(2(n-2)) & \hookrightarrow & T_{n-1} = T \cap \mathrm{Spin}(2(n-1)) & \hookrightarrow & T \\ \downarrow && \downarrow && \downarrow \\ \mathrm{Spin}(2(n-2)) & \hookrightarrow & \mathrm{Spin}(2(n-1)) & \hookrightarrow & \mathrm{Spin}(n) \end{array}$$

The subgroup S_{n-2} is of index 2 in S_{n-1}, itself of index 2 in S. Denote

$$u = e_{2n-5}e_{2n-4}e_{2n-3}e_{2n-2} \quad \text{and} \quad v = e_{2n-3}e_{2n-2}e_{2n-1}e_{2n} \in S\,;$$

we have $uv = -e_{2n-5}e_{2n-4}e_{2n-1}e_{2n}$ and the decompositions

$$\begin{aligned} S &= S_{n-1} \amalg vS_{n-1} \\ &= S_{n-2} \amalg uS_{n-2} \amalg vS_{n-2} \amalg uvS_{n-2}\,. \end{aligned}$$

We distinguish two cases, depending on whether the automorphism ψ preserves the normalizer of the maximal torus in $\mathrm{Spin}(2(n-1))$ or not.

Case 1: $\psi_{|N_{\mathrm{Spin}(2(n-1))}(T_{n-1})} \in \mathrm{Aut}(N_{\mathrm{Spin}(2(n-1))}(T_{n-1}))$
By induction, four sub-cases can occur.

- $\psi_{|N_{\mathrm{Spin}(2(n-1))}(T_{n-1})} = id_{N_{\mathrm{Spin}(2(n-1))}(T_{n-1})}$ and thus $t_j = 1$ for $j = 1, 2, \ldots, n-1$. We have to show that $t_n = 1$. The possible values for t_n are ± 1 or a product of the form $\pm \prod(e_{2j-1}e_{2j})e_{2n-3}e_{2n-2}e_{2n-1}e_{2n}$ of total length a multiple of 4. Let us treat the two non-trivial cases:

  ▷ $t_n = -1$. This case is ruled out because $\ell_{n-1,n} = 3$ and (**) gives the contradiction $1 = -1$.

  ▷ $t_n = \pm \prod(e_{2j-1}e_{2j})e_{2n-3}e_{2n-2}e_{2n-1}e_{2n}$. Let us show that this is equivalent to $t_n = 1$. First observe that for $j = 1, 2, \ldots, n-2$, we have $\ell_{j,n} = 2$ and thus
  $$\begin{aligned} & 1 = t_j(w_{n-1}\cdot t_j) = t_{n-1}(w_j\cdot t_{n-1}) \\ \Longrightarrow\quad & t_{n-1} = w_j\cdot t_{n-1}\,. \end{aligned}$$
  Therefore, either t_n does not contain the subword e_1e_2 and we have $t_n = -e_{2n-3}e_{2n-2}e_{2n-1}e_{2n}$, and this is equivalent to $t_n = 1$, or t_n contains the subword e_1e_2 and we have $t_n = \pm e_1e_2\cdots e_{2n}$. But the latter case is impossible: indeed as $\ell_{n-1,n} = 3$, applying (**) gives the contradiction $1 = \pm e_1e_2\cdots e_{2n}$.

- $\psi_{|N_{\mathrm{Spin}(2(n-1))}(T_{n-1})}$ is defined by $\psi(q_j) = -q_j$ for $j = 1, 2, \ldots, n-1$. The same arguments as in the previous case show that $t_n = -1$.

- In case n is odd, we can have $\psi_{|N_{\mathrm{Spin}(2(n-1))}(T_{n-1})}$ defined by $\psi(q_j) = tq_j$ with $t = e_1e_2\cdots e_{2n-2}$ for $j = 1, 2, \ldots, n-1$. We show that this case is impossible using $\ell_{1,n} = 2$. Formula (*) gives
  $$t_n(w_1\cdot t_n) = t(w_n\cdot t) = -e_{2n-3}e_{2n-2}e_{2n-1}e_{2n}\,.$$
  Now either t_n contains the subword $e_{2n-3}e_{2n-2}e_{2n-1}e_{2n}$ or none of the symbols $e_{2n-3}, e_{2n-2}, e_{2n-1}, e_{2n}$. In both cases $t_n(w_1\cdot t_n)$ does not contain the subword $e_{2n-3}e_{2n-2}e_{2n-1}e_{2n}$, in contradiction with the above calculation.

- In case n is odd, we can also have $\psi_{|N_{\mathrm{Spin}(2(n-1))}(T_{n-1})}$ defined by $\psi(q_j) = -tq_j$ with t as in the previous case for $j = 1, 2, \ldots, n-1$. The same arguments show that this case is impossible as well.

Case 2: $\psi_{|N_{\mathrm{Spin}(2(n-1))}(T_{n-1})} \notin \mathrm{Aut}(N_{\mathrm{Spin}(2(n-1))}(T_{n-1}))$
By hypothesis there exists $j_o \in \{1, 2, \ldots, n-1\}$ such that $t_{j_o} \notin S_{n-1}$. We claim that

$$t_j = vs_j\,, \quad \text{with } s_j \in S_{n-1} \text{ for all } j = 1, 2, \ldots, n-1\,.$$

First observe that $w_j \cdot v = v$ for all $j = 1, 2, \ldots, n-2$ and that the subword $e_{2n-1}e_{2n}$ is not exchanged by w_j for $j = 1, 2, \ldots, n-1$. Now α_{j_o} has a neighhour α_k for some $k \in \{1, 2, \ldots, n-1\}$ such that $\ell_{j_o,k} = 3$. Formula (**) and the fact that the subword $e_{2n-1}e_{2n}$ is not exchanged implies that $t_k = vs_k$ for some $s_k \in S_{n-1}$. By connectedness of the sub-diagram of Dynkin of vertices $\{\alpha_1, \ldots, \alpha_{n-1}\}$, repeating the argument demonstrates the claim. We now show that

$$t_j = vs_j\,, \quad \text{with } s_j \in S_{n-2} \text{ for all } j = 1, 2, \ldots, n-2\,.$$

The only case to exclude is the following: $t_j \in uS_{n-2}$ for some $j \in \{1, 2, \ldots, n-2\}$. Write $t_j = re_{2n-5}e_{2n-4}e_{2n-1}e_{2n}$ for some $r \in S_{n-2}$. We have $\ell_{j,n} = 2$ and thus

$$t_n(w_j \cdot t_n) = t_j(w_n \cdot t_j) = -e_{2n-3}e_{2n-2}e_{2n-1}e_{2n}\,.$$

This contradicts the fact that the element $t_n(w_j \cdot t_n)$ does not contain the subword $e_{2n-3}e_{2n-2}e_{2n-1}e_{2n}$.

Now, since v is invariant under w_j for $j = 1, 2, \ldots, n-2$, setting $\tilde{\psi}(q_j) = s_j q_j$ for all these j's defines an automorphism $\tilde{\psi}$ of $N_{\mathrm{Spin}(2(n-2))}(T_{n-2})$. By induction, four sub-cases can occur.

- $\tilde{\psi} = id_{N_{\mathrm{Spin}(2(n-2))}(T_{n-2})}$ and thus $t_j = v$ for $j = 1, 2, \ldots, n-2$. As $n \geqslant 5$, we have $\ell_{1,n-1} = 2$ and

  $$t_{n-1}(w_1 \cdot t_{n-1}) = v(w_{n-1} \cdot v) = -e_{2n-5}e_{2n-4}e_{2n-3}e_{2n-2}\,.$$

  This contradicts the fact that $t_{n-1}(w_j \cdot t_{n-1})$ does not contain the subword $e_{2n-5}e_{2n-4}e_{2n-3}e_{2n-2}$.

- $s_j = -1$, i.e. $t_j = -v$ for $j = 1, 2, \ldots, n-2$. This case is impossible by tbe same arguments.

- In case n is even, we can also have $s_j = e_1e_2 \cdots e_{2n-4}$, i.e. $t_j = t = e_1e_2 \cdots e_{2n} \in Z(\mathrm{Spin}(2n))$ for $j = 1, 2, \ldots, n-2$. We have to show that $t_{n-1} = t_n =$

t. We already know that $t_{n-1} = re_{2n-5}e_{2n-4}e_{2n-3}e_{2n-2}e_{2n-1}e_{2n}$ for some $r \in T(\mathrm{Spin}(2(n-3)))$. As $n \geqslant 5$, for $j = 1, 2, \ldots, n-3$, we have $\ell_{j,n-1} = 2$ and $(*)$ implies that $t_{n-1} = w_j \cdot t_{n-1}$ for these indices. Therefore, either $r = 1$, but this contradicts $t_{n-1} \in S$, or $r = e_1 e_2 \cdots e_{2n-6}$ and so $t_{n-1} = t$. The same arguments with $j = 1, 2, \ldots, n-2$ and $\ell_{j,n}$ shows that $t_n = t$ or $t_n = -e_{2n-3}e_{2n-2}e_{2n-1}e_{2n}$. But the latter case is equivalent to $t_n = 1$ and is ruled out by $\ell_{n-1,n}$ and formula $(**)$.

- In case n is even, we can also have $t_j = -t = -e_1 e_2 \cdots e_{2n} \in Z(\mathrm{Spin}(2n))$ for $j = 1, 2, \ldots, n-2$ and the same arguments shows that $t_{n-1} = t_n = -t$ and conclude the case Spin(2n).

(3): A diagonal element $t = diag(\epsilon_1 \cdot \mathbf{1}, \epsilon_2 \cdot \mathbf{1}, \ldots, \epsilon_n \cdot \mathbf{1})$ of $S \subset T \subset \mathrm{SO}(2n)$ will simply be denoted by $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$. For SO(6) the result follows by the same arguments as for SU(4). The inductive step is easier than for the previous cases; it suffices to consider the canonical inclusions

$$\begin{array}{ccc} S_{n-1} = S \cap \mathrm{SO}(2(n-1)) & \hookrightarrow & S \\ \downarrow & & \downarrow \\ T_{n-1} = T \cap \mathrm{SO}(2(n-1)) & \hookrightarrow & T \\ \downarrow & & \downarrow \\ \mathrm{SO}(2(n-1)) & \hookrightarrow & \mathrm{SO}(2n) \end{array}$$

The subgroup S_{n-1} is of index 2 in S. Denote $v = (1, \ldots, 1, -1) \in S$; we have the decomposition

$$S = S_{n-1} \amalg vS_{n-1}.$$

As before, we distinguish two cases, depending on whether the automorphism ψ preserves the normalizer of the maximal torus in $\mathrm{SO}(2(n-1))$ or not.

Case 1: $\psi_{|N_{\mathrm{SO}(2(n-1))}(T_{n-1})} \in \mathrm{Aut}(N_{\mathrm{SO}(2(n-1))}(T_{n-1}))$

By induction, only two sub-cases can occur.

- $\psi_{|N_{\mathrm{SO}(2(n-1))}(T_{n-1})} = id_{N_{\mathrm{SO}(2(n-1))}(T_{n-1})}$ and thus $t_j = \mathbf{1}$ for $j = 1, 2, \ldots, n-1$. For $j = 1, 2, \ldots, n-2$, we have $\ell_{j,n} = 2$ and therefore

$$\begin{aligned} & \mathbf{1} = t_j(w_{n-1} \cdot t_j) = t_{n-1}(w_j \cdot t_{n-1}) \\ \Longrightarrow \quad & t_{n-1} = w_j \cdot t_{n-1} \\ \Longrightarrow \quad & t_{n-1} = (\epsilon, \ldots, \epsilon, \epsilon_{n-1}, \epsilon_{n-1}) = (\epsilon, \ldots, \epsilon, 1, 1). \end{aligned}$$

Similarly, $\ell_{n-1,n} = 3$ and formula $(**)$ gives on the last three coordinates

$$(1,1,1) = (\epsilon, 1, 1) \cdot (1, \epsilon, 1) \cdot (1, 1, \epsilon) = (\epsilon, \epsilon, \epsilon),$$

and therefore $\psi = id$.

- $\psi_{|N_{SO(2(n-1))}(T_{n-1})}$ can also correspond to the non-trivial outer automorphism, i.e. for $j = 1, 2, \dots, n-1$, $\psi(q_j) = tq_j$ with $t = (-1, \dots, -1, 1)$. We exclude this case using $\ell_{1,n} = 2$; indeed, on the last two coordinates formula $(*)$ becomes

$$\begin{aligned} (-1,1)\cdot(1,-1) &= (\epsilon_{n-1}, \epsilon_{n-1})\cdot(\epsilon_{n-1}, \epsilon_{n-1}) \\ \Longrightarrow \quad (-1,-1) &= (1,1), \end{aligned}$$

  which is clearly impossible.

Case 2: $\psi_{|N_{SO(2(n-1))}(T_{n-1})} \notin \mathrm{Aut}(N_{SO(2(n-1))}(T_{n-1}))$
By hypothesis there exists $j_o \in \{1, 2, \dots, n-1\}$ such that $t_{j_o} \notin S_{n-1}$. By the same argument as for $\mathrm{Spin}(2n)$, we have

$$t_j = vs_j\,, \quad \text{with } s_j \in S_{n-1} \text{ for all } j = 1, 2, \dots, n-1\,.$$

However, in the present case we have $w_j \cdot v = v$ for $j = 1, 2, \dots, n-1$, and therefore setting $\tilde{\psi}(q_j) = s_j q_j$ for all these j's defines an automorphism $\tilde{\psi}$ of $N_{SO(2(n-1))}(T_{n-1})$. Again by induction, two sub-cases can occur.

- $\tilde{\psi} = id_{N_{SO(2(n-1))}(T_{n-1})}$ and thus $t_j = v$ for $j = 1, 2, \dots, n-1$. But $\ell_{1,n} = 2$ shows that this case is impossible by applying $(*)$ and inspecting the last two coordinates.

- We can also have $s_j = (-1, \dots, -1, 1)$, i.e. $t_j = -\mathbf{1}$ for $j = 1, 2, \dots, n-1$. Using $\ell_{n-1,n} = 3$, and $(**)$ on the last three coordinates shows that $t_n = -\mathbf{1}$. This concludes the proof for $SO(2n)$.

(4): For E_6, we will work in the Lie algebra LT of the maximal torus to prove that there are no non-trivial automorphism. We use the explicit description given in Bourbaki [11, Pl. V. pp. 260-262]. The Lie algebra LT is the subspace of $\mathbb{R}^8$ consisting of the vectors whose coordinates (x_j) satisfy $x_6 = x_7 = -x_8$. As E_6 is self-dual, we have $R^\vee \cong R$ and so we get the following basis of $R^\vee$:

$$\begin{aligned} \alpha_1^\vee &= \tfrac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8) \\ \alpha_2^\vee &= e_1 + e_2 \\ \alpha_3^\vee &= e_2 - e_1 \\ \alpha_4^\vee &= e_3 - e_2 \\ \alpha_5^\vee &= e_4 - e_3 \\ \alpha_6^\vee &= e_5 - e_4 \end{aligned}$$

Since E_6 is simply connected, the integral lattice $\Gamma(T)$ is equal to the lattice generated by the coroots, i.e. $\Gamma(T) = \mathbb{Z}R^\vee$ [10, Cor. 1, p. 35]. The strategy is rather clear: we are going to use the "central" situation of the coroot $\alpha_4^\vee$ in the Dynkin diagram. Working with preimages in LT of the elements t_j of order two, we will distinguish as many cases as there are values for t_4 and show that all but one are impossible. By symmetry of the Dynkin diagram, for the second Tits relation we can restrict our attention to $i, j \in 1,2,3,4$.

We proceed by explicitly describing what will be needed in the case-by-case checking. First let $(.,.)$ denote the usual inner product on $\mathbb{R}^8$. As $(\alpha^\vee, \alpha^\vee) = 2$ for all $\alpha^\vee \in R^\vee$, the formula for the symmetry s_α becomes

$$s_\alpha(X) = X - 2\frac{(X, \alpha^\vee)}{(\alpha^\vee, \alpha^\vee)}\cdot\alpha^\vee = X - (X, \alpha^\vee)\cdot\alpha^\vee,$$

for all $X \in LT$ and for all $\alpha \in R$. For instance, one gets

$$\begin{aligned}
s_{\alpha_4}(\alpha_1^\vee) &= \alpha_1^\vee\\
s_{\alpha_4}(\alpha_2^\vee) &= \alpha_2^\vee + \alpha_4^\vee\\
s_{\alpha_4}(\alpha_3^\vee) &= \alpha_3^\vee + \alpha_4^\vee\\
s_{\alpha_4}(\alpha_4^\vee) &= -\alpha_4^\vee\\
s_{\alpha_4}(\alpha_5^\vee) &= \alpha_4^\vee + \alpha_5^\vee\\
s_{\alpha_4}(\alpha_6^\vee) &= \alpha_6^\vee
\end{aligned}$$

We next rewrite the formulae $(*)$ and $(**)$ in the Lie algebra. Let $t_i, t_j \in S \subset T$, and choose $X_i, X_j \in LT$ such that $\exp(X_i) = t_i$ and $\exp(X_j) = t_j$; in particular $2X_i \equiv 2X_j \equiv 0 \mod \Gamma(T)$. In case $\ell_{ij} = 2$, $(*)$ becomes

$$\begin{aligned}
& & \psi(q_iq_j) &= \psi(q_jq_i) & &\\
&\Longleftrightarrow & t_i(w_j\cdot t_i) &= t_j(w_i\cdot t_j) & &\\
&\Longleftrightarrow & X_i + s_{\alpha_j}(X_i) &\equiv X_j + s_{\alpha_i}(X_j) & &\mod \Gamma(T)\\
&\Longleftrightarrow & X_i + X_i - (X_i, \alpha_j^\vee)\cdot\alpha_j^\vee &\equiv X_j + X_j - (X_j, \alpha_i^\vee)\cdot\alpha_i^\vee & &\mod \Gamma(T)\\
&\Longleftrightarrow & (X_i, \alpha_j^\vee)\cdot\alpha_j^\vee &\equiv (X_j, \alpha_i^\vee)\cdot\alpha_i^\vee & &\mod \Gamma(T)
\end{aligned}$$

In case $\ell_{ij} = 3$, using the fact that $2\big(X_i - (X_i, \alpha_j^\vee)\cdot\alpha_j^\vee\big) \equiv 0 \mod \Gamma(T)$, a straightforward computation shows that

$$X_i + s_{\alpha_j}(X_i) + s_{\alpha_i}\big(s_{\alpha_j}(X_i)\big) \equiv X_i + \big((X_i, \alpha_j^\vee)(\alpha_j^\vee, \alpha_i^\vee) - (X_i, \alpha_i^\vee)\big)\alpha_i^\vee \mod \Gamma(T)$$

So $(**)$ becomes

$$\begin{aligned}
& & \psi(q_iq_jq_i) &= \psi(q_jq_iq_j) & &\\
&\Longleftrightarrow & t_i(w_j\cdot t_i)((w_iw_j)\cdot t_i) &= t_j(w_i\cdot t_j)((w_jw_i)\cdot t_j) & &\\
&\Longleftrightarrow & X_i + s_{\alpha_j}(X_i) + s_{\alpha_i}\big(s_{\alpha_j}(X_i)\big) &\equiv X_j + s_{\alpha_i}(X_j) + s_{\alpha_j}\big(s_{\alpha_i}(X_j)\big) & &\mod \Gamma(T)
\end{aligned}$$

which is finally equivalent to

$$X_i + \big((X_i\,,\,\alpha_j^\vee)(\alpha_j^\vee\,,\,\alpha_i^\vee) - (X_i\,,\,\alpha_i^\vee)\big)\alpha_i^\vee \equiv X_j + \big((X_j\,,\,\alpha_i^\vee)(\alpha_i^\vee\,,\,\alpha_j^\vee) - (X_j\,,\,\alpha_j^\vee)\big)\alpha_j^\vee$$

$\mod \Gamma(T)$.

It is clear that $\exp^{-1}(S) = \frac{1}{2}\mathbb{Z}R^\vee$. Starting from the generators $\frac{1}{2}\alpha_j^\vee$, $j = 1,\dots,6$, one can find a representative for each of the 64 elements in S. For each j, the condition on $X \in \exp^{-1}(S)$ so that $\exp(X)$ is invariant under the action of $w_j = s_{\alpha_j}$ is the following:

$$X \equiv s_{\alpha_j}(X) \equiv X - (X\,,\,\alpha_j^\vee)\cdot\alpha_j^\vee \mod \Gamma(T).$$

As one can take $X = \frac{1}{2}\sum_{k\in K}\alpha_k^\vee$ for some $K \subset \{1,\dots,6\}$, this is equivalent to

$$(X\,,\,\alpha_j^\vee)\cdot\alpha_j^\vee \equiv (\frac{1}{2}\sum_k \alpha_k^\vee\,,\,\alpha_j^\vee)\cdot\alpha_j^\vee \equiv 0 \mod \Gamma(T).$$

and, finally, equivalent to

$$(\sum_k \alpha_k^\vee\,,\,\alpha_j^\vee) \in 2\mathbb{Z}.$$

From this, one gets a set of representatives in LT for the subgroup $F^{\alpha_j} \cap S \subset T$ of elements of order 2 fixed by $w_j = s_{\alpha_j}$, i.e. a set of representatives in LT for the possible values of t_j. For example, if $j = 4$, one finds that there are 32 elements in $F^{\alpha_4} \cap S$, and that $\exp^{-1}(F^{\alpha_4} \cap S)$ is generated by the 5 elements

$$\frac{1}{2}\alpha_1^\vee,\ \frac{1}{2}\alpha_4^\vee,\ \frac{1}{2}\alpha_6^\vee,\ \frac{1}{2}(\alpha_2^\vee + \alpha_3^\vee),\ \frac{1}{2}(\alpha_2^\vee + \alpha_5^\vee).$$

Recalling that possible values go by pairs, this gives the following 16 representatives $X_4 \in \exp^{-1}(F^{\alpha_4} \cap S)$:

1)	0	9)	$\frac{1}{2}(\alpha_1^\vee + \alpha_2^\vee + \alpha_5^\vee)$
2)	$\frac{1}{2}\alpha_1^\vee$	10)	$\frac{1}{2}(\alpha_1^\vee + \alpha_3^\vee + \alpha_5^\vee)$
3)	$\frac{1}{2}\alpha_6^\vee$	11)	$\frac{1}{2}(\alpha_2^\vee + \alpha_3^\vee + \alpha_6^\vee)$
4)	$\frac{1}{2}(\alpha_1^\vee + \alpha_6^\vee)$	12)	$\frac{1}{2}(\alpha_2^\vee + \alpha_5^\vee + \alpha_6^\vee)$
5)	$\frac{1}{2}(\alpha_2^\vee + \alpha_3^\vee)$	13)	$\frac{1}{2}(\alpha_3^\vee + \alpha_5^\vee + \alpha_6^\vee)$
6)	$\frac{1}{2}(\alpha_2^\vee + \alpha_5^\vee)$	14)	$\frac{1}{2}(\alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee + \alpha_6^\vee)$
7)	$\frac{1}{2}(\alpha_3^\vee + \alpha_5^\vee)$	15)	$\frac{1}{2}(\alpha_1^\vee + \alpha_2^\vee + \alpha_5^\vee + \alpha_6^\vee)$
8)	$\frac{1}{2}(\alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee)$	16)	$\frac{1}{2}(\alpha_1^\vee + \alpha_3^\vee + \alpha_5^\vee + \alpha_6^\vee)$.

In each case, the other value leading to an equivalent automorphism is simply obtained by adding $\frac{1}{2}\alpha_4^\vee$ (because $T_4 = \exp(\mathbb{R}\alpha_4^\vee)$). We are now distinguishing 16 cases according to the representatives X_4 just described. For the rest of the proof, X_j will always denote an element of LT such that $\exp(X_j) \in F^{\alpha_j} \cap S$. We still abuse notation and equalities are meant up to conjugation by an element of T, i.e. up to addition of $\frac{1}{2}\alpha_j^\vee$ to X_j.

Case 1: Claim: if there exists i such that $t_i = e$, then for j such that $\ell_{ij} = 3$, we have $t_j = e$. Indeed, $(**)$ implies that

$$X_j + \big((X_j\,,\,\alpha_i^\vee)(\alpha_i^\vee\,,\,\alpha_j^\vee) - (X_j\,,\,\alpha_j^\vee)\big)\alpha_j^\vee \equiv 0 \mod \Gamma(T)\,.$$

Now

$$\begin{aligned}\big((X_j\,,\,\alpha_i^\vee)(\alpha_i^\vee\,,\,\alpha_j^\vee) - (X_j\,,\,\alpha_j^\vee)\big)\alpha_j^\vee &= -(X_j\,,\,\alpha_i^\vee)\cdot\alpha_j^\vee - (X_j\,,\,\alpha_j^\vee)\cdot\alpha_j^\vee \\ &\equiv -(X_j\,,\,\alpha_i^\vee)\cdot\alpha_j^\vee \mod \Gamma(T)\,,\end{aligned}$$

so that the previous relation becomes

$$X_j \equiv (X_j\,,\,\alpha_i^\vee)\cdot\alpha_j^\vee \mod \Gamma(T)\,.$$

But this holds if and only if $X_j \in \frac{1}{2}\mathbb{Z}\alpha_j^\vee$, hence the claim.

By hypothesis, we have $t_4 = 1$, therefore, by the claim and by connectedness of the Dynkin diagram, we have $t_j = e$ for all $j = 1, \ldots, 6$. This finishes the first case.

Cases 2-3: By symmetry, we only consider $X_4 = \frac{1}{2}\alpha_1^\vee$. Let us first show that it implies $X_2 = \frac{1}{2}\alpha_1^\vee$. We have $\ell_{24} = 3$, and $(**)$ implies

$$\begin{aligned}\frac{1}{2}\alpha_1^\vee \equiv \frac{1}{2}\alpha_1^\vee + 0\cdot\alpha_4^\vee &\equiv X_2 + \big((X_2\,,\,\alpha_4^\vee)(\alpha_4^\vee\,,\,\alpha_2^\vee) - (X_2\,,\,\alpha_2^\vee)\big)\alpha_2^\vee \\ &\equiv X_2 - (X_2\,,\,\alpha_2^\vee)\cdot\alpha_2^\vee \mod \Gamma(T)\,.\end{aligned}$$

Therefore, we must have

$$X_2 \equiv \frac{1}{2}\alpha_1^\vee + (X_2\,,\,\alpha_2^\vee)\cdot\alpha_2^\vee \mod \Gamma(T)\,.$$

An analysis of the possible values for X_2 shows that this holds if and only if $X_2 = \frac{1}{2}\alpha_1^\vee$.

Now $\ell_{23} = 2$, and $(*)$ furnishes

$$\begin{aligned}(X_3\,,\,\alpha_2^\vee)\cdot\alpha_2^\vee &\equiv (X_2\,,\,\alpha_3^\vee)\cdot\alpha_3^\vee \\ &\equiv (\frac{1}{2}\alpha_1^\vee\,,\,\alpha_3^\vee)\cdot\alpha_3^\vee \\ &\equiv -\frac{1}{2}\alpha_3^\vee \mod \Gamma(T)\,,\end{aligned}$$

which is clearly impossible.

Case 4: The same arguments as in the previous case show that $X_4 = \frac{1}{2}(\alpha_1^\vee + \alpha_6^\vee)$ implies $X_2 = \frac{1}{2}(\alpha_1^\vee + \alpha_6^\vee)$. This case is then ruled out using $\ell_{23} = 2$ and $(*)$ as well.

Cases 5-6: By symmetry, we only consider $X_4 = \frac{1}{2}(\alpha_2^\vee + \alpha_3^\vee)$. This case is excluded by simply considering $\ell_{14} = 2$. Formula $(*)$ produces

$$\begin{aligned}(X_1, \alpha_4^\vee)\cdot\alpha_4^\vee &\equiv (X_4, \alpha_1^\vee)\cdot\alpha_1^\vee \\ &\equiv (\frac{1}{2}(\alpha_2^\vee + \alpha_3^\vee), \alpha_1^\vee)\cdot\alpha_1^\vee \\ &\equiv -\frac{1}{2}\alpha_1^\vee \mod \Gamma(T),\end{aligned}$$

which is clearly impossible.

Similarly, all the remaining cases are ruled out using a single Tits relation, and we simply indicate which one.

Case 7: Impossible by using $\ell_{14} = 2$.

Cases 8 and 12: It suffices to consider $X_4 = \frac{1}{2}(\alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee)$, which is impossible by using $\ell_{14} = 2$.

Cases 9 and 11: It suffices to consider $X_4 = \frac{1}{2}(\alpha_1^\vee + \alpha_2^\vee + \alpha_5^\vee)$, which is impossible by using $\ell_{46} = 2$.

Cases 10 and 13: It suffices to consider $X_4 = \frac{1}{2}(\alpha_1^\vee + \alpha_3^\vee + \alpha_5^\vee)$, which is impossible by using $\ell_{14} = 2$.

Cases 14-15: It suffices to consider $X_4 = \frac{1}{2}(\alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee + \alpha_6^\vee)$, which is impossible by using $\ell_{14} = 2$.

Case 16: Impossible by using $\ell_{14} = 2$.

□

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