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Université de Neuchâtel
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de mathématiques

Faculté des Sciences
Institut de mathématiques
Rue Emile-Argand 11, 2000 Neuchâtel

Analytic and algebraic aspects of solvable Baumslag-Solitar groups and of some crystallographic groups

Thèse

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par

Laurent Hayez

Jury :

Prof. Alain Valette, Université de Neuchâtel, directeur de thèse
Dr. Jérémie Brieussel, Université de Montpellier, rapporteur
Prof. Paul Jolissaint, Université de Neuchâtel, rapporteur
Dr. Ana Khukhro, EPFL, rapporteuse

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UNIVERSITÉ DE
NEUCHÂTEL

Faculté des Sciences
Secrétariat-décanat de Faculté
Rue Emile-Argand 11
2000 Neuchâtel – Suisse
Tél : + 41 (0)32 718 21 00
E-mail : secretariat.sciences@unine.ch

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Monsieur Laurent HAYEZ

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sur le rapport des membres du jury composé comme suit:

- Prof. Alain Valette, directeur de thèse, Université de Neuchâtel, Suisse
- Prof. tit. Paul Jolissaint, Université de Neuchâtel, Suisse
- Dr Anastasia Khukhro, EPF Lausanne, Suisse
- Dr Jérémie Brieussel, Université de Montpellier, France

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Abstract

The aspiration of this work is to further our understanding of some groups and their properties. The three parts therein are fairly distinct, even though two of them treat of the solvable Baumslag-Solitar groups.

The first part is concerned with the solvable Baumslag-Solitar groups $\mathrm{BS}(1,N) = \langle u,t | tut^{-1} = u^N \rangle$, $N \geq 2$. We study property D_α for these groups. Property D_α measures how close to linear the growth of the diameter of a sequence of graphs is: for $0 < \alpha \leq 1$, we say that a sequence $(X_k)_{k>0}$ of d-regular connected graphs has property D_α if there exists a constant $C > 0$ such that $\mathrm{diam}(X_k) \geq C \cdot |X_k|^\alpha$. We investigate this property for arithmetic box spaces of the groups $\mathrm{BS}(1,N)$: these are box spaces obtained by embedding $\mathrm{BS}(1,N)$ into the upper triangular matrices in $\mathrm{GL}_2(\mathbb{Z}[1/N])$ and intersecting with a family M_{N_k} of congruence subgroups of $\mathrm{GL}_2(\mathbb{Z}[1/N])$, where the levels N_k are coprime with N and $N_k | N_{k+1}$ for all $k \geq 1$. We prove that:

- if an arithmetic box space has D_α, then $\alpha \leq \frac{1}{2}$;
- if the family $(N_k)_k$ of levels is supported on finitely many primes, the corresponding arithmetic box space has $D_{1/2}$;
- if the family $(N_k)_k$ of levels is supported on a family of primes with positive analytic primitive density, then the corresponding arithmetic box space does not have $D_{\frac{1}{2}}$.

Next in order, we study wavelet representations π of $\mathrm{BS}(1,N)$. In particular, these representations include a notion of multiresolution analysis from which it is possible to obtain orthonormal wavelet bases on the representation space. We work with two representation spaces. The first one we consider is $L^2(\mathcal{R},\mu)$, where $\mathcal{R}$ is an inflated fractal set constructed from an iterated function system, and μ is the Hutchinson measure extended to $\mathcal{R}$. The second one is $L^2(\mathcal{S}_N, m)$ where $\mathcal{S}_N$ is an N-solenoid and m is a probability measure. Both representations are associated to the same low-pass filter $m_0 \in L^\infty(\mathbb{T})$ and to the same correlation function $h \in L^1(\mathbb{T})$, $h \geq 0$. This allows us to compare the two representations: we show that the two representations are equivalent for all $N \geq 2$. In addition, we show that on $L^2(\mathcal{S}_N, m)$, we can decompose the dilation operator $\pi(t) = T$ into partial isometries $(T_j)_{j=0}^{N-1}$. Futhermore, we compute the spectral measure of T, we construct a spectral measure on the set $(\mathbb{Z}/N\mathbb{Z})^{\mathbb{N}}$ and we prove that the canonical spectral measure on the solenoid is the product of the two previous spectral measures. Eventually, we compute the Fourier coefficients of the measure m for all $N \geq 2$.

The third and last part of the thesis is about computing Kesten spectral measure of wallpaper groups. We are able to compute it for ten of the seventeen wallpaper groups.

Keywords : Group theory, Cayley graph, solvable Baumslag-Solitar groups, wavelets, representations, Fourier coefficients, spectral measures, partial isometries, Kesten spectral measures, wallpaper groups.

Résumé

L'objectif de ce travail est d'améliorer notre compréhension de certains groupes et de leurs propriétés. Les trois parties qui le composent sont assez distinctes, même si deux d'entre elles traitent des groupes de Baumslag-Solitar résolubles.

Dans la première partie, nous étudions les groupes de Baumslag-Solitar résolubles $\mathrm{BS}(1,N) = \langle u,t | tut^{-1} = u^N \rangle$, $N \geq 2$, et plus précisément, nous nous intéressons à la propriété D_α pour ces groupes. La propriété D_α mesure à quel point la croissance du diamètre d'une suite de graphes est proche d'être linéaire : pour $0 < \alpha \leq 1$, une suite de graphes $(X_k)_{k>0}$ connexes d-réguliers, $d \geq 2$ possède la propriété D_α s'il existe une constante $C > 0$ telle que $\mathrm{diam}(X_k) \geq C \cdot |X_k|^\alpha$. Nous étudions cette propriété pour les box spaces arithmétiques des groupes $\mathrm{BS}(1,N)$: ces box spaces sont obtenus en plongeant $\mathrm{BS}(1,N)$ dans les matrices triangulaires supérieures de $\mathrm{GL}_2(\mathbb{Z}[\frac{1}{N}])$ et en l'intersectant avec une famille M_{N_k} de sous-groupes de congruence de $\mathrm{GL}_2(\mathbb{Z}[\frac{1}{N}])$, où les niveaux N_k sont premiers avec N, et tels que $N_k \mid N_{k+1}$ pour tout $k \geq 1$. Nous démontrons que :

- si un box space arithmétique a D_α, alors $\alpha \leq \frac{1}{2}$;
- si la famille $(N_k)_k$ de niveaux est supportée sur un nombre fini de nombres premiers, le box space correspondant a $D_{1/2}$;
- si la famille $(N_k)_k$ de niveaux est supportée sur une famille de nombres premiers de densité primitive analytique positive, le box space correspondant n'a pas $D_{1/2}$.

Dans la deuxième partie, nous étudions les représentations en ondelettes π des $\mathrm{BS}(1,N)$, qui contiennent la notion d'analyse multirésolution. Cette dernière notion nous permet d'obtenir des bases d'ondelettes orthonormales sur l'espace de représentation. Nous travaillons sur deux espaces de représentation. Le premier est $L^2(\mathcal{R},\mu)$, où $\mathcal{R}$ est un fractal "gonflé" construit à partir d'un système de fonctions itérées et μ est la mesure de Hutchinson étendue à $\mathcal{R}$. Le second est $L^2(\mathcal{S}_N, m)$ où $\mathcal{S}_N$ est un N-solénoïde et m est une mesure de probabilité. Ces deux représentations sont associées au même filtre passe-bas $m_0 \in L^\infty(\mathbb{T})$ ainsi qu'à la même fonction de corrélation $h \in L^1(\mathbb{T})$, $h \geq 0$. Ceci nous permet de comparer les deux représentations et de montrer qu'elles sont équivalentes pour tout $N \geq 2$. De plus, nous démontrons que sur $L^2(\mathcal{S}_N, m)$, l'opérateur $\pi(t) = T$ se décompose comme somme d'isométries partielles $(T_j)_{j=0}^{N-1}$. Nous calculons la mesure spectrale de T, construisons une mesure spectrale sur l'ensemble $(\mathbb{Z}/N\mathbb{Z})^{\mathbb{N}}$ et prouvons que la mesure spectrale canonique sur le solénoïde est le produit des mesures spectrales précédentes. Enfin, nous calculons les coefficients de Fourier de la mesure m pour tout $N \geq 2$.

La dernière partie de la thèse traite des mesures spectrales de Kesten des groupes cristallographiques en dimension 2. Nous calculons cette mesure pour dix des dix–sept groupes cristallographiques.

Mots clés : Théorie des groupes, graphes de Cayley, groupes de Baumslag-Solitar résolubles, ondelettes, représentations, coefficients de Fourier, isométries partielles, mesures spectrales, mesures spectrales de Kesten, groupes cristallographiques.

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1 Introduction

Two distinct parts cohabit in this thesis, yet the ambition remains the same: get a better understanding of some groups using various techniques to study these groups. In the first part, we focus on solvable Baumslag-Solitar groups, denoted by $\mathrm{BS}(1,n)$ for some $n \geq 2$, while in the second part, we direct our attention to the so-called wallpaper groups, or crystallographic groups.

We adopt two points of views to investigate Baumslag-Solitar groups. On the one hand, we embed them in linear groups and study the evolution of the diameter of some specific quotients of the embedding. This part mainly uses arithmetic techniques. The results in this chapter (see chapter 2) are obtained in collaboration with Alain Valette and Tom Kaiser and are in the process of being published. Note that the article contains further results than what is presented here, notably we discuss the effects of the choice of the embedding, which is a form of the *congruence subgroup property (CSP)*. On the other hand, we study *wavelet representations of* $\mathrm{BS}(1,n)$ which give an analytical point of view on Baumslag-Solitar groups. We are currently writing an article containing the results from this chapter (see chapter 3).

As for wallpaper groups, we are interested in their spectrum, and more specifically we compute the *Kesten spectral measures* for some of them. The results in this chapter (see chapter 4) are obtained in collaboration with Tom Kaiser.

1.1 Part I : arithmetic and analytical aspects of solvable Baumslag-Solitar groups

Can a finitely generated group be isomorphic to one of its proper quotient groups? This is the seemingly innocent question asked by topologist H. Hopf in 1932 [Mag04]. This question generated a substantial amount of research, so much that if a group does *not* possess this property, it is called **hopfian**, and should one group possess this property, it would then be called **non-hopfian**. If one stops here for a minute to think about this question a little deeper, one quickly realizes that this phenomenon would be very bizarre, almost absurd. Indeed, how could a group be isomorphic to an a priori strictly smaller subgroup? Yet, in 1950, B.H. Neumann exhibited such a group, with an infinite number of relations [Neu50] and asked whether every two-generators non-hopfian group is infinitely presented. The answer is no and was given by Higmann a year later [Hig51]: he shows that the same pathological pattern can happen with a finitely presented group with three generators and two relations. In 1962, G. Baumslag and D. Solitar were able to construct infinitely many two-generator one-relator groups that are isomorphic to a proper quotient in the very influential paper [BS62], thus answering negatively Neumann's question in a strong sense. In the aforementioned paper, they explicitly defined the groups that are now known as the Baumslag-Solitar groups:

$$\mathrm{BS}(m,n) = \langle u, t | t u^m t^{-1} = u^n \rangle \,. \tag{1.1}$$

It should be noted that the idea to use such a relation already appears in Higman's paper [Hig51] where the relation $tut^{-1} = u^2$ is present. This is the reason why G. Baumslag apparently complained that the group $\mathrm{BS}(1, 2)$, and more generally $\mathrm{BS}(m, n)$ is attributed to him and D. Solitar [Mei08]. As history shows, his efforts were vain. Tom Kaiser further explores the non-hopfianness of $\mathrm{BS}(2, 3)$ in his paper [Kai21].

From the moment they were introduced, Baumslag-Solitar groups served as great test beds for various properties in combinatorial and geometric group theory, often providing insightful examples or counter examples. The non-hopfianness of $\mathrm{BS}(2, 3)$ illustrates the previous statement, but I would like to sketch another example where Baumslag-Solitar groups arose as counter examples: in 1980, Mel'nikov asked [KM14, Problem 7.36]: "Is it true that every residually finite group in which every subgroup of finite index (including the group itself) is defined by a single defining relation is either free or isomorphic to the fundamental group of a compact surface?". According to the Kourovka notebook [KM14], Churkin observed that the solvable Baumslag-Solitar groups $\mathrm{BS}(1, n)$ are counter examples to Mel'nikov's question in 1982 already. Call a *Mel'nikov group* a non-free infinite one-relator group with every subgroup of finite index also a one-relator group. Without entering the details, one particular conjecture was made about surface groups [CFR13].

Conjecture 1.1 (SURFACE GROUP CONJECTURE)**.** *Let G be a residually finite Mel'nikov group. Then G is a surface group or* $\mathrm{BS}(1, n)$ *for some non-zero integer n.*

Gardam et al. proved that the conjecture holds, and more details concerning the surface group conjecture can be found in their paper [GKL22].

It is in this spirit of confronting properties to Baumslag-Solitar groups that we study them. The first property we investigate is denoted by D_α and measures the rate of growth of the diameter of a family of d-regular connected graphs. More precisely, for $0 < \alpha \leq 1$, we say that a sequence $(X_k)_{k>0}$ of d-regular connected graphs has property D_α if there exists a constant $C > 0$ such that $\mathrm{diam}(X_k) \geq C \cdot |X_k|^\alpha$. On the other hand, we study representations of Baumslag-Solitar groups that have wavelets, and in particular what is called a **wavelet representation**. Traditionally, a **wavelet** is defined to be a set $\{\psi_1, \dots, \psi_\ell\} \subset L^2(\mathbb{R})$ such that

$$\{\, T^j U^k \psi_i \mid j, k \in \mathbb{Z},\ i \in \{1, \dots, \ell\} \,\} \tag{1.2}$$

is an orthonormal basis for $L^2(\mathbb{R})$, where T and U are carefully chosen operators on $L^2(\mathbb{R})$. In this case, we can have a representation of the group $\mathrm{BS}(1, n)$ and the condition in eq. (3.1) translates to:

$$\left\{\, \pi(t^j u^k)\psi_i \;\middle|\; j, k \in \mathbb{Z},\ i \in \{1, \dots, \ell\} \,\right\} \tag{1.3}$$

shall be an orthonormal basis for $L^2(\mathbb{R})$ where $\pi : BS(1, n) \to \mathcal{U}(L^2(\mathbb{R}))$ is the corresponding representation.

For the moment, let us come back to property D_α, precise this notion for Cayley graphs and apply it to $\mathrm{BS}(1, n)$.

1.1.1 On arithmetic properties of solvable Baumslag-Solitar groups

Let G be a finitely generated, residually finite group. If $(H_k)_{k>0}$ is a decreasing sequence of finite index normal subgroups of G, with trivial intersection, and S is a finite generating set of G, then the **box space** $\boldsymbol{\square_{(H_k)}G}$ is the disjoint union of finite Cayley graphs

$$\square_{(H_k)}G = \bigsqcup_{k>0} \mathrm{Cay}(G/H_k, S);$$

here by abuse of notation we identify S with its image in G/H_k. Changing the generating set S does not change the coarse geometry of the box space[1] so we omit S from the notation.

In the dictionary between group-theoretical properties of G and metric properties of $\square_{(H_k)}G$ (see e.g. [Khu18]), it is natural to look at the behaviour of the diameter of the Cayley graphs $\mathrm{Cay}(G/H_k, S)$. Let $0 < \alpha \leq 1$. The box space $\square_{(H_k)}G$ satisfies **property** $\boldsymbol{D_\alpha}$ if there is some constant $C > 0$ such that for every $k > 0$:

$$\mathrm{diam}(\mathrm{Cay}(G/H_k, S)) \geq C|G/H_k|^\alpha. \tag{1.4}$$

Note that property D_α is a coarse geometry invariant of the box space. The following is known.

Theorem 1.2. *Let G be a finitely generated, residually finite group.*

1. *[BT16, see Corollary 1.7 and Lemma 5.1] If some box space of G has property D_α, for some $\alpha > 0$, then G virtually maps onto $\mathbb{Z}$.*

2. *[KV17, see Theorem 3] If G maps onto $\mathbb{Z}$, then for every $0 < \alpha < 1$, there exists a box space of G with property D_α.*

3. *[KV17, see Proposition 5] The group G is virtually cyclic if and only if some (hence any) box space of G has property D_1.*

As mentioned above, we consider the Baumslag-Solitar groups $\mathrm{BS}(m,n)$ $(m, n > 0)$. They all map onto $\mathbb{Z}$, by $u \mapsto 0, t \mapsto 1$. Moreover they are known to be residually finite if and only if $m = 1$ or $n = m$: [Mes72, Theorem C]. It turns out that the solvable Baumslag-Solitar groups $\mathrm{BS}(1,n)$, with $n \geq 2$, have interesting box spaces. Indeed it is well-known that $\mathrm{BS}(1,n)$ may be viewed as a semi-direct product

$$\mathrm{BS}(1,n) = \mathbb{Z}[1/n] \rtimes \mathbb{Z}$$

where the factor $\mathbb{Z}$ corresponds to the subgroup $\langle t \rangle$ acting by powers of n. We may identify this semi-direct product with the following subgroup G_n of upper triangular matrices in $\mathrm{GL}_2(\mathbb{Z}[1/n])$:

$$G_n = \left\{ \begin{pmatrix} n^k & r \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z},\ r \in \mathbb{Z}[1/n] \right\}.$$

---

[1] In the sense that the two families of graphs are quasi-isometrically equivalent, by a family of quasi-isometries with uniform constants.

The isomorphism is obtained by mapping u to $U = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and t to $T = \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The associated embedding of $\mathrm{BS}(1,n)$ into $\mathrm{GL}_2(\mathbb{Z}[1/n])$ is called the **standard embedding**.

In $\mathrm{GL}_m(\mathbb{Z}[1/n])$ we may define congruence subgroups: let $N > 0$ be coprime with n. The **principal congruence subgroup of level N** is the kernel M_N of the reduction modulo N:

$$M_N = \ker\left[\mathrm{GL}_m(\mathbb{Z}[1/n]) \to \mathrm{GL}_m(\mathbb{Z}/N\mathbb{Z})\right].$$

Definition 1.3. If G is any subgroup of $GL_m(\mathbb{Z}[1/n])$, and $N > 0$ is coprime to n, then the **congruence subgroup** $G(N)$ in G is

$$G(N) := G \cap M_N.$$

For a sequence of integers such that each one divides the next one, one obtains a sequence of nested congruence subgroups, and thus a box space of $\mathrm{BS}(1,n)$. Such box spaces deserve to be called **arithmetic box spaces**. We will study property D_α for the arithmetic box spaces of $\mathrm{BS}(1,n)$ through the standard embedding. From theorem 1.2, we know that for every $0 < \alpha < 1$, there exists a box space of $\mathrm{BS}(1,n)$ with property D_α, but what about arithmetic box spaces? We will prove that box spaces with D_α, for $\alpha > \frac{1}{2}$, can be distinguished from arithmetic box spaces by coarse geometry. More precisely:

Theorem 1.4 (SEE THEOREM 2.53). *For any $n \geq 2$, the following statements are true:*

1. *if an arithmetic box space $\square_{(G_n(N_k))_k} G_n$ has property D_α, then $\alpha \leq \frac{1}{2}$;*
2. *there exists an arithmetic box space of G_n with property $D_{1/2}$;*
3. *there exists an arithmetic box space of G_n without property D_α for any $\alpha \in]0, 1/2]$.*

In addition, we study how property D_α for an arithmetic box space depends on the prime factors of the N's in the sequence of congruence subgroups $(M_N)_N$. In fact, if we denote by $D'(P)$ the analytic primitive density of the prime factors (see Section 2.6.2), we prove the following.

Theorem 1.5 (SEE THEOREM 2.55). *Let $\square_{(G_n(N_k))_k} G_n$ be an arithmetic box space, and let P be the set of prime factors of the sequence $(N_k)_k$.*

1. *If $|P| < +\infty$, then $\square_{(G_n(N_k))_k} G_n$ has $D_{1/2}$;*
2. *If $D'(P) > 0$, then $\square_{(G_n(N_k))_k} G_n$ does not have $D_{1/2}$.*

These results are part of a paper in collaboration with Tom Kaiser and Alain Valette [HKV21], in which we additionally discuss the influence of the chosen embedding, naturally leading to a form of the *congruence subgroup property (CSP).*

1.1.2 Wavelet representations of Baumslag-Solitar groups on fractals

Baumslag-Solitar groups naturally appear in the context of wavelets. Indeed, recall that a wavelet is defined to be a set $\{\psi_1, \ldots, \psi_\ell\} \subset L^2(\mathbb{R})$ such that

$$\{\, T^j U^k \psi_i \mid j, k \in \mathbb{Z},\ i \in \{1, \ldots, \ell\} \,\}$$

is an orthonormal basis for $L^2(\mathbb{R})$, where $U : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ Uf(x) = f(x-1)$ is the translation operator, and $T : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ Tf(x) = \frac{1}{\sqrt{N}} f(\frac{x}{N})$ is the scaling operator. It turns out that these operators satisfy the relation

$$TUT^{-1} = U^N,$$

which is exactly the Baumslag-Solitar relation. Thus, when we deal with wavelets, we actually consider representations of the solvable Baumslag-Solitar group $\mathrm{BS}(1,N) = \langle u, t | tut^{-1} = u^N \rangle$. One of the questions that motivates this work is "are Fourier series typical for the Lebesgue measure, or are there other measures having orthogonal bases of exponential functions ?" [Dut09]. In other words, is there a measured space $(X, \mathcal{B}, \mu)$ different from $(\mathbb{R}, \mathrm{Bor}(\mathbb{R}), \lambda)$ such that we have two operators $U, T \in L^2(X,\mu)$ and functions $\{\psi_1, \dots, \psi_\ell\} \subset L^2(X,\mu)$ such that

$$\left\{ T^j U^k \psi_i \;\middle|\; j,k \in \mathbb{Z},\ i \in \{1, \dots, \ell\} \right\} \tag{1.5}$$

is an orthonormal basis for $L^2(X,\mu)$? In this context, it is natural to look at representations of $\mathrm{BS}(1,N)$ that have wavelets: for an arbitrary Hilbert space $\mathcal{H}$, if $\pi : \mathrm{BS}(1,N) \to \mathcal{U}(\mathcal{H}))$ is a unitary representation, a wavelet for the representation is then a finite set $\{\psi_1, \dots, \psi_\ell\} \subset \mathcal{H}$ such that

$$\left\{ \pi(t^j u^k) \psi_i \;\middle|\; j,k \in \mathbb{Z},\ i \in \{1, \dots, \ell\} \right\}$$

is an orthonormal basis for $\mathcal{H}$, $\pi(u) = U$ and $\pi(t) = T$ are respectively interpreted as a translation and scaling operator on $\mathcal{H}$.

The search for "good" wavelet bases in an ongoing research subject; here we only mention that the most useful tool to create orthonormal wavelet bases of $L^2(\mathbb{R})$ is what we call a Multiresolution Analysis (MRA). The definition is straightforward to adapt to a generic Hilbert space $\mathcal{H}$, the subtle task being to actually create one. To this end, we introduce the notion of wavelet representation of $\mathrm{BS}(1,N)$, which contains both the fact that we deal with a representation of $\mathrm{BS}(1,N)$ and the fact that we have a MRA, which ensures that we have an orthonormal wavelet base.

Dutkay and Jorgensen were able to prove that we can indeed have orthonormal wavelet bases on other spaces than $L^2(\mathbb{R})$. The framework they consider works particularly well for such constructions, it is that of fractals constructed through Iterated Function Systems (IFS) [DJ06b]. The IFS that they consider, and that we will consider as well are given by an endomorphism $\tau : [0,1] \to [0,1],\ \tau(x) = Nx \pmod 1$ and p of its N inverse branches $\tau_j(x) = \frac{x+j}{N}$ for $j = 0, \dots, N-1$. If we choose $1 \le p \le N$ of the N branches, we obtain a system $(\tau_{a_j})_{j=1}^p$ of p contractive maps where $(a_j)_{j=1}^p \subset \{0, \dots, N-1\}$ is a set of p distinct integers, and a result of Hutchinson [Hut81] tells us that there exists a unique compact subset $K \subset [0,1]$ and a unique probability measure μ on $[0,1]$ that is supported on K such that $K = \cup_{j=1}^p \tau_j(K)$ and

$$\mu = \frac{1}{p} \sum_{j=1}^{p} \mu \circ \tau_j^{-1}.$$

The set we consider is an inflated version of K which "covers" the real line:

$$\mathcal{R} := \left\{ \sum_{j\geq -m} b_j N^{-j} \;\middle|\; m \in \mathbb{Z}, b_j \in \{a_1, \dots, a_p\} \text{ for all but finitely many indices } j \right\}.$$

The measure μ can be extended to a Borel measure on $\mathcal{R}$ which we also call μ. Dutkay and Jorgensen studied wavelets on such fractals [DJ06b] and proved that there is a wavelet representation of $\mathrm{BS}(1, N)$ on $L^2(\mathcal{R}, \mu)$ associated to the low-pass filter $m_0 = \frac{1}{\sqrt{p}} \sum_{j=1}^{p} z^{a_j}$. The corresponding unitary operators on $L^2(\mathcal{R}, \mu)$ are $\hat{U} f(x) = f(x - 1)$ and $\hat{T} f(x) = \frac{1}{\sqrt{p}} f(\frac{x}{N})$.

Additionally, Dutkay constructed another wavelet representation of $\mathrm{BS}(1, N)$ associated to the same low-pass filter m_0, but on another representation space [Dut06]. Namely, he considered the N-solenoid

$$\mathcal{S}_N := \left\{ (z_n)_{n\in\mathbb{Z}} \mid z_{n+1}^N = z_n, \; |z_n| = 1 \text{ for } n \in \mathbb{Z} \right\}$$

and he constructed a probability measure m on $\mathcal{S}_N$ so that the representation space is $L^2(\mathcal{S}_N, m)$. He defines two operators U and T on $L^2(\mathcal{S}_N, m)$, where U is a multiplication operator and T is a multiplication composed with a shift. In the case $N = 3$, he is able to compute the Fourier coefficients of m, and proves that there is a unique isomorphism between the wavelet representations on $L^2(\mathcal{R}, \mu)$ and $L^2(\mathcal{S}_3, m)$. This isomorphism behaves similarly to a generalized Fourier transform in the sense that it maps the translation operator to the multiplication operator. Moreover, still in the case $N = 3$, Davison was able to decompose the operator T on $L^2(\mathcal{S}_3, m)$ as a sum of partial isometries, and compute the spectral measure of the operator U [Dav20].

We are able to generalize the previous results for any $N \geq 2$. More specifically, we prove the following.

Theorem 1.6 (SEE SECTION 3.3.1). *For $N \geq 2$, $T, T^* : L^2(\mathcal{S}_N, m) \to L^2(\mathcal{S}_N, m)$ can be decomposed as*

$$T = \sum_{j=0}^{N-1} T_j \quad \textit{and} \quad T^* = \sum_{j=1}^{N-1} T_j^*.$$

where $T_j : L^2(\mathcal{S}_N, m) \to L^2(\mathcal{S}_N, m)$ are linear, bounded operators and partial isometries for all $j = 0, ..., N - 1$, and they satisfy

$$\sum_{k=0}^{N-1} T_k T_k^* = \sum_{k=0}^{N-1} T_k^* T_k = \mathrm{Id}_{L^2(\mathcal{S}_N, m)}.$$

Furthermore, we also compute the spectral measure of the operator $U \in L^2(\mathcal{S}_N, m)$ for all $N \geq 2$. We denote by Δ a Borel set, χ_Δ is the indicator function and M_f is the multiplication operator by f, $\iota : [0, 1) \to \mathbb{T}, x \mapsto \mathrm{e}^{2\pi \mathrm{i} x}$. In addition, define $\sigma : \mathbb{Z}_N^{\mathbb{N}} \to \mathbb{Z}_N^{\mathbb{N}}$, $\sigma((\omega_1, \omega_2, \dots)) = (\omega_2, \omega_3, \dots)$ as well as the N inverse branches $\sigma_j : \mathbb{Z}_N^{\mathbb{N}} \to \mathbb{Z}_N^{\mathbb{N}}$, $\sigma_j((\omega_1, \omega_2, \dots)) = (j, \omega_1, \omega_2, \dots)$, $j \in \{0, \dots, N - 1\}$.

Theorem 1.7 (SEE THEOREM 3.45). *The map $E : \mathrm{Bor}(\mathbb{T}) \to L^2(\mathcal{S}_N, m)$ defined by*

$E(\Delta) = M_{\chi_\Delta}$ *is the unique spectral measure satifying*

$$E(\iota \circ \sigma^{-1} \circ \iota^{-1}(\Delta)) = \sum_{j=0}^{N-1} T_j E(\Delta) T_j^{-1},$$

and U *satisfies*

$$U = \int_{\mathbb{T}} z \mathrm{d}E.$$

Eventually, we compute the Fourier coefficients of the measure m. The wavelet representation we consider is associated to the low-pass filter $m_0(z) = \sum_{j=1}^{p} \frac{1}{\sqrt{p}} z^{a_j} \in L^\infty(\mathbb{T})$ where $(a_j)_{j=1}^p \subset \{0, \ldots, N-1\}$ consists of p distinct integers. From the $(a_j)_{j=1}^p$, we construct the following sets :

$$B := \{\, a_k - a_j \mid j, k \in \{1, \ldots, p\} \,\} \quad \text{and} \quad C := \{c_{-N+1}, c_{-N+2}, \ldots, c_{N-1}\}$$

where c_ℓ corresponds to the number of ways of obtaining ℓ from $a_k - a_j$, $j, k \in \{1, \ldots, p\}$.

Theorem 1.8 (SEE COROLLARY 3.56). *Let* $(a_j)_{j=1}^p \subset \{0, \ldots, N-1\}$ *be such that* $|b_k - b_j| > 1$ *for all* $b_k, b_j \in B$, $k \neq j$. *In this case we obtain that the Fourier coefficients of the measure* m *are*

$$\hat{m}(\lambda) = \begin{cases} \frac{1}{p^{\ell+1}} \prod_{j=0}^{\ell} c_{b_j} & \text{if } \lambda = \sum_{j=0}^{\ell} \frac{b_j}{N^j}, \quad b_j \in B \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

Moreover, in the case $N = 3$ *and* $a_1 = 0, a_2 = 2$ *we recover Dutkay's result [Dut06, Proposition 5.6].*

1.2 Spectral measures of certain wallpaper groups

For any locally finite, connected graph $\Gamma = (V, E)$, we can consider the Markov operator which acts on $\ell^2(V)$ by

$$Mf(v) = \frac{1}{\deg(v)} \sum_{u \sim v} f(u)$$

where deg(v) is the degree of $v \in V$, and $u \sim v$ means that $\{u, v\} \in E$. In particular, for the Cayley graph $\Gamma = \mathrm{Cay}(G, S) = (V(\Gamma), E(\Gamma))$ of a finitely generated group $G = \langle S | R \rangle$, $S = S^{-1} \subset G$, $|S| < +\infty$, the Markov operator takes the form

$$\begin{array}{rccl} M & : & \ell^2(G) & \rightarrow & \ell^2(G) \\ & & f & \mapsto & Mf(x) = \frac{1}{|S|} \sum_{s \in S} f(s^{-1}x). \end{array} \tag{1.7}$$

In this setting, the Markov operator induces a symmetric random walk on G: denote by $\delta_v \in \ell^2(G)$ the vector defined by $\delta_v(u) = 1$ if $u = v$, and 0 otherwise. The random walk is a sequence $(X_n)_n$ of random variables which take values in G, defined by

$$\mathbb{P}[X_n = v \mid X_k = u_k,\ k = 0, \ldots, n-1] = \mathbb{P}[X_n = v \mid X_{n-1} = u] = M_{u,v} \quad \forall u, v \in G.$$

Therefore, X_n gives us the random position after n steps, and the transition probabilities are given by $M_{u,v} = \frac{1}{|S|}$ if $v = us$ for some $s \in S$, and 0 otherwise.

Moreover, M is a linear, bounded operator ($\|M\| \leq 1$), and it is self-adjoint. Therefore its spectrum $\mathrm{Sp}(M)$ is real and contained in $[-1, 1]$, and there exists a spectral measure $E : \mathrm{Bor}(\mathrm{Sp}(M)) \to \mathcal{B}(\ell^2(G))$, where $\mathrm{Bor}(\mathrm{Sp}(M))$ denotes the Borel sets of $\mathrm{Sp}(M)$, which satisfies

$$M = \int_{\mathrm{Sp}(M)} \lambda \mathrm{d}E(\lambda).$$

The spectral measure E gives rise to an infinite family of measures on $\mathrm{Sp}(M)$, namely the measures

$$\mu_{u,v}(B) = \langle E(B)\delta_u | \delta_v \rangle, \quad B \in \mathrm{Bor}(\mathrm{Sp}(M)).$$

These measures satisfy in particular

$$\int_{\mathrm{Sp}(M)} \lambda^n \mathrm{d}\mu_{u,v}(\lambda) = \langle M^n \delta_u | \delta_v \rangle$$

which is the probability to go from u to v in n steps according to the random walk. We will be specifically interested in the measures $\mu_u := \mu_{u,u}$, which define a probability measure on $\mathrm{Sp}(M)$. For a vertex transitive graph such as a Cayley graph, the choice of u is irrelevant, thus we will consider μ_e, which is often called the **Kesten spectral measure** or the **Plancherel spectral measure**. The study of the spectrum of Markov operators was initiated by Kesten in 1959 [Kes59b]. He was the first to consider random walks on non-commutative groups, and to compute the spectrum of the associated Markov operator. In particular, he computes the spectrum of the simple random walk (or equivalently the spectrum of the Markov operator M) on the free group $\mathbb{F}_d$ with $1 \leq d < +\infty$ generators. Mohar and Woess wrote a detailed survey about spectra of infinite graphs in 1989 [MW89]. Some examples are mentionned in [MW89, chapter 7], but so far the amount of explicit examples of Kesten spectral measures we know is rather limited.

Understanding Kesten spectral measure can give greater insights about the groups we study and their Cayley graph. Indeed, it is easy to create non-isomorphic groups that are isospectral. For instance, $\mathrm{Sp}(\mathbb{Z}^n) = [-1, 1]$ for all $n \geq 1$, yet $\mathbb{Z}^n$ is not isomorphic to $\mathbb{Z}^m$ if $n \neq m$. However, the Kesten spectral measure gives us the repartition of the elements of the spectrum, and it is an open question whether it is possible to determine the Cayley graph of a finitely generated group G up to isometry from the (Kesten) spectral measure μ of the associated Markov operator [DG20, see Question 1].

In the quest of understanding completely Kesten spectral measure, we compute them for some wallpaper groups. Recall that the isometries of $\mathbb{R}^2$ form a group $\mathrm{Isom}(\mathbb{R}^2) = T \rtimes O_2$ where T is the subgroup of translations and O_2 is the subgroup of orthogonal transformations acting on T by conjugation. A wallpaper group is a discrete subgroup $G \leq \mathrm{Isom}(\mathbb{R}^2)$ that has the form

$$G = H \rtimes J$$

where $H := G \cap T$ is the translation subgroup of G, generated by two independent translations, and J is a finite group called the point group, acting on H by conjugation. Geometrically, H is a lattice, and depending on the form of the lattice, the action of J is different. Counting the different types of lattice and the different ways the point group can act on the lattices, we obtain the classification of the wallpaper groups, which amounts to

17 non-isomorphic finitely generated groups. This classification was first mathematically performed by Fedorov in 1891 [Fed91].

The cornerstone for our computations is the Kesten spectral measure of $\mathbb{Z} = \langle t \rangle = \langle \{\pm 1\} \rangle$. Of interest are some computations of exotic examples where $\mathbb{Z}$ has other generators than just $\{\pm 1\}$, for example:

Proposition 1.9 (SEE PROPOSITION 4.25). *The Kesten spectral measure of* $\Gamma = \mathrm{Cay}(\mathbb{Z}, \{\pm 1, \pm 2\})$ *is given by*

$$\mathrm{d}\mu_\Gamma(x) = \frac{4\mathrm{d}x}{\pi\sqrt{8x+\frac{9}{2}}} \left(\frac{1}{\sqrt{-8x-\sqrt{16x+9}+3}} \chi_{\left[\frac{-9}{16},0\right]}(x) + \frac{1}{\sqrt{-8x+\sqrt{16x+9}+3}} \chi_{\left[\frac{-9}{16},1\right]}(x) \right). \tag{1.8}$$

We often find that the Cayley graphs of the groups we consider have a structure isomorphic to that of $\mathrm{Cay}(\mathbb{Z}^2) \times \mathrm{Cay}(F)$ where $\mathbb{Z}^2$ is generated by the standard generators and relations, and F is a finite group, so that the Kesten spectral measure is a weighted sum of the Kesten spectral measure of $\mathbb{Z}^2$. We mention for instance:

Proposition 1.10 (SEE COROLLARY 4.23). *The Kesten spectral measure of*

$$cm = \langle \tau_1, \tau_2, \sigma | [\tau_1, \tau_2] = e, \tau_1^\sigma = \tau_2, \tau_2^\sigma = \tau_1, \sigma^2 = e \rangle .$$

is given by

$$\mu_{cm}(\Delta) = \frac{1}{2}\mu_{Z^2}((\Delta + 1) \cap [-1, 1]) + \frac{1}{2}\mu_{Z^2}((\Delta - 1) \cap [-1, 1]) \quad \forall \Delta \in \mathrm{Bor}([-1, 1]).$$

Finally, we also obtain as a corollary of our computations that some (non-isomorphic) wallpaper groups are isospectral, for instance cm, pm and $p2$, or $c2mm$, $p2mm$ and $p4$.

1.3 Organisation of the thesis

The thesis is organized in a similar fashion to the introduction. Chapter 2 is concerned with studying property D_α for the solvable Baumslag-Solitar groups. We start with some well-known background, and we prove some general facts about Baumslag-Solitar groups that are reused in Chapter 3. We then dive into the results in section 2.6, and conclude with some open questions in section 2.6.3.

In chapter 3, we consider wavelet representations of solvable Baumslag-Solitar groups. We introduce some necessary notions in section 3.2. Afterwards, we expose our different results in section 3.3. Section 3.3.4 concludes this chapter with an open question.

Finally, we compute the Kesten spectral measures of some wallpaper groups in chapter 4, where we first recall facts about wallpaper groups and Kesten spectral measures in section 4.2, and then perform the computations group by group in section 4.3.

2 On arithmetic properties of solvable Baumslag-Solitar groups

This chapter is based on our paper [HKV21], which is a collaboration with Tom Kaiser and Alain Valette.

2.1 Introduction

During the cold war, communication and particularly telecommunication was paramount for both blocks. This naturally raises the question *how can one have efficient telecommunication ?* Instinctively, we would like a telecommunication network which is well-connected, yet without too much links to minimize the costs. Utopia, you may think. However, both blocks theorized and proved that such networks exist: Barzdin and Kolmogorov in 1967 already [KB67], and Pinsker independently in 1973 [Pin73][1]. The two proofs are non-constructive, but Margulis is able to explicitly construct expanders [Mar73], giving hope to have efficient telecommunication networks in practice.

Since then, expanders have found their way into various applications, especially in computer science where they are used, for example, to model very efficient networks to spread information [DFS09, FP13, CRR16], to create efficient error correcting codes [DEL$^+$21], or they also appear in potential new cryptosystems [DFJP14] to mention only a few. A survey of expanders and their applications can be found in [HLW06].

Expanders have also sneaked into the landscape of pure mathematics, and particularly in the realm of (geometric) group theory (more examples and directions can be found in [Lub12]). Before giving one such example, let us introduce one definition of expanders. Let X be a finite, d-regular ($d \geq 2$) connected graph on n vertices, and denote by A its adjacency matrix. It is an $n \times n$ real-valued symmetric matrix, thus its spectrum $d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ is discrete and real. For $\varepsilon > 0$, we say that X is an ε-expander if $d - \lambda_2 \geq \varepsilon$. In practice, we don't consider one graph, but rather an infinite family of d-regular graphs (X_n) with $n \to \infty$ and we fix $\varepsilon > 0$. We say that such a family $(X_n)_n$ is a **family of expanders** or an **expanding family**[2] if there exists $\varepsilon > 0$ such that X_n is an ε-expander for all n.

The celebrated Baum-Connes conjecture, which is still open to this day, led to another related conjecture, the Baum-Connes conjecture for groups, with coefficients in commu-

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[1] I first learned about the paper [KB67] in a seminar given by H. Helfgott. Usually, the existence of expanders is attributed to Pinsker, and the explicit construction is attributed to Margulis [Mar73]. Gromov makes a similar observation [Gro10, p. 491]. It is interesting to see Pinsker and Margulis cite each other in their papers, yet they did not seem aware of the work of Barzdin and Kolmogorov, which is probably why their paper did not gain as much traction as the others. According to H. Helfgott, the name "expander" or "expanding graph" is not found in [KB67] and is due to Pinsker.

[2] This definition of expanders is sometimes referred to as the "spectral definition" of expanders. This is, in our opinion, the simplest definition to give, without having to introduce too much details. There are different ways to define expanders and we will give another definition of expanders later. For more equivalent definitions and the proofs that they are equivalent, see [Kow19].

tative C^*-algebras. Without entering the details, the conjecture states that a map μ between two spaces is an isomorphism. Gromov constructed certain groups, now called *Gromov's monsters*, which contain a weakly embedded expander, thus providing an obstruction to coarsly embed them in Hilbert spaces. Using this, Higson, Lafforgue and Skandalis [HLS02, see Section 7] were able to obtain a counterexample for the Baum-Connes conjecture with coefficients. More details on the Baum-Connes conjecture as well as this counterexample can be found in the recent survey [AJV19]. An introduction to the subject can be found in the PhD thesis of my colleague [Zum22].

Since expanders appear in numerous fields, it would be convenient to have a simple tool to distinguish graph sequences that may be expanders from graph sequences that are not expanders for sure. The diameter of graphs is a good indicator in this regard. Indeed, expanders are well-connected and every point are "close" in the sense that the diameter of an expander is logarithmic in the number of vertices [KS11]. Thus, if the diameter of a sequence of graphs grows faster than logarithmically in the number of vertices, we know that we are not in presence of an expander. This is exactly what property D_α measures: for $0 < \alpha \leq 1$, we say that a sequence $(X_k)_{k>0}$ of d-regular connected graphs has property D_α if there exists a constant $C > 0$ such that $\operatorname{diam}(X_k) \geq C|X_k|^\alpha$. Some expander families come from groups, and we have a neat procedure to describe a group G geometrically, through the use a of graph called the **Cayley graph** of G, which we denote by Γ for the moment. If G is in particular residually finite, we can approximate Γ through sequences $(\Gamma_j)_j$ which correspond to the Cayley graphs of the quotients G/H_j, where $(H_j)_j$ is a sequence of finite index normal subgroups of G whose intersection is trivial. Such constructions can yield exotic properties and we can use the group structure to manipulate the geometry of the sequence $(\Gamma_j)_j$. The coarse union $\sqcup_j \Gamma_j$ is what we will call the **box space** $\square_{(H_j)}G$.

The solvable Baumslag-Solitar groups $\mathrm{BS}(1,m) = \langle u, t | tut^{-1} = u^m \rangle$, $m \geq 2$ are residually finite, and can be embedded in $\mathrm{GL}_2(\mathbb{Z}[1/m])$ through an isomorphism which we call the **standard embedding** that maps u to $U = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and t to $T = \left(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Let us call $G_m \leq \mathrm{GL}_2(\mathbb{Z}[1/m])$ the group obtained this way (which is isomorphic to $\mathrm{BS}(1,m)$). We construct specific box spaces of G_m, namely **arithmetic box spaces** as follows: if $N > 0$ is coprime to m, then the **congruence subgroup** $G_m(N)$ in G_m is

$$G_m(N) := G_m \cap \ker\left[\mathrm{GL}_n(\mathbb{Z}[1/m]) \to \mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z})\right].$$

Finally, considering a sequence of integers $(N_j)_j$ such that $N_j \mid N_{j+1}$ and N_j is coprime to m for all j, we consider $\square_{(G_m(N_j))_j} G_m$ which is a particular box space of $\mathrm{BS}(1,m)$ that we call an arithmetic box space. We study property D_α for these box spaces. The following theorem follows from combining results of [BT16] and [KV17].

Theorem 2.1. *Let G be a finitely generated, residually finite group.*

1. *[BT16, see Corollary 1.7 and Lemma 5.1] If some box space of G has property D_α, for some $\alpha > 0$, then G virtually maps onto $\mathbb{Z}$.*
2. *[KV17, see Theorem 3] If G maps onto $\mathbb{Z}$, then for every $0 < \alpha < 1$, there exists a box space of G with property D_α.*
3. *[KV17, see Proposition 5] The group G is virtually cyclic if and only if some (hence any) box space of G has property D_1.*

This theorem shows that for every $0 < \alpha < 1$, there exists some box space of $\mathrm{BS}(1, m)$ with property D_α. We show that the arithmetic box spaces are distinguishable from box spaces with property D_α when $\alpha > \frac{1}{2}$.

Theorem 2.2. *For any $m \geq 2$, the following statements are true:*

1. *if an arithmetic box space $\Box_{(G_m(N_j))_j} G_m$ has property D_α, then $\alpha \leq \frac{1}{2}$;*
2. *there exists an arithmetic box space with property $D_{1/2}$;*
3. *there exists an arithmetic box space of G_m without property D_α for any $\alpha \in]0, 1/2]$.*

Furthermore, we show that the prime factors P in the sequence $(N_j)_j$ have an influence on property D_α. More specifically, denoting by $D'(P)$ the analytic primitive density of P, we prove:

Theorem 2.3. *Let $\Box_{(G_m(N_j))_j} G_m$ be an arithmetic box space, and let P be the set of prime factors of the sequence $(N_j)_j$.*

1. *If $|P| < +\infty$, then $\Box_{(G_m(N_j))_j} G_m$ has $D_{1/2}$;*
2. *If $D'(P) > 0$, then $\Box_{(G_m(N_j))_j} G_m$ does not have $D_{1/2}$.*

To prove these theorems, we will start with some background on geometric group theory in section 2.2. We recall facts about the semi-direct product structure of groups surjecting onto $\mathbb{Z}$, we introduce the notion of diameter and quasi-isometry. Furthermore, we introduce expanders and remind that they have a logarithmic diameter. We then go on to present box spaces and property D_α in section 2.3. In section 2.4 we introduce the notions we need for this article about Baumslag-Solitar groups in order to study their arithmetic box spaces. We conclude the background with section 2.5 by recalling some well-known facts about elementary number theory, especially concerning the multiplicative order of m in $\mathbb{Z}/N\mathbb{Z}$.

We then study property D_α for solvable Baumslag-Solitar groups in section 2.6. We begin with some metric aspects, where we estimate the diameter of the arithmetic box spaces of $\mathrm{BS}(1, m)$, which is what we use to prove theorem 2.2 (see theorem 2.53). To conclude section 2.6, we investigate the role of prime numbers in property D_α and we prove theorem 2.3 (see theorem 2.55).

2.2 A crash course in geometric group theory

The principal references for this section are J. Meier's book [Mei08], Krebs and Shaheen's book [KS11] as well as the notes from a class that I attended during my Master [Hay15].

I assume that the reader is familiar with group theory and most of the material presented here. My goal is not to make a self-contained introduction to geometric group theory, but rather present the results used later and fix notation. For a more in-depth introduction to this subject, I recommend Meier's book [Mei08].

2.2.1 The semi-direct product and groups surjecting onto $\mathbb{Z}$

One goal later is to prove a theorem for the property D_α for groups surjecting onto $\mathbb{Z}$. Thus we introduce the needed definitions here. In particular, we introduce tools to identify groups that are isomorphic to semi-direct products.

Definition 2.4 (Group surjecting onto $\mathbb{Z}$)**.** A group G **surjects onto** $\mathbb{Z}$ if there exists a surjective homomorphism $\varphi\colon G \to \mathbb{Z}$.

Definition 2.5 (Semi-direct product)**.** Let G_1, G_2 be two groups, and $\varphi\colon G_2 \to \operatorname{Aut}(G_1)$, $g_2 \mapsto \varphi_{g_2}$ be a homomorphism (i.e. φ is an action of G_2 on G_1). The **semi-direct product** of G_1 and G_2 is denoted by $G = G_1 \rtimes_\varphi G_2$ (or $G_1 \rtimes G_2$ if it is clear what φ is) and is the group whose elements are $G_1 \times G_2$, and whose internal operation is

$$(g_1, g_2).(g_1', g_2') = (g_1\varphi_{g_2}(g_1'), g_2g_2').$$

The neutral element in a semi-direct product is $e = (e_{G_1}, e_{G_2})$ and the inverse of (g_1, g_2) is $(g_1, g_2)^{-1} = (\varphi_{g_2^{-1}}(g_1^{-1}), g_2^{-1})$. Both of these assertions are easily verified.

We introduce the following notation for conjugation $g^h := hgh^{-1}$. The next proposition allows us to recognize semi-direct products.

Proposition 2.6. *Let G be a group, and let $H_1, H_2 \leq G$ be subgroups such that*

1. $H_1 \lhd G$;
2. $H_1 \cap H_2 = \{e\}$;
3. $G = H_1H_2 = \{h_1h_2 \mid h_1 \in H_1,\ h_2 \in H_2\}$.

Define $\varphi\colon H_2 \to \operatorname{Aut}(H_1)$, $h_2 \mapsto \varphi_{h_2}(h_1) = h_1^{h_2}$. In this case,

$$G \simeq H_1 \rtimes_\varphi H_2. \tag{2.1}$$

Proof. Since $G = H_1H_2$ and $H_1 \cap H_2 = \{e\}$, remark that every $g \in G$ has a unique representation $g = h_1h_2$ with $h_i \in H_i$, $i = 1, 2$. The natural isomorphism thus is

$$\psi\colon G \to H_1 \rtimes_\varphi H_2,\ g = h_1h_2 \mapsto (h_1, h_2).$$

Since the representation of $g \in G$ is unique, ψ is injective. Surjectivity results from the fact that every $g \in G$ has such a representation. Finally, the fact that ψ is a homomorphism is readily checked. ■

We use Proposition 2.6 to demonstrate the rigidity that comes with the hypothesis that a group surjects onto $\mathbb{Z}$.

Proposition 2.7. *G is a group that surjects itself onto $\mathbb{Z}$ if and only if there exists $H \lhd G$ such that $G \simeq H \rtimes \mathbb{Z}$.*

Proof. By definition, there exists a surjective homomorphism $\varphi\colon G \to \mathbb{Z}$. Let $H := \ker\varphi$. We immediately have that $H \lhd G$. Moreover, from the surjectivity, there exists $t \in G$ such that $\varphi(t) = 1$. Note that for any $k \in \mathbb{Z}$, $\varphi(t^k) = k$, thus $\langle t\rangle = \mathbb{Z}$. Note also that $H \cap \langle t\rangle = \{e\}$. Let now $g \in G$, then $\varphi(g) = k$ for some $k \in \mathbb{Z}$. We have that $gt^{-k} \in \ker\varphi$, and thus $g = gt^{-k} \cdot t^k$ with $gt^{-k} \in H$ and $t^k \in \langle t\rangle$. Since we can do this for all $g \in G$, we have that $G = H\langle t\rangle$. By Proposition 2.6, we conclude that $G \simeq H \rtimes \mathbb{Z}$.

Conversely, assume that there exists $H \lhd G$ such that $G \simeq H \rtimes \mathbb{Z}$. We need to show that there exists a surjective homomorphism $\pi\colon G \to \mathbb{Z}$. Define $\pi\colon H \rtimes \mathbb{Z} \to \mathbb{Z}$, $(h, k) \mapsto k$. Since π is a projection, it is a surjective homomorphism. ■

2.2.2 Geometric group theory

As explained in the introduction above, we have a neat way to geometrically represent a group G by a graph. In this subsection, the graphs we consider are arbitrary.

Definition 2.8 (Graph)**.** A **graph** $\Gamma = (V, E)$ is an ordered pair of disjoint sets (V, E) such that $E \subset V \times V$. The set $V = V(\Gamma)$ is the set of vertices of Γ, and $E = E(\Gamma)$ is the set of edges. An edge $\{x, y\} \in E$ is said to link x and y and is denoted by xy. Thus xy and yx denote the same edge, and we say that x and y are adjacent, or neighbors.

Definition 2.9 (Generating set)**.** If G is a group and $S \subset G$, we say that S **generates** G if every element of G can be expressed as a product of elements from S. The generating set S is symmetric if $S = S^{-1}$ and G is finitely generated if $|S| < +\infty$.

From now on, all the groups we will consider will be finitely generated with a symmetric generating set.

Definition 2.10 (Cayley graph)**.** The **Cayley graph** of G with respect to S, denoted by $\mathrm{Cay}(G, S)$ is the graph with vertex set $V = G$ and edge set $E = \{(g, gs) \mid g \in G,\ s \in S\}$.

The Cayley graph of G can be turned into a metric space. The **length** of a word $g \in G$ is the distance from g to ε (the empty word) in $\mathrm{Cay}(G, S)$, i.e.

$$|g|_S := \min\{n \in \mathbb{N} \mid g = s_1 \cdots s_n,\ s_i \in S\}.$$

For $g, h \in G$, the distance from g to h is the distance from g to h in $\mathrm{Cay}(G, S)$ and is denoted by $d_S(g, h) = |g^{-1}h|_S$. The generating set plays a role in the definition $\mathrm{Cay}(G, S)$ as a metric space, however if one changes S for another finite generating set T, the resulting spaces are not too different from one another, in the sense that they are “almost the same”. Intuitively, it means that both spaces are the same “up to small perturbations”. To make it precise, we say that two metric spaces (X, d_X) and (Y, d_Y) are **quasi-isometric** if there exists a map $f : X \to Y$ and a constant $C > 0$ such that

$$\frac{1}{C} d_X(x, x') - C \leq d_Y(f(x), f(x')) \leq C d_X(x, x') + C$$

for all $x, x' \in X$ and such that for all $y \in Y$, there exists $x \in X$ with $d_Y(y, f(x)) \leq C$. The map $f : X \to Y$ is called a **quasi-isometry**. The idea that the Cayley graphs $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, T)$ are quasi-isometric (with their respective word metrics) is due to the fact that every word in S can be written as a word in T and vice-versa, so the same remains true for words in G. Thus, we will morally study the properties of the quasi-isometry class of the Cayley graph of G. We summarize the previous discussion in the next proposition.

Proposition 2.11. *Let S, T be two finite generating sets of G. Then* $\mathrm{Cay}(G, S)$ *is quasi-isomorphic to* $\mathrm{Cay}(G, T)$*.*

Examples 2.12.

1. $\mathrm{Cay}(\mathbb{Z}, \{\pm 1\})$ is quasi-isometric to $\mathrm{Cay}(\mathbb{Z}, \{\pm 2, \pm 3\})$.
2. Every bounded space is quasi-isometric to a point (if you look at the space from far enough, it is similar to a point).
3. $f : \mathbb{R} \to \mathbb{Z}$, $x \mapsto \lfloor x \rfloor$ is a quasi-isometry.
4. "Being quasi-isometric" is an equivalence relation among metric spaces. ★

One interesting property of Cayley graphs is how large they are. One possible measure for this is the diameter of a graph, which is the distance between the farthest vertices of the graph.

Definition 2.13 (Diameter)**.** The **diameter** of a graph Γ is a positive integer or $+\infty$ and defined by

$$\mathrm{diam}(\Gamma) = \max_{v,v' \in V} \{d(v, v')\}$$

(with the word metric). In the case of a Cayley graph, we have

$$\mathrm{diam}(\mathrm{Cay}(G, S)) = \max_{g \in G} \{|g|_S\}.$$

Using Proposition 2.11, we immediately obtain that the diameter does not depend *too much* on the generating set. Taking

$$L = \max \left\{ \max\{|s|_T \mid s \in S\},\ \max\{|t|_S \mid t \in T\} \right\},$$

we have

$$\frac{1}{L} \mathrm{diam}(\mathrm{Cay}(G, S)) \leq \mathrm{diam}(\mathrm{Cay}(G, T)) \leq L \,\mathrm{diam}(\mathrm{Cay}(G, S)). \tag{2.2}$$

The constant L depends only of G. Thus we denote $\mathrm{diam}(G)$ rather than $\mathrm{diam}(\mathrm{Cay}(G, S))$.

It would be reasonable to think that, if a group surjects onto another group, the diameter of the first one grows at least as fast as the diameter of the latter. The next lemma shows that this is indeed the case.

Lemma 2.14. *Let G, H be two finitely generated groups such that G surjects onto H. In this case, there exists $L > 0$ such that* $\mathrm{diam}(G) \geq L \cdot \mathrm{diam}(H)$*.*

Proof. Let S be a (symmetrical) generating set for G. A well-known result is that if $\varphi\colon G \to H$ is a surjective homomorphism, then $\varphi(S)$ is a (symmetrical) generating set for H. To prove the diameter inequality, observe that if $h \in H$ realizes the diameter of H and $g \in G$ is such that $\varphi(g) = h$, we have

$$\mathrm{diam}(\mathrm{Cay}(H, \varphi(S))) = |h|_{\varphi(S)} \leq |g|_s \leq \mathrm{diam}(\mathrm{Cay}(G, S)). \qquad \blacksquare$$

Our goal is to study the diameter of groups that surject onto $\mathbb{Z}$. The following lemma helps us in this direction.

Lemma 2.15. *If $G = H_1 \rtimes H_2$ is a finitely generated group, there exists a finite generating set $S = T_1 \cup T_2$ with $T_1 \subset H_1$ and $T_2 \subset H_2$.*

Proof. The following short exact sequence splits :

$$1 \to H_1 \xrightarrow{\iota} G \xrightarrow{p} H_2 \to 1,$$

(where ι is the canonical injection and p is the projection onto the second factor), Thus, writing $H_1' = H_1 \times \{e_{H_2}\}$ and $H_2' = \{e_{H_1}\} \times H_2$, we obtain $G = H_1'H_2'$. Let S' be any finite generating set of G. Thus any $s' \in S'$ is written as $(h_1, e_{H_2})(e_{H_1}, h_2)$. We define $T_1 := \{h_1 \in H_1 \mid \exists s' \in S',\ s' = (h_1, e_{H_2})(e_{H_1}, h_2)\}$ and T_2 similarly. Let $S = T_1 \cup T_2$. By construction $\langle S \rangle = G$. ■

Combining the previous results we immediately obtain :

Theorem 2.16. *Let $G = H \rtimes \langle t \rangle$ be a finitely generated group ($\langle t \rangle$ is a cyclic group). Then there exists $L > 0$ such that $\operatorname{diam}(G) \geq L|\langle t \rangle|$. In particular, if $S = T \cup \{t\}$ is a generating set for G with $T \subset H$, we have*

$$\operatorname{diam}(\operatorname{Cay}(G, S)) \geq \frac{1}{2}|\langle t \rangle|. \tag{2.3}$$

Proof. From lemma 2.14, we only need to prove Equation (2.3). Let $S = T \cup \{t\}$ be a generating set for G with $T \subset H$. From Proposition 2.7, there exists a surjective homomorphism $\varphi\colon G \to \langle t \rangle$. Using the proof of Lemma 2.14 and that $\varphi(S) = \{t\}$, we obtain that $\operatorname{diam}(\operatorname{Cay}(G, S)) \geq \operatorname{diam}(\langle t \rangle, \{t\}) \geq \frac{1}{2}|\langle t \rangle|$[a]. ■

---

[a] It t has finite order, the diameter is really $\frac{1}{2}|\langle t \rangle|$ (+1 if the order is odd), while if t has infinite order, we have a group isomorphic to $\mathbb{Z}$ and the diameter is infinite.

2.2.3 Expanders

In this subsection, we are interested in families of finite d-regular graphs $(\Gamma_n)_n$ that are growing, and more specifically in how the diameter of each graph grows with respect to the family. If Γ is a graph, and A is a subset of $V(\Gamma)$, the **boundary** of A, written ∂A is the set of edges that connects A to $V(\Gamma) \setminus A$. Viewing Γ as a communication network, ∂A represents the set of edges to cut in order to disconnect A from the network. Intuitively, a well connected graph should have the property that a big subset of vertices should be hard to disconnect, or that ∂A is big when A is big. The Cheeger constant formalises this idea. When it is big, we know that our graph is well connected, and when it is small, we know that our graph can be easily disconnected.

Definition 2.17. The **Cheeger constant** (or isoperimetric constant) of a graph Γ is

$$h(\Gamma) := \inf_{\emptyset \neq A \subset V \neq V,\ |A| < \infty} \frac{|\partial A|}{\min\{|A|, |V \setminus A|\}}.$$

We will use it to define expanders, and their link with the growth of the diameter.

Remark 2.18 (ASYMPTOTIC NOTATIONS). We will use the Bachmann-Landau's notations $\mathcal{O}, o, \Omega, \Theta$ for the asymptotics, as defined by Knuth [Knu76]. Let f, g be real-valued functions :

- $\mathcal{O}(g)$ denotes the set of all f such that there exists $C, N_0 > 0$ such that $|f(x)| \leq Cg(x)$ for all $x \geq N_0$. In some computations, we will write $f = \mathcal{O}(g)$ as a shortcut, with the understanding that for x large enough, f is of order at most g, and that one could do the same computations with $C \cdot g$ for x large enough.
- $o(g)$ denotes the set of all f such that $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = 0$. As above, we will write $f = o(g)$.
- $\Omega(g)$ denotes the set of all f such that there exists $C, N_0 > 0$ such that $f(x) \geq Cg(x)$ for all $x \geq N_0$. Again, we write $f = \Omega(g)$.
- $\Theta(g)$ denotes the set of all f such that there exists $C, C', N_0 > 0$ such that $Cg(x) \leq f(x) \leq C'g(x)$ for all $x \geq N_0$. We also use the notation $f = \Theta(g)$.

Since our interest lies in finitely generated groups, the graphs we consider are $|S|$-regular. We have that for a family of finite d-regular connected graph $(\Gamma)_n$ with $d \geq 3$ and $|V(\Gamma_n)| \to \infty$, the diameter grows *at least* logarithmically in the number of vertices.

Theorem 2.19 ([KS11, EXAMPLE 4.4]). *Let $(\Gamma_n = (V_n, E_n))_n$ be a family of finite d-regular connected graph, with $d \geq 3$. Then*

$$\operatorname{diam}(\Gamma_n) \in \Omega(\log(|V_n|)).$$

Definition 2.20. Let (Γ_n) be a family of finite d-regular graphs such that $|V(\Gamma_n)| \to \infty$ as $n \to \infty$. We say that (Γ_n) is an **expander family** if there exists $\varepsilon > 0$ such that $h(\Gamma_n) > \varepsilon$ for every n (i.e. the sequence of isoperimetric constants is bounded away from 0).

Another desirable property for communication networks is to have a small diameter. In fact, expanders have a logarithmic diameter (i.e. $\operatorname{diam}(\Gamma_n) \in \mathcal{O}(\log(|V(\Gamma_n)|)))$ which is considered small in the computer science community.

Theorem 2.21 ([KS11, COROLLARY 4.8]). *If $(\Gamma_n)_n$ is a family of d-regular expanders, then $(\Gamma_n)_n$ has logarithmic diameter.*

Remark that this shows that expanders have the best possible diameter growth. Moreover, if the diameter grows faster than $\mathcal{O}(\log(|V(\Gamma_n)|))$, the family of graphs will not be an expander family.

2.3 Box spaces and property D_α

Although expander families are very interesting, it is also very interesting to look at non-expander families, and Theorem 2.21 gives us a nice tool to check if the family we consider is not an expander-family. The goal in the rest of this section is to define and characterize some families that are not expanders.

Definition 2.22. Let G be a group. A **filtration** of G is a sequence $(F_n)_n$ of normal subgroups of G such that

1. $F_n \lhd G$ and $[G : F_n] < \infty$;
2. $F_{n+1} \subset F_n$;
3. $\bigcap\limits_{n\in\mathbb{N}} F_n = \{e\}$.

Example 2.23. The easiest example is $\mathbb{Z}$. Take $F_n = (n!)\mathbb{Z}$. Then $(F_n)_n$ is a filtration of $\mathbb{Z}$. ★

Let (X, d_X), (Y, d_Y) be two metric spaces. A map $f : X \to Y$ is a **coarse embedding** if for any sequences $(x_n), (x'_n) \subset X$,

$$d_X(x_n, x'_n) \to \infty \iff d_Y(f(x_n), f(x'_n)) \to \infty.$$

We say that f is a **coarse equivalence** if f is almost surjective, i.e. there exists $R \geq 0$ such that Y is the R-neighborhood of $f(X)$.

Definition 2.24 ([KV17]). Let G be a group and let $(F_n)_n$ be a filtration of G. Let $\Gamma_n = \mathrm{Cay}\left({}^{G}\!/_{F_n}, {}^{S}\!/_{F_n}\right)$. The **box space** associated to $(F_n)_n$, written $\square_{(F_n)} G$, is

$$\square_{(F_n)}G = \bigsqcup_{n\geq 1} \Gamma_n$$

where $\square_{(F_n)} G$ is endowed with a metric d (well-defined up to coarse equivalence) inducing the word metric on each component Γ_n and such that

$$d\left(\Gamma_m, \Gamma_n\right) \geq \max\{\mathrm{diam}\left(\Gamma_m\right), \mathrm{diam}(\Gamma_n)\}$$

for $m \neq n$. We say that $\square_{(F_n)} G$ is the **metrized disjoint union** of the Γ_n.

In fact, a box space approximates the Cayley graph of our group with smaller graphs, obtained as quotients. Indeed, the condition $\cap F_n = \{e\}$ can be interpreted as $F_n \to \{e\}$, and thus ${}^{G}\!/_{F_n} \to {}^{G}\!/_{\{e\}} \simeq G$.

Since we want to work with filtrations, it is important to understand in which class of groups have filtrations. Proposition 2.26 gives us exactly the class of groups that admit such a filtration.

Definition 2.25. A group G is **residually finite** if for any $g \in G \setminus \{e\}$, there exists a normal subgroup of finite index $N \lhd G$ such that $g \notin N$.

Proposition 2.26. *Let G be a finitely generated group. Then G is residually finite if and only if there exists a filtration $(F_n)_n$.*

For a proof of this proposition, see for instance [DK18, Lemma 7.103].

Residual finiteness means that elements are distinguished by finite quotients of the group. In other words, each non-trivial element remains non-trivial in a finite quotient.

Definition 2.27 (Property D_α). For $0 < \alpha \leq 1$, we say that a sequence $(X_k)_{k>0}$ of d-regular connected graphs has **property $\boldsymbol{D_\alpha}$** if there exists a constant $C > 0$ such that $\mathrm{diam}(X_k) \geq C \cdot |X_k|^\alpha$. In particular, if G is a finitely generated, residually finite group and $(F_n)_n$ is a filtration of G, the box space $\square_{(F_n)} G$ has **property $\boldsymbol{D_\alpha}$** if there exists $C > 0$ such that

$$\operatorname{diam}\left({}^{G}\!/\!{}_{F_n}\right) \geq C \left|{}^{G}\!/\!{}_{F_n}\right|^{\alpha}. \tag{2.4}$$

Khukhro and Valette showed that property D_α is a coarse invariant of box spaces [KV17, Lemma 22].

In addition, the following is known for box spaces and property D_α.

Theorem 2.28. *Let G be a finitely generated, residually finite group.*

1. *[BT16, see Corollary 1.7 and Lemma 5.1] If some box space of G has property D_α, for some $\alpha > 0$, then G virtually maps onto $\mathbb{Z}$.*

2. *[KV17, see Theorem 3] If G maps onto $\mathbb{Z}$, then for every $0 < \alpha < 1$, there exists a box space of G with property D_α.*

3. *[KV17, see Proposition 5] The group G is virtually cyclic if and only if some (hence any) box space of G has property D_1.*

We conclude this section with a special example of filtration and relate this example with residually finite groups.

> **Example 2.29.** Consider $G = \mathrm{GL}_n(\mathbb{Z})$, and let $(N_n)_n \subset \mathbb{N}$ be a sequence of integers such that $N_n \mid N_{n+1}$. Let $\varphi_{N_n} \colon \mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n\left({}^{\mathbb{Z}}\!/\!{}_{N_n\mathbb{Z}}\right)$ be the reduction modulo N_n. Denote $\ker(\varphi_{N_n})$ by $G(N_n)$. Then $(G(N_n))_n$ is a filtration of G. ★

A normal subgroup of the form $G(N)$ as in the previous example is called a **principal congruence subgroup of level $\boldsymbol{N}$**. More generally speaking, if $H \leq \mathrm{GL}_n(\mathbb{Z})$, by defining $H(N_i) := H \cap G(N_i)$, we also obtain a filtration of H. This is a particular case of a more general construction, due to Mal'cev, that says that subgroups of linear groups are residually finite.

Theorem 2.30 (Mal'cev-Selberg). *Let $H \leq \mathrm{GL}_n(k)$ be a finitely generated subgroup, where k is a field. Then*

1. *(Mal'cev) H is residually finite ;*

2. *(Selberg) if k has characteristic zero, H is virtually torsion free (i.e. H has a subgroup of finite index whose elements all have infinite order).*

For a proof of theorem 2.30, consult [Nic13]. This theorem is important to us, since we will see that the solvable Baumslag-Solitar groups are embeddable in a linear group, thus they are residually finite and we can construct particular filtrations as in example 2.29.

2.4 Baumslag-Solitar groups, their box spaces and arithmetic box spaces

In this section, we introduce Baumslag-Solitar groups and give some useful characterizations before studying their box spaces.

Definition 2.31 (Bausmlag-Solitar groups). Let $m, n \in \mathbb{Z}$, the **Bausmlag-Solitar group** $\mathbf{BS}(\boldsymbol{n}, \boldsymbol{m})$ admitting the following presentation

$$\mathrm{BS}(n, m) = \langle u, t | tu^n t^{-1} = u^m \rangle \,. \tag{2.5}$$

Note that we have an isomorphism from $\mathrm{BS}(n, m)$ to $\mathrm{BS}(m, n)$ through the map $u \mapsto u$ and $t \mapsto t^{-1}$, and that $\mathrm{BS}(n, m) = \mathrm{BS}(-n, -m)$. When $m = 0$ or $n = 0$, we obtain a free product of $\mathbb{Z}$ with a cyclic group. With these considerations in mind, we will always assume that $0 < n < |m|$.

Even more specifically, we will restrict ourselves to the case $n = 1$, because of the following theorem due to Bausmlag, Solitar and Meskin [BS62, Mes72].

Theorem 2.32. *The group* $\mathrm{BS}(n, m)$ *is residually finite if and only if* $n = 1$ *or* $|m| = 1$ *or* $n = |m|$*.*

In this case, $\mathrm{BS}(1, m)$ is residually finite and we know that we can create box spaces.

Definition 2.33 (Graph isomorphism). Two graphs $\Gamma_1 = (V_1, E_1)$, $\Gamma_2 = (V_2, E_2)$ are **isomorphic** if there is a bijection $\varphi \; : V_1 \to V_2$ which preserves adjacency, i.e. $x \sim y \iff \varphi(x) \sim \varphi(y)$ for all $x, y \in V_1$.

Moreover, box spaces are geometric objects defined on Cayley graphs, and since $\mathrm{Cay}(\mathrm{BS}(1, m))$ is isomorphic to $\mathrm{Cay}(\mathrm{BS}(1, -m))$ for all m, we can assume $m \geq 2$.

We now give an equivalent characterization of $\mathrm{BS}(1, m)$. Let us define the $\boldsymbol{m}$**-adic rationals**

$$\mathbb{Z}[\tfrac{1}{m}] := \left\{ \frac{k}{m^\ell} \mid k, \ell \in \mathbb{Z} \right\}. \tag{2.6}$$

It is a ring whose structure is inherited from $\mathbb{Z}$.

Theorem 2.34. *The group* $\mathrm{BS}(1, m)$ *is isomorphic to* $\mathbb{Z}[\frac{1}{m}] \rtimes_m \mathbb{Z}$ *where* $\mathbb{Z}$ *acts on* $\mathbb{Z}$ *by multiplication by* m*.*

There must exist a (direct) proof of this fact in the literature since it is a well-known fact, yet we could not find any. The closest we could find is to combine the argument of Pooya and Valette [PV18, see beginning of section 2] with [Mei08, Proposition 4.1]. Therefore I will give a direct proof here, which uses a neat trick to determine when an application $\varphi \; : G \to H$ between two groups defined on the generators of G extends to an homomorphism, called Von Dyck's lemma (see [BDD19, Lemma 1.1]).

Lemma 2.35 (von Dyck's lemma). *Let* $X = \{x_1, x_2, \ldots\}$ *and suppose that* $G = \langle X | R \rangle$*. Let* H *be another group and define a map* $\bar{\varphi}$ *from* X *to* H*,* $\bar{\varphi}(x_j) = h_j \in H$*. Extend* $\bar{\varphi}$ *to be a correspondence* φ *from* G *to* H *such that given* $g \in G$ *with* $g = x_{j_1} \cdots x_{j_k}$ *(*$x_{j_\ell} \in X \cup X^{-1}$*), then*

$$\varphi(x_j^{-1}) = h_j^{-1} \quad \textit{and} \quad \varphi(g) = \bar{\varphi}(x_{j_1}) \cdots \bar{\varphi}(x_{j_k}).$$

If for every word $r = x_{j_1} \cdots x_{j_k} \in R$*,*

$$\varphi(r) = 1$$

in H*, then* φ *is a homomorphism from* G *to* H*.*

Before giving the proof, we also need the fact that there exists a unique normal form in $\mathrm{BS}(1,m)$: every word $w \in \mathrm{BS}(1,m)$ different from the identity can be written uniquely as

$$w = t^{-i}u^{\ell}t^{+j}$$

with $i, j \geq 0$, $\ell \in \mathbb{Z}$ and ℓ is a multiple of m only if either i or j is zero [BDD19, p. 137]. To see this, one can simply use the rewriting rules $tu = u^m t$, $tu^{-1} = u^{-m}t$, $ut^{-1} = t^{-1}u^m$, $u^{-1}t^{-1} = t^{-1}u^{-m}$. Under the conditions stated above, uniqueness is clear.

Proof (OF THEOREM 2.34)**.** Consider $\varphi : \mathrm{BS}(1,m) \to \mathbb{Z}[\frac{1}{m}] \rtimes \mathbb{Z}$ defined like in lemma 2.35, given by $\varphi(u) = (1,0)$ and $\varphi(t) = (0,1)$. It is straightforward to check that

$$\varphi(tut^{-1}u^{-n}) = (0,0) \in \mathbb{Z}[\tfrac{1}{m}] \rtimes \mathbb{Z}$$

so that φ is indeed a homomorphism. For any $w = (\frac{k}{m^\ell}, j) \in \mathbb{Z}[\frac{1}{m}] \rtimes \mathbb{Z}$ the element $t^{-\ell}u^k t^{j+\ell}$ gives w through φ. Now, taking any word $w \in \mathrm{BS}(1,m)$ different from the identity in normal form, we obtain $\varphi(w) \neq (0,0)$, which shows that $\ker\varphi = \{\varepsilon\}$. Thus φ is an isomorphism. ■

From theorem 2.34, we see that $1 \lhd \mathbb{Z}[\frac{1}{m}] \lhd \mathbb{Z}[\frac{1}{m}] \rtimes \mathbb{Z} \simeq \mathrm{BS}(1,m)$ thus $\mathrm{BS}(1,m)$ is metabelian and solvable.

We have another, convenient way to describe solvable Baumslag-Solitar groups.

Theorem 2.36. *The solvable Baumslag-Solitar group* $\mathrm{BS}(1,m)$ *is isomorphic to the subgroup*

$$G_m := \left\{ \begin{pmatrix} m^k & r \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z},\ r \in \mathbb{Z}[1/m] \right\} \tag{2.7}$$

of $\mathrm{GL}_2(\mathbb{Z}[\frac{1}{m}])$*. The isomorphism is obtained by mapping* u *to* $U = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ *and* t *to* $T = \left(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix}\right)$*. The associated embedding of* $\mathrm{BS}(1,m)$ *into* $\mathrm{GL}_2(\mathbb{Z}[\frac{1}{m}])$ *is called the* ***standard embedding****.*

Proof. It is readily checked that $G_m = \langle U, T \rangle$ and everything follows by construction. ■

With the previous results, observe $\mathrm{BS}(1,m) \simeq G_m$ is a finitely generated, residually finite group that surjects onto $\mathbb{Z}$ so that theorem 2.1 applies and we know that for every $0 < \alpha < 1$, there exists a box space of Γ with property D_α. We now construct particular box spaces of $\mathrm{BS}(1,m)$.

In $\mathrm{GL}_n(\mathbb{Z}[\frac{1}{m}])$ we may define congruence subgroups similarly to example 2.29. Let $N > 0$ be coprime with m. The **principal congruence subgroup of level N** is the kernel M_N of the reduction modulo N:

$$M_N = \ker\left[\mathrm{GL}_n(\mathbb{Z}[\tfrac{1}{m}]) \to \mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z})\right].$$

Definition 2.37. If G is any subgroup of $\mathrm{GL}_n(\mathbb{Z}[\frac{1}{m}])$, and $N > 0$ is coprime to m, then the **congruence subgroup** $G(N)$ in G is

$$G(N) := G \cap M_N.$$

Proposition 2.38. *If $(N_j)_j \subset \mathbb{N}$ is a strictly increasing sequence of integers such that $N_j \mid N_{j+1}$ for all j then $(G(N_j))_j$ is a filtration of G*

To prove this proposition, we need to study some properties of $\mathbb{Z}[\frac{1}{m}]$.

Lemma 2.39.

1. *$\mathbb{Z}[\frac{1}{m}]$ is a principal ideal domain,*
2. *$\mathbb{Z}[\frac{1}{m}]/N\mathbb{Z}[\frac{1}{m}]$ is isomorphic to $\mathbb{Z}/N\mathbb{Z}$ if N is coprime to m.*

Proof.

1. We first show that $R = \mathbb{Z}[\frac{1}{m}]$ is an integral domain. Let $x, y \in R \setminus \{0\}$. Then $x = \frac{k}{m^\ell}$ and $y = \frac{k'}{m^{\ell'}}$ for some $k, k' \in \mathbb{Z} \setminus \{0\}$ and $\ell, \ell' \in \mathbb{Z}$. Since $\mathbb{Z}$ is an integral domain, $k \cdot k' \neq 0$ and thus $xy = \frac{kk'}{m^{\ell+\ell'}} \neq 0$. Now we show that every ideal of R is principal. Let $I \subset A$ be an ideal. If $I = \{0\}$, then $I = \langle 0 \rangle$, thus we assume that $I \neq \{0\}$. Consider
$$J = \left\langle \left\{ k \in \mathbb{Z} \,\middle|\, \exists \ell \in \mathbb{Z} \text{ such that } \tfrac{k}{m^\ell} \in I \right\} \right\rangle.$$
By construction, J is an ideal of $\mathbb{Z}$, thus there exists $a \in \mathbb{Z}$ such that $\langle a \rangle = J$. We show that in fact, $\langle a \rangle = I$. Since $a \in J$, there exists $\ell \in \mathbb{Z}$ such that $\frac{a}{m^\ell} \in I$, and since I is an ideal, $\frac{a}{m^\ell} m^\ell = a \in I$, so that $\langle a \rangle \subset I$. For the converse, assume that $\frac{k}{m^\ell} \in I$. Since $k \in J$, there is some $x \in \mathbb{Z}$ such that $k = ax$, thus $a\frac{x}{m^\ell} = \frac{k}{m^\ell}$, showing that $I = \langle a \rangle$.
2. An element in $\mathbb{Z}[\frac{1}{m}]/N\mathbb{Z}[\frac{1}{m}]$ has the form $[k/m^\ell]$, where $[\cdot]$ signifies that we consider a representative of this element in the quotient. The application
$$\begin{array}{rccc} \varphi\colon & \mathbb{Z}[\frac{1}{m}]/N\mathbb{Z}[\frac{1}{m}] & \to & \mathbb{Z}/N\mathbb{Z} \\ & [k/m^\ell] & \mapsto & [k \cdot x^\ell] \end{array}$$
where $m \cdot x \equiv 1 \pmod N$ is a ring isomorphism. ■

Proof (OF PROPOSITION 2.38)**.** We obviously have that $G(N_j) \lhd G$ since $G(N_j)$ is the kernel of a homomorphism restricted to a subgroup of G. Moreover, $G/G(N_j) \leq \mathrm{GL}_n(\mathbb{Z}/N_j\mathbb{Z})$ which is a finite group, thus $G(N_j)$ is of finite index in G for all j. Now $\mathrm{GL}_n(\mathbb{Z}/N_j\mathbb{Z}) \subseteq \mathrm{GL}_n(\mathbb{Z}/N_{j+1}\mathbb{Z})$ so that $M_{N_{j+1}} \leq M_{N_j}$ and $G(N_{j+1}) \leq G(N_j)$. Assume that $A \in \mathrm{GL}_n(\mathbb{Z}[\frac{1}{m}]) \neq \mathrm{Id}$. If $A_{ab} = k \in \mathbb{Z}$ where $k \neq 1$ if $a = b$ or $k \neq 0$ if $a \neq b$, then any $N_j > k$ will satisfy $A \not\equiv \mathrm{Id} \pmod{N_j}$. If $A_{ab} = \frac{k_{ab}}{m^{\ell_{ab}}} \in \mathbb{Z}[\frac{1}{m}] \setminus \mathbb{Z}$ for every $a, b \in \{1, \dots, n\}$, define
$$L := \max_{a,b=1}^{n}\{\ell_{ab}\}, \qquad K := \max_{a,b=1}^{n}\{k_{ab}\} \quad \text{and} \quad \ell = \min_{a,b=1}^{n}\{\ell_{ab}\}.$$
Choose any $A_{ab} = \frac{k_{ab}}{m^\ell}$ and take any $N_j > KL$ such that $N_j \nmid k_{ab}$. Then A_{ab} is mapped to $k_{ab}x^\ell \pmod{N_j}$ in the quotient where $mx \equiv 1 \pmod{N_j}$. Note that $k_{ab}x^\ell \not\equiv 1 \pmod{N_j}$ otherwise k_{ab} would be equal to m^ℓ and A_{ab} would be equal to 1 which is a contradiction. Moreover, $k_{ab}x^\ell \not\equiv 0 \pmod{N_j}$ since $N_j \nmid x$ (otherwise $mx \equiv 0 \pmod{N_j}$ which is a contradiction) and we chose N_j such that $N_j \nmid k_{ab}$. Thus $A \not\equiv \mathrm{Id} \pmod{N_j}$ and this shows that $\cap_j G(N_j) = \{\mathrm{Id}\}$. This concludes the proof that $G(N_j)$ is a filtration. ■

Definition 2.40 (ARITHMETIC BOX SPACE). Let $G \leq \mathrm{GL}_n(\mathbb{Z}[\frac{1}{m}])$ and let $G(N_j)_j$ be the filtration from proposition 2.38. We call the associated box space $\square_{(G(N_j))_j} G$ an **arithmetic box space**.

As we mentioned above, $\mathrm{BS}(1,m)$ surjects onto $\mathbb{Z}$ therefore for every $0 < \alpha < 1$, there exists a box space of $\mathrm{BS}(1,m)$ with property D_α. We will show that box spaces with D_α, $\alpha > \frac{1}{2}$ can be distinguished from arithmetic box spaces by coarse geometry. To this end, we need to study the multiplicative order of m in $\mathbb{Z}/N\mathbb{Z}^\times$ viewed as a multiplicative group.

2.5 Some elementary number theory

We gather some technical lemmas. First we recall some facts about greatest common divisors and lowest common multiples, which will be respectively denoted by gcd and lcm.

Proposition 2.41. *Let* $\mu, a_1, \dots, a_n \in \mathbb{N}$*,*

1. $\gcd(\mu \cdot a_1, \mu \cdot a_2) = \mu \cdot \gcd(a_1, a_2)$,
2. $\gcd(a_1, \dots, a_n) = \gcd(\gcd(a_1, \dots, a_{n-1}), a_n)$,
3. $\gcd(a_1, \gcd(a_2, a_3)) = \gcd(\gcd(a_1, a_2), a_3)$,
4. $\gcd(a_1, a_2) \cdot \mathrm{lcm}(a_1, a_2) = a_1 \cdot a_2$,
5. $\mathrm{lcm}(a_1, ..., a_n) = \mathrm{lcm}(\mathrm{lcm}(a_1, \dots, a_{n-1}), a_n)$.

The following lemma generalises Proposition 2.41.(4).

Lemma 2.42. *Let* $a_1, \dots, a_n \in \mathbb{N}$*, then*

$$\mathrm{lcm}(a_1, \dots, a_n) = \frac{a_1 \cdots a_n}{\gcd(a_1 \cdots a_{n-1}, a_1 \cdots a_{n-2} a_n, \dots, a_2 \cdots a_n)}. \tag{2.8}$$

Proof. We use induction to show that eq. (2.8) is valid. If $n = 1$, the formula holds. Thus assume that the formula is true for $n \in \mathbb{N}$, and denote by Π_n the set $\{a_1 \cdots a_{n-1}, a_1 \cdots a_{n-2} a_n, \dots, a_2 \cdots a_n\}$. Using that

$$\gcd\left(\mathrm{lcm}(a_1, \dots, a_n), a_{n+1}\right) = \gcd\left(\frac{a_1 \cdots a_n}{\gcd(\Pi_n)}, a_{n+1}\right),$$

we obtain by a direct computation that

$$\begin{aligned}\operatorname{lcm}(a_1,\dots,a_{n+1}) &= \operatorname{lcm}(\operatorname{lcm}(a_1,\dots,a_n),a_{n+1})\\ &= \frac{\operatorname{lcm}(a_1,\dots,a_n)\cdot a_{n+1}}{\gcd\left(\operatorname{lcm}(a_1,\dots,a_n),a_{n+1}\right)}\\ &= \frac{a_1\cdots a_n\cdot a_{n+1}}{\gcd(\Pi_n)\gcd\left(\frac{a_1\cdots a_n}{\gcd(\Pi_n)},a_{n+1}\right)}\\ &= \frac{a_1\cdots a_{n+1}}{\gcd(a_1\cdots a_n,\gcd(\Pi_n)\cdot a_{n+1})}\\ &= \frac{a_1\cdots a_{n+1}}{\gcd(a_1\cdots a_n,\gcd(a_1a_3\cdots a_na_{n+1},\dots,a_2\cdots a_na_{n+1}))}\\ &= \frac{a_1\cdots a_{n+1}}{\gcd(a_1\cdots a_n,a_1a_3\cdots a_na_{n+1},\dots,a_2\cdots a_na_{n+1})}.\end{aligned}$$

■

Lemma 2.39 allows us to work with $\mathbb{Z}/N\mathbb{Z}$, which has a familiar ring structure. We will write $\mathbb{Z}/N\mathbb{Z}^\times$ for the multiplicative group of $\mathbb{Z}/N\mathbb{Z}$. We denote by $\operatorname{ord}_m(N)$ the **multiplicative order** of m in $\mathbb{Z}/N\mathbb{Z}^\times$. We define the following function.

Definition 2.43. Let $m,N\in\mathbb{N}$ be such that $\gcd(m,N)=1$. Write $m^{\operatorname{ord}_m(N)}=\mu N+1$ for some $\mu\in\mathbb{N}$ and let the function $\eta_N\colon\mathbb{N}\to\mathbb{N}$ be defined by

$$\eta_N(k)=\begin{cases}1 & \text{if } k=1\\ \frac{N^{k-1}}{\gcd(\mu,N)} & \text{if } k\geq 2.\end{cases}$$

Lemma 2.44. *Let* $m,N\in\mathbb{N}$ *be such that* $\gcd(m,N)=1$. *Write* $m^{\operatorname{ord}_m(N)}=\mu N+1$ *for some* $\mu\in\mathbb{N}$. *Then* $\operatorname{ord}_m(N^2)=\operatorname{ord}_m(N)\cdot\frac{N}{\gcd(\mu,N)}$, *and more generally*

$$\operatorname{ord}_m(N^k)=\operatorname{ord}_m(N)\cdot\eta_N(k),\quad\forall k\geq 1. \tag{2.9}$$

Proof. The case $k=1$ being obvious, let us consider $k=2$, and set $\beta=\operatorname{ord}_m(N)$. The smallest positive integer λ that satisfies $m^\lambda\equiv 1 \pmod{N^2}$ is $\lambda=\beta\frac{N}{\gcd(\mu,N)}$. Indeed, $\beta\mid\lambda$ thus $\lambda=\beta\tilde\lambda$ for some $\tilde\lambda\in\mathbb{N}$. From the fact that $(m^\beta)^{\tilde\lambda}\equiv 1+\tilde\lambda\mu N \pmod{N^2}$, we see that $\tilde\lambda=\frac{N}{\gcd(\mu,N)}$.

The same arguments can be applied to show that $\operatorname{ord}_m(N^k)=\beta\frac{N^{k-1}}{\gcd(\mu,N)}$ for $k\geq 2$. ■

Lemma 2.45. *Let* $k,N\in\mathbb{N}$. *Then* $\eta_N(k)\geq N^{k-2}$.

Proof. If $k=1$, then $1\geq N^{-1}$. If $k\geq 2$, observe that $\gcd(\mu,N)\leq N$ implies

$$\frac{N^{k-1}}{\gcd(\mu,N)}\geq N^{k-2}.$$

■

Denote by $\mathcal{P}\subset\mathbb{N}$ the set of prime numbers. The following lemma gives us a formula to compute the order of m in $\mathbb{Z}/N\mathbb{Z}$ for any $N\in\mathbb{N}$. It is an immediate consequence of the Chinese remainder theorem.

Lemma 2.46. *For every* $N \in \mathbb{N}$*, which we write* $N = p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}$ *with* $p_i \in \mathcal{P}$ *and* $\beta_i \in \mathbb{N}$ *for every* $i \in \{1, \ldots, n\}$*, we have*

$$\operatorname{ord}_m(N) = \operatorname{ord}_m\left(p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}\right) = \operatorname{lcm}\left(\operatorname{ord}_m\left(p_1^{\beta_1}\right), \operatorname{ord}_m\left(p_2^{\beta_2}\right), \ldots, \operatorname{ord}_m\left(p_n^{\beta_n}\right)\right). \tag{2.10}$$

Lemma 2.47. *Let* P *be a finite set of primes, not dividing* m*. There exists a constant* $C(m, P) > 0$ *such that, for every integer* N *with all prime factors in* P*, we have:*

$$\frac{\operatorname{ord}_m(N)}{N} \geq C(m, P).$$

Proof. Write $N = p_1^{\beta_1} \ldots p_k^{\beta_k}$, with $p_i \in P$, all different, $\beta_i > 0$ and η_{p_i} defined as in Definition 2.43. In addition, we define the set

$$\begin{aligned}\Pi_k := \{ & \operatorname{ord}_m(p_1)\eta_{p_1}(\beta_1) \cdots \operatorname{ord}_m(p_{k-1})\eta_{p_{k-1}}(\beta_{k-1}),\\ & \operatorname{ord}_m(p_1)\eta_{p_1}(\beta_1) \cdots \operatorname{ord}_m(p_{k-2})\eta_{p_{k-2}}(\beta_{k-2}) \operatorname{ord}_m(p_k)\eta_{p_k}(\beta_k),\\ & \ldots\\ & \operatorname{ord}_m(p_2)\eta_{p_2}(\beta_2) \cdots \operatorname{ord}_m(p_k)\eta_{p_k}(\beta_k)\}\end{aligned}$$

which contains all possible products with $k-1$ factors, each one of $\operatorname{ord}_m(p_i)\eta_{p_i}(\beta_i)$, $i = 1, \ldots, k$. Using Lemmas 2.42, 2.44, and 2.46, we obtain

$$\begin{aligned}\operatorname{ord}_m(N) &= \operatorname{lcm}\left(\operatorname{ord}_m\left(p_1^{\beta_1}\right), \operatorname{ord}_m\left(p_2^{\beta_2}\right), \ldots, \operatorname{ord}_m\left(p_k^{\beta_k}\right)\right)\\ &= \frac{\operatorname{ord}_m(p_1)\eta_{p_1}(\beta_1) \cdots \operatorname{ord}_m(p_k)\eta_{p_k}(\beta_k)}{\gcd(\Pi_k)},\end{aligned}$$

thus

$$\frac{\operatorname{ord}_m(N)}{N} = \frac{\operatorname{ord}_m(p_1)\eta_{p_1}(\beta_1) \cdots \operatorname{ord}_m(p_k)\eta_{p_k}(\beta_k)}{N \cdot \gcd(\Pi_k)}. \tag{2.11}$$

Moreover, $\operatorname{ord}_m(p_i) \geq 1$ for every i, and using Lemma 2.45 on each $\eta_{p_i}(\beta_i)$, we obtain from Eq. (2.11)

$$\frac{\operatorname{ord}_m(N)}{N} \geq \frac{p_1^{\beta_1-2} \cdots p_k^{\beta_k-2}}{p_1^{\beta_1} \cdots p_k^{\beta_k} \cdot \gcd(\Pi_k)} = \frac{1}{p_1^2 \cdots p_k^2 \cdot \gcd(\Pi_k)} > 0. \tag{2.12}$$

So we may take $C(m, P)$ as the minimum of the $\frac{1}{p_1^2 \cdots p_k^2 \cdot \gcd(\Pi_k)}$'s taken over all subsets $\{p_1, ..., p_k\}$ of P.

■

2.6 Property D_α for solvable Baumslag-Solitar groups

2.6.1 Metric aspects of solvable Baumslag-Solitar groups

We study the diameter of arithmetic box spaces of $\mathrm{BS}(1, m)$ according to eq. (2.4). In this section, we will always assume that $\gcd(m, N) = 1$. We recall that every element of $\mathrm{BS}(1, m)$ ($m > 1$) admits a unique normal form of the type $t^{-i} a^{\ell} t^{j}$ with $i, j \geq 0, \ell \in \mathbb{Z}$ and ℓ can be a multiple of m only if either i or j is zero.

The normal form of a word is usually not the geodesic form, and we want to estimate how well the normal form approximates the geodesic form.

Proposition 2.48 ([BE15, Prop. 2.1]). *There exist constants* $C_1, C_2, D_1, D_2 > 0$ *such that for any* $\omega = t^{-i}a^{\ell}t^{j} \in \mathrm{BS}$ *with* $\ell \neq 0$, *we have*

$$C_1(i + j + \log|\ell|) - D_1 \leq \|\omega\| \leq C_2(i + j + \log|\ell|) + D_2$$

where $\|\cdot\|$ *is the word metric with respect to* $\{u^{\pm 1}, t^{\pm 1}\}$. *Moreover we may take* $C_2 = D_2 = m$.

Let $U = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}[1/m])$, and recall from theorem 2.36 that $\mathrm{BS}(1,m) \simeq G_m = \langle U, T \rangle \leq \mathrm{GL}_2(\mathbb{Z}[1/m])$. As mentioned before, G_m is a finitely generated, residually finite group that surjects onto $\mathbb{Z}$ so that Theorem 1.2 applies and we know that for every $0 < \alpha < 1$, there exists a box space of G_m with property D_α. Here we study the arithmetic box spaces of G_m, i.e. box spaces of the form $\Box_{(G_m(N_k))_k} G_m$. To this end, we start by studying the quotients $G_m/G_m(N)$, and then explore how the diameters evolve.

Proposition 2.49. *Let* $N \in \mathbb{N}$ *be such that* $\gcd(m, N) = 1$. *Then*

$$G_m/G_m(N) \simeq \mathbb{Z}/N\mathbb{Z} \rtimes_m \mathbb{Z}/\operatorname{ord}_m(N)\mathbb{Z}, \qquad \text{and} \qquad |G_m/G_m(N)| = N \cdot \operatorname{ord}_m(N), \quad (2.13)$$

where $\mathbb{Z}/\operatorname{ord}_m(N)\mathbb{Z}$ *acts on* $\mathbb{Z}/N\mathbb{Z}$ *by multiplication by* m *(modulo* N*).*

Proof. Consider reduction modulo N:

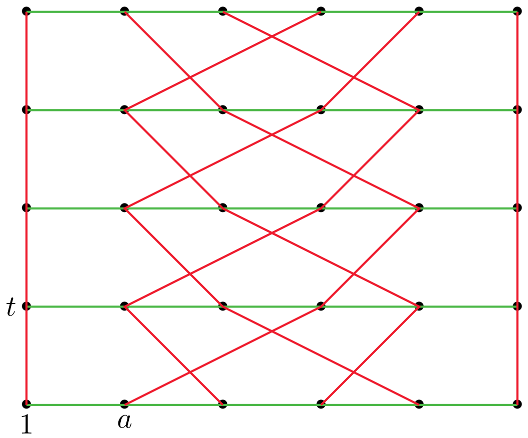
$$\begin{array}{rccc} \varphi\colon & G_m & \to & \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \\ & w = \begin{pmatrix} m^k & x \\ 0 & 1 \end{pmatrix} & \mapsto & \begin{pmatrix} [m^k] & [x] \\ [0] & [1] \end{pmatrix}. \end{array}$$

The image of φ is clearly isomorphic to $\mathbb{Z}/N\mathbb{Z} \rtimes_m \mathbb{Z}/\operatorname{ord}_m(N)\mathbb{Z}$, and of order $N \cdot \operatorname{ord}_m(N)$. Moreover we have

$$w = \begin{pmatrix} m^k & x \\ 0 & 1 \end{pmatrix} \in G_m(N) \iff m^k \equiv 1 \pmod{N} \text{ and } x \in N\mathbb{Z}\left[\tfrac{1}{m}\right] \iff \varphi(w) = 1.$$

So $\ker(\varphi) = G_m(N)$ and the result follows from the first isomorphism theorem. ■

Example 2.50. Consider $\mathrm{BS}(1,2)$ and $N = 5$. Then $\operatorname{ord}_2(5) = 4$, and $G_2/G_2(5) \simeq \mathbb{Z}/5\mathbb{Z} \rtimes_2 \mathbb{Z}/4\mathbb{Z}$. The Cayley graph of the quotient is the graph drawn below, using $a = (1,0)$ and $t = (0,1)$ as generators. Note that one still needs to identify the bottom line with the upper line, and the line to the left with the line to the right.



★

Thanks to the familiar structure of the quotient $G_m/G_m(N)$ and proposition 2.48, we are able to estimate the diameter of arithmetic box spaces of $\mathrm{BS}(1,m)$.

Lemma 2.51. *Let $N \geq 2$. Then*

$$\operatorname{diam}(\operatorname{Cay}(G_m/G_m(N))) = \Theta(\operatorname{ord}_m(N)). \tag{2.14}$$

More precisely, there exists a constant $C_m > 0$ such that

$$\frac{1}{3}\cdot \operatorname{ord}_m(N) \leq \operatorname{diam}(\operatorname{Cay}(G_m/G_m(N))) \leq C_m \cdot \operatorname{ord}_m(N). \tag{2.15}$$

Proof. Let $m \geq 2$ and consider $\mathrm{BS}(1,m) \simeq G_m \subset \mathrm{GL}_2(\mathbb{Z}[1/m])$. Recall that $G_m/G_m(N) \simeq \mathbb{Z}/N\mathbb{Z} \rtimes_m \mathbb{Z}/\operatorname{ord}_m(N)\mathbb{Z}$ so that $\operatorname{diam}(G_m/G_m(N)) = \operatorname{diam}(\mathbb{Z}/N\mathbb{Z} \rtimes_m \mathbb{Z}/\operatorname{ord}_m(N)\mathbb{Z}) \geq \operatorname{diam}(\mathbb{Z}/\operatorname{ord}_m(N)\mathbb{Z})$. Since the Cayley graph of $\mathbb{Z}/\operatorname{ord}_m(N)\mathbb{Z}$ is a cycle, we can roughly estimate the diameter to obtain

$$\operatorname{diam}(G_m/G_m(N)) \geq \frac{1}{3}\operatorname{ord}_m(N). \tag{2.16}$$

For the second inequality, let $([x],[k]) \in \mathbb{Z}/N\mathbb{Z} \rtimes_m \mathbb{Z}/\operatorname{ord}_m(N)\mathbb{Z}$ be an element realizing the diameter. We rewrite $([x],[k])$ as $\left(\begin{smallmatrix} m^k & x \\ 0 & 1\end{smallmatrix}\right) G_m(N)$. The induced metrics are always smaller or equal in a quotient, thus

$$\left\| \begin{pmatrix} m^k & x \\ 0 & 1 \end{pmatrix} G_m(N) \right\|_{G_m/G_m(N)} \leq \left\| \begin{pmatrix} m^k & x \\ 0 & 1 \end{pmatrix} \right\|_{G_m}.$$

In the quotient $G_m/G_m(N)$ every word can be written as $A^\ell T^j$ with $0 \leq \ell < N$ and $0 \leq j < \operatorname{ord}_m(N)$ by the semi-direct product stucture. With $\ell = x$ and $j = k$, we identify $\left(\begin{smallmatrix} m^k & x \\ 0 & 1\end{smallmatrix}\right)$ with the element $A^x T^k$ in normal form in G_m. If $x = 0$ we get T^k and

$$\|T^k\|_{G_m} = k < \operatorname{ord}_m(N).$$

Assume that $x \neq 0$. From Proposition 2.48, we obtain

$$\left\| \begin{pmatrix} m^k & x \\ 0 & 1 \end{pmatrix} \right\|_{G_m} = \|A^x T^k\|_{G_m} \leq 2m(k + \log x + 1). \tag{2.17}$$

Note that since $m^{\mathrm{ord}_m(N)} \geq N$ (equivalently $\log(N) \leq \mathrm{ord}_m(N) \cdot \log(m)$) and $\log x \leq \log(N)$, Eq. (2.17) becomes

$$\left\| \begin{pmatrix} m^k & x \\ 0 & 1 \end{pmatrix} \right\|_{G_m} \leq 2m(2 + \log(m))\, \mathrm{ord}_m(N). \tag{2.18}$$

Setting $C_m := 2m(2 + \log(m))$, we obtain

$$\mathrm{diam}(\mathrm{Cay}(G_m/G_m(N))) \leq C_m \cdot \mathrm{ord}_m(N). \tag{2.19}$$

■

Proposition 2.52. *An arithmetic box space $\Box_{(G_m(N_k))_k} G_m$ has property D_α if and only if*

$$\mathrm{ord}_m(N_k) = \Omega(N_k^{\frac{\alpha}{1-\alpha}}).$$

Proof. Using Lemma 2.51:

$$\Box_{(G_m(N_k))_k} G_m \,\text{has}\, D_\alpha \iff \mathrm{diam}(G_m/G_m(N_k)) = \Omega(|G_m/G_m(N_k)|^\alpha)$$

$$\iff \mathrm{ord}_m(N_k) = \Omega(N_k^\alpha \cdot \mathrm{ord}_m(N_k)^\alpha) \iff \mathrm{ord}_m(N_k) = \Omega(N_k^{\frac{\alpha}{1-\alpha}}).$$

■

We present here the main structure theorem for the arithmetic box spaces of $\mathrm{BS}(1, m)$.

Theorem 2.53. *For any $m \geq 2$, the following statements hold :*

1. *if an arithmetic box space $\Box_{(G_m(N_k))_k} G_m$ has property D_α, then $\alpha \leq \frac{1}{2}$,*
2. *there exists an arithmetic box space with property $D_{1/2}$,*
3. *there exists an arithmetic box space of G_m without property D_α for any $\alpha \in]0, 1/2]$.*

Proof.

1. If $\Box_{(G_m(N_k))_k} G_m$ has property D_α, using $N_k \geq \mathrm{ord}_m(N_k)$ and Proposition 2.52, we get $N_k = \Omega(N_k^{\frac{\alpha}{1-\alpha}})$, which forces $\alpha \leq \frac{1}{2}$.
2. Let $(N_k)_k \subset \mathbb{N}$ be the sequence defined by $N_k = (m^2 - 1)^k$. Clearly, $N_k \mid N_{k+1}$ for every k. We apply Lemma 2.44 with $N = m^2 - 1$, so that $\mathrm{ord}_m(m^2 - 1) = 2$ and $\mu = 1$. We thus obtain:

   $$\mathrm{ord}_m(N_k) = 2 \cdot (m^2 - 1)^{k-1}, \forall k \geq 1. \tag{2.20}$$

   i.e. $\mathrm{ord}_m(N_k) = \Omega(N_k)$. By Proposition 2.52 the box space $\Box_{(G_m(N_k))_k} G_m$ has property $D_{1/2}$.
3. We consider the sequence $(N_k)_k$ defined by $N_k = m^{2^k} - 1$ and prove that the

arithmetic box space $\square_{(G_m(N_k))_k} G_m$ does not have property D_α for any $\alpha \in]0, 1/2]$. It is straightforward that $N_k \mid N_{k+1}$ for every k, and $\mathrm{ord}_m(N_k) = 2^k$. We have

$$\lim_{k\to\infty} \frac{\mathrm{ord}_m(N_k)}{N_k^{\frac{\alpha}{1-\alpha}}} = \lim_{k\to\infty} \frac{2^k}{(m^{2^k}-1)^{\frac{\alpha}{1-\alpha}}} = 0,$$

i.e. $\mathrm{ord}_m(N_k) = o(N_k^{\frac{\alpha}{1-\alpha}})$. By Proposition 2.52 this shows that the arithmetic box space $\square_{(G_m(m^{2^k}-1))_k} G_m$ does not have property D_α for any $\alpha \in]0, 1/2]$.

■

2.6.2 Density results

A natural question after encountering the constructions of Theorem 2.53.2 and 2.53.3 is “how many arithmetic box spaces of $\mathrm{BS}(1, m)$ have $D_{1/2}$”? In the following paragraphs, we give a partial answer to this question.

Let $(N_k)_k \subset \mathbb{N}$ be such that $N_k \mid N_{k+1}$ for every $k > 0$, and denote by P_k the set of prime factors of N_k. Moreover, we define the set of prime factors of the sequence $(N_k)_k$ by

$$P := \bigcup_{k=1}^{+\infty} P_k. \tag{2.21}$$

Before stating our main result from this section, we need to introduce some definitions about the density of prime numbers. We follow Powell [Pow80] for the terminology.

Definition 2.54. Let $P \subset \mathcal{P}$ be a subset of the prime numbers. The **natural primitive density** of P is (if the limit exists)

$$d'(P) := \lim_{N\to\infty} \frac{|\{p \leq N \mid p \in P\}|}{|\{p \leq N \mid p \in \mathcal{P}\}|}.$$

The **analytic primitive density** of P is (if the limit exists)

$$D'(P) = \lim_{s\to 1^+} \frac{\sum_{p\in P} \frac{1}{p^s}}{\sum_{p\in\mathcal{P}} \frac{1}{p^s}}.$$

If P is finite then $d'(P) = D'(P) = 0$. Suppose now that $D'(P) > 0$. In this case, we see that $\sum_{p\in P} \frac{1}{p} = +\infty$, otherwise $D'(P)$ would be equal to 0. Observe that

$$\prod_{p\in P} \left(1 - \frac{1}{p}\right) = 0 \iff \sum_{p\in P} \ln\left(1 - \frac{1}{p}\right) = -\infty.$$

But using that $\ln(1 + x) \leq x$ for $x > -1$ we get

$$\sum_{p\in P} \ln\left(1 - \frac{1}{p}\right) \leq -\sum_{p\in P} \frac{1}{p} = -\infty.$$

Therefore, we obtain

$$\prod_{p\in P} \left(1 - \frac{1}{p}\right) = 0 \tag{2.22}$$

if $D'(P) > 0$.

Theorem 2.55. *Let $\Box_{(G_m(N_k))_k} G_m$ be an arithmetic box space, and let P be the set of prime factors of the sequence $(N_k)_k$.*

1. *If $|P| < +\infty$, then $\Box_{(G_m(N_k))_k} G_m$ has $D_{1/2}$;*
2. *If $D'(P) > 0$, then $\Box_{(G_m(N_k))_k} G_m$ does not have $D_{1/2}$.*

Proof. In view of Proposition 2.52, we must study the asymptotics of the quotient $\frac{\mathrm{ord}_m(N_k)}{N_k}$.

1. By Lemma 2.47, there exists a constant $C(m, P)$ such that $\frac{\mathrm{ord}_m(N_k)}{N_k} \geq C(m, P)$, i.e. $\mathrm{ord}_m(N_k) = \Omega(N_k)$. Proposition 2.52 applies to show that $\Box_{(G_m(N_k))_k} G_m$ has $D_{1/2}$.
2. Assume now $D'(P) > 0$, pick $N = p_1^{\beta_1}...p_k^{\beta_k}$ with $p_1, ..., p_k \in P$. We have $\mathrm{ord}_m(N) \leq \varphi(N) = |(\mathbb{Z}/N\mathbb{Z})^\times|$, where φ denotes Euler's totient function. Then

$$\frac{\mathrm{ord}_m(N)}{N} \leq \prod_{i=1}^{k} \frac{\varphi(p_i^{\beta_i})}{p_i^{\beta_i}} = \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right).$$

In view of eq. (2.22) we then get $\mathrm{ord}_m(N_k) = o(N_k)$, which proves the second part of the theorem thanks to Proposition 2.52.

■

2.6.3 Open questions

The following open questions are related to the previous theorem.

1. If we assume that $D'(P) > 0$, does the associated arithmetic box space have D_α for some $\alpha \in]0, 1/2[$ or not?
2. What happens in the case $|P| = +\infty$ and $D'(P) = 0$?
3. Given $\alpha \in]0, 1/2]$, can we create an arithmetic box space with exactly D_α?

3 Wavelet representations of Baumslag-Solitar groups on fractals

This chapter is based on an article that is currently under redaction.

3.1 Introduction

Wavelets have a rich history and their development benefited from the influence of various fields, such as harmonic analysis, approximation theory, geophysics, quantum mechanics or computer graphics [Dau96]. Initially developed to study phenomenons in a framework with good time-frequency localization properties, as opposed to the Fourier transform which only has good frequency localization properties, Meyer, Lemarié, Battle, Federbush and other researchers developed tools to obtain orthonormal wavelet bases of $L^2(\mathbb{R}^d)$ with good time-frequency localization properties. Practitioners have successfully used wavelets to solve applied problems, one of the most famous instance being the FBI fingerprint compression standard [PW12]. However, wavelets are at the intersection of many fields, whether applied or theoretical. As such, people from harmonic analysis are concerned with the theoretical side and questions such as "are Fourier series typical for the Lebesgue measure, or are there other measures having orthogonal bases of exponential functions ?" [Dut09]. A wavelet is defined to be a set $\{\psi_1, \dots, \psi_\ell\} \subset L^2(\mathbb{R})$ such that

$$\{\, T^j U^k \psi_i \mid j, k \in \mathbb{Z},\ i \in \{1, \dots, \ell\} \,\} \tag{3.1}$$

is an orthonormal basis for $L^2(\mathbb{R})$, where $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $Uf(x) = f(x-1)$ is the translation operator, and $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $Tf(x) = \frac{1}{\sqrt{N}} f(\frac{x}{N})$, $N \geq 2$ is the scaling operator. Our starting observation is that we have a representation of the group $\mathrm{BS}(1, N) = \langle u, t | tut^{-1} = u^N \rangle$ where u corresponds to translations in the group, and t corresponds to scaling by a factor N. Thus, the condition in eq. (3.1) translates to:

$$\left\{\, \pi(t^j u^k)\psi_i \;\middle|\; j, k \in \mathbb{Z},\ i \in \{1, \dots, \ell\} \,\right\} \tag{3.2}$$

shall be an orthonormal basis for $L^2(\mathbb{R})$ where $\pi : \mathrm{BS}(1, N) \to \mathcal{U}(L^2(\mathbb{R}))$ is the corresponding representation. One can obviously generalize this to arbitrary Hilbert spaces: if $\pi : \mathrm{BS}(1, N) \to \mathcal{U}(\mathcal{H})$ is a unitary representation, a wavelet for the representation is a finite set $\{\psi_1, \dots, \psi_\ell\} \subset \mathcal{H}$ such that

$$\left\{\, \pi(t^j u^k)\psi_i \;\middle|\; j, k \in \mathbb{Z},\ i \in \{1, \dots, \ell\} \,\right\}$$

is an orthonormal basis for $\mathcal{H}$, and the operators $\pi(u) = U$ and $\pi(t) = T$ are respectively interpreted as a translation and scaling operator on $\mathcal{H}$.

Nowadays, the most common tool to create orthonormal wavelet bases is called a Multiresolution Analysis (MRA), and in fact Lemarié and Auscher proved that any reasonable wavelet base comes from an MRA [LR92, Aus92]. Thus, we naturally extend this tool to arbitrary Hilbert spaces. Any MRA comes with a scaling function φ, which satisfies $T\varphi = \sum_{k\in\mathbb{Z}} a_k U^k \varphi$ for some coefficients $a_k \in \mathbb{C}$, from which we define a low-pass filter $m_0(z) = \sum_{k\in\mathbb{Z}} a_k z^k$. In most practical applications, m_0 is a trigonometric polynomial, i.e. a finite sum of complex exponentials. Cohen gave some explicit conditions on m_0 (and thus the sequence (a_k)) for m_0 to be issued from an MRA [Coh90, Theorem 2.1]. The generalization of an MRA to arbitrary Hilbert spaces is also straightforward, we discuss this further in definition 3.12. However, constructing an MRA is a rather subtle task, for which we introduce the notion of wavelet representation which encompasses on the one hand the notion of a representation of $\mathrm{BS}(1, N)$, and on the other hand the construction of suitable MRAs to obtain orthonormal wavelet bases of arbitrary Hilbert spaces.

A framework in which this generalization works particularly well is that of fractals coming from iterated function systems (IFS). The IFS we consider are given by an endomorphism $\tau\ : [0,1] \to [0,1]$, $\tau(x) = Nx \pmod 1$ and p of its N inverse branches $\tau_j(x) = \frac{x+j}{N}$ for $j = 0, \dots, N-1$. If we choose $1 \le p \le N$ of the N branches, we obtain a system $(\tau_{a_j})_{j=1}^{p}$ of p contractive maps where $(a_j)_{j=1}^{p} \subset \{0, \dots, N-1\}$ is a set of p distinct integers, and a result of Hutchinson [Hut81] tells us that there exists a unique compact subset $K \subset [0,1]$ such that $K = \cup_{j=1}^{p} \tau_j(K)$. This compact K is called the attractor, and satisfies

$$NK = \bigcup_{j=1}^{p} (K + a_j).$$

Hutchinson also proved that in this case, there is a unique probability measure μ on $[0,1]$ that is supported on K and which satisfies

$$\mu = \frac{1}{p} \sum_{j=1}^{p} \mu \circ \tau_j^{-1}.$$

We consider an inflated version of K: it is the set of real numbers that have a base N expansion containing a finite number of elements of the set $\mathbb{Z}/N\mathbb{Z} \setminus \{a_1, \dots a_p\}$, i.e.

$$\mathcal{R} := \left\{ \sum_{j\ge -m} b_j N^{-j} \;\middle|\; m \in \mathbb{Z}, b_j \in \{a_1, \dots, a_p\} \text{ for all but finitely many indices } j \right\}.$$

This set is invariant under N-adic translations, under dilation by N, K is contained in $\mathcal{R}$ and covers $\mathcal{R}$ by translations and dilations (see Proposition 3.18). Moreover, the measure μ can be extended to a Borel measure on $\mathcal{R}$ which we also call μ. Dutkay and Jorgensen studied wavelets on such fractals [DJ06b] and proved that there is a wavelet representation of $\mathrm{BS}(1, N)$ on $L^2(\mathcal{R}, \mu)$ associated to the low-pass filter $m_0 = \frac{1}{\sqrt{p}} \sum_{j=1}^{p} z^{a_j}$. The corresponding unitary operators on $L^2(\mathcal{R}, \mu)$ are $\hat{U} f(x) = f(x-1)$ and $\hat{T} f(x) = \frac{1}{\sqrt{p}} f(\frac{x}{N})$.

Dutkay proved that we can create another wavelet representation associated to the low-pass filter m_0 [Dut06]. The representation space will then be $L^2(\mathcal{S}_N, m)$ where $\mathcal{S}_N$ is the N-solenoid

$$\mathcal{S}_N := \left\{ (z_n)_{n\in\mathbb{Z}} \mid z_{n+1}^N = z_n, \ |z_n| = 1 \text{ for } n \in \mathbb{Z} \right\}.$$

Let $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$. The N-solenoid $\mathcal{S}_N$ is the dual group of $\mathbb{Z}[\frac{1}{N}]$ and is bimeasurably identified with $[0,1) \times \mathbb{Z}_N^{\mathbb{N}}$ by a bijection that we denote by Φ (see proposition 3.23) and we can create a probability measure P_x on $\mathbb{Z}_N^{\mathbb{N}}$ as follows. Let $\omega_1, \ldots, \omega_n \in \mathbb{Z}_N$, and consider the cylinder sets

$$A(\omega_1, \ldots, \omega_n) = \left\{ \tilde{\omega} = (\tilde{\omega_1}, \tilde{\omega_2}, \tilde{\omega_3}, \ldots) \in \mathbb{Z}_N^{\mathbb{N}} \;\middle|\; \tilde{\omega_1} = \omega_1, \ldots \tilde{\omega_n} = \omega_n \right\}.$$

They generate the topology of $\mathbb{Z}_N^{\mathbb{N}}$ and its Borel σ-algebra. If we denote $\frac{|m_0(z)|^2}{N}$ by $W(z)$, where $z \in \mathbb{T}$, Jorgensen [Jor06, Lemma 2.4.1] proved that there is a unique positive Radon probability

$$P_x(A(\omega_1, \ldots, \omega_n)) = W(\tau_{\omega_1} x) W(\tau_{\omega_2}\tau_{\omega_1} x) \cdots W(\tau_{\omega_n} \cdots \tau_{\omega_1} x).$$

The measure m on $\mathcal{S}_N$ is then defined by

$$\int_{\mathcal{S}_N} f \mathrm{d}m = \int_0^1 \int_\Omega f(\Phi(x,\omega)) \mathrm{d}P_x(\omega) \mathrm{d}x, \quad f \in C(\mathcal{S}_N)$$

using the Lebesgue measure $\mathrm{d}x$ on $[0,1)$, and where $C(\mathcal{S}_N)$ denotes the set of continuous functions on $\mathcal{S}_N$.

In the case $N = 3$, Dutkay computed the Fourier coefficients of the measure m [Dut06] and proved that there exists a unique isomorphism between the wavelet representations on $L^2(\mathcal{R}, \mu)$ and $L^2(\mathcal{S}_N, m)$. We generalize this work to the case $N \geq 2$, we compute the Fourier coefficients of m and prove the existence of a similar isomorphism for all $N \geq 2$. More specifically, the wavelet representation we consider is associated to the low-pass filter $m_0(z) = \sum_{j=1}^p \frac{1}{\sqrt{p}} z^{a_j} \in L^\infty(\mathbb{T})$ where $(a_j)_{j=1}^p \subset \{0, \ldots, N-1\}$ consists of p distinct integers. From the $(a_j)_{j=1}^p$, we construct the following sets :

$$B := \{ a_k - a_j \mid j,k \in \{1, \ldots, p\} \} \quad \text{and} \quad C := \{c_{-N+1}, c_{-N+2}, \ldots, c_{N-1}\}$$

where c_ℓ corresponds to the number of ways of obtaining ℓ from $a_k - a_j$, $j,k \in \{1, \ldots, p\}$.

Theorem 3.1 (THEOREM 3.55). *The Fourier coefficients of the measure m are*

$$\hat{m}(\lambda) = \begin{cases} \frac{1}{p^\ell} \prod_{j=1}^{\ell} c_{b_j} & \text{if } \lambda = \sum_{j=1}^{\ell} \frac{b_j}{N^j}, \ b_j \in B \\ 0 & \text{otherwise.} \end{cases}$$

Let $\hat{U}_\lambda : L^2(\mathcal{R}, \mu)$, $\hat{U}_\lambda f(x) = f(x - \lambda)$, and let U_λ be the corresponding operator on $L^2(\mathcal{S}_N, m)$ given by multiplication by the character ξ_λ. We are able to deduce the following:

Corollary 3.2 (COROLLARY 3.54). *There exists a unique isomorphism $\mathcal{F}_N : L^2(\mathcal{R}, \mu) \to L^2(\mathcal{S}_N, m)$ which satisfies*

1. $\mathcal{F}_N \hat{U}_\lambda \mathcal{F}_N^{-1} f = U_\lambda f = \xi_\lambda f$;
2. $\mathcal{F}_N \hat{T} \mathcal{F}_N^{-1} f = Tf = m_0 f \circ S$;

3. $\mathcal{F}_N \chi_K = 1$.

Furthermore, building on the work of Davison [Dav20] who focused on the case $N = 3$, we decompose the dilation operator $T : L^2(\mathcal{S}_N, m) \to L^2(\mathcal{S}_N, m)$ as well as its adjoint into sums of partial isometries $(T_j)_{j=0}^{N-1}$ and we prove that the operators $T_j T_j^*$ and $T_j^* T_j$ are multiplication operators that add up to the identity operator on $L^2(\mathcal{S}_N, m)$.

Theorem 3.3 (PROPOSITION 3.35 AND PROPOSITION 3.38). *For $N \geq 2$, $T, T^* : L^2(\mathcal{S}_N, m) \to L^2(\mathcal{S}_N, m)$ can be decomposed as*

$$T = \sum_{j=0}^{N-1} T_j \quad \textit{and} \quad T^* = \sum_{j=1}^{N-1} T_j^*.$$

where $T_j : L^2(\mathcal{S}_N, m) \to L^2(\mathcal{S}_N, m)$ are linear, bounded operators and partial isometries for all $j = 0, ..., N-1$, and they satisfy

$$\sum_{k=0}^{N-1} T_k T_k^* = \sum_{k=0}^{N-1} T_k^* T_k = \mathrm{Id}_{L^2(\mathcal{S}_N, m)}.$$

Additionally we analyze the spectral measure E of the operator U on $L^2(\mathcal{S}_N, m)$. This is done by Davison in the case $N = 3$ [Dav20] and we generalize this result for all $N \geq 2$. We denote by Δ a Borel set, χ_Δ is the indicator function, M_f is the multiplication operator by f, $\iota : [0,1) \to \mathbb{T}, x \mapsto \mathrm{e}^{2\pi \mathrm{i} x}$. In addition, define $\sigma : \mathbb{Z}_N^{\mathbb{N}} \to \mathbb{Z}_N^{\mathbb{N}}$, $\sigma((\omega_1, \omega_2, \ldots)) = (\omega_2, \omega_3, \ldots)$ as well as the N inverse branches $\sigma_j : \mathbb{Z}_N^{\mathbb{N}} \to \mathbb{Z}_N^{\mathbb{N}}$, $\sigma_j((\omega_1, \omega_2, \ldots)) = (j, \omega_1, \omega_2, \ldots)$, $j \in \{0, \ldots, N-1\}$.

Theorem 3.4 (THEOREM 3.45). *The map $E : \mathrm{Bor}(\mathbb{T}) \to L^2(\mathcal{S}_N, m)$ defined by $E(\Delta) = M_{\chi_\Delta}$ is the unique spectral measure satifying*

$$E(\iota \circ \sigma^{-1} \circ \iota^{-1}(\Delta)) = \sum_{j=0}^{N-1} T_j E(\Delta) T_j^*,$$

and U satisfies

$$U = \int_{\mathbb{T}} z \mathrm{d}E.$$

In addition, we exhibit a spectral measure F on $\mathbb{Z}_N^{\mathbb{N}}$ which can be recovered from the partial isometries $(T_j)_{j=0}^{N-1}$ (Theorem 3.46), and we prove as well that the canonical spectral measure G on $L^2(\mathcal{S}_N, m)$ is the product of the two previous spectral measures (Theorem 3.47). Finally, we show that the unique operator on $L^2(\mathcal{S}_N)$ having this spectral measure is the multiplication operator.

Corollary 3.5 (COROLLARY 3.49). *Let f be a complex-valued, bounded, measurable function $f : \mathcal{S}_N \to \mathbb{C}$. The unique operator M_f such that $M_f = \int_{\mathcal{S}_N} f \mathrm{d}G$ is the multiplication operator $M_f : L^2(\mathcal{S}_N, m) \to L^2(\mathcal{S}_N, m)$, $M_f g = fg$.*

This chapter is structured as follows. In section 3.2, we introduce the relevant background for this article. Section 3.2.1 are quick recalls on Baumslag-Solitar groups to fix the notations, since most material can be found in the previous chapter (see section 2.4), and the N-solenoid. In Section 3.2.2 we explain how one can define wavelets on an arbitrary Hilbert space, with the use of representations of $\mathrm{BS}(1, N)$. We then go on to explain how to construct fractals from iterated function systems in Section 3.2.3, and we focus in particular on the IFS that we will use throughout this article, namely the IFS whose inverse branches can be chosen from the endomorphism $\tau(x) = Nx \pmod 1$. From these inverse branches, we can perform random walks on $\mathbb{Z}_N^{\mathbb{N}}$, which leads to the definition of probability measures on the solenoid. This is explained in Section 3.2.4. Eventually we define wavelet representations in Section 3.2.5, and explain in details how to create a wavelet representation on fractals as well as on the solenoid in Section 3.2.5.

We then proceed to our results, and start by the decomposition of the dilation operator on $L^2(\mathcal{S}_N, m)$ into partial isometries and analyze the properties of the latter in Section 3.3.1. In Section 3.3.2, we perform the analysis of the spectral measure of T as well as of the canonical spectral measures on $\mathbb{Z}_N^{\mathbb{N}}$ and $\mathcal{S}_N$. For this, we need to adapt some results of Jorgensen [Jor05] to the case of partial isometries, instead of isometries. This is also done in this section. Subsequently, we compute the Fourier coefficients of the measure m and show the existence of an isomorphism between $L^2(\mathcal{R}, \mu)$ and $L^2(\mathcal{S}_N, m)$ in Section 3.3.3. Finally, we conclude with an open question in section 3.3.4.

3.2 Framework and relevant background

3.2.1 Baumslag-Solitar groups and solenoid

The so-called Baumslag-Solitar groups were first discovered and studied as a mean to better understand non-Hopfian groups (see [BS62, BDD19]). Graham Higman conjectured that every finitely generated group with one defining relation is Hopfian [Hig51] ; which turned out to be false, as $\mathrm{BS}(2, 3)$ shows [BDD19]. Although the Baumslag-Solitar groups have a simple description, they have a rich structure and interesting properties. We will see that they are closely connected to wavelets, and describe this connection.

Definition 3.6. Let $M, N \in \mathbb{Z}$, the **Baumslag-Solitar group** $\mathrm{BS}(M, N)$ is the group defined over two generators by

$$\mathrm{BS}(M, N) = \langle u, t | tu^M t^{-1} = u^N \rangle . \tag{3.3}$$

For more properties and their proofs on Baumslag-Solitar groups, we refer to section 2.4. We will only recall the results we need in this chapter to fix the notations.

In the study of wavelets, we will see that we deal with representations of the groups $\mathrm{BS}(1, N)$, $N \geq 2$. These groups are thus going to be of particular interest to us. They are known to be solvable, and more importantly they can be embedded isometrically in a linear group.

Proposition 3.7 (SEE THEOREM 2.34 AND THEOREM 2.36). *Let $N \geq 2$, and let*

$$\mathbb{Z}[\tfrac{1}{N}] := \left\{ \frac{k}{N^\ell} \mid k, \ell \in \mathbb{Z} \right\}.$$

We have

$$\mathrm{BS}(1, N) \simeq \mathbb{Z}[\tfrac{1}{N}] \rtimes_\alpha \mathbb{Z} \hookrightarrow \mathrm{GL}_2(\mathbb{Q}),$$

where the action α of $\mathbb{Z}$ on $\mathbb{Z}[\frac{1}{N}]$ is the left multiplication by N, i.e. for $(a, b), (c, d) \in$

$\mathbb{Z}[\frac{1}{N}] \rtimes \mathbb{Z}$, *the group operation is defined by*

$$(a,b).(c,d) = (a + \alpha^b(c), b+d) = (a + N^b \cdot c, b+d).$$

We now turn our attention specifically to $\mathbb{Z}[\frac{1}{N}]$ and to its dual group, which will play a central role in this article. Recall that for a locally compact abelian group G, a continuous homomorphism $\xi\colon G \to \mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ is called a **character**, and the set of all characters is called the **dual group** of G and denoted by $\hat{G}$. We use the notation $\langle x, \xi \rangle = \xi(x)$, $x \in G$, $\xi \in \hat{G}$ to express the duality.

Definition 3.8. Let $N \geq 2$, and define the ***N*-solenoid** by

$$\mathcal{S}_N := \left\{ (z_n)_{n \geq 0} \mid z_{n+1}^N = z_n,\ |z_n| = 1 \text{ for } n \geq 0 \right\}.$$

We can also use an indexation by $\mathbb{Z}$ when it is more convenient

$$\mathcal{S}_N := \left\{ (z_n)_{n \in \mathbb{Z}} \mid z_{n+1}^N = z_n,\ |z_n| = 1 \text{ for } n \in \mathbb{Z} \right\},$$

where $(z_n)_{n \geq 0}$ is represented by $(z_n)_{n \in \mathbb{Z}}$ with $z_{-n} = z_0^{N^n}$ for $n > 0$.

Remark that $\mathcal{S}_N$ is a group for the pointwise multiplication which is compact as a closed subset of $\mathbb{T}^{\mathbb{N}}$.

Proposition 3.9. *Let $N \geq 2$, the dual group of $\mathbb{Z}[\frac{1}{N}]$ is $\mathcal{S}_N$.*

One can find a proof of this proposition when N is a prime in [BdlH19, see Proposition 3.D.2], and it is also proved for all $N \geq 2$ in [Dav20]. We give a third proof here.

Proof. Let us define the following pairing between $\mathbb{Z}[\frac{1}{N}]$ and $\mathcal{S}_N$:

$$\langle \tfrac{k}{N^\ell}, (z_n)_n \rangle = z_\ell^k$$

for $k/N^\ell \in \mathbb{Z}[\frac{1}{N}]$ and $(z_n)_{n \in \mathbb{Z}} \in \mathcal{S}_N$. Moreover, define

$$\begin{array}{rccccl} \varphi: & \mathcal{S}_N & \to & \widehat{Z[\frac{1}{N}]} & & \\ & (z_n)_n & \mapsto & \varphi((z_n)): & \mathbb{Z}[\frac{1}{N}] \to \mathbb{T} & \\ & & & & \lambda = \frac{k}{N^\ell} \mapsto \varphi((z_n))(\lambda) = \langle \frac{k}{N^\ell}, (z_n) \rangle = z_\ell^k. & \end{array} \tag{3.4}$$

We need to show that φ is a bijective homomorphism, and that $\varphi((z_n))$ is a well-defined continuous homomorphism (i.e. a character) for all $(z_n) \in \mathcal{S}_N$. Let us start by showing that $\varphi((z_n)_n)$ is well-defined, i.e. it does not depend of the representative $\lambda \in \mathbb{Z}[\frac{1}{N}]$: let $z = (z_n)_n \in \mathcal{S}_N$ and let $\lambda = \frac{k}{N^\ell} = \frac{k'}{N^{\ell'}} = \lambda'$, and without loss of generality, assume that $\ell \geq \ell'$. In this case, $\frac{k'}{N^{\ell'}} = \frac{k' N^{\ell - \ell'}}{N^\ell}$ and $k = k' N^{\ell - \ell'}$, which in turn implies

$$\varphi(z)(\lambda) = \langle \tfrac{k}{N^\ell}, z \rangle = z_\ell^k = (z_\ell^{k'})^{N^{\ell-\ell'}} = z_{\ell-(\ell-\ell')}^{k'} = z_{\ell'}^{k'} = \langle \tfrac{k'}{N^{\ell'}}, z \rangle = \varphi(z)(\lambda').$$

The fact that $\varphi((z_n))$ is a continuous homomorphism is due to the fact that $\mathbb{Z}[\frac{1}{N}]$ is a discrete group, and if $\lambda = k/N^\ell, \tilde{\lambda} = \tilde{k}/N^{\tilde{\ell}} \in \mathbb{Z}[\frac{1}{N}]$, and $z = (z_n) \in \mathcal{S}_N$, we have

$$\begin{aligned} \varphi(z)(\lambda + \tilde{\lambda}) &= \langle k/N^\ell + \tilde{k}/N^{\tilde{\ell}}, z\rangle \\ &= z_{\ell+\tilde{\ell}}^{kN^{\tilde{\ell}}+\tilde{k}N^\ell} \\ &= z_{\ell+\tilde{\ell}}^{kN^{\tilde{\ell}}} z_{\ell+\tilde{\ell}}^{\tilde{k}N^\ell} \\ &= \langle k/N^\ell, z\rangle \langle \tilde{k}/N^{\tilde{\ell}}, z\rangle \\ &= (\varphi(z)(\lambda))(\varphi(z)(\tilde{\lambda})). \end{aligned}$$

We now show that φ is a bijective homomorphism. For $\lambda = k/N^\ell \in \mathbb{Z}[\frac{1}{n}]$ and $z, z' \in \mathcal{S}_N$, we have

$$\varphi(zz')(\lambda) = \langle k/N^\ell, zz'\rangle = (zz')_\ell^k = z_\ell^k {z'}_\ell^k = (\varphi(z)\varphi(z'))(\lambda).$$

Moreover, if $\varphi((z_n))(\lambda) = 1$ for every $\lambda \in \mathbb{Z}[\frac{1}{N}]$, then $z_n = 1$ for all $n \in \mathbb{Z}$, so that φ is a one-to-one homomorphism. Finally, if ξ is a character of $\mathbb{Z}[\frac{1}{N}]$, let us use the indexation by $\mathbb{N}$ and define $(z_n)_{n\geq 0} \in \mathcal{S}_N$ by

$$z_n = \xi(N^{-n}).$$

Then we obtain

$$\varphi((z_n))(\lambda) = \langle \lambda, (z_n)\rangle = \xi(N^{-\ell})^k = \xi(k/N^\ell) = \xi(\lambda).$$

This shows that φ is onto and completes the proof. ■

If $\lambda = \frac{k}{N^\ell} \in \mathbb{Z}[\frac{1}{N}]$, the character $\xi_\lambda : \mathcal{S}_N \to \mathbb{T}$, $(z_n) \mapsto \langle \lambda, (z_n)\rangle = z_\ell^k$ is the **character on $\mathcal{S}_N$ attached to λ**.

Remark 3.10. Observe that the dual action $\hat{\alpha}$ of $\mathbb{Z}$ on $\mathcal{S}_N$ is given by the inverse shift

$$\begin{array}{rccl} S^{-1} : & \mathcal{S}_N & \to & \mathcal{S}_N \\ & (z_n)_{n\in\mathbb{N}} & \mapsto & S^{-1}((z_n)_{n\in\mathbb{N}}) = (z_{n+1})_{n\in\mathbb{N}} = (z_1, z_2, z_3, \ldots). \end{array} \tag{3.5}$$

and we have

$$\hat{\alpha}^j (z_n)_{n\in\mathbb{Z}} = S^{-j}(z_n)_{n\in\mathbb{Z}},$$

where the **shift on $\mathcal{S}_N$** is:

$$\begin{array}{rccl} S : & \mathcal{S}_N & \to & \mathcal{S}_N \\ & (z_n)_{n\in\mathbb{N}} & \mapsto & S((z_n)_{n\in\mathbb{N}}) = (z_{n-1})_{n\in\mathbb{N}} = (z_0^N, z_0, z_1, \ldots). \end{array} \tag{3.6}$$

3.2.2 Wavelets on arbitrary Hilbert spaces

In textbooks, wavelet theory is customarily presented for $L^2(\mathbb{R})$, in order to construct orthonormal bases that have good localization properties [Dau92, PW12]. In the classical theory, we define two operators[1] on $L^2(\mathbb{R})$,

$$U\ : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ Uf(x) = f(x-1) \quad \text{and} \quad T\ : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ Tf(x) = \tfrac{1}{\sqrt{N}} f(\tfrac{x}{N}),$$

where U is the translation operator (or shift operator), T is the dilation operator (or scaling operator) and $N \geq 2$ is an integer usually called the scale. We define a **wavelet** to be a finite set $\{\psi_1, \ldots, \psi_\ell\} \subset L^2(\mathbb{R})$ such that

$$\{\, T^j U^k \psi_i \mid j,k \in \mathbb{Z},\ i \in \{1,\ldots,\ell\}\,\}$$

is an orthonormal basis for $L^2(\mathbb{R})$. Note that for the operators U, T defined above, we have

$$\begin{aligned} U^N &\ : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ U^N f(x) = f(x-N), \\ T^{-1} &\ : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ T^{-1} f(x) = \sqrt{N} f(Nx), \end{aligned}$$

and

$$\begin{aligned} TUT^{-1}f(x) &= \tfrac{1}{\sqrt{N}} UT^{-1} f(\tfrac{x}{N}) \\ &= \tfrac{1}{\sqrt{N}} T^{-1} f(\tfrac{x}{N} - 1) \\ &= \tfrac{1}{\sqrt{N}} T^{-1} f(\tfrac{x-N}{N}) \\ &= \tfrac{1}{\sqrt{N}} \sqrt{N} f(\tfrac{x-N}{N} N) \\ &= f(x-N) \\ &= U^N f(x), \end{aligned}$$

thus the operators U, T satisfy the Baumslag-Solitar relation for $\mathrm{BS}(1,N)$, and we are dealing with a representation of $\mathrm{BS}(1,N)$.

From the previous observation, one can define wavelets on other Hilbert spaces than $L^2(\mathbb{R})$ using representations of $\mathrm{BS}(1,N)$. In this article, we follow [Dut06] for the definition of wavelets on an arbitrary Hilbert space $\mathcal{H}$. This gives us a more general and abstract setting to work with.

Definition 3.11. Let $\pi\ : \mathrm{BS}(1,N) \to \mathcal{U}(\mathcal{H})$ be a unitary representation of $\mathrm{BS}(1,N)$ in an arbitrary Hilbert space $\mathcal{H}$, which will be denoted by $(\pi, \mathcal{H})$. We use capital letters to denote the corresponding operators, $\pi(u) = U$ and $\pi(t) = T$. A **wavelet for the representation $\boldsymbol{\pi}$** is a finite set $\{\psi_1, \ldots, \psi_\ell\} \subset \mathcal{H}$ such that

$$\left\{\, \pi(t^j u^k)\psi_i \;\middle|\; j,k \in \mathbb{Z},\ i \in \{1,\ldots,\ell\} \,\right\}$$

is an othonormal basis for $\mathcal{H}$.

The search for good wavelet bases is a very active area of research since the work of Morlet in the late 1970's (for a wonderful report on the development of wavelets, we recommend Daubechies' article [Dau96]). It was not until Mallat and Meyer finalized

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[1] We warn the reader that in this chapter, the role of U and T is reversed compared to Dutkay's article [Dut06], the reason being that we chose to view $\mathrm{BS}(1,N)$ as an HNN-extension whose stable letter is t, and we obviously want $\pi(u) = U$ and $\pi(t) = T$ rather than the converse.

the details of the so-called multiresolution analysis that we had a powerful tool to create “good” wavelet bases [Mal87]. A few years later, Lemarié and Auscher proved that any reasonable wavelet base comes from an MRA [LR92, Aus92]. Thus, we naturally extend this tool to Hilbert spaces $\mathcal{H}$ such that $(\pi, \mathcal{H})$ is a representation of $\mathrm{BS}(1, N)$.

Definition 3.12. An **(orthogonal) multiresolution analysis** for $\mathcal{H}$, which we often abbreviate by MRA, with **scaling vector** φ is a collection of closed subspaces $\{V_j\}_{j\in\mathbb{Z}} \subset \mathcal{H}$ such that

1. $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$ (increasing subspaces), thus
$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots ,$$
2. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ (trivial intersection),
3. $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $\mathcal{H}$ (closure is $\mathcal{H}$),
4. $f \in V_{j+1} \iff Tf \in V_j$ (scale invariance), which can be rewritten as $TV_{j+1} = V_j$,
5. $\varphi \in V_0$ and $\{U^k\varphi \mid k \in \mathbb{Z}\}$ forms an orthonormal basis for V_0.

Any MRA naturally comes with a **scaling vector** φ, which satisfies $T\varphi = \sum_{k\in\mathbb{Z}} a_k U^k \varphi$ for some coefficients $a_k \in \mathbb{C}$ (the a_k’s exist because $T\varphi \in V_{-1} \subset V_0$), from which we define a function called the **low-pass filter** $m_0(z) = \sum_{k\in\mathbb{Z}} a_k z^k$. In the $L^2(\mathbb{R})$ case, Cohen gave some explicit conditions on m_0 (and thus the sequence (a_k)) for m_0 to be issued from an MRA [Coh90, Theorem 2.1].

Let us quickly come back to the case $L^2(\mathbb{R})$. Given $N \geq 2$ and an MRA for $L^2(\mathbb{R})$ with scaling function φ, we have a well-oiled routine to construct functions $\psi_1, \dots, \psi_{N-1} \in L^2(\mathbb{R})$ such that $\{\psi_1, \dots, \psi_{N-1}\}$ is a wavelet. If $N = 2$, we have a direct formula for ψ, and if $N > 2$, the functions $\{\psi_1, \dots, \psi_{N-1}\}$ result from the solution of a matrix completion problem (see [DJ06b, BJ00, BJ02]). The general idea is to solve the matrix completion problem, from which we obtain subband filters $m_1, \dots, m_{N-1} \in L^\infty(\mathbb{T})$ and we construct the wavelet functions in the Fourier domain from these filters as [BJ02, Thm 2.2.1]

$$\hat{\psi}_i(\lambda) = \frac{1}{\sqrt{N}} m_i(\tfrac{\lambda}{N})\hat{\varphi}(\tfrac{\lambda}{N}), \quad \lambda \in \mathbb{R}, i \in \{1, \dots, N-1\} \tag{3.7}$$

where the Fourier transform is defined as

$$\mathcal{F}\ : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ f \mapsto \mathcal{F}(f)(\lambda) = \hat{f}(\lambda) = \int_{\mathbb{R}} f(t)\mathrm{e}^{-2\pi \mathrm{i}\lambda t}\mathrm{d}t.$$

Using the usual time-frequency dictionary of the Fourier transform, one can check that equation (3.7) is in fact equivalent in the time domain to

$$\psi_i(t) = (T^{-1}m_i(U)\varphi)(t) = \sqrt{N}\sum_{k\in\mathbb{Z}} c_k(m_i)U^k\varphi(Nt) = \sqrt{N}\sum_{k\in\mathbb{Z}} c_k(m_i)\varphi(Nt-k) \tag{3.8}$$

where $m_i(z) = \sum_{k\in\mathbb{Z}} c_k(m_i)z^k$ is the Fourier decomposition of the filter m_i. The conclusion of Theorem 2.2.1 in [BJ02] is that

$$\left\{ T^j U^k \psi_i \,\middle|\, j,k \in \mathbb{Z}\, i \in \{1,\dots,N-1\} \right\}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

It turns out that a similar construction works for a larger class of Hilbert spaces, namely the class $L^2(\mathcal{R},\mathcal{H}^s)$ where $\mathcal{R}$ is a fractal set coming from an iterated function system, and $\mathcal{H}^s$ its Hausdorff measure [DJ06b], which we describe in the following subsections. The fact that this works on specific Hilbert spaces raises the question "if we have a representation of $\mathrm{BS}(1,N)$ that admits a wavelet, what restrictions do we have on the representation ?" Dutkay [Dut06] generalized a result of Martin and Valette [MV00] that answers this question.

Theorem 3.13 ([Dut06, Theorem 2.1]). *Let π be a representation of* $\mathrm{BS}(1,N)$ *that has a wavelet. Then π is faithful on $C_r^*(\mathrm{BS}(1,N))$ (the reduced C^*-algebra of* $\mathrm{BS}(1,N)$*, i.e. the C^*-algebra generated by the right regular representation) and weakly equivalent to the right regular representation of the group. Each element in the C^*-algebra generated by π has a connected spectrum.*

Let $\pi : \mathrm{BS}(1,N) \to \mathcal{U}(\mathcal{H})$ be a unitary representation of $\mathrm{BS}(1,N)$. Since $U \in \mathcal{U}(\mathcal{H})$, we can define a representation ρ of $L^\infty(\mathbb{T})$ on $\mathcal{H}$ using Borel functional calculus by

$$\rho(f) = f(U), \quad f \in L^\infty(\mathbb{T}).$$

This representation thus satisfies the relation

$$T\rho(f(z))T^{-1} = \rho(f(z^N)), \quad f \in L^\infty(\mathbb{T}),\, \rho(z \mapsto z) = U.$$

From Definition 3.12.4, we know that $T\varphi \in V_{-1} \subset V_0$ and since $\left\{ U^k\varphi \,\middle|\, k \in \mathbb{Z} \right\}$ must form an orthonormal basis of V_0, φ must satisfy the so-called **scaling equation**

$$T\varphi = \sum_{k\in\mathbb{Z}} a_k U^k \varphi = \rho(m_0)\varphi \tag{3.9}$$

where $m_0(z) = \sum_{k\in\mathbb{Z}} a_k z^k \in L^\infty(\mathbb{T})$ for some coefficients $a_k \in \mathbb{C}$ is the low-pass filter. When the $a_k \in \mathbb{C}$ are chosen in advance, i.e. when the low-pass filter is prescribed, it is well-understood when equation (3.9) admits solutions $\varphi \in L^2(\mathbb{R})$. It is a fairly non-trivial question to solve the analogue question, but in an arbitrary Hilbert space. Dutkay and Jorgensen studied this question for Hilbert spaces constructed from a Hausdorff measure [DJ06b]. We will be interested in a subset of those Hilbert spaces, and in the next sections we explain how to construct them through the use of iterated function systems.

3.2.3 Iterated Function Systems (IFS)

We turn to a specific construction of fractals, which uses Iterated Function Systems (IFS). IFS have been studied in a variety of contexts (see for instance [AST04, ALTW04, DJ06a, Hut81, Jor05, JP96, JP98, Str00]). We first recall two theorems from Hutchinson [Hut81] that will be useful to construct the spaces we need.

Theorem 3.14 ([HUT81]). *Let $(\tau_i)_{i=1}^p$ be a system of p contractive maps on a complete metric space X. Then there is a unique non empty compact subset $K \subset X$ such that*

$$K = \bigcup_{i=1}^{p} \tau_i(K). \tag{3.10}$$

Theorem 3.15 ([HUT81]). *Let $(\tau_j)_{j=1}^p$ be a system of p contractive maps on a complete metric space X. There is a unique probability measure μ on X such that for every Borel subsets E*

$$\mu(E) = \frac{1}{p}\sum_{j=1}^{p} \mu \circ \tau_j^{-1}(E). \tag{3.11}$$

Moreover, μ is compactly supported on K as defined in eq. (3.10). *This measure is called the* ***Hutchinson measure****.*

We leave the general case of metric spaces to restrict ourselves to $\mathbb{R}^d$. The usual definition of an IFS is the following.

Definition 3.16 ([DJ08]). Let A be a $d \times d$ expansive integer matrix where we say that a matrix is **expansive** if all its eigenvalues have absolute value strictly greater than 1. Let $B \subset \mathbb{Z}^d$ be a finite set of points, of cardinality $|B| = p$. For each $b \in B$, define the following affine maps on $\mathbb{R}^d$,

$$\tau_b(x) = A^{-1}(x + b), \quad (x \in \mathbb{R}^d). \tag{3.12}$$

The family of functions $(\tau_b)_{b\in B}$ is called an **affine iterated function system (affine IFS)**.

Note that we can apply theorem 3.14 in this case because A is an expansive matrix, thus we can find a norm on $\mathbb{R}^d$ such that all the maps τ_b are contractive.

Definition 3.17. The compact set K defined by equation (3.10) is called the **attractor** of the IFS $(\tau_i)_{i=1}^p$.

We can give a more explicit representation of the attractor K, namely

$$K = \left\{ \sum_{j\geq 1} A^{-j} b_j \;\middle|\; b_j \in B \right\} \tag{3.13}$$

or

$$AK = \bigcup_{b\in B} (K + b).$$

If we assume that the sets $K + b$, $b \in B$ are disjoint from one another, then the function $\varphi := \chi_K$ satisfies a form of scaling equation, namely

$$\varphi(A^{-1}x) = \sum_{b\in B} \varphi(x - b), \quad x \in \mathbb{R}^d.$$

In fact, the role played by the scaling function in the classical wavelet theory is played here by an invariant measure.

We further restrict ourselves to the case that we are most interested in here, i.e. the case of the real line. We introduce the map

$$\begin{array}{rcl} \tau \ : \ [0,1] & \to & [0,1] \\ x & \mapsto & Nx \pmod 1 \end{array} \tag{3.14}$$

This map has N inverse branches, namely the maps $(\tau_j)_{j=0}^{N-1}$ defined by $\tau_j(x) = \frac{x+j}{N}$ (see Fig. 3.1 for an example).

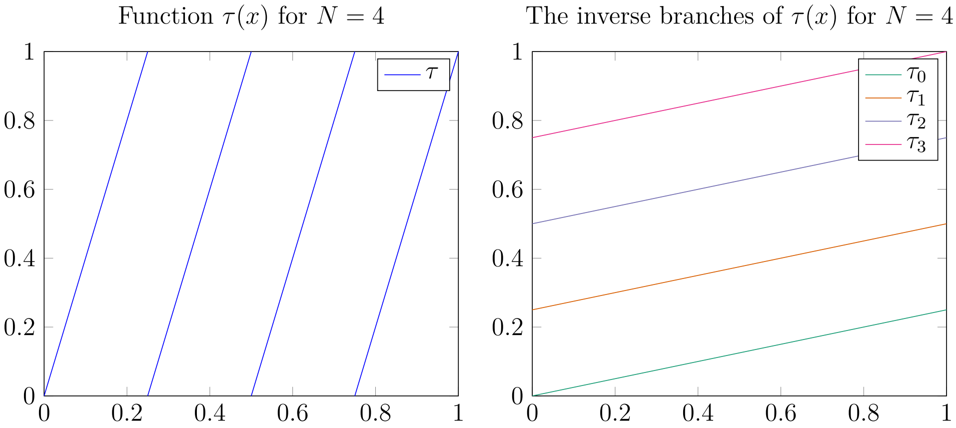

Figure 3.1: Example of the map τ and its inverse branches

The IFS we consider are of the form $(\tau_{a_j})_{j=1}^p$ where $N \geq 2$ is an integer and $(a_j)_{j=1}^p \subset \{0, 1, \dots, N-1\}$ are p distinct integers. The p maps τ_{a_j} correspond to p different inverse branches of the endomorphism τ. From Theorem 3.14, we know that there exists a unique compact $K \subset [0,1]$ which is the attractor of the IFS. Equation (3.13) gives us the representation

$$K = \left\{ \sum_{j\geq 1} b_j N^{-j} \;\middle|\; b_j \in \{a_1, \dots, a_p\},\, j \geq 1 \right\}. \tag{3.15}$$

We also obtain a probability measure μ supported on K from Theorem 3.15. Note that we have $|\tau_{a_i}(K) \cap \tau_{a_j}(K)| \leq 1$ for any $i, j \in \{1, \dots, p\}$, $i \neq j$, i.e. they have at most one point in common, namely those of the form $\frac{a_k}{N}$ for some $k \in \{1, \dots, p\}$.

In order to work with wavelet representation, we need translations as well as a form of scaling. Thus we consider an inflated version of the attractor K, which is, at its core, a union of scaled, and translated copies of K. We follow [DJ06b] for the definition :

$$\mathcal{R} := \left\{ \sum_{j\geq -m} b_j N^{-j} \;\middle|\; m \in \mathbb{Z}, b_j \in \{a_1, \dots, a_p\} \text{ for all but finitely many indices } j \right\}. \tag{3.16}$$

It is the set of real numbers that admit a base N expansion containing a finite number of elements of the set $\mathbb{Z}/N\mathbb{Z} \setminus \{a_1, \dots a_p\}$. We have the following properties for the set $\mathcal{R}$.

Proposition 3.18 ([DJ06b, Prop. 2.1]). *1. Invariance under N-adic translation : $\mathcal{R} + \frac{k}{N^n} = \mathcal{R}$, $k \in \mathbb{Z}$, $n \in \mathbb{Z}$.*

2. Invariance under dilation by N : $N^n\mathcal{R} = \mathcal{R}$, $n \in \mathbb{Z}$.

3. The attractor K is contained in $\mathcal{R}$ and moreover it covers $\mathcal{R}$ by translations and dilations

$$\mathcal{R} := \bigcup_{n\geq 0}\bigcup_{k\in\mathbb{Z}} N^{-n}(K+k) = \bigcup_{n\in\mathbb{Z}}\bigcup_{k\in\mathbb{Z}} N^{-n}(K+k). \tag{3.17}$$

From [DJ06b, Prop. 2.3] we can extend μ to a Borel measure on $\mathcal{R}$. We also call this measure μ. This measure is invariant under translations by elements of $\mathcal{R}$, i.e for any μ-measurable set E and any $t = \frac{a_i}{N^k} \in \mathcal{R}$,

$$\mu(E) = \mu(E + \tfrac{a_i}{N^k}), \tag{3.18}$$

and in addition, for every μ-measurable set E and any $c > 0$

$$\mu(cE) = c^h \mu(E)$$

where h is the Hausdorff dimension of K[2]. This last equation implies that

$$\mu(NE) = p\mu(E). \tag{3.19}$$

Finally, from [DJ06b, Prop. 2.3], we have that μ is not regular from above (i.e. all open sets have infinite measure).

3.2.4 Random walks and measures on the solenoid

We temporarily leave the fractal world to go back to the solenoid. In this section, we will see a first connection between the fractal $\mathcal{R}$ and the solenoid. This connection will come from the fact that we use the inverse branches $(\tau_{a_j})_{j=1}^p$ to define a random walk on the solenoid. To do this, we will need to work with another representation of $\mathcal{S}_N$, namely $[0,1) \times \mathbb{Z}_N^{\mathbb{N}}$ and define a probability measure on $\mathbb{Z}_N^{\mathbb{N}}$ which is constructed using the low-pass filter $m_0(z) = \frac{1}{\sqrt{p}}\sum_{j=1}^p z^{a_j}$, where $(a_j)_{j=1}^p$ is the same subset of $\{0,\dots,N-1\}$ that we chose in the previous section.

Lemma 3.19. *The low-pass filter m_0 satisfies for all $z \in \mathbb{T}$*

$$\frac{1}{N}\sum_{y^N=z} |m_0(y)|^2 = 1.$$

Proof. This is a direct computation, using that $z = \mathrm{e}^{-2\pi \mathrm{i} x}$ for some $x \in [0,1]$, and

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[2] Since we don't need to work with the Hausdorff dimension here, we don't define it. The interested reader can consult [Fal86] [Jor06, p.72]. Here, we have $h = \frac{\ln p}{\ln N} = \log_N(p)$.

$\tau_j(x) = \frac{x+j}{N}$ is the j-th inverse branch of the endomorphism τ defined in Eq. (3.14).

$$\begin{aligned}
\frac{1}{N}\sum_{y^N=z}|m_0(y)|^2 &= \frac{1}{N}\sum_{j=0}^{N-1}|m_0(\mathrm{e}^{-2\pi\mathbf{i}\tau_j(x)})|^2 \\
&= \frac{1}{Np}\sum_{j=0}^{N-1}\left|\sum_{k=1}^{p}\mathrm{e}^{-2\pi\mathbf{i}\tau_j(x)a_k}\right|^2 \\
&= \frac{1}{Np}\sum_{j=0}^{N-1}\sum_{k=1}^{p}\sum_{\ell=1}^{p}\mathrm{e}^{-2\pi\mathbf{i}\tau_j(x)a_k}\mathrm{e}^{2\pi\mathbf{i}\tau_j(x)a_\ell} \\
&= 1+\frac{1}{Np}\sum_{j=0}^{N-1}\sum_{k=1}^{p}\sum_{\substack{\ell=1\\ \ell\neq k}}^{p}\mathrm{e}^{2\pi\mathbf{i}\tau_j(x)(a_\ell-a_k)} \\
&= 1+\frac{1}{Np}\sum_{k=1}^{p}\sum_{\substack{\ell=1\\ \ell\neq k}}^{p}\mathrm{e}^{2\pi\mathbf{i}x(a_\ell-a_k)/N}\sum_{j=0}^{N-1}(\mathrm{e}^{2\pi\mathbf{i}(a_\ell-a_k)/N})^j \\
&= 1+\frac{1}{Np}\sum_{k=1}^{p}\sum_{\substack{\ell=1\\ \ell\neq k}}^{p}\mathrm{e}^{2\pi\mathbf{i}x(a_\ell-a_k)/N}\frac{\overbrace{1-\mathrm{e}^{2\pi\mathbf{i}(a_\ell-a_k)N/N}}^{=0}}{1-\mathrm{e}^{2\pi\mathbf{i}(a_\ell-a_k)/N}} \\
&= 1.
\end{aligned}$$

■

Let $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$, and let $\Omega := \mathbb{Z}_N^{\mathbb{N}} = \{\,\omega = (\omega_1, \omega_2, \dots) \mid \omega_j \in \mathbb{Z}_N\,\forall j\,\}$. Ω is a compact set for Tychonoff's topology. Let $\omega_1, \dots, \omega_n \in \mathbb{Z}_N$, we call the subsets

$$A(\omega_1, \dots, \omega_n) = \{\,\tilde{\omega} = (\tilde{\omega_1}, \tilde{\omega_2}, \tilde{\omega_3}, \dots) \in \Omega \mid \tilde{\omega_1} = \omega_1, \dots \tilde{\omega_n} = \omega_n\,\} \tag{3.20}$$

cylinder sets. The cylinder sets generate the topology of Ω and its Borel σ-algebra. For convenience, let us denote $\frac{|m_0(z)|^2}{N}$ by $W(z)$, where $z = \mathrm{e}^{-2\pi\mathbf{i}x} \in \mathbb{T}$ for some $x \in [0,1]$, and observe that Lemma 3.19 implies

$$\sum_{y^N=z} W(y) = 1 \iff \sum_{j=0}^{N-1} W(\tau_j(x)) = 1 \quad \forall x \in [0,1].$$

Proposition 3.20 ([Jor06, Lemma 2.4.1])**.** *For every $x \in [0,1]$, there is a unique positive Radon probability measure P_x on Ω such that*

$$P_x(A(\omega_1, \dots, \omega_n)) = W(\tau_{\omega_1}x)W(\tau_{\omega_2}\tau_{\omega_1}x)\cdots W(\tau_{\omega_n}\cdots\tau_{\omega_1}x). \tag{3.21}$$

For a Radon measure μ on Ω, we set

$$\mu[f] := \int_\Omega f(\omega)\mathrm{d}\mu(\omega) \quad \forall f \in C(\Omega).$$

Thus if $f \in C(\Omega)$ depends only on a finite number of coordinates $\omega_1, \ldots, \omega_n$, we obtain

$$P_x[f] = \int_\Omega f \mathrm{d}P_x = \sum_{(\omega_1,\ldots,\omega_n)\in\mathbb{Z}_N^n} W(\tau_{\omega_1}x)\cdots W(\tau_{\omega_n}\cdots\tau_{\omega_1}x) f(\omega_1,\ldots,\omega_n). \tag{3.22}$$

Hidden above is an implicit application of Riesz's theorem, because when we write $\mu[f] = \int_\Omega f \mathrm{d}\mu$, we identify positive linear functionals μ on $C(\Omega)$ with the corresponding Radon measures $\tilde{\mu}$ on Ω, i.e. $\mu[f] = \int_\Omega f \mathrm{d}\tilde{\mu}$ (see [Jor06, Sec 2.5], [Wal87, Chap. 1] for more details).

Denote by $M(\Omega)$ the σ-algebra of measurable subsets of Ω generated by the cylinders. Observe that the subsets $\tau_{\omega_n}\cdots\tau_{\omega_1}([0,1])$ also generate a σ-algebra of measurable subsets, namely that of $[0,1]$ which we denote by $M([0,1])$. Note that, in our case, $\tau_{\omega_n}\cdots\tau_{\omega_1}([0,1])$ are in fact N-adic subintervals, i.e.

$$\tau_{\omega_n}\cdots\tau_{\omega_1}([0,1]) = \left[\frac{\omega_1}{N} + \cdots + \frac{\omega_n}{N^n}, \frac{\omega_1}{N} + \cdots + \frac{\omega_n}{N^n} + \frac{1}{N^n}\right].$$

We identify the measurables subsets of Ω generated by the cylinders with characteristic functions, so that $M([0,1])$ is the σ-algebra generated by characteristic functions of N-adic intervals.

Lemma 3.21 ([Jor06, Lemma 2.6.2])**.** *There is a unique mapping $\Theta : M(\Omega) \to M([0,1])$ which satisfies*

$$\Theta(fg) = \Theta(f)\Theta(g)$$

and

$$\Theta\left(\chi_{A(\omega_1,\ldots,\omega_n)}\right) = \chi_{\tau_{\omega_1}\cdots\tau_{\omega_n}}([0,1]).$$

The mapping Θ is an isomorphism of $M(\Omega)$ onto $M([0,1])$.

Lemma 3.22 ([Jor06, Lemma 2.6.3])**.** *Let $\{ P_x \mid x \in [0,1] \}$ be the process obtained in proposition 3.20. Then*

$$\sum_{j=0}^{N-1} W(\tau_j x) P_{\tau_j x}[f(j,\cdot)] = P_x[f] \quad \forall f \in C(\Omega) \tag{3.23}$$

where $C(\Omega)$ is the space of continuous functions on Ω.

This last lemma means that W induces a Markov chain on $[0,1]$ with transition probabilities $W(\tau_j x)$ for all $x \in [0,1]$.

We can now relate the set Ω with the solenoid. The next results come from [Dut06].

Proposition 3.23. *The map*

$$\begin{array}{rrcl} \Phi : & [0,1)\times\Omega & \to & \mathcal{S}_N \\ & (x,\omega) & \mapsto & (\mathrm{e}^{-2\pi\mathbf{i}x}, \mathrm{e}^{-2\pi\mathbf{i}\tau_{\omega_1}x}, \mathrm{e}^{-2\pi\mathbf{i}\tau_{\omega_2}\tau_{\omega_1}x}, \ldots) \end{array} \tag{3.24}$$

is a bimeasurable bijection.

Recall that the shift on $\mathcal{S}_N$ was defined in Eq. (3.6). With the identification from Eq.

(3.24), the shift on $[0,1)\times\Omega$ becomes

$$\Phi^{-1}S\Phi(x,\omega)=(\tau(x),i_x,\omega_1,\omega_2,\ldots)$$

where i_x is the class $\pmod N$ of k if $x\in[\frac{k}{N},\frac{k+1}{N}[$. The inverse shift is

$$\Phi^{-1}S^{-1}\Phi(x,\omega)=(\tau_{\omega_1}x,\omega_2,\omega_3,\ldots).$$

Since we have the Lebesgue measure on $[0,1)$ and the measures P_x on Ω, it is now straightforward to define a measure m on $\mathcal{S}_N$ using the identification above. We define

$$\int_{\mathcal{S}_N} f\mathrm{d}m=\int_0^1\int_\Omega f(\Phi(x,\omega))\mathrm{d}P_x(\omega)\mathrm{d}x,\quad f\in C(\mathcal{S}_N). \tag{3.25}$$

As previously, we identify the Radon measure m with the positive linear functional m on $C(\mathcal{S}_N)$ defined by $m[f]=\int_{\mathcal{S}_N} f\mathrm{d}m$. In addition, note that a function f on $\mathbb{T}$ can be identified as a function on $\mathcal{S}_N$ that depends only on the first coordinate, that is $f((z_n)_{n\in\mathbb{Z}})=f(z_0)$ for $(z_n)_{n\in\mathbb{Z}}\in\mathcal{S}_N$.

Dutkay proved that the measure m has the following properties.

Proposition 3.24 ([Dut06, Proposition 4.2]). *1. The measure m is a probability measure, i.e.*

$$m(\mathcal{S}_N)=1;$$

2. if $f\in L^1(m)$ depends only on the first n coordinates then

$$\int_{\mathcal{S}_N} f\mathrm{d}m=\int_{\mathbb{T}}\sum_{y^{N^n}=z} f(y^{N^{n-1}},y^{N^{n-2}},\ldots,y^N,y)W^{(n)}(y)\mathrm{d}z$$

where $W^{(n)}(y)=W(y)W(y^N)\cdots W(y^{N^{n-1}})$;

3. m is the unique probability measure on $\mathcal{S}_N$ satisfying

$$\int_{\mathcal{S}_N} f\mathrm{d}m=\int_{\mathbb{T}} f(z)\mathrm{d}z\quad\forall f\in L^1(\mathbb{T}) \tag{3.26}$$

(where $\mathrm{d}z$ is the standard Haar measure on $\mathbb{T}$), and

$$\int_{\mathcal{S}_N} f\circ S^{-1}\mathrm{d}m=\int_{\mathcal{S}_N} NWf\mathrm{d}m\quad\forall f\in L^1(m)$$

(where $W\in L^\infty(\mathbb{T})$ is identified as a function on $\mathcal{S}_N$) i.e. the Radon-Nikodym derivative of $m\circ S$ with respect to m is

$$\frac{\mathrm{d}(m\circ S)}{\mathrm{d}m}=NW$$

and $m\circ S$ is absolutely continuous with respect to m;

4. If $W(z)=\sum_{k\in\mathbb{Z}}c_k(W)z^k$ (where $c_k(W)$ are the Fourier coefficients of W) then for *all* $\lambda = \frac{k}{N^\ell} \in \mathbb{Z}[\frac{1}{N}]$

$$\hat{m}(\lambda) := \int_{\mathcal{S}_N} \xi_\lambda \mathrm{d}m = \int_{\mathbb{T}} z^k N^\ell W^{(\ell)} \mathrm{d}z$$

where ξ_λ *is the character on* $\mathcal{S}_N$ *attached to* λ*. Moreover* $\hat{m}(k) = \delta_0(k)$ *for* $k \in \mathbb{Z}$ *and* $\hat{m}$ *satifies the following scaling equation :*

$$\hat{m}(\lambda) = N \sum_{k \in \mathbb{Z}} c_k(W) \hat{m}(N\lambda + k) \quad \forall \lambda \in \mathbb{Z}[\tfrac{1}{N}]. \tag{3.27}$$

Davison [Dav20] proves the following proposition in the case $N = 3$, but it can be applied to the case $N \geq 2$.

Proposition 3.25. *Let* $m_0 \in L^\infty(\mathbb{T})$ *be non-singular. The measure* m *is absolutely continuous with respect to* $m \circ S$ *and the Radon-Nikodym derivative is*

$$\frac{1}{NW} = \frac{1}{|m_0|^2} = \frac{\mathrm{d}m}{\mathrm{d}(m \circ S)} \in L^1(\mathcal{S}_N, m \circ S).$$

Proof. Since m_0 is non-singular, $W = \frac{|m_0|^2}{N}$ is positive and non-singular, i.e. $W(z) \geq 0$ for all $z \in \mathbb{T}$ and W vanishes on a set of measure m zero. Indeed, it is clear that W vanishes on a set of Haar measure 0 on $\mathbb{T}$, and if we view it as a function on $\mathcal{S}_N$, it depends only on the first coordinate and we conclude by eq. (3.26) applied to the indicator function $\chi_{\{\, z \in \mathbb{T} | W(z) = 0 \,\}}$. Now let $\Delta \in \mathrm{Bor}(\mathcal{S}_N)$ be such that $m \circ S(\Delta) = 0$ and let $\varepsilon > 0$,

$$\begin{aligned}
m(\Delta) &= \int_{\mathcal{S}_N} \chi_{\Delta \cap \{\, z \in \mathbb{T} | |m_0(z)|^2 < \varepsilon \,\}} + \chi_{\Delta \cap \{\, z \in \mathbb{T} | |m_0(z)|^2 \geq \varepsilon \,\}} \mathrm{d}m \\
&\leq m(\{\, z \in \mathbb{T} \mid |m_0(z)|^2 < \varepsilon \,\}) + \int_{\Delta \cap \{\, z \in \mathbb{T} | |m_0(z)|^2 \geq \varepsilon \,\}} \frac{1}{|m_0|^2} \mathrm{d}m \circ S \\
&\leq m(\{\, z \in \mathbb{T} \mid |m_0(z)|^2 < \varepsilon \,\}) + \frac{1}{\varepsilon} \int_\Delta \mathrm{d}m \circ S \\
&= m(\{\, z \in \mathbb{T} \mid |m_0(z)|^2 < \varepsilon \,\}) \\
&\xrightarrow{\varepsilon \to 0} 0.
\end{aligned}$$

This proves that m is absolutely continuous with respect to $m \circ S$ and the Radon-Nikodym derivative is

$$\frac{1}{NW} = \frac{1}{|m_0|^2} = \frac{\mathrm{d}m}{\mathrm{d}(m \circ S)} \in L^1(\mathcal{S}_N, m \circ S).$$

■

3.2.5 Wavelet representations

We now have all the required tools to study wavelet representations of $\mathrm{BS}(1, N)$ on the spaces that interest us. We will study two particular representations, both associated to the same low-pass filter. Later in Section 3.3 we will explore the connections between these two representations.

We first give the definition of a spectral measure and state a general functional analysis fact before giving the definition of a wavelet representation. Then we explain how we can create wavelet representations on fractals and on the N-solenoid.

Definition 3.26. Let $(X, \mathcal{M})$ be a measurable space and let $\mathcal{H}$ be a Hilbert space. A **spectral measure** on $(X, \mathcal{M}; \mathcal{H})$ is an application $E : \mathcal{M} \to \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the C^*-algebra of linear bounded operators on $\mathcal{H}$, which satisfies

1. $E(\Delta)$ is a projection for all $\Delta \in \mathcal{M}$;
2. $E(\emptyset) = 0$ and $E(X) = \mathrm{Id}$;
3. $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$ for all $\Delta_1, \Delta_2 \in \mathcal{M}$;
4. E is σ-additive, i.e. for all disjoint family $\{\Delta_n\}$ of $\mathcal{M}$,

$$E\left(\bigcup_{n\in\mathbb{N}} \Delta_n\right) = \sum_{n\in\mathbb{N}} E(\Delta_n)$$

(where the equality is to be understood in the strong operator topology).

Let us do a quick heuristic digression. Spectral measures are introduced to generalize the spectral decomposition in finite dimension, which tells us that a diagonalizable matrix $A \in \mathcal{M}_n(\mathbb{C})$ can be written by means of its eigenvalues as $A = \lambda_1 E_1 + \cdots + \lambda_r E_r$ where the E_j's are $n \times n$ matrices satisfying the properties of a resolution of the identity (see [BW89, Chapter 10.20]), to the case of the infinite dimension. One could be tempted to write $A = \sum_{\lambda\in\mathrm{Sp}(\lambda)} \lambda E_\lambda$, where the E_λ are some operators satisfying similar properties to those in finite dimension. If A is "nice", the spectrum is compact, and we may have an infinite amount of eigenvalues, thus leading to a sum where the gap between the elements is infinitesimaly small. Intuitively, this can not be something else than an integral, and at this point, it is tempting to write

$$A = \int_{\mathrm{Sp}(A)} \lambda \mathrm{d}E(\lambda).$$

Being able to write such an integral is one of the ideas behind spectral measures. To give some sense to the previous integral, we use ideas similar to that of the construction of the Lebesgue integral with step functions, sum of step functions and then continuous functions, as illustrated in the proof of proposition 3.27. For a more in-depth spectral heuristic, see [Hal17, Chapter 35], and for the precise construction, see [Hal17, Chapter 36-40, 43-44].

We now come back to the main matter. The next proposition is well-known, and perhaps the first example of a spectral measure one sees when first introduced to the subject. Yet it is an important one for this chapter, thus we also prove it.

Proposition 3.27. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and consider the spectral measure $E : \mathcal{M} \to \mathcal{B}(L^2(X, \mathcal{M}, \mu))$ given by*

$$E(\Delta) = M_{\chi_\Delta}$$

where M_g *is the multiplication operator by some measurable function* $g\ : X \to \mathbb{C}$*. Then*

$$M_g = \int_X g\mathrm{d}E.$$

Proof. This is obviously true if $g = \chi_\Delta$ for some $\Delta \in \mathcal{M}$, since

$$\int_X \chi_\Delta \mathrm{d}E = E(\Delta) = M_{\chi_\Delta}$$

by definition of the spectral measure. By linearity of the integral, it is also true for step functions. Now if g is any measurable function, let $\varepsilon > 0$ and let g_ε be a step function such that $\|g - g_\varepsilon\| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned}\left\| M_g - \int_X g\mathrm{d}E \right\| &\leq \|M_g - M_{g_\varepsilon}\| + \left\| \int_X g_\varepsilon \mathrm{d}E - \int_X g\mathrm{d}E \right\| \\ &\leq \|g - g_\varepsilon\| + \|g - g_\varepsilon\| \|E(X)\| \\ &= \varepsilon.\end{aligned}$$

■

Definition 3.28. Let $\mathcal{H}$ be a Hilbert space, $T \in \mathcal{U}(\mathcal{H})$, $\rho\ : L^\infty(\mathbb{T}) \to \mathcal{B}(\mathcal{H})$ a C^*-algebra representation and $\varphi \in \mathcal{H}$. In addition, let $m_0(z) = \sum_{k\in\mathbb{Z}} a_k z^k \in L^\infty(\mathbb{T})$. The data $(\mathcal{H}, T, \rho, \varphi)$ is a **wavelet representation** corresponding to the low-pass filter m_0 if the following are satisfied.

1. Covariance : $T\rho(f(z))T^{-1} = \rho(f(z^N))$ for all $f \in L^\infty(\mathbb{T})$.

2. Scaling : $T\varphi = \sum_{k\in\mathbb{Z}} a_k U^k \varphi = \rho(m_0)\varphi$, where $U = \rho(z \mapsto z)$.

3. Orthogonality : $\langle \rho(f)\varphi | \varphi \rangle = \int_\mathbb{T} f(z)\mathrm{d}z$ for all $f \in L^\infty(\mathbb{T})$, where $\mathrm{d}z$ is the standard Haar measure on $\mathbb{T}$.

4. Density : $\mathcal{H} = \overline{\operatorname{span}}\,\{\, T^n\rho(f)\varphi \mid n \in \mathbb{Z},\ f \in L^\infty(\mathbb{T}) \,\}$.

The first requirement is simply an extension of the Baumslag-Solitar relation. Indeed, if we have a representation π of $\mathrm{BS}(1, N)$ on $\mathcal{H}$, call $\pi(t) = T$ and $\pi(u) = U$. The operators U and T we will consider are unitaries, thus in particular normal. Therefore we can apply Borel functional calculus and define a representation $\rho\ : L^\infty(\mathbb{T}) \to \mathcal{B}(\mathcal{H})$ by $\rho(f(z)) = f(U)$ for $f \in L^\infty(\mathbb{T})$. This representation will satisfy

$$T\rho(f(z))T^{-1} = \rho(f(z^N))$$

and we recover the usual Baumslag-Solitar relation with $\rho(z \mapsto z) = U$. As for the second requirement, it means that φ must be the scaling function of some MRA. The third and fourth assumptions are concerned with creating the subspaces $\{V_j\}_j \subset \mathcal{H}$ of some MRA. One can set $V_0 = \overline{\operatorname{span}}\left\{\, T^k\varphi \;\middle|\; k \in \mathbb{Z} \,\right\}$ and $V_n = U^n(V_0)$ (an explicit example for fractals is done in [DJ06b, Prop. 2.8]), and these subspaces satisfy the requirements to have a MRA. Thus, if one has a representation of a Baumslag-Solitar group that is nice enough, it is possible to then create a MRA, and then obtain an orthonormal wavelet base of the representation space.

From now on, let $(a_j)_{j=1}^{p} \subset \{0, \dots, N-1\}$ be a set of p distinct integers and $m_0(z) = \frac{1}{\sqrt{p}} \sum_{j=1}^{p} z^{a_j}$ be the low-pass filter. We will study two wavelet representations associated to m_0.

Wavelet representations on fractals

Let K be the attractor of the IFS $(\tau_{a_j})_{j=1}^{p}$, and let $\mathcal{R}$ be the inflated version of K, endowed with the Hausdorff measure μ.

We first consider the Hilbert space $\hat{\mathcal{H}} = L^2(\mathcal{R}, \mu)$, on which we define two operators

$$\begin{array}{rccl} \hat{U} : & \hat{\mathcal{H}} & \to & \hat{\mathcal{H}} \\ & f & \mapsto & \hat{U}f(x) = f(x-1), \end{array} \tag{3.28}$$

$$\begin{array}{rccl} \hat{T} : & \hat{\mathcal{H}} & \to & \hat{\mathcal{H}} \\ & f & \mapsto & \hat{T}f(x) = \frac{1}{\sqrt{p}} f\left(\frac{x}{N}\right). \end{array} \tag{3.29}$$

Proposition 3.29. *The map* $\hat{\pi}\colon \mathrm{BS}(1,N) \to \mathcal{U}(\hat{\mathcal{H}})$ *defined by* $\hat{\pi}(u) = \hat{U}$ *and* $\hat{\pi}(t) = \hat{T}$ *is a unitary representation.*

Proof. The fact that $\hat{\pi}$ is a homomorphism is straightforward.

We claim that the operators $\hat{U}$ and $\hat{T}$ are unitaries. Indeed, $\hat{U}$ is surjective by the invariance of $\mathcal{R}$ and μ under N-adic translation (see Prop. 3.18 and Eq. (3.18)), along with the fact that $1 \in \mathcal{R}$ (observe that 1 can be written as $1/N^0$ and thus satisfies the definition $1 = \sum_{j \geq -m} b_j N^{-j}$, $m \in \mathbb{Z}, b_j \in \{a_1, \dots, a_p\}$ for all but finitely many indices j). Let $g \in \hat{\mathcal{H}}$, then $f(x) = g(x+1)$ belongs to $\hat{\mathcal{H}}$ since

$$\int_{\mathcal{R}} |f(x)|^2 \mathrm{d}\mu(x) = \int_{\mathcal{R}} |g(x+1)|^2 \mathrm{d}\mu(x) = \int_{\mathcal{R}+1} |g(x)|^2 \mathrm{d}\mu(x+1) = \int_{\mathcal{R}} |g(x)|^2 \mathrm{d}\mu(x),$$

and $\hat{U}f = g$. Note that the previous computation shows that $\|\hat{U}f\|_{\hat{\mathcal{H}}} = \|f\|_{\hat{\mathcal{H}}}$, which shows that $\hat{U}$ is unitary. Applying a similar reasoning to $\hat{T}$, we obtain that $\hat{T}$ is surjective, and

$$\|\hat{T}f\|_{\hat{\mathcal{H}}}^2 = \int_{\mathcal{R}} \left| \frac{1}{\sqrt{p}} f(x/N) \right|^2 \mathrm{d}\mu(x) = \frac{1}{p} \int_{\mathcal{R}} |f(x/N)|^2 p \mathrm{d}\mu(x/N) = \int_{\mathcal{R}} |f|^2 \mathrm{d}\mu = \|f\|_{\hat{\mathcal{H}}}^2,$$

which proves that $\hat{T}$ is also unitary. Moreover, note that for any $f \in \hat{\mathcal{H}}$

$$\hat{T}\hat{U}\hat{T}^{-1} f(x) = \tfrac{1}{\sqrt{p}} \hat{U}\hat{T}^{-1} f(x/N) = \tfrac{1}{\sqrt{p}} \hat{T}^{-1} f(x/N - 1) = f((x/N-1)N) = \hat{U}^N f,$$

so we indeed deal with a unitary representation of $\mathrm{BS}(1,N)$. ■

In addition, we can define a representation of $L^\infty(\mathbb{T})$ on $\hat{\mathcal{H}}$ by

$$\begin{array}{rccl} \hat{\rho} : & L^\infty(\mathbb{T}) & \to & \mathcal{B}(\hat{\mathcal{H}}) \\ & \xi & \mapsto & \hat{\rho}(\xi) f = \xi(\hat{U}) f. \end{array}$$

Indeed, since $\hat{U}$ is unitary, its spectrum $\mathrm{Sp}(\hat{U})$ is contained in $\mathbb{T}$ and using the spectral theorem, we know that there exists a spectral measure $E\ :\mathrm{Bor}(\mathbb{T})\to\mathcal{B}(\hat{\mathcal{H}})$ such that

$$\hat{U}=\int_{\mathbb{T}} z\mathrm{d}E(z).$$

Using Borel functional calculus, for all $\xi\in L^\infty(\mathbb{T})$, we can define $\xi(\hat{U})$ and it satisfies

$$\xi(\hat{U})=\int_{\mathbb{T}}\xi(z)\mathrm{d}E(z).$$

Thus we have

$$\hat{\rho}(\xi)f=\xi(\hat{U})f=\int_{\mathbb{T}}\xi(z)\mathrm{d}E(z)f$$

and this is the C^*-algebra representation we need to have a wavelet representation. In particular, $m_0(\hat{U})=\frac{1}{\sqrt{p}}\sum_{j=1}^{p}\hat{U}^{a_j}$ is well defined as an operator in $\mathcal{B}(\hat{\mathcal{H}})$. The following properties are satisfied.

Proposition 3.30 ([DJ06b, Prop. 2.8])**.** *Let $\varphi=\chi_K$ be the indicator function of the attractor K.*

1. *the scaling equation : $\hat{T}\varphi=\frac{1}{\sqrt{p}}\sum_{i=1}^{p}\hat{U}^{a_i}\varphi=m_0(\hat{U})\varphi$,*
2. *the translates are orthogonal : $\langle\hat{U}^k\varphi|\varphi\rangle=\delta_k$, $k\in\mathbb{Z}$,*
3. *cyclicity : $\overline{\mathrm{span}}\left\{\hat{T}^n\hat{U}^k\varphi\;\middle|\;n,k\in\mathbb{Z}\right\}=L^2(\mathcal{R},\mu)$.*

The previous proposition is the stepping stone to prove that we have a wavelet representation on $L^2(\mathcal{R},\mu)$, and it shows that φ is a scaling function for the wavelet representation.

Theorem 3.31 ([DJ06b])**.** *The data $(\hat{\mathcal{H}},\hat{T},\hat{\rho},\varphi=\chi_K)$ as defined above is a wavelet representation for m_0.*

Wavelet representations on the solenoid

The goal now is to create a wavelet representation on $L^2(\mathcal{S}_N,m)$.

Theorem 3.32 ([Dut06, Theorem 4.3])**.** *Let $W=\frac{|m_0|^2}{N}$, m the measure associated to W and $\varphi=1$. Consider $\mathcal{H}=L^2(\mathcal{S}_N,m)$ and the operators*

$$\begin{array}{rcccl} T & : & \mathcal{H} & \to & \mathcal{H} \\ & & \xi & \mapsto & T\xi((z_n)_n)=m_0((z_n))\cdot\xi\circ S((z_n)_n), \end{array} \tag{3.30}$$

$$\begin{array}{rcccl} U & : & \mathcal{H} & \to & \mathcal{H} \\ & & \xi & \mapsto & U\xi((z_n)_n)=z_0\cdot\xi((z_n)_n), \end{array} \tag{3.31}$$

The map $\pi\ :\mathrm{BS}(1,N)\to\mathcal{U}(\mathcal{H})$, $\pi(t)=T$, $\pi(u)=U$ is a unitary representation. In addition, by setting

$$\begin{array}{rcccl} \rho & : & L^\infty(\mathbb{T}) & \to & \mathcal{B}(\mathcal{H}) \\ & & f & \mapsto & \rho(f)\xi=f\xi, \end{array}$$

the data $(\mathcal{H},T,\rho,\varphi)$ is a wavelet representation.

The proof below is different from the one Dutkay gives in [DJ06b].

Proof. To prove that $T \in \mathcal{U}(\mathcal{H})$, note that since m is absolutely continuous with respect to $m \circ S$ (c.f. proposition 3.25), the inverse of T is well-defined as

$$\begin{array}{rccl} T^{-1} & : & \mathcal{H} & \to \mathcal{H} \\ & & \xi & \mapsto T^{-1}\xi((z_n)_n) = \frac{1}{m_0 \circ S^{-1}((z_n)_n)} \cdot \xi \circ S^{-1}((z_n)_n), \end{array}$$

thus $T(\mathcal{H})$ is dense in $\mathcal{H}$. Moreover, let $\xi, \eta \in \mathcal{H}$, we have

$$\begin{aligned} \langle T\xi | T\eta \rangle_{\mathcal{H}} &= \int_{\mathcal{S}_N} |m_0|^2 (\xi \circ S) \overline{(\eta \circ S)} \mathrm{d}m \\ &= \int_{\mathcal{S}_N} (\xi \circ S) \overline{(\eta \circ S)} \mathrm{d}m \circ S \\ &= \int_{\mathcal{S}_N} \xi \overline{\eta} \mathrm{d}m \\ &= \langle \xi | \eta \rangle_{\mathcal{H}} \end{aligned}$$

which proves that T is unitary.

U is simply a multiplication operator by the function $g : \mathcal{S}_N \to \mathbb{C}$, $g((z_n)_n) = z_0$ which has absolute value 1. Thus U is also unitary. Note that, since $|z_0| = 1$, the adjoint and the inverse of U are well-defined and equal to each other :

$$\begin{array}{rccl} U^* & : & \mathcal{H} & \to \mathcal{H} \\ & & \xi & \mapsto U^*\xi((z_n)_n) = \overline{z_0} \cdot \xi((z_n)_n), \end{array}$$

$$\begin{array}{rccl} U^{-1} & : & \mathcal{H} & \to \mathcal{H} \\ & & \xi & \mapsto U^{-1}\xi((z_n)_n) = z_0^{-1} \cdot \xi((z_n)_n). \end{array}$$

We now show that U and T satisfy the Baumslag-Solitar relation. We write $z_0 \circ S$ for $g \circ S$, and thus we have $z_0 \circ S = z_0^N$ by definition of the solenoid. Thus

$$\begin{aligned} TUT^{-1}\xi &= TU \tfrac{1}{m_0 \circ S^{-1}} \xi \circ S^{-1} \\ &= T z_0 \tfrac{1}{m_0 \circ S^{-1}} \xi \circ S^{-1} \\ &= m_0 \cdot (z_0 \circ S) \tfrac{1}{m_0 \circ S^{-1} \circ S} \xi \circ S^{-1} \circ S \\ &= z_0^N \xi \\ &= U^N \xi. \end{aligned}$$

This proves that we indeed have a unitary representation of $\mathrm{BS}(1, N)$ on $\mathcal{H}$.

We now show that U generates a representation of $L^\infty(\mathbb{T})$ on $\mathcal{H}$. Note that, since U is unitary we know that its spectrum $\mathrm{Sp}(U)$ is contained in $\mathbb{T}$. Moreover, $(U - \lambda I)\xi = (z_0 - \lambda)\xi$ is not invertible if $z_0 = \lambda$. Thus the point spectrum $\mathrm{Sp}_p(U)$ is equal to $\mathbb{T}$, and we have $\mathrm{Sp}(U) = \mathbb{T}$. Now, by the spectral theorem, we know that there exists a unique spectral measure $E : \mathrm{Bor}(\mathbb{T}) \to \mathcal{B}(\mathcal{H})$ such that

$$U = \int_{\mathbb{T}} z \mathrm{d}E(z).$$

But since U is a multiplication operator and the spectral measure defined in Proposition 3.27 satisfies this requirement, we deduce that $E : \mathrm{Bor}(\mathbb{T}) \to \mathcal{B}(L^2(\mathcal{S}_N, m))$ given by $E(\Delta) = M_{\chi_\Delta}$ is the required spectral measure. Thus using Proposition 3.27

along with Eq. (3.26) we obtain that for any $f \in L^\infty(\mathbb{T})$,

$$M_f = \int_{\mathbb{T}} f \mathrm{d}E.$$

Thus $\rho\ : L^\infty(\mathbb{T}) \to \mathcal{B}(L^2(\mathcal{S}_N, m))$ defined by

$$\rho(f) = f(U) = M_f = \int_{\mathbb{T}} f \mathrm{d}E$$

is a representation of $L^\infty(\mathbb{T})$ on $\mathcal{H}$.

We show that the covariance relation is satisfied. We have

$$\begin{aligned} T\rho(f(z))T^{-1}\xi &= T\rho(f)\frac{\xi \circ S^{-1}}{m_0 \circ S^{-1}} \\ &= Tf \cdot \frac{\xi \circ S^{-1}}{m_0 \circ S^{-1}} \\ &= m_0 \cdot \underbrace{f \circ S}_{=f(z^N)} \frac{\xi \circ S^{-1} \circ S}{m_0 \circ S^{-1} \circ S} \\ &= \rho(f(z^N)), \end{aligned}$$

which can be translated as $TM_fT^{-1} = M_{f\circ S}$.

Recall that we view $m_0 \in L^\infty(\mathbb{T})$ as a function on $\mathcal{S}_N$ by defining $m_0((z_n)_n) = m_0(z_0)$. With what we have above, we obtain

$$T\varphi = m_0 \cdot 1 \circ S = \rho(m_0)\varphi,$$

which is the scaling equation for the wavelet representation.

For the orthogonality, we have for all $f \in L^\infty(\mathbb{T})$

$$\langle \rho(f)1|1\rangle_{\mathcal{H}} = \int_{\mathcal{S}_N} f(z_0)\mathrm{d}m((z_n)) = \int_{\mathbb{T}} f(z)\mathrm{d}z$$

where the last equality comes from Eq. (3.26).

As for the density, note that for all $n \in \mathbb{Z}$

$$T^n\rho(f)T^{-n}\varphi = f \circ S^n,$$

and since the functions that depend only on finitely many coordinates are dense in $L^2(\mathcal{S}_N, m)$, it follows that φ is cyclic for the representation, and we have

$$\mathcal{H} = \overline{\operatorname{span}}\left\{ T^n\rho(f)\varphi \mid n \in \mathbb{Z},\ f \in L^\infty(\mathbb{T}) \right\}.$$

■

3.3 Contributions

3.3.1 Decomposition of the dilation operator T

We generalize the work of Davison [Dav20] to the case $N \geq 2$ and decompose the dilation operator $T\ : L^2(\mathcal{S}_N, m) \to L^2(\mathcal{S}_N, m)$ into a sum of partial isometries $(T_j)_{j=0}^{N-1}$. Recall that we have a bijection $\Phi\ : [0,1) \times \Omega \to \mathcal{S}_N$.

Definition 3.33. Let $N \geq 2$. We write $\mathbf{z}$ for $(z_n)_{n\in\mathbb{Z}} \in \mathcal{S}_N$ and define for $j = 0, \ldots, N-1$

$$\begin{array}{rccl} T_j & : & L^2(\mathcal{S}_N, m) & \to & L^2(\mathcal{S}_N, m) \\ & & f & \mapsto & T_j f(\mathbf{z}) = m_0(\mathbf{z}) \cdot (f \circ S)(\mathbf{z}) \cdot (\chi_{\tau_j([0,1))\times\Omega} \circ \Phi^{-1})(\mathbf{z}). \end{array}$$

In addition, define $\sigma \ : \Omega \to \Omega$, $\sigma((\omega_1, \omega_2, \ldots)) = (\omega_2, \omega_3, \ldots)$ as well as the N inverse branches $\sigma_j \ : \Omega \to \Omega$, $\sigma_j((\omega_1, \omega_2, \ldots)) = (j, \omega_1, \omega_2, \ldots)$, $j \in \{0, \ldots, N-1\}$ and

$$\begin{array}{rccl} T_j^* & : & L^2(\mathcal{S}_N, m) & \to & L^2(\mathcal{S}_N, m) \\ & & f & \mapsto & T_j^* f(\mathbf{z}) = \frac{1}{m_0 \circ S^{-1}(\mathbf{z})} \cdot (f \circ S^{-1})(\mathbf{z}) \cdot (\chi_{[0,1)\times\sigma_j(\Omega)} \circ \Phi^{-1})(\mathbf{z}). \end{array}$$

Proposition 3.34. *The operators T_j and T_j^* are linear, bounded operators on $\mathcal{H} = L^2(\mathcal{S}_N, m)$ and T_j^* is the adjoint of T_j.*

Proof. Linearity is clear. As for boundedness, note that

$$\|T_j f\|_{\mathcal{H}}^2 = \int_{\mathcal{S}_N} |m_0|^2 |f \circ S|^2 |\chi_{\tau_j([0,1))\times\Omega} \circ \Phi^{-1}|^2 \mathrm{d}m \leq \int_{\mathcal{S}_N} |m_0|^2 |f \circ S|^2 \mathrm{d}m = \|Tf\|_{\mathcal{H}}^2$$

which is bounded since $T \in \mathcal{B}(\mathcal{H})$. Recall from Theorem 3.32 that $T^* f = T^{-1} f = \frac{f \circ S^{-1}}{m_0 \circ S^{-1}}$. Thus

$$\|T_j^* f\|_{\mathcal{H}}^2 = \int_{\mathcal{S}_N} \frac{|f \circ S^{-1}|^2}{|m_0 \circ S^{-1}|^2} |\chi_{[0,1)\times\sigma_j(\Omega)} \circ \Phi^{-1}|^2 \mathrm{d}m \leq \int_{\mathcal{S}_N} \frac{|f \circ S^{-1}|^2}{|m_0 \circ S^{-1}|^2} \mathrm{d}m = \|T^* f\|_{\mathcal{H}}^2$$

which is also bounded. Finally, we have

$$\begin{aligned} \langle T_j f | g \rangle &= \int_{\mathcal{S}_N} m_0 (f \circ S)(\chi_{\tau_j([0,1))\times\Omega} \circ \Phi^{-1}) \overline{g} \mathrm{d}m \\ &= \int_{\mathcal{S}_N} |m_0|^2 (f \circ S) \overline{(\chi_{\tau_j([0,1))\times\Omega} \circ \Phi^{-1}) \frac{g}{m_0}} \mathrm{d}m \\ &= \int_{\mathcal{S}_N} (f \circ S) \overline{(\chi_{\tau_j([0,1))\times\Omega} \circ \Phi^{-1}) \frac{g}{m_0}} \mathrm{d}m \circ S \\ &= \int_{\mathcal{S}_N} f \overline{(\chi_{\tau_j([0,1))\times\Omega} \circ \Phi^{-1} \circ S^{-1}) \frac{g \circ S^{-1}}{m_0 \circ S^{-1}}} \mathrm{d}m. \end{aligned}$$

It remains to see that

$$\chi_{\tau_j([0,1))\times\Omega} \circ \Phi^{-1} \circ S^{-1} = \chi_{[0,1)\times\sigma_j(\Omega)} \circ \Phi^{-1} \tag{3.32}$$

for T_j^* to be the adjoint of T_j. We have

$$\begin{aligned} \chi_{\tau_j([0,1))\times\Omega} \circ \Phi^{-1} \circ S^{-1}((z_0, z_1, \ldots)) &= \chi_{\tau_j([0,1))\times\Omega} \circ \Phi^{-1}((z_1, z_2, \ldots)) \\ &= \chi_{\tau_j([0,1))\times\Omega}(\tau_{\omega_1} x, (\omega_2, \omega_3, \ldots) \\ &= \begin{cases} 1 & \text{if } j = \omega_1, \\ 0 & \text{otherwise,} \end{cases} \\ &= \chi_{[0,1]\times\sigma_j(\Omega)}(x, (\omega_1, \omega_2, \ldots)) \\ &= \chi_{[0,1]\times\sigma_j(\Omega)} \circ \Phi^{-1}(z_0, z_1, \ldots) \end{aligned}$$

which concludes. ∎

We can now decompose the dilation operator T and its adjoint as follows.

Proposition 3.35. *For $N \geq 2$, $T, T^*\ : L^2(\mathcal{S}_N, m) \to L^2(\mathcal{S}_N, m)$ can be decomposed as*

$$T = \sum_{j=0}^{N-1} T_j \quad \textit{and} \quad T^* = \sum_{j=1}^{N-1} T_j^*.$$

Proof. Note that $\bigsqcup_{j=0}^{N-1} \tau_j([0,1)) = [0,1)$, thus

$$\begin{aligned}\sum_{j=0}^{N-1} T_j \circ \Phi(x,\omega) &= \sum_{j=0}^{N-1} m_0 \circ \Phi(x,\omega) \cdot (f \circ S \circ \Phi)(x,\omega) \cdot (\chi_{\tau_j([0,1))\times\Omega})(x,\omega) \\ &= m_0 \circ \Phi(x,\omega) \cdot (f \circ S \circ \Phi)(x,\omega) \left(\sum_{j=0}^{N-1} (\chi_{\tau_j([0,1))\times\Omega})(x,\omega) \right) \\ &= m_0 \circ \Phi(x,\omega) \cdot (f \circ S \circ \Phi)(x,\omega) \cdot (\chi_{[0,1)\times\Omega})(x,\omega) \\ &= T \circ \Phi(x,\omega).\end{aligned}$$

Similarly, $\bigsqcup_{j=0}^{N-1} \sigma_j(\Omega) = \Omega$ and a similar computation shows that

$$\sum_{j=0}^{N-1} T_j^* \circ \Phi(x,\omega) = T^* \circ \Phi(x,\omega).$$

∎

We now examine how products of the T_j's and T_j^*'s behave. We start by proving a couple of useful lemmas. Denote by M_f the multiplication operator by f on $L^\infty(\mathcal{S}_N)$.

Lemma 3.36. *We have for all $f \in L^\infty(\mathcal{S}_N)$*

$$TM_f = M_{f\circ S}T \quad \textit{and} \quad T^*M_f = M_{f\circ S^{-1}}T^* \tag{3.33}$$

Proof. Let $f, g \in L^\infty(\mathcal{S}_N)$. We have

$$\begin{aligned}TM_f g = Tfg &= m_0(fg \circ S) \\ &= (f \circ S) m_0 (g \circ S) \\ &= M_{f\circ S} T g\end{aligned}$$

and

$$\begin{aligned}T^*M_f g = T^* fg &= \frac{fg \circ S^{-1}}{m_0 \circ S^{-1}} \\ &= f \circ S^{-1} \frac{g \circ S^{-1}}{m_0 \circ S^{-1}} \\ &= M_{f\circ S^{-1}} T^* g.\end{aligned}$$

∎

Lemma 3.37. *For all* $0 \leq i, j \leq N-1$, *we have*

$$M_{\chi_{\tau_i([0,1))\times\Omega}\circ\Phi^{-1}\circ S} M_{\chi_{\tau_j([0,1))\times\Omega}\circ\Phi^{-1}} = M_{\chi_{\tau_j\circ\tau_i([0,1))\times\Omega}\circ\Phi^{-1}}, \tag{3.34}$$

and more generally,

$$M_{\chi_{\tau_{i_1}\circ\ldots\tau_{i_n}([0,1))\times\Omega}\circ\Phi^{-1}\circ S} M_{\chi_{\tau_j([0,1))\times\Omega}\circ\Phi^{-1}} = M_{\chi_{\tau_j\circ\tau_{i_1}\circ\ldots\tau_{i_n}([0,1))\times\Omega}\circ\Phi^{-1}}. \tag{3.35}$$

In addition, we have

$$M_{\chi_{[0,1)\times\sigma_{k_2}\circ\cdots\circ\sigma_{k_n}(\Omega)}\circ\Phi^{-1}\circ S^{-1}} M_{\chi_{[0,1)\times\sigma_{k_1}(\Omega)}\circ\Phi^{-1}} = M_{\chi_{[0,1)\times\sigma_{k_1}\circ\cdots\circ\sigma_{k_n}(\Omega)}\circ\Phi^{-1}}. \tag{3.36}$$

Proof. We have

$$\begin{aligned}
&\chi_{\tau_i([0,1))\times\Omega}\circ\Phi^{-1}\circ S\cdot\chi_{\tau_j([0,1))\times\Omega}\circ\Phi^{-1}((z_n)_{n\in\mathbb{N}})\\
&\qquad\qquad = \chi_{\tau_i([0,1))\times\Omega}(\tau(x),(i_x,\omega_1,\omega_2,\ldots))\chi_{\tau_j([0,1))\times\Omega}(x,(\omega_1,\omega_2,\ldots))\\
&\qquad\qquad = \begin{cases} 1 & \text{if } \tau(x)\in\tau_i([0,1)) \text{ and } x\in\tau_j([0,1))\\ 0 & \text{otherwise,}\end{cases}\\
&\qquad\qquad = \begin{cases} 1 & \text{if } x\in\tau_j\circ\tau_i([0,1)),\\ 0 & \text{otherwise}\end{cases}\\
&\qquad\qquad = \chi_{\tau_j\circ\tau_i([0,1))\times\Omega}\circ\Phi^{-1}((z_n)_{n\in\mathbb{N}})
\end{aligned}$$

where the penultimate equality is due to the fact that if $x \in \tau_j\circ\tau_i([0,1))$, then $\tau(x)\in\tau(\tau_j\circ\tau_i([0,1))) = \tau_i([0,1))$.

The same reasoning can be applied to prove Eq. (3.35).

To prove Eq. (3.36), note that

$$\begin{aligned}
&\chi_{[0,1)\times\sigma_{k_2}\circ\cdots\circ\sigma_{k_n}(\Omega)}\circ\Phi^{-1}\circ S^{-1}\cdot\chi_{[0,1)\times\sigma_{k_1}(\Omega)}\circ\Phi^{-1}((z_n)_{n\in\mathbb{N}})\\
&\qquad\qquad = \chi_{[0,1)\times\sigma_{k_2}\circ\cdots\circ\sigma_{k_n}(\Omega)}(\tau_{\omega_1}x,(\omega_2,\omega_3,\ldots))\\
&\qquad\qquad\quad \cdot\chi_{[0,1)\times\sigma_{k_1}(\Omega)}\circ\Phi^{-1}(x,\omega)\\
&\qquad\qquad = \begin{cases} 1 & \text{if } \omega_2 = k_2,\ldots,\omega_n = k_n \text{ and } \omega_1 = k_1\\ 0 & \text{otherwise,}\end{cases}\\
&\qquad\qquad = M_{\chi_{[0,1)\times\sigma_{k_1}\circ\cdots\circ\sigma_{k_n}(\Omega)}\circ\Phi^{-1}}.
\end{aligned}$$

■

For more comfort, we introduce the following notation. Let $a = (a_1,\ldots,a_n)\in\mathbb{Z}_N^n$. We write $T_a = T_{a_1}\cdots T_{a_n}$, $T_a^* = T_{a_n}^*\cdots T_{a_1}^*$, $\tau_a = \tau_{a_1}\circ\cdots\circ\tau_{a_n}$, and $\sigma_a = \sigma_{a_1}\circ\cdots\circ\sigma_{a_n}$.

Proposition 3.38. *The operators* $\{T_j\}_{j=0}^{N-1}$ *are partial isometries that satisfy*

1. $T_a T_a^* = M_{\chi_{\tau_a([0,1))\times\Omega}\circ\Phi^{-1}}$,
2. $T_a^* T_a = M_{\chi_{[0,1)\times\sigma_a(\Omega)}\circ\Phi^{-1}}$,
3. $T_k T_j^* = \delta_{kj}\cdot M_{\chi_{\tau_k([0,1))\times\Omega}\circ\Phi^{-1}}$ *and* $T_k^* T_j = \delta_{kj}\cdot M_{\chi_{[0,1)\times\sigma_k(\Omega)}\circ\Phi^{-1}}$,
4. $\sum_{k=0}^{N-1} T_k T_k^* = \sum_{k=0}^{N-1} T_k^* T_k = \mathrm{Id}_{\mathcal{H}}$.

Proof. Let $\ker(T_k)$ be the kernel of T_k. Since $T_k(\ker(T_k)^\perp) = T|_{\ker(T_k)^\perp}$ by definition of the T_k's, they are indeed partial isometries.

1. Note that
$$T_j = TM_{\chi_{\tau_j([0,1))\times\Omega}\circ\Phi^{-1}\circ S^{-1}} = M_{\chi_{\tau_j([0,1))\times\Omega}\circ\Phi^{-1}}T,$$
and
$$T_j^* = T^*M_{\chi_{[0,1)\times\sigma_j(\Omega)}\circ\Phi^{-1}\circ S} = M_{\chi_{[0,1)\times\sigma_j(\Omega)}\circ\Phi^{-1}}T^*,$$
and we proved in eq. (3.32) that
$$\chi_{\tau_j([0,1))\times\Omega}\circ\Phi^{-1}\circ S^{-1} = \chi_{[0,1)\times\sigma_j(\Omega)}\circ\Phi^{-1}.$$
Thus we have
$$\begin{aligned} T_jT_j^* = T_jM_{\chi_{[0,1)\times\sigma_j(\Omega)}\circ\Phi^{-1}}T^* &= M_{\chi_{\tau_j([0,1))\times\Omega}\circ\Phi^{-1}}\underbrace{M_{\chi_{[0,1)\times\sigma_j(\Omega)}\circ\Phi^{-1}\circ S}}_{=M_{\chi_{\tau_j([0,1))\times\Omega}\circ\Phi^{-1}}}\underbrace{TT^*}_{=\mathrm{Id}_{\mathcal{H}}} \\ &= M_{\chi_{\tau_j([0,1))\times\Omega}\circ\Phi^{-1}}. \end{aligned}$$
Assume the formula is true for all $a' = (a_2, \dots, a_n) \in \mathbb{Z}_N^{n-1}$. Then for $a = (a_1, \dots, a_n) \in \mathbb{Z}_N^n$ we have
$$\begin{aligned} T_aT_a^* &= T_{a_1}M_{\chi_{\tau_{a'}([0,1))\times\Omega}\circ\Phi^{-1}}T_{a_1}^* \\ &= M_{\chi_{\tau_{a'}([0,1))\times\Omega}\circ\Phi^{-1}\circ S}M_{\chi_{\tau_{a_1}([0,1))\times\Omega}\circ\Phi^{-1}}M_{\chi_{[0,1)\times\sigma_{a_1}(\Omega)}\circ\Phi^{-1}\circ S}TT^* \\ &= M_{\chi_{\tau_{a'}([0,1))\times\Omega}\circ\Phi^{-1}\circ S}M_{\chi_{\tau_{a_1}([0,1))\times\Omega}\circ\Phi^{-1}} \\ &= M_{\chi_{\tau_a([0,1))\times\Omega}\circ\Phi^{-1}} \end{aligned}$$
where the last equality comes from Eq. (3.35).

2. This relation is proved like the previous one. We have
$$\begin{aligned} T_j^*T_j = T_j^*M_{\chi_{\tau_j([0,1))\times\Omega}\circ\Phi^{-1}}T &= \underbrace{M_{\chi_{\tau_j([0,1))\times\Omega}\circ\Phi^{-1}\circ S^{-1}}}_{=M_{\chi_{[0,1)\times\sigma_j(\Omega)}\circ\Phi^{-1}}}M_{\chi_{[0,1)\times\sigma_j(\Omega)}\circ\Phi^{-1}}T^*T \\ &= M_{\chi_{[0,1)\times\sigma_j(\Omega)}\circ\Phi^{-1}} \end{aligned}$$
for the base case, and for the induction step, assuming the formula holds for all $a' = (a_2, \dots, a_n) \in \mathbb{Z}_N^{n-1}$, for $a = (a_1, \dots, a_n) \in \mathbb{Z}_N^n$ we have
$$\begin{aligned} T_a^*T_a &= T_{a_1}^*M_{\chi_{[0,1)\times\sigma_{a'}(\Omega)}\circ\Phi^{-1}}T_{a_1} \\ &= M_{\chi_{[0,1)\times\sigma_{a'}(\Omega)}\circ\Phi^{-1}\circ S^{-1}}M_{\chi_{\tau_{a_1}([0,1))\times\Omega}\circ\Phi^{-1}\circ S^{-1}}M_{\chi_{[0,1)\times\sigma_{a_1}(\Omega)}\circ\Phi^{-1}}T^*T \\ &= M_{\chi_{[0,1)\times\sigma_{a'}(\Omega)}\circ\Phi^{-1}\circ S^{-1}}M_{\chi_{[0,1)\times\sigma_{a_1}(\Omega)}\circ\Phi^{-1}} \\ &= M_{\chi_{[0,1)\times\sigma_{a_1}\circ\cdots\circ\sigma_{a_n}(\Omega)}\circ\Phi^{-1}}. \end{aligned}$$
where the last equality is Eq. (3.36).

3. Since $\tau_k([0,1))\cap\tau_j([0,1)) = \emptyset$ and $\sigma_k(\Omega)\cap\sigma_j(\Omega) = \emptyset$ if $k \neq j$, we obtain on the

one hand

$$T_kT_j^* = M_{\chi_{\tau_k([0,1))\times\Omega}\circ\Phi^{-1}} \underbrace{M_{\chi_{[0,1)\times\sigma_j(\Omega)}\circ\Phi^{-1}\circ S}}_{=M_{\chi_{\tau_j([0,1))\times\Omega}\circ\Phi^{-1}}} TT^* = \delta_{kj}\cdot M_{\chi_{\tau_k([0,1))\times\Omega}\circ\Phi^{-1}}$$

and on the other hand

$$T_k^*T_j = \underbrace{M_{\chi_{\tau_k([0,1))\times\Omega}\circ\Phi^{-1}\circ S^{-1}}}_{=M_{\chi_{[0,1)\times\sigma_k(\Omega)}\circ\Phi^{-1}}} M_{\chi_{[0,1)\times\sigma_j(\Omega)}\circ\Phi^{-1}} T^*T = \delta_{kj}\cdot M_{\chi_{[0,1)\times\sigma_k(\Omega)}\circ\Phi^{-1}}.$$

4. Since $\cup_{k=0}^{N-1}\tau_k([0,1)) = [0,1)$ and $\cup_{k=0}^{N-1}\sigma_k(\Omega) = \Omega$, it is clear that on the one hand,

$$\sum_{k=0}^{N-1} T_kT_k^* = \sum_{k=0}^{N-1} M_{\chi_{\tau_k([0,1))\times\Omega}\circ\Phi^{-1}} = M_{\chi_{\cup_{k=0}^{N-1}\tau_k([0,1))\times\Omega}\circ\Phi^{-1}} = \mathrm{Id}_{\mathcal{H}}$$

and on the other hand

$$\sum_{k=0}^{N-1} T_k^*T_k = \sum_{k=0}^{N-1} M_{\chi_{[0,1)\times\sigma_k(\Omega)}\circ\Phi^{-1}} = M_{\chi_{[0,1)\times\cup_{k=0}^{N-1}\sigma_k(\Omega)}\circ\Phi^{-1}} = \mathrm{Id}_{\mathcal{H}}\,.$$

■

3.3.2 Spectral measures of the wavelet representation on the Solenoid

In this subsection, we analyze different spectral measures arising from the wavelet representation of $\mathrm{BS}(1,N)$ on the solenoid $\mathcal{S}_N$. To this end, we need two definitions and to adapt a few results from [Jor05].

Definition 3.39 ([Jor05, Definition 3.1])**.** Let (X,d) be a compact metric space. The **diameter** of a subset $A \subset X$ is

$$\mathrm{diam}(A) := \sup_{x,y\in A}\{d(x,y)\}.$$

An ***N*-adic partition** of X is a family $\{A_k(a)\}_{k>0,a\in\mathbb{Z}_N^k}$, $a=(a_1,\dots,a_k)\in\mathbb{Z}_N^k$ such that

1. $\cup_a A_k(a) = X$ and $A_k(a)\cap A_k(a') = \emptyset$ if $a \neq a'$ for all $k>0$;
2. $\mathrm{diam}(A_k(a)) \in \mathcal{O}(N^{-ck})$ for all $k>0$, $a\in\mathbb{Z}_N^k$ and some $c>0$;
3. for all $k>0$, $a\in\mathbb{Z}_N^{k+1}$, there exists $b\in\mathbb{Z}_N^k$ such that $A_{k+1}(a)\subset A_k(b)$.

Definition 3.40 ([Jor05, Definition 3.2])**.** Let $\mathcal{H}$ be a complex Hilbert space. An ***N*-adic system of partitions of $\mathrm{Id}_{\mathcal{H}}$ into projections**, or an ***N*-adic system of projections** when the context is clear, is a system $\{P_k(a)\}_{k>0,a\in\mathbb{Z}_N^k}$ such that

1. $P_k(a)$ is a projection for all $k>0$, $a\in\mathbb{Z}_n^k$, i.e. $P_k(a) = P_k(a)^* = P_k(a)^2$;

2. $\sum_{a\in\mathbb{Z}_N^k} P_k(a) = \mathrm{Id}_{\mathcal{H}}$ and $P_k(a)P_k(a') = 0$ if $a \neq a'$ for all $k > 0$;

3. for all $k > 0$, $a \in \mathbb{Z}_N^{k+1}$, there exists $b \in \mathbb{Z}_N^k$ such that $P_{k+1}(a)$ is contained in $P_k(b)$, i.e. $P_k(b)P_{k+1}(a) = P_{k+1}(a)$.

Below, we adapt a few lemmas from Jorgensen [Jor05] to the case where we don't have isometries (and thus a representation of the Cuntz algebra), but merely partial isometries. Nevertheless, the ideas are the same, and we will sketch the proofs rapidly. We refer to Jorgensen's article for more details.

We replace the isometries $\{S_j\}_{j=0}^{N-1}$ with partial isometries $\{T_j\}_{j=0}^{N-1}$ and we assume that they satisfy

Property 3.41. *1.* $\sum_{j=0}^{N-1} T_jT_j^* = \mathrm{Id}_{\mathcal{H}}$ *;*

2. $T_j^*T_k = \delta_{jk}M_{\chi_{\ker(T_j)^\perp}}$*, i.e. it is the identity on the initial subspace of* T_j *and* 0 *on the rest of* $\mathcal{H}$*.*

With these assumptions, the similarity with the Cuntz relations is obvious. The first result that we prove is that if we have an N-adic partition, we can associate a system of partition of $\mathrm{Id}_{\mathcal{H}}$ into projections, where $\mathcal{H}$ is the Hilbert space on which the partial isometries act. It will be convenient to use the following result.

Lemma 3.42 ([Con00])**.** *For an operator* $T \in \mathcal{B}(\mathcal{H})$*, the following are equivalent :*

1. *T is a partial isometry ;*
2. *T^* is a partial isometry ;*
3. *T^*T is a projection (onto the initial subspace of T) ;*
4. *TT^* is a projection (onto the final subspace of T) ;*
5. *$TT^*T = T$;*
6. *$T^*TT^* = T^*$.*

Lemma 3.43. *Let* $N \geq 2$*,* (X, d) *a compact metric space. Let* $\{A_k(a)\}_{k>0,a\in\mathbb{Z}_N^k}$ *be an* N*-adic system of partitions of* X*. Let* $\{T_j : \mathcal{H} \to \mathcal{H}\}_{j=0}^{N-1}$ *be a family of partial isometries on a complex Hilbert space* $\mathcal{H}$ *satisfying the properties 3.41. Then for* $a = (a_1, \ldots, a_k) \in \mathbb{Z}_N^k$*, define*

$$T_a = T_{a_1} \cdots T_{a_k} \quad \textit{and} \quad P_k(a) = T_aT_a^*.$$

Then $\{P_k(a)\}_{k>0,a\in\mathbb{Z}_N^k}$ *is an* N*-adic system of partition of* $\mathrm{Id}_{\mathcal{H}}$ *into projections. We call it the corresponding* N*-adic system of partitions.*

Proof. 1. The composition of partial isometries remains a partial isometry, thus T_a is a partial isometry. By Lemma 3.42, we directly have that $P_k(a) = T_aT_a^*$ are projections.

2. We prove this by induction on $k \geq 1$. The base case holds by assumption on the T_j's. Assume that $\sum_a P_j(a) = \mathrm{Id}_{\mathcal{H}}$ and $P_j(a)P_j(a') = 0$ if $a \neq a'$ for all

$j < k$ for some $k > 1$, then

$$\begin{aligned}\sum_{a\in\mathbb{Z}_n^k} P_k(a) &= \sum_{a_1\in\mathbb{Z}_N}\sum_{a'\in\mathbb{Z}_N^{k-1}} T_{a_1}T_{a'}T_{a'}^*T_{a_1}^* \qquad\qquad a=(a_1,a')\\ &= \sum_{a_1\in\mathbb{Z}_N} T_{a_1}\left(\sum_{a'\in\mathbb{Z}_N^{k-1}} T_{a'}T_{a'}^*\right)T_{a_1}^*\\ &= \sum_{a_1\in\mathbb{Z}_N} T_{a_1}T_{a_1}^*\\ &= \mathrm{Id}_{\mathcal{H}}\,.\end{aligned}$$

In addition, we have for $a \neq b \in \mathbb{Z}_N^k$

$$P_k(a)P_k(b) = T_aT_a^*T_bT_b^* = T_{a_1}T_{a'}T_{a'}^*T_{a_1}^*T_{b_1}T_{b'}T_{b'}^*T_{b_1}^*.$$

If $a_1 \neq b_1$ we directly obtain $T_{a_1}^*T_{b_1} = 0$. Otherwise we obtain the identity operator on $\ker(T_{a_1})^\perp$ and a' must be different from b'. From the induction hypothesis, we know that $T_{a'}^*T_{a_1}^*T_{b_1}T_{b'}T_{b'}^* = T_{a'}^*\,\mathrm{Id}\,|_{\ker(T_{a_1})^\perp}T_{b'} = T_{a'}^*T_{b'}|_{\ker(T_{a_1})^\perp} = 0$, and so

$$P_k(a)P_k(b) = 0.$$

3. Note that

$$\begin{aligned}P_k(a)P_{k+1}((a,j)) &= T_aT_a^*T_{(a,j)}T_{(a,j)}^*\\ &= T_a\underbrace{T_a^*T_a}_{=\mathrm{Id}\,|_{\ker(T_a)^\perp}}T_jT_j^*T_a^*\\ &= T_aT_jT_j^*T_a\\ &= P_{k+1}((a,j))\end{aligned}$$

since $T_a\,\mathrm{Id}\,|_{\ker(T_a)^\perp} = T_a$.

■

The following adapts [Jor05, Lemma 3.5 and Corollary 3.11] to the case of partial isometries.

Lemma 3.44. *Let $N \geq 2$, (X,d) be a compact metric space and $\mathcal{H}$ be a complex Hilbert space. Let $\{A_k(a)\}_{k>0,a\in\mathbb{Z}_N^k}$ be an N-adic system of partitions of X, and $\{P_k(a)\}_{k>0,a\in\mathbb{Z}_N^k}$ the corresponding N-adic system of partitions of $\mathrm{Id}_{\mathcal{H}}$ into projections. Then there is a unique spectral measure $E\,:\mathrm{Bor}(X)\to\mathcal{B}(\mathcal{H})$ such that $E(A_k(a)) = P_k(a)$ for all $k>0$ and $a\in\mathbb{Z}_N^k$. Moreover, if the partition $\{A_k(a)\}_{k,a}$ is affiliated with an iterated function system $\sigma,(\sigma_k)_{k=1}^N$ acting on (X,d) (i.e. $A_k(a)=\sigma_a(X)$), then E satisfies*

$$T_aE(\Delta)T_a^* = E(\sigma_a(\Delta)) \quad \forall k>0,\ \forall a\in\mathbb{Z}_N^k,\ \forall\Delta\in\mathrm{Bor}(X), \tag{3.37}$$

$$\sum_{j=0}^{N-1} T_jE(\Delta)T_j^* = E(\sigma^{-1}(\Delta)) \quad \forall\Delta\in\mathrm{Bor}(X). \tag{3.38}$$

Proof. The details to construct E are done in [Jor05] and are the same here. We only sketch the main idea so that the reader understands how we prove Eqs. (3.37) and (3.38).

The proof is based on a three steps approximation. Let $\mathcal{A}_k$ be the algebra of functions of the form $\sum_{a\in\mathbb{Z}_N^k} C_a\chi_{A_k(a)}$ and let $\tilde{\mathcal{A}}_k$ be the algebra of operators of the form $\sum_{a\in\mathbb{Z}_N^k} C_aP_k(a)$. The first step is to show that the map

$$\begin{array}{rccc} \vartheta : & \mathcal{A}_k & \to & \tilde{\mathcal{A}}_k \\ & \sum_{a\in\mathbb{Z}_N^k} C_a\chi_{A_k(a)} & \mapsto & \sum_{a\in\mathbb{Z}_N^k} C_aP_k(a) \end{array}$$

extends uniquely to the algebra $\mathcal{A} = \cup_k\mathcal{A}_k$. Then, one shows that this algebra is dense in $\mathcal{C}(X)$, the space of continuous functions on X, and thus ϑ extends once again uniquely to $\mathcal{C}(X)$. This is where the hypothesis that $\mathrm{diam}(A_k(a)) \in \mathcal{O}(N^{-k})$ is used to uniformly approximate functions $f \in \mathcal{C}(X)$. Then a classical argument allows to extend ϑ uniquely to the space of Baire functions on X. Since all these extensions are unique, the map

$$E \ : \mathrm{Bor}(X) \to \mathcal{B}(\mathcal{H}),\ \Delta \mapsto \vartheta(\chi_\Delta)$$

is the unique spectral measure we were looking for, and it is determined by $E(A_K) = P_k$.

Thus, by the previous approximations, we only need to check that Eqs. (3.37) and (3.38) hold for all $A_k(a)$. We have for all $a \in \mathbb{Z}_N^k$ and $b \in \mathbb{Z}_N^n$

$$\begin{aligned} T_aE(A_n(b))T_a^* &= T_aP_n(b)T_a^* \\ &= T_aT_bT_b^*T_a^* \\ &= P_{k+n}(ab) \\ &= E(A_{k+n}(ab)) \\ &= E(\sigma_a(A_n(b))). \end{aligned}$$

Furthermore, we have

$$\sum_{j=0}^{N-1} T_jE(A_k(a))T_j^* = \sum_{j=0}^{N-1} E(A_{k+1}(ja)) = E(\sigma^{-1}(A_k(a)))$$

where $\sigma^{-1}(\Delta) = \{\, x \in X \mid \sigma(x) \in \Delta \,\}$. ■

Recall that for the wavelet representation $(\mathcal{H} = L^2(\mathcal{S}_N, m), T, \pi, 1)$, the operator U satisfies $U = \int_{\mathbb{T}} z\mathrm{d}E$ where $E \ : \mathrm{Bor}(\mathbb{T}) \to \mathcal{B}(\mathcal{H})$ is the spectral measure defined by $E(\Delta) = M_{\chi_\Delta}$. Moreover, let $\iota \ : [0,1) \to \mathbb{T}$, $x \mapsto \iota(x) = \mathrm{e}^{2\pi\mathbf{i}x}$.

Theorem 3.45. *The map E is the unique spectral measure satisfying*

$$E(\iota \circ \sigma^{-1} \circ \iota^{-1}(\Delta)) = \sum_{j=0}^{N-1} T_jE(\Delta)T_j^*.$$

Proof. Let $a = (a_1, \dots, a_n) \in \mathbb{Z}_N^n$ and define $A_n(a) := \tau_{(a)}([0,1)) = \tau_{a_1} \circ \cdots \circ \tau_{a_n}([0,1))$. The families $\{A_n(a)\}_{n>0,a\in\mathbb{Z}_N^n}$ and $\{\iota(A_n(a))\}_{n>0,a\in\mathbb{Z}_N^n}$ are N-adic partitions of $[0,1)$ and $\mathbb{T}$ respectively. In addition, define

$$P_n(a) = T_a T_a^*$$

the corresponding N-adic system of partitions of $\mathrm{Id}_{\mathcal{H}}$ into projections. By Proposition 3.38 we have

$$P_n(a) = M_{\chi_{\tau_{(a)}([0,1))\times\Omega}\circ\Phi^{-1}}$$

but since

$$\begin{aligned} P_n(a)f((z_j)_{j\in\mathbb{N}}) &= \chi_{A_n(a)}(x)f((z_j)_{j\in\mathbb{N}}) \\ &= \chi_{\iota(A_n(a))}(\iota(x))f((z_j)_{j\in\mathbb{N}}) \\ &= \chi_{\iota(A_n(a))}(z_0)f((z_j)_{j\in\mathbb{N}}) \\ &= E(\iota(A_n(a)))f((z_j)_j), \end{aligned}$$

we obtain by Lemma 3.44 that E is the unique spectral measure with the property

$$E(\iota(A_n(a))) = P_n(a) = T_a T_a^*.$$

The fact that

$$E(\iota\circ\sigma^{-1}\circ\iota^{-1}(\Delta)) = \sum_{j=0}^{N-1} T_j E(\Delta) T_j^{-1}.$$

is directly from Lemma 3.44. ■

Theorem 3.46. *Let* $B_n(a) = \sigma_{(a)}(\Omega)$ *and let* $Q_n(a) = T_a^* T_a$. *There exists a unique spectral measure* $F\ :\mathrm{Bor}(\Omega)\to\mathcal{B}(\mathcal{H})$ *such that*

1. $F(B_n(a)) = Q_n(a)$;
2. $F(\sigma^{-1}(\Delta)) = \sum_{j=0}^{N-1} T_j^* F(\Delta) T_j$.

Proof.

1. The existence and uniqueness of F is guaranteed by Lemma 3.44 since $\{B_n(a)\}_{n,a}$ is an N-adic partition on Ω and and $\{Q_n(a)\}_{n,a}$ is the corresponding N-adic system of projections.
2. This is by Lemma 3.44.

■

Note that E and F extend to $\mathcal{S}_N$ as follows, for all $\Delta_1\in\mathrm{Bor}([0,1))$, for all $\Delta_2\in\mathrm{Bor}(\Omega)$, we have

$$E(\Delta_1) = M_{\chi_{\Delta_1\times\Omega}\circ\Phi^{-1}} \quad\text{and}\quad F(\Delta_2) = M_{\chi_{[0,1)\times\Delta_2}\circ\Phi^{-1}}.$$

Theorem 3.47. *The unique spectral measure* $G\ :\mathrm{Bor}(S_N)\to\mathcal{B}(\mathcal{H})$ *which satisfies*

$$G(\Delta) = M_{\chi_\Delta}$$

is the product spectral measure $G = E\cdot F$.

Proof. The spectral measures E and F commute since they are real-valued multiplication operators, hence, using [Ber09, Theorem 33], we know that there exists a unique spectral measure $\tilde{G} : [0,1) \times \Omega \to \mathcal{B}(\mathcal{H})$ such that $\tilde{G}(\Delta_1 \times \Delta_2) = E(\Delta_1)F(\Delta_2)$ for all measurable rectangles $\Delta_1 \times \Delta_2$. We define $G : \mathcal{S}_N \to \mathcal{B}(\mathcal{H})$ by $G = \tilde{G} \circ \Phi$. Note that, if $\Delta = \Phi(\Delta_1 \times \Delta_2) \in \mathrm{Bor}(\mathcal{S}_N)$ for some $\Delta_1 \in \mathrm{Bor}([0,1))$ and some $\Delta_2 \in \mathrm{Bor}(\Omega)$, then

$$\begin{aligned} G(\Delta) = E(\Delta_1)F(\Delta_2) &= M_{\chi_{\Delta_1 \times \Omega} \circ \Phi^{-1}} M_{\chi_{[0,1) \times \Delta_2} \circ \Phi^{-1}} \\ &= M_{\chi_{\Delta_1 \times \Delta_2} \circ \Phi^{-1}} \\ &= M_{\chi_{\Phi(\Delta_1 \times \Delta_2)}} \\ &= M_{\chi_\Delta}. \end{aligned}$$

Since $\tilde{G}$ is unique, by construction we obtain that G is the unique spectral measure satisfying

$$G(\Delta) = M_{\chi_\Delta}.$$

■

Corollary 3.48. *For the associated scalar measure* $G_\xi(\Delta) = \langle G(\Delta)\xi | \xi \rangle$ $(\xi \in \mathcal{H})$, *we have*

$$G_1(\Delta) = m(\Delta) \quad \text{and} \quad G_{m_0}(\Delta) = m(S(\Delta))$$

so that

$$G_1(S(\Delta)) = G_{m_0}(\Delta).$$

Proof. A direct computation shows that

$$G_1(\Delta) = \int_{\mathcal{S}_N} G(\Delta)\mathrm{d}m = \int_{\mathcal{S}_N} M_{\chi(\Delta)}(1)\mathrm{d}m = \int_\Delta \mathrm{d}m = m(\Delta).$$

Similarly, since $|m_0|^2 = NW$, from Proposition 3.24 we have

$$G_{m_0}(\Delta) = \int_{\mathcal{S}_N} G(\Delta)|m_0|^2\mathrm{d}m = \int_{\mathcal{S}_N} M_{\chi_\Delta}(1) \circ S^{-1}\mathrm{d}m = \int_{\mathcal{S}_N} \chi_\Delta \circ S^{-1}\mathrm{d}m = m(S(\Delta)).$$

■

From [Hal17, Theorem 37.1] we know that for any complex-valued, bounded, measurable function $f : \mathcal{S}_N \to \mathbb{C}$, there exists a unique operator M_f such that

$$M_f = \int_{\mathcal{S}_N} f\mathrm{d}G.$$

Using Proposition 3.27, we know that M_f has to be the multiplication operator $M_f : L^2(\mathcal{S}_N, m) \to L^2(\mathcal{S}_N, m)$, $M_f g = fg$. This proves the following.

Corollary 3.49. *Let f be a complex-valued, bounded, measurable function $f : \mathcal{S}_N \to \mathbb{C}$. The unique operator M_f such that $M_f = \int_{\mathcal{S}_N} f\mathrm{d}G$ is the multiplication operator $M_f : L^2(\mathcal{S}_N, m) \to L^2(\mathcal{S}_N, m)$, $M_f g = fg$.*

3.3.3 Fourier coefficients of the measure m

This section generalizes the work from Dutkay (see [Dut06, Section 5.2]). From Section 3.2.5, we know that we have two wavelet representations of BS$(1, N)$, both associated to the same filter

$$m_0(z) = \sum_{j=1}^{p} \frac{1}{\sqrt{p}} z^{a_j}$$

where $(a_j)_{j=1}^{p} \subset \{0, \dots, N-1\}$ consists of p distinct integers. The first representation is on the space $L^2(\mathcal{S}_N, m)$ (c.f. Theorem 3.32) and the second one is on the space $L^2(\mathcal{R}, \mu)$ (c.f. Theorem 3.31). The goal here is to identify both representations through an isomorphism that behaves similarly to a generalized Fourier transform (in the sense that translations on one space become multiplications on the other space). To this end, we compute the Fourier coefficients of the measure m, and we relate these coefficients to the measure μ.

We know from Proposition 3.24 that for all $\lambda = \frac{j}{N^\ell} \in \mathbb{Z}[\frac{1}{N}]$,

$$\hat{m}(\lambda) = \int_{\mathbb{T}} z^j N^\ell W^{(\ell)} \mathrm{d}z$$

and $\hat{m}(j) = \delta_0(j)$ for all $j \in \mathbb{Z}$.

We can now relate $\hat{m}$ and μ, which gives us insights on the geometry of the attractor K.

Lemma 3.50. *We have for all* $\lambda \in \mathbb{Z}[\frac{1}{N}]$

$$\hat{m}(\lambda) = \mu((K + \lambda) \cap K)$$

where K *is the attractor of the IFS* $(\tau_{a_j})_{j=1}^{p}$.

To prove this lemma, we need to introduce correlation functions. Let $\xi, \eta \in \mathcal{H}$, the **correlation function** $h_{\xi,\eta} \in L^1(\mathbb{T})$ associated to ξ and η is defined by the moments

$$\langle U^k \xi | \eta \rangle = \int_{\mathbb{T}} z^k h_{\xi,\eta} \mathrm{d}z, \quad (k \in \mathbb{Z})$$

where $\mathrm{d}z$ is the Lebesgue measure on $\mathbb{T}$. If $\xi = \eta$, we write $h_\xi := h_{\xi,\xi}$.

We introduce the transfer operator, also known as the Ruelle operator, or Perron-Frobenius operator, or Perron-Frobenius-Ruelle operator, which has been extensively studied and may also be known under other names [BJ02]. We follow [Jor06] for the definition.

Definition 3.51. The **transfer operator** is defined by

$$\begin{array}{rccl} R_{m_0} : & L^\infty(\mathbb{T}) & \to & L^\infty(\mathbb{T}) \\ & f & \mapsto & R_{m_0} f(z) = \sum_{y^N = z} W(y) f(y) \end{array}$$

and maps $L^\infty(\mathbb{T})$ to itself.

If φ is a scaling vector with filter m_0, its correlation function satisfies the equation

$$R_{m_0} h_\varphi = h_\varphi$$

(see [Jør01, Lemma 3.3]). We also have a converse to this fact, in the sense that if we prescribe a filter m_0 and a correlation function h which satisfies $R_{m_0}h = h$, we can find a Hilbert space $\mathcal{H}$ and scaling vector $\varphi \in \mathcal{H}$ that satisfies the scaling equation (3.9) to obtain a wavelet representation, which is unique up to isomorphism. We summarize this in the next theorem, which is a combination of the following results [Jør01, Theorem 2.4] and [Dut04, Theorem 2.14].

Theorem 3.52. *Let $m_0 \in L^\infty(\mathbb{T})$ and $h \in L^1(\mathbb{T})$ be such that $h \geq 0$ and $R_{m_0}h = h$. Then there exists a wavelet representation $(\mathcal{H}, T, \rho, \varphi)$ of* $\mathrm{BS}(1, N)$*, and this representation is unique up to isomorphism.*

Proof (OF LEMMA 3.50)**.** Denote by π the wavelet representation on $L^2(\mathcal{S}_N, m)$ and by $\hat{\pi}$ the wavelet representation on $L^2(\mathcal{R}, \mu)$. The two representations are associated to the same filter m_0 and the same correlation function $h = 1$. For π, this is obvious since $\langle U^k 1|1\rangle = \int_{\mathcal{S}_N}((U^k 1)(z_n))1\mathrm{d}m = \int_{\mathcal{S}_n} z_0^k \mathrm{d}m = \int_{\mathbb{T}} z^k \mathrm{d}z$. For $\hat{\pi}$, we need to observe that $\langle \hat{U}^k \chi_K|\chi_K\rangle = \int_K \chi_K(x-k)\mathrm{d}\mu(x) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases} = \int_{\mathbb{T}} z^k \mathrm{d}z$. Thus, using theorem 3.52, the two representations are equivalent and we have an isomorphism $\mathcal{F}_N : L^2(\mathcal{R}, \mu) \to L^2(\mathcal{S}_N, m)$. Recall that we denoted by $\hat{U}$ and $\hat{T}$ the translation operator and dilation operator on $\hat{\mathcal{H}} = L^2(\mathcal{R}, \mu)$, and by U and T the corresponding operators on $\mathcal{H} = L^2(\mathcal{S}_N, m)$. Let $\hat{U}_\lambda f(x) = f(x - \lambda)$ be the operator “translation by λ” on $\hat{\mathcal{H}}$. The corresponding operator U_λ on $\mathcal{S}_N$ is given by multiplication by the character ξ_λ. Moreover, let us write $\varphi_{\mathcal{R}} = \chi_K$ and $\varphi_{\mathcal{S}_N} = 1$ for the respective scaling functions. By definition, we have

$$\hat{m}(\lambda) = \int_{\mathcal{S}_N} \xi_\lambda \mathrm{d}m = \langle U_\lambda \varphi_{S_N}|\varphi_{S_N}\rangle_{L^2(\mathcal{S}_N, m)}$$

But under the isomorphism $\mathcal{F}_N$, by definition of equivalent representations, $\hat{U}_\lambda$ becomes U_λ and $\varphi_{\mathcal{R}}$ becomes $\varphi_{\mathcal{S}_N}$. Hence

$$\begin{aligned}
\hat{m}(\lambda) &= \langle U_\lambda \varphi_{S_N}|\varphi_{S_N}\rangle_{L^2(\mathcal{S}_N, m)} \\
&= \langle \mathcal{F}_N^{-1} U_\lambda \varphi_{S_N}\}|\mathcal{F}_N^{-1}\varphi_{S_N}\rangle_{L^2(\mathcal{R}, \mu)} \\
&= \langle \hat{U}_\lambda \chi_K|\chi_K\rangle_{L^2(\mathcal{R}, \mu)} \\
&= \int_{\mathcal{R}} \chi_K(x - \lambda)\chi_K(x)\mathrm{d}\mu(x) \\
&= \mu((K + \lambda) \cap K).
\end{aligned}$$

■

Remark 3.53. The fact that $\hat{m}(\lambda) = \mu((K + \lambda) \cap K) \geq 0$ implies that m is a measure of positive type on $\mathcal{S}_N$, from Bochner’s theorem [BdLHV08, Theorem D.2.2] and Pontrjagin’s duality [BdLHV08, Theorem D.1.3] since $\hat{\hat{m}}$ must be of positive type and is equal to m.

From the previous proof, one can deduce the following.

Corollary 3.54. *There exists a unique isomorphism $\mathcal{F}_N : L^2(\mathcal{R}, \mu) \to L^2(\mathcal{S}_N, m)$ which satisfies*

1. $\mathcal{F}_N \hat{U}_\lambda \mathcal{F}_N^{-1} f = U_\lambda f = \xi_\lambda f$;
2. $\mathcal{F}_N \hat{T} \mathcal{F}_N^{-1} f = T f = m_0 f \circ S$;
3. $\mathcal{F}_N \chi_K = 1$.

The fact that $\hat{U}_\lambda$, which is a translation, is transformed into a multiplication tells us that $\mathcal{F}_N$ behaves like a generalized Fourier transform.

From the $(a_j)_{j=1}^p$, we construct the following sets :

$$B := \{\, a_k - a_j \mid j, k \in \{1, \dots, p\} \,\} \qquad \text{and} \qquad C := \{c_{-N+1}, c_{-N+2}, \dots, c_{N-1}\} \tag{3.39}$$

where c_ℓ corresponds to the number of ways of obtaining ℓ from $a_k - a_j$, $j, k \in \{1, \dots, p\}$. For instance, if $(a_j)_{j=1}^3 = (0, 2, 4)$, then $c_2 = 2$ since 2 can be obtained from $2-0$ or $4-2$, and $c_{-2} = 2$ for similar reasons, but $c_0 = 3$ and $c_1 = 0$. Note that $c_j = c_{-j}$ for all j since $a_k - a_\ell = -(a_\ell - a_k)$. Moreover, let us define for $\lambda \in \mathbb{Z}[\frac{1}{N}]$ the set of representations of λ in base N with coefficients in B

$$\mathfrak{R}(\lambda) := \left\{ (b_0, \dots, b_\ell) \;\middle|\; \sum_{j=0}^{\ell} \tfrac{b_j}{N^j} = \tfrac{\alpha}{N^\ell} = \lambda,\ b_j \in B \right\}$$

as well as

$$\mathfrak{R}_{\pm\mathbb{Z}_N}(\lambda) := \left\{ (b_0, \dots, b_\ell) \;\middle|\; \sum_{j=0}^{\ell} \tfrac{b_j}{N^j} = \tfrac{\alpha}{N^\ell} = \lambda,\ b_j \in \{-N+1, -N+2, \dots, N-1\} \right\}$$

the set of representations of λ in base N with coefficients in $\{-N+1, \dots, N-1\}$. Note that $\mathfrak{R}(\lambda) \subset \mathfrak{R}_{\pm\mathbb{Z}_N}(\lambda)$ and $|\mathfrak{R}(\lambda)|$ can be greater than 1 since we allow the b_j's to take negative values.

Theorem 3.55. *The Fourier coefficients of the measure m are*

$$\hat{m}(\lambda) = \begin{cases} \sum_{(b_0,\dots,b_\ell)\in\mathfrak{R}(\lambda)} \frac{1}{p^{\ell+1}} \prod_{j=0}^{\ell} c_{b_j} & \text{if } \lambda = \frac{\alpha}{N^\ell} \text{ and } |\mathfrak{R}(\lambda)| \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.40}$$

Proof. We have

$$W = \frac{|m_0|^2}{N} = \frac{1}{Np} \sum_{k,j=1}^{p} z^{a_k - a_j} = \sum_{j=-N+1}^{N-1} \frac{c_j}{Np} z^j$$

where $c_j \in C$ is the number of ways of obtaining j from $a_k - a_\ell$, $k, \ell \in \{1, \dots, p\}$. Thus $\hat{m}$ satisfies the scaling equation (3.27), which takes the form

$$\hat{m}(\lambda) = \frac{1}{p} \sum_{j=-N+1}^{N-1} c_j \hat{m}(N\lambda + j), \quad \lambda \in \mathbb{Z}[\tfrac{1}{N}]. \tag{3.41}$$

Consider $\lambda = \frac{k}{N^\ell}$. We prove Eq. (3.42) by induction on $\ell \geq 0$. Assume $\ell = 0$, i.e. $\lambda = k$ for some $k \in \mathbb{Z}$. In this case, the result follows from proposition 3.24, the fact that if $|k| \geq 1$, $\hat{m}(k) = \mu((K+k) \cap k) = 0$ by lemma 3.50 and the fact that $c_0 = p$.

Let $\ell \geq 1$ so that $\lambda = \frac{k}{N^\ell}$ and assume that $\hat{m}(\lambda) \neq 0$. We have

$$\hat{m}(\lambda) = \frac{1}{p} \sum_{j_0=-N+1}^{N-1} c_{j_0} \hat{m}(\tfrac{k}{N^{\ell-1}} + j_0)$$

Since we assumed $\hat{m}(\lambda) \neq 0$, at least one of the terms in the right-hand side has to be different that 0. We can apply the induction hypothesis to these terms since they are non-zero and of the form $\frac{k'}{N^{\ell-1}}$, and we obtain

$$\hat{m}(\lambda) = \frac{1}{p} \sum_{j_0 | c_{j_0} \hat{m}(N\lambda + j_0) \neq 0} c_{j_0} \sum_{(b_1,\dots,b_\ell) \in \mathfrak{R}(\frac{k}{N^{\ell-1}} + j_0)} \frac{1}{p^\ell} \prod_{j=1}^{\ell} c_{b_j} = \sum_{(b_0,\dots,b_\ell) \in \mathfrak{R}(\lambda)} \frac{1}{p^{\ell+1}} \prod_{j=0}^{\ell} c_{b_j}.$$

■

Corollary 3.56. *Let $(a_j)_{j=1}^p \subset \{0, \dots, N-1\}$ be such that $|b_k - b_j| > 1$ for all $b_k, b_j \in B$, $k \neq j$. In this case we obtain that the Fourier coefficients of the measure m are*

$$\hat{m}(\lambda) = \begin{cases} \frac{1}{p^{\ell+1}} \prod_{j=0}^{\ell} c_{b_j} & \text{if } \lambda = \sum_{j=0}^{\ell} \frac{b_j}{N^j}, \quad b_j \in B \\ 0 & \text{otherwise.} \end{cases} \tag{3.42}$$

Moreover, in the case $N = 3$ and $a_1 = 0, a_2 = 2$ we recover Dutkay's result [Dut06, Proposition 5.6].

Proof. Recall that if $|\lambda| > 1$, $\hat{m}(\lambda) = 0$. Now assume that $\lambda = \frac{k}{N^\ell} \in \mathbb{Z}[\frac{1}{N}]$ is such that $\hat{m}(\lambda) \neq 0$. We once again argue by induction over $\ell \geq 0$. The case $\ell = 0$ being covered already, assume $\ell \geq 1$. We have that $\lambda \in (-1, 1)$, and eq. (3.41) tells us that

$$\hat{m}(\lambda) = \frac{1}{p} \sum_{j_0=-N+1}^{N-1} c_{j_0} \hat{m}(N\lambda + j_0).$$

First, note that only the $j_0 \in B$ can contribute to the sum, for otherwise $c_{j_0} = 0$ and $c_{j_0} \hat{m}(N\lambda + j_0) = 0$. Secondly, since $\hat{m}(\lambda) \neq 0$, at least one term in the right-hand side is non-zero. Take any $j_0 \in \{-N+1, \dots, N-1\}$ such that $c_{j_0} \hat{m}(N\lambda + j_0) \neq 0$. This implies that $N\lambda + j_0 \in (-1, 1)$. From the hypothesis that $|b_k - b_j| > 1$ if $k \neq j$, we have that $j_0 \pm 1 \notin B$ and $c_{j_0 \pm 1} \hat{m}(N\lambda + j_0 \pm 1) = 0$. Finally, for any $|k| \geq 2$, $|N\lambda + j_0 + k| > 1$, thus $\hat{m}(N\lambda + j_0 + k) = 0$. Therefore, j_0 is in fact the unique element such that $N\lambda + j_0 \in (-1, 1)$, and we have

$$\hat{m}(\lambda) = \frac{1}{p} c_{j_0} \hat{m}(N\lambda + j_0), \quad j_0 \in B.$$

But $N\lambda + j_0$ is of the form $\frac{k'}{N^{\ell-1}}$ so we can apply the induction hypothesis and we obtain

$$\hat{m}(\lambda) = \frac{1}{p} c_{j_0} \cdot \frac{1}{p^\ell} \prod_{j=1}^{\ell} c_{b_j} = \frac{1}{p^{\ell+1}} \prod_{j=0}^{\ell} c_{b_j}.$$

In the case $N = 3$ and $a_1 = 0$, $a_2 = 2$, we obtain $B = \{-2, 0, 2\}$ and $C = \{1, 0, 2, 0, 1\}$. Note that in Dutkay's case, b_0 must be equal to 0. Otherwise, $b_0 = \pm 2$, we have

$$-2 + \sum_{j=1}^{\ell} \frac{b_j}{3^j} \leq -2 + 2 \sum_{j=1}^{\ell} \frac{1}{3^j} = -2 + \frac{3^\ell - 1}{3^\ell} < -1,$$

and similarly

$$2 + \sum_{j=1}^{\ell} \frac{b_j}{3^j} \geq 1,$$

so that if $\lambda = \sum_{j=0}^{\ell} \frac{b_j}{3^j}$ with $b_0 \neq 0$, $\hat{m}(\lambda) = 0$. Thus, Dutkay's formula simply counts the number of non-zero b_j's in the representation of $\lambda = \sum_{j=0}^{\ell} \frac{b_j}{3^j}$, $b_j \in B$. With our formula, we obtain for such λ

$$\hat{m}(\lambda) = \frac{1}{2^{\ell+1}} \cdot \prod_{j=0}^{\ell} c_{b_j}.$$

The elements in the product on the right-hand side can be either 1 when $b_j = \pm 2$, or 2 when $b_j = 0$. Hence we obtain

$$\hat{m}(\lambda) = 2^{-\ell-1+\#\{b_j=0\}} = 2^{-\#\{b_j \neq 0\}}$$

as in Dutkay's formula. ■

3.3.4 Open question

Since the measure m on $\mathcal{S}_N$ is defined with respect to the low-pass filter m_0, one can wonder what happens to the measure m if we change m_0.

Definition 3.57. Moreover, we call a bounded measurable function $V : [0,1) \times \Omega \to \mathbb{C}$ a **cocycle** if it satisfies

$$V(x, (\omega_1, \omega_2, \ldots)) = V(\tau_{\omega_1} x, (\omega_2, \omega_3, \ldots))$$

for all $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$.

With this definition, we can define invariant measures and harmonic functions.

Definition 3.58. We say that $h \in L^\infty(\mathbb{T})$ is **harmonic** (with respect to R_{m_0}) if

$$R_{m_0} h = h \iff \sum_{y^N = z} W(y) h(y) = h(y) \, \forall z \in \mathbb{T}. \tag{3.43}$$

We say that a probability measure ν on $\mathbb{T}$ is **invariant** (with respect to R_{m_0}) if $\nu(R_{m_0} f) = \nu(f)$ for all bounded measurable functions f on $\mathbb{T}$ or equivalently if

$$\int_{\mathbb{T}} R_{m_0} f \mathrm{d}\nu = \int_{\mathbb{T}} f \mathrm{d}\nu$$

for all bounded measurable functions f on $\mathbb{T}$.

Note that if $h = 1$, then by definition of m_0, we have $R_{m_0} 1 = 1$ thus $h = 1$ is harmonic. This condition, that $R_{m_0} 1 = 1$ is called the **quadrature mirror filter (QMF)** condition.

The following is known for bounded harmonic functions.

Theorem 3.59 ([Jor06, Theorem 2.7.1]). *Let* $\{ P_x \mid x \in [0,1] \}$ *be the measures on* Ω *obtained in proposition 3.20. Then there is a* $1-1$ *correspondence between bounded harmonic functions* h *and cocycles* V *as follows : let* V *be a bounded and measurable cocycle, then* $h(x) = h_V(x) = P_x[V(x,\cdot)]$ *is harmonic, i.e.*

$$R_{m_0} h = h.$$

Conversely, V *may be recovered from* h *as a martingale limit.*

For the case of invariant probability measures, the following is known.

Proposition 3.60 ([Jor06, Prop. 2.8.2]). *Let* ν *be an* R_{m_0}*-invariant probability measure, and let* h *be an harmonic function. Then the measure*

$$\mathrm{d}\nu_h = h\mathrm{d}\nu$$

is invariant.

Conversely, if ν_1 *is a Borel probability measure on* $\mathbb{T}$ *satisfying* $\nu_1 \circ \tau^{-1} = \nu_1$ *(*ν_1 *is invariant under* τ*), and if* $\nu_1 \ll \nu$*, then the Radon-Nikodym derivative* $h = \frac{\mathrm{d}\nu_1}{\mathrm{d}\nu}$ *satisfies*

$$R_{m_0} h = h \quad \nu - a.e. \ on\ \mathbb{T}.$$

The following results come from [DJ06a].

Proposition 3.61 ([DJ06a, Proposition 5.3]). *Let* W *be continuous with* $R_W 1 = 1$ *and suppose* $R_W f = f$ *is continous whenever* f *is. Then the set*

$$M_{\mathrm{inv}} := \{ \nu \mid \nu \text{ is a probability measure on } X,\ \nu \circ R_W = \nu \}$$

is a nonempty convex set, compact in the weak topology. In the case of an endomorphism, if $\nu \in M_{inv}$*, then* $\nu = \nu \circ r^{-1}$*. The extreme points of* M_{inv} *are the ergodic invariant measures.*

Proposition 3.62 ([DJ06a, Proposition 5.14]). *In the case of an endomorphism system* (X, r)*, let* $W, W' \in C(X)$*,* $W, W' \geq 0$*,* $R_W 1 = R_{W'} 1 = 1$*. Suppose* ν *is an extreme point of the probability measures which are invariant for* R_W*, and similarly for* ν' *and* $R_{W'}$ *Then, if* $\nu \neq \nu'$*, then* ν *and* ν' *are mutually singular.*

Proposition 3.63 ([DJ06a, Corollary 5.15]). *Take* $r(z) = z^N$ *on* $\mathbb{T}$*. Suppose* $W, W' \in C(\mathbb{T})$ *are Lipschitz,* $R_W 1 = R_{W'} 1 = 1$ *and suppose they have no cycles (i.e. for all* $z \in \mathbb{T}$ *such that there exists* $n \in \mathbb{N}$ *such that* $r^n(z) = z$*, there is at least one* $y \in \{z, r(z), \ldots, r^{n-1}(z)\}$ *such that* $W(y) \neq 1$*), and they have finitely many zeroes. If* $W \neq W'$*, then their invariant measures are mutually singular. In particular, if* W *is not constant* $\frac{1}{N}$*, then* ν *is singular with respect to the Haar measure on* $\mathbb{T}$*.*

Theorem 3.64 ([DJ06b, Theorem 7.7]). *Let* m_0*,* m_0' *be two filters that satisfy*

1. $m_0, m_0' \in \mathrm{Lip}(\mathbb{T})$ *(they are Lipschitz functions on* $\mathbb{T}$*);*
2. m_0, m_0' *have a finite number of zeros;*

3. $R_{m_0}(1) = 1$ *and* $R_{m_0'}(1) = 1$.

Suppose that 1 *is a simple eigenvalue for* R_{m_0} *and* $R_{m_o'}$ *on* $C(\mathbb{T})$. *Let* ν *and* ν' *be the invaraiant measures for* R_{m_0} *and* $R_{m_0'}$ *respectively and let* $(\mathcal{H}, T, \pi, \varphi)$, $(\mathcal{H}', T', \pi', \varphi')$ *be the wavelet representations associated to* $(m_0, 1)$ *and* $(m_0', 1)$ *respectively. Then if* $\nu \neq \nu'$, *the two wavelet representations are disjoint.*

Conjecture 3.65. *Let* $m_0(z) = \frac{1}{\sqrt{p}} \sum_{j=1}^{p} z^{a_j}$, $m_0'(z) = \frac{1}{\sqrt{q}} \sum_{j=1}^{q} z^{a_j}$ *such that* $m_0 \neq m_0'$. *The two measures* m, m' *on* $\mathcal{S}_N$ *corresponding to* m_0 *and* m_0' *are not mutually absolutely continuous.*

4 Spectral measures of certain wallpaper groups

The results in this chapter are obtained in collaboration with Tom Kaiser.

4.1 Introduction

From a finitely generated group $G = \langle S|R\rangle$, $S = S^{-1} \subset G$, $|S| < +\infty$, we can consider its Cayley graph $\Gamma = \mathrm{Cay}(G, S) = (V(\Gamma), E(\Gamma))$ whose vertex set is $V(\Gamma) = G$ and whose edge set is $E(\Gamma) = \{\ (g, sg) \mid g \in G, s \in S\ \}$. There are three similar operators associated with Γ: the adjacency operator A, the Markov operator M and the discrete Laplace operator Δ. The operator A can be thought of as a matrix which describes how the vertices are interconnected, $M = \frac{1}{|S|}A$ is the normalized version of A, that can be thought of as the probability to walk from one vertex to another by using the generator $s \in S$ with probability $\frac{1}{|S|}$, and $\Delta = I - M$ where I is the identity operator is a discrete analogue of the continuous Laplace operator, which can be thought of as how a process diffuses on the Cayley graph. From these relations, we can see that A, M and Δ are interconnected and that studying one is practically equivalent to studying the others. For this reason, we will focus on the Markov operator M, and in particular its spectrum.

The study of the spectrum of Markov operators was initiated by Kesten in 1959 [Kes59b]. He was the first to consider random walks on non-commutative groups, and to compute the spectrum of the associated Markov operator. More precisely, the operator M is defined as

$$\begin{array}{rccl} M: & \ell^2(G) & \to & \ell^2(G) \\ & f & \mapsto & Mf(x) = \frac{1}{|S|}\sum_{s\in S} f(s^{-1}x). \end{array} \tag{4.1}$$

It is a linear, bounded operator ($\|M\| \leq 1$), and it is self-adjoint. Therefore its spectrum $\mathrm{Sp}(M)$ is real and contained in $[-1, 1]$, and there exists a spectral measure $E : \mathrm{Bor}(\mathrm{Sp}(M)) \to \mathcal{B}(\ell^2(G))$, where $\mathrm{Bor}(\mathrm{Sp}(M))$ denotes the Borel sets of $\mathrm{Sp}(M)$, which satisfies

$$M = \int_{\mathrm{Sp}(M)} \lambda \mathrm{d}E(\lambda).$$

Denote by $\delta_v \in \ell^2(G)$ the vector defined by $\delta_v(u) = 1$ if $u = v$, and 0 otherwise. The Markov operator produces a Markov chain on G, which we call the **simple random walk (SRW)** on G. It is a sequence $(X_n)_n$ of random variables which take values in G, defined by

$$\mathbb{P}[X_n = v \mid X_k = u_k,\ k = 0, \dots, n-1] = \mathbb{P}[X_n = v \mid X_{n-1} = u] = M_{u,v} \quad \forall u, v \in G.$$

Therefore, X_n gives us the random position after n steps, and the transition probabilities are given by $M_{u,v} = \frac{1}{|S|}$ if $v = us$ for some $s \in S$, and 0 otherwise. The spectral measure

E gives rise to an infinite family of measures on $\mathrm{Sp}(M)$, namely the measures

$$\mu_{u,v}(B) = \langle E(B)\delta_u | \delta_v \rangle , \quad B \in \mathrm{Bor}(\mathrm{Sp}(M)).$$

These measures satisfy in particular

$$\int_{\mathrm{Sp}(M)} \lambda^n \mathrm{d}\mu_{u,v}(\lambda) = \langle M^n \delta_u | \delta_v \rangle$$

which is the probability to go from u to v in n steps according to the simple random walk. We will be specifically interested in the measures $\mu_u := \mu_{u,u}$, which define a probability measure on $\mathrm{Sp}(M)$. For a vertex transitive graph such as a Cayley graph, the choice of u is irrelevant, thus we will consider μ_e, which is often called the **Kesten spectral measure** or the **Plancherel spectral measure**.

So far, the amount of computed spectra of finitely generated groups is rather scarce, and even fewer examples of computed spectral measures are known. As mentioned previously, Kesten initiated the study of spectra of finitely generated groups [Kes59b]. In particular, he computes the spectrum of the simple random walk (or equivalently the spectrum of the Markov operator M) on the free group $\mathbb{F}_d$ with $1 \leq d < +\infty$ generators. In another pioneering paper, Kesten proves the following result [Kes59a]. Recall that a group G is **amenable** if and only if there exists a Følner sequence, i.e. a sequence of finite sets $\{F_n\}_{n\geq 1}$ such that for all $g \in G$, $\lim_{n\to\infty} \frac{|F_n g \Delta F_n|}{|F_n|} = 0$ (Følner condition).

Theorem 4.1 ([Kes59a]). *1. $1 \in \mathrm{Sp}(M) \iff G$ is amenable.*

2. If G is amenable, $-1 \in \mathrm{Sp}(M))$ if and only if Γ is bipartite if and only if there exists an homomorphism $\varphi\colon G \to \{\pm 1\}$ and $\varphi(S) = -1$.

Mohar and Woess wrote a detailed survey about spectra of infinite graphs in 1989 [MW89]. Some examples are mentionned in [MW89, chapter 7], and some references to applications in chemistry and physics are given in [MW89, chapter 8]. An earlier monograph by Mohar contains some general results about the spectrum of infinite graphs [Moh82]. In [dLHRV93b, dlHRV93a], the authors investigate the link between the spectral properties of finitely generated groups G and a unitary representation π of G, and the spectral properties of $\pi(h)$ where h is the average of a finite set S of generators of G. Béguin, Valette and Żuk compute the spectrum of the Heisenberg group with respect to its standard presentation in [BVZ97]. In 1999, Bartholdi and Grigorchuk give methods to approximate the spectrum of fractal groups (more precisely the spectrum of the Markov operator, as well as Hecke type operators) [BG99]. Martin and Valette show in 2000 that the spectrum of the solvable Baumslag-Solitar groups $\mathrm{BS}(1, N)$ is $[-1, 1]$ for N odd, and $[r_N, 1]$ for N even, with some lower and upper bounds on r_N [MV00]. Grigorchuk and Żuk computed the spectral measure of the Lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ for a specific set of generators, and proved that this measure is discrete and concentrated on a dense subset of points in $[-1, 1]$ [GŻ01]. In 2002, Dicks and Schick computed the Kesten spectral measure of groups of the form $U \wr \mathbb{Z}$, where U is a discrete group with torsion [DS02], recovering the result of Grigorchuk and Żuk in the case $U = \mathbb{Z}/2\mathbb{Z}$. Related work on Lamplighter groups include [KSS04, LNW08, GLN16]. Grigorchuk and Krylyuk computed the Kesten spectral measure of Grigorchuk's group in [GK12]. More recently, Perez studied spectral properties of spinal groups in his thesis [PP20], and in a recent article, Grigorchuk and Simanek are able to exhibit a system of generators of the Lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ for which the spectrum of the associated Laplacian is a union of an interval with a countable set of isolated points accumulating to a point outside this interval [GS21], a rather surprising result since in all previous results, the spectrum was either connected, or completely disconnected.

Another line of research, inspired by Mark Kac's celebrated question "can one hear the shape of a drum?" [Kac66] was explored by, among others, Alain Valette in [Val94] where he asks "can one hear the shape of a group?". It is much easier to see why the answer is "no" in the case of groups than in the case of a plane domain: any non-isomorphic pair (G, S), (G', S') with isomorphic Cayley graphs will be isospectral. Another easy example is that for all $n \geq 1$, the spectrum of the group $\mathbb{Z}^n$ with the standard generators is $[-1, 1]$, yet $\mathbb{Z}^n$ is not isomorphic to $\mathbb{Z}^m$ for $n \neq m$. Dudko and Grigorchzuk provide a continuum of pairwise non-quasi-isometric isospectral groups [DG20]. Thus, the spectrum alone loses some information about the group and we need more information about the spectrum to distinguish isospectral groups. An open question nowadays is "is it possible to determine the Cayley graph of a finitely generated group G up to isometry from the (Kesten) spectral measure μ of the associated Markov operator?" [DG20, see Question 1].

Our goal in this chapter is to study wallpaper groups, and for some of them, compute their Kesten spectral measure. We are able to compute it for ten of the seventeen groups. The cornerstone of these computations is the Kesten spectral measure of $\mathbb{Z} = \langle t \rangle = \langle \{\pm 1\} \rangle$. Of interest are some computations of exotic examples where $\mathbb{Z}$ has other generators than just $\{\pm 1\}$, for example:

Proposition 4.2 (SEE PROPOSITION 4.25). *The Kesten spectral measure of* $\Gamma = \mathrm{Cay}(\mathbb{Z}, \{\pm 1, \pm 2\})$ *is given by*

$$\mathrm{d}\mu_\Gamma(x) = \frac{4\mathrm{d}x}{\pi\sqrt{8x+\frac{9}{2}}} \left(\frac{1}{\sqrt{-8x-\sqrt{16x+9}+3}} \chi_{\left[\frac{-9}{16},0\right]}(x) + \frac{1}{\sqrt{-8x+\sqrt{16x+9}+3}} \chi_{\left[\frac{-9}{16},1\right]}(x) \right). \tag{4.2}$$

We often find that the Cayley graphs of the groups we consider have a structure isomorphic to that of $\mathrm{Cay}(\mathbb{Z}^2) \times \mathrm{Cay}(F)$ where $\mathbb{Z}^2$ is generated by the standard generators and relations, and F is a finite group, so that the Kesten spectral measure is a weighted sum of the Kesten spectral measure of $\mathbb{Z}^2$. We mention for instance:

Proposition 4.3 (SEE COROLLARY 4.23). *The Kesten spectral measure of*

$$cm = \langle \tau_1, \tau_2, \sigma | [\tau_1, \tau_2] = e, \tau_1^\sigma = \tau_2, \tau_2^\sigma = \tau_1, \sigma^2 = e \rangle .$$

is given by

$$\mu_{cm}(\Delta) = \frac{1}{2}\mu_{\mathbb{Z}^2}((\Delta + 1) \cap [-1, 1]) + \frac{1}{2}\mu_{\mathbb{Z}^2}((\Delta - 1) \cap [-1, 1]) \quad \forall \Delta \in \mathrm{Bor}([-1, 1]).$$

Finally, we also obtain as a corollary of our computations that some (non-isomorphic) wallpaper groups are isospectral, for instance cm, pm and $p2$, or $c2mm$, $p2mm$ and $p4$.

We start by introducing some background on wallpaper groups in section 4.2.1, where we recall the classification of crystallographic groups in two dimensions as well as how to generate these groups. We go on in section 4.2.3 to introduce spectral measures, and more specifically Kesten spectral measure. In particular, we give some tools to explicitly compute some of these measures. The computations are then performed group by group in section 4.3.

4.2 Background

4.2.1 Wallpaper groups

The content of this section comes mostly from the book [Arm97].

An **isometry** of the plane is a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which preserves the distance, i.e.

$$\|f(x) - f(y)\| = \|x - y\| \qquad \forall x, y \in \mathbb{R}^2,$$

where $\| \cdot \|$ denotes the usual euclidean norm. An isometry of the plane is necessarily surjective, so that the isometries of the plane form a group under the composition of functions which is called the **Euclidean group Isom($\mathbb{R}^2$)**. This group has a rigid structure in the sense that every element of $\mathrm{Isom}(\mathbb{R}^2)$ is either a rotation about the origin followed by a translation, or a reflection in a line through the origin followed by a translation. To make this statement precise, denote by $T \leq \mathrm{Isom}(\mathbb{R}^2)$ the subgroup of translations, $O_2 \leq \mathrm{Isom}(\mathbb{R}^2)$ the subgroup of orthogonal transformations, which consists of rotations about the origin and reflections in lines through the origin. It's rather easy to see that $T \cap O_2$ consists only of the identity transformation. Thus, the result says that $\mathrm{Isom}(\mathbb{R}^2) = TO_2$, but we can be more precise. Let $\tau \in T$ be a translation and $o \in O_2$ an orthogonal transformation. If $f = \tau o$, f is called a **direct isometry**, and in the other case, f is an **opposite isometry**. Finally, a reflection in a line followed by a translation parallel to the same line is a **glide reflection**. We can now state the theorem more precisely.

Theorem 4.4 ([Arm97, Theorem 24.1])**.** *Every direct isometry is a translation or a rotation. Every opposite isometry is a reflection or a glide reflection.*

In fact, we have an isomorphism between $\mathrm{Isom}(\mathbb{R}^2)$ and the semi-direct product $T \rtimes_\alpha O_2$ where $\alpha : O_2 \to \mathrm{Aut}(T)$ is the action by conjugation. However, to introduce wallpaper groups, another representation is more convenient: note that if $f = \tau o$, with $\tau \in T, o \in O_2$, let $v = \tau(0)$ and let M be the orthogonal matrix representing o in the standard basis of $\mathbb{R}^2$, then for all $x \in \mathbb{R}^2$,

$$f(x) = v + o_M(x) = v + xM^\top. \tag{4.3}$$

Conversely, such a $v \in \mathbb{R}^2$ and $M \in O$, eq. (4.3) determines an element of $\mathrm{Isom}(\mathbb{R}^2)$. Thus, we can also think about $f \in \mathrm{Isom}(\mathbb{R}^2)$ as an ordered pair (v, M), with multiplication given by $(v, M)(v', M') = (v + Mv', MM')$, but this simply the semi-direct product $\mathbb{R}^2 \rtimes_\psi O_2$ where ψ is the usual action of O_2 on $\mathbb{R}^2$.

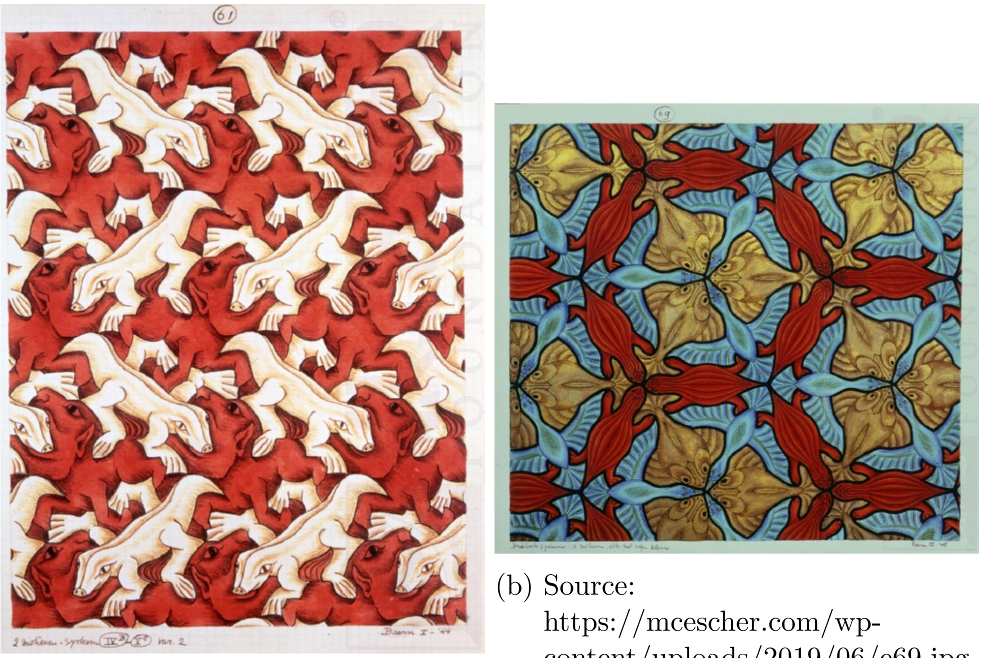
Define the projection $\pi : \mathrm{Isom}(\mathbb{R}^2) \to O_2$, $\pi((v, M)) = M$. We have $\ker \pi = T$ since the elements of the kernel have the form (v, Id), $v \in \mathbb{R}^2$. Let $G \leq \mathrm{Isom}(\mathbb{R}^2)$, and let

$$H := G \cap T, \quad J := \pi(G).$$

We call H the **translation subgroup** of G and J the **point group** of G.

Definition 4.5 (Wallpaper group)**.** A subgroup of $\mathrm{Isom}(\mathbb{R}^2)$ is a **wallpaper group**

if its translation subgroup is generated by two independent translations, and its point group is finite.



(a) Source: https://mcescher.com/wp-content/uploads/2019/06/e61.jpg

(b) Source: https://mcescher.com/wp-content/uploads/2019/06/e69.jpg

Figure 4.1: Two representations of some wallpaper groups, by M.C. Escher.

Example 4.6. Consider fig. 4.2 below, it is a wallpaper group where the translation subgroup H is generated by the translations τ_1, τ_2, and the point group J is the dihedral group $D_6 \leq O_2$ determined by the matrices

$$\begin{pmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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It is a priori not clear from definition 4.5 how many wallpaper groups exist. In fact, there are only seventeen wallpaper groups. This classification was first mathematically performed by Fedorov in 1891 [Fed91]. We explain below the idea behind this result.

A **lattice** L in $\mathbb{R}^2$ is the orbit of the origin under the action of H. There exists a non-zero vector $\tau_1 \in \mathbb{R}^2$ which has minimum length in L, and another vector $\tau_2 \in \mathbb{R}^2$ which is also in L, is skew to τ_1 and whose length is minimal.

Theorem 4.7 ([Arm97, Theorem 25.1])**.** *The set L is the lattice spanned by τ_1 and τ_2, i.e.*

$$L = \{\, m\tau_1 + n\tau_2 \mid m, n \in \mathbb{Z} \,\}.$$

The different lattices spanned by two translations are classified into five categories:

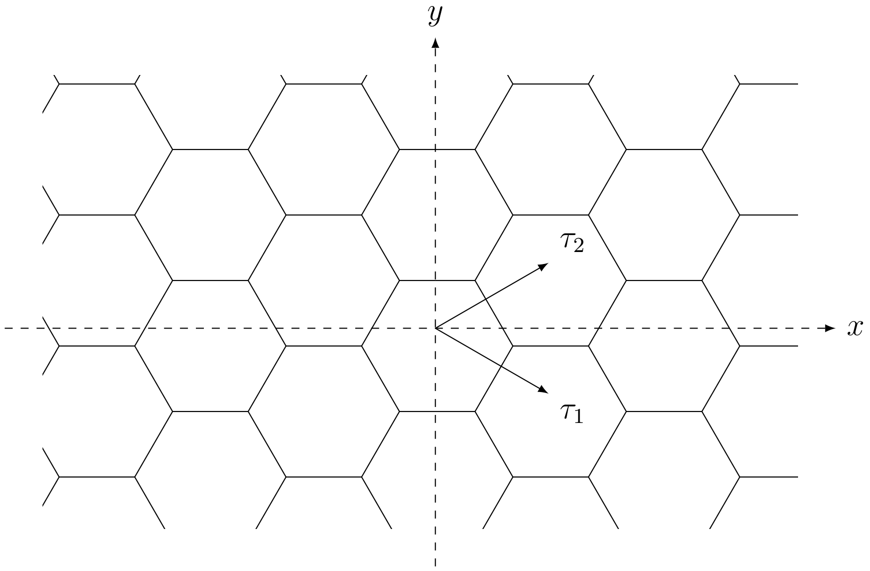


Figure 4.2: Example of a wallpaper pattern

oblique (parallelogram), rectangular, centered rectangular, square or hexagonal [Arm97, see p. 149].

The next restrictions we have is on the point group, and sometimes referred to as the “crystallographic restriction”.

Theorem 4.8 ([Arm97, Theorem 25.3]). *The order of a rotation in a wallpaper group can only be* 2, 3, 4 *or* 6.

The lattice must be preserved under the action of the point group. Thus the strategy to enumerate the wallpaper groups is to consider every type of lattice, and check which point groups preserve the lattice.

Corollary 4.9 ([Arm97, Theorem 25.4][Zav18, see p. 5]). *If J is the point group of some wallpaper group, J is isomorphic to one of the following group*

$$C_1,\ C_2,\ C_3,\ C_4,\ C_6,\ D_1,\ D_2,\ D_3,\ D_4,\ D_6 \tag{4.4}$$

where C_j is the cyclic group of order j, and D_j is the dihedral group of order $2j$.

Another description of wallpaper groups is as follows: a wallpaper group G contains the abelian group of translations T as a normal subgroup, and G/T is isomorphic to J, the point group. This means that there is an exact sequence

$$1 \to T \to G \to J \to 1.$$

More generally, if T is an arbitrary abelian group, and J is a fixed group, a **group extension** of T by J is an exact sequence

$$1 \to T \to G \xrightarrow{\pi} J \to 1.$$

Since there are ten different point groups, we have at least ten group extensions. Some point groups can have different actions on T, hence counting the number of extensions gives us the number of wallpaper groups. Therefore, to get the presentation of a wallpaper group, we only need the generators and relations of T, the generators and relations of J and the action(s) of J on T. The following result allows us to write down the elements of a wallpaper group from the point group and some vectors from $\mathbb{R}^2$. Recall that an isometry $f \in \operatorname{Isom}(\mathbb{R}^2)$ can be identified with (v, M), $v \in \mathbb{R}^2$ and $M \in O_2$.

Lemma 4.10 ([Mor03, Lemma 5.1])**.** *Let G be a wallpaper group with point group J. For each $M \in J$, there exists $v_M \in \mathbb{R}^2$ such that $(v_M, M) \in G$. Furthermore, v_M is uniquely determined up to addition by an element of T, and*

$$G = \{ (v_M + v, M) \mid M \in J,\ v \in T \}.$$

Therefore, we can describe a wallpaper group using only the v_M's. Moreover, this also gives us the generators of G: they will be τ_1, τ_2 and $\{v_M, M\}$ such that the M's generate the point group. We then only need to consider all possible values of v_M for each point group. Note that v_M need not belong to T, as we will see. The work of finding the generators is done in [Arm97, Zav18, Mor03] and we simply use them as presented in these references. For more details on the classification, see [Mor03, Chapter 5].

4.2.2 Spectral measures

Consider a finitely generated group $G = \langle S|R \rangle$ with $S = S^{-1}$, $|S| < +\infty$ and its Cayley graph $\Gamma = \operatorname{Cay}(G, S)$. Recall that

$$\ell^2(G) = \left\{ f \colon G \to \mathbb{C} \mid \sum_{\gamma \in G} |f(\gamma)|^2 < +\infty \right\}.$$

In addition, the **left regular representation** of G, denoted by λ_G, is defined by

$$\begin{array}{llllllll}
\lambda_G : & G & \to & \mathcal{B}(\ell^2(G)) & & & & \\
& s & \mapsto & \lambda_G(s) : & \ell^2(G) & \to & \ell^2(G) & \\
& & & & \xi & \mapsto & \lambda_G(s)(\xi) : & G \to \mathbb{C} \\
& & & & & & & t \mapsto \xi(s^{-1}t).
\end{array}$$

Definition 4.11 (Markov operator)**.** The **Markov operator** is the operator on $\ell^2(G)$ defined by

$$Mf(g) = \frac{1}{|S|} \sum_{s \in S} \lambda_G(s) f(g) = \frac{1}{|S|} \sum_{s \in S} f(s^{-1}g).$$

On $\ell^2(G)$, we have the canonical basis $(\delta_g)_{g \in G}$ where

$$\delta_g(h) = \begin{cases} 1 & \text{if } h = g, \\ 0 & \text{if } h \neq g \end{cases}, \quad \forall g, h \in G.$$

Thus, when we view M in this basis, $p_{gh} := M_{gh}$ represents the probability to go from g to h. The Markov operator, sometimes referred to as the transfer operator, induces a simple random walk on Γ. It is a Markov chain $(X_n)_{n \geq 0}$ with state space $V(\Gamma) = G$ which satisfies

$$\mathbb{P}[X_n = h \mid X_k = g_k,\ k = 0, \dots, n-1] = \mathbb{P}[X_n = h \mid X_{n-1} = g] = p_{gh}$$

for all $g_0, \dots, g_{n-1} = g, h \in G$.

Two related operators are the **adjacency operator** and the **laplacian operator** which also act on $\ell^2(G)$ and are defined by

$$A = |S|M \qquad \text{and} \qquad \Delta = \mathrm{Id} - M$$

respectively. However, we shall only consider the Markov operator here.

More generally, for any locally finite, connected graph $\Gamma = (V, E)$, we can consider the Markov operator which acts on $\ell^2(V)$ by

$$Mf(v) = \frac{1}{\deg(v)} \sum_{u \sim v} f(u)$$

where deg(v) is the degree of $v \in V$, and $u \sim v$ means that $\{u, v\} \in E$. Therefore, if $\Gamma = \mathrm{Cay}(G, S)$ for some finitely generated group G, we can in particular consider the Markov operator on subgraphs of Γ, or on **Schreier graphs** $\mathcal{S}(G, H, S)$ (if $H \leq G$, the Schreier graph $\mathcal{S}(G, H, S)$ is the oriented graph with vertex set G/H and edge set $\{\{gH, sgH\} \mid s \in S\}$).

We come back to the case where $\Gamma = \mathrm{Cay}(G, S)$ and G is a finitely generated group with symmetric generating set. In this case, M is a linear bounded self-adjoint operator with $\|M\| \leq 1$ (see [Moh82, MW89]). Therefore we have that the spectrum of M is real and contained in $[-1, 1]$, i.e. $\mathrm{Sp}(M) \subset [-1, 1]$. Moreover, since M is a linear bounded self-adjoint operator, the spectral theorem applies. Let us quickly recall it (see [Hal17] for more details).

Definition 4.12. Let $(X, \mathcal{M})$ be a measure space and let $\mathcal{H}$ be a Hilbert space. A **spectral measure** on $(X, \mathcal{M}; \mathcal{H})$ is an application $E\ : \mathcal{M} \to \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the C^*-algebra of linear bounded operators on $\mathcal{H}$, which satisfies

1. $E(\Delta)$ is a projection for all $\Delta \in \mathcal{M}$;

2. $E(\emptyset) = 0$ and $E(X) = \mathrm{Id}$;

3. $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$ for all $\Delta_1, \Delta_2 \in \mathcal{M}$;

4. E is σ-additive, i.e. for all disjoint family $\{\Delta_n\}$ of $\mathcal{M}$,

$$E\left(\bigcup_{n \in \mathbb{N}} \Delta_n\right) = \sum_{n \in \mathbb{N}} E(\Delta_n)$$

(where the equality is to be understood in the strong operator topology).

Theorem 4.13 (Spectral theorem (for bounded self-adjoint operators))**.** *Let T be a linear bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. There exists a unique* *compact, real spectral measure* $E : \mathrm{Bor}(\mathrm{Sp}(T)) \to \mathcal{B}(\mathcal{H})$ *such that*

$$T = \int_{\mathrm{Sp}(T)} \lambda \mathrm{d}E(\lambda).$$

In addition, a spectral measure on $(X, \mathcal{M}; \mathcal{H})$ gives rise to an infinite family of complex measures: for all $\xi, \eta \in \mathcal{H}$, the mapping

$$E_{\xi,\eta} : \mathcal{M} \to \mathbb{C},\ \Delta \mapsto \langle E(\Delta)\xi | \eta\rangle$$

is a complex measure on $(X, \mathcal{M})$ with total variation $\|E_{\xi,\eta}\| \leq \|\xi\|\|\eta\|$, and $E_{\xi,\xi}$ is a finite positive measure.

Since M satisfies the hypotheses of theorem 4.13, we know that there exists a unique spectral measure $E : \mathrm{Bor}([-1, 1]) \to \mathcal{B}(\ell^2(G))$ such that

$$M = \int_{\mathrm{Sp}(M)} \lambda \mathrm{d}E(\lambda),$$

and there is an infinite family of complex measures

$$\mu_{x,y}(\Delta) := E_{\delta_x,\delta_y}(\Delta) = \langle E(\Delta)\delta_x | \delta_y\rangle, \quad \forall \Delta \in \mathrm{Bor}(\mathrm{Sp}(M)).$$

Let $x, y \in G$ and let $n \in \mathbb{N}$. Denote by $p^n_{x,y}$ the probability to go from x to y in n steps following the random walk prescribed by the Markov operator M. Then

$$p^n_{x,y} = \langle M^n \delta_x | \delta_y\rangle$$

or equivalently

$$p^n_{x,y} = \int_{\mathrm{Sp}(M)} \lambda^n \mathrm{d}\mu_{x,y}(\lambda).$$

Of particular interest to us are the probability of returns $p^n_{x,x}$.

Definition 4.14 (Kesten spectral measure). The finite measure $\mu_x := \mu_{x,x}$ is called the **Kesten spectral measure**.

The Kesten spectral measure μ_x is a probability measure on $\mathrm{Sp}(M)$. They can be recovered through the probabilities of return with the Stieltjes inversion formula (see [MW89, eq. (4.19)]). For a vertex-transitive graph such as a Cayley graph, the probabilities of return are the same at every vertex. Therefore, $\mu_x = \mu_y$ for every $x, y \in G$ and we will be interested only in computing μ_e, where e is the neutral element of the group. In the rest of the paper, we will usually denote the measure μ_e by μ_G to explicitly state of which group we are computing the measure.

The Stieltjes inversion formula is one way to compute the Kesten spectral measure. Below we present some other ways of computing the Kesten spectral measure.

Theorem 4.15 ([MW89, Theorem 4.10], [Bor16, Section 1.3.1]). *Let* Γ *be the cartesian product of two graphs* Γ_1 *and* Γ_2. *Moreover, denote by* E, E^{Γ_1} *and* E^{Γ_2} *the spectral measures of* Γ, Γ_1 *and* Γ_2 *respectively. Then* E *is the convolution of* E^{Γ_1} *and* E^{Γ_2}. *In other words, if* $x, y \in V(\Gamma)$ *with* $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$, $x_i, y_i \in V(\Gamma_i)$, *then*

$$\mu_{x,y} = \mu^{\Gamma_1}_{x_1,y_1} * \mu^{\Gamma_2}_{x_2,y_2}.$$

The previous theorem assumes that we already know the spectral measures of the graphs Γ_1 and Γ_2. Another classical way to compute Kesten spectral measures is the "finite approximation" method. We say that a sequence of subgraphs $(\Gamma_n)_n$ converges to Γ if each edge of Γ is contained in all but finitely many of the Γ_n.

Theorem 4.16 ([Moh82, Theorem 4.14])**.** *Let $(\Gamma_n)_n$ be a sequence of subgraphs converging to Γ, and let M_n the truncation of M with respect to Γ_n. Then the spectral measures of M_n converge to the spectral measure of M at all points of continuity of the latter (i.e. the spectral measures converge weakly).*

Another notion of convergence for graphs is as follows. Let $\{(X_n, v_n)\}$ be a family of marked graphs, i.e. graphs X_n with a chosen vertex $v_n \in V(X_n)$ for each $n \geq 1$. The space of marked graphs can be turned into a metric space with the following distance:

$$\operatorname{Dist}((X_1, v_1), (X_2, v_s)) = \inf\left\{\frac{1}{n+1},\ B_{X_1}(v_1, n) \text{ is isometric to } B_{X_2}(v_2, n)\right\}$$

where $B_{X_i}(v_i, n)$ denotes the ball of radius n around v_i in X_i for the usual graph distance. We say that a sequence of marked graphs $\{(X_n, v_n)\}$ converges to (X, v) if

$$\lim_{n\to\infty} \operatorname{Dist}((X, v), (X_n, v_n)) = 0.$$

The limit graph is unique up to isometry (see [GZ99] for more details).

Now, let P_n be subgroups of finite index of G, and let $P = \cap_{n=1}^{\infty} P_n$. The sequence of finite Schreier graphs $\Gamma_n = \mathcal{S}(G, P_n, S)$ converges to $\Gamma = \mathcal{S}(G, P, S)$ in the space of marked graphs and Grigorchuk and Żuk proved that the Kesten spectral measure μ_e^n of Γ_n converges weakly to the Kesten spectral measure μ_e of Γ. In other words,

$$\lim_{n\to\infty} \int_{-1}^{1} f \mathrm{d}\mu_e^n = \int_{-1}^{1} f \mathrm{d}\mu_e$$

for any $f \in C[-1, 1]$. We summarize this discussion in the following theorems.

Theorem 4.17 ([GZ99])**.** *Suppose that the sequence of marked graphs (X_n, v_n) converges to the marked graphs (X, v). Then the Kesten spectral measure μ_v^n of the graph X_n converges weakly to the Kesten spectral measure μ_v of the graph X*

Corollary 4.18. *Let μ_e^n and μ_e be the Kesten spectral measures of the Schreier graphs $\Gamma_n = \mathcal{S}(G, P_n, S)$ and $\Gamma = \mathcal{S}(G, P, S)$. Then $(\Gamma_n, P_n) \to (\Gamma, P)$ in the space of marked graphs and μ_e^n converges weakly to μ_e. In particular, if $P = 1$, then μ_e^n converges weakly to the Kesten spectral measure of the Cayley graph* $\operatorname{Cay}(G, S)$*.*

4.2.3 Spectral measures

4.3 Computations of the Kesten spectral measure of certain wallpaper groups

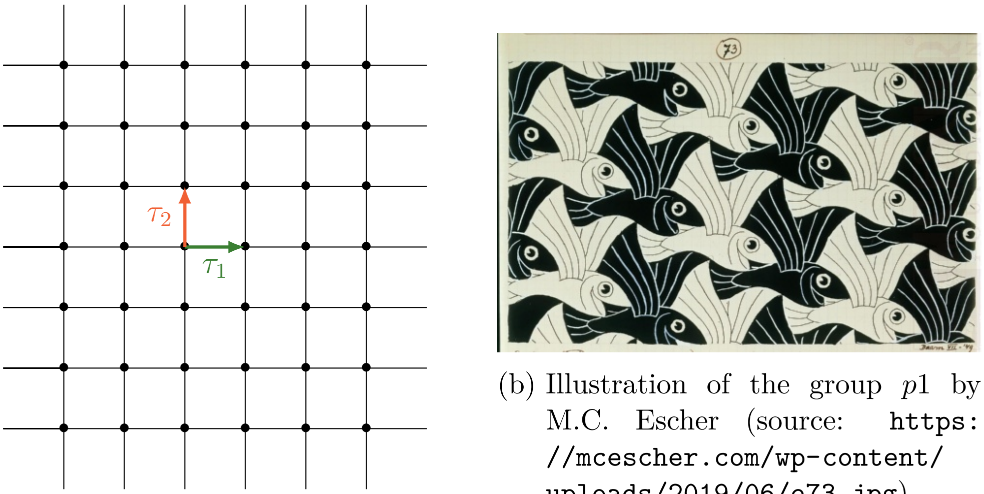
In this section, we compute the Kesten spectral measure of some wallpaper groups. We use the generators as prescribed in [Arm97, Zav18, Mor03]. There are other possible presentations for the wallpaper groups, see [CM13] for instance. Recall that elements of wallpaper groups can be described by (v_M, M) where $M \in J$ and $v_M \in \mathbb{R}^2$ and that the multiplication is given by $(v_M, M)(v'_M, M') = (v_M + Mv'_M, MM')$. Moreover, we consider all generating sets to be symmetric (if we write $G = \langle x_1, \dots, x_n | R\rangle$ with the x_j's generating the group G, we consider that the x_j^{-1}s also belong to the generating set), and the Cayley graph are thus viewed as non-oriented graphs.

4.3.1 $p1$

The group p_1 is simply generated by two translations. Its point group is C_1, the cyclic group of order 1 which acts trivially on the lattice T. Thus, denoting by $[x, y] = xyx^{-1}y^{-1}$ the commutator of two elements, p_1 admits the following presentation:

$$p_1 = \langle \tau_1, \tau_2 | [\tau_1, \tau_2] = e\rangle \tag{4.5}$$

where $e \in G$ is the neutral element.



(a) Cayley graph of p_1.

(b) Illustration of the group $p1$ by M.C. Escher (source: `https://mcescher.com/wp-content/uploads/2019/06/e73.jpg`)

Figure 4.3: Two geometric representations of $p1$.

Remark 4.19. To be very precise, we should write (τ_1, Id_2), (τ_2, Id_2) where τ_1 has the minimal length in the lattice, τ_2 is skew to τ_1 and its length is minimal, and Id_2 is the neutral element of the point group J. We will make this slight abuse of notation for every lattice.

Note that with this presentation, $\mathrm{Cay}(p_1, \{\tau_1^{\pm1}, \tau_2^{\pm1}\}) = \mathrm{Cay}\left(\mathbb{Z}^2, \left\{\left(\begin{smallmatrix}\pm1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\\pm1\end{smallmatrix}\right)\right\}\right)$. Therefore, the Kesten spectral measure of p_1 is that of $\mathbb{Z}^2$ with the standard generators. We can use theorem 4.15 to compute it, since the Cayley graph of $\mathbb{Z}^2$ is the cartesian product of $\mathbb{Z}$ (generated by $S = \{\pm 1\}$) with itself. Thus, the problem is reduced to computing the spectral measure of $\mathbb{Z}$. This problem is known and solved (see [MW89, Section 7.A] for the result and references). Still, we feel that it is worth showing how one explicitly computes the Kesten spectral measure on an easy example. Moreover, this result is the cornerstone for all subsequent results in this chapter. For this, we use the method of finite approximation given in corollary 4.18.

An $n \times n$ matrix C is **circulant** if C or $C^\top$ has the form

$$\begin{pmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{pmatrix}, \quad c_j \in \mathbb{C},\ j = 0, \dots, n-1.$$

It is known that its eigenvalues $\{\lambda_k\}_{k=1}^n$ are given by

$$\lambda_k = \sum_{j=1}^{n} c_j \mathrm{e}^{\frac{2\pi \mathbf{i} kj}{n}}, \quad k = 1, \dots, n, \tag{4.6}$$

(see [G$^+$06, Section 3.1] for instance).

Theorem 4.20. *The Kesten spectral measure of* $\mathbb{Z} = \langle \pm 1 \rangle$ *is given by the arcsine distribution with density on* $[-1, 1]$*:*

$$\mathrm{d}\mu_{\mathbb{Z}} = \frac{1}{\pi\sqrt{1-x^2}} \chi_{|x| \le 1} \mathrm{d}x. \tag{4.7}$$

Proof. Consider the sequence $\Gamma_n = \mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, \{\pm 1\})$. We have that $\Gamma_n \to \mathrm{Cay}(\mathbb{Z}, \{\pm 1\}) = \Gamma$ by corollary 4.18, so the Kesten spectral measure of Γ_n (denoted by μ_{Γ_n}) will converge weakly to the spectral measure of Γ (denoted by $\mu_{\mathbb{Z}}$). Therefore, we compute μ_{Γ_n} and deduce $\mu_{\mathbb{Z}}$. Let M_n be the Markov operator of Γ_n, represented as a matrix. It has the form

$$M_n = \begin{pmatrix} 0 & \frac{1}{2} & \cdots & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & & 0 \\ \vdots & \frac{1}{2} & 0 & \ddots & \vdots \\ 0 & & \ddots & \ddots & \frac{1}{2} \\ \frac{1}{2} & 0 & \cdots & \frac{1}{2} & 0 \end{pmatrix}$$

From eq. (4.6) we know that the eigenvalues of M_n are given by

$$\lambda_k(M_n) = \frac{1}{2}\mathrm{e}^{\frac{2\pi \mathbf{i} k}{n}} + \frac{1}{2}\mathrm{e}^{\frac{-2\pi \mathbf{i} k}{n}} = \cos\left(\frac{2\pi k}{n}\right), \quad k = 1, \dots, n.$$

For a Borel set Δ in $\mathrm{Sp}(M_n)$, the Kesten spectral measure of Γ_n is given by

$$\mu_{\Gamma_n}(\Delta) = \frac{1}{n}\sum_{\lambda\in\Delta} m_\lambda$$

where m_λ is the multiplicity of λ as an eigenvalue. In this case, we obtain

$$\mu_{\Gamma_n}(\Delta) = \frac{1}{n}\sum_{k=1}^{n} \chi_\Delta\left(\cos\left(\frac{2\pi k}{n}\right)\right)$$

where $\chi_\Delta(\cdot)$ is the characteristic function of Δ. Now, for the weak convergence, we must have that

$$\lim_{n\to\infty}\int_{-1}^{1} f \mathrm{d}\mu_{\Gamma_n} = \int_{-1}^{1} f \mathrm{d}\mu_{\mathbb{Z}} \tag{4.8}$$

for any $f \in C[-1,1]$, thus we only need to find $\mathrm{d}\mu_{\mathbb{Z}}$. The left-hand side is

$$\lim_{n\to\infty}\int_{-1}^{1} f \mathrm{d}\mu_{\Gamma_n} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\cos\left(\frac{2\pi k}{n}\right)\right) = \int_{0}^{1} f(\cos(2\pi t))\mathrm{d}t. \tag{4.9}$$

Observe that, for $\tilde{t} \in [1/2, 1]$, there exists $t \in [0, 1/2]$ such that $\tilde{t} = t + 1/2$ and we have

$$\cos(2\pi\tilde{t}) = \cos(2\pi(t+1/2)) = \cos(2\pi t + \pi) = -\cos(2\pi t).$$

Thus the right-hand side of eq. (4.9) becomes

$$\begin{aligned}\int_{0}^{1} f(\cos(2\pi t))\mathrm{d}t &= \int_{0}^{1/2} f(\cos(2\pi t))\mathrm{d}t + \int_{1/2}^{1} f(\cos(2\pi t))\mathrm{d}t \\ &= \int_{0}^{1/2} f(\cos(2\pi t))\mathrm{d}t + \int_{0}^{1/2} f(-\cos(2\pi t))\mathrm{d}t.\end{aligned}$$

For the first integral, we apply the following change of variables:

$$s = \cos(2\pi t) \implies t = \frac{1}{2\pi}\arccos(t) \text{ and } \mathrm{d}t = \frac{1}{2\pi}\frac{-1}{\sqrt{1-s^2}}\mathrm{d}s.$$

For the second one, we apply the following change of variables:

$$x = -\cos(2\pi t) \implies t = \frac{1}{2\pi}\arccos(-x) \text{ and } \mathrm{d}t = \frac{1}{2\pi}\frac{1}{\sqrt{1-x^2}}\mathrm{d}x.$$

We obtain

$$\begin{aligned}
\int_0^1 f(\cos(2\pi t))\mathrm{d}t &= \int_0^{1/2} f(\cos(2\pi t))\mathrm{d}t + \int_0^{1/2} f(-\cos(2\pi t))\mathrm{d}t \\
&= \int_{\cos(2\pi\cdot 0)}^{\cos(2\pi\cdot 1/2)} f(s)\frac{1}{2\pi}\frac{-1}{\sqrt{1-s^2}}\mathrm{d}s \\
&\quad + \int_{-\cos(2\pi\cdot 0)}^{-\cos(2\pi\cdot 1/2)} f(x)\frac{1}{2\pi}\frac{1}{\sqrt{1-x^2}}\mathrm{d}x \\
&= \frac{-1}{2}\int_1^{-1} f(x)\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\mathrm{d}x + \frac{1}{2}\int_{-1}^1 f(x)\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\mathrm{d}x \\
&= \frac{1}{2}\int_{-1}^1 f(x)\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\mathrm{d}x + \frac{1}{2}\int_{-1}^1 f(x)\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\mathrm{d}x \\
&= \int_{-1}^1 f(x)\frac{1}{\pi\sqrt{1-x^2}}\mathrm{d}x.
\end{aligned}$$

Using eq. (4.8) we conclude that

$$\mathrm{d}\mu_{\mathbb{Z}} = \frac{1}{\pi\sqrt{1-x^2}}\chi_{|x|\leq 1}\mathrm{d}x.$$

■

Using theorem 4.15, we directly obtain the following result.

Theorem 4.21. *The Kesten spectral measure of* p_1 *as generated in eq.* (4.5) *is the same as that of* $\mathbb{Z}^2$ *with its standard generators and is given by*

$$\mathrm{d}\mu_{p_1} = \mathrm{d}\mu_{\mathbb{Z}} * \mathrm{d}\mu_{\mathbb{Z}},$$

or in other words, for any $\Delta \in \mathrm{Bor}(\mathrm{Sp}(\mathbb{Z}^2)) = \mathrm{Bor}([-1,1])$,

$$\mathrm{d}\mu_{p_1}(\Delta) = \int_{[-1,1]^2} \chi_\Delta(x+y)\frac{1}{\pi^2\sqrt{(1-x^2)(1-y^2)}}\mathrm{d}x\mathrm{d}y.$$

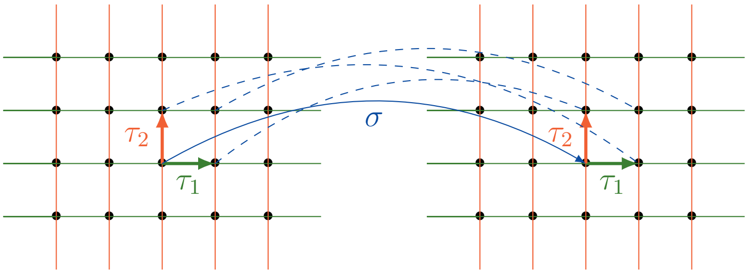
4.3.2 *cm*

The group *cm* is generated by two translations τ_1, τ_2 and its point group is D_1. A possible presentation for *cm* is

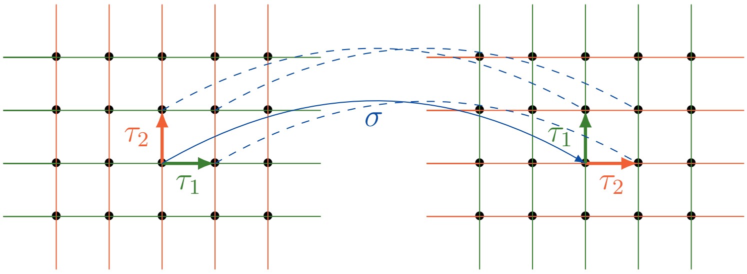
$$cm = \left\langle \tau_1, \tau_2, \sigma | [\tau_1, \tau_2] = e, \tau_1^\sigma = \tau_2, \tau_2^\sigma = \tau_1, \sigma^2 = e \right\rangle. \tag{4.10}$$

This is realized for instance by $\tau_1 = \left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)$, $\tau_2 = \left(\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)$ and $\sigma = \left(\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\right)$. Its Cayley graph is illustrated in fig. 4.4.

The colors and arrows in fig. 4.4 are here to simplify the visualization of the Cayley graph, but as a non-oriented, non-labelled graph, we see that it is isomorphic to $\mathrm{Cay}(\mathbb{Z}^2) \times \mathrm{Cay}(D_1)$. Indeed, by re-drawing the copy of $\mathbb{Z}^2$ on the right-hand side of fig. 4.4a by reversing the roles of τ_1 and τ_2, and keeping all the other edges between the vertices the same, we obtain an isomorphism in the sense of definition 2.33 between $\mathrm{Cay}(cm)$ and $\mathrm{Cay}(\mathbb{Z}^2, \{\tau_1^{\pm 1}, \tau_2^{\pm 1}\}) \times \mathrm{Cay}(D_1, \{\sigma^{\pm 1}\})$. This is illustrated in fig. 4.4b. Therefore, combining theorem 4.21 and theorem 4.15 we only need to compute the Kesten spectral measure of D_1 to obtain the Kesten spectral measure of *cm*.

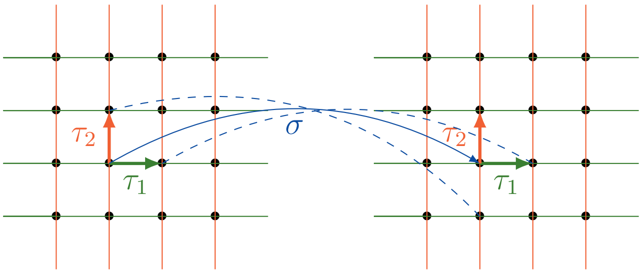


(a) Cayley graph of cm.

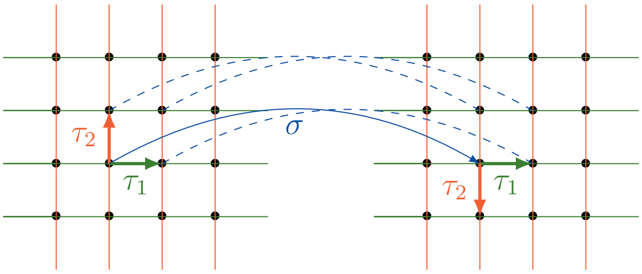


(b) Cayley graph of cm, redrawn to show the isomorphism between Cay(cm) and Cay($\mathbb{Z}^2$) × Cay(D_1).

Figure 4.4: Cayley graph of cm.



(a) Cayley graph of pm.



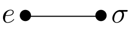
(b) Cay(pm), redrawn to illustrate the isomorphism with Cay($\mathbb{Z}^2$)×Cay(D_1).

Figure 4.5: Cayley graph of pm.

Proposition 4.22. *The Kesten spectral measure of* $D_1 = \langle \sigma^{\pm 1} | \sigma^2 = e \rangle$ *is given by*

$$\mathrm{d}\mu_{D_1}(x) = \frac{1}{2}\delta_{-1}(x) + \frac{1}{2}\delta_1(x). \tag{4.11}$$

Proof. The Cayley graph of D_1 being



its Markov operator in matrix form is $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and its eigenvalues are ± 1, hence the conclusion. ■

For a Borel set $\Delta \in \mathrm{Bor}(\mathrm{Sp}(M))$ and $t \in \mathbb{R}$, denote by $\Delta + t$ the set

$$\Delta + t := \{ \delta + t \mid \delta \in \Delta \} .$$

Corollary 4.23. *The Kesten spectral measure of cm generated by eq.* (4.10) *is given by*

$$\mu_{cm}(\Delta) = \frac{1}{2}\mu_{p_1}((\Delta+1)\cap[-1,1]) + \frac{1}{2}\mu_{p_1}((\Delta-1)\cap[-1,1]) \quad \forall \Delta \in \mathrm{Bor}([-1,1]). \tag{4.12}$$

Proof. From theorem 4.15, we know that

$$\mathrm{d}\mu_{cm} = \mathrm{d}\mu_{p1} * \mathrm{d}\mu_{D_1}.$$

Thus, using the definition of the convolution of measures and letting $\Delta \in \mathrm{Bor}([-1,1])$, we obtain

$$\begin{aligned}
\mathrm{d}\mu_{cm}(\Delta) &= \int_{[-1,1]^2} \chi_\Delta(x+y)\mathrm{d}\mu_{p_1}(x)\mathrm{d}\mu_{D_1}(y) \\
&= \int_{[-1,1]^2} \chi_\Delta(x+y)\mathrm{d}\mu_{p_1}(x)\left(\frac{1}{2}\delta_{-1}(y) + \frac{1}{2}\delta_1(y)\right) \\
&= \frac{1}{2}\int_{[-1,1]} \chi_\Delta(x-1)\mathrm{d}\mu_{p_1}(x) + \frac{1}{2}\int_{[-1,1]} \chi_\Delta(x+1)\mathrm{d}\mu_{p_1}(x) \\
&= \frac{1}{2}\mathrm{d}\mu_{p_1}((\Delta+1)\cap[-1,1]) + \frac{1}{2}\mathrm{d}\mu_{p_1}((\Delta-1)\cap[-1,1]).
\end{aligned}$$

■

4.3.3 *pm*

The group *pm* is generated by two translations τ_1, τ_2 and its point group is also D_1. A possible presentation for *pm* is

$$pm = \langle \tau_1, \tau_2, \sigma | [\tau_1, \tau_2] = e, \tau_1^\sigma = \tau_1, \tau_2^\sigma = \tau_2^{-1}, \sigma^2 = e \rangle . \tag{4.13}$$

This is realized for instance by $\tau_1 = (\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right))$, $\tau_2 = (\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right))$ and $\sigma = (\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right))$. Its Cayley graph is illustrated in fig. 4.5a. Arguing as in section 4.3.2, we obtain that $\mathrm{Cay}(pm)$ is isomorphic as a non-oriented, non-labeled graph to $\mathrm{Cay}(\mathbb{Z}^2, \{\tau_1^{\pm 1}, \tau_2^{\pm 1}\}) \times \mathrm{Cay}(D_1, \{\sigma^{\pm 1}\})$. This is illustrated in fig. 4.5b. Therefore, the Kesten spectral measure of *pm* is the same as that of *cm*.

Proposition 4.24. *The Kesten spectral measure of pm generated by eq.* (4.13) *is given by*

$$\mu_{pm}(\Delta) = \frac{1}{2}\mu_{p_1}((\Delta+1)\cap[-1,1]) + \frac{1}{2}\mu_{p_1}((\Delta-1)\cap[-1,1]) \quad \forall \Delta \in \mathrm{Bor}([-1,1]). \tag{4.14}$$

This shows that we have two (in fact three as we will see) non-isomorphic groups (cm, pm and $p2$) that are isospectral.

4.3.4 *pg*

The group pg is generated by two translations τ_1, τ_2 and its point group is also D_1. It contains a glide reflection. A possible presentation for pg is

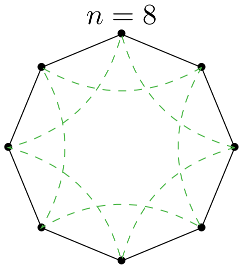
$$pg = \langle \tau_1, \tau_2, \gamma | [\tau_1, \tau_2] = e, \tau_1^\gamma = \tau_1, \tau_2^\gamma = \tau_2^{-1}, \gamma^2 = \tau_1 \rangle \,. \tag{4.15}$$

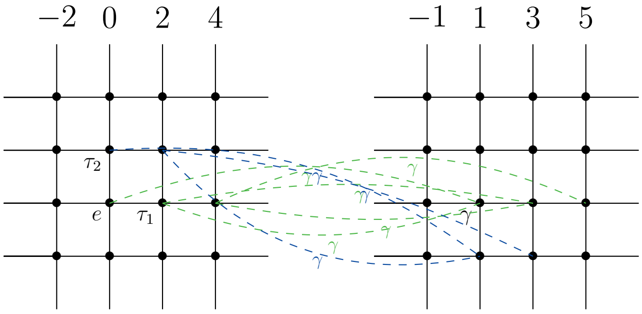
This is realized, for instance, by taking $\tau_1 = \left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)$, $\tau_2 = \left(\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)$ and $\gamma = \left(\left(\begin{smallmatrix}1/2\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)\right)$. Note that pg is the fundamental group of the Klein bottle (see [Flo07, p. 56]). Its Cayley graph is illustrated in fig. 4.6a. Using the same trick as the one we used in section 4.3.2 and section 4.3.3, we can redraw it (by having the role of τ_2 being played by τ_2^{-1} in the right-hand side copy) as in fig. 4.6b and both graphs are still isomorphic. Now we re-index the columns of both copies of $\mathbb{Z}^2$ according to the labeling we chose in fig. 4.6a and fig. 4.6b, and we keep the same edges between all vertices, yielding another isomorphism. This is illustrated in fig. 4.6c. We see that this graph is isomorphic to $\mathrm{Cay}(\mathbb{Z}, \{\pm 1\}) \times \mathrm{Cay}(\mathbb{Z}, \{\pm 1, \pm 2\})$. Therefore, since we computed the Kesten spectral measure of $\mathrm{Cay}(\mathbb{Z}, \{\pm 1\})$ in eq. (4.7), we only need to compute the Kesten spectral measure of $\mathrm{Cay}(\mathbb{Z}, \{\pm 1, \pm 2\})$ to use theorem 4.15 and find the Kesten spectral measure of pg.

Proposition 4.25. *The Kesten spectral measure of* $\Gamma = \mathrm{Cay}(\mathbb{Z}, \{\pm 1, \pm 2\})$ *is given by*

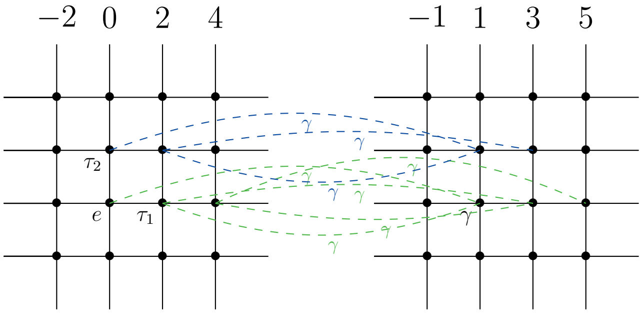
$$\mathrm{d}\mu_\Gamma(x) = \frac{4\mathrm{d}x}{\pi\sqrt{8x+\frac{9}{2}}} \left(\frac{1}{\sqrt{-8x-\sqrt{16x+9}+3}} \chi_{\left[\frac{-9}{16},0\right]}(x) + \frac{1}{\sqrt{-8x+\sqrt{16x+9}+3}} \chi_{\left[\frac{-9}{16},1\right]}(x) \right) . \tag{4.16}$$

Proof. We first consider the Schreier graphs $\Gamma_n = \mathcal{S}(\mathbb{Z}, n\mathbb{Z}, \{\pm 1, \pm 2\})$ which are equal to $\mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, \{\pm 1, \pm 2\})$. These graphs look as follows:

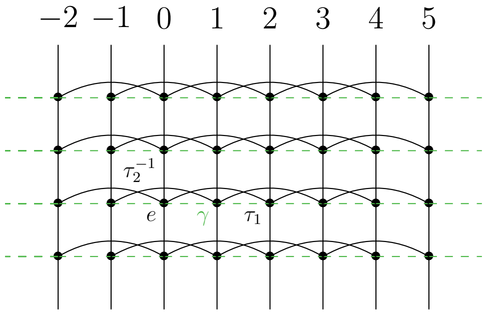




(a) Cayley graph of *pg*.



(b) Cayley graph of *pg*, redrawn to illustrate a first graph isomorphism.



(c) Cayley graph of *pg*, redrawn to show the isomorphism between $\mathrm{Cay}(pg)$ and $\mathrm{Cay}(\mathbb{Z},\{\pm1\})\times\mathrm{Cay}(\mathbb{Z},\{\pm1,\pm2\})$.

Figure 4.6: Cayley graph of *pg*.

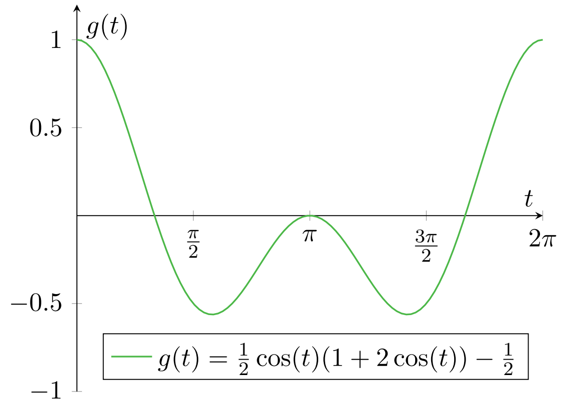
Their Markov operator in matrix form is a circulant matrix which has the form

$$M_n = \frac{1}{4}\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \end{pmatrix}$$

Using eq. (4.6) we obtain that the eigenvalues of M_n are given by

$$\lambda_k = \frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{2}\cos\left(\frac{4\pi k}{n}\right) = \frac{1}{2}\cos\left(\frac{2\pi k}{n}\right)\left(1 + 2\cos\left(\frac{2\pi k}{n}\right)\right) - \frac{1}{2}. \tag{4.17}$$

As $n \to \infty$, the eigenvalues in eq. (4.17) take all possible values $\frac{1}{2}\cos(t)(1+2\cos(t))-\frac{1}{2}$ for $t \in [0, 2\pi]$. The function $g : [0, 2\pi] \to \mathbb{R}$, $t \mapsto \frac{1}{2}\cos(t)(1 + 2\cos(t)) - \frac{1}{2}$ has three zeros: $z_1 = \frac{\pi}{3}$, $z_2 = \pi$ and $z_3 = \frac{5\pi}{3}$. It has three local maxima, $M_1 = 0$, $M_2 = \pi$, $M_3 = 2\pi$ and two local minima, $m_1 = \arccos(\frac{-1}{4})$ and $m_2 = 2\pi - \arccos(\frac{-1}{4})$. We illustrate this in the plot below.



We now apply corollary 4.18 to study the weak convergence of the Kesten spectral measure of Γ_n and find the Kesten spectral measure of Γ. Let $f : [0, 2\pi] \to \mathbb{R}$ be a continuous function. Then, as a Riemann sum with subdivisions of length $\frac{2\pi}{n}$, the convergence is

$$\lim_{n\to\infty} \int_{\mathrm{Sp}(\Gamma_n)} f \mathrm{d}\mu_{\Gamma_n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(g(t)) = \frac{1}{2\pi} \int_0^{2\pi} f(g(t))\mathrm{d}t.$$

The function g being 2π-periodic and even, we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} f(g(t))\mathrm{d}t = \frac{1}{\pi} \int_0^{\pi} f(g(t))\mathrm{d}t,$$

which can be split into two parts using the maxima and minima computed above. We obtain

$$\frac{1}{\pi} \int_0^{\pi} f(g(t))\mathrm{d}t = \frac{1}{\pi} \int_0^{\arccos(\frac{-1}{4})} f(g(t))\mathrm{d}t + \int_{\arccos(\frac{-1}{4})}^{\pi} f(g(t))\mathrm{d}t \tag{4.18}$$

so that g is bijective on both intervals and we can apply a change of variable to each integral. The change of variable is given by $x = \frac{1}{2}\cos(t)(1 + 2\cos(t)) - \frac{1}{2}$. On the interval $[0, m_1]$ this gives us $x \in g([0, m_1]) = [\frac{-9}{16}, 1]$ (note that this will reverse the sign of the integral) and

$$t = \arccos\left(\frac{-1 + \sqrt{9 + 16x}}{4}\right) \implies \mathrm{d}t = \frac{-4\mathrm{d}x}{\sqrt{8x + \frac{9}{2}}\sqrt{-8x + \sqrt{16x + 9} + 3}}$$

and on the interval $[m_1,\pi]$ we obtain $x \in g([m_1,\pi]) = [-9/16,0]$

$$t = \arccos\left(\frac{-1-\sqrt{9+16x}}{4}\right) \implies \mathrm{d}t = \frac{4\mathrm{d}x}{\sqrt{8x+\frac{9}{2}}\sqrt{-8x-\sqrt{16x+9}+3}}$$

Therefore, eq. (4.18) becomes

$$\begin{aligned}\frac{1}{\pi}\int_0^\pi f(g(t))\mathrm{d}t = \frac{1}{\pi}\int_1^{-9/16} f(x)\frac{-4\mathrm{d}x}{\sqrt{8x+\frac{9}{2}}\sqrt{-8x+\sqrt{16x+9}+3}} \\ + \frac{1}{\pi}\int_{-9/16}^{0} f(x)\frac{4\mathrm{d}x}{\sqrt{8x+\frac{9}{2}}\sqrt{-8x-\sqrt{16x+9}+3}}\end{aligned}$$

and we finally obtain that the Kesten spectral measure of Γ is

$$\mathrm{d}\mu_\Gamma(x) = \frac{4\mathrm{d}x}{\pi\sqrt{8x+\frac{9}{2}}}\left(\frac{1}{\sqrt{-8x-\sqrt{16x+9}+3}}\chi_{\left[\frac{-9}{16},0\right]}(x) + \frac{1}{\sqrt{-8x+\sqrt{16x+9}+3}}\chi_{\left[\frac{-9}{16},1\right]}(x)\right).$$

■

Therefore, using theorem 4.15 applied to theorem 4.20 and proposition 4.25 the following result is immediate.

Corollary 4.26. *The Kesten spectral measure of pg generated by eq.* (4.15) *is given by the convolution of the Kesten spectral measure* $\mathrm{d}\mu_{\mathbb{Z}}$ *of* $\mathbb{Z}$ *generated by* $\{\pm 1\}$ *and the Kesten spectral measure* $\mathrm{d}\mu_{(\mathbb{Z},\{\pm 1,\pm 2\})}$ *of* $\mathbb{Z}$ *generated by* $\{\pm 1,\pm 2\}$*, i.e.*

$$\mathrm{d}\mu_{pg} = \mathrm{d}\mu_{\mathbb{Z}} * \mathrm{d}\mu_{(\mathbb{Z},\{\pm 1,\pm 2\})}.$$

4.3.5 *p*2

The group $p2$ is generated by two translations τ_1, τ_2 and its point group is C_2. The element that generates C_2 is a rotation of π around the origin. A possible presentation for $p2$ is

$$p2 = \left\langle \tau_1,\tau_2,\rho \middle| [\tau_1,\tau_2] = e, \tau_1^\rho = \tau_1^{-1}, \tau_2^\rho = \tau_2^{-1}, \rho^2 = e \right\rangle. \tag{4.19}$$

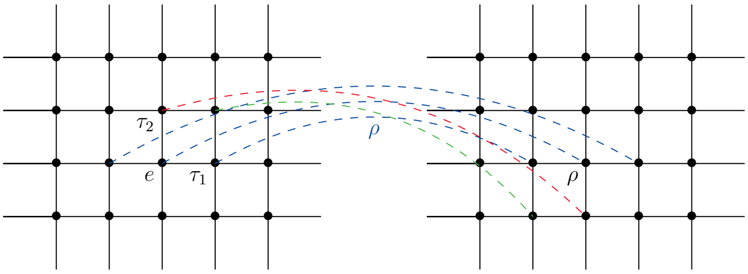
This is realized, for instance, by choosing $\tau_1 = \left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)$, $\tau_2 = \left(\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)$ and $\rho = \left(\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)\right)$. Its Cayley graph is illustrated in fig. 4.7a. Arguing as in section 4.3.2, we obtain that $\mathrm{Cay}(p2)$ is isomorphic as a non-oriented, non-labeled graph to $\mathrm{Cay}(\mathbb{Z}^2, \{\tau_1^{\pm 1}, \tau_2^{\pm 1}\}) \times \mathrm{Cay}(C_2, \{\rho\})$. This is illustrated in fig. 4.7b. Therefore, the Kesten spectral measure of $p2$ is the same as that of cm and pm since $C_2 \simeq D_1$.

Proposition 4.27. *The Kesten spectral measure of* $p2$ *generated as in eq.* (4.19) *is given by*

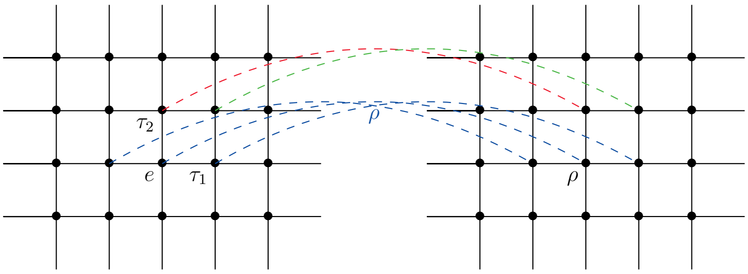
$$\mu_{p_2}(\Delta) = \frac{1}{2}\mu_{p_1}((\Delta+1)\cap[-1,1]) + \frac{1}{2}\mu_{p_1}((\Delta-1)\cap[-1,1]) \quad \forall \Delta \in \mathrm{Bor}([-1,1]). \tag{4.20}$$

4.3.6 *c*2*mm*

The group $c2mm$ has D_2 as its point group, generated by $\rho = \left(\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&-1\\-1&0\end{smallmatrix}\right)\right)$ and $\sigma = \left(\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\right)$. Recall that D_2 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, the Klein four-group where each



(a) Cayley graph of $p2$.



(b) Cayley graph of $p2$, redrawn to show the isomorphism between $\mathrm{Cay}(p2)$ and $\mathrm{Cay}(\mathbb{Z}^2) \times \mathrm{Cay}(C_2)$.

Figure 4.7: Cayley graph of $p2$.

non-identity element is its own inverse. The action of ρ and σ on τ_1 and τ_2 gives us an abstract presentation for $c2mm$:

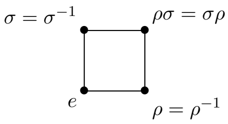
$$c2mm = \langle \tau_1, \tau_2, \rho, \sigma | [\tau_1, \tau_2] = \rho^2 = \sigma^2 = (\rho\sigma)^2 = e, \tau_1^\rho = \tau_2^{-1}, \tau_2^\rho = \tau_1^{-1}, \tau_1^\sigma = \tau_2, \tau_2^\sigma = \tau_1 \rangle . \tag{4.21}$$

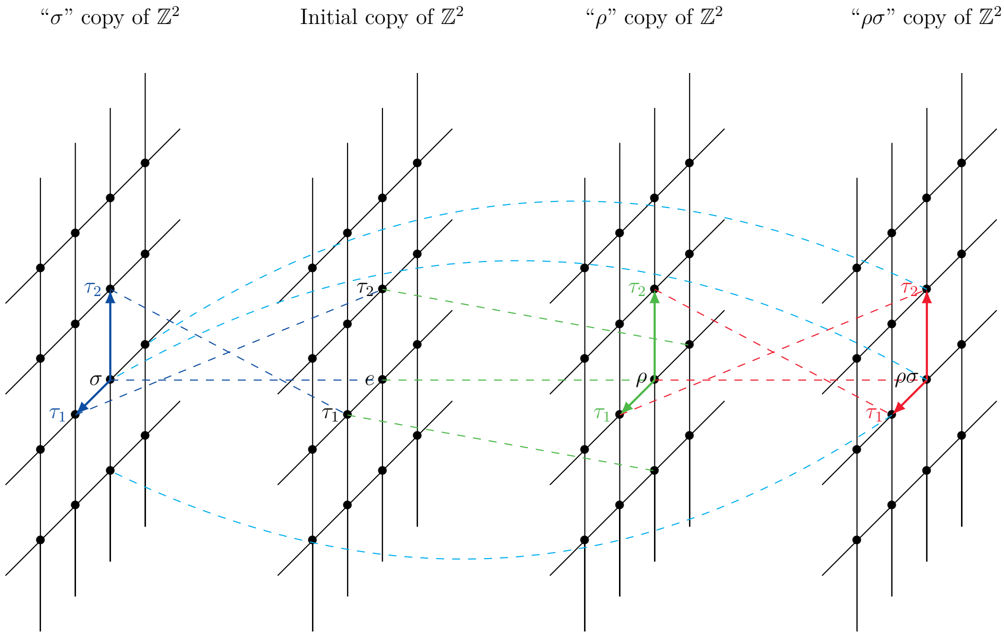
Its Cayley graph is illustrated in fig. 4.8a. Redrawing the different copies of $\mathbb{Z}^2$ as illustrated in fig. 4.8b (i.e. $\tau_1 \mapsto \tau_2$, $\tau_2 \mapsto \tau_1$ in the "σ" copy, $\tau_1 \mapsto \tau_2^{-1}$, $\tau_2 \mapsto \tau_1^{-1}$ in the "ρ" copy and $\tau_1 \mapsto \tau_1^{-1}$, $\tau_2 \mapsto \tau_2^{-1}$ in the "$\rho\sigma$" copy) we obtain a graph that is isomorphic (as a non-oriented, non-labeled graph) to $\mathrm{Cay}(\mathbb{Z}^2, \{\tau_1^{\pm 1}, \tau_2^{\pm 1}\}) \times \mathrm{Cay}(D_2, \{\rho, \sigma\})$. Therefore, to compute the Kesten spectral measure of $c2mm$, we only need to compute the Kesten spectral measure of D_2 and use theorem 4.15.

Proposition 4.28. *The Kesten spectral measure of* $D_2 = \langle \rho, \sigma | \rho^2 = \sigma^2 = (\rho\sigma)^2 = e \rangle$ *is*

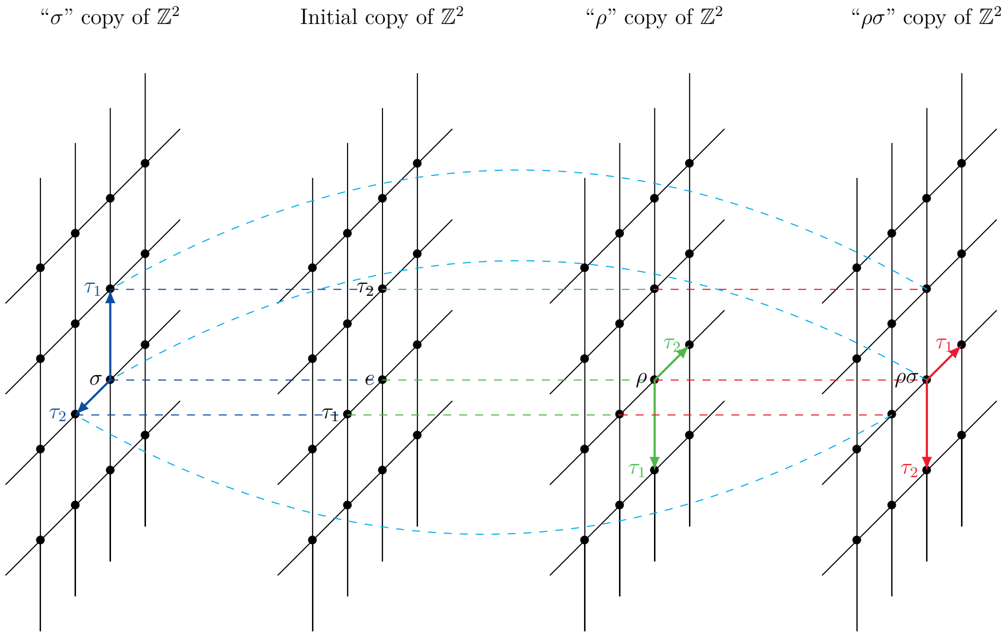
$$\mathrm{d}\mu_{D_2}(x) = \frac{1}{4}\delta_1(x) + \frac{1}{2}\delta_0(x) + \frac{1}{4}\delta_1(x). \tag{4.22}$$

Proof. The Cayley graph of D_2 is





(a) Cayley graph of $c2mm$.



(b) Cayley graph of $c2mm$, redrawn to show the isomorphism between $\mathrm{Cay}(c2mm)$ and $\mathrm{Cay}(\mathbb{Z}^2) \times \mathrm{Cay}(D_2)$.

Figure 4.8: Cayley graph of $c2mm$.

and thus its Markov operator in matrix form is

$$M = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues are

$$\lambda_1 = -1,\ \lambda_2 = 0,\ \lambda_3 = 0,\ \lambda_4 = 1$$

and therefore the Kesten spectral measure is

$$\mathrm{d}\mu_{D_2}(x) = \frac{1}{4}\delta_{-1}(x) + \frac{1}{2}\delta_0(x) + \frac{1}{4}\delta_1(x).$$

$\blacksquare$

Corollary 4.29. *The Kesten spectral measure of c2mm generated by eq.* (4.21) *is given, for all* $\Delta \in \mathrm{Bor}([-1,1])$, *by*

$$\mu_{c2mm}(\Delta) = \tfrac{1}{4}\mu_{p1}((\Delta+1)\cap[-1,1]) + \tfrac{1}{2}\mu_{p1}(\Delta) + \frac{1}{4}\mu_{p1}((\Delta-1)\cap[-1,1]). \tag{4.23}$$

Proof. Let $\Delta \in \mathrm{Bor}([-1,1])$. Using theorem 4.15 and the definition of the convolution of measures, we obtain

$$\begin{aligned}\mu_{c2mm}(\Delta) = (\mu_{p_1} * \mu_{D_2})(\Delta) &= \int_{[-1,1]^2} \chi_\Delta(x+y)\mathrm{d}\mu_{p_1}(x)\mathrm{d}\mu_{D_2}(y)\\ &= \int_{-1}^{1} \tfrac{1}{4}\chi_\Delta(x-1) + \tfrac{1}{2}\chi_\Delta(x) + \tfrac{1}{4}\chi_\Delta(x+1)\mathrm{d}\mu_{p_1}(x)\\ &= \tfrac{1}{4}\mu_{p_1}((\Delta+1)\cap[-1,1]) + \tfrac{1}{2}\mu_{p_1}(\Delta)\\ &\quad + \frac{1}{4}\mu((\Delta-1)\cap[-1,1]).\end{aligned}$$ $\blacksquare$

4.3.7 *p2mm*

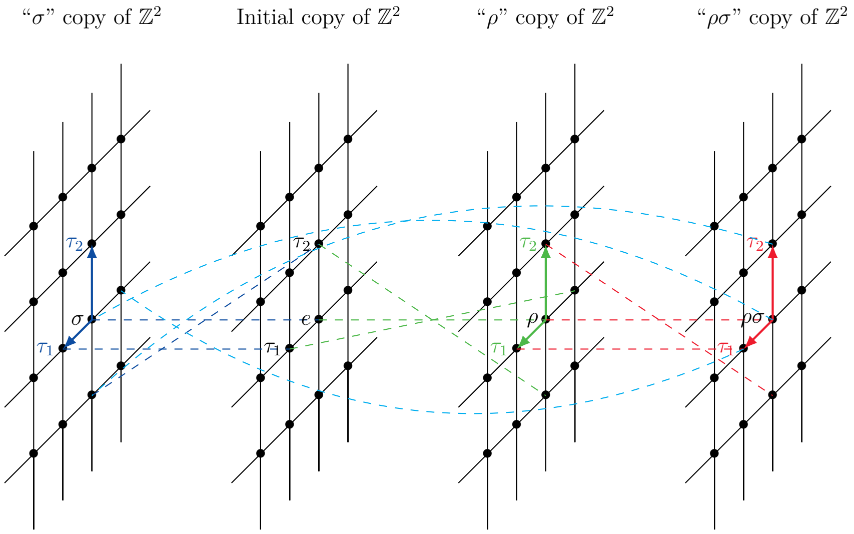
The group *p2mm* also has D_2 as its point group, generated by $\rho = \left(\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)\right)$ and $\sigma = \left(\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)\right)$. The action of ρ and σ on τ_1 and τ_2 gives us an abstract presentation for *p2mm*:

$$p2mm = \langle \tau_1, \tau_2, \rho, \sigma | [\tau_1, \tau_2] = \rho^2 = \sigma^2 = [\rho, \sigma] = e, \tau_1^\rho = \tau_1^{-1}, \tau_2^\rho = \tau_2^{-1}, \tau_1^\sigma = \tau_1, \tau_2^\sigma = \tau_2^{-1} \rangle . \tag{4.24}$$

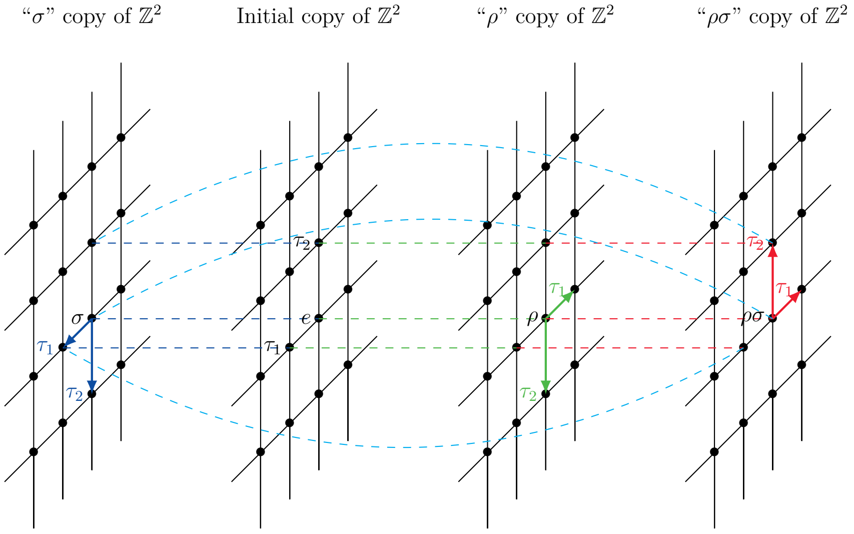
Its Cayley graph is illustrated in fig. 4.9a. Redrawing the different copies of $\mathbb{Z}^2$ as illustrated in fig. 4.9b (i.e. $\tau_1 \mapsto \tau_1$, $\tau_2 \mapsto \tau_2^{-1}$ in the "σ" copy, $\tau_1 \mapsto \tau_1^{-1}$, $\tau_2 \mapsto \tau_2^{-1}$ in the "ρ" copy and $\tau_1 \mapsto \tau_1^{-1}$, $\tau_2 \mapsto \tau_2$ in the "$\rho\sigma$" copy) we obtain a graph that is isomorphic (as a non-oriented, non-labeled graph) to $\mathrm{Cay}(\mathbb{Z}^2, \{\tau_1^{\pm1}, \tau_2^{\pm1}\}) \times \mathrm{Cay}(D_2, \{\rho^{\pm1}, \sigma^{\pm1}\})$. Therefore, the Kesten spectral measure of *p2mm* is the same as that of *c2mm*.

Proposition 4.30. *The Kesten spectral measure of p2mm generated by eq.* (4.24) *is given, for all* $\Delta \in \mathrm{Bor}([-1,1])$, *by*

$$\mu_{p2mm}(\Delta) = \tfrac{1}{4}\mu_{p1}((\Delta+1)\cap[-1,1]) + \tfrac{1}{2}\mu_{p1}(\Delta) + \frac{1}{4}\mu_{p1}((\Delta-1)\cap[-1,1]). \tag{4.25}$$



(a) Cayley graph of $p2mm$.



(b) Cayley graph of $p2mm$, redrawn to show the isomorphism between $\operatorname{Cay}(p2mm)$ and $\operatorname{Cay}(\mathbb{Z}^2) \times \operatorname{Cay}(D_2)$.

Figure 4.9: Cayley graph of $p2mm$.

4.3.8 *p2mg*

The point group of $p2mg$ is D_2. This time, we use the presentation as given in [HP18]:

$$\begin{aligned} p2mg = \Big\langle \tau_1, \tau_2, \rho, \sigma \;\Big|\; & [\tau_1, \tau_2] = \rho^2 = \sigma^2 = e, \tau_1^\rho = \tau_1^{-1}, \tau_2^\rho = \tau_2^{-1}, \\ & \tau_1^\sigma = \tau_1, \tau_2^\sigma = \tau_2^{-1}, (\rho\sigma)^2 = \tau_2 \Big\rangle. \end{aligned} \tag{4.26}$$

This presentation is realized, for example, with the elements $\rho = \left(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \right)$ and $\sigma = \left(\left(\begin{smallmatrix} 0 \\ -1/2 \end{smallmatrix} \right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \right)$.

The Cayley graph of $p2mg$ is illustrated in fig. 4.10. Let X be the following (infinite) graph.

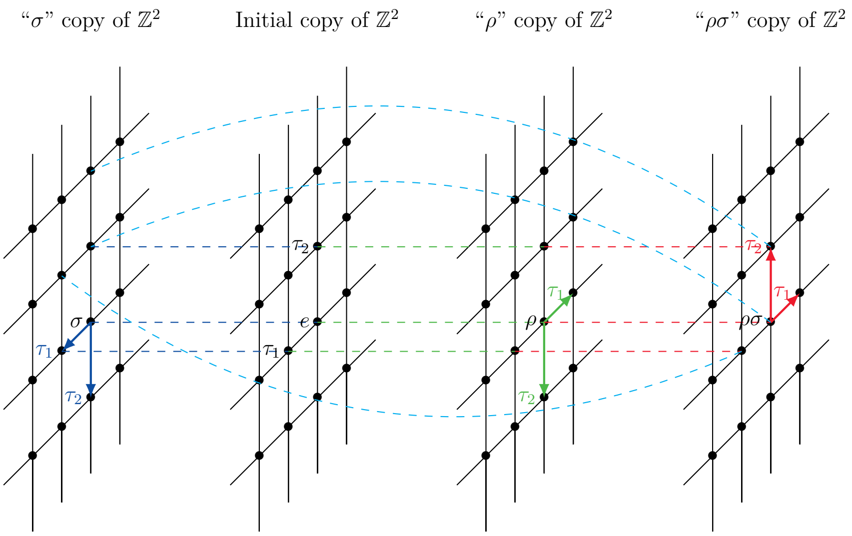
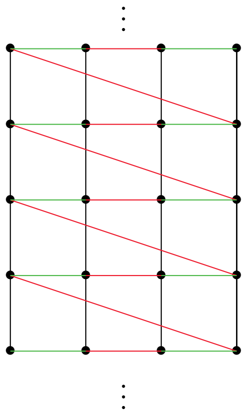


Figure 4.10: Cayley graph of $p2mg$.



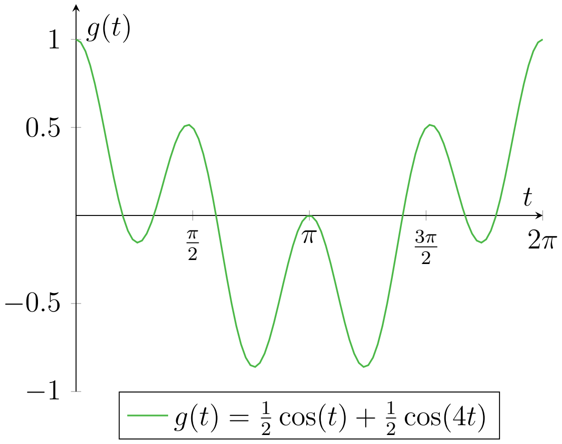
The red (respectively green, black) edges correspond to generators ρ (respectively σ and τ_2). This graph is unoriented, as each edge is seen as a generator and its inverse merged. We see that $\mathrm{Cay}(p2mg) \simeq X \times \mathrm{Cay}(\mathbb{Z}, \{\tau_1\})$ as an non-oriented, non-labeled graph. Moreover, it's easy to see that $X \simeq \Gamma = \mathrm{Cay}(\mathbb{Z}, \{\pm 1, \pm 4\})$, so that $\mathrm{Cay}(p2mg) \simeq \mathrm{Cay}(\mathbb{Z}, \{\pm 1, \pm 4\}) \times \mathrm{Cay}(\mathbb{Z}, \{\tau_1\})$ Therefore, we need to compute the Kesten spectral measure of $\mathbb{Z}$ generated by ± 1 and ± 4. We do this by computing the Kesten spectral measure of $\Gamma_n = \mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, \{\pm 1, \pm 4\})$ (for $n \geq 5$) and studying the weak convergence. The Markov operator of Γ_n is a circulant matrix with coefficients $c_1 = c_4 = c_{n-4} = c_{n-1} = 1$ and all other c_j's equal to 0. Thus using eq. (4.6) we find that the eigenvalues of M_n are

$$\lambda_k = \frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{2}\cos\left(\frac{8\pi k}{n}\right), \qquad k = 1, \ldots, n.$$

As $n \to \infty$, the eigenvalues take all possible values $\frac{1}{2}\cos(t) + \frac{1}{2}\cos(4t)$ for $t \in [0, 2\pi]$. Let $g : [0, 2\pi] \to \mathbb{R}$, $t \mapsto \frac{1}{2}\cos(t) + \frac{1}{2}\cos(4t)$. Using twice that $\cos(2t) = 2\cos^2(t) - 1$, we obtain

$$g(t) = 4\cos^4(t) - 4\cos^2(t) + \frac{1}{2}\cos(t) + \frac{1}{2}$$

which is illustrated below.



Let us already do the change of variable

$$x = \cos(t) \quad \Longrightarrow \quad \mathrm{d}t = \tfrac{-1}{\sqrt{1-x^2}}\mathrm{d}x, \tag{4.27}$$

and let $\varphi : [-1, 1] \to \mathbb{R}, x \mapsto 4x^4 - 4x^2 + \frac{1}{2}x + \frac{1}{2}$. This function has 1 local maximum M and 2 local maxima m_1 and m_2 on $(-1, 1)$. Using Viète's formula for cubic equations, we obtain that the values of M, m_1 and m_2 are given by

$$\begin{cases} m_1 &= \sqrt{\frac{2}{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{-3\sqrt{6}}{32}\right) - \frac{4\pi}{3}\right), \\ M &= \sqrt{\frac{2}{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{-3\sqrt{6}}{32}\right) - \frac{2\pi}{3}\right), \\ m_2 &= \sqrt{\frac{2}{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{-3\sqrt{6}}{32}\right)\right). \end{cases}$$

The second change of variable we will apply is

$$z = \varphi(x). \tag{4.28}$$

We have four inverse branches which respectively have $[-1, m_1]$, $[m_1, M]$, $[M, m_2]$ and $[m_2, 1]$ as their domain of definition. Using the method described in [Wik22] to solve the quartic equation, we obtain the different branches as

$$\begin{cases} \varphi_1 : [-1, m_1] \to \mathbb{R}, & \varphi_1(z) = \frac{1}{2}\left(-\sqrt{\frac{2b}{\sqrt{2y(z)-a}} - a - 2y(z)} - \sqrt{2y(z) - a}\right) \\ \varphi_2 : [m_1, M] \to \mathbb{R}, & \varphi_2(z) = \frac{1}{2}\left(\sqrt{\frac{2b}{\sqrt{2y(z)-a}} - a - 2y(z)} - \sqrt{2y(z) - a}\right) \\ \varphi_3 : [M, m_2] \to \mathbb{R}, & \varphi_3(z) = \frac{1}{2}\left(\sqrt{2y(z) - a} - \sqrt{-\frac{2b}{\sqrt{2y(z)-a}} - a - 2y(z)}\right) \\ \varphi_4 : [m_2, 1] \to \mathbb{R}, & \varphi_4(z) = \frac{1}{2}\left(\sqrt{-\frac{2b}{\sqrt{2y(z)-a}} - a - 2y(z)} + \sqrt{2y(z) - a}\right) \end{cases} \tag{4.29}$$

where

$$\begin{cases} a & = -1, \\ b & = \frac{1}{8}, \\ c(z) & = \frac{1}{8} - \frac{z}{4}, \\ p(z) & = -\frac{a^2}{12} - c(z), \\ q(z) & = -\frac{a^3}{108} + \frac{1}{3}ac(z) - \frac{b^2}{8}, \\ w(z) & = \sqrt[3]{\sqrt{\frac{p(z)^3}{27} + \frac{q(z)^2}{4}} - \frac{q(z)}{2}}, \\ y(z) & = \frac{a}{6} - \frac{p(z)}{3w(z)} + w(z). \end{cases}$$

Mathematica is able to get a closed form for each φ_j, $j = 1, \ldots, 4$ but it is very long and we feel like it does not bring much more knowledge than the form we present here. With these considerations, we study the weak convergence of the Kesten spectral measure. For all $f \in C([-1,1])$, we must have that

$$\begin{aligned} \int_{\mathrm{Sp}(\Gamma_n)} f \mathrm{d}\mu_{\Gamma_n} &= \frac{1}{n} \sum_{j=1}^{n} f\left(\frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{2}\cos\left(\frac{8\pi k}{n}\right)\right) \\ &\xrightarrow{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} f(g(t)) \mathrm{d}t \end{aligned}$$

using that g is 2π-periodic and even, we get

$$= \frac{1}{\pi} \int_0^{\pi} f(g(t)) \mathrm{d}t$$

with eq. (4.27) we obtain

$$\begin{aligned} &= \frac{1}{\pi} \int_{-1}^{1} f(\varphi(x)) \frac{1}{\sqrt{1-x^2}} \mathrm{d}x \\ &= \frac{1}{\pi} \int_{-1}^{m_1} f(\varphi(x)) \frac{1}{\sqrt{1-x^2}} \mathrm{d}x + \frac{1}{\pi} \int_{m_1}^{M} f(\varphi(x)) \frac{1}{\sqrt{1-x^2}} \mathrm{d}x \\ &\quad + \frac{1}{\pi} \int_{M}^{m_2} f(\varphi(x)) \frac{1}{\sqrt{1-x^2}} \mathrm{d}x + \frac{1}{\pi} \int_{m_2}^{1} f(\varphi(x)) \frac{1}{\sqrt{1-x^2}} \mathrm{d}x \end{aligned}$$

and using the second change of variable in eq. (4.28) we get

$$\begin{aligned} &= \frac{1}{\pi} \int_{\varphi(-1)}^{\varphi(m_1)} f(z) \frac{1}{\sqrt{1-\varphi_1(z)^2}} \frac{\mathrm{d}\varphi_1(z)}{\mathrm{d}z} \mathrm{d}z + \frac{1}{\pi} \int_{\varphi(m_1)}^{\varphi(M)} f(z) \frac{1}{\sqrt{1-\varphi_2(z)^2}} \frac{\mathrm{d}\varphi_2(z)}{\mathrm{d}z} \mathrm{d}z \\ &\quad + \frac{1}{\pi} \int_{\varphi(M)}^{\varphi(m_2)} f(z) \frac{1}{\sqrt{1-\varphi_3(z)^2}} \frac{\mathrm{d}\varphi_3(z)}{\mathrm{d}z} \mathrm{d}z + \frac{1}{\pi} \int_{\varphi(m_2)}^{\varphi(1)} f(z) \frac{1}{\sqrt{1-\varphi_4(z)^2}} \frac{\mathrm{d}\varphi_4(z)}{\mathrm{d}z} \mathrm{d}z. \end{aligned}$$

Since $\varphi(m_1) < \varphi(-1)$, $\varphi(m_2) < \varphi(M)$ and $\varphi(1) = 1$, the previous equation is equal to

$$\begin{aligned} &\int_{\varphi(m_1)}^{\varphi(1)} f(x) \frac{-1}{\pi\sqrt{1-\varphi_1(x)^2}} \frac{\mathrm{d}\varphi_1(x)}{\mathrm{d}x} \mathrm{d}x + \int_{\varphi(m_1)}^{\varphi(M)} f(x) \frac{1}{\pi\sqrt{1-\varphi_2(x)^2}} \frac{\mathrm{d}\varphi_2(x)}{\mathrm{d}x} \mathrm{d}x \\ &\quad + \int_{\varphi(m_2)}^{\varphi(M)} f(x) \frac{-1}{\pi\sqrt{1-\varphi_3(x)^2}} \frac{\mathrm{d}\varphi_3(x)}{\mathrm{d}x} \mathrm{d}x + \int_{\varphi(m_2)}^{1} f(x) \frac{1}{\pi\sqrt{1-\varphi_4(x)^2}} \frac{\mathrm{d}\varphi_4(x)}{\mathrm{d}x} \mathrm{d}x. \end{aligned}$$

Therefore, we obtain that the Kesten spectral measure of $\mathbb{Z}$ generated by $\{\pm1, \pm4\}$ is equal to

$$\begin{aligned}\mathrm{d}\mu_{\mathbb{Z},\{\pm1,\pm4\}}(x) = {} & \tfrac{-1}{\pi\sqrt{1-\varphi_1(x)^2}}\tfrac{\mathrm{d}\varphi_1(x)}{\mathrm{d}x}\chi_{[\varphi(m_1),\varphi(1)]}(x)\mathrm{d}x + \tfrac{1}{\pi\sqrt{1-\varphi_2(x)^2}}\tfrac{\mathrm{d}\varphi_2(x)}{\mathrm{d}x}\chi_{[\varphi(m_1),\varphi(M)]}(x)\mathrm{d}x \\ & \tfrac{-1}{\pi\sqrt{1-\varphi_3(x)^2}}\tfrac{\mathrm{d}\varphi_3(x)}{\mathrm{d}x}\chi_{[\varphi(m_2),\varphi(M)]}(x)\mathrm{d}x + \tfrac{1}{\pi\sqrt{1-\varphi_4(x)^2}}\tfrac{\mathrm{d}\varphi_4(x)}{\mathrm{d}x}\chi_{[\varphi(M),1]}(x)\mathrm{d}x.\end{aligned}$$

We proved the following result.

Proposition 4.31. *The Kesten spectral measure of* $\Gamma = \mathrm{Cay}(\mathbb{Z}, \{\pm1, \pm4\})$ *is given by*

$$\begin{aligned}\mathrm{d}\mu_{\Gamma}(x) = {} & \tfrac{-1}{\pi\sqrt{1-\varphi_1(x)^2}}\tfrac{\mathrm{d}\varphi_1(x)}{\mathrm{d}x}\chi_{[\varphi(m_1),\varphi(1)]}(x)\mathrm{d}x + \tfrac{1}{\pi\sqrt{1-\varphi_2(x)^2}}\tfrac{\mathrm{d}\varphi_2(x)}{\mathrm{d}x}\chi_{[\varphi(m_1),\varphi(M)]}(x)\mathrm{d}x \\ & \tfrac{-1}{\pi\sqrt{1-\varphi_3(x)^2}}\tfrac{\mathrm{d}\varphi_3(x)}{\mathrm{d}x}\chi_{[\varphi(m_2),\varphi(M)]}(x)\mathrm{d}x + \tfrac{1}{\pi\sqrt{1-\varphi_4(x)^2}}\tfrac{\mathrm{d}\varphi_4(x)}{\mathrm{d}x}\chi_{[\varphi(M),1]}(x)\mathrm{d}x.\end{aligned} \tag{4.30}$$

where the φ_j*'s are defined in eq.* (4.29) *and* φ *is defined by* $\varphi(x) = 4x^4 - 4x^2 + \frac{1}{2}x + \frac{1}{2}$.

As a corollary of theorem 4.15 and our observation that $\mathrm{Cay}(p2mg, \{\tau_1, \tau_2, \rho, \sigma\}) \simeq \mathrm{Cay}(\mathbb{Z}, \{\pm1\}) \times \mathrm{Cay}(\mathbb{Z}, \{\pm1, \pm4\})$, we obtain the Kesten spectral measure of $p2mg$ for the set of chosen generators.

Corollary 4.32. *The Kesten spectral measure of* $p2mg$ *as generated by eq.* (4.26) *is*

$$\mathrm{d}\mu_{p2mg} = \mathrm{d}\mu_{\mathbb{Z},\{\pm1\}} * \mathrm{d}\mu_{\mathbb{Z},\{\pm1,4\}}. \tag{4.31}$$

Remark 4.33. Proposition 4.25 and proposition 4.31 both show the Kesten spectral measure of $\mathbb{Z}$ with some different sets of generators. More generally, one can generate $\mathbb{Z}$ with two coprime integers a and b (and their inverses) using Bézout's identity, and one could wonder what the Kesten spectral measure of $\mathrm{Cay}(\mathbb{Z}, \{\pm a, \pm b\})$ is. It is hopeless to find it for any such a and b with the method of finite approximation we used: it is easy to see that the eigenvalues of a finite cycle generated by a and b are given by $\lambda_k = \frac{1}{2}\cos\left(\frac{2\pi ka}{n}\right) + \frac{1}{2}\cos\left(\frac{2\pi kb}{n}\right)$ and thus, as $n \to \infty$, take all possible values $g(t) = \frac{1}{2}\cos(at) + \frac{1}{2}\cos(bt)$ for $t \in [0, 2\pi]$. With the weak convergence of the Kesten spectral measure, we must have

$$\int_{\mathrm{Sp}(\Gamma_n)} f \mathrm{d}\mu_{\Gamma_n} \xrightarrow{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} f(g(t))\mathrm{d}t.$$

Since g is 2π-periodic and even, we can restrict ourselves to $[0, \pi]$, but we still need to inverse it. Using multiple-angle formulas such as the ones given in [Wei00], we see that if $|a| + |b| \geq 5$, we obtain a polynomial in $\cos(t)$ whose degree is ≥ 5, therefore we have no chance of finding its inverse, unless we are lucky.

4.3.9 $p4$

The group $p4$ has C_4 as its point group, generated by $\rho = \left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0&-1\\1&0\end{pmatrix}\right)$. The action of ρ on τ_1 and τ_2 gives us an abstract presentation for $p4$:

$$p4 = \left\langle \tau_1, \tau_2, \rho \middle| [\tau_1, \tau_2] = \rho^4 = e, \tau_1^\rho = \tau_2, \tau_2^\rho = \tau_1^{-1} \right\rangle. \tag{4.32}$$

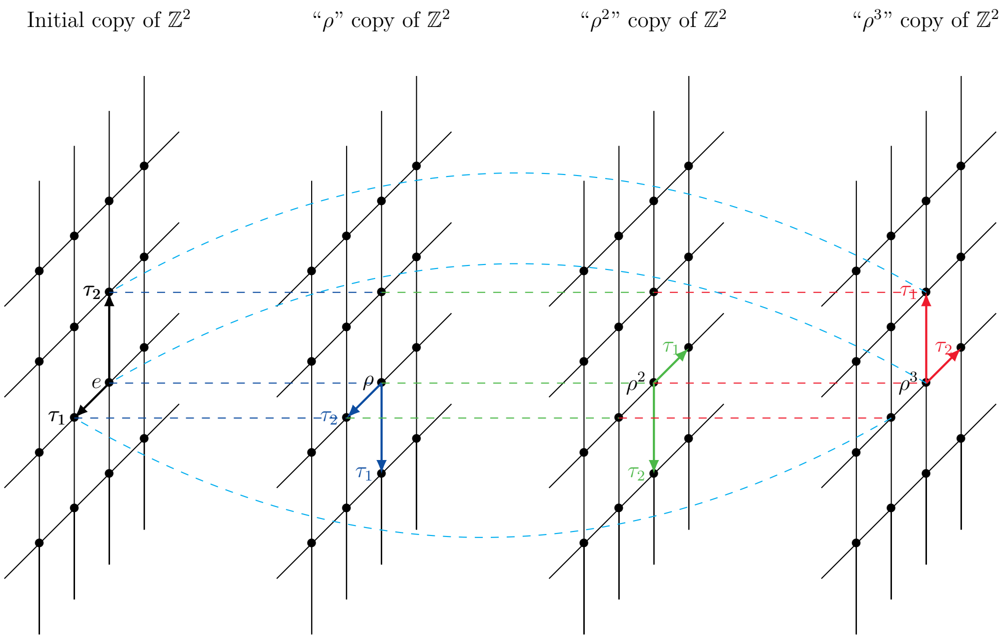


Figure 4.11: Cayley graph of $p4$, redrawn to show the isomorphism between Cay($p4$) and Cay($\mathbb{Z}^2$) × Cay(C_4).

Its Cayley graph is illustrated in fig. 4.11, where we directly drew the generators in each copy of $\mathbb{Z}^2$ in a convenient way to illustrate that we have a graph isomorphism between $\mathrm{Cay}(p4, \{\tau_1, \tau_2, \rho\})$ and $\mathrm{Cay}(\mathbb{Z}^2, \{\tau_1, \tau_2\}) \times \mathrm{Cay}(C_4, \{\rho\})$. Thus, we will once again use theorem 4.15. Observe that that $\mathrm{Cay}(D_2, \{\rho, \sigma\}) \simeq \mathrm{Cay}(C_4, \{\rho\})$. Therefore, using the same proof as proposition 4.30 mutatis mutandis, we obtain the Kesten spectral measure of $p4$.

Corollary 4.34. *The Kesten spectral measure of* $p4$ *is given, for all* $\Delta \in \mathrm{Bor}([-1,1])$, *by*

$$\mu_{p4}(\Delta) = \tfrac{1}{4}\mu_{p1}((\Delta+1)\cap[-1,1]) + \tfrac{1}{2}\mu_{p1}(\Delta) + \frac{1}{4}\mu_{p1}((\Delta-1)\cap[-1,1]). \tag{4.33}$$

Note that this shows that $c2mm$, $p2mm$ and $p4$ are pairwise non-isomorphic yet isospectral.

4.3.10 *p4m*

The group $p4m$ has D_4 as its point group, generated by $\rho = \left(\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & -1\\1 & 0\end{smallmatrix}\right)\right)$ and $\sigma = \left(\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1 & 0\\0 & -1\end{smallmatrix}\right)\right)$. The action of ρ and σ on τ_1 and τ_2 gives us an abstract presentation for $p4m$:

$$p4m = \langle \tau_1, \tau_2, \rho, \sigma \,|\, [\tau_1, \tau_2] = \rho^4 = \sigma^2 = (\rho\sigma)^2 = e, \tau_1^\rho = \tau_2, \tau_2^\rho = \tau_1^{-1}, \tau_1^\sigma = \tau_1, \tau_2^\sigma = \tau_2^{-1} \rangle\,. \tag{4.34}$$

We illustrate part of its Cayley graph in fig. 4.12. Each copy of $\mathbb{Z}^2$ is drawn with the generators such that the isomorphism between $\mathrm{Cay}(p4m, \{\tau_1, \tau_2, \rho, \sigma\})$ and $\mathrm{Cay}(\mathbb{Z}^2, \{\tau_1, \tau_2\}) \times \mathrm{Cay}(D_4, \{\rho, \sigma\})$ is clear. We routinely perform the computations of the Kesten spectral measures of D_4 and $p4m$.

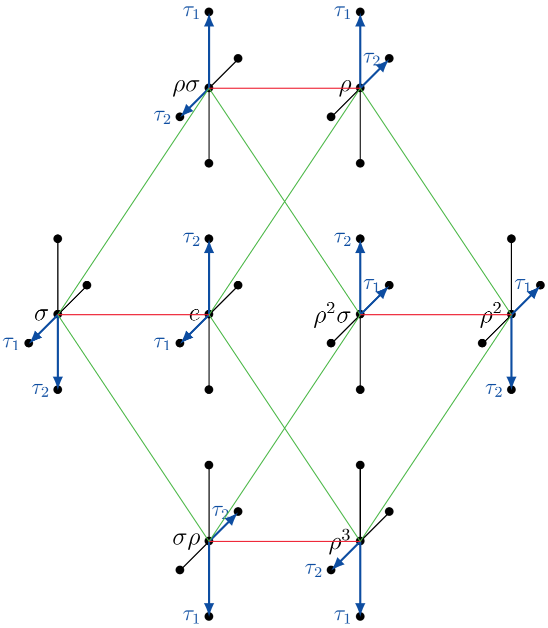
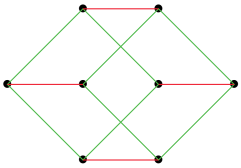


Figure 4.12: Cayley graph of $p4m$ drawn to illustrate the action of D_4 on the generators of $\mathbb{Z}^2$ and the isomorphism between the Cayley graph of $p4m$ and $\mathrm{Cay}(\mathbb{Z}^2) \times \mathrm{Cay}(D_4)$.

Proposition 4.35. *The Kesten spectral measure of* $D_4 = \langle \rho, \sigma | \rho^4 = \sigma^2 = (\rho\sigma)^2 = e \rangle$ *is*

$$\mathrm{d}\mu_{D_4}(x) = \frac{1}{8}\delta_{-1}(x) + \frac{3}{8}\delta_{-\frac{1}{3}}(x) + \frac{3}{8}\delta_{\frac{1}{3}}(x) + \frac{1}{8}\delta_1(x). \tag{4.35}$$

Proof. The Cayley graph of $D_4 = \langle \rho, \sigma | \rho^4 = \sigma^2 = (\rho\sigma)^2 = e \rangle$ is



The Markov operator of this graph in matrix form is

$$M = \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and the eigenvalues are $-1, 1$ with multiplicity 1, and $\frac{-1}{3}$ and $\frac{1}{3}$ with multiplicity 3.

Thus the Kesten spectral measure of D_4 is

$$\mathrm{d}\mu_{D_4}(x) = \frac{1}{8}\delta_{-1}(x) + \frac{3}{8}\delta_{-\frac{1}{3}}(x) + \frac{3}{8}\delta_{\frac{1}{3}}(x) + \frac{1}{8}\delta_{1}(x).$$

■

Once again using theorem 4.15 and applying the proof of proposition 4.30 mutatis mutandis, we obtain the Kesten spectral measure of $p4m$.

Corollary 4.36. *The Kesten spectral measure of* p4m *is given, for all* $\Delta \in \mathrm{Bor}([-1,1])$, *by*

$$\begin{aligned}\mu_{p4m}(\Delta) = {} & \tfrac{1}{8}\mu_{p1}((\Delta+1)\cap[-1,1]) + \tfrac{3}{8}\mu_{p1}((\Delta+\tfrac{1}{3})\cap[-1,1]) \\ & + \tfrac{3}{8}\mu_{p1}((\Delta-\tfrac{1}{3})\cap[-1,1]) + \tfrac{1}{8}\mu_{p1}((\Delta-1)\cap[-1,1]).\end{aligned} \tag{4.36}$$

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