UNIVERSITÉ DE NEUCHATEL
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THEORY AND APPLICATIONS OF A SYNTACTICAL NOTION OF THE EQUIVALENCE OF FORMAL LOGICAL SYSTEMS

THÈSE

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La Faculté des Sciences de l'Université de Neuchâtel, sur le rapport de Messieurs les professeurs J.-B. GRIZE, F. FIALA et W. SÖRENSEN autorise l'impression de la présente thèse sans exprimer d'opinion sur les propositions qui y sont contenues.

Neuchâtel, le 13 novembre 1963.

Le doyen :
CH. TERRIER

THEORY AND APPLICATIONS OF A SYNTACTICAL NOTION OF THE EQUIVALENCE OF FORMAL LOGICAL SYSTEMS

TO MY WIFE, JUDY

PREFACE

The general purpose and nature of this work are set forth in the Introduction. The contents to be found here are a result of independent study engaged in by the author during the years 1962-63.

During the same period of study, the author collaborated with Professor Jean-Blaise Grize of the University of Neuchâtel, Switzerland, who kindly consented to be the director of this thesis, on another study concerning the semantical aspects of logical implication, and he has had the benefit of many helpful suggestions from Professor Grize concerning the contents of the present volume. Professor Felix Fiala, also of the University of Neuchâtel, has added many valuable comments and both Professors Grize and Fiala have read and criticized various parts of the manuscript in different phases of its development.

But, beyond any technical help, I would like to thank Professors Grize, Fiala, and Sörenson for their constant encouragement which has meant much to me in my efforts. A sincere word of thanks is also due to Professor E. P. Specker of the Swiss Federal Institute of Technology who was kind enough to comment on certain aspects of this thesis, and whose suggestions have served to better the final form of its exposition.

William S. Hatcher

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GLOSSARY OF SPECIAL SYMBOLS

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NOTE ON THE EQUALITY SIGN :

It was noticed after the composition of this book was well under way that, by typographical accident, the equality sign had been rendered sometimes by « $=$ » and sometimes by « $=$ », these two signs being typographically different. The reader is hereby informed that there is absolutely no difference in meaning attached to these two signs. He is consequently asked to consider the two signs in question as the same throughout the book, whether the equality sign be employed in a formal system (see Chapter II, Section B), or informally to render the intuitive notion of identity.

INTRODUCTION

This study undertakes to define syntactically a relation of equivalence between formal logical systems, and to consider certain applications of this definition. For purposes of precising here what we intend to do, let us call the ordered pair $F = \langle S, R\rangle$ a *formal system*, where S is a set and $R \subset \bigcup_{n=1}^{\infty} S^n$ (where S^n is the set of all n-tuples of elements of S). Given a set $X \subset S$, we say that $y \in S$ is *inferred from* X if there is a finite sequence $\langle x_1, \ldots, x_{n-1}\rangle$ of elements of X such that $\langle x_1, \ldots, x_{n-1}, y\rangle \in R$. Given $X \subset S$, $\mathbf{K}_F(X)$ is defined as the smallest subset of S which contains X and which is closed with respect to the operation of inference. If $y \in \mathbf{K}_F(X)$, we can say that y is *deduced from* (or *deducible from*) X *in* F.

The principle interpretation of the above scheme is clearly that in which S is a set of statements and R is a set of primitive rules of inference.

For a given formal system F, $\mathbf{K}_F$ is a mapping from P(S) to P(S) which satisfies all of the properties of a topological closure except that (1) $\mathbf{K}_F(\Lambda) \neq \Lambda$ in general, and (2) $\mathbf{K}_F(X) \cup \mathbf{K}_F(Y) \subset \mathbf{K}_F(X \cup Y)$ but not conversely. Such a closure relation is a Moore closure and $\langle S, \mathbf{K}_F\rangle$ is thus a Moore space uniquely associated with $F = \langle S, R\rangle$. We say that F and F′ are *isomorphic* if $\langle S, \mathbf{K}_F\rangle$ and $\langle S', \mathbf{K}_{F'}\rangle$ are isomorphic.

Let $\mathbf{e}$ be the equivalence relation on S defined by $x\,\mathbf{e}\,y \longleftrightarrow \mathbf{K}_F(\{x\}) = \mathbf{K}_F(\{y\})$. Consider the quotient Moore space $F^* = \langle S/\mathbf{e}, \mathbf{K}^*_F\rangle$ with respect to the relation $\mathbf{e}$. We call F^* the *quasi-system* associated with F. We say that two given formal systems F and F′ are *equivalent* if their respectively associated quasi-systems F^* and F'^* are isomorphic.

Several sets of necessary and sufficient conditions for equivalence are found and the comparison between equivalence in our sense and certain other relationships between formal systems is discussed. Also, certain applications of the notion of equivalence are given in order to clarify its nature. In particular, it is seen that two systems must have the same *deductive structure* in order to be equivalent in our sense, and it is not sufficient (though it is necessary) that they have «the same theorems» in some precise sense of that term. For example, there exist different systems which are traditionally considered as formulations of the propositional calculus but which are not equivalent in our sense. Other examples are given.

There are several other interesting definitions of the equivalence of formal systems which have appeared in the literature, and it would perhaps be worthwhile to mention some of these and to indicate briefly the way in which these definitions differ from the one presented in this study.

There is the definition of Kemeny [1] based on the concept of *model* which, in the particular form of Kemeny's article, is due to him. Here Kemeny requires that a formal system be formalized by the aid of variables and constants to which type symbols are permanently attached. The equivalence of two systems, in Kemeny's sense of the concept, requires that the well-formed formulae of the two systems be capable of « expressing » the same objects in a sense which Kemeny defines quite precisely. However, it is possible, with Kemeny's notion of equivalence, that two systems with precisely the same deductive structure be non-equivalent.

As a way of illustrating this point, suppose that we take, for example, the calculus of propositions in Kemeny's formulation P (see Kemeny [1], p. 22). Now suppose that we construct another system P′ from P in the following way : We add one primitive symbol, the variable x_ι (of type « ι »). Instead of propositional variables a_o, b_o, etc. of type « o », we take an infinite list of one-place function variables $a_{o\iota}$, $b_{o\iota}$, etc. These predicate variables are not well-formed standing alone (see Kemeny [1], p. 18), but where $A_{o\iota}$ is any predicate variable, $A_{o\iota}x_\iota$ is well-formed. From then on, the construction of P′ follows precisely the construction of P (see Kemeny [1] for this and other details). P′ is then the same as P except that now we have replaced every occurrence of A_o, where A is some letter of the latin alphabet (barred or not), by an occurrence of $A_{o\iota}x_\iota$.

Now, according to Kemeny's definition, the two systems P and P′ are not equivalent. This is immediately clear when we realize that, for P′, any non-empty set whatsoever can be the domain of the free variable x_ι, whereas, by Kemeny's restrictions, every model of P contains only the two elements « t » and « f » (or, otherwise said, there is only this one model for P). Again, the reader is referred to Kemeny [1] for details.

Yet, no one would argue that P and P′ do not have precisely the same deductive structure. In fact, P and P′ are exactly the same system except for the symbolism chosen. [+] This is clear from the way in which P′ was constructed. By the definition of equivalence given in the present study, the two systems P and P′ above are equivalent.

On the other hand, it will also be seen that certain systems which would be equivalent under Kemeny's definition are not equivalent under ours (see Chapter III, Section C, of this work).

[+] It is interesting, through totally irrelevant to the argument here, that we have even a sensible interpretation of P′ which is « equivalent » to our interpretation of P. Consider, for each small letter A of the latin alphabet (barred or not) that $A_{o\iota}$ represents a predicate which is either true of every object in the universe or false of every object in the universe. Then, $A_{o\iota}\,x_\iota$ represents a constant propositional function or, briefly said, a proposition. But A_o, which is the counterpart in P of $A_{o\iota}\,x_\iota$, also represents a proposition and so we find equivalent interpretations. These are not the only interpretations possible, of course, as is clear.

Another interesting point here is that our definition of equivalence will not depend in any essential way on the set of signs which are used to construct the set S of well-formed formulae, for the deductive structure of a formal system is defined in terms of axioms (which constitute a subset A of S) and rules of inference (which are relations over the set S). Thus, for the sake of rigor and clarity, we will make a distinction right from the beginning between a *calculus of signs* and a *formal system.*

More recent notions of equivalence are to be found in such works as Wang [1], Kreisel [1], and Myhill [1]. With Wang's notion we are dealing essentially with relations between the Gödel numbers of the two systems. The notion of one system having a model in another which is discussed in Kreisel [1] supposes that the systems in question are formalized within the predicate calculus. Myhill requires that there exists a 1—1 recursive mapping such that theorems correspond to theorems and non-theorems to non-theorems. The chief application of Myhill's notion is to questions of decidability.

In this work, formal systems are regarded as abstract structures and the theory is developed by the general methods of modern mathematics. In particular, no arbitrary restriction to constructive methods is sought and no recourse to Gödel numbers is necessary. There is no necessity to « mathematize » a formal system in order to deal with it as it will be regarded as an abstract mathematical entity from the beginning.

On the other hand, the notion of the recursiveness of sets and functions is needed for the *definition* of the notion of formal system. The notion of recursiveness thus enters in as a *definitory concept* (whose goal is obviously to yield applications to logic under the interpretations of our theory) but not as a methodological tool.

A word should also be said concerning our way of defining the primitive rules of inference. These are defined in a somewhat unnatural way in order to simplify the ensuing exposition. An appendix is added to Chapter I which shows that our theory is the same as that which would have been obtained by defining the primitive rules of inference (more naturally) as recursive relations over the set S of well-formed formulae.

Finally, we make the following conventions. Theorems are numbered consecutively throughout a given chapter. Definitions are also so numbered with the exception of Chapter II where we start a fresh numbering of definitions for each one of the specific systems which we consider in that chapter.

Footnotes are numbered consecutively throughout the book (with the exception of the Introduction). We will designate theorems and definitions, and their parts, by a method of decimals. Thus, I.Th.25.3 means the third hypothesis of the 25th theorem of the first chapter. When reference is made within a given chapter, the chapter indication can be optionally dropped. Thus, Df.15 means the 15th definition of the chapter in which the reference « Df.15 » is made.

The system of giving references is one in which works are given a number under the name of the author. Thus, the reader may always find the precise work cited by referring to the bibliography.

We will use the following symbols as *abbreviations* of English usage : the arrow « $\rightarrow$ » for « if ... then ——— », and the double arrow « $\longleftrightarrow$ » for « if and only if » ; « & » for « both ... and ——— » ; « $\vee$ » for « either ... or ——— » ; « $(\exists x)$ » for « there is an x such that » ; « (x) » for « for all x such that ». In the last two formulations, it is clear that « x » is any variable whatever. Parentheses will be used in the usual way to indicate the scope of these quantifiers and to group expressions. It is clear that these signs, in their above-indicated use, are not part of any formal system but are simply abbreviations for vernacular expressions.

On the other hand, certain of these signs will be used in particular formal systems of Chapter II and Chapter III. The informal and the formal use of these signs will be clearly separated and no confusion will result from such double usage.

Usual mathematical symbolism, such as that for the application of a function to its argument, etc. will be used without explicit convention.

A final word should be said concerning our use of quotation marks. Though this work appears in English, the fact of its being published in continental Europe (French Switzerland) has made it much simpler, from a practical point of view, to use French quotation marks as replacing both single and double English quotation marks in all of their uses. In so doing, we are following the lead of several other works in English published in Europe. In particular, the French quotation marks will be used (1) in citing the works of other authors, (2) in forming the name of a sign. Questions of tradition aside, readability is, if anything, heightened by such usage.

CHAPTER I

FORMAL SYSTEMS

SECTION A

THE BASIC STRUCTURE AND PROPERTIES OF FORMAL SYSTEMS

§ I. BASIC DEFINITIONS

Df.1. — A *calculus of signs* C is an ordered quadruple $\langle L, S, A, R \rangle$ which satisfies the following conditions : 1) L is a recursive set whose elements are called signs.[1] 2) S is a recursive subset of the set G of all mappings f for which the domain is a finite subset of the positive integers 1, 2, . . ., n and for which the range is a subset of L. (G is, in short, the set of all *finite sequences* or *words* of L. It is the « free semi-group » generated by the given set L, see Bourbaki [2], p. 7.) S is called the set of *well-formed formulas* of C (abbreviated as «wffs» after Church [4], p. 70). Members of S will also be referred to as the *expressions* of C. 3) A is a non-void, recursive subset of S, called the *axioms* of C. 4) R is a finite family δ_i $(0 \leqslant i \leqslant p)$ of single-valued functions (called *primitive rules of inference*) which satisfy the following conditions : a) For each i, the domain of δ_i is a recursive subset of the set of all finite subsets of S. b) For each i, the range of δ_i is a recursively enumerable class of recursive subsets of S. c) For each i, there exists an $n > 0$ such that, for every member X of the domain of δ_i. Card $(X) = n$. n is called the *order* of the rule δ_i. d) For each i, δ_i is a recursive function.

Df.2. — A *formal system* or *formal logical system* F is an ordered triple $\langle S, A, R \rangle$ such that : 1) S is a recursive set whose elements are called *expressions*. 2) A is a recursive subset of S called the *axioms* of F.

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[1] Here and elsewhere, when we speak of a recursive set (and thus a set which is also recursively enumerable) or of a recursively enumerable set, we will assume that there is given some canonic enumeration of the set which thus allows us to speak of sets and functions as « recursive » *in the strict sense of the term, instead of simply* « effective » in a more intuitive meaning.

3) R satisfies the same conditions as those of Df.I.4 above with respect to S. The members of the family R are called the *primitive rules of inference* of F.

Th.1. — If F = ⟨S, A, R⟩ is a formal system, then ⟨S, S, A, R⟩ is a calculus of signs C.

Proof : Immediate from the definitions involved.

Thus, a formal system can always be thought of as a calculus of signs if one so wishes. With this in view, we shall often use the terms « formula » or « wff » to refer to the expressions of a formal system F even when we have not explicitly constructed the set S from an alphabet L.

Th.2. — If C = ⟨L, S, A, R⟩ is a calculus of signs, then F = ⟨S, A, R⟩ is a formal system called the *resulting formal system of* C.

Proof : Again immediate from the definitions in question.

It is possible, of course, for a given formal system F to be the resulting formal system of more than one calculus of signs.

It is evident that the logic which is characterized by the above definitions is *constructive* (it will be shown later that the notion of *proof* in a formal system, a notion which we have not yet defined, is effective). However, many of the various restrictions involving the recursiveness of sets and functions are not essential, from a purely mathematical point of view, to the theory that will be developed. Thus, generalization to non-constructive logics will often (but not always) be immediate.

In general, we will not explicitly designate those theorems which would or would not remain true under certain generalizations (such as admitting rules of infinite order or proofs of infinite length, etc.).

It is possible to admit $n = 0$ in Df.I.4.c, suppress Df.I.3 and mention of the set A altogether, and thus assimilate axioms to rules (see Carnap [1], § 10). In order to reestablish completely our formulation, which requires that the axiom set be non-void, we would have to add also that $0 < i \leqslant p$ in D.1.4, that the null set appear in the domain of δ_i for at least one i, and that for at least one δ_i for which the null set appears in the domain, the image of the null set is not the null set itself.

With such a formulation, we could consider a calculus C as an ordered triple ⟨L, S, R⟩ and a formal system as an ordered pair ⟨S, R⟩. It would be possible, of course, to make the above changes without assimilating axioms to rules.

Such questions are obviously a matter of presentation and no difficulties are present with either choice. But, for the sake of precision, a choice must be made and we have thus chosen the formulation represented by Df.1 and Df.2.

Our choice to require the set A of axioms, and hence the set T of *theorems* (not yet defined) to be non-void is obviously to avoid having to consider trivial cases in the proofs of our (meta)theorems about formal systems and calculi.

Some logicians might contend that we should have defined the primitive rules of inference as recursive relations over S instead of (not everywhere defined) functions from P(S) to P(S),[2] feeling that our definition does not take adequate account of the order of occurrence of the premisses in an instance of primitive (and thus of general) inference. (See Davis [1], p. 117 for an example of rules defined in this manner. In Smullyan [1], rules are not directly required to be recursive, and the requirement of effectiveness is introduced in a somewhat different manner). It is interesting that our definition is not defective in this regard, and a discussion of this will be found in the Appendix to Chapter I of this thesis. The reader who is interested in this question might find it interesting and worthwhile to read this Appendix immediately.

§ 2. THE CONSEQUENCE RELATION $\mathbf{K}$ FOR A FORMAL SYSTEM F

Let any formal system $F = (S, A, R)$ be given. We will now construct a relation $\mathbf{K}$ called the *consequence relation of* F. We will alternatively write « $\mathbf{K}_F$ » whenever it is necessary to indicate precisely the system in which we are working.

$\mathbf{K}$ is a function from P(S) to P(S) which assigns to every subset X of S the set $\mathbf{K}(X)$ of all the *consequences* of X. The final definition of $\mathbf{K}$ involves several preliminary definitions.

Df.3. — K is the mapping from P(S) to P(S) which satisfies the following conditions : for every subset X of S, 1) $X \subset K(X)$, 2) $K(X)$ contains all expressions y of F such that $y \in \delta_i(Y)$ for some δ_i and for some $Y \subset X$, 3) $K(X)$ contains no other elements.

Otherwise said, the K-image of every member X of P(S) is the set which is the union of all sets which can be obtained from subsets of X by a single application of a) the identity function from P(S) to P(S), or b) one of the δ_i, applied to a subset of X.

Df.4. — Let K^1 represent K and let K^n represent the n—1th application of K to itself. Then, the mapping K° is the mapping from P(S) to P(S) which is defined as follows : for every subset X of S, $K^\circ(X) = \{z \mid (\exists n)(n \geqslant 1 \,\&\, z \in K^n(X)\} = \bigcup_{n=1}^{\infty} K^n(X)$, this last equality by definition.

Df.5. — $\mathbf{K}$ is the mapping from P(S) to P(S) which is defined as follows : for every subset X of S, $\mathbf{K}(X) = K^\circ(X \cup A)$.

It is clear from the definitions that, for any given formal system F, the consequence relation $\mathbf{K}_F$ is uniquely defined.

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[2] Following Bourbaki [1], p. 4, we will use the notation « P (S) » to represent the set of all subsets of S, though we will not use a script « P ».

Df.6. — The set $\mathrm{T} = \mathbf{K}(\Lambda)$ is called the set of *theorems* of F.[3]

Df.7. — A *derived rule* of F is a single-valued function Q which is a subset of $\mathbf{K}$ and such that 1) every member of the domain of the function Q is a finite set with the same cardinality n, 2) the domain of Q is a recursive subset of the set of all finite subsets of S, 3) the range of Q is a recursively enumerable class of recursive subsets of S, 4) Q is a recursive function.

We will now derive several properties of the relation $\mathbf{K}$ which will be needed later in the proofs of theorems.

Th.3. — Let m and n be any two positive integers. Then, if $n \leqslant m$, then $K^n(X) \subset K^m(X)$ where X any subset of S.

Proof : If $m = 1$, then $n = m = 1$ and the theorem is trivial (n is a positive integer).

Assume the property to be true for m. Now, consider $m + 1$. $K^{m+1}(X) = K(K^m(X))$. But, by definition, $K^m(X) \subset K(K^m(X))$. Now, for all $n \leqslant m$, $K^n(X) \subset K^m(X)$ by the hypothesis of the induction. Thus, $K^n(X) \subset K^m(X) \subset K^{m+1}(X)$ where $n \leqslant m$, and the theorem is thus true for $m + 1$. Thus, the theorem is true for all m.

Th.4. — For all subsets X of S, $X \subset \mathbf{K}(X)$.

Proof : By Df.3, $X \subset K(X)$. Thus, $X \subset K^{*}(X)$. But, $X \subset X \cup \mathrm{A} \subset K^{*}(X \cup \mathrm{A}) = \mathbf{K}(X)$ which gives $X \subset \mathbf{K}(X)$.

Th.5. — 1) $\mathbf{K}(X \cup \mathrm{A}) = \mathbf{K}(X)$; 2) $\mathrm{T} = \mathbf{K}(\mathrm{A}) = \mathbf{K}(\Lambda)$; 3) $\mathrm{A} \subset \mathbf{K}(X)$.

Proof : — 1) $\mathbf{K}(X \cup \mathrm{A}) = K^{*}((X \cup \mathrm{A}) \cup \mathrm{A}) = K^{*}(X \cup \mathrm{A}) = \mathbf{K}(X)$. 2) $\mathrm{T} = \mathbf{K}(\Lambda) = K^{*}(\Lambda \cup \mathrm{A}) = K^{*}(\mathrm{A} \cup \mathrm{A}) = \mathbf{K}(\mathrm{A})$. 3) $\mathrm{A} \subset X \cup \mathrm{A} \subset \mathbf{K}(X \cup \mathrm{A}) = \mathbf{K}(X)$.

Th.6. — $\mathbf{K}$ is monotone, that is : $X \subset Y \rightarrow \mathbf{K}(X) \subset \mathbf{K}(Y)$.

Proof: Since $X \subset Y$, we have that $Z \subset X$ and $b \in \delta_i(Z) \rightarrow b \in K(Y)$, where δ_i is some primitive rule of F. Thus, K is monotone. This proves the monotonicity of K^n where $n = 1$. Suppose that, for $n \geqslant 1$, $X \subset Y \rightarrow K^n(X) \subset K^n(Y)$. Then $K^{n+1}(X) = K(K^n(X))$ and $K^{n+1}(Y) = K(K^n(Y))$. Thus, $X \subset Y \rightarrow K^n(X) \subset K^n(Y)$ which gives $K(K^n(X)) \subset K(K^n(Y))$ (since K is monotone), which gives $K^{n+1}(X) \subset K^{n+1}(Y)$ and the property is true for $n+1$. Thus, K^{*} is monotone, for consider $X \subset Y$ and $b \in K^{*}(X)$. Then, there exists $n \geqslant 1$ such that $b \in K^n(X)$ which gives $b \in K^n(Y) \subset K^{*}(Y)$. Finally, $X \subset Y \rightarrow X \cup \mathrm{A} \subset Y \cup \mathrm{A}$ which implies that $K^{*}(X \cup \mathrm{A}) \subset K^{*}(Y \cup \mathrm{A})$ which gives $\mathbf{K}(X) \subset \mathbf{K}(Y)$.

Corollary. $\mathbf{K}(X) \cup \mathbf{K}(Y) \subset \mathbf{K}(X \cup Y)$.

Proof : Immediate by the monotonicity of $\mathbf{K}$.

That the converse of the above corollary to Th.6 is not true can be seen by counterexample.

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[3] The sign « Λ » here and in all of the following will be our symbol for the null set.

Th.7. — $\mathbf{K}$ is idempotent, that is : $\mathbf{K}(\mathbf{K}(X)) = \mathbf{K}(X)$.

Proof : $X \subset \mathbf{K}(X)$, by Th.4, which implies, by Th.6 that $\mathbf{K}(X) \subset \mathbf{K}(\mathbf{K}(X))$.

Conversely, consider any $b \in \mathbf{K}(\mathbf{K}(X))$. Now $b \in \mathbf{K}(\mathbf{K}(X)) \rightarrow (\exists n)$ $b \in K^n(\mathbf{K}(X) \cup \mathrm{A}) = K^n(\mathbf{K}(X))$, this last equality by Th.5. If $n = 1$, then $b \in K(\mathbf{K}(X))$ which means that either 1) $b \in \mathbf{K}(X)$ or 2) b $\in \delta_i(Y)$ where $Y \subset \mathbf{K}(X)$. If 1), then our theorem is satisfied. If 2), then for all $y \in Y$, $y \in \mathbf{K}(X)$ which implies that there exists an $m \geqslant 1$ such that $y \in K^m(X \cup \mathrm{A})$. Now, δ_i is a rule of finite order which implies that Y has finite cardinality j. Thus, for each $y \in Y$, let m_y be the smallest m such that $y \in K^m(X \cup \mathrm{A})$. There are only j such m_y's. Thus, let $t = \max(m_{y_1}, m_{y_2}, \ldots, m_{y_j})$ which is well-defined. Thus, by Th.3, $Y \subset K^t(X \cup A)$ which implies that $b \in K^{t+1}(X \cup \mathrm{A})$ and thus $b \in \mathbf{K}(X)$, and the theorem holds for n = 1.

Assume that, for $n \geqslant 1$, $b \in K^n(\mathbf{K}(X) \cup \mathrm{A}) \rightarrow b \in \mathbf{K}(X)$. Then $b \in K^{n+1}(\mathbf{K}(X) \cup \mathrm{A}) = K(K^n(\mathbf{K}(X)) \rightarrow$ either 1) $b \in K^n(\mathbf{K}(X))$ or 2) $b \in \delta_i(Y)$ where $Y \subset K^n(\mathbf{K}(X))$. If 1), then, by the induction hypothesis, $b \in \mathbf{K}(X)$. If 2), then $Y \subset K^n(\mathbf{K}(X))$ which, by the induction hypothesis, implies that $Y \subset \mathbf{K}(X)$. But, by the same reasoning as above, this gives $b \in \mathbf{K}(X)$. Thus, the property is true for $n + 1$.

Thus, the property holds for all n and our theorem is established.

Corollary. If $Y \subset \mathbf{K}(X)$, then $\mathbf{K}(Y) \subset \mathbf{K}(X)$.

Proof : $Y \subset \mathbf{K}(X) \rightarrow \mathbf{K}(Y) \subset \mathbf{K}(\mathbf{K}(X)) = \mathbf{K}(X)$.

Th.8. — $\mathbf{K}(X \cup \mathrm{T}) = \mathbf{K}(X)$.

Proof : Since $\Lambda \subset X$, then $\mathbf{K}(\Lambda) = \mathrm{T} \subset \mathbf{K}(X)$. But $X \subset \mathbf{K}(X)$ also. Thus, $X \cup \mathrm{T} \subset \mathbf{K}(X)$ which gives $\mathbf{K}(X \cup \mathrm{T}) \subset \mathbf{K}(\mathbf{K}(X)) = \mathbf{K}(X)$.

Conversely, $X \subset X \cup \mathrm{T} \rightarrow \mathbf{K}(X) \subset \mathbf{K}(X \cup \mathrm{T})$.

Thus, $\mathbf{K}(X \cup \mathrm{T}) = \mathbf{K}(X)$.

There is, of course, an obvious analogy between the properties of the relation $\mathbf{K}$ and the properties of a topological closure defined on the set S. As will shortly appear, however, the relation $\mathbf{K}$ is much more special in nature, due to its constructive character. Nevertheless, we shall devote a later section of this chapter to a discussion of this analogy between the closure relation of topology and the consequence relation for formal systems. Others, such as Tarski [1], pp. 421-454, have explored the relationship between topology and certain particular formal systems (in the case of Tarski which we have cited it is the sentential calculus). Our discussion, however, will concern not a particular system but formal systems in general (this discussion to be found in Section C of Chapter I of this study).

We now introduce another (but, as will be seen, equivalent) relation defined within a formal system F.

§ 3. DEDUCTIONS FROM HYPOTHESES

Again, let it be understood that we are working with respect to an arbitrary formal system $F = \langle S, A, R \rangle$.

Df.8. — When, for any $Y \subset S$ and any δ_i, $b \in \delta_i(Y)$, then we say that « b is inferred from the members of Y » or « b is inferred from Y » by the primitive rule δ_i.

Df.9. — Let X be some subset of S. If there exists a finite ordered list of members of S such that each member of the list is either 1) a member of X, 2) a member of A, or 3) inferred from prior members of the list by a primitive rule of inference, then we say that the list in question is a « formal deduction in F from the hypotheses X » or, when no ambiguity is possible, « a formal deduction ». We say that a formal deduction is a formal deduction *of* its last member b, and we write « $X \vdash b$ » as an abbreviation for « there exists a formal deduction of b from the hypotheses X ». We alternatively write « $X \vdash_F b$ » when it is necessary to indicate in which system the deduction exists. If $X = \Lambda$, then we write alternatively « $\vdash b$ », respectively « $\vdash_F b$ ».

Df.10. — Let H be any finite ordered list of elements of S (allowing repititions). Then the number of members of the list (counting each different mention of the same element as another member of the list) is called the *length* of H.

Df.11. — If H and B are each lists of length n and m respectively, then the list HB of length $n + m$ is called the *concatenation* of H and B.

The operation of concatenation is the operation of the free semigroup of the words of S. It is not, in general, a commutative operation.

Th.9. — If H and B are each deductions from the hypotheses X, then the concatenation of H and B, HB, is also a deduction from the hypotheses X.

Proof : This is immediate from the definitions involved.

Th.9 is a lemma for the following more general theorem :

Th.10. — If, for each i, $1 \leqslant i \leqslant n$, H_i is a deduction from the hypotheses X, then the list $H_1 H_2 \dots H_n$ is a deduction from the hypotheses X.

Proof : If $n = 1$, then the theorem is trivially true. Assume that the property holds for n. Now, consider a list $H_1 H_2 \dots H_n H_{n+1}$. The list $H_1 H_2 \dots H_n$ is a deduction from X by the induction hypothesis (where H_i is a deduction from X for each i, $1 \leqslant i \leqslant n + 1$). But, $H_1 H_2 \dots H_n H_{n+1}$ is the concatenation of $H_1 H_2 \dots H_n$ and H_{n+1}. Thus, by Th.9, $H_1 H_2 \dots H_n H_{n+1}$ is also a deduction from the hypotheses X. Thus, the property is true for $n + 1$ and hence for all n.

Our theorem is thus established.

Df.12. — Let H be a formal deduction from the hypotheses X and let b be a member of H. *Then* we say that the presence of b in H is *justified* by the property (or properties) Df.9.1-Df.9.3 which are true of b, and we call any particular one of these properties which are true of b *a justification of b in H.*

The above definition gives us convenient terminology. With it, we can state the definition of a formal deduction thusly : a formal deduction is a finite ordered list of elements of S such that each member of the list has a justification.

Th.11. — If H is a deduction from the hypotheses X, and B is a deduction from the hypotheses Y, and if every member of Y either appears in the list H or is a member of X, then HB is a deduction from the hypotheses X.

Proof : Let H be of length n and B of length m. We will prove the theorem by induction on m. Suppose that $m = 1$. Then B reduces to a single element b. Since there are no prior members of B, then either 1) b is in Y or 2) b is in A. If 1), then by hypothesis either b is in X or b appears in H. If b is in X, then b is a deduction from X and thus the concatenation Hb is also a deduction from X. If b appears in H, then b must have a justification. It is clear, from Df.9, that if b has justification at line i of a formal deduction, then b has the same justification at any line $k \geqslant i$. Thus, in the list Hb, b is the n + 1st line and thus has a justification. Thus, Hb is a deduction from X.

If 2), then, by Df.9, b has a justification at any line of any deduction. Thus, Hb is a deduction from the hypotheses X.

Suppose that, for all $p \leqslant m$, the property holds. Now consider any $B = b_1 b_2 \ldots b_m b$ of length $m + 1$, a deduction of b from the hypotheses Y where Y satisfies the hypotheses of this theorem. By Df.9, either 1) $b \in Y$, 2) $b \in A$, or 3) b is inferred from prior members of B. Now, by the hypothesis of our induction, the list $b_1 b_2 \ldots b_m$ is a deduction from X since it is of length less than or equal to m. Thus, $Hb_1 b_2 \ldots b_m$ is a deduction from X by Th.10. Now, if either 1) or 2), then by the same reasoning as in the above, HB is a deduction from X. If 3), then b has a justification. And, since $b_1 b_2 \ldots b_m$ is a deduction from X, then so is $B = b_1 b_2 \ldots b_m b$ and thus, the concatenation HB is also a deduction from the hypotheses. Thus, the property is true for $m + 1$ and the theorem is true for all m.

Lemma. Let X be any subset of S. Then, for all n, $b \in K^n(X \cup \mathrm{A}) \rightarrow X \vdash b$.

Proof : Let $n = 1$. Then, $K^n(X \cup \mathrm{A}) = K(X \cup \mathrm{A})$. If b is an element of $K(X \cup \mathrm{A})$, then by definition, either 1) $b \in X \cup \mathrm{A}$ or 2) there exists a subset Y of $X \cup \mathrm{A}$ and an i such that $b \in \delta_i(Y)$. If 1), then either $b \in X$ or $b \in A$. In either case, the list consisting of the single element b is a deduction from X.

If 2), then let $y_1 y_2 \ldots y_p$ be an ordering of the elements of Y. Y is finite, since every rule δ_i is of finite order. Y is a subset of $X \cup \mathrm{A}$. Thus, every element y_j of Y is either in X or in A. Thus, by Df.9, every member of the list $y_1 y_2 \ldots y_p$ is justified and this list is thus a deduction from the hypotheses X. Thus, for the list $y_1 y_2 \ldots y_p b$, b is justified, since it is inferred from prior members of the list by the rule δ_i, and thus the list is a deduction of b from the hypotheses X.

Suppose that, for $n \geqslant 1$, the property holds. Now consider $b \in K^{n+1} (X \cup \mathrm{A}) = K(K^n(X \cup \mathrm{A})$. Now, $b \in K(K^n(X \cup \mathrm{A}))$ means that either 1) $b \in K^n(X \cup \mathrm{A})$ or 2) there is a subset Y of $K^n(X \cup \mathrm{A})$ such that $b \in \delta_i(Y)$ for some δ_i. If 1), then by the hypothesis of induction the theorem holds. If 2), then consider the subset Y of $K^n(X \cup \mathrm{A})$ in question. Each member y_j of Y has a deduction H_j from the hypotheses X since each member of Y is in $K^n(X \cup \mathrm{A})$ (hypothesis of induction). Now, the list $H_1 H_2 \ldots H_p$ is thus a deduction from the hypotheses X by Th.10 since each H_j is a deduction from X. But, since each H_j is a deduction of y_j, the elements of Y all appear in the list $H_1 H_2 \ldots H_p$. Now, consider the list $H_1 H_2 \ldots H_p b$. The last member b has a justification in this list since it is inferred from prior members by the primitive rule δ_i. Thus, the whole list is justified and the list is thus a deduction of b from the hypotheses X.

The property is true for all n, and the lemma is established.

Th.12. — Given any formal system F, then we have $b \in \mathbf{K}_F(X) \longleftrightarrow X \vdash_F b$ where X is any subset of S and b is any element of S.

Proof: Suppose that $b \in \mathbf{K}_F(X)$. By definition this means that $b \in K^{\circ}{}_F(X \cup \mathrm{A})$ which, by definition, means that there exists $n \geqslant 1$ such that $b \in K^n_F(X \cup \mathrm{A})$. But, by the lemma proved above, this gives $X \vdash_F b$.

Conversely, suppose that $X \vdash_F b$. Let J represent any deduction of b from the hypotheses X of length $p \geqslant 1$. Then we will prove the theorem by induction on p.

Suppose that $p = 1$. Then J is b, since b must be the last line of J. Since there are no prior members of the list, then either 1) $b \in X$ or 2) $b \in \mathrm{A}$. If either 1) or 2), then $b \in X \cup \mathrm{A}$ which implies that $b \in \mathbf{K}_F(X \cup \mathrm{A})$ (by Th.4). But $\mathbf{K}_F(X \cup \mathrm{A}) = \mathbf{K}_F(X)$, by Th.5, and thus $b \in \mathbf{K}_F(X)$.

Suppose that for all deductions of length $k \leqslant p$, the property holds. Now consider a deduction from the hypotheses X of length $p + 1$ whose last member is b. It is the case that either 1) $b \in X$, 2) $b \in \mathrm{A}$, or 3) $b \in \delta_i(Y)$ where the members of Y are prior to b in the list J, Y has finite cardinality $m \geqslant 1$, and δ_i is a rule of order m. If either 1) or 2), then $b \in \mathbf{K}_F(X \cup \mathrm{A}) = \mathbf{K}_F(X)$. If 3), then consider the set Y. Every member of Y appears in the list J before b, and J is of length $p + 1$. Thus, every member of Y is the last member of some list of length $k \leqslant p$ which is a deduction from X. Thus, by the hypothesis of our induction, every member of Y is in $\mathbf{K}_F(X)$. Thus, for every member y of Y, there exists an n such that $y \in K^n_F(X \cup \mathrm{A})$ by definition of $\mathbf{K}$. For each y, let n_y be the smallest number such that $y \in K^{n_y}_F(X \cup \mathrm{A})$. By the postulates for the natural numbers; n_y exists and is unique for each y. Thus, for each y we have associated a *unique number* n_y such that $y \in K_F^{n_y}(X \cup \mathrm{A})$. Now, let $t = \max(n_{y_1}, n_{y_2}, \ldots, n_{y_m})$, where here the y_j is an enumeration of the members of Y. The element t is well-defined since the set n_{y_j} in question is finite. Thus, by Th.3, $y \in K_F^t(X \cup \mathrm{A})$ for all y in Y. Thus, $Y \subset K_F^t(X \cup \mathrm{A})$. But, $b \in \delta_i(Y)$. Thus, by definition, $b \in K_F^{t+1}(X \cup \mathrm{A})$ which means that $b \in K^{\circ}{}_F(X \cup \mathrm{A}) = \mathbf{K}_F(X)$.

Thus, the property in question is true for deductions of any length whatever.

Thus, the theorem is true.

Corollary. — $\vdash_F b$ if and only if b is a theorem of F.

Proof: $\vdash_F b$ means $\Lambda \vdash_F b$ which implies, by Th.12, that $b \in \mathbf{K}_F(\Lambda)$ which is, by definition, the set of theorems T of F.

Conversely, $b \in \mathrm{T} \rightarrow b \in \mathbf{K}_F(\Lambda)$ which, by Th.12, gives $\Lambda \vdash_F b$ or $\vdash_F b$.

Th.12 is, in a certain sense, a fundamental theorem of the relation $\mathbf{K}$, for with it we can translate statements about the relation $\mathbf{K}$ into statements about formal deductions which is perhaps the more traditional way of handling the idea of « logical consequence » for formal systems. Also, some theorems might prove easier to demonstrate in terms of formal deductions than in terms of $\mathbf{K}$ (or vice-versa).

Needless to say, we will not mention explicitly the translation of every theorem concerning $\mathbf{K}$ into its corresponding form involving deductions from hypotheses. The reader might sometimes find it helpful to make the translation mentally.

Df.13. — Let X be any finite subset of S and let $x_1, x_2, \ldots, x_n$ be any finite sequence of the elements of X (allowing the possibility of repetitions of the same element). Then, where $X \vdash_F b$, we will sometimes write equivalently « $x_1, x_2, \ldots, x_n \vdash_F b$ ».

With definition Df.13, we finally arrive at the more traditional use of the symbol « $\vdash$ » to indicate deductions from hypotheses. A brief account of the history of this usage is to be found in Kleene [5], p. 88 and also in several brief notes in the bibliography of Kleene [5]. In these notes, Kleene traces the origins of his usage to Kleene [1] and to Rosser [1].

§ 4. THE EFFECTIVENESS OF THE NOTION OF FORMAL DEDUCTION

We are here concerned to show, briefly, that the notion of formal deduction as here presented is « effective » in the sense that, given any finite list of elements of S and any recursive subset X of S, there is a finite number of operations which will tell us whether or not the list in question is a deduction from the hypotheses X.

Suppose that a recursive subset X of S be given and let any finite list $J = b_1 b_2 \dots b_n$ elements of S be given. We will now describe an operation which is to be applied in turn to each member of the list J, starting with the first member and continuing in order until the last, which will tell whether or not the member in question has a justification.

Consider a member b_i of the list J. We want to know if b_i has a justification or not. We first ascertain if b_i is in X or not. Since X is a recursive subset of S and b_i is an element of S, then there is an effective method for resolving this question. If b_i is in X, then b_i has a justification and we proceed to examine b_{i+1}. If b_i is not in X, then we ascertain if b_i is in A or not. Again the process is effective since A is recursive subset of S. If b_i is in A, then it is justified and we stop and go on to b_{i+1}. If not, then we must ascertain whether or not b_i is inferred from prior members of the list by a primitive rule of inference. If, of course, $i = 1$ and b_i is thus the first member in the list, then it is not inferred from prior members and it has no justification and the list J is not a deduction from the hypotheses X.

If b_i is not the first member of the list, then consider the set P of all subsets of S whose elements occur as members of the list J before b_i. Since the list J is finite (and i is finite), the set P is also finite. In fact its cardinality is less than or equal to 2^{i-1}. And the sets which are elements of P are also finite. Now, let us consider, in turn, each member of P which is in the domain of some primitive rule of inference. (We can trivially disregard all sets whose cardinality is not the order of some primitive rule of inference. In particular, in view of Df.I.4 and Df.2.3, the null set Λ can always be disregarded). Since, by Df.2.3 and Df.I.4, the domain of each primitive rule δ_j is a recursive subset of the set of all finite subsets of S, then given any set which is an element of P, there is an effective method of deciding whether or not it is in the domain of a given δ_j. And since the number of primitive rules is finite, we can thus effectively decide whether or not any given element of P is in the domain of any of the δ_j.

Thus, we take in turn each element of P which is in the domain of a δ_j and we calculate its image which is also a subset of S. There is an effective method for calculating the image since each δ_j is a recursive function (Df.1.4 and Df.2.3). And since the image by δ_j of any set is a recursive subset of S, then there is an effective method for deciding whether or not b_i is in the set of elements which are « immediately inferred » from any given set.

If for at least one δ_j we find that b_i is the δ_j-image of some element of P, then we stop, for b_i has a justification. If not, then b_i has no justification and J is not a deduction from the hypotheses X.

If b_i, by any part of the above process, has a justification, then we apply the same algorithim to b_{i+1}. If, for all i, b_i is justified, then J is a deduction from the hypotheses X. Otherwise, J is not a deduction, and we stop our process as soon as we find an unjustified member of the list.

It is clear from the above that, in general, no algorithim will work where X is not a recursive subset of S. However, the ensuing theory will show that the consequence relation **K** for a formal system F, though defined for every subset X of S, is actually determined once it is defined for all finite subsets of S.

There is, of course, no algorithim for deciding, given a subset X of S and an element b of S, whether or not b is in the set of consequences $\mathbf{K}(X)$, at least in the general case. The problem of finding the solution to various special cases of the decision problem which are solvable bas been much discussed and this problem lies outside the scope of this study (see, for discussions of the decision problem, Ackermann [1]; Church [1], [2], and [3] ; Davis [1] ; Gödel [1] ; Kleene [2], [3], and [4] ; Post [2]).

SECTION B

RELATIONSHIPS BETWEEN FORMAL SYSTEMS

We will now be concerned with developing several relations of different force between two arbitrary formal systems. Through an examination of these relationships, we will eventually be led to pose a definition of *equivalence* between two formal systems.

In the following, we will need to be clear about our use of words, and we wish to make certain explicit conventions. The terms *mapping*, and *function* will be used synonymously to designate a single-valued *relation*. A relation is a set of ordered pairs or, synonymously, a correspondence between two sets which is not necessarily (but may be) single-valued. We will adopt the terminology of Bourbaki [3] concerning functions. More precisely, given a function g from a set Y to a set Y', if every member of Y' is the image of some element of Y, then the mapping will be called *onto* or *surjective* or *a surjection*. If a function g from Y to Y' carries distinct elements into distinct elements (i.e $g(x) = g(y) \rightarrow x = y$) then it will be called *injective* or *an injection*. A mapping which is both injective and surjective is called bijective. As is usual in English, a mapping g from a set Y into (or to) a set Y' is single-valued, but ambiguous as to its other properties unless specifically stated.

The advantage of the above terminology is that it allows us to designate various properties of a mapping g, such as being 1—I for instance, without our having to consider the inverse relation g^{-1}. Other well-known terminology and symbolism may be used without explicit conventions governing their usage.

In the following we will be frequently speaking of the relationship between two formal systems. We will use a method of primes to distinguish between the two systems so that continual verbal comment, indicating the system to which certain sets and functions belong, will not be necessary. Thus, A will be the set of axioms of the one and A′ the set of axioms of the other and so on with the other sets and relations.

We will use capital letters for classes and sets (and occasionally for sequences) and lower case letters for elements. (This has already, in effect, been our usage in the above.) Finally, where we are concerned with a mapping g from the elements of one set Y to the elements of another set Y', we will « abuse our language » by also using « g » to designate the cannonically induced mapping from $\mathrm{P}(Y)$ to $\mathrm{P}(Y')$ which assigns to each subset X of Y the subset $\mathrm{g}(X)$ of Y' which consists of all the g-images of elements of X. (In particular, $\mathrm{g}(\Lambda) = \Lambda$.)

§ I. DEFINITIONS OF VARIOUS RELATIONSHIPS

Df.14. — Consider two formal systems $\mathrm{F} = \langle \mathrm{S}, \mathrm{A}, \mathrm{R} \rangle$ and $\mathrm{F}' = \langle \mathrm{S}', \mathrm{A}', \mathrm{R}' \rangle$, and a mapping g from S to S'. Then we say that g *preserves the consequence relation from* F *to* F' if, for any $b \in \mathrm{S}$ and any subset X of S, $b \in \mathbf{K}_F(X) \rightarrow g(b) \in \mathbf{K}_{F'}(\mathrm{g}(X))$.

There are, of course, some very trivial mappings which preserve the consequence relation from one system to another. The mapping g which maps all of the elements of S onto some one arbitrarily chosen theorem of S' is seen to preserve the consequence relation. We will shortly develop some relationships between formal systems which are not satisfied by trivial mappings, but it is often worthwhile to consider properties in their most general setting before proceeding to more specific and useful cases.

Df.15. — Consider two formal systems F and F' and a mapping g from S to S'. We say that g *preserves primitive inference from* F *to* F' if, for any $b \in \mathrm{S}$, any primitive rule δ_i, and any $Y \subset \mathrm{S}$, we have that $b \in \delta_i(Y) \rightarrow \mathrm{g}(b) \in \mathbf{K}_{F'}(g(Y))$.

Obviously, the property of preserving primitive inference is even less strict than that of preserving the consequence relation. However, we have :

Th.13. — In order that a mapping g from S to S' preserve the consequence relation from F to F', it is necessary and sufficient that g be such that 1) $b \in \mathrm{A} \rightarrow \mathrm{g}(b) \in \mathrm{T}'$, 2) g preserves primitive inference from F to F'.

Proof : Suppose that g preserves the consequence relation from F to F'. Now, suppose that $b \in \mathrm{A}$. Then we have $b \in \mathrm{A} \rightarrow b \in \mathbf{K}_F(\mathrm{A}) = \mathbf{K}_F(\Lambda)$ (Th.5) and this gives $g(b) \in \mathbf{K}_{F'}(\mathrm{g}(\Lambda)) = \mathbf{K}_{F'}(\Lambda) = \mathrm{T}'$ since g preserves the consequence relation from F to F'. Thus, Th.13.1 holds.

Th.13.2 is trivial since we have that $b \in \delta_i(Y) \rightarrow b \in \mathbf{K}_F(Y)$ which immediately gives $g(b) \in \mathbf{K}_{F'}(g(Y))$ since g preserves the consequence relation from F to F'.

Conversely, suppose that the properties Th.13.1 and Th.13.2 hold for g. Now, suppose that $b \in \mathbf{K}_F(X)$ where b is an element of S and X

is a subset of S. Then, there exists an $n \geqslant 1$ such that $b \in K_F^n(X \cup A)$ by definition of $\mathbf{K}_F$.

If $n = 1$, then $b \in K_F(X \cup A)$ which means that either i) $b \in X \cup A$ or ii) $b \in \delta_i(Y)$ where $Y \subset X \cup A$, and $\delta_i \in R$. If i), then either $b \in X$ or $b \in A$. If $b \in X$, then $g(b) \in g(X) \subset \mathbf{K}_{F'}(g(X))$. If $b \in A$, then $g(b) \in T' \subset \mathbf{K}_{F'}(g(X))$, by Th.13.1 and Th.8.

If ii), then $g(b) \in \mathbf{K}_{F'}(g(Y))$ by Th.13.2. Now, $Y \subset X \cup A$ which means that $g(Y) \subset g(X \cup A) = g(X) \cup g(A)$ by the definition of g. Thus, by the monotone character of $\mathbf{K}_{F'}$ (Th.6), we have $g(b) \in \mathbf{K}_{F'}(g(Y)) \subset \mathbf{K}_{F'}(g(X) \cup g(A)) \subset \mathbf{K}_{F'}(g(X) \cup T') = \mathbf{K}_{F'}(g(X))$, using again Th.13.1 and Th.8.

Thus, our property holds for $n = 1$.

Suppose that, for $n \geqslant 1$, $g(b) \in \mathbf{K}_{F'}(g(X))$ where b is any element of $K_F^n(X \cup A)$. Now, consider $b \in K_F^{n+1}(X \cup A)$. By definition, either i) $b \in K_F^n(X \cup A)$ or ii) $b \in \delta_i(Y)$ where $Y \subset K_F^n(X \cup A)$. If i), then by the hypothesis of our induction, $g(b) \in \mathbf{K}_{F'}(g(X))$. If ii), then, by Th.13.2, $g(b) \in \mathbf{K}_{F'}(g(Y))$.But $Y \subset K_F^n(X \cup A)$ which implies that $g(Y) \subset \mathbf{K}_{F'}(g(X))$ (again by the hypothesis of the induction). But this gives $\mathbf{K}_{F'}(g(Y)) \subset \mathbf{K}_{F'}(g(X))$, by Th.7, and thus $g(b) \in \mathbf{K}_{F'}(g(X))$. Thus, the property holds for $n + 1$.

The property is true for all n and the theorem is proved.

By Th.12, a mapping which preserves the consequence relation is the same thing as a mapping which *preserves deductions from hypotheses*, i. e. a mapping g such that $X \vdash_F b \rightarrow g(X) \vdash_{F'} g(b)$, and so by Th.12 we can, as already mentioned above, translate our results into terminology and concepts involving the notion of deductions from hypotheses.

We now have the following interesting theorem :

Th.14. — In order that a mapping g from S to S′ preserve the consequence relation from F to F′, it is necessary and sufficient that, for $n \geqslant 0$, $b_1, b_2, \ldots, b_n \vdash_F b \rightarrow g(b_1), g(b_2), \ldots, g(b_n) \vdash_{F'} g(b)$.

Proof : Suppose that g preserves the consequence relation. Then, by Th.12, g preserves deductions from hypotheses, that is : $X \vdash_F b \rightarrow g(X) \vdash_{F'} g(b)$ where X is any subset of S. Thus, in particular, this is true of all finite sets and thus the condition of Th.14 holds.

Conversely, suppose that the condition of Th.14 holds. We will show that the conditions Th.13.1 and Th.13.2 hold. Suppose that $b \in A$. Then, $\vdash_F b$ which gives $\vdash_{F'} g(b)$ which, by Th.12 gives $g(b) \in T'$.

Suppose that $b \in \delta_i(Y)$. Now, δ_i is of finite order which means that Y is finite with m elements where $m \geqslant 1$ is some natural number. We thus have $y_1, y_2, \ldots, y_m \vdash_F b$ where the y_i represent an arbitrarily chosen enumeration of the elements of Y (Df.13). But this gives us $g(y_1), g(y_2), \ldots, g(y_m) \vdash_{F'} g(b)$, or equivalently, $g(Y) \vdash_{F'} g(b)$ which, by Th.12 gives $g(b) \in \mathbf{K}_{F'}(g(Y))$. Thus, the conditions of Th.13 hold and, by Th.13, g preserves the consequence relation from F to F′.

This theorem emphasises the constructive nature of the consequence relation for it shows that the consequence relation for a formal system F really depends only on deductions from finite sets of hypotheses. Once the consequence relation **K** is defined for all *finite* subsets of S, it is completely determined for *all* subsets of S. We can formulate this result more strikingly in the following theorem :

Th.15. — Consider two formal system $F = \langle S, A, R\rangle$ and $F' = \langle S', A', R'\rangle$. Suppose that $S = S'$, that is, the set of expressions for the two systems is the same. Suppose further that $\mathbf{K}_F(X) = \mathbf{K}_{F'}(X)$ where X is any *finite* subset of S (and thus of $S' = S$). Then $\mathbf{K}_F(X) = \mathbf{K}_{F'}(X)$ where X is any subset of S whatever.

Proof : Consider the identity mapping g from S onto S′ which maps every element of S onto *itself*. For the sake of precision, we will speak of g when we are going from S to S′ and of g^{-1} when we are going from S′ to S, though we are speaking, in each case, of precisely the same mapping.

Now, the mapping g is seen to trivially satisfy the hypotheses of Th.14. Thus, by Th.14, g preserves the consequence relation from F to F′. But g^{-1} also satisfies the hypotheses of Th.14. Thus, g^{-1} preserves the consequence relation from F′ to F.

Now, consider any element $b \in \mathbf{K}_F(X)$ where X is *any* subset of S, finite or not. Since g preserves the consequence relation from F to F′, then $g(b) \in \mathbf{K}_{F'}(g(X))$. But g is the identity. Thus, $b \in \mathbf{K}_{F'}(X)$ and we have that, for all subsets X of S, $\mathbf{K}_F(X) \subset \mathbf{K}_{F'}(X)$ and thus the desired equality follows from the symmetry of the situation.

Thus, our theorem is proved.

It is, of course, quite possible for two formal systems F and F′ to satisfy the hypotheses of Th.15 and yet have different axioms and different rules of inference. This is precisely what is done when we choose two different formulations of the propositional calculus, for instance, but guard the same symbolism (and thus the same set of wffs) in both cases. However, what about the case when we choose different sets of expressions S and S′ ? The method of proof Th.15 suggests the following definition :

Df.16. — Two formal systems F and F′ are said to be *isomorphic* if there exists a bijective mapping g from S *onto* S′ such that 1) g preserves the consequence relation from F to F′, 2) g^{-1} (which is a mapping) preserves the consequence relation from F′ to F.

With two arbitrary formal systems we cannot speak of identy between the respective consequence relations since the two sets S and S′ of expressions are in general different. But the existence of an isomorphism assures us the intuitive property of « identity except for symbolism » which is the intuitive property that isomorphism tries to render precise. An isomorphism assures us that, though the expressions may be different, there is, for every expression b of S a corresponding expression $g(b)$ of S′ which « plays the same rôle » in F′ as b plays in F, at least from a *deductive* or *syntactical* point of view. The respective rôles which these or other expressions might play under some *interpretation* of the system is another question, namely a question of *semantics*, or if not semantics, a question of intended usage under interpretation.

It is our feeling that it is this question, the question of the relative strength of formal systems under certain interpretations, that Kemeny's definition of equivalence in terms of models really treats (see Kemeny [*1*]).

This opinion seems to be supported by the fact, which should by now be evident, that the signs (i.e. the set L) of a calculus of signs C really play no rôle whatsoever in the deductive structure of the resulting formal system F of C. The reason is not far to seek. It is simply that the set A is a subset of the set of expressions S and the primitive rules R are certain relations defined between sets of members of S (or, if one takes the definition of the Appendix to Chapter I, the rules ares recursive relations over the set S). And it is the set A of axioms together with the rules R which determine the deductive structure of a formal system.

It is only under some consideration of the meaning or intended usage of the particular symbolism chosen that the signs which are used to construct the expressions of a formal system are seen to play any rôle whatever.

Thus, though Kemeny's definition of equivalence by models does not appeal to the *meaning* of the symbols used, in a strictly intuitive sense, there is certainly a very strong appeal to *intended usage* of symbols, for definitions of such concepts as « bound variable », « free variable », etc., defined for all systems, are necessary to Kemeny's treatment. And the definition of such concepts, even though formal or syntactical, is really quite meaningless unless one has constantly in mind the usage which is to be made of these notions in interpreted systems (see the Introduction of this study for an example of this).

However, such is not at all the case for the definitions of *isomorphism* and *equivalence* which we have given and will give, respectively (the notion of equivalence between formal systems will be defined in § 2 of this Section). No appeal whatsoever is made to the usage of the signs which may be used to construct the set S in our definitions of these concepts.

Thus, we pose also the following :

Df.17. — Two calculi of signs C and C′ are said to be *deductively* or *syntactically isomorphic* if their resulting formal systems F and F′ are isomorphic.

As was just noticed above, we have not, as yet, given a definition of equivalence between formal systems. Our reasons for not choosing the relation of isomorphism as a definition of equivalence will become clearer upon supplementary exposition. But first it will be useful to consider several other types of relationships between formal systems.

Df.18. — Given two formal systems F and F′, we say that F *has a model in* F′ if there exists an *injective* mapping g from S into S′ such that g preserves the consequence relation from F to F′.

Clearly, if F and F′ are isomorphic, then each system has a model in the other. However, counterexamples can be constructed which show that the converse of this statement is not true. Such counterexamples will be considered in a later section of this Chapter.

Df.19. — Given two formal systems F and F′, a *homomorphism* from F to F′ is a *surjection* g from S onto S′ such that g preserves the consequence relation from F to F′. If there exists a homomorphism from F to F′, we say that F′ is a *homomorphic image of* F.

The relationship of homomorphism between formal systems admits of certain trivial cases of realization. For instance, the system W with $S = \{1, 0\}$; $A = \{1\}$; and $R = \Lambda$ is a homomorphic image of any consistent system F whatever,[4] for, given F, we simply map all theorems onto 1 and all non-theorems onto 0 and it is seen that this mapping is a homomorphism.

However, the algebraic notion of homomorphism for groups, rings and other such structures (to which our notion for formal systems is an intended analogy to some degree) admits also of such trivial cases of realization. It will be seen further on that when certain other fairly natural hypotheses are added to that of the existence of a homomorphism, the resulting relation will not be so trivial (Th.27).

Before proceeding to the next paragraph, one further remark of a general nature is perhaps worthwhile. From the definitions of our various relationships between formal systems, it is clear that no requirement of the recursiveness of the mappings is involved. If the assumption that two systems F and F′ are not isomorphic, to take an example, leads to a contradiction, then the two systems F and F′ are isomorphic for the existence of the mapping in question will have been proved. The question of recursiveness enters in as part of the definition of a formal system and the ensuing theory has shown the rôle that this aspect of the definition plays. But our methods or tools of research are those of mathematics in general and no arbitrary restriction to recursive methods is sought (this will become even clearer when, in the following theory we will make direct appeal to the axiom of choice). The theory of our study is, then, a general mathematical theory *of* formal logical systems.

§ 2. THE EQUIVALENCE RELATION BETWEEN FORMAL SYSTEMS

Df.20. — For any formal system F, two expressions $b, d \in S$ are said to be **e**-*equivalent* and we write $b \mathbf{e} d$ if and only if $b \in \mathbf{K}_F(\{d\})$ and $d \in \mathbf{K}_F(\{b\})$ (or, equivalently, $b \vdash_F d$ and $d \vdash_F b$).

The above is a relation of equivalence defined on the set S of expressions of a system F and it partitions S into equivalence classes.

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[4] A system F is said to be consistent if not every member of S is a theorem, i. e. if T is a proper subset of S.

That the relation Df.20 is indeed an equivalence relation follows trivially from properties of $\mathbf{K}$ that we have already proved. Reflexivity follows from Th.4, symmetry from the definition of $\mathbf{e}$-equivalence, and transitivity from Th.7 and its corollary.

Th.16. — Given any formal system F and b, d any elements of S, the set of expressions of F, then we have that $b\,\mathbf{e}\,d \longleftrightarrow \mathbf{K}_F(\{b\}) = \mathbf{K}_F(\{d\})$.

Proof : If $b \in \mathbf{K}_F(\{d\})$, then $\mathbf{K}_F(\{b\}) \subset \mathbf{K}_F(\mathbf{K}_F(\{d\})) = \mathbf{K}_F(\{d\})$. Similarly, $\mathbf{K}_F(\{d\}) \subset \mathbf{K}_F(\{b\})$, and so the desired equality holds.

Conversely, suppose that $\mathbf{K}_F(\{b\}) = \mathbf{K}_F(\{d\})$. Then, since $b \in \mathbf{K}_F(\{b\})$, we have $b \in \mathbf{K}_F(\{d\})$. Similarly, $d \in \mathbf{K}_F(\{b\})$ and thus $b\,\mathbf{e}\,d$.

Th.17. — For any system F, the class T of theorems is one of the $\mathbf{e}$-equivalence classes.

Proof : Consider any two expressions b, $d \in \mathrm{T}$. By Th.8, $\mathrm{T} \subset \mathbf{K}_F(\{d\})$ which gives $b \in \mathbf{K}_F(\{d\})$. Similarly, $d \in \mathbf{K}_F(\{b\})$. Thus, $b, d \in \mathrm{T} \rightarrow b\,\mathbf{e}\,d$. Hence, T is a subset of some $\mathbf{e}$-equivalence class.

Now, for any $b \in \mathrm{T}$, suppose that $x\,\mathbf{e}\,b$. Then $x \in \mathbf{K}_F(\{b\}) \subset \mathbf{K}_F(\mathrm{T}) = \mathrm{T}$ (since the relation $\mathbf{K}_F$ is idempotent). Thus, $x \in \mathrm{T}$ also and thus the set T is an $\mathbf{e}$-equivalence class and our theorem is established.

Th.18. — Given two subsets X and Y of S, in any system F. If for every expression $x \in X$, there exists an expression $y \in Y$ such that $x\,\mathbf{e}\,y$, then $\mathbf{K}_F(X) \subset \mathbf{K}_F(Y)$.

Proof : Assume that for all $x \in X$ there exists $y \in Y$ such that $x\,\mathbf{e}\,y$. Then, for all $x \in X$, $x \in \mathbf{K}_F(\{y\}) \subset \mathbf{K}_F(Y)$. Thus, $X \subset \mathbf{K}_F(Y)$ and thus $\mathbf{K}_F(X) \subset \mathbf{K}_F(Y)$ by Th.7 and its corollary. Thus, our theorem is true.

That the converse of Th.18 is not true can be easily seen by counterexample.

Df.21. — Where F is a formal system and X and Y are any two subsets of S, we write $X\,\mathbf{E}\,Y$ and we say that X and Y are $\mathbf{E}$-equivalent if it is case that 1) $(x)\,(x \in X \rightarrow (\exists\, y)\,(y \in Y \,\&\, y\,\mathbf{e}\,x))$ and 2) $(y)\,(y \in Y \rightarrow (\exists\, x)\,(x \in X \,\&\, x\,\mathbf{e}\,y))$.

That Df.21 does indeed define an equivalence relation between the *classes* of S (and thus *on* the set P(S)) can be trivially verified. Df.21 poses as equivalent two classes which draw their members only from the same $\mathbf{e}$-equivalence classes.

Th.19. — For any two subsets X and Y of S, $X\,\mathbf{E}\,Y \rightarrow \mathbf{K}_F(X) = \mathbf{K}_F(Y)$, where F is any formal system.

Proof : Immediate from Th.18 and Df.21.

The above shows that two sets of hypotheses can have different consequences only if one of the sets has some elements which are not equivalent to any expressions of the other set. Sets of hypotheses which draw their elements from the same $\mathbf{e}$-equivalence classes have the same set of consequences.

This shows the way in which the relation of isomorphism is « too strong » to be taken as the definition of equivalence between two formal systems. Wffs which are $\mathbf{e}$-equivalent may be said to « play the same rôle » in the deductive structure of the formal system (again, it is not meant that under interpretation there can be no distinction between them). Thus, what seems to be indicated for the equivalence of two formal systems is a mapping or correspondence such that $\mathbf{e}$-equivalent elements of one system go into $\mathbf{e}$-equivalent elements of the other system and such that the consequence relation is preserved in both directions.

Such a relation will be forthcoming, but the following theorems will help to illuminate the form in which the definition will be couched.

Th.20. — Consider two formal systems F and F′. Then g, a mapping from S to S′, preserves the consequence relation from F to F′ if and only if the following schema is true : where X and Y are any subsets of S and b and d are any elements of S, $b \in \mathbf{K}_F(X) \mathbin{\&} d \mathbin{\mathbf{e}} b \mathbin{\&} Y \mathbin{\mathbf{E}} X \rightarrow g(d) \in \mathbf{K}_{F'}(g(Y))$.

Proof : If the above schema holds, then, since $b \mathbin{\mathbf{e}} b$ *and* $X \mathbin{\mathbf{E}} X$ for all b and X, then g preserves the consequence relation from F to F′.

Conversely, if g preserves the consequence relation from F to F′, then suppose that $b \in \mathbf{K}_F(X)$, that $d \mathbin{\mathbf{e}} b$, and that $Y \mathbin{\mathbf{E}} X$ for some b, d, X and Y which satisfy the hypotheses of the schema of this theorem. Now, since $d \mathbin{\mathbf{e}} b$, we have that $d \in \mathbf{K}_F(X)$. And since $Y \mathbin{\mathbf{E}} X$, we have $\mathbf{K}_F(X) = \mathbf{K}_F(Y)$ and thus $d \in \mathbf{K}_F(Y)$ which gives $g(d) \in \mathbf{K}_{F'}(g(Y))$ since by hypothesis the consequence relation is preserved from F to F′. Thus, the schema in question is seen to hold as this is the desired conclusion.

The convention concerning the use of primes to distinguish between sets which belong to F and those which belong to F′, a convention which was explicitly stated at the beginning of Section B of this Chapter, will now be extended to cover our newly-defined notions represented by « $\mathbf{e}$ » and « $\mathbf{E}$ ». Thus, without auxiliary comment, the reader will be able to know which system the relationship is said to hold.

Th.21. — Consider two formal systems F and F′ and a mapping g from S to S′. Then, the schema $b' \in \mathbf{K}_{F'}(X') \mathbin{\&} g(b) = b' \mathbin{\&} g(X) = X' \rightarrow b \in \mathbf{K}_F(X)$ is true if and only if the following schema is true : $b' \in \mathbf{K}_{F'}(X') \mathbin{\&} g(b) \mathbin{\mathbf{e}'} b' \mathbin{\&} g(X) \mathbin{\mathbf{E}'} X' \rightarrow b \in \mathbf{K}_F(X)$.

Proof : Assume the second schema. Then the first schema is a special case since $b' \mathbin{\mathbf{e}'} b'$ and $X' \mathbin{\mathbf{E}'} X'$. Thus, if the second schema is true, then so is the first.

Conversely, suppose that the first schema is true. Now, assume the hypotheses of the second schema, that is that $b' \in \mathbf{K}_{F'}(X')$ and $g(b) \mathbin{\mathbf{e}'} b'$ and $g(X) \mathbin{\mathbf{E}'} X'$. Now, from $b' \in \mathbf{K}_{F'}(X')$ we have that $\mathbf{K}_{F'}(\{b'\}) \subset \mathbf{K}_{F'}(X')$. Thus, since $g(b) \in \mathbf{K}_{F'}(\{b'\})$ by the definition of $\mathbf{e}$-equivalence, we have, by Th.19, $g(b) \in \mathbf{K}_{F'}(g(X))$ which in turn gives us that $b \in \mathbf{K}_F(X)$ by hypothesis (i. e. by the first schema which is supposed true). Thus, the first schema implies the second and our theorem is seen to hold.

*Th.*22. — F is isomorphic to F′ if and only if there exists a bijective mapping g from S onto S′ such that 1) $b \in \mathbf{K}_F(X) \,\&\, d \,\mathbf{e}\, b \,\&\, Y \,\mathbf{E}\, X \rightarrow g(d) \in \mathbf{K}_{F'}(g(Y))$, and 2) $b' \in \mathbf{K}_{F'}(X') \,\&\, g(b) \,\mathbf{e}'\, b' \,\&\, g(X) \,\mathbf{E}'\, X' \rightarrow b \in \mathbf{K}_F(X)$.

Proof : Immediate from the definition of isomorphism and the two preceeding theorems Th.20 and Th.21.

At this point we would be tempted to define the relation of equivalence between formal systems as a mapping from S to S′ such that **e**-equivalent elements go into **e**-equivalent elements, the consequence relation is preserved from F to F′, and such that one of the schemas of Th.21 holds. But what kind of mapping ? Since we are concerned only with **e**-equivalent elements, a bijection is too strong. But a surjection has the unfortunate drawback of not defining an equivalence relation (symmetry fails). We admit, then, simply a single-valued mapping *into* S′, but we add the requirement that every **e**-equivalence class of S′ be represented in g(S).

*Df.*22. — Two formal systems F and F′ are said to be *equivalent* if and only if there exists a single-valued mapping g from S into S′ such that

1) $x' \in S' \rightarrow (\exists y)\,(y' \in g(S) \,\&\, y' \,\mathbf{e}'\, x')$,
2) the consequence relation is preserved from F to F′, by the mapping g,
3) $b' \in \mathbf{K}_{F'}(X') \,\&\, g(b) = b' \,\&\, g(X) = X' \rightarrow b \in \mathbf{K}_F(X)$.

That the relation defined by Df.22, which is defined on the class of all formal systems, is reflexive and transitive is immediate. Symmetry can be seen as follows : given g from S to S′ satisfying the conditions of Df.22, we construct a mapping h from S′ to S which satisfies these conditions. First, for each expression b' of g(S), we pick some one, arbitrarily chosen member b of the set of inverse images of b'. b is the (unique) h-image of b'. Secondly, for any expression d' of S′ which is not in g(S), we consider the class of all elements $\mathbf{e}'$-equivalent to d' in S′. From this class, which must contain at least one member of g(S), we pick some one member $p' \in g(S)$. The h-image of d' and of every member of this class which is not in g(S) is the h-image of p' which is already defined by the first part of the process.

That h does indeed satisfy the requisite conditions of Df.22 is clear. Df.22.1 for h follows from the fact that only **e**-equivalent members of S have the same g-images in S′ (this fact follows from the property Df.22.3 for g). Df.22.2 for h follows from Df.22.3 for the mapping g; theorem Th.21 (applied to the property Df.22.3 of g), and from the definition of h. Finally, Df.22.3 for h follows from Df.23.2 for g, Th.20 (applied to Df.22.2 as a property of g), and the definition of h. Thus, the symmetry of the relation defined in Df.22 is guaranteed.

It is in the above that the appeal to the axiom of choice is made (see our remarks concerning method at the end of the first paragraph of this section). However, since the **e**-equivalence classes and the inverse image classes of g partition the set S (and similarly the $\mathbf{e}'$-equivalence classes the set S′) and since the sets S and S′ denumerable, then we need here only the denumerable axiom of choice.

In any case, as already indicated by remarks which we have made in the Introduction and in the last part of § 1 of this section, we will not be at all disturbed by any appeals to non-constructive mathematics as the theory here developed is an abstract mathematical one, even though the principal applications envisaged are to logical systems.

Th.23. — If two formal systems F and F′ are equivalent, then the cardinality of the equivalence classes on S, induced by the relation $\mathbf{e}$, is the same as the cardinality of the equivalence classes on S′, induced by the relation $\mathbf{e}'$.

Proof : Assume F and F′ equivalent, and let g be the mapping satisfying Df.22. Let « $S/\mathbf{e}$ » represent the set of $\mathbf{e}$-equivalence classes on S and « $S'/\mathbf{e}'$ » represent the set of $\mathbf{e}'$-equivalence classes on S′. Then we must show that there is a bijection from $S/\mathbf{e}$ to $S/\mathbf{e}'$. Consider the mapping g from S to S′, and consider two elements b and d of S such that $b\ \mathbf{e}\ d$. Then, by Df.22.2, $g(b)\ \mathbf{e}'\ g(d)$. Moreover, if, for any two elements b and d of S, $g(b)\ \mathbf{e}'\ g(d)$, then $b\ \mathbf{e}\ d$, and this by Df.22.3. From these two facts, we have immediately that, for any $\mathbf{e}$-equivalence class X of S, $g(X) \subset Y$ where Y is some *one* equivalence class of S′ under the relation $\mathbf{e}'$. Morever, by Df.22.1, every $\mathbf{e}'$-equivalence class of S′ has some members which appear in $g(S)$. Thus, for every $\mathbf{e}'$-equivalence class Y of S′ there exists an $\mathbf{e}$-equivalence class X of S such that $g(X) \subset Y$, and there is only one such class for each Y since $g(b)\ \mathbf{e}'\ g(d) \rightarrow b\ \mathbf{e}\ d$.

Thus, our bijection from $S/\mathbf{e}$ to $S/\mathbf{e}'$ is clearly established : to every $\mathbf{e}$-equivalence class X of S we correspond the unique $\mathbf{e}'$-equivalence class Y of S such that $g(X) \subset Y$. Thus, our theorem is seen to hold.

In the above exposition, there should be no confusion. Given any equivalence class X of S (by the relation $\mathbf{e}$), there are obviously many different classes of elements of S′ which contain $g(X)$ (at least in the general case). But there is only *one* $\mathbf{e}'$*-equivalence class* which contains it uniquely as we have shown.

Some mathematicians may wonder why we have not defined a quotient structure by the $\mathbf{e}$-equivalence relation and thus attempted to define equivalence between formal systems as an isomorphism of the quotient structures. The sense in which such a procedure will work and a discussion of the potentialities of such an approach will be found in Section C of this chapter of our study.

Th.24. — F is equivalent to F′ if and only if there exists a mapping g from S to S′ such that :

1) $x' \in S' \rightarrow (\exists y')(y' \in g(S)\ \&\ y'\ \mathbf{e}'\ x')$,
2) $b \in \mathbf{K}_F(X) \rightarrow g(b) \in \mathbf{K}_{F'}(g(X))$,
3) $b' \in \mathbf{K}_{F'}(X')\ \&\ g(b)\ \mathbf{e}'\ b'\ \&\ g(X)\ \mathbf{E}'\ X' \rightarrow b \in \mathbf{K}_F(X)$.

Proof : Immediate from Df.22 and Th.21.

In view of Th.20 and Th.21, we have immediately four different, logically equivalent ways of stating definition Df.22. The form Th.24 is wanted for its utility in the following :

*Th.*25. — F is equivalent to F′ if and only if there exists a mapping g from S to S′ such that :

1) $x' \in S' \rightarrow (\exists y')\,(y' \in g(S)\ \&\ y'\,\mathbf{e}'\,x')$,
2) g preserves the consequence relation from F to F′,
3) $g(x)\,\mathbf{e}'\,g(y) \rightarrow x\,\mathbf{e}\,y$, for x, $y \in S$,
4) For $x \in S$, $g(x) \in T' \rightarrow x \in T$,
5) $b' \in \delta_i'(X')\ \&\ g(b)\,\mathbf{e}\,b'\ \&\ g(X)\,\mathbf{E}'\,X' \rightarrow b \in \mathbf{K}_F(X)$.

Proof : Suppose that F is equivalent to F′. Then this is equivalent to the conditions Th.24.1-Th.24.3. Thus, Th.25.1 and Th.25.2 are immediate.

For Th.25.3, we have : $g(x)\,\mathbf{e}'\,g(y) \rightarrow g(x) \in \mathbf{K}_{F'}(\{g(y)\})$ which gives, by Th.24.3, $x \in \mathbf{K}_F(\{y\})$. Similarly, $y \in \mathbf{K}_F(\{x\})$ and thus $x\,\mathbf{e}\,y$.

For Th.25.4 : if $g(x) \in \mathbf{K}_{F'}(\Lambda)$, then, since $g(\Lambda)\,\mathbf{E}\,\Lambda$ and $g(x)\,\mathbf{e}'\,g(x)$, we have $x \in \mathbf{K}_F(\Lambda)$ by Th.24.3.

For Th.25.5 : if $b' \in \delta_i'(X)$, then $b' \in \mathbf{K}_{F'}(X')$; and the other hypotheses of Th.24.3 are satisfied. Thus, $b \in \mathbf{K}_F(X)$.

Conversely, suppose that Th.25.1-Th.25.5 hold. Then, Th.24.1 and Th.24.2 are immediate, To prove Th.24.3, we will proceed by induction.

Assume the hypotheses of the schema Th.24.3, that is, suppose that b', X', b, and X are any objects for which the conjunction of the properties $b' \in \mathbf{K}_{F'}(X')$, $g(b)\,\mathbf{e}'\,b'$, and $g(X)\,\mathbf{E}\,'X'$ holds (where g is here the mapping from S to S′ in question). We must establish that the conjunction of these properties implies that $b \in \mathbf{K}_F(X)$.

Now, $b' \in \mathbf{K}_{F'}(X') \longleftrightarrow X' \vdash_{F'} b'$. Let J be the formal deduction (in the system F′) of b' from the hypotheses X', and let J be of length p.

If $p = 1$, then $J = b'$. But b' must be justified. Since there are no prior members of the list, then either $b' \in X'$ or $b' \in A'$. If $b' \in X'$, then, since $g(X)\,\mathbf{E}\,X'$ by hypothesis, $(\exists x)\,(x \in X\ \&\ g(x)\,\mathbf{e}'\,b')$. But, by hypothesis, $g(b)\,\mathbf{e}'\,b'$ which, by transitivity of $\mathbf{e}'$, gives us $g(x)\,\mathbf{e}'\,g(b)$ which gives, by Th.25.3, $x\,\mathbf{e}\,b$. But, $x \in X$ which gives $x \in \mathbf{K}_F(X)$ and thus $b \in \mathbf{K}_F(X)$ (by the definition of $\mathbf{e}$-equivalence and by the corollary to Th.7) which is the desired conclusion.

If $b' \in A'$, then $b' \in T'$ which gives $g(b) \in T'$ since $g(b)\,\mathbf{e}'\,b'$. But $g(b) \in T' \rightarrow b \in T$ by Th.25.4 and thus $b \in \mathbf{K}_F(X)$ since $T \subset \mathbf{K}_F(X)$ for all X.

Thus, the property in question is true for $p = 1$. Suppose now that for all $k \leqslant p$, it is the case that if $X' \vdash_F b'$ where the deduction J is of length k, and $g(b)\,\mathbf{e}'\,b'$, and $g(X)\,\mathbf{E}'\,X'$, then $b \in \mathbf{K}_F(X)$. This is the hypothesis of our induction.

Now consider a deduction $J = b'_1 b'_2 \ldots b'_p b'$ of length $p + 1$ of b' from the hypotheses X'. Again, there must be a justification of b'. If $b' \in X'$ or if $b' \in A'$, then by the same reasoning as was used above in the basis of our induction, $b \in \mathbf{K}_F(X)$.

Suppose then, that b' is inferred from prior members of the list J by a primitive rule of inference δ_i'. Let Y' be a set of prior members of the list such that $b' \in \delta_i'(Y')$. Now, since every element $y' \in Y'$ is prior to b' in the list J, then every $y' \in Y'$ is the final member of a deduction of length $k \leqslant p$ from the hypotheses X'. Thus, by the hypothesis of our induction, we have that $g(y)\,\mathbf{e}'\,y' \rightarrow y \in \mathbf{K}_F(X)$ (since $g(X)\,\mathbf{E}'\,X'$). But this gives immediately the result that $g(Y)\,\mathbf{E}'\,Y' \rightarrow Y \subset \mathbf{K}_F(X)$. But, by Th.25.1, there exists at least one Y such that $g(Y)\,\mathbf{E}'\,Y'$. Thus, we have the conjunction of the properties $b' \in \delta_i'(Y')$, $g(b)\,\mathbf{e}\,b'$ and $g(Y)\,\mathbf{E}\,Y'$ which gives, by Th.25.5, $b \in \mathbf{K}_F(Y)$. But, as we have shown, it must be the case that $Y \subset \mathbf{K}_F(X)$ and thus $b \in \mathbf{K}_F(X)$ by our various general properties of the relation $\mathbf{K}_F$. Thus, our property of induction holds for $p + 1$ and is thus true for deductions of any length whatever. Thus, our theorem is true.

The following theorem shows that the relation of equivalence between formal systems, like that of isomorphism, depends only on the consequence relation from finite sets.

Th.26. — Two formal systems F and F′ are equivalent if and only if there exists a mapping g from S to S′ such that :

1) $x' \in S' \rightarrow (\exists y')\,(y' \in g(S)\ \&\ y'\,\mathbf{e}'\,x')$,
2) if $b_1, b_2, \ldots, b_n \vdash_F b$, then $g(b_1), g(b_2), \ldots, g(b_n) \vdash_{F'} g(b)$, $n \geqslant 0$, and 3) if $g(b_1), g(b_2), \ldots, g(b_n) \vdash_{F'} g(b)$, then $b_1, b_2, \ldots, b_n \vdash_F b$, $n \geqslant 0$.

Proof: If F is equivalent to F′, then, in view of Th.12, Th.26.1-Th.26.3 are automatically fulfilled.

Conversely, suppose the conditions Th.26.1-Th.26.3 to be true for two formal systems F and F′ and let g be the mapping in question. Now, Th.26.1 $\longleftrightarrow$ Th.25.1 and so Th.25.1 holds.

Now, by Th.14, Th.26.2 $\longleftrightarrow$ Th.25.2. Thus, Th.25.2 holds also.

For Th.25.3 we have the following : if $g(x)\,\mathbf{e}'\,g(y)$, then $g(x) \vdash_{F'} g(y)$ and $g(y) \vdash_{F'} g(x)$. But, by Th.26.3, $g(x) \vdash_{F'} g(y)$ gives $x \vdash_F y$. Similarly, $g(y) \vdash_{F'} g(x)$ gives $y \vdash_F x$ and thus we have $x\,\mathbf{e}\,y$.

For Th.25.4 we have : $g(x) \in T' \rightarrow\ \vdash_{F'} g(x)$ and $\vdash_{F'} g(x) \rightarrow\ \vdash_F x$ and thus $x \in T$.

Finally, for Th.25.5, we have the following: suppose that $b' \in \delta_i'(X')$ and $g(b)\,\mathbf{e}'\,b'$ and $g(X)\,\mathbf{E}\,X'$. Then let $a'_1, a'_2, \ldots, a'_n$ be a permutation of the members of X' (X' is finite since all primitive rules are of finite order). Then we have $a_1', a_2', \ldots, a_n' \vdash_{F'} b'$.

Now, $g(X)\,\mathbf{E}'\,X' \longleftrightarrow (a_i')\,(\exists x)\,(x \in X\ \&\ g(x)\,\mathbf{e}'\,a_i')\ \&\ (x)\,(x \in X \rightarrow (\exists a_i')\,(a_i'\,\mathbf{e}'\,g(x)))$. Now if, for $x, y \in X$, it is the case that $g(x)\,\mathbf{e}'\,a_i'$ and $g(y)\,\mathbf{e}'\,a_i'$, then we have that $g(x)\,\mathbf{e}'\,g(y)$ and thus $x\,\mathbf{e}\,y$ (by Th.25.3 which has already been established above on the basis of Th.26.3). Thus, there exists a set $Z \subset X$ of cardinality n and a permutation $z_1, z_2, \ldots, z_n$ of elements of Z such that $g(Z)\,\mathbf{E}'\,X'$ and such that if $g(z_i)\,\mathbf{e}'\,a_k$ and if $g(z_j)\,\mathbf{e}'\,a_k$, then $z_j = z_i$.

Now, $g(Z)\,\mathbf{E}'\,X'$ and $b' \in \mathbf{K}_{F'}(X')$ which gives $g(b) \in \mathbf{K}_{F'}(g(Z))$ which means the same as $g(z_1), g(z_2), \ldots, g(z_n) \vdash_{F'} g(b)$ which, by Th.26.3, gives $z_1, z_2, \ldots, z_n \vdash_F b$ and thus $b \in \mathbf{K}_F(Z)$. But, $Z \subset X$ and thus $b \in \mathbf{K}_F(X)$.

Thus, our theorem is established.

The conditions Th.26.1-Th.26.3 were taken as the definition of equivalence in Hatcher [1]. Thus, Th.26 justifies that definition in terms of the general theory which has been presented in this study.

None of the conditions for equivalence so far developed would be very easy to apply as a practical test for the equivalence between two formal systems. The following theorem establishes a sufficient condition for equivalence which constitutes a somewhat more practical test.

Th.27. — Let g be a homomorphism from F to F' such that 1) $g(x) = g(y) \rightarrow x\,\mathbf{e}\,y$, 2) $g(x) \in A' \rightarrow x \in T$, 3) if $b' \in \delta_i'(Y')$, where δ_i' is a primitive rule of F', then there exists $b \in S$ and $X \subset S$, such that $b \in \mathbf{K}_F(X)$ and such that $g(Y) = Y' \rightarrow X\,\mathbf{E}\,Y$ and $g(z) = b' \rightarrow b\,\mathbf{e}\,z$. If there exists such a homomorphism from F to F', then F and F' are equivalent.

Proof : We assume the hypotheses of Th.27. Df.22.1 and Df.22.2 are trivial and immediate.

In order to establish Df.22.3, we first demonstrate a schema. Let $b' \in \delta_i'(Y')$ and $g(Y) = Y'$ and $g(z) = b'$. Then $z\,\mathbf{e}\,b \in \mathbf{K}_F(X) = \mathbf{K}_F(Y)$, and thus $z \in \mathbf{K}_F(Y)$, where b and X are the element and the set whose existence is guaranteed by Th.27.3 (which is thus part of our hypotheses). Thus, the schema which follows is true : $b' \in \delta_i'(Y') \,\&\, g(Y) = Y' \,\&\, g(z) = b' \rightarrow z \in \mathbf{K}_F(Y)$.

Now, let us assume the hypotheses of the schema of Df.22.3, that is we assume that $b' \in \mathbf{K}_{F'}(Y')$ and $g(Y) = Y'$ and $g(z) = b'$. Now $b' \in \mathbf{K}_{F}(Y) \longleftrightarrow Y' \vdash_{F'} b'$ by Th.12. Let J be a formal deduction in F' of b' from the hypotheses Y' and let J be of length p.

If $p = 1$, then $J = b'$ which means that either $b' \in Y'$ or $b' \in A'$. If $b' \in Y'$, then $g(z) = b' \in Y' = g(Y)$ and thus $z \in Y$ which gives $z \in \mathbf{K}_F(Y)$.

If $b' \in A'$, then $g(z) = b' \in A'$ which gives $z \in T$ and thus $z \in \mathbf{K}_F(Y)$.

Thus, our property of induction holds true for $p = 1$.

Suppose, now, that for all $k \leqslant p$, if an element $b' \in S'$ is the last member of a deduction J of length k from the hypotheses Y', and if $g(z) = b'$, and if $g(Y) = Y'$, then $z \in \mathbf{K}_F(Y)$. This is the hypothesis of the induction step.

Now, consider a deduction $J = b_1', b_2', \ldots, b_p' b'$ of length $p + 1$ of b' from the hypotheses Y'. If $b' \in Y'$ or $b' \in A'$, then by the same reasoning as is found above in the basis of our induction, $z \in \mathbf{K}_F(Y)$.

Suppose, then, that b' is inferred from prior members of the list, say the set Z'. Then each member of Z' is the last member of a formal deduction of length $k \leqslant p$ from the hypotheses Y'. Thus, if $g(y) \in Z'$, then $y \in \mathbf{K}_F(Y)$ by the hypothesis of our induction step. Thus, where $g(Z) = Z'$, it is also the case that $Z \subset \mathbf{K}_F(Y)$. But g is a surjective mapping and thus there exists a set Z such that $g(Z) = Z'$. Thus, we have the conjunction of the conditions $g(z) = b'$, $g(Z) = Z'$ and $b' \in \delta_i'(Z')$ which are all true. The conjunction of these conditions gives, by the schema which we demonstrated in the first part of this proof, that $z \in \mathbf{K}_F(Z)$. But, as we have shown, $Z \subset \mathbf{K}_F(Y)$. Thus, $z \in \mathbf{K}_F(Y)$ and the property bolds true for $p + 1$. Thus, the property holds true for all p and our theorem is proved.

Later on, when we consider applications of our theory (Chapter II especially), we will have occasion to use Th.27.

Df.23. — Two calculi of signs C and C′ are said to be *deductively equivalent* or *syntactically equivalent* if and only if their resulting formal systems F and F′ are equivalent.

In later chapters of this thesis, we will consider general definitions of such notions as « consistency » and « completeness » and we will consider several theorems concerning the relationship between these concepts and the various relations between formal systems which we have defined in this section.

SECTION C

FORMAL SYSTEMS, TOPOLOGY, AND THE QUESTION OF FORMING QUOTIENT SYSTEMS

§ 0. INTRODUCTION

Our procedure of defining the equivalence between two formal systems strongly suggests the often-employed mathematical procedure of considering, for a given mathematical structure, the « quotient » structure formed with respect to an equivalence relation Q. More precisely, given a set E on which is defined a certain mathematical structure (such as an algebra or a topology, for example) we can consider an equivalence relation Q and the set E/Q whose members are the equivalence classes of Q on E. It is often possible, in such a situation, to « induce », in some natural way, a structure on E/Q of the same mathematical species as that defined on E. The new structure thus defined on E/Q is a *quotient* structure because it bears some sort of intuitively natural relationship to the structure defined on E, instead of being defined on E/Q in a completely arbitrary manner. One of the uses of quotient structures is that certain properties of the structure defined on E can be more simply expresses as properties of the quotient structure.

In the case of algebras, a definite « compatibility » requirement must be satisfied or the quotient structure is not forthcoming. In topology, the quotient structure can be defined for any equivalence relation, but certain conditions are required of the quotient structure.

There are thus, generally speaking, two basic aspects to a quotient structure defined on E/Q : 1) it is of the same mathematical species as the structure defined on E, 2) it bears some sort of intuitively natural relationship to the structure defined on E.

In § 2 of this section, we will discuss the possibility of defining a quotient structure for a formal system with respect to an arbitrary equivalence relation Q. In § 1 which follows, however, we will consider only the relation of **e**-equivalence and we will define a certain *quasi-system* of a formal system. We will see that the quasi-system satisfies the second of the two above requirements but not the first ; it bears a certain intuitively natural relation to its formal system, but it is not itself a formal system.

§ 1. THE *QUASI-SYSTEM OF A FORMAL SYSTEM*

Consider a formal system $\mathrm{F} = \langle \mathrm{S}, \mathrm{A}, \mathrm{R} \rangle$ and the relation of **e**-equivalence defined on S. Let $\mathrm{S}/\mathbf{e}$ denote the set whose members are the equivalence classes on S. Let $\mathbf{K}$ represent the consequence relation for F, as usual. We say that a subset X of S is *saturated* if X is a union of **e**-equivalence classes. The *saturate* of a subset X of S is the smallest saturated subset of S which contains X. Thus, if Y is the saturate of X, then $Y = \{y \mid (\exists x)(x \in X \,\&\, x \,\mathbf{e}\, y)\}$. A saturated subset of S, because it is composed of **e**-equivalence classes, can be considered also a subset of $\mathrm{S}/\mathbf{e}$. Thus, we define the *cannonical mapping* from S to $\mathrm{S}/\mathbf{e}$ as follows : the cannonical mapping g assigns to each $x \in \mathrm{S}$ the saturate X of $\{x\}$, where now X is considered as a subset of $\mathrm{S}/\mathbf{e}$.

Df.24. — Given a formal system F, the *quasi-system* F° of F is the ordered pair $\langle \mathrm{S}/\mathbf{e}, \mathbf{K}^\circ \rangle$ where the relation $\mathbf{K}^\circ$ is defined as follows : for $X \subset \mathrm{S}/\mathbf{e}$, $\mathbf{K}^\circ(X) = \mathbf{K}(X)$ where X is considered a subset of S and $\mathbf{K}(X)$ as a subset of $\mathrm{S}/\mathbf{e}$. More precisely stated : $\mathbf{K}^\circ(X) = \{Y \mid Y \in \mathrm{S}/\mathbf{e} \,\&\, Y \subset \mathbf{K}(X)\}$.

In view of the notation « $\mathbf{K}^\circ$ » which Df.24 introduces to represent the consequence relation of the quasi-system, care should be taken not to confuse « $\mathbf{K}^\circ$ » with « K° », the latter having been defined in Df.4. The concepts respectively associated with these two notations are, of course, quite distinct.

In our statement of Df.24, we have tacitly assumed that our two different statements of the definition of $\mathbf{K}^\circ(X)$ are, in fact, equivalent in that they define the same relation in both cases. If, of course, this were not the case, then the relation $\mathbf{K}^\circ$ would not be well-defined by Df.24. These two statements are equivalent, however, only if it is true that, where X is any saturated subset of S, $\mathbf{K}(X)$ is also saturated. We will establish that this property does hold by proving the following theorem which is even stronger :

Th.28. — Let F be any formal system and let X be any subset of S. Then $\mathbf{K}(X)$ is saturated by the relation of **e**-equivalence.

Proof : Let X be any subset of S and let Y be the saturate of $\mathbf{K}(X)$. By the definition of the saturate, $Y \,\mathbf{E}\, \mathbf{K}(X)$ which gives, by Th.19, $\mathbf{K}(Y) =$

$\mathbf{K}(\mathbf{K}(X)) = \mathbf{K}(X)$ (this last equality by Th.7). But $Y \subset \mathbf{K}(Y)$ and thus, by the above equality, $Y \subset \mathbf{K}(X)$. But Y is the saturate of $\mathbf{K}(X)$ and thus we have also $\mathbf{K}(X) \subset Y$ since every set is contained in its saturate by definition. Thus, $\mathbf{K}(X)$ is equal to its own saturate Y which means that $\mathbf{K}(X)$ is saturated. Thus, our theorem is seen to hold.

Our statement of Df.24 is now justified.

Clearly it is the above property of the relation $\mathbf{K}$ and the $\mathbf{e}$-equivalence relation which permits us to define the relation $\mathbf{K}^{\circ}$ in so natural a manner. On the other hand, it is also clear that the quasi-system F° of a formal system F is not, in general, a formal system itself. In particular, the set $\mathrm{S}/\mathbf{e}$ may not be recursive at all (and in general will not be) and we are not assured either of a *non-void* recursive subset of $\mathrm{S}/\mathbf{e}$ which can serve as axioms even if $\mathrm{S}/\mathbf{e}$ is recursive. And, more fundamentally, we are not assured of the existence of a finite set of recursive functions which serve as primitive rules of inference in the sense of I.Df.I.

Briefly said, the relation $\mathbf{K}^{\circ}$ satisfies some of the general properties of a consequence relation for a formal system such as those of Th.5, Th.6 and Th.7, but it does not satisfy the constructive properties of the consequence relation exemplified by such fundamental theorems as Th.12.

Df.25. — Given two formal systems F and F′, their respective quasi-systems F° and F'° are said to be *isomorphic* if and only if there exists a bijective mapping g from $\mathrm{S}/\mathbf{e}$ to $\mathrm{S}'/\mathbf{e}'$ such that 1) $Y \in \mathbf{K}^{\circ}_{F}(X) \rightarrow \mathrm{g}(Y) \in \mathbf{K}^{\circ}_{F'}(\mathrm{g}(X))$, where X is any subset of $\mathrm{S}/\mathbf{e}$ and Y is any element of $\mathrm{S}/\mathbf{e}$, 2) $\mathrm{g}(Y) \in \mathbf{K}^{\circ}_{F'}(\mathrm{g}(X)) \rightarrow Y \in \mathbf{K}^{\circ}_{F}(X)$, again where $X \subset \mathrm{S}/\mathbf{e}$ and $Y \in \mathrm{S}/\mathbf{e}$.

Finally, we have the following theorem :

Th.29. — Two formal systems F and F′ are equivalent if and only if their quasi-systems are isomorphic.

Proof : Assume the two formal systems F and F′ to be equivalent. Let g be a mapping from $\mathrm{S}/\mathbf{e}$ to S which assigns to each $\mathbf{e}$-equivalence class some arbitrary one of its members. Such a mapping exists by the axiom of choice. Let h be the mapping from S to S′ which satisfies the conditions of Df.22, the definition of equivalence between formal systems. Let j be the cannonical mapping from S′ to $\mathrm{S}'/\mathbf{e}'$. Now consider the mapping $jh\mathrm{g}$. It is a bijection from $\mathrm{S}/\mathbf{e}$ to $\mathrm{S}'/\mathbf{e}'$. That it satisfies the conditions of the above definition Df.25 follows immediately from the requirements embodied in Df.22, the properties of $\mathbf{e}$-equivalence, and the definition Df. 24 together with Th.28. Hence, the quasi-systems are isomorphic.

Conversely, assume that the quasi-systems F° and F'° of two formal systems F and F′ are isomorphic. Let g be the cannonical mapping from S to $\mathrm{S}/\mathbf{e}$ and let h be the bijection from $\mathrm{S}/\mathbf{e}$ to $\mathrm{S}'/\mathbf{e}'$. Let j be any mapping from $\mathrm{S}'/\mathbf{e}'$ to S′ which assigns to each $\mathbf{e}'$-equivalence class some one of its members. Then $jh\mathrm{g}$ is a mapping from S to S′ which clearly satisfies the conditions of Df.22. Thus, F and F′ are equivalent, and our theorem is seen to hold.

Our reason for not choosing Th.29 as the definition itself of the notion of equivalence is two-fold. Firstly, from a theoretical point of view, we are constrained by the fact that the quasi-system of a formal system is not itself a formal system and it thus becomes somewhat unnatural to treat the theory of formal systems by means of quasi-systems.

Secondly, and from the pragmatic viewpoint, it is clear that one of the utilities of our notion of equivalence lies precisely in the fact that it can be reduced to a conjunction of more or less finitistic conditions such as found in Th.25 and this because of the constructive nature of the consequence relation $\mathbf{K}$. Thus, the quasi-system, for which the consequence relation lacks this constructive character (in the above sense of the word), is not so directly amenable to practical considerations.

On the other hand, Th.29 shows that we could have reconstructed all that we have found since the isomorphism of the quasi-system is equivalent to our definition. Thus, in the end, the choice is simply one of presentation and approach and not a choice between two different theories, and we will not hesitate to use Th.29 as a characterization of the notion of equivalence whenever it is useful to us.

§ 2. TOPOLOGY AND THE CONSEQUENCE RELATION FOR FORMAL SYSTEMS

Although it is out of the main line of interest of this study, it might be of interest to devote a brief paragraph to an examination of the question of defining a quotient system with respect to an *arbitrary* equivalence relation Q on the set S of wffs of a formal system. This discussion will also afford us a look at the relationship between topology and the consequence relation, and indeed this comparison will be our very manner of proceeding.

In the foregoing theory (see Section A, § 2 of this chapter) we have already mentioned the analogy between certain properties of the relation $\mathbf{K}$ for formal systems and the closure relation for topologies. More precisely, given a formal system F, the relation $\mathbf{K}$ has all of the properties of a topological closure with the following exceptions : 1) in topology, $\overline{\Lambda} = \Lambda$ whereas for formal systems, $\mathbf{K}(\Lambda) = T$ and T is never void, 2) in topology, $\overline{X} \cup \overline{Y} = \overline{X \cup Y}$ whereas for formal systems $\mathbf{K}(X) \cup \mathbf{K}(Y) \subset \mathbf{K}(X \cup Y)$ but not conversely. If we consider a topology defined by closed sets and if we relax the usual conditions so that the closure relation satisfies the weaker of each of the two pairs of conditions above, the resulting structure is called a *Moore family* (see Dubreil and Dubreil-Jacotin [1], p. 10).

Now, with Moore families, as with topologies, one can define the closure relation, which is unique for any given set of sets constituting a Moore family, and one can define such concepts as *open set, closed set, open mapping, closed mapping, continuous mapping*, etc. in analogy with the usual way of defining such concepts for a topological space. It is then possible to define a quotient Moore family for any Moore family for which an arbitrary equivalence relation is given on the space. Dubreil indicates some of this development in the work cited above. We have, in an unpublished paper, carried out this development for Moore families in detail and we have been unable to find a reference in which this detailed development has been published. Bourbaki [4], pp. 15-85, indicates such a development for topologies and the corresponding development for Moore families follows this by analogy (with, of course, certain limitations and changes) and the reader is referred to this latter work.

Now if, given a formal system F, we consider the pair (S, **K**), we see that this pair is a Moore family which is subject to certain further conditions, namely that the closure relation **K** has a certain constructive nature characterized by such theorems as Th.12. We could, thus, consider a formal system as a special case of a Moore family and a Moore family as a non-constructive formal system.

Similarly, given an equivalence relation Q on S, the quotient Moore family, which is always defined, could be considered as the *quasi-system relative to* Q of F. *The* quasi-system, defined in § 1 of this section, would thus be the quasi-system relative to **e**-equivalence. The notion of quasi-system would thus be defined relative to any equivalence relation on S and the quasi-system relative to **e**-equivalence would appear as a particular one of these.

From this brief discussion, we see clearly the relationship between topology and a formal system and in particular, the relative simplicity of the structure of a Moore family illustrates the significance of the constructive requirements which characterize the structure of a formal system and differentiate it from Moore families in general.

SECTION D

THE RELATIVE FORCE OF VARIOUS EQUIVALENCE RELATIONS DEFINED BETWEEN FORMAL SYSTEMS

In the foregoing sections of this study, we have formally or informally considered a number of different relationships between formal systems. A large number of these relationships can be construed as equivalence relations and thus could have been chosen as our definition of the equivalence of two formal systems. Our purpose in this section is to consider certain of the more plausible of these alternatives and to explain, by example and counterexample, the reason for our choice.

§ 1. EQUIVALENCE AND ISOMORPHISM

Clearly the most natural alternative to our definition of equivalence is the concept of isomorphism. We have already seen, from a theoretical point of view, the reason for not choosing isomorphism as the definition of equivalence. It is, simply stated, a result of the realization that **e**-equivalent wffs play the same deductive rôle in a formal system.

When we regard the realm of applications, we have an even clearer realization of the fact that our definition is a much more accurate representation of the intuitive notion of equivalence than is the notion of isomorphism. Briefly stated, we see that certain systems which we intuitively regard as equivalent are not isomorphic, but are equivalent in our sense of the term. Chapter II of this study contains several such examples and we will not go into detail here. But we will mention one interesting example which is worthy of note.

It was Russell who remarked that there was a certain lack of elegance in the predicate calculus due to the fact that the assertion of a formula with free variables is always equivalent to the assertion of the universal closure of the same formula (see Whitehead and Russell [1], the introduction to the second edition). Because of the equivalence of these two forms of assertion, Russell's (and other authors') formulations of the predicate calculus usually involve a primitive rule of universal generalization which permits us, in effect, to infer, from a given formula, its universal closure. Now it was Quine [1] which showed how to avoid this formulation of the predicate calculus. In Quine's formulation only closed formulae are well-formed. Other formulae, called *matrices* by Quine, are not well-formed, though their universal closures are. Quine's system has only the rule of *Modus Ponens*, no rule of generalization being necessary (the other substitution rules are avoided, as is often done, by the use of schemata and syntactical variables).

Now, intuitively speaking, the Quine formulation should be equivalent to the Russell formulation as this is the very purpose of the Quine formulation. But is it equivalent according to our definition ? The answer is « yes » and the correspondence g which satisfies the requirements of Th.27 is easily forthcoming. Taking Russell's set of wffs (let us imagine that we are using Quine's notation), we let every closed formula correspond to itself. Every open formula corresponds to its universal closure where we now stipulate that we add universal quantifiers in the names of the free variables as they occur in *alphabetic order*. This stipulation is necessary so that our function g be single-valued. Our function g is now well-defined for every wff of Russell's system. Moreover, g is a surjection since Quine's set of wffs is a proper subset of Russell's, all of Quine's wffs are closed, and all closed wffs of Russell correspond to themselves. We are thus assured that every wff of Quine's system is the image of some Russell wff under g. That the inverse images of an element are **e**-equivalent among themselves follows immediately from the primitive rule of generalization which holds in the Russell formulation ! It is evident, from the definition of the function g, that the other conditions of Th.27 are satisfied. Thus, the two formulations are equivalent.

Now, are these two systems isomorphic ? Though a theorem of impossibility of isomorphism is not easily forthcoming, it is immediately clear that one would have great difficulty in establishing a *bijection* from one set of wffs to the other which would preserve the consequence relation in both directions. Moreover, the analogy between our syntactical considerations (such as our use of **e**-equivalence and the hypotheses of Th.27, etc.) on the one hand and the semantical considerations which led to Quine's formulation, on the other hand, is remarkably close. This example alone would almost suffice to justify our choice of definition as opposed to the concept of isomorphism, as the definition of the equivalence of formal systems.

§ 2. EQUIVALENCE AND THE NOTION OF ONE SYSTEM HAVING A MODEL IN ANOTHER

I.Df.18 above defines the notion « F has a model in F′ ». We might be tempted, in the light of this, to define equivalence of formal system by the concept « each system has a model in the other ». It might seem, at superficial glance, that this definition would be stronger than our definition of equivalence. This is not so and we will now construct a counter-example of two systems, each of which has a model in the other, but such that the two are not equivalent.

Consider two systems F and F′ defined in the following manner. $S = S' =$ the set of positive integers. $A = A' = \{2\}$. R, the set of rules of F, contains one rule of order one which can be defined as follows: From $\{1\}$ we can infer $\{3\}$ and, for all $n \geqslant 3$, from $\{n\}$ we can infer $\{n+1\}$. It is clear that the consequence relation $\mathbf{K}_F$ can be described as follows: given any subset X of S, consider the smallest member y of X *other than* 2. Then $\mathbf{K}_F(X) = \{n \mid n \geqslant y\} \cup \{2\}$. R′, the set of rules of F′, contains the one following rule: for all $n \geqslant 3$, from $\{n\}$ we may infer $\{n+1\}$. Let X' be any subset of S′ and let y' be the smallest element of X' which is neither 1 nor 2. Then, $\mathbf{K}_{F'}(X') = \{n \mid n \geqslant y'\} \cup \{2\}$.

Now each of these two systems has a model in the other and we will describe the two mappings which shows this to be true.

Consider the mapping g from S into S′ defined as follows: $g(1) = 3$, $g(2) = 2$ and, for all $n > 2$, $g(n) = n+1$. This is clearly an injection from S to S′. Moreover, it is clear that the consequence relationship is preserved from F to F′ by g as can be seen by regarding the structure of the consequence relation for each of the two systems. Thus, F has a model in F′.

Conversely, consider the mapping g' from S′ to S which is defined as follows: for all n, $g'(n) = n$. Again the consequence relationship is preserved and we clearly have an injection. Thus, F′ has a model in F. Thus, each of the two systems has a model in the other. But the two systems are *not* equivalent as we will now prove.

First consider the quasi-system F^* of F. It is clear from the definition of F that no two distinct elements of S are **e**-equivalent. Thus, in this case, the quasi-system F^* of F is the system F itself. Now the same is clearly true of F′. Now, as we have seen, the equivalence of two systems is the isomorphism of their quasi-systems which, in this case, means the isomorphism of the systems themselves. We will not show that isomorphism is impossible.

Let h be any bijection from S to S'. Then, there is an $n \in S$ such that $h(n) = 1$. Now this n cannot be 2 since $g(2) = 2$ (because the theorems of F must go into the theorems of F' if h is to preserve the consequence relation from F to F' and the axioms of the two systems are the only theorems), or else h does not preserve the consequence relation from F to F'. Now, for every $n \neq 2$, there exists an $m \neq 2$ such that m is distinct from n and such that $m \in \mathbf{K}_F(\{n\})$. Now, is it the case that $h(m) \in \mathbf{K}_{F'}(\{h(n)\})$? Since $h(n) = 1$, then $\mathbf{K}_{F'}(\{h(n)\}) = \mathbf{K}_{F'}(\{1\}) = \{1, 2\}$. Now $m \neq n$ and so $h(m) \neq 1$ since h is an injection. But $m \neq 2$ and so $h(m) \neq h(2) = 2$. Thus, $h(m)$ is not an element of $\{1, 2\} = \mathbf{K}_{F'}(\{h(n)\})$ and the consequence relation is not preserved. But h was any bijection from S to S'. Thus, no bijection from S to S' preserves the consequence relation from F to F' and the two systems are not isomorphic. Thus they are not, in this case, equivalent.

The above counterexample shows that the concept « each system has a model in the other » is not stronger than our definition of equivalence. It is true, of course, that equivalence is not strictly stronger than this concept either, since the injection in both directions requires that S and S' have the same cardinality, whereas equivalence requires only that the respective quasi-systems have the same cardinality. However, whenever it is the case that the quasi-system is the same as the system itself for each of the two systems involved, then equivalence is strictly stronger than the property of each system having a model in the other.

In any case, the general theory of the foregoing sections illustrates amply our justification for choosing our definition and the counterexample just considered suffices to exclude the alternative here considered as a reasonable choice for the equivalence of formal systems since the two systems F and F' above are not equivalent either intuitively or, as we have just proved, according to our syntactical definition.

§ 3. EQUIVALENCE AND DOUBLE HOMOMORPHISM

A third alternative to our definition might seem plausible when one regards the definition of homomorphism. Especially in the light of Th.27, one might feel inclined to define equivalence of two formal systems by the definition that each is a homomorphic image of the other. Again we have a counterexample to show that this concept is not stronger than the definition of equivalence (and thus not as strong as the hypotheses of Th.27).

Consider the two systems F and F' defined as follows. $S = S' =$ the set of non-negative integers. $A = A' = \{0\}$. R consists of one rule of order one which is the following : from $\{1\}$ we can infer $\{2\}$ and from $\{3\}$ we can infer $\{4\}$. R' consists of one rule of order one which is as follows : from $\{1\}$ we can infer $\{2\}$.

We must now that each of these two systems is the homomorphic image of the other. The homomorphism g' from F' to F is simply the identity mapping. The homomorphism from F to F' is the mapping g defined as follows : $g(0) = 0$, $g(1) = 1$, $g(2) = 2$, $g(3) = 1$, $g(4) = 2$, and for all $n > 4$, $g(n) = n—2$. Thus, we see that each of these systems is a homomorphic image of the other.

But are these systems equivalent ? The answer is « no ». As in the preceeding paragraph, the respective quasi-systems of F and F' are the systems F and F' themselves. And it is again clear that there is no isomorphism between the two systems. Thus, we see that the two systems are not equivalent. Moreover, from an intuitive point of view, it is clear that the systems do not have the same deductive structure.

Thus, we see that this concept of « double homomorphism » is not stronger than our definition of equivalence and that it admits as equivalent certain intuitively unequivalent systems.

Again, our definition of equivalence is not stronger than the double homomorphism because the double surjection also requires that S and S' have the same cardinality. But it is clear that our definition is much closer to the intuitive notion than that of double homomorphism as the above counterexample shows.

As with the notion of the preceeding paragraph, if the quasi-systems of the two systems are equal to the systems themselves, then the notion of equivalence is strictly stronger then that of double homomorphism.

This completes our consideration of particular alternatives to our definition and counterexamples concerning such. Extended consideration of other possible cases seems unnecessary, especially in view of the fact that Chapter II is entirely devoted to applications of the theory of this chapter.

SECTION E

THE DEDUCTION THEOREM FOR FORMAL SYSTEMS

Df.26. — Given a formal system F, we say that *the deduction theorem is true for* F if there exists a recursive subset Z of S and a recursive mapping g from $S \times S$ *onto* Z such that 1) for every element $z \in Z$, the set $\{\langle x, y\rangle \mid g(\langle x, y\rangle) = z\}$ is a recursive subset of $S \times S$ ($S \times S$ is a recursive set since S is also), 2)for every ordered pair $\langle x, y\rangle$ of wffs of F, $x \vdash_F y$ if and only if $\vdash_F g(\langle x, y\rangle)$.

Df.27. — Given a calculus of signs C, we say that *the deduction theorem is true for* C if there exists an element k of the alphabet L of C such that 1) for every ordered pair $\langle x, y\rangle$ of wffs of C, the finite sequence of elements of L represented by xky is also a wff of C.[5] 2) for

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[5] Here and in the following we use the notation of Df.10 and Df.11 to express the concatenation of finite sequences of objects, the objects in question being drawn from any definite set which may be designated (here the set L).

every ordered pair $\langle x, y\rangle$ of wffs of C, $x \vdash_F y$ if and only if $\vdash_F xky$ where F is the resulting formal system of the calculus C.

We have immediately the following theorem :

Th.30. — If the deduction theorem is true for a calculus of signs C, then it is true for the resulting formal system F of C.

Proof : Assume that the deduction theorem is true in a calculus C and let k be the sign of C which satisfies the requirements of Df.27. Then the mapping g which satisfies Df.26 is the mapping which assigns to each ordered pair $\langle x, y\rangle$ of elements of S the wff xky. This mapping satisfies Df.26.2 and is clearly recursive. Moreover, the set Z which is the range of the function g is a recursive subset of S as easily follows from the way in which the set Z is constructed.

Finally, to see that Df.26.1 holds, we observe that the sign k only occurs a finite number of times in any wff z. Thus, we can test successively each separate occurrence of k in z to see if the sequence x which precedes k and the sequence y which follows k are both wffs of C. If they are, then $g(\langle x, y\rangle) = z$ by the definition of g. If not, then $\langle x,y\rangle$ is not a pair of elements of S and is thus not in the domain of g. Because S is a recursive set, this test is effective for each occurrence of k. Thus, by applying the test to each occurrence of k, we can effectively calculate the (finite) set of all ordered pairs of wffs whose g-image is the wff z.

Thus, all of the conditions of Df.26 are satisfied by our mapping g and our theorem is seen to hold.

The following theorem is, perhaps, somewhat less obvious :

Th.31. — If the deduction theorem is true for a formal system F, then there exists a calculus of signs C′ such that the deduction theorem is true for C′ and such that the resulting formal system F′ of C′ is equivalent to F.

Proof : Given a formal system $F = \langle S, A, R\rangle$ for which the deduction theorem is true, let g be the mapping which satisfies the conditions of Df.26. Now, consider the calculus of signs C which is constructed as follows : 1) the alphabet L′ of C′ is the set $S \cup \{k\}$ where k is any object which is not in the set S, 2) the set S′ of wffs of C′ is defined as follows : i) every element of S is an element of S′, ii) if x and y are members of S′, then the finite sequence of members of L′ represented by xky is also a member of S′, iii) S′ has no other members. 3) The axioms A′ of C′ are the following : i) every element of A is also an element of A′, ii) all members of S′ which have the form xky where not both x and y are elements of S, iii) A′ contains no other members. 4) The set R′ of rules of C′ is as follows : i) every rule of F is also a rule of C′, ii) we add the following rule Q : for all $x, y \in S$, from xky we can infer $g(\langle x, y\rangle)$ and from $g(\langle x, y\rangle)$ we can infer xky. iii) R′ contains no other members.

That the various constructive requirements for a primitive rule are satisfied by our new rule Q follows from the various constructive conditions of Df.26 which are imposed on the mapping g. If these conditions were not part of Df.26, Q would not be a primitive rule.

Now, we will show that the resulting formal system F′ of C′ is equivalent to F and we will use Th.27 to establish this result. We now define a mapping h from S′ to S and we will show that this mapping satisfies the hypotheses of Th.27. a) For all members z of S′ which are also members of S, $h(z) = z$. b) For all members z of S′ which are of the form xky where x and y are both members of S, $h(xky) = g(\langle x, y\rangle)$. c) By definition of S′, all members of S′ which do not have h-images under a) and b) must be of the form xky where not both x and y are elements of S. But, by our definition of A′, all such elements are axioms of C′. Thus, we pick some one axiom of F and we map all of these remaining wffs of F′ onto this one axiom of F. The mapping h is now defined for every member of the set S′.

We must now show that h does indeed satisfy the requirements of Th.27. First, h is surjective as is trivially clear. Secondly, we wish to show that only **e**′-equivalent wffs have the same h-image. Under a) of the definition of h, distinct elements have distinct images. Under b), however, we have that $x, y \in \mathrm{S} \rightarrow h(xky) = g(\langle x, y\rangle)$. But $g(\langle x, y\rangle)$ is in S and so we have also that $h(g(\langle x, y\rangle)) = g(\langle x, y\rangle)$. Thus, we must show that $x, y \in \mathrm{S} \rightarrow xky\ \mathbf{e}'\ g(\langle x, y\rangle)$. But this follows immediately by our primitive rule Q. Finally, under c), we have mapped axioms of F′ onto an axiom of F. Since axioms are always **e**′-equivalent among themselves, then again we have that only **e**′-equivalent wffs of F′ have identical h-images.

That the other hypotheses of Th.27 are satisfied follows immediately from the way in which we have constructed the system F′ and the mapping h. In particular, we remark that the mapping h is the identity when it is restricted to the subset S of S′ and that the rules of inference of F′ apply only to members of S except for the rule Q whose effect has already been clearly accounted for.

Thus, our theorem is true.

From the above theorem, we see that the notion of having a theorem of deduction is essentially the same whether it be taken with respect to a formal system or a calculus of signs. Indeed, it has been one of the purposes of our study to show that, in the literature, it is with formal systems in our sense that one deals essentially for syntactical considerations and that the primitive signs of a calculus play a rôle mainly when we consider interpretations of formal systems. The deduction theorem, as it is usually conceived, is one case in which one might have contended that an appeal to the primitive signs was inescapable. This section has the virtue of showing that this is not so.

Finally, one naturally poses the question of whether or not the property of having a deduction theorem is a property which is preserved by equivalence, that is, whether, if the deduction theorem is true in F and if F is equivalent to F′, then the deduction theorem is true in F′. The answer in the general case is « no » and this because of the constructive requirements in Df.26. However, the following statement is easily verified :

Suppose that the deduction theorem is true for F and F is equivalent to F′. Then there exists a mapping g' from $S' \times S'$ to S′ which satisfies Df.26.2. Whether or not the function g' will satisfy the other requirements of Df.26 obviously depends on the particular systems involved.

A final word concerning terminology is perhaps worthwhile. In the case of Df.26, we will permit ourselves to speak of « a deduction theorem relative to the mapping g », and in the case of Df.27 we will speak of « a deduction theorem relative to the sign k ». The reason for this convention is obvious. Df.26 (respectively, Df.27) requires the existence of a mapping (sign) satisfying certain conditions, but there might well be more than one such mapping (sign) for any given formal system F (calculus of signs C). Thus, if there is at least one such mapping (sign) we say that the deduction theorem is true for F (for C) and we allow ourselves to indicate which mapping (sign) we are thinking of by speaking of the « deduction theorem relative to » the mapping (sign) in question.

This terminology should lead to no confusion.

APPENDIX TO CHAPTER I

ON THE DEFINITION OF THE PRIMITIVE RULES OF INFERENCE AND THE QUESTION OF ORDER IN INFERENCE

It is difficult to say, from a strictly intuitive point of view, whether the order of occurrence of the premisses of an instance of inference affects (or should affect) the conclusion that one is allowed to draw from the premisses in question. The commutativity and associativity of the conjunction in the classic calculus of propositions assures that the question of the order of a set of premisses is irrelevant in that theory. On the other hand, the classic theory of the syllogism clearly considers as different certain inferences in which the same conclusion is drawn from the same premisses but in which the premisses occur in a different order. And it is clear that we can construct a situation in which a rule, say a rule Q of order 2, for example, allows us to infer only c from the premisses a, b and only d from the premisses b, a and it is the case that $c \neq d$.

However, as soon as we consider the case of formal systems, there is no ambiguity. For the definition of formal deduction or proof in a formal system renders irrelevant considerations of the order of occurrence in a formal deduction of the premisses of an instance of primitive (and thus of general) inference. We can, in fact, prove a rule of « repetition » as a derived rule (metatheorem) in any formal system.

As a way of illustrating and precising this, let us take as example the rule Q of order 2 mentioned above. Suppose that, in some formal deduction $b_1 b_2 \ldots b_m$, we have $b_i = a$ and $b_j = b$ and $i < j$. Now we can immediately infer c since the premisses occur in the order a, b. But what about d? Since a has a justification at line i of the deduction, it has the same justification at any later line. This fact follows from the definition of formal deduction. Thus, we can « repeat » a and we have the deduction $b_1 b_2 \ldots b_m b_{m+1}$ where $b_{m+1} = a$. Now, $j < m+1$ and we now have the order b, a and can thus infer d. Thus, because we can always repeat a or b, it is clear that whatever can be inferred from a and b in a certain order of initial occurrence can be inferred just as well if the initial occurrence is in any order whatever.

Obviously a rigorous proof of the above for rules of general order would be by induction, but the example clearly illustrates the point in question and indicates the manner of proof.

For deduction in a formal system, then, we are not essentially concerned with the *order* of occurrence in a formal deduction of the members of a set X of premisses but rather with the *fact* of all the members of X having occurred at some particular point in the deduction (this is not to be confused with the fact that a formal deduction is, itself, an *ordered list*).

Consider, then, the definitions Df.1 and Df.2 of this chapter and let us suppose that we keep them the same except that now we define the set R of rules as being a finite set of (not necessarily symmetrical) recursive relations (each relation being of degree greater than 1) over the set S of wffs of F (or of C). Now, given a set of premisses $X \subset S$, X of cardinality $n - 1$, and given an n-placed recursive relation Q_n over S, we can always consider « the set Y of all expressions which can be immediately inferred from X by the rule Q_n » as the set of all wffs p which bear the relation Q_n to some ordered $(n-1)$-tuple of (not necessarily distinct) elements of X. But since X is finite and the relation Q_n is recursive, this set Y is recursive. Thus, for every rule considered as a recursive relation Q_n, we have a *uniquely* associated recursive function δ_i of the type described in Df.1.4 which is, in the sense of Df.1.4 a rule of order $n - 1$. (That this function δ_i does indeed satisfy all of the requirements of Df.1.4 is easily seen from the way in which we have constructed δ_i from Q_n and from the fact that Q_n is a recursive relation).

If we are given, on the other hand, a function of the type Df.1.4, a rule of order n in the sense of that definition, there is more than one recursive relation Q_{n+1} which will yield δ_i by the above process. Thus, for each δ_i of order n, we must consider the *set* of all recursive relations with which δ_i is uniquely associated by the above process.

However, in view of our above considerations on the question of order, it is clear that all of the relations which yield the same δ_i by our process have the same deductive power. (For two relations Q and M to have the same deductive power means that they each yield the same relation K°, i. e., that anything which can be inferred from any set X of premisses by Q can also be inferred by M and conversely). Thus, if we identify two relations over S which yield the same δ_i, then we have an equivalence relation on the set of all recursive relations over S, two relations being equivalent when and only when they have the same deductive power (sufficiency has been proved, necessity clearly bolds).

Our choice by which we identify primitive inference with the functions δ_i of the type of Df.1.4 is thus a choice to work with the equivalence classes and not with the relations directly since the result is the same. We have, thereby, avoided a step in our construction of the consequence relation and thus simplified somewhat the proofs of certain theorems (though the essentials are no different in any case).

There is, however, the question of elegance, and the definition of primitive rules as recursive relations is clearly more aesthetic than is I.Df.1.4. We can consider, if we please, that we have so defined the rules and that we have constructed, for each rule, the associated function δ_i in the above indicated manner. From then on, our theory rests the same (with some minor details being different).

In any case, we are always aware that a structure such as Df.2, but with rules defined as recursive relations, constitutes a formal system, and we feel *no restraint in specifying a system in this* manner *if it* suits our purposes.

CHAPTER II

SOME APPLICATIONS AND USES OF THE SYNTACTICAL DEFINITION OF EQUIVALENCE BETWEEN FORMAL SYSTEMS

The way in which I.Df.22 expresses syntactically the intuitive idea of the equivalence of formal systems should be fairly clear from the exposition of Chapter I. However, it is perhaps worth while to take several specific examples by way of illustration. We have chosen, for the Section A of this chapter, the consideration of the equivalence of several different formulations of the propositional calculus, since there are so many different formulations which have appeared over the years most of which are considered intuitively as « equivalent » (see Kleene [5], p. 140, for example).

SECTION A

THE EQUIVALENCE OF VARIOUS FORMULATIONS OF THE PROPOSITIONAL CALCULUS

§ 0. PRELIMINARY DISCUSSION

In our dealings with specific formal systems, we will often use a device known as *abbreviative definition*. When we use the word « definition » it will always be understood that we are speaking of abbreviative definition.

Abbreviative definition enables us to talk about a formal system more easily by using metalinguistic *names* or abbreviations for certain expressions of the formal system. The names are abbreviative in the sense that they will often be shorter or in some way easier to use than other names such as the expressions themselves placed in quotations. In stating definitions, we will often use definition schemata which give abbreviations for each member of an infinite set of expressions (see Church [4], pp. 74-81).

§ 1. THE SYSTEM P_1

The system P_1 is the resulting formal system of the calculus C_1 which we will now describe.

(i) L_1 consists of the following signs :

a) « $\sim$ », « $\vee$ », « (», «) », and « * » which is called a *star.*

b) small italic letters of the latin alphabet which may be followed by any finite number of occurrences of the star. Such a letter followed by any finite number of occurrences of the star is called a *variable.*

c) L_1 has no other members.

(ii) S_1 is defined as follows :

a) Any variable is in S_1.

b) If X is any member of S_1, then $(\sim X)$ is also a member of S_1.

c) If X and Y are any two members of S_1, then $(X \vee Y)$ is also a member of S_1.

d) S_1 has no other members except by the above.

When speaking of a calculus C, we will consistently employ the convention whereby the capital latin letters V, W, X, Y, and Z, starred or unstarred, will be variables whose domain is the set of all words (finite sequences of members) of L. The other capital latin letters will be used for our various other purposes, including their use as variables whose domain is the set of wffs S of C (here the term « variable » has its usual mathematical significance. These are the so-called « syntactical variables » of modern syntactics. There should be no confusion between this general use of the term and the technical use which we make in connection with the calculus C_1 as found in i.b above.)

In order to render less complicated our designation of certain wffs of P_1, we make the following definitions :

Df.1. — $(B \,\&\, D)$ for $(\sim((\sim B) \vee (\sim D)))$.

Df.2. — $(B \supset D)$ for $((\sim B) \vee D)$.

Df.3. — $(B \equiv D)$ for $((B \supset D) \,\&\, (D \supset B))$;

where B and D represent any arbitrary members of S_1.

The above are examples of the use of definition schemata of which we have already spoken. Expanded in full, a schema such as Df.2, to take an instance, means : When any expressions of S_1 are substituted for the (syntactical) variables « B » and « D » of the forms « $(B \supset D)$ » and « $((\sim B) \vee D)$ », the (metalinguistic) expression resulting from the form « $(B \supset D)$ » (which we will call an *instance* of the form in question) will be used as a name (abbreviation) of the corresponding instance of the form « $((\sim B) \vee D)$ ».

(iii) A_1 is an infinite recursive set of axioms which we will now designate.

Where B, D, and H represent any members of S_1 whatever, those wffs which are named by the instances of the following forms are axioms :

a) $((B \vee B) \supset B)$

b) $(B \supset (D \vee B))$

c) $((B \vee D) \supset (D \vee B))$

d) $((B \supset H) \supset ((D \vee B) \supset (D \vee H)))$

e) A_1 contains no other members.

(iv) The set R_1 consists of one rule of order 2 which can be stated as follows :

Where B and D represent any arbitrary expressions of S_1, then from $((\sim B) \vee D)$ and B we can infer D.

In the following, we will abuse our language and refer to the above rule as « the rule R_1 » instead of « the member of the set R_1 ».

The system P_1, which is the resulting formal system of the above calculus C_1, is the Whitehead-Russell-Bernays system (see Whitehead and Russell [1], and Bernays [1]). We have used the method of Von Neumann [1] of designating an infinite recursive axiom set and have thus avoided the use of a rule of substitution.

§ 2. THE SYSTEM P_2

P_2 is the resulting formal system of the calculus C_2 which we will now describe.

(i) L_2 has the following members :

a) The signs « | », « (», «) », and « ° » which is called a *star*.

b) Same as i.b of C_1.

c) Same as i.c of C_1.

(ii) S_2 can be described as follows :

a) Any variable is in S_2.

b) If X and Y are in S_2, then $(X \mid Y)$ is in S_2.

c) S_2 contains no other members except by the above.

We now introduce the following definitions for the calculus C_2 :

Df.1. — $(\sim B)$ for $(B \mid B)$.

Df.2. — $(B \vee D)$ for $((\sim B) \mid (\sim D))$.

Df.3. — $(B \supset D)$ for $((\sim B) \vee D)$.

Df.4. — $(B \,\&\, D)$ for $(\sim((\sim B) \vee (\sim D)))$.

Df.5. — $(B \equiv D)$ for $((B \supset D) \,\&\, (D \supset B))$.

(iii) A_2 is an infinite recursive set. Where B, D, H and N represent any members of S_2 whatever, those wffs which are instances of the following form are axioms :

a) $((B \mid (D \mid H)) \mid ((B \mid (H \mid B)) \mid ((N \mid D) \mid ((B \mid N) \mid (B \mid N)))))$

b) A_2 contains no other elements.

(iv) The set R_2 consists of the one rule of order 2 which can be stated as follows :

Where B and D and H represent any arbitrary expressions of P_2, then from $(B \mid (D \mid H))$ and B we can infer H.

As in the case of C_1, we will abuse our language and speak of the rule R_2.

The system P_2 which is the resulting formal system of the calculus C_2 is a formulation of the propositional calculus using a single primitive connective, the functor of incompatibility of Sheffer (see Sheffer [1] and Peirce [1], 4.12 and 4.264). P_2 uses the single axiom of Lukasiewicz which is to be found in the article of Wajsberg [1], and it uses Nicod's strong rule of inference (see Nicod [1]).

§ 3. THE EQUIVALENCE OF THE SYSTEMS P_1 AND P_2.

From now on, we will assume, for each of the systems P_1 and P_2, many familiar metatheorems which could be reproduced here, but whose proof can be found in many standard works. Kleene [5] and Dopp [1] contain several important ones including most of those we will use, and we will not feel bound to give a separate reference for every metatheorem that we use.

Also, we will henceforth feel free, in our metalinguistic discussions of these systems, to drop parentheses from wffs and to use dots in the familiar manner, guarding always against ambiguity. We will not bother to make explicit here precise rules governing the usage of these conventions. We will also continue to use parentheses informally in the vernacular, as we have done in the foregoing, to indicate the application of a function to its argument, etc. Thus, when we have a function whose arguments are wffs of P_1 or P_2, there is a slight equivocation in the use of parentheses. It is clear that this abuse of language should cause no confusion.

To facilitate our discussion about our systems, we give the following vernacular names to some of the signs. The term « negation sign » will designate « $\sim$ ». The term « disjunction sign » will designate « $\vee$ » ; « conjunction sign » will designate « & » ; « conditional sign » will designate « $\supset$ » ; « equivalence sign ; will designate « $\equiv$ » ; « stroke sign » will designate « $|$ ».

We will now establish a mapping g from S_1 to S_2 and then show that this mapping has the properties required by the hypotheses of I.Th.27, thus proving the equivalence of P_1 and P_2.

Bearing in mind the recursive definition of S_1 and S_2, we will give a recursive definition of the mapping g. Every wff H of P_1 has one of three forms. I) If H is a variable, then $g(H) = H$. 2) If H is of the form $(\sim B)$, then H is of the form $(\sim(\sim(\ldots(\sim D)\ldots)))$ where D does not begin with « $(\sim$ » (in other words, D is either a variable or D is of the form $(J \vee N)$), and where the number n of the occurrences of « $\sim$ » which precede D is finite, greater than or equal to 1. Given H, D is the largest wff contained in H which does not begin with the negation sign, and n is the number of negation signs preceding D. D is called the *kernel* of H and n is called the *index of D in H*. In general we allow also the case $n = 0$ and $D = H$.

Now, to return to the definition of the function g in the case where H begins with a negation sign, if the index n of H is even, then $g(H) = g(D)$ where D is the kernel of H. If the index n of H is odd, then, $g(H) = (g(D) \mid g(D))$, where D is the kernel of H.

3) Finally, if H is of the form $(B \vee D)$, where B and D are in S_1, then $g(H) = (g(\sim B) \mid g(\sim D))$. 4) g is defined only for members of S_1.

Now, we must show that g has the desired properties. g is clearly defined for every member of S_1 since every member of S_1 has one of the three forms treated above. It is necessary to show also that the image $g(H)$ of any wff H of P_1 is, in fact a member of S_2 (and not simply some other word defined on the alphabet L_2). The rigorous proof of this is by induction on the structure of the wffs of P_1, but the recursive definition of g makes the theorem apparent. First we establish the property for 0 occurrences of « $\vee$ », which is immediate by parts 1) and 2) of the definition of g (every variable of P_1 is a variable of P_2 and conversely). By part 3) of the definition of g, the induction step is immediate.

Before passing to the proof of the other specific properties of g which relate directly to I.Th.27, it might be well to observe several general properties of g. From the definitions Df.1-Df.3 for P_1 and Df.1-Df.5 for P_2, it is clear that every wff of P_1 is, at the same time, a metalinguistic *name* of some uniquely defined wff of P_2. Thus, we can consider another functional relation h which assigns, to each member of S_1, the member of S_2 of which it is the name. In fact, the recursive definition of the set S_1 and the definitions Df.1 and Df.2 of P_2 constitute a recursive definition of this function h, provided it is understood that $h(H) = H$ where H is any *variable* of P_1. More explicitly, the definition of h is as follows: 1) if H is a variable, $h(H) = H$. 2) if H is of the form $(\sim B)$, then $h(H) = (h(B) \mid h(B))$. 3) If H is of the form $(B \vee D)$, then $h(H) = (h(\sim B) \mid h(\sim D))$.

It is often by general observations concerning the name relation h (though things are not usually considered quite so formally) that the equivalence between P_1 and P_2 is viewed from the intuitive standpoint, and one might wonder why we have not chosen h as our mapping in the case of our syntactical definition of equivalence. One of the main reasons is that h is not a surjective mapping from S_1 to S_2, and thus does not lend itself to an application of I.Th.27. To take an example, the wff « $(a \mid b)$ » of P_2 is not named by any wff of P_1 (and has no metalinguistic abbreviation under Df.1-Df.5 for P_2), but this wff, as well as all other wffs of P_2, *is* the g-image of some wff of P_1, as will be proved shortly.

However, it is true that every **e**-equivalence class of S_2 is represented in $h(S_1)$ and considerations of this nature will lead to an interesting relationship between the two functions g and h. But first, we must consider some metatheorems for our two systems of which we will have need.

First, let us make the convention, which is to hold only for this paragraph § 3 of Section A of Chapter II, that we use primes when speaking of **e**-equivalence in P_2 and not so in P_1, thus allowing us to avoid verbally designating our system of reference each time.

Now, as is well known (see Kleene [5], p. 90) the deduction theorem is true in both P_1 and P_2. The mapping *f* which shows this to be true in P_1 is the mapping which assigns, to each ordered pair $\langle B, D\rangle$ of wffs of P_1, the wff $((\sim B) \vee D)$. Using the definition Df.2 of P_1, we thus have the property that $B \vdash_{P_1} D$ if and only if $\vdash_{P_1} B \supset D$. Using Df.3 for P_1, this gives us the property $B\,\mathbf{e}\,D$ if and only if $\vdash_{P_1} B \equiv D$. Similarly, and bearing in mind the definitions Df.1-Df.5 for P_2, we have that $B \vdash_{P_2} D$ if and only if $\vdash_{P_2} B \supset D$ and thus $B\,\mathbf{e}'\,D$ if and only if $\vdash_{P_2} B \equiv D$, where *B* and *D* are any wffs of P_2.

Now, as the above indicates, the relation « $\vdash \ldots \equiv$ ——— » is an equivalence relation in both of the systems P_1 and P_2. And it is also a well-known metatheorem for each of the two systems that the substitutivity of this relation of equivalence holds. Thus, in view of the above considerations in which we see that **e**-equivalence is, for these two systems, the same as the relation « $\vdash \ldots \equiv$ ——— », we have immediately the substitutivity of the relation of **e**-equivalence (respectively, **e**′-equivalence).

With the aid of these metatheorems, we will now be able to prove that, where *B* is any wff of P_1, $g(B)\,\mathbf{e}'\,h(B)$. This relation, together with the surjective character of *g* (which will also be proved), yields the result that $h(S_1)$ contains representatives from every **e**′-equivalence class, a fact that was mentioned above. Before establishing this relationship between the two functions *g* and *h*, we will need two other theorems.

Th.1. — Let *B* be any element of S_1. Then $g(\sim\sim B) = g(B)$.

Proof : This is seen immediately by taking cases under the definition of *g* and the structure of *B*. If *B* is a variable, then *B* is the kernel of $\sim\sim B$ and the index of *B* in $\sim\sim B$ is 2, thus an even number. Thus, $g(\sim\sim B) = g(B)$ by part 2) of the definition of *g*, and our theorem holds in this case.

If *B* begins with an occurrence of the negation sign, then let *D* be the kernel of *B*, and let *n* be the index of *D* in *B*. Now *D* is also the kernel of $\sim\sim B$ and $n + 2$ is the index of *D* in $\sim\sim B$. And $n + 2$ is odd or even according to whether n is odd or even. Thus, $g(\sim\sim B) = g(D) = g(B)$ if *n* and $n + 2$ are even. If, on the other hand, n is odd, then $n + 2$ is odd and $g(\sim\sim B) = (g(D) \mid g(D)) = g(B)$.

Finally, if *B* is of the form $(D \vee H)$, then $(D \vee H)$ is the kernel of $\sim\sim B$ and the index of $(D \vee H)$ in $\sim\sim B$ is 2, thus an even number. Thus, $g(\sim\sim B) = g(B)$ and our theorem is seen to hold.

Th.2. — Let *B* be any wff of P_1. Then $g(\sim B)\,\mathbf{e}'\,(g(B) \mid g(B))$.

Proof : Again we must take cases. If *B* is a variable, then *B* is the kernel of $\sim B$ and the index of *B* in $\sim B$ is odd. Thus, $g(\sim B) = (g(B) \mid g(B))$ by the definition of *g*. If, in second place, *B* begins with an occurrence of the negation sign, then let *D* be the kernel of *B* and let *n* be the index of D in B. If n is odd, then $g(B) = (g(D) \mid g(D))$. But, since n is odd, $n + 1$, which is the index of D in $\sim B$, is even (D being also the kernel of $\sim B$). Thus, $g(\sim B) = g(D)$. Now, $g(B) \mid g(B) = (g(D) \mid g(D)) \mid (g(D) \mid g(D))$. Let us use « H' » to denote this latter formula. Now, it is a well-known theorem of P_2 that $\vdash_{P_2} H' \equiv g(D)$ where $g(D)$ is any wff of P_2. Thus, we have $g(\sim B) = g(D)\, \mathbf{e}'\, H' = (g(B) \mid g(B))$.

If n is even, then, with an argument similar to the above, the theorem is again seen to hold.

If, thirdly, B is of the form $(D \vee H)$, then $(D \vee H)$ is the kernel of $\sim B$ and so $g(\sim B) = (g(B) \mid g(B))$ and the theorem again holds.

Thus, our theorem is true.

We now have our desired theorem :

Th.3. — Let B be any wff of P_1. Then $g(B)\, \mathbf{e}'\, h(B)$.

Proof : If B is a variable, then $g(B) = B = h(B)$. If B begins with a negation sign, then B is of the form $\sim D$. Now, $g(\sim D)\, \mathbf{e}'\, (g(D) \mid g(D))$ by Th.2. And, by definition, $h(\sim D) = (h(D) \mid h(D))$. If, to take the third case, $B = (D \vee H)$, then we have $g(B) = (g(\sim D) \mid g(\sim H))$ and $h(B) = (h(\sim D) \mid h(\sim H))$.

From the above facts, it is immediately clear that an induction on the structure of the wff B (more precisely, on the number of occurrences of « $\vee$ » in B) immediately yields the desired conclusion. Thus, our theorem holds.

We will now proceed to show that g fulfills the specific properties of Th.27 of Chapter I and we will see that these general theorems which we have just proved will be useful to us in our analysis.

Let us establish that g is surjective. Let B' be any wff of P_2, and let n be the number of occurrences of the stroke sign « $\mid$ » in B'. If $n = 0$, then B' is a variable (see the definition of S_2). But every variable of S_2 is also a variable of S_1 and conversely and, for all wffs X of S_1 which are variables, $g(X) = X$. Thus, $g(B') = B'$ and B' is thus the g-image of some wff of S_1. Thus, our property holds for $n = 0$.

Now, suppose that, for all wffs of P_2 with $n \leqslant k$ occurrences of the stroke sign, the property holds. Now consider a wff B' of P_2 with $k + 1$ occurrences of the stroke sign. Now, B' is of the form $(D' \mid H')$ where D' and H' each have less than $k + 1$ occurrences of the stroke sign (the fact that we here appeal to is clear from the definition of S_2 and can be easily proved by induction). Thus, by the hypothesis of the induction step, there exist D and H such that $g(D) = D'$ and $g(H) = H'$. Now consider the wff of P_1, $B = ((\sim D) \vee (\sim H))$ (this is wf since D and H are wf). Then, by definition of g and by Th.1 above, $g(B) = (g(\sim \sim D) \mid g(\sim \sim H)) = (g(D) \mid g(H)) = (D' \mid H') = B'$ and our property holds for $k + 1$. Thus, the property holds for all n and every wff of P_2 is the image of at least one wff of P_1 and g is thus surjective.

Now, we wish to establish that only $\mathbf{e}$-equivalent wffs have identical g-images, thus, that $g(B) = g(D) \rightarrow B\, \mathbf{e}\, D$. We take cases under the definition of g. All wffs of P_1 which are transformed by 1) have distinct images if they are distinct since here the function is the identity. Under 2), distinct wffs give rise to distinct images except that $g(B) = g(D)$ where D is the kernel of B and the index of D in B is even. But a well-known metatheorem of P_1 is $\vdash_{P_1} (\sim(\sim D)) \equiv D$ where D is any wff. Thus, by induction, we can prove that $\vdash_{P_1} B \equiv D$ where B consists of D preceeded by an even number occurrences of the negation sign. Thus, under 2), two wffs have the same image only if they are $\mathbf{e}$-equivalent.

Under 3), two wffs $(B \vee D)$ and $(H \vee J)$ have the same image only if $g(B) = g(H)$ and $g(D) = g(J)$. But, the above cases which we have already treated show that this is true only if $B \,\mathbf{e}\, H$ and $D \,\mathbf{e}\, J$ (as in other similar cases, the rigorous form of this theorem is by induction and we here only sketch the indications). But, by the substitutivity of $\mathbf{e}$-equivalence which we have established above, if $B \,\mathbf{e}\, H$ and $D \,\mathbf{e}\, J$, then $(B \vee D) \,\mathbf{e}\, (H \vee J)$. Thus, under 3) only $\mathbf{e}$-equivalent wffs have identical images. Thus, it is clear from the above that an induction on the structure of the wffs of P_1 yields the desired conclusion and our property is thus established.

Now let us show that g preserves the consequence relation from P_1 to P_2. First we must show that the image of every axiom of P_1 is a theorem of P_2. Now it is well-known, and we will not take the trouble to derive the necessary theorems here, that every wff of P_2 which is named by an axiom of P_1 is a theorem of P_2. Thus, for every axiom B of P_1, $h(B)$ is a theorem of P_2. But for all B in S_1, $h(B) \,\mathbf{e}'\, g(B)$ (by Th.3) and hence $g(B)$ is also a theorem of P_2. Thus, the g-image of every axiom of P_1 is a theorem of P_2.

Now let us show that primitive inference is preserved from P_1 to P_2. The rule R_1 allows us to infer D from $(\sim B \vee D)$ and B where B and D are any wffs of P_1, and all primitive inference is of this form. Now, by the definition of g, $g((\sim B \vee D)) = g(\sim \sim B) \mid g(\sim D) = g(B) \mid g(\sim D)$ (by Th.1). Now, by Th.2, $g(\sim D) \,\mathbf{e}'\, (g(D) \mid g(D))$ which, by the substitutivity of $\mathbf{e}'$-equivalence, gives us that $(g(B) \mid (g(D) \mid g(D))) \,\mathbf{e}'\, (g(B) \mid g(\sim D))$. But, the rule R_2 permits us to infer $g(D)$ from $g(B)$ and $(g(B) \mid (g(D) \mid g(D)))$. Thus, from $g((\sim B \vee D))$ and $g(B)$ we can deduce $g(D)$ and thus g preserves primitive inference from P_1 to P_2.

Thus, g preserves the consequence relation from P_1 to P_2.

Next we will show that every axiom of P_2 is the image of a theorem of P_1 and hence, since only $\mathbf{e}$-equivalent wffs of P_1 have identical images under g, that every axiom has *only* theorems in its inverse image set.

Let J' be any axiom of P_2. J' must be an instance of the schema iii. a of P_2. Thus, there are wffs B', D', H' and N' such that J' is of the form which follows: $((B' \mid (D' \mid H')) \mid ((B' \mid (H' \mid B')) \mid ((N' \mid D') \mid ((B' \mid N') \mid (B' \mid N')))))$. Now, since B', D', H', and N' are all wffs of P_2, there exist wffs B, D, H, and N of P_1 such that $g(B) = B'$, $g(D) = D'$, $g(H) = H'$ and $g(N) = N'$. Thus, by the method used in the proof of the surjective character of g, we can construct a wff J of P_1 such that $g(J) = J'$. This wff is: $\sim(\sim B \vee (D \,\&\, H)) \vee ((\sim B \vee (H \,\&\, B)) \,\&\, ((N \,\&\, D) \vee \sim((B \,\&\, N)$

$\vee (B \& N))))$ (here we have used Df.I of P_1 to help abbreviate J). That J is indeed a theorem of P_1 will not be formally demonstrated here but can be easily checked by the normal form method. Thus, every axiom J' of P_2 has only theorems in its inverse image set.

Finally, suppose that from $(B' \mid (D' \mid H'))$ and B' we infer H' by the rule R_2, where B', D', and H' are any wffs of P_2. All primitive inference of P_2 is of this form. Now, let B, D, and H, be wffs of P_1 whose g-images are B', D', and H' respectively. Let $J = \sim B \vee (D \& H)$. Then $g(J) = g(\sim B \vee \sim(\sim D \vee \sim H)) = g(\sim\sim B) \mid g(\sim\sim(\sim D \vee \sim H)) = g(B \mid (g(D) \mid g(H)) = B' \mid (D' \mid H')$ (applying Th.I several times). We also have $g(B) = B'$ and $g(H) = H'$. Now, by the rule R_1, we can immediately infer $D \& H$ from B and J, and it is a well-known metatheorem of P_1 that from $D \& H$ we can deduce H. Thus, the last of the hypotheses of I.Th.27 is seen to hold. Thus, by I.Th.27, P_1 and P_2 are equivalent.

It is interesting to notice that, though P_1 and P_2 are equivalent, there is no apparent isomorphism between them. It might, of course, be possible to establish an isomorphism, but the required mapping is not immediately forthcoming in any case.

If one were interested in seeing, as quickly as possible, whether or not two systems were equivalent, many of the details we have supplied (in particular certain proofs by induction) could often be omitted as is currently done in the applications of any mathematical theory (in modern algebra, one almost never takes the trouble to demonstrate generalized substitutivity principles or generalized associative and distributive laws).

The often used method of proving equivalence between two formal systems F and F′ by deducing the axioms of F′ as theorems of F and reciprocally is a method which is limited at best. Usually this method presumes that the set of wffs is the same for both systems, or at least that there is some obvious sameness between the two sets (such as where one sign such as « & » is consistently replaced by another such as « $\wedge$ »). Moreover, this method is only concerned with showing that the set of theorems is somehow « the same » in both cases and thus takes no account of the deductive structure of the system as a whole. Thus, with our definition, two systems F and F′ which have the same set of wffs, the same axioms and (or) the same set of theorems may very well not be equivalent. (We will see in Chapter III, Section C, that another well-known system, usually considered as another formulation of the propositional calculus, is *not* equivalent to P_1 and P_2 and thus constitutes, for us, a different system from P_1 and P_2).

Before passing to other considerations, we will briefly consider a third formal system P_3 which will be useful in later developments and for which the equivalence to P_1 and P_2 is immediate.

§ 4. THE SYSTEM P_3.

The system P_3 is the resulting formal system of the calculus C_3 which we will now describe.

(i) L_3 consists of the same members as L_1 except that it includes also the sign « & ».

(ii) ii.a, ii.b, and ii.c for P_3 are the same as the respective (corresponding) definitions for P_1. We add, however :

d) If X and Y are any members of S_3, then $(X \& Y)$ is also a member of S_3.

e) Same as ii.d for P_1.

Df.1. — Same as Df.2 for P_1.

Df.2. — Same as Df.3 of P_1.

(iii) iii.a-iii.d for P_3 are the same as iii.a-iii.d respectively for P_1. For P_3 we add a fifth schema which designates other wffs of P_3 as axioms :

e) $((\sim((\sim B) \vee (\sim D))) \equiv (B \& D))$

f) A_3 contains no other members.

(iv) The set of rules for P_3 is the same as for P_1 and we will refer to this one rule as « the rule R_3 ».

To see the equivalence of P_3 and P_1, we observe that $S_1 \subset S_3$. Our mapping g from S_3 to S_1 (which is to satisfy the hypotheses of I.Th.27) is defined as follows : 1) If B is a variable, then $g(B) = B$. 2) If B is of the form $\sim D$, then $g(B) = \sim g(D)$. 3) If B is of the form $(D \vee H)$, then $g(B) = (g(D) \vee g(H))$. 4) If B is of the form $(D \& H)$, then $g(B) = (\sim(\sim g(D) \vee \sim g(H)))$.

It is clear, from an intuitive point of view, that P_3 simply assimilates one of the definitions of P_1 into the formal system itself. To see that g satisfies the hypotheses of I.Th.27, we again use several metatheorems for P_3. Again, due to the deduction theorem which is true in P_3, we have that the relation « $\vdash_{P3} \ldots \equiv$ — — — » is the same as the relation of **e**-equivalence for P_3 and we also have the substitutivity of this relation. These two facts, together with part 4) of the definition of g and the schema iii.e of P_3, give us that unique wffs of P_1 have only **e**-equivalent wffs of P_3 as inverse images. That the other hypotheses of I.Th.27 are true is clear and can be easily checked in view of the definition of g (in particular it is clear that g, when restricted to the subset S_1 of S_3, is simply the identity mapping). Thus, P_3 is equivalent to P_1 and thus to P_2.

In all of our subsequent exposition in this thesis, we will use « **P** » to stand ambiguously for any one of the three equivalent systems P_1, P_2, and P_3. In the following section, we will find that it simplifies matters somewhat to make use of P_3. For us, the equivalence class of formal systems determined by **P** is the definitive form of the propositional calculus and all systems or « formulations » which are not equivalent to some representative of this class are not, for us, formulations of the propositional calculus.

SECTION B

THE EQUIVALENCE BETWEEN BOOLEAN ALGEBRA AND THE CALCULUS OF PROPOSITIONS

§ 1. INTRODUCTORY DISCUSSION OF THE QUESTION

Since the very beginning of modern logic it has been a standard, and supposedly trivial, observation that the propositional calculus and the Boolean algebra of « classes » or « terms » are in some sense « equivalent » (see Boole [1], pp. 159-184). Indeed, the equivalence between the two systems is rather trivially clear, provided one is clear and concise as to the mathematical framework in which the two systems are placed and to the notion of equivalence which is used. But such conciseness has, for one reason or another, been quite often lacking in treatments by various authors of the relationship between these two systems.

If, then, we discuss, in this section, the equivalence of the Boolean algebra and the propositional calculus, it is not with the illusion of bringing a profoundly new idea to the literature, but rather in the spirit of knowing precisely in what sense this « equivalence » is meaningful, and with the purpose of using these newly-clarified concepts as a basis of further discussion of certain related questions in later sections of this study. Thus, we proceed with our discussion.

What has often been meant by the « equivalence » of Boolean algebra and the propositional calculus is that both of the semantic notions associated respectively with these names can somehow be considered as interpretations of a single « system », or that certain signs of the respective systems « obey the same laws ». This idea has been clouded over, however, by the fact that, since Frege [1], the calculus of propositions is most often treated as a formal system by the logistic method whereas the Boolean algebra is most often treated as an « algebra defined on an arbitrary set ». Thus, as long as the propositional calculus and the Boolean algebra are considered as two different kinds of entities, a precise notion of « equivalence » is not easily forthcoming.

One answer to this is to consider the propositional calculus as an algebra. This is the approach of Tarski [1] (p. 30 ff.) and has been used by other authors such as Huntingdon [1] and Rosenbloom [1]. With this approach, the « equivalence » of the two concepts can be established in the following sense : If $P = \langle S, \vee, \sim, T \rangle$ is a propositional calculus (T is a proper subset of S, $\vee$ and $\sim$ are respectively a binary and a singulary operation on S, these operations satisfying certain laws with respect to the set T), then $\langle S/E, \vee, \sim \rangle$ is a Boolean algebra where E is the equivalence relation on S defined by $(x \equiv y) \in T$ (it must be remembered that $x \equiv y$ is here an element of S and not a relation). And, conversely, if $\langle M, \cup, ' \rangle$ is a Boolean algebra whose maximal element we will denote by

« 1 », then $\langle M, \cup, ', T \rangle$ is a propositional calculus where $T = \{1\}$ (or, indeed, where T is any *filter* of the algebra, see Rosenbloom [1], pp. 38 and 44).

It is clear that equivalence in the above sense is not a relation between two specific systems, but between two *definition schemata*, because we have, in each case, free variables which are left unspecified. Two specific Boolean algebras, in the above sense, may well be non-isomorphic. Thus, the above notion does not necessarily express the relation between the propositional calculus as a specific (constant) mathematical entity (namely, a formal system) and the Boolean algebra so considered.

Thus, the above notion of equivalence is precise, but it does not throw light on the relationship between the two systems in the particular form that we desire to have it.

Another method of treating the Boolean algebra is by formalizing it in the first order predicate calculus with identity. Here, however, we are constrained by the fact that there is no corresponding formulation of the propositional calculus within the predicate calculus with identity. In the propositional calculus, individual variables « x », « y » etc. represent propositions and thus constitute wffs alone, whereas in the predicate calculus, regardless of what constant predicates we choose as primitive, this is never the case.

The above schema P uses the membership relation « $\in$ » and can thus only be formalized within set theory (such as that of Bernays in Bernays and Fraenkel [1], for example, or any other formal set theory).

The only remaining alternative would seem to be that of considering the Boolean algebra itself as an independent logistic system and then of proving its equivalence to the propositional calculus so conceived, in some appropriate sense of the term « equivalence ». But what sense ? The sense of this present work is the one we propose.

Thus, the remainder of this section will consist of considering the Boolean algebra as a formal system and then proving its equivalence with our system **P**.

§ 2. THE SYSTEM **B**.

The system **B** is the resulting formal system of the following calculus :

(i) The let L_b contains the following members :

a) The signs « $\cup$ », « $\cap$ », « $'$ », « (», «) », « $=$ » called the « equality sign » and « * » called « star ».

b) small italic latin letters which may be followed by any finite number of occurrences of the star. A small italic latin letter followed by any finite number of occurrences of the star is called a *variable*.

c) L_b contains no other members.

(ii) S_b is defined as follows :

a) A *term* is defined as follows :

1) Any variable is a term.
2) If X is a term, then (X') is a term.
3) If X and Y are terms, then $(X \cup Y)$ and $(X \cap Y)$ are terms.
4) Nothing else is a term except by the above.

b) If X and Y are terms, then $X = Y$ is in S_b.

c) Nothing else is in S_b except by the above.

We now make several definitions.

Df.1. — « 0 » for « $(d \cap (d'))$ ».

Df.2. — « I » for « $(d \cup (d'))$ ».

Df.3. — $X \subset Y$ for $X = X \cap Y$, where X and Y represent any terms of **B**.

It should be noted that Df.1 and Df.2 are not definition schemata, but proper definitions. They give a specific symbol as an abbreviation for a particular term of **B** (and thus indirectly for an infinite class of wffs, namely all those which contain the term in question).

(iii) Where X, Y, and Z represent any terms whatever, those wffs which are instances (or, where definitions are used, those wffs which are *named* by instances) of the following forms are axioms :

a) $(X \cap Y) = (Y \cap X)$
b) $((X \cap Y) \cap Z) = (X \cap (Y \cap Z))$
c) $(X \cap (X')) = 0$
d) $(X \cap (0')) = X$
e) $(X \cap X) = X$
f) $(X \cap (Y \cup Z)) = ((X \cap Y) \cup (X \cap Z))$
g) $(((X') \cap (Y'))') = (X \cup Y)$
h) $(X \cup 0) = X$
i) $X = X$
j) There are no other members of A_b except by the above.

(iv) **B** has one rule of order 2 which can be stated as follows :

Let X and Y be any two terms and let D be any wff except the wff $X = Y$. Then, from $X = Y$ and D we can infer B where B is like D except that it contains Y in at least one place (not necessarily all) where D contains X. This rule will be referred to as the « rule of replacement ».

The above axioms are our modification of a set due to Byrne [1]. Our modifications have served to decrease the number of primitive rules by augmenting the set of axioms and the number of primitive signs.

As with the system **P**, we will here also feel free to drop parentheses and we will also use certain theorems and metatheorems of **B** without proving them.

Since the Boolean algebra is not treated quite as often as a formal system, we will recall a few of its usual properties.

Th.4. — From $X = Y$, where X and Y are any terms of **B**, we can deduce $(X') = (Y')$, $(X \cap Z) = (Y \cap Z)$, and $(X \cup Z) = (Y \cup Z)$ where Z is any term of **B**.

Proof : This follows immediately by the rule of replacement and use of the schema iii.i.

Likewise, the transitive, reflexive, and symmetric properties of « = » are directly forthcoming from the replacement rule and we will freely use the properties in our proofs.

Th.5. — Where X is any term of **B**, $((X')') = X$ is a theorem.

Proof : Where X is any term of **B**, $X' \cap 0' = X'$ is an axiom by iii.d. By Th.4 this gives $(X' \cap 0')' = X''$. Then, by iii.g and the replacement rule we infer $X \cup 0 = X''$ and thus $X'' = X$ by symmetry of equality and iii.h.

Notice that whereas both of the above theorems are metatheorems and not theorems of the system **B**, yet statements such as Th.4 have the form of *derived rules* in the precise sense of I.Df.7.

Th.6. — $\vdash_{\mathbf{B}} \cdot 1 = 0'$.

Proof : By the rule of replacement, from $(d' \cap d'')' = d \cup d'$, which is an instance of the schema iii.g, and $d'' = d$, which is a theorem of **B** by Th.5, we can infer $(d' \cap d)' = d \cup d'$. Now, by iii.a and the replacement rule we infer $(d \cap d')' = d \cup d'$ which, in view of Df.1 and Df.2 for **B**, gives us $0' = 1$ and thus $1 = 0'$ by the symmetry of « = ».

Th.7. — Where X and Y are any terms of **B**, $X \cap 0 = 0$, $X \cup 1 = 1$, $X \cup X' = 1$, $(X \cap Y)' = X' \cup Y'$ and $(X \cup Y)' = X' \cap Y'$ are all theorems of **B**.

Proof : These properties are immediate from the above theorems and properties.

The usual lattice identities can be proved taking into account Df.3 of **B**.

This terminates our general discussion of **B**, but we will not hesitate to use other well-known theorems and metatheorems when this is useful to us.

§ 3. THE EQUIVALENCE OF P AND B

We will prove the equivalence of **B** and **P** by proving the equivalence of the system **B** with the system P_3. We define a surjective mapping g from S_b to S_3 in two stages. First we will define a bijective mapping h from the terms of **B** to the wffs of P_3 and then we will define g by the aid of h. We will then show that g satisfies the hypotheses of I.Th.27 and that **B** and **P** are thus equivalent.

Since the sign « $=$ » is now being used as a sign of one of our systems, we risk confusion if we continue to use this sign, as we have in the foregoing, to render the intuitive notion of identity between two objects of some set. In this paragraph, § 3, the notion of identity will be rendered by the vernacular « is » and no special sign will be used to render this idea.

The mapping h is defined for all terms of **B** as follows : 1) If X is a variable of **B**, then $h(X)$ is X. This is well-defined since every variable of **B** is a variable of P_3 and reciprocally. 2) if X is a term of **B** and if X has the form (Y'), where Y is a term of **B**, then $h(X)$ is $({\sim}h(Y))$. 3) If X is of the form $(Y \cup Z)$, then $h(X)$ is $(h(Y) \vee h(Z))$, where X, Y, and Z are all terms of **B**. 4) Where X, Y, and Z are all terms of **B**, if X is of the form $(Y \cap Z)$, then $h(X)$ is $(g(Y) \,\&\, g(Z))$.

It is easy to verify that h is a bijection from the terms of **B** onto S_3 since it is simply an algorithim which leaves variables and parentheses unchanged and which replaces each occurrence of « $\cup$ » by « $\vee$ », « $\cap$ » by « & », and « $'$ » by « $\sim$ ».

We now define the mapping g as follows : 1) If D is a wff of **B** and D has the form $X = Y$ where Y is *not* the wff «$(d \cup (d'))$», then $g(D)$ is $(h(X) \equiv h(Y))$. 2) If D does not have the above form, then D must be of the form $X = (d \cup (d'))$ (no restrictions on X). In this case, $g(D)$ is $h(X)$.

We must now show that g satisfies the hypotheses of 1.Th.27 and thus that **B** and **P** are equivalent.

It is clear that the domain of g is the set S_b and that g is surjective onto the set S_3. For the latter property, consider any wff D of P_3. There is a unique term X of **B** such that $h(X)$ is D. Now consider the wff of **B** $X = 1$. $g(X = 1)$ is $h(X)$ which is D and so g is surjective.

Now let us show that $g(D)$ is $g(H) \rightarrow D \,\mathbf{e}\, H$ in **B**. By the definition of g, two distinct wffs D and H of **B** can have the same image only when D is of the form $X = Y$, H is of the form $Z = 1$, and it is the case that $h(Z)$ is the same expression as $(h(X) \equiv h(Y))$. For convenience, let N represent $h(Z)$, M represent $h(X)$, and J represent $h(Y)$. Now we have that N is the same expression as $M \equiv J$ which means that N is of the form $({\sim}M \vee J) \,\&\, ({\sim}J \vee M)$ (see Df.2 for P_3). But this means that Z is of the form $(X' \cup Y) \cap (Y' \cup X)$. Thus, two distinct wffs D and H of **B** have the same image if and only if D is of the form $X = Y$ and H is of the form $(X' \cup Y) \cap (Y' \cup X) = 1$. We must now prove the metatheorem of **B** that any two such wffs D and H are in the same **e**-equivalence class of **B**.

Th.8. — In **B**, where X and Y are any terms, we have that $(X' \cup Y) \cap (Y' \cup X) = 1 \,\mathbf{e}\, X = Y$.

Proof : Where X and Y are any terms of **B**, assume the wff $X = Y$ as hypothesis. Then, by the rule of replacement, Th.4, and Th.7, we can infer $Y' \cup X = Y' \cup Y$ and then $Y' \cup X = 1$. Similarly, we can infer $X' \cup Y = 1$. By Th.4 and the replacement rule we then infer $(X' \cup Y) \cap (Y' \cup X) = 1$. Thus, $X = Y \vdash_{\mathbf{B}} (X' \cup Y) \cap (Y' \cup X) = 1$.

Conversely, take as hypothesis the wff $(X' \cup Y) \cap (Y' \cup X) = \mathrm{I}$, where X and Y are any terms. We infer, by various well-known laws, $(X' \cup Y) \cap (Y' \cup X) = (X \cap Y) \cup (X' \cap Y')$ (distributive law and regrouping) which yields in turn the two wffs $(X \cap Y) = X$ and $(X \cap Y) = Y$ (multiplying first by X, then by Y according to Th.4). Finally we infer from these two wffs the wff $X = Y$ by the transitivity of « = ». Thus, $(X' \cup Y) \cap (Y' \cup X) = 1 \vdash_{\mathbf{B}} X = Y$, and our theorem is proved.

Thus, only **e**-equivalent wffs of **B** have identical g-images.

That the axioms of **B** have only theorems of P_3 as images under g and that every axiom of P_3 is the image of a theorem (and thus, in view of the above, has only theorems in its inverse image set) follows from well-known theorems of the two systems. We will not take the trouble to sketch the formal deductions of these theorems here.

To see that primitive inference is preserved from **B** to P_3, let us first notice that the rule of replacement with respect to the relation « $\vdash_{\mathrm{P3}} \ldots \equiv - - -$ » (and thus the relation of **e**-equivalence in P_3) is a derived rule of P_3. Now, suppose that we have two wffs $X = Y$ and D of **B**. Suppose that D contains X and that from these two wffs we infer H where H contains Y at none, or more places where D contains X, and is otherwise like D. Consider first the case where Y is not the wff I of **B** and where D is of the form $V = \mathrm{Z}$ where Z is not I. Then, $g(X = Y)$ is $(h(X) \equiv h(Y))$ and $g(D)$ is $(h(V) \equiv h(\mathrm{Z}))$. From the structural way in which h is defined and from the fact the replacement rule is true in P_3, it is clear that we can deduce $g(H)$ from $g(X = Y)$ and $g(D)$.

In the other cases, when either Y is the wff I or Z is I, we have that $g(X = Y)$ is $h(X)$ or respectively, $g(D)$ is $h(V)$. But it is easy to show that $(h(X) \equiv (d \vee (\sim d)))$ is in the **e**-equivalence class of $h(X)$ and that $(h(V) \equiv (d \vee (\sim d)))$ is in the same **e**-equivalence class as $h(V)$. Thus, again by the structural character of h, it is clear that inference is preserved from **B** to P_3.

It is evident that rigorous proofs of the above would involve long and tedious inductions, but the structural correspondence h renders the truth of the assertions immediately apparent. If our mapping h were not so clear, we could resort to other methods such as « atomizing » our rule of replacement by finding a number of simpler rules which are equivalent to it or sufficient for it and showing that these rules are preserved from one system to the other. It is well-known, for example, that a slight variant of Th.4 plus the symmetrical, reflexive, and transitive properties of « = » yields our rule of replacement as a metatheorem.

This technique of finding a conjunction of simpler rules equivalent to or sufficient for a given rule is another technique which is open to us within the framework of our general method of proving the equivalence of two formal systems.

Now, consider two wffs of P_3 which have the forms $\sim D \vee H$ and D respectively. From these, we can infer H and all primitive inference is of this form. Now, in the set of inverse g-images of $\sim D \vee H$ is the wff of **B** which as the form $(X' \cup Y) = 1$ where $h(X)$ is D and $h(Y)$ is H. Similarly,

$g(Y=1)$ is H, and $g(X=1)$ is D. It must be shown, then, that from $X' \cup Y = 1$ and $X = 1$, we can deduce $Y = 1$ in **B**. That this inference holds where X and Y are any terms whatever can be seen as follows: from $X = 1$ and $X' \cup Y = 1$ we can infer $1' \cup Y = 1$ by the rule of replacement. But this gives $0 \cup Y = 1$ or $Y = 1$ which is the desired wff. Thus, the last of the properties of I.Th.27 holds and the equivalence of **B** and P_3 is established. Thus, **B** and **P** are equivalent.

SECTION C

C. I. LEWIS' CRITICISMS OF THE PROPOSITIONAL CALCULUS

§ 1. THE STRUCTURE OF THE PROPOSITIONAL CALCULUS

In this section we are concerned to refute certain well-known criticisms of the classic calculus of propositions which have been made by C. I. Lewis. Before considering these criticisms directly, we wish to establish briefly a theorem concerning the system **P**.

Th.9. — The number of **e**-equivalence classes for the system **P** is (denumerably) infinite.

Proof: Let us say that a wff of **P** which contains n variables is *of order n*. By a well-known theorem of Post [1] (see p. 172 of that work), it is possible to find, for every $n \geqslant 1$, 2^{2^n} wffs of order n no two of which are equivalent, equivalence being here understood as the relation « $\vdash_{\mathbf{P}} \ldots \equiv$ — — — ».[6] However, in view of the fact that the deduction theorem is true for **P**, the above relation is the same as **e**-equivalence for **P** and thus Post's theorem is true where « equivalence » is understood as **e**-equivalence. (Again, it is to be remarked that this is true for our formulations P_1, P_2, P_3 which we have taken as being the definitive form of the propositional calculus. Chapter III, Section C, contains a discussion of the relationship between **P** and other « formulations » of the propositional calculus.)

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[6] Post's otherwise clear exposition is, perhaps, slightly clouded by failure to distinguish clearly the difference between a *form* and an *associated function* of a form in the sense of Church [4], pp. 19-21. This imprecision can be rectified by agreeing on some *order* of the variables, say the order of first mention from left to right (or alphabetic order), and thus considering that each form (wff) of order n has a uniquely associated *representative* function from E^n to E where $E = \{\mathrm{t}, \mathrm{f}\}$. Without doubt, this imprecision in exposition is due to the fact that this article of Post was written by him as a pioneer in the field at that time.

Thus, suppose that **P** has a finite number m of **e**-equivalence classes, $m \geqslant 1$. Then there exist $2^{2^m} > m$ wffs of order m which are pairwise non-equivalent and which cannot, consequently, be found in the same **e**-equivalence class. Thus, there are at least 2^{2^m} equivalence classes which is a contradiction. Thus, there is no finite number m which represents the number of equivalence classes and the classes are thus infinite in number.

That the set of classes is denumerable follows from the fact that the equivalence classes form a partition of the set of wffs and this set is denumerable.

§ 2. A CONSIDERATION OF LEWIS' CRITICISMS

In the well-known works of Lewis [1] and Lewis and Langford [1], Lewis criticises as being inadequate the traditional calculus of propositions which we have formalized as **P**, and it is, at least partly, on the basis of this criticism that Lewis constructs (or at least seeks justification for) his famous systems of modal logic. Modal logic has a legitimate interest of its own and nothing that we say here invalidates the study of modal logic as modal logic. However, we feel that several of Lewis' criticisms of classical logic are based on certain imprecisions or confusions and it is our purpose to clarify several of these through the use of the methods of our study.

As is well-known, much of the controversy revolves around discussion of the « true » meaning of *implication* and of *material implication* in particular. Many of Lewis' criticisms are thus semantic in nature and consequently do not concern formal systems directly. Quine (in Quine [1], pp. 27-33 and more recently in Quine [3], pp. 195-200) has done much to clarify the semantic aspect of the question by pointing out the confusion between the various *relations* of implication on the one hand and the (truth-functional or other) *meaning* of the *conditional* sentence connective on the other. This confusion would seem to go back at least to Whitehead and Russell [1] (see Quine [2], pp. 140-42 for a discussion of this point in connection with the work of Whitehead and Russell).

Pushing further in the direction of Quine's analysis, we have, in a study conducted in collaboration with Jean-Blaise Grize, found at least seven distinct logical concepts which have, in the literature, passed under the appellation « material implication ».

But there are criticisms of Lewis which can be treated by the method of formal systems and it is these that we wish to examine briefly.

In Lewis [1] and Lewis and Langford [1], Lewis refers to the propositional calculus as the « two-valued » algebra and trys to obtain this « algebra » from the « Boole-Schröder » algebra of classes by a two-fold process which Lewis ascribes to Schröder [1], vol 2. First, Lewis reinterpretes this algebra as a propositional calculus, changing the operational signs (substituting « **v** » for « + », « ∼ », for « — », etc.). Then, he adds an axiom whose intended purpose, according to Lewis, is to restrict the number of « elements » in the « calculus » to two. Lewis says, for instance, « Thus, in strict accuracy, it should be said that there are only two elements in this Two-valued Algebra ; the two truth-values or extensions of propositions, 0 and 1. Every element p simply represents, ambiguously, one or other of these ». (Lewis and Langford [1], p. 88).

Now, **P** is *saturated* or *complete* in the *particular* sense that, for every wff D of **P**, either $\vdash_{\mathbf{P}} D$ or, if D is added as an *axiom schema* (that is, if we permit a substitution rule in connection with the variables of D), **P** becomes inconsistent in the strong sense that every wff of **P** becomes a theorem[7] (see Kleene [5], § 29, Corollary 2°, p. 134). Moreover, it is quite clear that Lewis treats the added axiom as a schema or general law and not as proper axiom for he immediately uses it in connection with his proofs in a completely general form, substituting any wff for the variables in the new axiom schema. Thus, Lewis' added axiom should render the resulting system inconsistent ! (There is further confusion in the fact that the added axiom contains a sign of equality which is not a primitive sign of the calculus. Thus, the added axiom would seem not to be a wff of the system.)

Lewis might respond to this by saying that he is not here considering the Boolean algebra as a formal system in the sense of **B** (or **P**), but is using instead one of the other forms of Boolean algebra which we considered above in Section B, § 1 of this chapter, and that the added axiom is thus legitimate. Let us consider this possibility.

If Lewis' Boolean algebra is (implicitly) formalized in the first-order predicate calculus with identity, then his new axiom, properly added, is not contradictory. But we are here troubled by the fact that there is no corresponding formulation of the propositional calculus and Lewis' reinterpretation is thus not possible.

If, on the other hand, Lewis considers that he is treating Boolean algebra as a schema in set theory (which would seem to be the most probable contention), thus, as an « algebra defined on an arbitrary set », then he is quite free to consider a model for the schema in which the algebra is restricted to a set with only two members. However, it must then be realized that whatever criticisms Lewis makes of such a *literally* two-valued algebra are *not valid* for the propositional calculus considered as a formal system ! (This point is, we feel, often overlooked.) And, since Lewis' systems of modal logic are formal systems, comparisons made between the formal modal systems and a particular set-theoretical model of Boolean algebra are not enlightening as to the true structure of **P**.

Because language and metalanguage seem sometimes confused in Lewis' treatment of these questions (at least in the two works here cited), the method involved is, perhaps, too unclear to find a consistent expression in any of the above frameworks.

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[7] The *general* notions of completeness and of consistency will be discussed in Chapter III of this study.

Lewis does seem to draw a distinction between the method outlined above and that of *Principia Mathematica* (see Lewis and Langford [1], p. 118, for example), but then he makes such statements as : « ... older developments such as the Two-valued Algebra of Schröder and newer ones like the calculus of elementary propositions in *Principia Mathematica* have exactly the same import and the same limitations ». (Lewis and Langford [1], p. 119) ; or, « It is ... important ... that the *agreement in results* and the *difference of method*, of these two procedures should be understood ». (Lewis [1], p. 222.) Here in the last quotation, the two methods of which Lewis speaks are the one outlined above on the one hand, and that of the logistic method of *Principia* on the other.

From this, we see that Lewis expects his criticisms to hold valid even in the case of the formal system **P** (which in its representation as P_1, is even the very system of *Principia* of which Lewis is here speaking). Yet, we have clearly seen that Lewis' added axiom and his criticism of the resulting system are not valid with respect to **P**.

Finally, Lewis continually refers to the relation « $\vdash_{\mathbf{P}} \ldots \supset$ — — — » as the relation of « material implication ». Now, if this relation is the relation of material implication, then the derivative equivalence « $\vdash_{\mathbf{P}} \ldots \equiv$ — — — » must be the relation of material equivalence. As we have seen, the above relation of equivalence is, in the case of **P**, the same as the relation of **e**-equivalence.

Now, as is well-known, material implication is characterized by the fact that, given any two propositions, one must materially imply the other. This in turn yields the result that the relation of material equivalence divides the set of all propositions into only two equivalence classes, the true and the false (might this not be a more precise way of stating what Lewis seems to mean by saying that every wff « simply represents, ambiguously » one of the « two truth-values » ?). But as we have demonstrated in § 1 of this section, the set of wffs of **P** has an infinite number of **e**-equivalence classes. Thus, the relation « $\vdash_{\mathbf{P}} \ldots \supset$ — — — » *cannot* represent « material implication » on the set of wffs of **P** (thus the set of all propositions). Chapter III, Section C, contains further discussion of certain aspects of this question.

Moreover, as we saw in Chapter I, theorem I.Th.23, a necessary condition that two systems be equivalent is that they have the same number of **e**-equivalence classes. Thus we have the result : any system F which can be constructed for which the relationship « $x \vdash_F y$ » is the relationship of material implication *cannot* be equivalent to **P**.

To make the point in a simpler way, even if we agree to consider two **e**-equivalent wffs of **P** as the same, we still have an infinite number of wffs. Thus, Lewis' statement that every wff represents « ambiguously » one of two values is false with respect to **P** and thus any conclusions which he draws from such an assumption are not valid criticisms of **P**.

The often-used term « two-valued » logic is thus a misnomer of **P**. The term obviously owes its use to the fact that, under its usual interpretation, the *variables* of **P** are interpreted as variables in the mathematical sense whose domain is a set with two members.

Semantically speaking, the relation « $\vdash_P \ldots \supset$ ——— » is, of course, the much stronger relation of tautological implication (see Quine [I], p. 28 ff.). The basic confusion here is between the material or truth-functional *meaning* of the conditional « $\supset$ » and the *induced relation* of implication (our terminology) « $\vdash_P \ldots \supset$ ——— ». It can be seen that it is possible to *induce* many different relations of implication while guarding the same definition of the conditional.

Further discussion of this question, though interesting, is semantical in nature and thus out of the main line of development of this study. Grize and Hatcher hope to publish the results of their study of this question in a semantical and historico-critical treatise on implication.

CHAPTER III

CONSISTENCY AND COMPLETENESS FOR FORMAL SYSTEMS

SECTION A

CONSISTENCY

Df.1. — A formal system F is said to be *consistent* if not all wffs are theorems, thus if $\mathbf{K}_F(\Lambda)$ is a proper subset of S. If F is not consistent it is called *inconsistent.*

The above is the usual syntactical formulation of the intuitive notion of consistency. The intuitive notion is that a system is consistent if nothing which is a theorem is, at the same time, a falsehood. But, where there is no semantic notion of falsebood, we must content ourselves with some reasonable abstraction of the semantic notion.

Kemeny [I] attacks the above definition as being inadequate, exhibiting a system which he feels is intuitively inconsistent but which is consistent according to the above criterion. However, where we restrain ourselves to purely syntactical considerations, Df.1 represents, we feel, a reasonable and useful generalization of the semantic notion, and we will accept it as our working definition of consistency.

Th.1. — A system F is inconsistent if and only if S has only one **e**-equivalence class, the set S itself.

Proof : If F is inconsistent, then every wff of F is a theorem. But, by I.Th.17 above, the set of theorems is an **e**-equivalence class is any system whatever. Thus, there is only one **e**-equivalence class, the set S itself.

Conversely, suppose that S has only one **e**-equivalence class, the set S. Now, by I.Df.2, the set of axioms and thus the set of theorems of F is non-void. Thus, every wff of F is equivalent to a theorem. Thus, every wff of F is a theorem and F is thus inconsistent.

Corollary. If F and F′ are two inconsistent systems, then F is equivalent to F′.

Proof : Let the mapping g map all of the wffs of F onto some arbitrarily chosen wff of F′. This mapping satisfies I.Df.22 and thus F and F′ are equivalent.

Th.2. — If F is equivalent to F′ and if F is inconsistent then F′ is also.

Proof : By I.Th.23, if two systems are equivalent, then they have the same number of **e**-equivalence classes. But F is inconsistent and thus has only one **e**-equivalence class. Thus, F′ has only one **e**-equivalence class and is thus inconsistent by Th.I.

Hence, we see that, if we consider two equivalent formal systems as identical, there is only one inconsistent system which is represented by any particular inconsistent system. This reminds one very much of the null set of set theory.

Th.3. — If F is equivalent to F′ and if F is consistent, then F′ is also.

Proof : If F′ is inconsistent, then F is also. But F is consistent, thus so is F′.

Thus, consistency is preserved by equivalence.

We could proceed to a number of different results concerning consistency and the various other relations that we have defined between formal systems. We will not go into such detail, but the following theorem is perhaps interesting.

Th.4. — If there exists a homomorphism g from F to F′, then if F is inconsistent, so is F′.

Proof : Consider any wff b' of F′. Since g is surjective, there exists a wff $b \in S$ such that $g(b) = b'$. Now F is inconsistent. Thus, every wff of F is a theorem. Thus, $b \in \mathbf{K}_F(\Lambda)$. But g preserves the consequence relation. Thus, $b' = g(b) \in \mathbf{K}_{F'}(g(\Lambda)) = T'$. Thus, every wff of F′ is a theorem of F′ and F′ is also inconsistent.

Th.5. — If g is a homomorphism from F to F′ and if F′ is consistent, then F is also.

Proof : If F is inconsistent, then, by Th.4. F′ is also inconsistent. But F′ is consistent, thus F is also.

Finally, it is perhaps worth the effort to have the following theorem :

Th.6. — Any system F which is formalized within the predicate calculus of first (or higher) order is inconsistent if and only if we can deduce a contradiction $\sim B \,\&\, B$ in F where B is any wff of F.

Proof : If F is inconsistent, then every wff is a theorem of F. In particular, wffs of the form $\sim B \,\&\, B$ exist and are thus theorems of F.

Conversely, suppose that some wff of the form $\sim B \,\&\, B$ is a theorem of F. Now, $\sim B \,\&\, B . \supset . H$ is a theorem of F for every wff H of F. Thus, by *Modus Ponens*, every wff H of F is a theorem.

SECTION B

COMPLETENESS

Of all the semantic notions, completeness is undoubtedly one of the most difficult for which to find a meaningful syntactical generalization. Intuitively, a system is complete if it is consistent and if there is some set of truths such that a wff of F is a theorem if and only if it represents a member of the given set of truths. Obviously the notion is relative to the set of truths we choose and this obviously depends on semantic interpretations of the system. However, we can consider the notion of *saturation* which is at least as old as Post [I], p. 177.

Df.2. — A system F is *saturated* if F consistent and if, for every $x \in S$, either $\vdash_F x$ or it is the case that the system F′, obtained from F by adding x as an axiom but guarding the same sets S and R, is inconsistent.

We will always use the term « saturated » in the sense of Df.2 and will thus guard the term « complete » for the semantical sense described above. This usage should be born in mind in the following exposition.

Th.7. — A system F is saturated if and only if S has exactly two **e**-equivalence classes, the set of theorems and the set of non-theorems of F.

Proof : If F is saturated, then F is consistent and thus S has more than one **e**-equivalence class. Now, where x is any non-theorem of F, consider the system $F' = \langle S, A', R\rangle$ where $A' = A \cup \{x\}$. Since F is saturated, F′ is inconsistent which means that $\mathbf{K}_{F'}(\Lambda) = S$. Now, $\mathbf{K}_{F'}(\Lambda) = K^{\circ}_{F'}(A' \cup \Lambda) = K^{\circ}_{F'}(A \cup \{x\}) = K^{\circ}_{F}(A \cup \{x\})$ (since the set of rules is the same for both F and F′). But $K^{\circ}_{F}(A \cup \{x\}) = \mathbf{K}_F(\{x\})$. Hence, for every $x \in S$, $\mathbf{K}_F(\{x\}) = S$. Thus, any two non-theorems of F are each in the other's set of consequences and thus all non-theorems are **e**-equivalent. Since the set of theorems also forms an **e**-equivalence class, we have that F has no more than two **e**-equivalence classes. Thus, S has exactly two **e**-equivalence classes, the theorems and the non-theorems of F.

Conversely, suppose that S has exactly two **e**-equivalence classes. Since the set of theorems is always non-void and always constitutes an **e**-equivalence class, it follows that the two classes are the set of theorems and the set of non-theorems of F. Now, consider any $x \in S$. If x is a non-theorem, then x is equivalent to every other non-theorem. Thus, the set N of non-theorems is contained in $\mathbf{K}_F(\{x\})$. But also $T \subset \mathbf{K}_F(\{x\})$. Thus, $S = T \cup N \subset \mathbf{K}_F(\{x\})$ and thus $\mathbf{K}_F(\{x\}) = S$ for all $x \in S$. Reversing the string of equalities which appear in the first part of this proof, we find that $F' = \langle S, A \cup \{x\}, R\rangle$ is inconsistent and thus that F is saturated.

Th.8. — If F and F′ are saturated, then they are equivalent.

Proof : Consider the System $W = \langle \{1, 0\}, \{1\}, \Lambda\rangle$ of which we have already spoken in Chapter I, Section B, § 1 above. This system has exactly two **e**-equivalence classes and is thus saturated. Now consider any saturated system F, and consider the mapping g which maps all of the theorems of F onto I and all of the non-theorems of F onto 0. This mapping is easily seen to satisfy the requirements of I.Df.22. Thus F is equivalent to W. Thus, by the transitivity of equivalence, F is equivalent to any saturated system F′.

Th. 9. — If F is equivalent to F′ and if F is saturated, then F′ is also.

Proof : If F is equivalent to F′, then both have the same number of **e**-equivalence classes by I.Th.23. Since F is saturated, S has exactly two **e**-equivalence classes and so S′ have exactly two **e**-equivalence classes. Hence, by Th.7, F′ is saturated.

As with inconsistency, we again see that the notion of saturation uniquely determines an equivalence class on the set of all formal systems. The system W can be thought of as a representative of this class and this shows us the trivial nature of the notion of saturation, and thus of completeness in this sense.

We also see that the notion of equivalence between formal systems reduces, in the case of one or two **e**-equivalence classes, to the notion « to have the same number of **e**-equivalence classes ». However, it is *only* in these two cases that we have this reduction. The property of having the same number of **e**-equivalence classes, though always necessary in view of I.Th.23, is not sufficient in any other case. This could be proven by transfinite induction, but the few counterexamples and examples that we have already considered in Chapter I should suffice to show this.

SECTION C

SATURATION AND THE PROPOSITIONAL CALCULUS

One might recall that in such works as Post [1], p. 173, and Church [4], p. 110, the propositional calculus is proved saturated in our sense and thus would have two **e**-equivalence classes. Yet we have demonstrated that **P** has and infinite number of **e**-equivalence classes. Is this not a contradiction ?

The difference lies in the formulation of the propositional calculus. The formulations of Church and Post use proper axioms, and a general rule of substitution is taken as primitive. By this rule of substitution, we can infer any wff from any non-theorem. Their formulations are thus not equivalent to **P** (which we have taken as the definitive form of the propositional calculus) nor are they equivalent to other formulations which use axiom schemata as we have done (though, as Church demonstrates in Church [4], pp. 149-150, the set of *theorems* is « the same » in both cases). The formulation with proper axioms can be made equivalent to

P by an extremely simple device. Instead of taking a general rule of substitution as primitive, we can instead allow substitution only from the set of wffs designated as axioms, though bere we need a stronger notion of substitution, that of *simultaneous* substitution (see Church [4], p. 82 ff). Of course there is the drawback that, in this form, our rule practically amounts to assuming schemata in the first place, and we have never seen this rule taken as primitive in a formulation of the propositional calculus. However, it is easily seen that this rule satisfies all of the requirements of I.Df.2.3 and is thus quite legitimate as a primitive rule of inference.

Moreover, this new formulation enjoys, with **P**, an advantage which is not shared by the formulations of Church and Post mentioned above; the deduction theorem is true for this new formulation as it is for **P**.[8] But, with the systems of Church and Post, the deduction theorem is *not* true, unless, of course, we place certain *ad hoc* restrictions on the notion of « deduction from hypotheses » (see Church [4], p. 86 ff and compare with Kleene [5], p. 87 ff). It was, in fact, the deduction theorem which allowed us to identify **e**-equivalence with « $\vdash_{\mathbf{P}} \ldots \equiv$ — — — » and thus facilitate the proof of many of our theorems and it was this identification which allowed us to prove that **P** has an infinite number of **e**-equivalence classes. Kleene [5], p. 140 contains a brief discussion of certain of these points which we have discussed. In particular, he points out that *ad hoc* restrictions are necessary for the deduction theorem to be true for formulations which take the general substitution rule as primitive. As is clear from the above theory, and as we have clearly stated, however, we cannot, as does Kleene, consider these different formulations as determining « essentially the same calculus » (see Kleene [5], p. 140). In fairness, it must be said that Kleene is perfectly aware of the differences of deductive structure of these two approaches and is thus aware of the differences which he is willingly ignoring in considering these various formulations as determining the same system.

By way of further comparison between our definition and that of Kemeny [1] (see the Introduction of this study), it should be noted that, by Kemeny's definition of equivalence by models, our **P** would be equivalent to formulations such as those of Church and Post.

In Hatcher [2], we have given the necessary and sufficient conditions for a formal system F to be equivalent to our **P**.

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[8] It should be mentioned again that « **P** » stands ambiguously for P_1, P_2, or P_3 and *not* for the equivalence class of formal systems determined by these three systems (see § 4, Section A, Chapter II). Otherwise, in view of the discussion following I.Th.31, there would be somewhat of an equivocation in speaking of a deduction theorem for **P**. The equivocation involved would be slight, however, since it would concern only certain constructive features of the deduction theorem, but not the existence of the mapping I.Df.26.2. And it is only the *existence* of this mapping which has ever been essential to the theory of Chapters II and II, the constructive features being incidentally true.

BIBLIOGRAPHY

Ackermann, W.
[1] *Solvable cases of the decision problem.* North-Holland Publishing Co., Amsterdam, 1954.

Bernays, P.
[1] « Axiomatische Untersuchung des Aussagenkalküls der *Principia Mathematica.* » *Mathematische Zeitschrift,* vol. 25 (1926), pp. 305-320.

Bernays, P. and Fraenkel, A.
[1] *Axiomatic set theory.* North-Holland Publishing Co., Amsterdam, 1958.

Boole, G.
[1] *An investigation of the laws of throught.* Dover Publications, New York, re-edition of original version first published in 1854.

Bourbaki, N.
[1] *Eléments de mathématique.* Livre I, Théorie des ensembles ; fascicule des résultats. Hermann et Cie, Paris, 1939.
[2] *Eléments de mathématique.* Livre II, Algèbre ; chapitre I, Structures algèbriques. Hermann et Cie, Paris, 1942.
[3] *Eléments de mathématique.* Livre I, Théorie des ensembles ; chapitre I, Description de la mathématique formelle, et chapitre II, Théorie des ensembles. Hermann et Cie, Paris, 1954.
[4] *Eléments de mathématique.* Livre III, Topologie ; chapitre I, Structures topologiques ; chapitre II, Structures uniformes. Troisième édition entièrement refondue. Hermann et Cie, Paris, 1961.

Byrne, L.
[1] « Two brief formulations of Boolean algebra. » *Bulletin of the American Mathematical Society,* vol. 52 (1946), pp. 269-272.

Carnap, R.
[1] *The logical syntax of language.* Second printing. Routledge and Kegan Paul, London, 1949.

Church, A.
[1] « An unsolvable problem of elementary number theory. » *American journal of mathematics,* vol. 58 (1936), pp. 345-363.
[2] « A note on the Entscheidungsproblem. » *The journal of symbolic logic,* vol. 1 (1936), pp. 40-41. Correction, *ibid.,* pp. 101-102.
[3] « Special cases of the decision problem. » *Revue philosophique de Louvain,* vol. 49 (1951), pp. 203-221. A correction, *ibid.,* vol. 50 (1952), pp. 270-272.
[4] *Introduction to mathematical logic.* Princeton University Press, Princeton, New Jersey, 1956.

Davis, M. [1] *Computability and unsolvability*. McGraw-Hill, New York, 1958.

Dopp. J. [1] *Leçons de logique formelle*. Deuxième partie, Logique moderne I. Institut supérieur de philosophie, Louvain, 1950.

Dubreil, P. and Dubreil-Jacotin, M. L. [1] *Leçons d'algèbre moderne*. Dunod, Paris, 1961.

Frege, G. [1] *Begriffsschrift*. Halle, 1879.

Gödel, K. [1] *On formally undecidable propositions of Principia Mathematica and related systems*. Oliver and Boyd, London, 1962. Translation from the German by B. Meltzer of Gödel's article in *Monatshefte für Mathematik und Physik*, vol. 38 (1931), pp. 173-198.

Hatcher, W. [1] « La notion d'équivalence entre systèmes formels et une généralisation inductive du système dit *New Foundations* de Quine. » *Comptes rendus hebdomadaires des séances de l'Académie des Sciences* (Paris), vol. 256 (1963), pp. 563-566.

[2] « La notion d'équivalence entre systèmes formels. » *L'enseignement mathématique*, fascicule 3-4, vol. 10 (1964), pp. 8-9.

Huntingdon, E.V. [1] « Mathematical postulates for the logical operations of assertion, conjunction, negation and equality. » *Proceedings of the National Academy of Sciences*, vol. 22 (1936), pp. 291-296.

Kemeny, J. [1] « Models of logical systems. » *The journal of symbolic logic*, vol. 13 (1948), pp. 16-30.

Kleene, S. C. [1] « Proof by cases in formal logic. » *Annals of mathematics*, vol. 35 (1934), pp. 529-544.

[2] « λ-definability and recursiveness. » *Duke mathematical journal*, vol. 2 (1936), pp. 340-353.

[3] « General recursive functions of natural numbers. » *Mathematische Annalen*, vol. 112 (1936), pp. 727-742.

[4] « A note on recursive functions. » *Bulletin of the American Mathematical Society*, vol. 42 (1936), pp. 544-546.

[5] *Introduction to metamathematics*. D. Van Nostrand Co., Princeton, New Jersey, 1952.

Kreisel, G. [1] « Models, translations, and interpretations. » *Mathematical interpretation of formal systems*, pp. 26-50. North-Holland Publishing Co., Amsterdam, 1955.

Lewis, C. I. [1] *A survey of symbolic logic*. Dover Publications, New York, 1960, a corrected republication of the original work of the same name which was published by the University of California Press in 1918. The present edition omits chapters V and VI.

Lewis, C. I. and Langford, C. H. [1] *Symbolic logic*. Second edition. Dover Publications, 1959.

Myhill, J. [1] « Creative sets », *Zeitschrift für mathematische Logik und Grundlagen der Mathematik,* vol. 1 (1955), pp. 97-108.

Neumann, J. v. [1] « Zur Hilbertschen Beweistheorie. » *Mathematische Zeitschrift,* vol. 27 (1927), pp. 1-46.

Nicod, J. [1] « A reduction in the number of primitive propositions of logic. » *Proceedings of the Cambridge Philosophical Society,* vol. 19 (1917), pp. 32-41.

Peirce, C. S. [1] *Collected papers.* Edited by C. Hartshorne and P. Weiss, 6 vols. ; Belknap Press of Harvard University, Cambridge, Mass., 1931-35.

Post, E. L. [1] « Introduction to a general theory of elementary propositions. » *American journal of mathematics,* vol. 43 (1921), pp. 163-185.

[2] « Recursively enumerable sets of positive integers and their decision problems. » *Bulletin of the American Mathematical Society,* vol. 50 (1944), pp. 284-316.

Quine, W. V. [1] *Mathematical logic.* Revised edition, Harvard University Press, Cambridge, Mass., 1955.

[2] « Whitehead and the rise of modern logic. » Article in *The philosophy of Alfred North Whitehead,* edited by P. A. Schlipp, pp. 125-163 ; Tudor Publishing Co., New York, 1941.

[3] *Word and object.* The Technology Press of the Massachusetts Institute of Technology and John Wiley and Sons, New York, 1960.

Rosenbloom, P. C. [1] *The elements of mathematical logic.* Dover Publications, New York, 1950.

Rosser, J. B. [1] « A mathematical logic without variables. » *Annals of mathematics,* vol. 36 (1935), pp. 127-150 ; *Duke mathematical journal,* vol. 1 (1935), pp. 328-355.

Sheffer, H. M. [1] « A set of five independent postulates for Boolean algebras. » *Transactions of the American Mathematical Society,* vol. 14 (1913), pp. 481-488.

Schröder, E. [1] *Vorlesungen über die Algebra der Logik.* 3 vols. Teubner, Leipzig, 1890-1905.

Smullyan, R. M. [1] *Theory of formal systems.* Princeton University Press, Princeton, New Jersey, 1961.

Tarski, A. [1] *Logic, semantics, metamathematics.* Oxford at the Clarendon Press, England, 1956.

Wajsberg, M. [1] « Uber Axiomensysteme des Aussagenkalküls. » *Monatshefte für Mathematik und Physik,* vol. 39 (1932), pp. 259-282.

Wang, H. [1] « Arithmetic translations of axiom systems. » *Transactions of the American Mathematical Society,* vol. 71 (1951), pp. 283-293.

Whitehead, A. N. and Russell, B. [1] *Principia mathematica.* 3 vols. Cambridge University Press, Cambridge, England, 1910-1913 ; Second edition, 1925-1927.

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