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Faculté des Sciences
Institut de mathématiques
Rue Emile-Argand 11, 2000 Neuchâtel

Inégalités géométriques pour des valeurs propres de Steklov de graphes et de surfaces

Thèse

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par

Hélène Perrin

Acceptée sur proposition du jury :

Prof. Bruno Colbois, directeur de thèse, Université de Neuchâtel, Suisse
Prof. Alexandre Girouard, rapporteur, Université Laval, Québec, Canada
Prof. Asma Hassannezhad, rapporteure, University of Bristol, Royaume-Uni
Prof. Alain Valette, Université de Neuchâtel, Suisse

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unine
Université de Neuchâtel
Faculté des sciences

Rue Emile-Argand 11
CH-2000 Neuchâtel
doctorat.sciences@unine.ch

IMPRIMATUR POUR THESE DE DOCTORAT

La Faculté des sciences de l'Université de Neuchâtel autorise l'impression de la présente thèse soutenue par

Madame Hélène PERRIN

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"Inégalités géométriques pour des valeurs propres de Steklov de graphes et de surfaces"

sur le rapport des membres du jury composé comme suit :

- Prof. Bruno Colbois, directeur de thèse, Université de Neuchâtel, Suisse
- Prof. Alain Valette, Université de Neuchâtel, Suisse
- Prof. Alexandre Girouard, Université Laval, Québec, Canada
- Prof. Asma Hassannezhad, University of Bristol, UK

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Résumé

Cette thèse est consacrée à l'obtention d'inégalités géométriques pour des valeurs propres de Steklov de variétés riemanniennes de dimension 2 et de graphes. Les résultats obtenus concernent différentes situations. D'un côté, je m'intéresse à la géométrie de la première valeur propre non nulle de Steklov σ_1 d'un graphe à bord. Pour cette valeur propre, je donne une borne inférieure qui dépend d'une borne supérieure sur le diamètre extrinsèque du bord et d'une borne supérieure sur le nombre de sommets du bord. Un autre résultat est une borne supérieure pour certains sous-graphes d'un graphe de Cayley à croissance polynomiale, qui montre en particulier que σ_1 tend vers 0 lorsque le nombre de sommets du sous-graphe tend vers l'infini et généralise ainsi un résultat de Han et Hua obtenu pour des sous-graphes de $\mathbb{Z}^n$. Un deuxième but de la thèse est d'obtenir des bornes inférieures pour la première valeur propre non nulle de Steklov σ_1 d'une variété riemannienne M dont le bord a plusieurs composantes connexes. Dans ce cas, la géométrie de M loin du bord peut avoir une forte influence sur σ_1. Afin de préciser la forme de cette relation on étudie les variétés riemanniennes dont le bord a un voisinage cylindrique. En dimension 2, en supposant que la courbure de Gauss de M est bornée inférieurement, je donne une borne inférieure qui dépend d'une borne supérieure sur le diamètre extrinsèque du bord, d'une borne supérieure sur la longueur du bord et d'une borne inférieure sur la rayon d'injectivité des points d'un certain sous-ensemble de M. Finalement, je donne des bornes inférieure et supérieure pour les premières valeurs propres de Steklov d'une surface hyperbolique à bord géodésique en fonction de la longueur de certaines familles de géodésiques qui séparent le bord. Ce résultat est similaire à un résultat classique de Schoen, Wolpert et Yau pour les valeurs propres du laplacien d'une surface hyperbolique fermée.

Mots-clés : géométrie spectrale, problème de Steklov, opérateur Dirichlet-Neumann, valeur propre, borne inférieure, borne supérieure, graphe à bord, surface hyperbolique.

Abstract

The aim of this thesis is to obtain geometric inequalities for Steklov eigenvalues of 2-dimensional Riemannian manifolds and graphs. The results obtained relate to different situations. On the one hand, our interest focuses on the geometry of the first non-zero Steklov eigenvalue σ_1 of a graph with boundary. For this eigenvalue, we give a lower bound which depends on an upper bound on the extrinsic diameter of the boundary and on an upper bound on the number of vertices of the boundary. Another result is an upper bound for some subgraphs of a Cayley graph with polynomial growth, which shows in particular that σ_1 tends to 0 when the number of vertices of the subgraph tends to infinity and thus generalizes a result of Han and Hua obtained for subgraphs of $\mathbb{Z}^n$. A second goal of the thesis is to obtain lower bounds for the first non-zero Steklov eigenvalue σ_1 of a Riemannian manifold M whose boundary has several connected components. In this case, the geometry of M far from the boundary can have a strong influence on σ_1. In order to specify the form of this relation we study Riemannian manifolds whose boundary has a cylindrical neighborhood. In dimension 2, assuming that the Gaussian curvature of M is bounded below, we give a lower bound which depends on an upper bound on the extrinsic diameter of the boundary, an upper bound on the length of the boundary and a lower bound on the radius of injectivity at the points of a certain subset of M. Finally, we give lower and upper bounds for the first Steklov eigenvalues of hyperbolic surfaces with geodesic boundary, which depend on the length of some families of geodesics that separate the boundary. This result is similar to a classical result of Schoen, Wolpert and Yau for Laplace eigenvalues of a closed hyperbolic surface.

Keywords : spectral geometry, Steklov problem, Dirichlet-to-Neumann operator, eigenvalue, lower bound, upper bound, graph with boundary, hyperbolic surface.

Table des matières

1. Introduction

1.1 Valeurs propres de Steklov et inégalités géométriques

On connaît sous le nom de problème de Steklov sur une variété riemannienne compacte M à bord ∂M le système d'équations différentielles

$$\begin{cases} \triangle u = 0 & \text{dans } M \\ \partial_\nu u = \sigma u & \text{sur } \partial M \end{cases}$$

où $\triangle$ est l'opérateur de Laplace-Beltrami et ∂_ν est la dérivée normale extérieure. Les nombres $\sigma \in \mathbb{R}$ pour lesquels ce problème admet une solution non nulle sont en fait les valeurs propres de l'opérateur Dirichlet-Neumann $\Lambda : C^\infty(\partial M) \to C^\infty(\partial M)$ qui à une fonction $f \in C^\infty(\partial M)$ associe $\Lambda f = \partial_\nu u_f$ où u_f signifie l'extension harmonique de f à l'intérieur de M. Le spectre des valeurs propres forme la suite $0 = \sigma_0(M) \leq \sigma_1(M) \leq \sigma_2(M) \leq \cdots \nearrow \infty$ où elles sont répétées en fonction de leur multiplicité. Ces valeurs propres sont communément appelées valeurs propres de Steklov.

Excepté la première valeur propre 0 qui est commune à toutes, le reste du spectre dépend de la variété riemannienne considérée. Le but des inégalités géométriques est de comparer les valeurs propres à d'autres grandeurs géométriques afin de comprendre leur signification. Ces inégalités sont aussi souvent appelées inégalités isopérimétriques en référence à l'inégalité isopérimétrique classique $L^2 \geq 4\pi A$ qui exprime le fait que le disque est le domaine du plan de périmètre fixé L qui a la plus grandre aire A.

Dans cette thèse, je donne des résultats, principalement pour la première valeur propre non nulle de Steklov, dans deux contextes différents : les variétés riemanniennes dont la géométrie au voisinage du bord est contrôlée et les graphes. Le problème de Steklov peut en effet être envisagé dans une version discrète, où la variété riemannienne est remplacée par un graphe sur lequel des sommets sont distingués pour former un bord. Ces deux approches ne sont pas complètement indépendantes, car, comme nous le verrons, les questions qui se posent sur les variétés riemanniennes ont souvent leur équivalent sur les graphes et un résultat obtenu dans un des contextes permet parfois d'en déduire un dans l'autre.

Les qualités que nous recherchons dans une inégalité géométrique, c'est-à-dire dans une borne inférieure (ou supérieure) du type $\sigma_1(M) \geq A(M)$ (respectivement $\sigma_1(M) \leq A(M)$), où $A(M)$ est une combinaison de grandeurs géométriques, sont de plusieurs sortes.

1. Il existe une variété riemannienne pour laquelle l'égalité est réalisée. Cela est vrai pour l'inégalité isopérimétrique classique mentionnée plus haut puisque l'égalité est réalisée par le disque. Ce dernier résultat possède une qualité supplémentaire de rigidité puisque le disque est l'unique domaine pour lequel cela se produit. J'obtiens des cas d'égalité pour les théorèmes 2.1 et 2.2 qui sont des résultats sur les graphes.

2. Nous avons deux bornes, inférieure et supérieure, pour une valeur propre $\sigma_1(M)$ qui dépendent de la même quantité géométrique $A(M)$. Un tel résultat a été obtenu pour la constante de Cheeger et la première valeur propre non nulle du laplacien sur une variété

riemannienne fermée : Cheeger a d'abord donné une borne inférieure dépendant de la constante qui porte son nom [Che70] et Buser a ensuite obtenu une borne supérieure en ajoutant une hypothèse sur la courbure de Ricci de la variété [Bus82]. Il semble cependant difficile en général de produire une grandeur géométrique explicite qui soit ainsi équivalente à une valeur propre donnée. Un des moyens pour s'approcher d'un tel résultat est de se restreindre à certaines familles de variétés riemanniennes. C'est ce que je fais au théorème 4.3.

3. L'exposant des quantités géométriques utilisées dans l'expression A(M) ne peut pas être amélioré. Etant donné une borne inférieure $\sigma_1(M) \geq \alpha(M)^j B(M)$, c'est ce que l'on montre en donnant une famille de variétés $(M_n)_{n>0}$ pour lesquelles on a des constantes C_1 et C_2 telles que $C_1\alpha(M_n)^j \leq \sigma_1(M_n) \leq C_2\alpha(M_n)^j$ et $\alpha(M_n) \to 0$ lorsque $n \to \infty$. Les bornes inférieures données aux théorèmes 4.1 et 4.2 possèdent cette propriété.

Pour obtenir de bonnes bornes, une des difficultés consiste à choisir les grandeurs géométriques avec lesquelles on compare les valeurs propres. Dans cette thèse, j'en introduis plusieurs qui dépendent de la géométrie extrinsèque du bord dans la variété et semblent pertinentes pour comprendre le comportement des valeurs propres de Steklov.

Avant de passer aux principaux résultats, je rappelle deux propriétés importantes des valeurs propres de Steklov. Les valeurs propres sont caractérisées par la formulation variationnelle

$$\sigma_k(M) = \min_{E \in V_k} \max_{0 \neq u \in E} R(u),$$

où V_k est l'ensemble des sous-espaces de dimension $k+1$ de l'espace de Sobolev $H^1(M)$, et $R(u)$ est le quotient de Rayleigh associé au problème de Steklov,

$$R(u) = \frac{\int_M |\nabla u|^2 dv_g}{\int_{\partial M} u^2 dS_g}.$$

En calculant le rapport entre les quotients de Rayleigh d'une fonction f de $H^1(M)$ pour la métrique g et la métrique $c^2 g$, on établit facilement un premier lien entre spectre et géométrie : le rapport entre le spectre d'une variété riemannienne (M, g) et de son homothétique $(M, c^2 g)$, avec $c > 0$, est donné par la relation

$$\sigma_k(M, c^2 g) = \frac{1}{c}\sigma_k(M, g).$$

Le reste de cette introduction est consacré à présenter les principaux résultats de la thèse qui, en plus de l'introduction, est composée de trois chapitres. Les deux premiers chapitres sont des articles qui ont déjà été publiés (voir [Per19] et [Per21]) et dont le texte est repris sans modification. Le troisième chapitre est un article qui n'est pas encore publié mais dont la version actuelle du texte est disponible sur arXiv (voir [Per23]). J'avertis ici encore du fait que les notations utilisées peuvent varier d'un chapitre à l'autre en raison du caractère composite de cette thèse, qui est une réunion d'articles destinés à être publiés indépendamment. Lorsqu'une notion n'est pas définie dans l'introduction, je renvoie au chapitre en question pour plus de détails.

1.2 Le problème de Steklov sur les graphes

1.2.1 Définition du problème

L'étude du problème de Steklov sur les graphes a été initiée récemment dans [CGR18], [HHW17] et [HM20] indépendamment. Dans [CGR18], les auteurs l'utilisent comme moyen en vue d'obtenir des résultats sur les variétés riemanniennes. Dans [HHW17] et [HM20], le problème de Steklov sur les graphes est établi comme objet d'étude à part entière.

Un graphe est une paire (V, E) où V est un ensemble d'éléments, appelés sommets, et E un ensemble de paires de sommets, appelées arêtes. Etant donnés $i, j \in V$, on utilise la notation $i \sim j$ pour signifier que $(i, j) \in E$. $E(\Omega_1, \Omega_2) := \{\{i, j\} \in E : i \in \Omega_1, j \in \Omega_2\}$ est l'ensemble des arêtes entre deux sous-ensembles $\Omega_1, \Omega_2 \subset V$.

Définition 1. *Un graphe à bord est une paire* (Γ, B) *où* $\Gamma = (V, E)$ *est un graphe simple et* $B \subset V$ *est un sous-ensemble de* V *tel que* $B \neq \emptyset$ *et* $E(B, B) = \emptyset$*. On appelle* B *le bord du graphe et* $V \setminus B$ *l'intérieur du graphe.*

Dans cette thèse, nous étudions uniquement le problème de Steklov sur des graphes à bord finis. L'espace des fonctions sur l'ensemble des sommets V est identifié à $\mathbb{R}^{|V|}$, et l'espace des fonctions sur le sous-ensemble B est identifié à $\mathbb{R}^{|B|}$. Sur un graphe $\Gamma = (V, E)$, l'opérateur de Laplace $\triangle : \mathbb{R}^{|V|} \to \mathbb{R}^{|V|}$ est défini par

$$(\triangle v)(i) = \sum_{j \sim i} (v(i) - v(j)).$$

La dérivée normale sur le bord $\frac{\partial}{\partial n} : \mathbb{R}^{|V|} \to \mathbb{R}^{|B|}$ est définie par

$$\left(\frac{\partial v}{\partial n}\right)(i) = \sum_{j \sim i} (v(i) - v(j)).$$

Définition 2. *Le problème de Steklov sur un graphe à bord* (Γ, B) *fini est de trouver les* $\sigma \in \mathbb{R}$ *pour lesquels le système*

$$\begin{cases} (\triangle v)(i) = 0 & si\ i \notin B \\ (\frac{\partial v}{\partial n})(i) = \sigma v(i) & si\ i \in B \end{cases}$$

admet une solution $v \in \mathbb{R}^{|V|}$ *non nulle.*

Dans [Per19], j'ai montré que les solutions de ce problème coïncident avec les valeurs propres de l'opérateur Dirichlet-Neumann discret défini dans [HHW17]. Ces valeurs propres forment une suite $0 = \sigma_0(\Gamma, B) \leq \sigma_1(\Gamma, B) \leq ... \leq \sigma_{|B|-1}(\Gamma, B)$, où elles sont répétées en fonction de leur multiplicité. Lorsque le graphe est connexe, ce que nous supposerons toujours, la multiplicité de la valeur propre 0 est simple, ce qui fait de σ_1 la première valeur propre non nulle.

Comme sur une variété riemannienne, les valeurs propres sont caractérisées par une formulation variationnelle. On a

$$\sigma_j = \min_E \max_{v \in E, v \neq 0} R(v),$$

où E est l'ensemble des sous-espaces vectoriels de dimension $j + 1$ de $\mathbb{R}^{|V|}$ et $R(v)$ est le quotient de Rayleigh associé à l'opérateur Dirichlet-Neumann discret

$$R(v) = \frac{\sum_{i\sim j}(v(i) - v(j))^2}{\sum_{i\in B} v(i)^2}.$$

Deux exemples simples permettent de se rendre compte que la première valeur propre non nulle d'une famille de graphes peut tendre vers 0 ou vers l'infini.

Exemple 1. *On considère la famille de graphes linéaires $(L_m)_{m\geq 2}$ à m sommets, dont les extrémités forment le bord. Par calcul, on obtient que $\sigma_1(L_m, B_{L_m}) = \frac{2}{m-1}$, qui tend vers 0 lorsque m tend vers l'infini. Considérons maintenant la famille $(G_n, B_{G_n})_{n\geq 1}$ de graphes formés de n copies de (L_3, B_{L_3}) dont on a identifié les bords comme indiqué à la figure 3.1.. On a alors $\sigma_1(G_n, B_{G_n}) = n$ et donc $\sigma_1 \to \infty$ lorsque $n \to \infty$.*

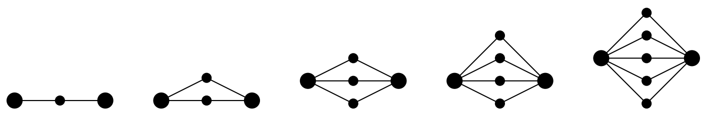

FIGURE 1.1. – $(G_1, B_{G_1}), (G_2, B_{G_2}), (G_3, B_{G_3}), (G_4, B_{G_4})$, et (G_5, B_{G_5}).

1.2.2 Estimations de σ_1

En observant la caractérisation variationnelle des valeurs propres d'un graphe à bord (Γ, B), on constate que $\sigma_k(\Gamma, B) \geq \lambda_k(\Gamma)$, où $\lambda_k(\Gamma)$ est la k-ème valeur propre du laplacien sur Γ. Cela a pour conséquence que toutes les bornes inférieures connues pour la première valeur propre non nulle du laplacien sur un graphe deviennent des bornes inférieures pour la première valeur propre non nulle de Steklov. Il est par exemple bien connu que pour un graphe connexe $\Gamma = (V, E)$, $\lambda_1(\Gamma) \geq \frac{1}{d|V|}$ (voir [Chu97], lemme 1.9) où d est le diamètre du graphe. Cette borne inférieure n'est cependant pas satisfaisante pour la première valeur propre non nulle de Steklov comme le suggère l'exemple ci-dessous.

Exemple 2. *On considère la famille de graphes à bord $(D_{n+3}, B_{D_{n+3}})_{n\in\mathbb{N}}$ formés d'un graphe linéaire à trois sommets (les deux extrémités forment le bord) auquel on ajoute une tige de longueur n (voir figure 2.1.). Le diamètre du graphe tend vers l'infini lorsque n tend vers l'infini mais $\sigma_1(D_{n+3}, B_{D_{n+3}}) = 1$ $\forall n \in \mathbb{N}$.*

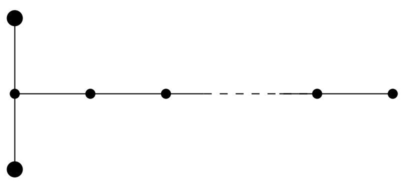

FIGURE 1.2.

Dans [Per19], j'ai pu obtenir une borne inférieure pour σ_1 qui dépend du diamètre extrinsèque et du nombre de sommets du bord.

Définition 3. *Soit* (Γ, B) *un graphe à bord. Le diamètre extrinsèque du bord est*

$$d_B := \max\{d(i,j)|i,j \in B\}$$

où la distance $d(i,j)$ *entre deux sommets est le nombre d'arêtes du plus court chemin qui les relie.*

Théorème 1 (Théorème 2.2)**.** *Soit* (Γ, B) *un graphe à bord, connexe, dont le diamètre du bord est* d_B *et tel que* $|B| = b$*. On a*

$$\sigma_1 \geq \frac{b}{\lfloor \frac{b}{2} \rfloor \lceil \frac{b}{2} \rceil \cdot d_B}.$$

Lorsque $b = 2$*, l'égalité est réalisée pour chaque* d_B *par le graphe linéaire de diamètre* d_B*. Lorsque* $b > 2$*, il existe une famille de graphes* $((H^b)_{d_B})_{d_B \in \mathbb{N}}$ *telle que* $\sigma_1((H^b)_{d_B} = \frac{b}{\lfloor \frac{b}{2} \rfloor \lceil \frac{b}{2} \rceil \cdot d_B} + O(\frac{1}{d_B^2})$ *lorsque* $d_B \to \infty$*.*

Ce résultat est à comparer avec les bornes inférieures données dans [HHW17] qui dépendent de constantes isopérimétriques. Une de ces bornes inférieures, que l'on appelle estimation du type Jammes, est la version discrète d'un résultat de Jammes [Jam15] dont nous reparlerons. Dans certains cas, comme pour la famille de graphes donnée à l'exemple 2.1, le théorème ci-dessus montre que σ_1 est bornée inférieurement alors que l'estimation du type Jammes ne le détecte pas. On remarque encore que par la même observation que nous faisons pour les constantes isopérimétriques sur une variété riemannienne à bord à la section 4.2.2, on peut améliorer les constantes isopérimétriques discrètes de la borne inférieure du type Jammes de [HHW17].

Il y a cependant des situations où le diamètre extrinsèque du bord tend vers l'infini sans que σ_1 tende vers 0. C'est le cas, par exemple, pour la famille de graphes ci-dessous.

Exemple 3. *On considère la famille de graphes à bord* $(A_n, B_{A_n})_{n\in\mathbb{N}}$ *formés de deux arbres réguliers de hauteur n dont on identifie les feuilles, et dont les deux racines forment le bord, comme illustré à la figure 1.3. ci-dessous. En utilisant la proposition 2.24 et l'exemple 3.12 de [Per17], il est facile de calculer que* $\sigma_1(A_n, B_{A_n}) = \frac{2^n}{2^n-1}$*.*

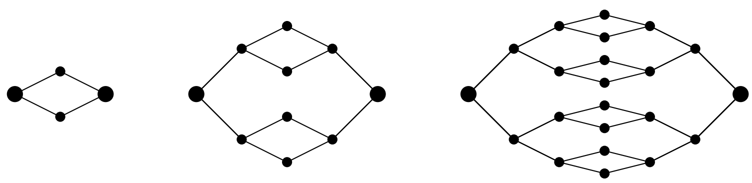

Figure 1.3. – $(A_1, B_{A_1}), (A_2, B_{A_2}), (A_3, B_{A_3})$.

1.2.3 Résultats pour les sous-graphes

Une inégalité classique de Brock affirme que pour un domaine Ω de $\mathbb{R}^n$, $\sigma_1(\Omega) \leq \frac{\omega_n^{1/n}}{|\Omega|^{1/n}}$, où ω_n est le volume de la boule unité de dimension n, avec égalité si et seulement si Ω est une boule. Une borne supérieure similaire à celle de Brock a pu être obtenue par Han et Hua [HH19] pour certains sous-graphes de $\mathbb{Z}^n$, que l'on appelle graphes à bord induits par un sous-ensemble de $\mathbb{Z}^n$.

Définition 4. *Soit* $\Gamma = (V, E)$ *un graphe et* $\Omega \subset V$*.*

1. *Le bord de* Ω *dans* Γ *est*

$$\delta\Omega := \{i \in V \setminus \Omega : \exists j \in \Omega, i \sim j\}.$$

2. *Soit* Γ' *le sous-graphe formé des sommets* $\bar{\Omega} := \Omega \cup \delta\Omega$ *et des arêtes* $E(\Omega, \bar{\Omega})$*, et* $B = \delta\Omega$*. On appelle* (Γ', B) *le graphe à bord induit par* Ω*.*
3. *On note* $\sigma_1(\Omega)$ *la valeur propre* σ_1 *du graphe à bord induit par* Ω*.*

Théorème 2 (Han, Hua [HH19])**.** *Soit* Ω *un sous-ensemble fini de* $\mathbb{Z}^n$*. Il existe des constantes* $C_1(n)$ *et* $C_2(n)$ *telles que*

$$\sigma_1(\Omega) \leq \frac{n}{C_1(n)|\Omega|^{\frac{1}{n}} - \frac{C_2(n)}{|\Omega|}}.$$

Dans [Per21], j'ai généralisé ce résultat à des graphes à bords induits par un sous-ensemble d'un graphe de Cayley à croissance polynomiale. Un graphe de Cayley associé au groupe de Heisenberg discret de dimension 3, dont la croissance est polynomiale d'ordre 4, est un exemple de graphe de Cayley à croissance polynomiale différent de $\mathbb{Z}^n$.

Théorème 3 (Corollaire 3.2)**.** *Soit* $\Gamma = (V, E)$ *un graph de Cayley à croissance polynomiale d'ordre* $n \geq 2$*. Il existe une constante* $C(\Gamma) > 0$ *telle que pour* $\Omega \subset V$ *fini et connexe, on a*

$$\sigma_1(\Omega) \leq C(\Gamma)\frac{1}{|\bar{\Omega}|^{\frac{1}{n}}},$$

où $\bar{\Omega} = \delta\Omega \cup \Omega$*.*

J'obtiens aussi le résultat suivant qui évoque un résultat de Colbois, El Soufi et Girouard [CESG11] qui dit que pour $n \geq 2$ il existe une constante $C(n)$ telle que pour tout domaine Ω de $\mathbb{R}^n$, de l'espace hyperbolique $\mathbb{H}^n$, ou d'un hémisphère de $\mathbb{S}^n$, on a $\sigma_k(\Omega) \leq \frac{C(n)k^{2/n}}{|\partial\Omega|^{1/n-1}}$.

Théorème 4 (Corollaire 3.1)**.** *Soit* $\Gamma = (V, E)$ *un graphe de Calyey à croissance polynomiale d'ordre* $n \geq 2$*. Il existe une constante* $C(\Gamma) > 0$ *telle que pour tout sous-ensemble fini et connexe* Ω *de* V *on a*

$$\sigma_1(\Omega) \leq C(\Gamma)\frac{1}{|\delta\Omega|^{\frac{1}{n-1}}}.$$

Ces deux résultats sont en fait des corollaires d'un résultat plus général pour les graphes à bords inclus dans un graphe de Cayley à croissance polynomiale, qui ne sont pas nécessairement induits par un sous-ensemble. La figure 1.4. illustre la différence entre les deux objets.

Théorème 5 (Théorème 3.1)**.** *Soit* $\Gamma = (V, E)$ *un graphe de Cayley à croissance polynomiale d'ordre n. Il existe une constante* $C(\Gamma) > 0$ *telle que pour tout graphe à bord* $(\Gamma' = (V', E'), B)$ *inclu dans* Γ *et tel que* $|B| > 1$*, on a*

$$\sigma_1(\Gamma', B) \leq \begin{cases} C(\Gamma)\frac{1}{|B|} & si\ n \leq 2, \\ C(\Gamma)\frac{|V'|^{\frac{n-2}{n}}}{|B|} & si\ n \geq 2. \end{cases}$$

Ces résultats sur les sous-graphes soulèvent d'intéressantes questions et ont déjà donné lieu à plusieurs développements. Nous avons vu à la section précédente qu'il n'était en général pas

FIGURE 1.4. – Le graphe de gauche est induit par un sous-ensemble de $\mathbb{Z}^2$ alors que le graphe de droite est uniquement inclu dans $\mathbb{Z}^2$ (les sommets du bord sont plus gros).

vrai que σ_1 tend vers 0 lorsque le diamètre extrinsèque du bord tend vers l'infini. Les résultats ci-dessus montrent que c'est le cas pour des graphes induits par un sous-ensemble d'un graphe de Cayley à croissance polynomiale puisque si le diamètre du bord tend vers l'infini, le nombre de sommets du graphe aussi. Cependant, lorsque le nombre de sommets du bord est fixé, la relation entre σ_1 et le diamètre du bord pour des graphes à bord simplement inclus dans $\mathbb{Z}^2$ n'est pas établie. On se demande aussi si les théorèmes 3 et 4 peuvent être généralisés à des graphes de Cayley dont la croissance dépasse une croissance polynomiale ; des progrès sur cette question ont été obtenus récemment dans [HH22] et [Tsc23]. On remarque enfin que des résultats similaires aux théorèmes 3 et 4 ont été donnés dans [Tsc22] pour les valeurs propres supérieures.

1.3 Bornes inférieures pour la première valeur propre non nulle de Steklov d'une variété riemannienne

1.3.1 Lien entre petite valeur propre et géométrie

Un des buts de cette thèse est de donner des bornes inférieures pour la première valeur propre non nulle de Steklov σ_1 d'une variété riemannienne connexe M à bord ∂M. Lorsque le bord est connexe, un résultat de Colbois, Girouard et Hassannezhad [CGH20] montre qu'avec des hypothèses sur la géométrie de ∂M et la géométrie de M proche du bord, σ_1 est bornée inférieurement par la première valeur propre non nulle du laplacien sur ∂M. D'autres résultats sont connus lorsque le bord satisfait en plus une condition de convexité (voir, par exemple, [Esc97], [Esc99] et [Xio22]) . Ils procèdent d'une volonté de généraliser un résultat de Payne datant de 1970 [Pay70] qui affirme que σ_1 d'un domaine convexe du plan est bornée inférieurement par le minimum de la courbure de son bord.

Dans le cas où M est une variété riemannienne quelconque, Escobar [Esc97] a donné une borne inférieure qui dépend d'une constante isopérimétrique et de la première valeur propre non nulle d'un problème auxiliaire. Ensuite, une borne inférieure qui dépend uniquement de constantes isopérimétriques a été obtenue par Jammes.

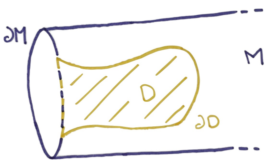

FIGURE 1.5. – Un domaine D de M qui intersecte ∂M.

Théorème 6 (Jammes [Jam15]). *Soit (M, g) une variété riemannienne à bord ∂M. Etant donné un domaine D de M, on pose*

$$h_c(M) = \inf_{|D| \leq \frac{|M|}{2}} \frac{|\partial D|}{|D|} \quad et \quad h_j(M) = \inf_{|D| \leq \frac{|M|}{2}} \frac{|\partial D|}{|D \cap \partial M|}.$$

On a alors

$$\sigma_1(M) \geq \frac{h_c(M) \cdot h_j(M)}{4}.$$

J'ai montré à la section 4.2.2 que le résultat est toujours vrai si, dans la définition des constantes isopérimétriques, on ne considère que les domaines D de M qui, en plus de satisfaire $|D| \leq \frac{|M|}{2}$, satisfont $D \cap \partial M \neq \emptyset$, et tels que $M \setminus D$ est aussi connexe et intersecte ∂M (un domaine D qui intersecte ∂M est représenté à la figure 1.5.). Cela constitue une amélioration en dimension 2 où il existe des familles de variétés, comme celles de l'exemple 4 ci-dessous, pour lesquelles les constantes de Jammes tendent vers 0 alors que celles que je définis sont bornées inférieurement.

Exemple 4. *Soit C un cylindre droit de dimension 2 dans $\mathbb{R}^3$ dont la base contient un segment de droite. On considère les surfaces obtenues en collant sur la partie du cylindre qui est plate visuellement une surface de révolution contenant un cylindre de plus en plus fin, comme illustré à la figure 1.6.. Ces surfaces sont toutes Steklov isospectrales à C (voir [CGG19], appendice A, et [Bri19] pour plus de détails). Les constantes utilisées par Jammes tendent vers 0 lorsque la circonférence du cylindre mince tend vers 0 alors que les constantes modifiées que je définis restent bornées inférieurement (voir lemme 4.2).*

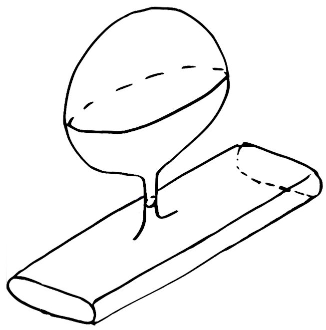

FIGURE 1.6. – Un cylindre sur lequel on a collé une surface de révolution.

Il est souhaitable d'obtenir dans le cas général une borne inférieure dépendant de grandeurs géométriques plus explicites que des constantes isopérimétriques comme celle qui a pu être obtenue par Li et Yau [LY80] pour la première valeur propre non nulle du laplacien sur une variété fermée. En plus d'impliquer des contraintes sur la géométrie du bord et la géométrie de M proche du bord, le comportement de la première valeur propre non nulle d'un cylindre droit de longueur L qui tend vers 0 lorsque L tend vers l'infini montre que de telles contraintes ne sont pas suffisantes pour borner inférieurement σ_1 et qu'il faut également imposer des contraintes sur la géométrie de M loin du bord.

Afin de préciser la relation entre σ_1 et des grandeurs géométriques globales, nous nous intéressons aux variétés riemanniennes à bord cylindrique car cela permet d'éliminer les perturbations du spectre venant de la géométrie du bord. On remarque que les résultats pour les variétés riemanniennes à bord cylindrique peuvent être utilisés pour déduire des résultats plus généraux grâce aux quasi-isométries, comme cela a été fait dans [CESG19] (voir théorème 1.1).

1.3.2 Les variétés riemanniennes à bord cylindrique

Une variété riemannienne à bord cylindrique est définie de la façon suivante.

Définition 5. *On dit que le bord ∂M d'une variété riemannienne (M, g) est à voisinage cylindrique, ou simplement cylindrique, si, pour un certain $L > 0$, il admet un voisinage isométrique au produit $(\partial M \times [0, L), \tilde{g} \oplus g_{eucl})$ où $\tilde{g}$ est la métrique induite par g sur ∂M et g_{eucl} est la métrique euclidienne sur $[0, L) \subset \mathbb{R}$.*

En comparant les valeurs propres de Steklov d'une variété à bord cylindrique avec les valeurs propres des problèmes mixtes Steklov-Neumann et Steklov-Dirichlet sur le voisinage cylindrique (voir section 4.2.1 et [CGG19] pour plus de détails), on obtient des bornes inférieure et supérieure pour les valeurs propres de Steklov.

Lemme 1. *Soit (M, g) une variété riemannienne à bord cylindrique ∂M. On a*

$$\sqrt{\lambda_k} \tanh(\sqrt{\lambda_k} L) \leq \sigma_k(M) \leq \sqrt{\lambda_k} \coth(\sqrt{\lambda_k} L),$$

où les λ_k sont les valeurs propres du laplacien sur ∂M.

Cette inégalité montre que lorsque le bord d'une variété riemannienne à bord cylindrique est connexe, σ_1 est bornée inférieurement par la première valeur propre non nulle du laplacien sur le bord. On remarque aussi que les perturbations de la géométrie hors du voisinage cylindrique du bord ne peuvent affecter fortement que les b premières valeurs propres contrairement à d'autres perturbations géométriques, comme celles décrites dans [CGM20] par exemple.

Dans [CGR18], Colbois, Girouard et Raveendran ont montré qu'on pouvait associer à une variété riemannienne à bord cylindrique un graphe à bord de manière à ce que les valeurs propres de la variété soient bien estimées par les valeurs propres du graphe. Cela rend possible l'utilisation d'une borne inférieure que j'ai obtenue sur les graphes pour obtenir une borne inférieure pour σ_1 d'une variété riemannienne à bord cylindrique lorsque le bord a plusieurs composantes connexes. Avant d'énoncer le résultat, j'introduis deux grandeurs géométriques qui apparaissent dans celui-ci.

Définition 6. *Soit (M, g) une variété riemannienne à bord ∂M.*

1. *Le diamètre extrinsèque du bord est*

$$\mathrm{diam}_M(\partial M) = \max\{d(x, y) | x, y \in \partial M\},$$

où $d(x, y)$ désigne la distance sur M induite par g.

Supposons maintenant que ∂M a un voisinage cylindrique $V(\partial M)$ de largeur L.

2. *Le rayon d'injectivité de M est*

$$\mathrm{inj}_M(M) := \mathrm{inj}_M(M \setminus V(\partial M)) = \inf\{\mathrm{inj}_M(x) : x \in M \setminus V(\partial M)\}.$$

On remarque que $\mathrm{inj}_M(M) \leq L$.

Le théorème est suivi de sa preuve car il n'apparaît que dans cette introduction.

Théorème 7. *Soit (M, g) un variété riemannienne de dimension $n \geq 2$ à voisinage cylindrique ∂M de largeur $L \leq 1$. Supposons aussi qu'il existe $\kappa \geq 0$ tel que la courbure de Ricci de M est bornée inférieurement par $-(n-1)\kappa$ et la courbure de Ricci de ∂M est bornée inférieurement par $-(n-2)\kappa$. Alors il existe une constante $C(n, \kappa)$ qui ne dépend que de n et de κ telle que*

$$\sigma_1(M) \geq C(n, \kappa) \frac{\mathrm{inj}_M(M)^{n-1}}{|\partial M| \, \mathrm{diam}_M(\partial M)}.$$

Démonstration. On commence par poser $\tilde{L} = \frac{L}{2}$ et on voit à partir de maintenant M comme variété riemannienne dont le bord a un voisinage cylindrique de largeur $\tilde{L}$ (cela peut changer la valeur du rayon d'injectivité). On note i_0 le rayon d'injectivité de M vue comme variété à bord cylindrique de largeur $\tilde{L}$. On remarque que comme $L \leq 1$, on a $i_0 \leq \tilde{L} < 1$. On considère la variété riemannienne $(M, \frac{1}{i_0^2} g)$ que l'on note $M_{\tilde{g}}$. On a alors $\sigma_1(M) = \frac{1}{i_0} \sigma_1(M_{\tilde{g}})$.

Voyons que $M_{\tilde{g}}$ satisfait les hypothèses du théorème 3 de [CGR18]. On remarque d'abord que le bord de $M_{\tilde{g}}$ a un voisinage cylindrique de largeur $\frac{1}{i_0}\tilde{L} \geq 1$ et que $\mathrm{inj}_{M_{\tilde{g}}}(M_{\tilde{g}}) = \frac{1}{i_0} i_0 = 1$. De plus, on a $\mathrm{inj}_{\partial M_{\tilde{g}}}(\partial M_{\tilde{g}}) \geq \mathrm{inj}_{M_{\tilde{g}}}(M_{\tilde{g}})$ et donc $\mathrm{inj}_{\partial M_{\tilde{g}}}(\partial M_{\tilde{g}}) \geq 1$. Pour s'en convaincre, supposons que $\mathrm{inj}_{\partial M_{\tilde{g}}}(\partial M_{\tilde{g}}) < \mathrm{inj}_{M_{\tilde{g}}}(M_{\tilde{g}})$ et montrons qu'on arrive à une contradiction. Soit $p \in \partial M_{\tilde{g}}$. On se rappelle que $M_{\tilde{g}}$ a en fait un voisinage cylindrique de largeur $\frac{1}{i_0} L$ et on considère le point $x = (p, \frac{3}{4}\frac{1}{i_0} L) \in \partial M_{\tilde{g}} \times [0, \frac{1}{i_0} L)$. Alors $\mathrm{inj}_{M_{\tilde{g}}}(x) \geq \mathrm{inj}_{M_{\tilde{g}}}(M_{\tilde{g}})$ et donc $\mathrm{inj}_{\partial M_{\tilde{g}}}(p) \geq \mathrm{inj}_{M_{\partial g}}(M_{\tilde{g}})$ ce qui contredit l'hypothèse puisque c'est vrai pour tout $p \in \partial M_{\tilde{g}}$. Enfin comme $0 < i_0 < 1$, les courbures de Ricci de $M_{\tilde{g}}$ et de son bord restent bornées inférieurement par $-(n-1)\kappa$ et $-(n-2)\kappa$.

On note $(\Gamma_{M_{\tilde{g}}}, B_{\partial M_{\tilde{g}}})$ une ϵ-discrétisation de $M_{\tilde{g}}$ avec $\epsilon = \frac{1}{16}$. En utilisant le théorème 3 de [CGR18] avec $r_0 = \frac{1}{2}$, on obtient qu'il existe une constante $\alpha(\kappa)$ telle que

$$\sigma_1(M_{\tilde{g}}) > \alpha(\kappa) \sigma_1(\Gamma_{M_{\tilde{g}}}, B_{\partial M_{\tilde{g}}}), \tag{1.3.1}$$

où $\sigma_1(\Gamma_{M_{\tilde{g}}}, B_{\partial M_{\tilde{g}}})$ est la première valeur propre non nulle du graphe à bord $(\Gamma_{M_{\tilde{g}}}, B_{\partial M_{\tilde{g}}})$. On note $d_{B_{\partial M_{\tilde{g}}}}$ le diamètre extrinsèque de $B_{\partial M_{\tilde{g}}}$ que l'on a vu à la définition 3. Par le théorème 1 (théorème 2.2), on obtient

$$\sigma_1(\Gamma_{M_{\tilde{g}}}, B_{\partial M_{\tilde{g}}}) \geq \frac{1}{|B_{\partial M_{\tilde{g}}}| d_{B_{\partial M_{\tilde{g}}}}}. \tag{1.3.2}$$

Par un résultat de Croke ([Cro80], proposition 14), une boule de $\partial M_{\tilde{g}}$ de rayon $r \leq \frac{\mathrm{inj}_{\partial M_{\tilde{g}}}(\partial M_{\tilde{g}})}{2}$ centrée en $p \in \partial M_{\tilde{g}}$ satisfait $|B_r(p)| \geq \beta(n-1) r^{n-1}$ où $\beta(n-1)$ est une constante qui dépend de la dimension de $\partial M_{\tilde{g}}$. Comme on a une ϵ-discrétisation, les boules de rayon $\frac{\epsilon}{2}$ centrées aux points qui forment $B_{\partial M_{\tilde{g}}}$ sont disjointes. En observant aussi que $\frac{\epsilon}{2} = \frac{1}{32} \leq \frac{1}{2} \leq \frac{\mathrm{inj}_{\partial M_{\tilde{g}}}(\partial M_{\tilde{g}})}{2}$, on obtient

$$|\partial M_{\tilde{g}}| \geq \sum_{p \in B_{\partial M_{\tilde{g}}}} |B_{\frac{\epsilon}{2}}(p)| \geq \gamma(n) |B_{\partial M_{\tilde{g}}}|$$

avec $\gamma(n) = \beta(n)(\frac{\epsilon}{2})^{n-1}$. Par le lemme 12 de [CGR18] la distance sur le graphe et la distance sur $M_{\tilde{g}}$ sont liées par la relation

$$d_{M_{\tilde{g}}}(x, y) \geq \frac{\epsilon}{4} d_{\Gamma_{M_{\tilde{g}}}}(x, y) - 10.$$

Soient $p, q \in B_{\partial M_{\tilde g}}$ tels que $d_{B_{\partial M_{\tilde g}}} = d_{\Gamma_{M_{\tilde g}}}(p,q)$. Alors on a

$$d_{B_{\partial M_{\tilde g}}} = d_{\Gamma_{M_{\tilde g}}}(p,q) \leq \frac{4}{\epsilon} d_{M_{\tilde g}}(p,q) + 10 \leq \frac{4}{\epsilon} \operatorname{diam}_{M_{\tilde g}}(\partial M_{\tilde g}) + 10$$

et donc, comme $\operatorname{diam}_{M_{\tilde g}}(\partial M_{\tilde g}) \geq 2$,

$$\frac{1}{d_{B_{\partial M_{\tilde g}}}} \geq \frac{\epsilon}{24} \frac{1}{\operatorname{diam}_{M_{\tilde g}}(\partial M_{\tilde g})}.$$

En combinant ces observations avec les inégalités 1.3.1 et 1.3.2, on obtient que

$$\sigma_1(M_{\tilde g}) > a(\kappa) \frac{1}{|B_{\partial M_{\tilde g}}| d_{B_{\partial M_{\tilde g}}}} \geq \alpha(\kappa)\gamma(n) \frac{\epsilon}{24} \frac{1}{|\partial M_{\tilde g}| \operatorname{diam}_{M_{\tilde g}}(\partial M_{\tilde g})}.$$

Enfin, comme $|\partial M_{\tilde g}| = \frac{1}{i_0^{n-1}} |\partial M|$ et $\operatorname{diam}_{M_{\tilde g}}(\partial M_{\tilde g}) = \frac{1}{i_0} \operatorname{diam}_M(\partial M)$, on trouve

$$\sigma_1(M_{\tilde g}) \geq \alpha(\kappa)\gamma(n) \frac{\epsilon}{24} \frac{i_0^n}{|\partial M| \operatorname{diam}_M(\partial M)}$$

et finalement, en remarquant que $i_0 \geq \frac{\operatorname{inj}_M(M)}{2}$,

$$\sigma_1(M) = \frac{1}{i_0} \sigma_1(M_{\tilde g}) \geq C(n,\kappa) \frac{\operatorname{inj}_M(M)^{n-1}}{|\partial M| \operatorname{diam}_M(\partial M)},$$

où $C(n, \kappa)$ est une constante qui dépend de n et de κ. □

En dimension $n = 2$, la condition sur la courbure de Ricci du bord est trivialement satisfaite et le résultat devient que lorsque la courbure de Gauss de M est bornée inférieurement par $-\kappa$ avec $\kappa > 0$, il existe une constante $C(\kappa)$ telle que $\sigma_1(M) \geq C(\kappa) \frac{\operatorname{inj}_M(M)}{|\partial M| \operatorname{diam}_M(\partial M)}$. Un exemple comme celui de la figure 1.6. montre que le rayon d'injectivité peut être petit en certains points de M sans que cela influence σ_1. A la section suivante, je présente une borne inférieure semblable mais qui n'implique que le rayon d'injectivité des points d'un sous-ensemble de M, et pour laquelle la constante est explicite. Je donne aussi une borne inférieure qui ne requière pas d'hypothèse sur la courbure de M.

1.3.3 Bornes inférieures explicites en dimension 2

Pour une variété riemannienne de dimension 2 à bord cylindrique, l'estimation des constantes isopérimétriques améliorées permet d'obtenir une borne inférieure explicite sans faire d'hypothèse sur la courbure de Gauss de la variété. Elle implique la grandeur géométrique suivante.

Définition 7. *Soit (M, g) une variété riemannienne compacte et connexe de dimension 2 avec un bord formé de $b \geq 2$ composantes connexes. On considère la famille des courbes qui, sans intersecter ∂M, divisent M en deux composantes connexes contenant chacune au moins une composante de ∂M (comme à la figure 1.7.). On note $C(M)$ cette famille de courbes et on définit*

$$\mathrm{l}(M) := \inf\{l(c) : c \in C(M)\}$$

où $l(c)$ est la longueur de la courbe c.

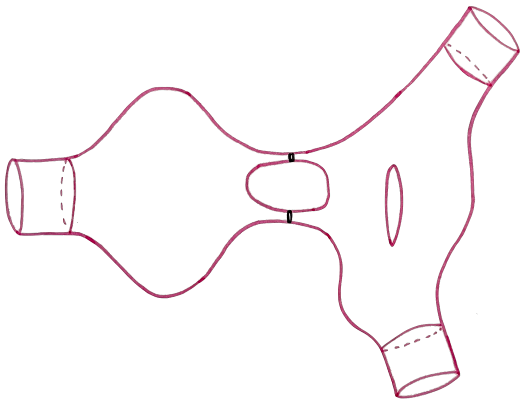

Figure 1.7. – Une surface à bord cylindrique partagée en deux par une courbe.

On peut maintenant énoncer le résultat.

Théorème 8 (Théorème 4.1)**.** *Soit (M, g) une variété riemannienne compacte et connexe de dimension 2 avec un bord à voisinage cylindrique de largeur L formé de $b \geq 2$ composantes connexes, chacune de longueur a. On a alors*

$$\sigma_1(M) \geq \frac{\min\{\mathrm{l}(M), L\}^2}{2(b-1)a|M|}.$$

Cette borne inférieure a un caractère optimal du fait que je montre que les exposants des grandeurs géométriques utilisées ne peuvent pas être améliorés. Cependant, la présence du volume de M au dénominateur la rend imprécise dans certaines situations. Si on considère, par exemple, une famille de surfaces du type de celle illustrée à la figure 1.6., telle que le volume de la surface de révolution collée tend vers l'infini, ce résultat ne permet pas de détecter que σ_1 est bornée inférieurement.

En ajoutant une hypothèse sur la courbure de Gauss, j'ai pu obtenir une borne inférieure qui ne dépend pas du volume, similaire à celle donnée au théorème 7, mais plus précise, car elle ne nécessite que de tenir compte que du rayon d'injectivité des points d'un sous-ensemble de $M \setminus V(\partial M)$.

Définition 8. *Soit (M, g) une variété riemannienne dont le bord ∂M a un voisinage cylindrique $V(\partial M)$ de largeur L. Soit Γ le sous-ensemble de M*

$$\Gamma = \{x \in M, \exists p, q \in \partial M \text{ et une géodésique minimisante } \gamma \text{ entre } p \text{ et } q \text{ telle que } x \in \gamma\}.$$

On définit

$$\mathrm{inj}_{\partial M}(M) := \mathrm{inj}(\Gamma \setminus V(\partial M)) = \min\{\mathrm{inj}_M(x) : x \in \Gamma \setminus V(\partial M)\}.$$

On remarque que $\mathrm{inj}_{\partial M}(M) \leq L$.

Théorème 9 (Théorème 4.2)**.** *Soit (M, g) une variété riemannienne compacte et connexe de dimension 2 avec un bord à voisinage cylindrique de largeur $L \leq 1$ formé de $b \geq 2$ composantes connexes, chacune de longueur a. Supposons qu'il existe $\kappa < 0$ tel que la courbure de Gauss de*

M satisfait $K(p) \geq \kappa \ \forall \ p \in M$ *et supposons que* $a \leq \mathrm{diam}_M(\partial M)$. *Alors on a*

$$\sigma_1(M) \geq C(\kappa, b) \frac{\mathrm{inj}_{\partial M}(M)}{a \, \mathrm{diam}_M(\partial M)},$$

où $C(\kappa, b) = \frac{1}{16b^2 \cosh(\sqrt{-\kappa})}$.

Comme pour le théorème précédent, l'exposant des grandeurs géométriques présentes ne peut pas être amélioré. L'hypothèse que $a \leq \mathrm{diam}_M(\partial M)$ permet d'obtenir cette borne inférieure à l'expression simple. Sans cette hypothèse, la borne inférieure a une forme a priori un peu plus compliquée, cependant, une question accessoire se pose : peut-on déduire des autres hypothèses que a est bornée supérieurement par le diamètre du bord multiplié par une constante ?

La preuve du théorème 9, contrairement à celle du théorème 7, ne nécessite pas de comparaison avec les graphes. Elle repose sur une version plus forte du théorème 8 (le théorème 4.4) où M peut être remplacée par n'importe quel domaine A de M qui contient le voisinage cylindrique de ∂M. Le résultat est ensuite obtenu en construisant un domaine A de M qui permet de faire apparaître les grandeurs géométriques voulues.

1.4 Les premières valeurs propres de Steklov d'une surface hyperbolique à bord géodésique

Un résultat classique de Schoen, Wolpert et Yau montre que les premières valeurs propres du laplacien d'une surface hyperbolique fermée sont bien approchées par la longueur de certaines familles de géodésiques fermées simples.

Théorème 10 (Schoen, Wolpert, Yau [SWY80]). *Soit M une surface hyperbolique compacte de genre g, alors il existe des constantes α_1 et α_2 dépendant uniquement de g telles que pour* $1 \leq n \leq 2g - 3$, *on a*

$$\alpha_1 \, \mathrm{L}_n \leq \lambda_n \leq \alpha_2 \, \mathrm{L}_n \quad \text{et} \quad \alpha_1 \leq \lambda_{2g-2} \leq \alpha_2$$

où L_n *est le minimum de la longueur d'une union de géodésiques fermées simples qui divisent M en* $n + 1$ *composantes connexes.*

J'ai obtenu un résultat similaire pour les b premières valeurs propres de Steklov d'une surface hyperbolique avec un bord géodésique formé de b composantes connexes. Les familles de géodésiques pertinentes pour estimer les valeurs propres sont différentes.

Définition 9. *Soit M une surface hyperbolique compacte avec un bord géodésique formé de* $b \geq 2$ *composantes connexes. Pour* $1 \leq n \leq b - 1$, *on considère la famille de courbes consistant en une union de géodésiques fermées simples qui, sans intersecter ∂M, divisent M en* $n + 1$ *composantes connexes contenant chacune au moins une composante connexe de ∂M. On note $C_n(M)$ l'ensemble de ces courbes. Si $C_n(M) \neq \emptyset$, on définit*

$$\mathrm{l}_n(M) := \min\{l(c) : c \in C_n(M)\}$$

où $l(c)$ est la longueur de la courbe c.

Le théorème suivant est prouvé dans la troisième partie de cette thèse. Comme pour les résultats de la section précédente, la preuve repose sur l'estimation des constantes isopérimétriques.

Ces estimations sont ici possibles grâce à une propriété importante des surfaces hyperboliques qui est d'être isométriques à un produit tordu autour de leurs géodésiques fermées simples.

Théorème 11 (Théorème 4.3)**.** *Soit M une surface hyperbolique compacte de genre g avec un bord géodésique formé de $b \geq 2$ composantes connexes, chacune de longueur $a \leq 2\operatorname{arcsinh}(1)$. Supposons que $g \neq 0$ ou $b > 3$. Alors il existe une constante C_1 dépendant uniquement de g et b et une constante universelle C_2 telles que pour $1 \leq n < \lceil \frac{b}{2} \rceil$ on a*

$$C_1 \, \mathrm{l}_n^2 \leq \sigma_n \leq C_2 \frac{\mathrm{l}_n}{a}.$$

L'inégalité est aussi vraie pour $\lceil \frac{b}{2} \rceil \leq n < b$ si $C_n(M) \neq \emptyset$ et s'il existe $c \in C_n(M)$ telle que chaque géodésique fermée simple de c a une longueur $l \leq L_{g+b}$, où $L_{g+b} = 4(3(g+b) - 3)\log(\frac{8\pi(g+b-1)}{3(g+b)-3})$.

Lorsque a devient petit, la borne supérieure devient grande, mais, avec un résultat similaire au lemme 1 (le lemme 4.3) qui utilise que chaque composante de ∂M a un voisinage isométrique à un demi-cylindre hyperbolique, on montre que pour $0 \leq n < b$, σ_n est bornée supérieurement par $\frac{1}{\arctan(\frac{1}{\sinh \frac{a}{2}})} \leq \frac{2}{\pi}$. On remarque aussi que dans le théorème 11, qui est le résultat que je peux donner actuellement, l'exposant de l_n dans la borne inférieure est 2 alors qu'on peut souhaiter obtenir 1 comme au théorème 10.

2. Lower bounds for the first eigenvalue of the Steklov problem on graphs

This article has been previously published in Calculus of Variations and Partial Differential Equations, see [Per19].

Abstract. We give lower bounds for the first non-zero Steklov eigenvalue on connected graphs. These bounds depend on the extrinsic diameter of the boundary and not on the diameter of the graph. We obtain a lower bound which is sharp when the cardinal of the boundary is 2, and asymptotically sharp as the diameter of the boundary tends to infinity in the other cases. We also investigate the case of weighted graphs and compare our result to the Cheeger inequality.

2.1 Introduction

The Steklov problem on a compact Riemannian manifold M with boundary ∂M is known to be

$$\begin{cases} \triangle u = 0 & \text{in } M \\ \frac{\partial u}{\partial n} = \sigma u & \text{on } \delta M \end{cases}$$

where $\triangle$ is the Laplace-Beltrami operator and $\frac{\partial u}{\partial n}$ is the outward normal derivative along the boundary δM. It is a classical result that if the boundary is sufficiently regular, the spectrum of the Steklov problem is discrete and its eigenvalues form a sequence $0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \nearrow \infty$. It is also known that this spectrum coincides with the one of the Dirichlet-to-Neumann operator.

The Steklov problem has been pointed out to be of particular interest for spectral geometry because it has features that distinguish it from the Dirichlet and Neumann problems for the Laplacian (see [GP17]). For the Laplacian, it is classical to try to understand spectral properties of a manifold via a discretization (see e.g. [Cha01] and [Man05]). The eigenvalues of the Laplacian on M are related to those of its discretization and the discrete Laplacian is studied in order to get information about the spectral geometry of M. This approach has been recently done for the Steklov operator in [CGR18]. In consequence, it becomes very relevant to study the spectrum of the discrete Steklov operator.

The object of this paper is to study lower bounds for the first non-zero eigenvalue of the Steklov problem on graphs. The study of the spectral geometry associated to this eigenvalue has been initiated in a recent article by Hua, Huang and Wang [HHW17]. They define two Cheeger-type constants for the Steklov problem on graphs, based on the constants by Jammes and Escobar already existing for the continuous case, and give two very interesting lower bound estimates depending on these constants. For estimating the first non-zero eigenvalue, the so called Jammes-type Cheeger constant is not as relevant as the Cheeger constant for the combinatorial Laplacian because it does not provide an upper bound estimate. Indeed, often the Jammes-type Cheeger constant is small while the first Steklov eigenvalue is not (a typical example of this phenomenon is given in Example 2.1.).

In order to better understand the geometry of the graph captured by σ_1, we investigate a different lower bound, depending on geometric features of the boundary. In particular, it depends on the extrinsic diameter of the boundary, but the diameter of the whole graph is not involved. It is sharp when there are only two vertices in the boundary and, otherwise, it becomes asymptotically sharp as the diameter of the boundary goes to infinity. Our bound is, in some cases, more accurate than the Jammes-type Cheeger estimate of Hua et al. We draw here the reader's attention to the fact that, while Hua et al. are considering weighted graphs, we focus in this paper on graphs with weight one on all the edges, which are relevant in the context of discretization. The definition of the Steklov problem, or Dirichlet-to-Neumann operator, is also slightly different. In analogy to the Laplacian, we could say that we use an unnormalized form of the Steklov problem on graphs (for a discussion about the several definitions of the combinatorial Laplacian, see e.g. [But08]). However, in the examples that we give for comparison, the two definitions coincide and in our last section we will adapt a part of our result to the settings of the article by Hua et al.

The paper is structured as follows: in section 2, we give the definitions necesary to state the Steklov problem on graphs; the main lower bound results, Theorem 2.1 and Theorem 2.2, are given in section 3, respectively in part I and II of this section; in the last section, we extend the first result to weighted graphs. Theorem 2.1 can be deduced from Theorem 2.2, but we begin with it since the proof is very simple and we use it as a starting point for the more technical proof of Theorem 2.2.

This article resulted from a master's thesis realized in 2017 at the University of Neuchâtel under the direction of Professor Bruno Colbois.

2.2 Preliminaries

In a graph $\Gamma = (V, E)$, for a subset $S \subset V$, the boundary of S in Γ is defined in the following way. The edge boundary of S is $\partial S := E(S, S^c)$, where $E(\Omega_1, \Omega_2)$ is the set of edges between the two subsets $\Omega_1, \Omega_2 \subset V$, i.e. the set $\{e = \{i, j\} \in E | i \in \Omega_1, j \in \Omega_2\}$. The vertex boundary of S is $\delta S := \{i \in S^c | i \sim j \text{ for some } j \in S\}$. We use $i \sim j$ to signify that $\{i, j\} \in E$. The degree of a vertex i is denoted $d(i)$.

Definition 2.1. *A graph with boundary is a pair* (Γ, B)*, where* $\Gamma = (V, E)$ *is a simple graph, that is, without loops and multiple edges, and* $B \subset V$ *is a subset of* V *such that* $\delta(B^c) = B$ *and* $E(B, B) = \emptyset$*. We call* B *the boundary and* $I := B^c$ *the interior of the graph.*

In this paper, we always consider graphs with boundary, connected, and finite. The space of all real functions defined on the vertices V, denoted $\mathbb{R}^V$, is the Euclidean space of dimension $|V|$. Similarly, the space of real functions defined on the vertices of the boundary, denoted $\mathbb{R}^B$, is the Euclidean space of dimension $|B|$. The scalar product in an Euclidean space is denoted by $\langle \cdot, \cdot \rangle$. In addition to the scalar product, we introduce on $\mathbb{R}^V$ the following bilinear form: for functions $v, w \in \mathbb{R}^V$, we define $\langle v, w \rangle_B := \langle v|_B, w|_B \rangle$, we write $\|v\|_B$ for $\langle v, v \rangle_B^{1/2}$, and $v \perp_B w$ if $\langle v, w \rangle_B = 0$. We denote by 1_B the matrix of the orthogonal projection onto $\mathbb{R}^B \subset \mathbb{R}^V$.

The Laplacian operator $\triangle : \mathbb{R}^V \to \mathbb{R}^V$ is defined by

$$(\triangle v)_i = \sum_{j \sim i} (v_i - v_j)$$

and its matrix is

$$(\triangle)_{ij} = \begin{cases} d(i) & \text{if } i = j \\ -1 & \text{if i and j are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

A function $v = (v_1, ..., v_{|V|})$ on (Γ, B) is called harmonic if

$$(\triangle v)_i = \sum_{j\sim i}(v_i - v_j) = 0 \quad \forall i \in I$$

The restriction of $\triangle v$ to B can be seen as the image of v under the normal derivative operator $\frac{\partial v}{\partial n} : \mathbb{R}^V \to \mathbb{R}^B$ defined by

$$(\frac{\partial v}{\partial n})_i = \sum_{j\in I, j\sim i}(v_i - v_j) \quad i \in B$$

In analogy to the Riemannian case, we give the following definition:

Definition 2.2. *The Steklov problem on a graph with boundary is the eigenvalue problem*

$$\triangle v = \sigma 1_B v$$

where $v \not\equiv 0$ and σ is a spectral parameter.

The eigenvalues of this problem are the same as the eigenvalues of the discrete Dirichlet-to-Neumann operator defined by [HHW17] in the case that the weights of the edges are all 1 and the degree of the boundary vertices is 1. In order to see this, we need this useful linear algebra lemma:

Lemma 2.1. *For a connected graph with boundary (Γ, B), given any $\varphi \in \mathbb{R}^B$, there is a unique function $\tilde{\varphi} \in \mathbb{R}^V$, called the harmonic extension of φ, which satisfies*

$$(\triangle\tilde{\varphi})_i = 0 \qquad \text{if } i \in I \tag{2.2.1}$$

$$\tilde{\varphi}_i = \varphi_i \qquad \text{if } i \in B \tag{2.2.2}$$

Proof. Putting 2.2.2 in 2.2.1 and developing, we obtain the following linear system of equations

$$(\triangle[I])\tilde{\varphi}\big|_I)_i = \sum_{j\sim i, j\in B} \varphi_j \quad \forall i \in I$$

where $\triangle[I]$ denotes the principal submatrix of $\triangle$ with rows and columns corresponding to the vertices of the interior. If the graph is connected, this matrix is invertible and thus there is a unique solution of the system. □

The Dirichlet-to-Neumann operator $\Lambda : \mathbb{R}^B \to \mathbb{R}^B$ maps φ to $\Lambda\varphi := \frac{\partial\tilde{\varphi}}{\partial n}$. Let σ be an eigenvalue of the Dirichlet-to-Neumann operator and φ an associated eigenfunction, then we have that the pair $(\sigma, \tilde{\varphi})$ is solution of the Steklov problem. In the other direction, if (σ, v) is solution of the Steklov problem, it is also clear that σ is an eigenvalue of the Dirichlet-to-Neumann operator, associated to $v|_B$. Thus, the spectra are equivalent.

This problem has b solutions $\sigma_0 \leq \sigma_1 \leq ... \leq \sigma_{b-1}$, called the eigenvalues, and there exist b associated eigenfunctions $v^0, v^1, ..., v^{b-1}$, defined on Γ, which are mutually orthogonal for the bilinear form $\langle \cdot, \cdot \rangle_B$ and can be choosen such that $||v^i||_B = 1$. This affirmation results from the fact that Λ is a non-negative self-adjoint operator as explained in [HHW17]. A proof using basic analysis and linear algebra tools is also given in [Per17]. The Rayleigh quotient associated to Λ is, for a function $v \in \mathbb{R}^V$,

$$\frac{\langle v, \triangle v \rangle}{||v||_B^2} = \frac{\sum_{i\sim j}(v_i - v_j)^2}{\sum_{i\in B} v_i^2}$$

and there are variational characterizations for the eigenvalues

$$\sigma_j = \min_E \max_{v\in E, v\neq 0} \left\{ \frac{\langle v, \triangle v \rangle}{||v||_B^2} \right\} \tag{2.2.3}$$

where E is the set of all linear subspaces of $\mathbb{R}^V$ of dimension $j+1$. For σ_1, we have

$$\sigma_1 = \min_{v\in\mathbb{R}^V, v\in[v^0]^{\perp_B}} \left\{ \frac{\langle v, \triangle v \rangle}{||v||_B^2} \right\} \tag{2.2.4}$$

Since a constant function is an eigenfunction associated to the eigenvalue 0, we can rewrite equation (2.2.4) in the following way

$$\sigma_1 = \min_{v\in\mathbb{R}^V} \left\{ \sum_{i\sim j}(v_i - v_j)^2 : \sum_{i\in B} v_i^2 = 1, \sum_{i\in B} v_i = 0 \right\} \tag{2.2.5}$$

Remark 2.1. *From equation (2.2.3), we see that if* (Γ, B) *is a graph with boundary, then* $\sigma_k(\Gamma, B) \geq \lambda_k(\Gamma)$ *where* $\lambda_k(\Gamma)$ *is the kth-eigenvalue of the combinatorial Laplacian on* Γ.

2.3 Lower bound for σ_1

Let (Γ, B) be a graph with boundary. The distance between two vertices i and j is the number of edges in the shortest path joining i and j. We will denote it $d(i, j)$.

Definition 2.3. *The diameter* d *of* (Γ, B) *is the maximum distance between any two vertices of* (Γ, B)*, i.e.* $d = \max\{d(i, j) | i, j \in V\}$.

Definition 2.4. *The diameter of the boundary* d_B *is the extrinsic diameter of* B *in* Γ *or, in other words, it is the maximum distance between any two vertices of* B*, i.e.* $d_B = \max\{d(i, j) | i, j \in B\}$.

2.3.1 Lower bound for σ_1 (I)

We give now a first lower bound in terms of the diameter of the boundary and of the number of vertices of the boundary.

Theorem 2.1. *Let* (Γ, B) *be a connected graph with diameter of the boundary* d_B *and* $|B| = b$*. We have*

$$\sigma_1 \geq \frac{b}{(b-1)^2 \cdot d_B}$$

The bound is optimal when $b = 2$.

Proof. Let v be an eigenfunction of σ_1 normalized as in (2.2.5). We consider the vertices α and β of the boundary such that $\max_{i\in B} v_i = v_\alpha$ and $\min_{i\in B} v_i = v_\beta$. Without loss of generality, we can assume $v_\alpha \geq |v_\beta|$.

From $\sum_{i\in B} v_i^2 = 1$, we get

$$1 \leq \sum_{i\in B} v_\alpha^2 \Rightarrow v_\alpha \geq \frac{1}{\sqrt{b}}$$

and from $\sum_{i\in B} v_i = 0$, we get

$$-\frac{1}{\sqrt{b}} \geq -v_\alpha = \sum_{i\in B, i\neq\alpha} v_i \geq (b-1)v_\beta \Rightarrow v_\beta \leq -\frac{1}{(b-1)\sqrt{b}}$$

so finally we have

$$v_\alpha - v_\beta \geq \frac{b}{(b-1)\sqrt{b}}$$

Since the graph is connected and the diameter of the boundary is d_B, there exists a path of length $c \leq d_B$ joining α and β. We label the $c+1$ vertices of the path by $1, ..., c+1$, with $1 = \alpha$ and $c+1 = \beta$. Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\sigma_1 &= \sum_{i\sim j}(v_i - v_j)^2 \geq \sum_{k=1}^{c}(v_k - v_{k+1})^2 \\ &\geq \frac{(v_\alpha - v_\beta)^2}{c} \geq \frac{1}{d_B}\left(\frac{b}{(b-1)\sqrt{b}}\right)^2 \\ &= \frac{b}{(b-1)^2 \cdot d_B}\end{aligned}$$

Furthermore, using (2.2.5), it is easy to compute that $\frac{2}{d_B}$ is the first non-zero eigenvalue of the graph (P_n, B_{P_n}) where P_n is the path of length n and B_{P_n} the two extremities of the path. So the bound is optimal when $b = 2$. □

Remark 2.2. *This result is analog to Lemma 1.9. for the combinatorial normalized Laplacian in [Chu97].*

As we will see in the following examples, this lower bound reflects other aspects of the geometrical meaning of σ_1 than the one given by Theorem 1.3 of [HHW17]. This is due to the fact that the diameter of the boundary does not depend on the total number of vertices of the graph.

Example 2.1. *We consider the family of graphs $\{(D_{n+3}, B_{D_{n+3}})\}_{n\in\mathbb{N}}$ as shown in Figure 2.1., which have two boundary vertices (the two bigger vertices).*

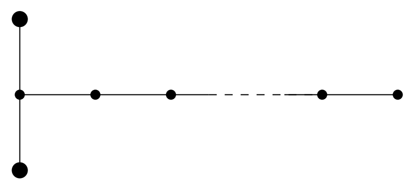

Figure 2.1.

If we extend an eigenfunction of (P_2, B_{P_2}) associated to σ_1 to a function on $(D_{n+3}, B_{D_{n+3}})$ by giving the value of the interior vertex of (P_2, B_{P_2}) to all other vertices of the interior, the

Rayleigh quotient remains unchanged and it is clear that $\sigma_1(D_{n+3}, B_{D_{n+3}}) = \frac{2}{d_B}$. *We note here that the minimizer for* σ_1*, if the number of vertices of the boundary and the diameter of the boundary are fixed, is not unique.*

By computation, we obtain that for this family of graphs, the lower bound of Theorem 1.3 in [HHW17] tends to 0 *as n goes to infinity. So, in this case where the diameter is unbounded but the diameter of the boundary is bounded, the Jammes-type Cheeger estimate fails to see that* σ_1 *is bounded.*

The contrary happens in the next example:

Example 2.2. *Let* $\{\Gamma_n\}_{n\in\mathbb{N}}$ *be a family of expanders (on expanders, see e.g. [Chu97]). By choosing a pair of vertices* $B_{\Gamma_n} := \{i, j\}$ *in* (Γ_n) *that are at distance n, we obtain the family of graphs with boundary* $(\Gamma_n, B_{\Gamma_n})_{n\in\mathbb{N}}$. *From Remark 2.1, we deduce that* σ_1 *is bounded below. On this example, the Jammes-type Cheeger constant of Hua et al. is better than our bound which tends to* 0 *as n goes to infinity.*

2.3.2 Lower bound for σ_1 (II)

In this part, we prove the following improvement of Theorem 2.1:

Theorem 2.2. *Let* (Γ, B) *be a connected graph with diameter of the boundary* d_B *and* $|B| = b$. *We have*

$$\sigma_1 \geq \frac{b}{\left\lfloor \frac{b}{2} \right\rfloor \left\lceil \frac{b}{2} \right\rceil \cdot d_B} \tag{2.3.6}$$

Moreover, the bound is sharp for any b as $d_B \to \infty$.

We begin with the proof of (2.3.6). The proof of the sharpness will follow.

Proof of (2.3.6). Let v be an eigenfunction of σ_1 normalized as in (2.2.5). In the proof of Theorem 2.1, we approximate the difference between v_α and v_β, defined respectively as the largest and the smallest value of v on the boundary, by $v_\alpha - v_\beta \geq \frac{b}{\sqrt{b(b-1)}}$. However, this bound is not sharp as soon as $b > 2$. We improve it by solving a constrained optimization problem.

Consider the map $f_b : \mathbb{R}^b \to \mathbb{R} : (x_1, ..., x_b) \mapsto x_1 - x_b$. Define $D_b := \{x \in \mathbb{R}^b : x_1 \geq x_2 \geq ... \geq x_b\}$ and $S_b := \{\sum_{i=1}^b x_i^2 = 1\} \cap \{\sum_{i=1}^b x_i = 0\}$.

Proposition 2.1. *Let* f_b, D_b, *and* S_b *be as defined above, then*

$$\min_{x\in D_b\cap S_b} f_b(x) = \frac{\sqrt{b}}{\sqrt{\left\lfloor \frac{b}{2} \right\rfloor}\sqrt{\left\lceil \frac{b}{2} \right\rceil}}$$

The proof of (2.3.6) is now immediate by replacing in the proof of Theorem 2.1 the inequality $v_\alpha - v_\beta \geq \frac{b}{\sqrt{b(b-1)}}$ by $v_\alpha - v_\beta \geq \frac{\sqrt{b}}{\sqrt{\lfloor \frac{b}{2} \rfloor}\sqrt{\lceil \frac{b}{2} \rceil}}$. □

Proof of Proposition 2.1. The proof is by induction over the number b of vertices in the boundary and uses the method of Lagrange multipliers.

Base case For $b = 2$, we have to show that $min_{x\in D_2\cap S_2} f_2(x) = \sqrt{2}$. Since $S_2 \cap D_2 = \{(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\}$, $x = (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$ achieves the minimum which is $\sqrt{2}$.

Inductive step We show now that if the result holds for $b-1$, it holds for b.

We define $g_b(x) = \sum_{i=1}^b x_i^2 - 1$, $\tilde{g}_b(x) = \sum_{i=1}^b x_i$, and $G_b(x) = (g_b(x), \tilde{g}_b(x))$. Therefore, we have

$$G_b^{-1}(0) = \{\sum_{i=1}^b x_i^2 = 1\} \cap \{\sum_{i=1}^b x_i = 0\} = S_b$$

We make now a partition of D_b. In order to do this, let us consider the set $M := \{1, 2, .., b-1\}$ and $M_j \subset \mathcal{P}(M)$ the family of sets over M that have cardinality j, $M_0 = \{\emptyset\}$. For a set m_j in the family M_j, we define $E_{m_j} := \{x \in \mathbb{R}^b : x_i > x_{i+1} \text{ if } i \notin m_j; x_i = x_{i+1} \text{ if } i \in m_j\}$. For instance, if $b = 8$ and $m_j = \{1, 5, 6\} \in M_3$, $E_{\{1,5,6\}} = \{x \in \mathbb{R}^b : x_1 = x_2 < x_3 < x_4 < x_5 = x_6 = x_7 < x_8\}$. We have the following partition of D_b

$$D_b = \bigcup_{j=0}^{b-1} \bigcup_{m_j \in M_j} E_{m_j}$$

We remark that E_{m_j} is an open subset of the linear subspace of $\mathbb{R}^b$ $\{x \in \mathbb{R}^b : x_i = x_{i+1} \text{ if } i \in m_j\}$ which is of dimension $b-j$. We will say that E_{m_j} is of dimension $b-j$.

On each E_{m_j} intersected with S_b, we will look for a minimum of f_b. Since the minimum of f_b over $D_b \cap S_b$ must be one of them, we will compare them and find it. We divide these subsets in three categories: the one of dimension $b-j=1$, those of dimension $b-j=2$ and those of dimension $b-j>2$.

Case 1: j = b − 1 The only subset of our partition of D_b of dimension 1 is $\{x \in \mathbb{R}^b : x_1 = ... = x_b\}$ and its intersection with S_b is empty.

Case 2: j = b − 2 Since the sets in the family M_2 only contain one element between 1 and $b-1$, we denote the subsets of dimension 2 $E_k := \{x \in \mathbb{R}^b : x_1 = ... = x_{b-k} > x_{b-k+1} = ... = x_b\}$ for $k = 1, ..., b-1$.

The intersection $E_k \cap S_b$ contains only one vector $y^k = (y_1, ..., y_b)$ where $y_i = \frac{k\sqrt{b-k}}{(b-k)\sqrt{b}\sqrt{k}}$ if $1 \le i \le (b-k)$ and $y_i = \frac{-\sqrt{b-k}}{\sqrt{b}\sqrt{k}}$ if $(b-k) < i \le b$. Hence, the values possible for the minimum are

$$f_b(y^k) = \frac{k\sqrt{b-k}}{(b-k)\sqrt{b}\sqrt{k}} - \frac{-\sqrt{b-k}}{\sqrt{b}\sqrt{k}} = \frac{\sqrt{b}}{\sqrt{b-k}\sqrt{k}} \quad \text{for } k = 1, ..., b-1$$

In order to see for which k the value $f_b(y^k)$ is minimal, we can study the function $h :]0, b[\to \mathbb{R}^+ : k \mapsto h(k) = \frac{\sqrt{b}}{\sqrt{b-k}\sqrt{k}}$. We obtain that the better k is $k = \lfloor \frac{b}{2} \rfloor$ and

$$f_b(y^{\lfloor \frac{b}{2} \rfloor}) = \frac{\sqrt{b}}{\sqrt{\lfloor \frac{b}{2} \rfloor}\sqrt{\lceil \frac{b}{2} \rceil}}$$

Case 3: j < b − 2 It remains to examine the minima of f_b on the subsets of dimension greater than 2 intersected with S_b. This is the most technical part of the proof. We show that if there is such a minimum at $y = (y_1, ..., y_b)$, then one of the coefficients of y must be 0. This allows us to reduce to a situation with $b-1$ vertices. We use then the induction hypothesis and compare the value of f_b at y with the candidate obtained in case 2. The result is that the minimum is the candidate obtained in case 2.

Let $m_j \in M_j$ for $j < b-2$ and E_{m_j} be its corresponding subset of D_b.

If $j > 0$, we number the elements in m_j such that to each $k \in \{1, ..., j\}$ corresponds an element i:=n(k) in m_j. We define $h_k(x) = (x_{n(k)} - x_{n(k)+1})$ for each $k \in \{1, ..., j\}$ and $H(x) = (h_1(x), ..., h_j(x))$. We have $H^{-1}(0) = \{x \in \mathbb{R}^b : x_i = x_{i+1}$ if $i \in m_j\}$. Although we write it without indice, the function H depends on the set E_{m_j} that we are looking at. We note that if a is a minimum of f_b on E_{m_j}, it is a relative minimum on $H^{-1}(0)$. We set $K(x) := (G_b(x), H(x))$. Then $K^{-1}(0) = G_b^{-1}(0) \cap H^{-1}(0)$. If $j = 0$, we define $K(x) = G_b(x)$. We note that in both cases 0 is a regular value of K.

We introduce the Lagrangian defined by

$$L(x, \alpha, \beta) = f_b(x) - \alpha \cdot G_b(x) - \beta \cdot H(x)$$

Let y be a point in $E_{m_j} \cap S_b$ such that $f_b|_{E_{m_j} \cap S_b}(y)$ is a minimum. In particular, it is a local extremum of $f_b|_{K^{-1}(0)}$. Then, by the Lagrange multipliers theorem, there exist $\lambda \in \mathbb{R}^2, \mu \in \mathbb{R}^j$ such that $\nabla L(y, \lambda, \mu) = 0$. This is a system of $b + 2 + j$ linear equations. The first one is

$$\begin{aligned} &1 - 2\lambda_1 y_1 - \lambda_2 - \mu_{n(1)} = 0 && \text{if } 1 \in m_j \\ &1 - 2\lambda_1 y_1 - \lambda_2 = 0 && \text{if } 1 \notin m_j \end{aligned}$$

The following $b - 2$ are for $1 < i < b$

$$\begin{aligned} &- 2\lambda_1 y_i - \lambda_2 + \mu_{n(i-1)} = 0 && \text{if } i - 1 \in m_j \text{ and } i \notin m_j \\ &- 2\lambda_1 y_i - \lambda_2 + \mu_{n(i-1)} - \mu_{n(i)} = 0 && \text{if } i - 1 \in m_j \text{ and } i \in m_j \\ &- 2\lambda_1 y_i - \lambda_2 - \mu_{n(i)} = 0 && \text{if } i - 1 \notin m_j \text{ and } i \in m_j \\ &- 2\lambda_1 y_i - \lambda_2 = 0 && \text{if } i - 1 \notin m_j \text{ and } i \notin m_j \end{aligned}$$

The b^{th} equation is

$$\begin{aligned} &- 1 - 2\lambda_1 y_b - \lambda_2 - \mu_{n(b)} = 0 && \text{if } (b - 1) \in m_j \\ &- 1 - 2\lambda_1 y_b - \lambda_2 = 0 && \text{if } (b - 1) \notin m_j \end{aligned}$$

The two following are

$$\sum_{i=1}^{b} y_i^2 = 1 \quad \text{and} \quad \sum_{i=1}^{b} y_i = 0$$

The last j equations are

$$y_i = y_{i+1} \qquad \text{if } i \in m_j$$

We use now that $j < b - 2$ to show that one of the coefficients of $y = (y_1, ..., y_b)$ must be 0.

By adding the b first equations and using $\sum_{i=1}^{b} y_i = 0$, we get $\lambda_2 = 0$. If $0 < j < b - 2$, there exist $r > 1$ and $s \geq 1$ such that $y_{r-1} < y_r = y_{r+1} = ... = y_{r+s-1} < y_{r+s}$. We note that $r > 1$ and

$r + s - 1 < b$. Therefore, by adding the equations r to $r + s - 1$, we obtain

$$-2\lambda_1(\sum_{i=r}^{i=r+s-1} y_i) = 0 \quad \Rightarrow -2\lambda_1(s-1)y_r = 0$$

This implies $\lambda_1 = 0$ or $y_r = 0$. If $j = 0$, the equations 1 to $b - 1$ show that this is true for any $1 < r < b$. Let k be the smallest integer $\leq b$ such that $k \notin m_j$. It exists since $j < b - 2$. We have now $y_1 = ... = y_k < y_{k+1}$. By adding the k first equations, we get

$$1 - 2\lambda_1(\sum_{i=1}^{k} y_i) = 0 \Rightarrow 1 - 2\lambda_1 k y_1 = 0$$

Therefore, $\lambda_1 \neq 0 \Rightarrow y_r = 0$.

Let p be the projection $p : \mathbb{R}^b \to \mathbb{R}^{b-1} : (x_1, ..., x_b) \mapsto (x_1, ..., x_{r-1}, x_{r+1}, ..., x_b)$. Since $y_r = 0$ for some $r \in \{1, ..., b\}$, $f_b(y) = (f_{b-1} \circ p)(y)$. Moreover, $p(y) \in D_{b-1} \cap S_{b-1}$, hence we have

$$f_b(y) = f_{b-1}(p(y)) \geq \min_{x \in D_{b-1} \cap S_{b-1}} f_{b-1}(x)$$

Using the induction hypothesis $\min_{x \in D_{b-1} \cap S_{b-1}} f_{b-1}(x) = \frac{\sqrt{b-1}}{\sqrt{\lfloor \frac{b-1}{2} \rfloor}\sqrt{\lceil \frac{b-1}{2} \rceil}}$ and the fact that $\frac{\sqrt{b-1}}{\sqrt{\lfloor \frac{b-1}{2} \rfloor}\sqrt{\lceil \frac{b-1}{2} \rceil}} > \frac{\sqrt{b}}{\sqrt{\lfloor \frac{b}{2} \rfloor}\sqrt{\lceil \frac{b}{2} \rceil}}$, we find

$$f_b(y) > \frac{\sqrt{b}}{\sqrt{\lfloor \frac{b}{2} \rfloor}\sqrt{\lceil \frac{b}{2} \rceil}}$$

This concludes the proof of the proposition because it shows that the minimum must be the candidate obtained in case 2 which is $\frac{\sqrt{b}}{\sqrt{\lfloor \frac{b}{2} \rfloor}\sqrt{\lceil \frac{b}{2} \rceil}}$. □

Proof of the sharpness result. For $b = 2$, we already proved in part I that the bound is sharp for all d_B. For $b > 2$, we define a family of graphs $\{(H^b)_{d_B}\}_{d_B \in \mathbb{N}}$ such that $\sigma_1((H^b)_{d_B}) = \frac{b}{\lfloor \frac{b}{2} \rfloor \lceil \frac{b}{2} \rceil (d_B - 2) + b}$ This will complete the proof of Theorem 2.2 because a short calculation shows that $\frac{b}{\lfloor \frac{b}{2} \rfloor \lceil \frac{b}{2} \rceil (d_B - 2) + b} = \frac{b}{\lfloor \frac{b}{2} \rfloor \lceil \frac{b}{2} \rceil \cdot d_B} + O(\frac{1}{d_B^2})$ as $d_B \to \infty$.

Definition 2.5 (The family of graphs $\{(H^b)_{d_B}\}_{d_B \in \mathbb{N}}$)**.** *If b is even, the family of graphs $\{(H^b)_{d_B}\}_{d_B \in \mathbb{N}}$ is defined as shown in figure 2.2., that is, there are $\frac{b}{2}$ boundary vertices on each side of the graph. If b is odd it is defined as shown in figure 2.3.: there is always a vertex more on one of the two sides. In both cases, the path in the middle increases as d_B increases.*

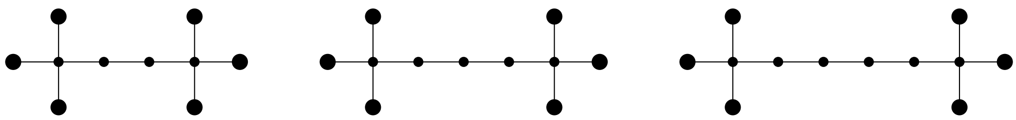

Figure 2.2. – $(H^6)_5$, $(H^6)_6$ and $(H^6)_7$

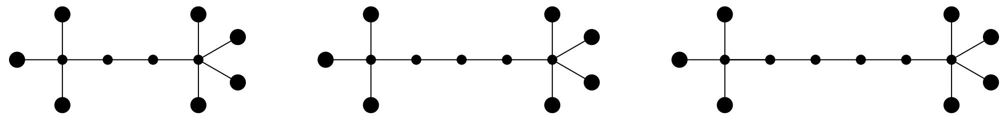

Figure 2.3. – $(H^7)_5$, $(H^7)_6$ and $(H^7)_7$

Lemma 2.2.

$$\sigma_1((H^b)_{d_B}) = \frac{b}{\left\lfloor \frac{b}{2} \right\rfloor \left\lceil \frac{b}{2} \right\rceil (d_B - 2) + b}$$

Proof. For the convenience of the proof, we will assume that b is even. In this case, the lemma can be rewritten $\sigma_1((H^b)_{d_B}) = \frac{4}{b(d_B-2)+4}$. If b is odd, the proof is the same but because of the loss of symmetry the calculations are less pleasant.

We number the vertices of the boundary on the left side from 1 to $\frac{b}{2}$ and those on the right side from $\frac{b}{2} + 1$ to b. We call l the vertex of the interior connected to the left boundary vertices and r the one connected to the right boundary vertices.

Let v be an eigenfunction of $\sigma_1((H^b)_{d_B})$ normalized as in (2.2.5). The coefficients of v corresponding to the b vertices of the boundary are $v_1, ..., v_b$ and those corresponding to l and r are v_l and v_r. Assuming that $\sigma_1 \neq 1$, we have

$$v_i - v_l = \sigma_1 v_i \quad \text{if } i \leq \frac{b}{2} \qquad \Rightarrow v_1 = v_2 = ... = v_{\frac{b}{2}} = \frac{v_l}{1 - \sigma_1}$$

$$v_i - v_r = \sigma_1 v_i \quad \text{if } \frac{b}{2} \leq i \leq b \qquad \Rightarrow v_{\frac{b}{2}} = ... = v_b = \frac{v_r}{1 - \sigma_1}$$

Moreover, using that $\sum_{i=1}^{b} v_i = 0$ and $\sum_{i=1}^{b} v_i^2 = 1$, we get $v_1 = ... = v_{\frac{b}{2}} = \frac{1}{\sqrt{b}}$ and $v_{\frac{b}{2}} = ... = v_b = -\frac{1}{\sqrt{b}}$. Note here that if b were odd, the value on one side would not be exactly the opposite of the value on the other side.

Since v is harmonic on the interior, its energy on the path of length $d_B - 2$ between the vertices l and r is $\frac{(v_l - v_r)^2}{d_B - 2} = \frac{(2v_l)^2}{d_B-2}$. Thus, the total energy of v on the graph, $\sum_{i \sim j}(v_i - v_j)^2$, can be written as a function depending only on the variable v_l

$$\sum_{i \sim j}(v_i - v_j)^2 = \frac{b}{2}(\frac{1}{\sqrt{b}} - v_l)^2 + \frac{(2v_l)^2}{d_B - 2} + \frac{b}{2}(v_l - \frac{1}{\sqrt{b}})^2 =: Q(v_l)$$

From equation (2.2.5), we know that v_l must be the value where Q reaches its minimum. After calculation we find $v_l = \frac{\sqrt{b}(d_B-2)}{b(d_B-2)+4}$. Therefore

$$\sigma_1((H^b)_{d_B}) = Q(v_l) = \frac{4}{b(d_B - 2) + 4}$$

□

□

Remark 2.3. *In analogy to the Riemannian case, Faber-Krahn type inequalities are also interesting on graphs (see [BLS07]). For the Steklov problem, if b = 2, the graphs $(H^2)_{d_B}$ (which are the paths (P_n, B_{P_n}) in the proof of Theorem 2.1) are minimizers for σ_1 among all graphs with* *two vertices in the boundary and the diameter of the boundary equal to d_B. We already noted in example 2.1 that they are not unique. If $b > 2$, we conjecture that $(H^b)_{d_B}$ are minimizers under the same conditions.*

2.4 Weighted graphs

We state finally an equivalent of Theorem 2.1 in the context of the Dirichlet-to-Neumann operator on weighted graphs as defined in [HHW17].

Let (Γ, B) be a graph with boundary (we do not more assume that $\Gamma = (V, E)$ is simple).

Let μ be a symmetric weight function given by

$$\begin{aligned} \mu : V \times V &\to [0, \infty) \\ (i, j) &\mapsto \mu_{ij} = \mu_{ji} \end{aligned}$$

with $\mu_{ij} = 0$ if $\{ij\} \notin E$. We recall that in the definition of a graph with boundary there is no edge between two vertices of the boundary.

Define the measure on V, $m : V \to (0, \infty)$ as follows:

$$m_i = \sum_{i \sim j} \mu_{ij}$$

We will denote by (Γ, B, μ) a weighted graph with boundary.

Recall the variational characterization of the first non-zero eigenvalue for the Dirichlet-to-Neumann operator in this setting according to the definition of Hua et al. (see [HHW17] p.17)

$$\sigma_1 = \min_{v \in \mathbb{R}^b} \left\{ \frac{\sum_{i \sim j} \mu_{ij}(v_i - v_j)^2}{\sum_{i \in B} v_i^2 m_i} : \sum_{i \in B} v_i m_i = 0 \right\} \tag{2.4.7}$$

We keep the definition of the distance between two vertices and the one of the diameter of the boundary unchanged. We define $Vol(B) := \sum_{i \in B} m_i$.

Proposition 2.2. *Let (Γ, B, μ) be a connected weighted graph with diameter of the boundary d_B. We have*

$$\sigma_1 \geq \frac{c}{d_B \cdot Vol(B)}$$

where $c := \min_{i \sim j} \mu_{ij}$.

Remark 2.4. *This result is analog to the the estimate given for the Laplacian in Theorem 3.5. of [Gri09]; the proof is also analog but we use the diameter of the boundary instead of using the diameter of the graph. We recall it here.*

Proof of the proposition. Suppose v is an eigenfunction achieving σ_1 in (2.4.7). Let α be the vertex where $|v_\alpha| = \max_{i \in B} |v_i|$. Since $\sum_{i \in B} v_i m_i = 0$ and $m_i > 0$, there exists a vertex $\beta \in B$ satisfying $v_\alpha v_\beta < 0$. Let P denote a shortest path in Γ joining α and β. Then, by (2.4.7) and using Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sigma_1 &= \frac{\sum_{i\sim j}\mu_{ij}(v_i - v_j)^2}{\sum_{i\in B} v_i^2 m_i} \geq \frac{\sum_{\{ij\}\in P}\mu_{ij}(v_i - v_j)^2}{Vol(B)\cdot v_\alpha^2} \\ &\geq \frac{c\sum_{\{ij\}\in P}(v_i - v_j)^2}{Vol(B)\cdot v_\alpha^2} \geq \frac{c(v_\alpha - v_\beta)^2}{d_B\cdot Vol(B)\cdot v_\alpha^2} \\ &\geq \frac{c}{d_b\cdot Vol(B)}\end{aligned}$$

with $c := \min_{i\sim j}\mu_{ij}$. □

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3. Isoperimetric upper bound for the first eigenvalue of discrete Steklov problems

This article has been previously published in The Journal of Geometric Analysis, see [Per21].

Abstract. We study upper bounds for the first non-zero eigenvalue of the Steklov problem defined on finite graphs with boundary. For finite graphs with boundary included in a Cayley graph associated to a group of polynomial growth, we give an upper bound for the first non-zero Steklov eigenvalue depending on the number of vertices of the graph and of its boundary. As a corollary, if the graph with boundary also satisfies a discrete isoperimetric inequality, we show that the first non-zero Steklov eigenvalue tends to zero as the number of vertices of the graph tends to infinity. This extends recent results of Han and Hua, who obtained a similar result in the case of $\mathbb{Z}^n$. We obtain the result using metric properties of Cayley graphs associated to groups of polynomial growth.

3.1 Introduction

Let M be a compact Riemannian manifold of dimension $n \geq 2$ with boundary ∂M. The Steklov problem on M is

$$\begin{cases} \triangle u = 0 & \text{in } M \\ \frac{\partial u}{\partial n} = \sigma u & \text{on } \partial M \end{cases}$$

where $\triangle$ is the Laplace-Beltrami operator and $\frac{\partial u}{\partial n}$ is the outward normal derivative along the boundary ∂M. It is a well known result that if the boundary is sufficiently regular, the spectrum of the Steklov problem is discrete and its eigenvalues form a sequence $0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \nearrow \infty$.

An important question in studying the spectral geometry of the Steklov problem is to maximize its eigenvalues under a constraint on the volume of the boundary or on the volume of the manifold. For simply-connected planar domains of prescribed perimeter, it has been shown by R. Weinstock that the disk maximizes σ_1 (see [Wei54]). For bounded Lipschitz domains of fixed volume in $\mathbb{R}^n$, F. Brock proved that the ball maximizes σ_1 (see [Bro01]). Several upper bounds have also been obtained for different families of manifolds where the volume or the volume of the boundary is fixed. In 2017, a survey of the literature on this question has been given in [GP17]. More recently, it was shown in [BFNT21] that a Weinstock-type inequality holds in $\mathbb{R}^n$ in the class of convex sets, that is, that among all bounded convex sets in $\mathbb{R}^n$ with prescribed volume of the boundary, the ball maximizes σ_1.

In this article, we investigate isoperimetric upper bounds for σ_1 of the Steklov problem on graphs. The Steklov problem on graphs is a discrete analogue of the Steklov problem and has recently received attention in the literature. In [HHW17] and [Per19], lower bounds for the first non-zero eigenvalue are given. A lower bound for higher eigenvalues is given in [HM20]. For subgraphs of integer lattices, an upper bound has been obtained by W. Han and B. Hua [HH19]. In [CGR18], a relation between the eigenvalues of the Steklov problem on a manifold and the eigenvalues of a discrete problem is established. Hence, results in the discrete and in the Riemannian settings are closely related and the study of the discrete problem is a possible approach to understand the spectral geometry of the Steklov problem.

Definition 3.1. *A graph with boundary is a pair (Γ, B), where $\Gamma = (V, E)$ is a simple, that is without loops or multiple edges, connected graph and $B \subset V$ is a subset of V such that $B \neq \emptyset$ and $E(B, B) = \emptyset$. We call B the boundary of the graph and B^c the interior.*

In this paper, we always consider graphs with boundary that are finite. The space of all real functions defined on the vertices V, denoted by $\mathbb{R}^V$, is the Euclidean space of dimension $|V|$. Similarly, the space of real functions defined on the vertices of the boundary, denoted $\mathbb{R}^B$, is the Euclidean space of dimension $|B|$.

The Laplacian $\triangle$ of a function $v \in \mathbb{R}^V$ is defined by

$$(\triangle v)(i) = \sum_{j\sim i}(v(i) - v(j))$$

where $i \sim j$ signifies that $\{i, j\} \in E$.

A function $v \in \mathbb{R}^V$ is called harmonic if

$$(\triangle v)(i) = \sum_{j\sim i}(v(i) - v(j)) = 0 \quad \forall i \notin B.$$

The normal derivative operator $\frac{\partial}{\partial n} : \mathbb{R}^V \to \mathbb{R}^B$ is defined by

$$\left(\frac{\partial v}{\partial n}\right)(i) = \sum_{j\in B^c, j\sim i}(v(i) - v(j)) \quad i \in B.$$

Definition 3.2. *The Steklov problem on a finite graph with boundary (Γ, B) is the eigenvalue problem*

$$\begin{cases} (\triangle v)(i) = 0 & \text{if } i \notin B \\ (\frac{\partial v}{\partial n})(i) = \sigma v(i) & \text{if } i \in B \end{cases}$$

where $v \not\equiv 0$ and σ is a spectral parameter.

As shown in [Per19], the solutions of this problem coincide with the eigenvalues of the discrete Dirichlet-to-Neumann operator defined in [HHW17]. They form a finite sequence $0 = \sigma_0 \leq \sigma_1 \leq ... \leq \sigma_{b-1}$, where $b = |B|$.

We recall that we are interested in upper bounds for σ_1. Therefore, we will always assume that $|B| > 1$ because if not, σ_1 is not defined. A first remark is that without any additional geometric constraint on (Γ, B), σ_1 may become unbounded. This occurs in the following example.

Example 3.1. *We consider the family of graphs with boundary $\{(G_n, B)\}_{n\in\mathbb{N}}$ as shown in Figure 3.1., that is, two boundary vertices (the bigger vertices) are connected by n paths of length 2. By computation, we obtain that $\sigma_1(G_n, B) = n$ and hence σ_1 tends to $+\infty$ as n tends to $+\infty$.*

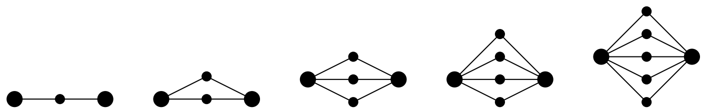

Figure 3.1. – $(G_1, B), (G_2, B), (G_3, B), (G_4, B)$ and (G_5, B).

If we assume that the degree of the graph, d, is bounded, it is easy to obtain that $\sigma_1 \leq d$. The goal of this paper is to show isoperimetric upper bounds for the Steklov eigenvalues of graphs with boundary that are included in a Cayley graph with polynomial growth (we recall the notions of geometric group theory that we use in Section 2).

Definition 3.3. *A graph with boundary* $(\Gamma' = (V', E'), B)$ *is included in a graph* $\Gamma = (V, E)$ *if* $V' \subset V$ *and* $E' \subset E$.

Remark 3.1. *The Steklov problem is defined on finite graphs with boundary. In contrast, the Cayley graphs with polynomial growth that we use as host graph are infinite.*

The main result is the following.

Theorem 3.1. *Let* $\Gamma = (V, E)$ *be a Cayley graph with polynomial growth of order D. There exists* $C(\Gamma) > 0$ *such that for any finite graph with boundary* $(\Gamma' = (V', E'), B)$ *included in* Γ *and such that* $|B| > 1$, *we have*

$$\sigma_1(\Gamma', B) \leq \begin{cases} C(\Gamma)\frac{1}{|B|} & \text{if } D \leq 2, \\ C(\Gamma)\frac{|V'|^{\frac{D-2}{D}}}{|B|} & \text{if } D \geq 2. \end{cases}$$

This result can be pushed further for a particular class of graph with boundary included in the Cayley graph $\Gamma = (V, E)$, *graphs with boundary induced by a subset* $\Omega \subset V$.

Definition 3.4. *Let* $\Gamma = (V, E)$ *be a graph.*

1. *The vertex boundary of a subset* $\Omega \subset \Gamma$ *is*

$$\delta\Omega := \{i \in V \setminus \Omega : \exists j \in \Omega, i \sim j\}$$

   where $i \sim j$ *signifies that* $\{i, j\} \in E$.

2. *The set of edges between two subset* $\Omega_1, \Omega_2 \subset V$ *is*

$$E(\Omega_1, \Omega_2) := \{\{i, j\} \in E : i \in \Omega_1, j \in \Omega_2\}.$$

3. *Given* $\Omega \subsetneq V$, *consider the graph* Γ' *with vertex set* $\bar{\Omega} := \Omega \cup \delta\Omega$ *and edge set* $E(\Omega, \bar{\Omega})$. *This defines a graph with boundary, with* $B = \delta\Omega$, *which is called graph with boundary induced by a subset* $\Omega \subsetneq V$.
4. *Given a subset* $\Omega \subsetneq V$, $\sigma_1(\Omega)$ *is the eigenvalue* σ_1 *of the graph with boundary induced by* Ω.

Because graphs with boundary induced by a finite subset Ω of the set of vertices of a Cayley graph with polynomial growth satisfy a discrete isoperimetric inequality, we can deduce the following two corollaries of Theorem 3.1 for this particular case.

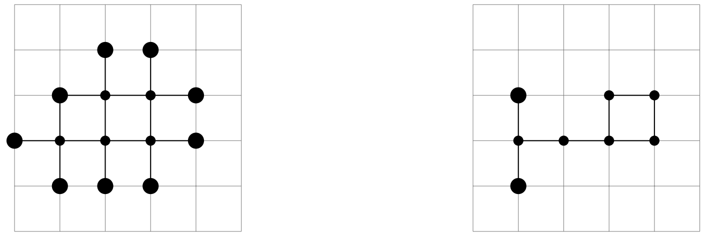

Figure 3.2. – Two graphs with boundary included in $\mathbb{Z}^2$, but only the first one is induced by a subset of vertices of $\mathbb{Z}^2$ (the bigger vertices are boundary vertices).

Corollary 3.1. *Let* $\Gamma = (V, E)$ *be a Cayley graph with polynomial growth of order* $D \geq 2$*. There exists* $C(\Gamma) > 0$ *such that for any finite connected subset* Ω *of the set of vertices* V *we have*

$$\sigma_1(\Omega) \leq C(\Gamma)\frac{1}{|\delta\Omega|^{\frac{1}{D-1}}}.$$

Corollary 3.2. *Let* $\Gamma = (V, E)$ *be a Cayley graph with polynomial growth of order* $D \geq 2$*. There exists* $C(\Gamma) > 0$ *such that for any finite connected subset* Ω *of the set of vertices* V *we have*

$$\sigma_1(\Omega) \leq C(\Gamma)\frac{1}{|\bar{\Omega}|^{\frac{1}{D}}},$$

where $\bar{\Omega} = \delta\Omega \cup \Omega$*.*

A direct consequence is that for a sequence of graphs with boundary induced by subsets in a Cayley graph with polynomial growth such that the number of vertices tends to infinity, σ_1 tends to zero. It is easy to find examples (see, e.g., Example 1 in [Per19]) showing that this is not true if we do not assume that the graphs with boundary are induced by subsets of the Cayley graph.

In $\mathbb{Z}^n$, with $n \geq 2$, Corollary 2 corresponds to a recent result of Han and Hua (see Corollary 1.4 in [HH19]). They show it using a very interesting method to reduce to the case of domains in $\mathbb{R}^n$, which also allows them to give explicit constants. In the contrast to the proof of the result of Han and Hua, the proof of our main result is direct because it does not use known results for domains in Euclidean space. It essentially uses the control of the growth function of the Cayley graph. The method was inspired by the methods used in [CESG11]. A straightforward example of a Cayley graph of a group of polynomial growth that is different from $\mathbb{Z}^n$ is a Cayley graph associated to the discrete Heisenberg group of dimension 3, which has polynomial growth of order 4. Many other examples exist (see Example 3.4) where the result holds.

3.2 Groups with polynomial growth and Cayley graphs

In this article, we work in the setting of Cayley graphs of groups with polynomial growth. We recall here the definitions and the geometric group theory notions that we will use. For further details on this topic, one can see e.g. [dlH00].

Let G be a finitely generated infinite discrete group and S=$\{g_1, ..., g_k\}$ a generating set of G. For $n \in \mathbb{N}^*$, we denote the ball of radius n $B(n) := \{x \in G : x = g_{i_1}^{\epsilon_1}...g_{i_n}^{\epsilon_n}, i_1, ..., i_n \in \{1, ..., k\}, \epsilon_j =$

$\pm 1\}$. The growth function of G is $V(n) := |B(n)|$. If there exist $D \in \mathbb{N}^*$ and $C > 0$ such that

$$C^{-1}n^D \leq V(n) \leq Cn^D,$$

we say that the growth rate is polynomial of order D. Since the growth rate does not depend on the choice of generating set, we can speak of the growth type of a group.

Let G be a group and S a generating set that does not contain the identity element of the group and is symmetric, that is, satisfies $S = S^{-1}$. The Cayley graph $\Gamma = \Gamma(G, S)$ associated to (G, S) is the graph with vertices $V = G$ and edges $E = \{\{x, y\} : x, y \in V \text{ and } \exists s \in S \text{ such that } y = xs\}$. Since S is symmetric and does not contain the identity element, the graph is simple, and since S is a generating set of G, the graph is connected. We say that a Cayley graph has polynomial growth of order D if it is associated to a group with polynomial growth of order D.

We now give two properties of Cayley graphs with polynomial growth that we will need to prove our results.

Lemma 3.1. *Let $\Gamma = (V, E)$ be a Cayley graph with polynomial growth of order D. Let $a, b \in \mathbb{R}_+^*$ and $B(x, aR)$ be a ball in Γ of radius aR. Then $\exists N \in \mathbb{N}^*$ such that $B(x, aR)$ is the union of N balls of radius bR and this number does not depend on R. More precisely, we can take $N = \lceil C^2(\frac{2a+b}{b})^D \rceil$ where C is a constant satisfying $C^{-1}n^D \leq V(n) \leq Cn^D$.*

Proof. Let $\{y_i\}_{i=1}^m$ be a maximal subset of vertices in $B(x, aR)$ such that $d(y_i, y_j) \geq bR$ for $i \neq j$. Then $\cup_{i=1}^n B(y_i, bR) \supset B(x, aR)$ and, by the triangle inequality, $B(y_i, \frac{bR}{2}) \cap B(y_j, \frac{bR}{2}) = \emptyset$. This implies

$$\sum_{i=1}^m |B(y_i, \frac{bR}{2})| \leq |B(x, (a + \frac{b}{2})R)|. \tag{3.2.1}$$

Since the graph has polynomial growth of order D, we know that there exists C such that $C^{-1}n^D \leq |B(z, n)| \leq Cn^D \; \forall z \in V$. We approximate the volume of the balls in equation (3.2.1) using the latter inequality and we obtain that $m \leq C^2(\frac{2a+b}{b})^D$. □

The second property is a discrete isoperimetric inequality.

Proposition 3.1. *Let $\Gamma = (V, E)$ be a Cayley graph with polynomial growth of order D. There exists C such that for any finite subset $\Omega \subset V$, $\delta\Omega$ its boundary, and $\bar{\Omega} := \Omega \cup \delta\Omega$, we have that*

$$\frac{|\bar{\Omega}|^{\frac{(D-1)}{D}}}{|\delta\Omega|} \leq C. \tag{3.2.2}$$

For the proof of this proposition, we refer to [CSC93]. In fact, the result that we give corresponds to the first particular case of Theorem 1 of [CSC93], but formulated in the setting of Cayley graphs.

3.3 Isoperimetric upper bound for σ_1 in Cayley graphs with polynomial growth

In this section, we prove the results presented in the introduction and give examples of application.

The following variational characterization of the Steklov eigenvalues on graphs with boundary is important for the proof of our main result, Theorem 3.1.

$$\sigma_j = \min_{E} \max_{v \in E, v \neq 0} R(v), \tag{3.3.3}$$

where E is the set of all linear subspaces of $\mathbb{R}^V$ of dimension $j+1$, and $R(v)$ is the Rayleigh quotient associated to the Dirichlet-to-Neumann operator (see [HHW17])

$$R(v) := \frac{\sum_{i \sim j}(v(i) - v(j))^2}{\sum_{i \in B} v(i)^2}.$$

3.3.1 Proof of Theorem 3.1

The proof consists of finding two regions of the graph with boundary with a sufficient number of vertices of the boundary, then building test functions, evaluating their Rayleigh quotient, and using the variational characterization in order to obtain an upper bound for σ_1.

Proof. By Lemma 3.1, there exists c_1 such that a ball of radius $3R$ in Γ is the union of c_1 balls of radius $\frac{1}{2}R$.

If $|B| \leq c_1 + 1$, it is easy to show that the result is true using that σ_1 is bounded from above by d, the degree of the host Cayley graph: we have that

$$\sigma_1 \leq d = \frac{d|B|}{|B|} \leq \frac{d(c_1+1)}{|B|} =: c_2 \frac{1}{|B|}$$

and, if $D \geq 2$,

$$\sigma_1 \leq c_2 \frac{1}{|B|} \leq c_2 \frac{|V'|^{\frac{D-2}{D}}}{|B|}.$$

From now on, we will assume $|B| > c_1 + 1$. We define

$$\alpha := \frac{|B|}{c_1 + 1}.$$

Let $x \in V$. We set

$$r_x := \min\{r \in \mathbb{N} : |B(x,r) \cap B| \geq \alpha\}$$

and

$$R := \min_{x \in V} r_x.$$

Then, we have that $\forall x \in V$, $|B(x, R-1) \cap B| < \alpha$ and there exists x_0 such that $|B(x_0, R) \cap B| \geq \alpha$. We remark that $R \geq 1$. Since $B(x, R-1) \geq B(x, \frac{1}{2}R)$ we have that $B(x_0, 3R)$ is the union of c_1 balls of radius $R-1$. This implies

$$|B(x_0, 3R) \cap B| < c_1 \alpha$$

and consequently

$$\begin{aligned}|B(x_0, 3R)^c \cap B| &= |B| - |B(x_0, 3R) \cap B| \\ &> |B| - c_1\alpha \\ &= |B| - c_1\frac{|B|}{c_1+1} \\ &= \frac{|B|}{c_1+1} = \alpha.\end{aligned}$$

Hence, we have found two regions, $B(x_0, R)$ and $B(x_0, 3R)^c$, such that

$$|B(x_0, R) \cap B| \geq \alpha$$

and

$$|B(x_0, 3R)^c \cap B| > \alpha.$$

We define two test functions, one with support $B(x_0, 2R)$, and the other with support $B(x_0, 2R)^c$.

$$f_1(y) = \begin{cases} 1 & \text{if } y \in B(x_0, R) \\ 1 - \frac{k}{R} & \text{if } k := d(y, B(x_0, R)) \leq R \\ 0 & \text{otherwise,} \end{cases}$$

$$f_2(y) = \begin{cases} 1 & \text{if } y \in B(x_0, 3R)^c \\ 1 - \frac{k}{R} & \text{if } k := d(y, B(x_0, 3R)^c) \leq R \\ 0 & \text{otherwise.} \end{cases}$$

We consider the linear subspace W of $\mathbb{R}^V$ generated by f_1 and f_2. The variational characterization of equation (3.3.3) gives

$$\sigma_1 \leq \max_{v \in W} R(v).$$

Since f_1 and f_2 have disjoint support, it implies

$$\sigma_1 \leq \max\{R(f_1), R(f_2)\}.$$

$R(f_1)$ can be evaluated in the following way. The denominator is

$$\sum_{i \in B} f_1(i)^2 \geq |B(x_0, R) \cap B| \geq \alpha = \frac{|B|}{c_1+1}.$$

The only edges contributing to the sum in the numerator $\sum_{i\sim j}(f_1(i) - f_1(j))^2$ are the ones in $B(x_0, 2R) \setminus B(x_0, R)$. In this annulus, for two adjacent vertices, we have that $(f_1(i) - f_1(j))^2 \leq \frac{1}{R^2}$. Moreover, the number of edges in this annulus is smaller than or equal to the number of edges in $B(x_0, 3R)$. Hence we have

$$\sum_{i\sim j}(f_1(i) - f_1(j))^2 \leq \sum_{i\sim j, i,j \in B(x_0,3R)} \frac{1}{R^2}.$$

Because the graph has polynomial growth of order D, there exists $c_3 > 0$ such that $|B(x_0, 3R)| \leq c_3(3R)^D$. We recall that the graph is the Cayley graph defined by a group G and a generating set S of G. The degree of the graph is $|S| = |B(y, 1)| \leq c_3$. By the handshaking lemma, $|E(B(x_0, 3R), B(x_0, 3R))| \leq \frac{1}{2}|B(x_0, 3R)||S| \leq \frac{1}{2}c_3^2(3R)^D := c_4R^D$. Consequently, for $D = 1$ or $D = 2$, we have

$$\sum_{i\sim j, i,j\in B(x_0,3R)} \frac{1}{R^2} \leq c_4\frac{R^D}{R^2} \leq c_4$$

and the Rayleigh quotient of f_1 becomes

$$R(f_1) = \frac{\sum_{i\sim j}(f_1(i) - f_1(j))^2}{\sum_{i\in B} f_1(i)^2} \leq \frac{(c_1 + 1)c_4}{|B|} =: \frac{c_5}{|B|}$$

If $D \geq 2$, we note that we have the following equality

$$\sum_{i\sim j, i,j\in B(x_0,3R)} \frac{1}{R^2} = (\sum_{i\sim j, i,j\in B(x_0,3R)} \frac{1}{R^D})^{\frac{2}{D}}(\sum_{i\sim j, i,j\in B(x_0,3R)} 1)^{\frac{D-2}{D}}$$

The left factor is bounded by a constant:

$$(\sum_{i\sim j, i,j\in B(x_0,3R)} \frac{1}{R^D})^{\frac{2}{D}} \leq c_4^{\frac{2}{D}} =: c_6.$$

For the right factor, we have

$$(\sum_{i\sim j, i,j\in B(x_0,3R)} 1)^{\frac{D-2}{D}} \leq (\frac{c_3}{2}|V'|)^{\frac{D-2}{D}},$$

and we obtain

$$\sum_{i\sim j, i,j\in B(x_0,3R)} \frac{1}{R^2} \leq c_6(\frac{c_3}{2})^{\frac{D-2}{D}}|V'|^{\frac{D-2}{D}} =: c_7|V'|^{\frac{D-2}{D}}.$$

Hence, if $D \geq 2$, the numerator of the Rayleigh quotient satisfies

$$\sum_{i\sim j}(f_1(i) - f_1(j))^2 \leq c_7|V'|^{\frac{D-2}{D}}.$$

The Rayleigh quotient of f_1 becomes

$$R(f_1) = \frac{\sum_{i\sim j}(f_1(i) - f_1(j))^2}{\sum_{i\in B} f_1(i)^2} \leq \frac{(c_1 + 1)c_7|V'|^{\frac{D-2}{D}}}{|B|} =: c_8\frac{|V'|^{\frac{D-2}{D}}}{|B|}.$$

By the definition of the test functions, the same upper bound can be obtained for f_2. We conclude that

$$\sigma_1 \leq \max\{R(f_1), R(f_2)\} \leq \begin{cases} c_5\frac{1}{|B|} & \text{if } D \leq 2, \\ c_8\frac{|V'|^{\frac{D-2}{D}}}{|B|} & \text{if } D \geq 2. \end{cases}$$

In order to unify the case $|B| \leq c_1 + 1$ and the general case, we take $C := \max\{c_5, c_2\}$ if $D \leq 2$ or $C := \max\{c_8, c_2\}$ if $D \geq 2$. This completes the proof. □

Remark 3.2. *The proof is qualitative rather than quantitative since the goal here is not to find an optimal constant (the constant depends on the generating set of the group).*

Example 3.2. *An example of a group with polynomial growth of order D is $\mathbb{Z}^D$.*

Example 3.3. *The Heisenberg group over $\mathbb{Z}$,*

$$Heis(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\},$$

is an example of a group with polynomial growth of order 4, which is not quasi-isometric to $\mathbb{Z}^4$ (on this affirmation, see [GdlH90], p. 13, for example). Hence, for the Steklov problem on a graph with boundary $(\Gamma' = (V', E'), B)$ included in a Cayley graph associated to the Heisenberg group, σ_1 is bounded from above by $C(Heis(\mathbb{Z}))\frac{|V'|^{1/2}}{|B|}$.

Example 3.4. *An important theorem due to M. Gromov characterizes finitely generated groups of polynomial growth (see [Gro81]). It says that that a group is of polynomial growth if and only if it has a nilpotent subgroup of finite index. Lattices in nilpotent Lie groups, which are finitely generated and themselves nilpotent are other examples where the theorem holds (for the existence of such lattices, see e.g. [Rag72] and [Ebe94]).*

Remark 3.3. *Given a Cayley graph $\Gamma = (V, E)$ with polynomial growth of order 1 or 2, Theorem 3.1 shows that for a sequence $\{(\Gamma'_n, B_n)\}_{n\in\mathbb{N}}$ of graphs with boundary included in Γ and satisfying $|B_n| \to \infty$, we have that σ_1 tends to 0 as n tends to infinity.*

3.3.2 Application to subgraphs

Proof of Corollary 3.1. By the isoperimetric inequality in Proposition 3.1, there exists $c_1 > 0$ such that $\frac{|\bar{\Omega}|^{\frac{(D-1)}{D}}}{|\delta\Omega|} \le c_1$, where $\bar{\Omega} = \delta\Omega \cup \Omega$. We raise the latter inequality to the power of $\frac{D-2}{D-1}$ and obtain $|\bar{\Omega}|^{\frac{D-2}{D}} \le (c_1|\delta\Omega|)^{\frac{D-2}{D-1}} =: c_2|\delta\Omega|^{\frac{D-2}{D-1}}$. By Theorem 3.1, there exists c_3 such that $\sigma_1 \le c_3\frac{|\bar{\Omega}|^{\frac{D-2}{D}}}{|\delta\Omega|}$. Consequently,

$$\sigma_1 \le c_3\frac{|\bar{\Omega}|^{\frac{D-2}{D}}}{|\delta\Omega|} \le c_3c_2\frac{|\delta\Omega|^{\frac{D-2}{D-1}}}{|\delta\Omega|} = c_3c_2\frac{1}{|\delta\Omega|^{\frac{1}{D-1}}} =: c_4\frac{1}{|\delta\Omega|^{\frac{1}{D-1}}}.$$

□

Remark 3.4. *For $D = 1$, we remark that by Theorem 3.1, we have that $\sigma_1(\Omega) \le C(\Gamma)\frac{1}{|\delta\Omega|}$.*

Proof of Corollary 3.2. By the isoperimetric inequality in Proposition 3.1, there exists $c_1 > 0$ such that $\frac{|\bar{\Omega}|^{\frac{(D-1)}{D}}}{|\delta\Omega|} \le c_1$. By Theorem 3.1, there exists c_2 such that $\sigma_1 \le c_2\frac{|\bar{\Omega}|^{\frac{D-2}{D}}}{|\delta\Omega|}$. Hence, we have

$$\sigma_1 \le c_2\frac{|\bar{\Omega}|^{\frac{D-2}{D}}}{|\delta\Omega|} = c_2\frac{|\bar{\Omega}|^{\frac{D-1}{D}}|\bar{\Omega}|^{\frac{-1}{D}}}{|\delta\Omega|} \le c_2c_1|\bar{\Omega}|^{\frac{-1}{D}} =: c_3\frac{1}{|\bar{\Omega}|^{\frac{1}{D}}}.$$

□

Remark 3.5. *Since $\bar{\Omega} = \delta\Omega \cup \Omega$, we also have $\sigma_1 \le C(\Gamma)\frac{1}{|\Omega|^{\frac{1}{D}}}$ and $\sigma_1 \le C(\Gamma)\frac{1}{|\delta\Omega|^{\frac{1}{D}}}$ but this last bound is weaker than Corollary 3.1.*

Remark 3.6. *In a Cayley graph with polynomial growth of order $D \geq 2$, for a sequence $\{\Omega_n\}_{n\in\mathbb{N}}$ of finite subsets satisfying $|\Omega_n| \to \infty$, we have that $\sigma_1(\Omega_n)$ tends to* 0 *as n tends to infinity.*

Remark 3.7. *For graphs with boundary induced by a finite subset of $\mathbb{Z}^n$, the result of Corollary 3.2 was recently obtained by Han and Hua (see Corollary 1.4 in [HH19]), who also give an explicit constant.*

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4. Estimates for low Steklov eigenvalues of surfaces with several boundary components

This article is available on ArXiv, see [Per23].

Abstract. In this article, we give computable lower bounds for the first non-zero Steklov eigenvalue σ_1 of a compact connected 2-dimensional Riemannian manifold M with several cylindrical boundary components. These estimates show how the geometry of M away from the boundary affects this eigenvalue. They involve geometric quantities specific to manifolds with boundary such as the extrinsic diameter of the boundary. In a second part, we give lower and upper estimates for the low Steklov eigenvalues of a hyperbolic surface with a geodesic boundary in terms of the length of some families of geodesics. This result is similar to a well known result of Schoen, Wolpert and Yau for Laplace eigenvalues on a closed hyperbolic surface.

4.1 Introduction

We study lower bounds for low Steklov eigenvalues of a compact connected 2-dimensional Riemannian manifold with several boundary components. Few lower bounds are known for the first non-zero Steklov eigenvalue σ_1. For a Riemannian manifold with connected boundary, there are generalizations (see e.g. [Esc97], [Esc99] and [Xio22]) of a result of Payne [Pay70] of 1970 saying that σ_1 of a convex domain in the plane is bounded from below by the minimum curvature of its boundary. In a general setting, Escobar [Esc97] has given a lower bound depending on an isoperimetric constant and the first non-zero eigenvalue of a Robin problem (see also [HM20] for lower bounds depending on eigenvalues of auxiliary problems). In [Jam15], Jammes gives lower bounds in terms of isoperimetric constants. This result has been generalized by Hassannezhad and Miclo [HM20] for higher eigenvalues. These lower bounds however are not easily computable. In [CGH20], the authors show that under some assumptions on the geometry of the boundary and near the boundary, Steklov eigenvalues are well approximated by the Laplace eigenvalues of the boundary. But when a connected Riemannian manifold has $b \geq 2$ boundary components, such estimates do not give lower bounds for the b first eigenvalues of the Steklov problem.

For obtaining lower bounds, conditions on the intrinsic geometry of the boundary as well as conditions on the geometry near the boundary are expected. But even if the boundary and the geometry of M near the boundary are fixed, σ_1 is not bounded below if the boundary has multiple connected components, as shows the case of a right cylinder whose first eigenvalue tends to zero as its height goes to infinity.

In this article, we give explicit estimates for the b first Steklov eigenvalues of some families of compact connected 2-dimensional Riemannian manifolds with $b \geq 2$ boundary components having each one a neighborhood which is a right or a hyperbolic cylinder. This strong assumption on the geometry near the boundary allows us to focus on how the geometry of the manifold away from the boundary affects these eigenvalues. The first result is an explicit lower bound for σ_1 of a 2-dimensional Riemannian manifold with cylindrical boundary. It does not require any assumption on the Gaussian curvature and involves the following quantity.

Definition 4.1. *Let (M, g) be a compact connected 2-dimensional Riemannian manifold with $b \geq 2$ boundary components. We consider the family of curves (not necessarily connected) not intersecting ∂M and dividing M into two connected components, each containing at least one connected component of ∂M. We let $C(M)$ denote this family of curves and define*

$$\mathrm{l}(M) := \inf\{l(c) : c \in C(M)\}$$

where $l(c)$ is the length of the curve c.

We can now state the result.

Theorem 4.1. *Let (M, g) be a compact connected 2-dimensional Riemannian manifold with a boundary having $b \geq 2$ components of length a. Assume that the boundary $\partial M = \Sigma_1 \cup \cdots \cup \Sigma_b$ has a neighborhood $V(\partial M)$ which is isometric to the union of disjoint right cylinders $\cup_{i=1}^{b} \Sigma_i \times [0, L)$. We have*

$$\sigma_1(M) \geq \frac{\min\{\mathrm{l}(M), L\}^2}{2(b-1)a|M|}.$$

Examples 4.2 and 4.3 show that the exponent of the geometric quantities involved in the lower bound are optimal. Another natural question to ask for evaluating a lower bound is how close to σ_1 it is. We construct a family of surfaces which shows that the presence of the area of the manifold in the denominator makes the lower bound given in Theorem 4.1 sometimes inaccurate since it can go to zero while σ_1 is constant.

For surfaces whose Gaussian curvature is bounded below, we succeeded in removing the depency of the area from the lower bound. This estimate involves the extrinsic diameter of the boundary and the injectivity radius of a certain subset of M.

Definition 4.2. *Let (M, g) be a compact connected Riemannian manifold with boundary ∂M.*

1. *The extrinsic diameter of the boundary is*

$$\mathrm{diam}_M(\partial M) = \max\{d(x, y) | x, y \in \partial M\},$$

where $d(x, y)$ denotes the distance on M induced by g.

For simplification, we will omit the term "extrinsic" and call it the diameter of the boundary. Assume now that the boundary $\partial M = \Sigma_1 \cup \cdots \cup \Sigma_b$ has a neighborhood $V(\partial M)$ which is isometric to the union of disjoint right cylinders $\cup_{i=1}^{b} \Sigma_i \times [0, L)$.

2. *Let Γ be the subset of M*

$$\Gamma = \{x \in M, \exists p, q \in \partial M \text{ and a length minimising geodesic } \gamma \text{ between } p \text{ and } q \text{ such that } x \in \gamma\}.$$

We denote $\mathrm{inj}_{\partial M}(M)$ *the injectivity radius of* $\Gamma \setminus V(\partial M) \subset M$, *that is*

$$\mathrm{inj}_{\partial M}(M) = \mathrm{inj}(\Gamma \setminus V(\partial M)) = \min\{\mathrm{inj}_M(x) : x \in \Gamma \setminus V(\partial M)\}.$$

We note that $\mathrm{inj}_{\partial M}(M) \leq L$.

Theorem 4.2. *Let* (M, g) *be a compact connected 2-dimensional Riemannian manifold with a boundary having* $b \geq 2$ *boundary components of length* a. *Assume that the boundary* $\partial M = \Sigma_1 \cup \cdots \cup \Sigma_b$ *has a neighborhood* $V(\partial M)$ *which is isometric to the union of disjoint right cylinders* $\cup_{i=1}^{b} \Sigma_i \times [0, L)$ *with* $L \leq 1$. *Suppose there exists* $\kappa < 0$ *such that the Gaussian curvature of* M *satisfies* $K(p) \geq \kappa$ *for all* $p \in M$ *and suppose* $a \leq \mathrm{diam}_M(\partial M)$. *Then we have an explicit constant* $C(\kappa, b)$ *such that*

$$\sigma_1(M) \geq C(\kappa, b) \frac{\mathrm{inj}_{\partial M}(M)}{a \, \mathrm{diam}_M(\partial M)}.$$

As for Theorem 4.1, we show that the exponent of the geometric quantities involved in Theorem 4.2 cannot be improved (see Remark 4.5). With the stronger assumption that the injectivity radius is bounded from below at each point of M outside the cylindrical neighborhood of the boundary, Theorem 4.2 can also be obtained from the combination of the lower bound given in [Per19] for σ_1 of the Steklov problem on graphs and the discretization process described in [CGR18].

We note that results for surfaces with a cylindrical boundary are significant since they can be used for deducing results for any manifolds with boundary by using quasi-isometries as it has been done in [CESG19] (see Theorem 1.1).

In a second part, we give an upper and lower estimate for the b first Steklov eigenvalues of compact hyperbolic surfaces with b geodesic boundary components. It shows that these eigenvalues are equivalent to the length of some separating curves of the manifold. The result is similar to a result of Schoen, Wolpert and Yau [SWY80] for Laplace eigenvalues. However, the family of curves that are relevant is different.

Definition 4.3. *Let* M *be a compact hyperbolic surface with* $b \geq 2$ *geodesic boundary components. For* $1 \leq n \leq b - 1$, *we consider the family of curves which consist of a union of disjoint simple closed geodesics, not intersecting* ∂M, *and dividing* M *into* $n + 1$ *connected components, each containing at least one connected component of* ∂M. *We denote* $C_n(M)$ *the family of such curves. If* $C_n(M) \neq \emptyset$, *we define*

$$\mathrm{l}_n(M) := \min\{l(c) : c \in C_n(M)\}$$

where $l(c)$ *is the length of the curve* c.

We have the following result.

Theorem 4.3. *Let* M *be a hyperbolic surface of genus* g *with* $b \geq 2$ *geodesic boundary components, each of them having length* $a \leq 2\,\mathrm{arcsinh}(1)$. *Assume that* $g \neq 0$ *or* $b > 3$. *There exists a constant* C_1 *depending only on* g *and* b *and a universal constant* C_2 *such that for* $1 \leq n < \lceil \frac{b}{2} \rceil$ *we have*

$$C_1 \, \mathrm{l}_n^2 \leq \sigma_n \leq C_2 \frac{\mathrm{l}_n}{a}.$$

The inequality is also true for $\lceil \frac{b}{2} \rceil \leq n < b$ *if* $C_n(M) \neq \emptyset$ *and there exists* $c \in C_n(M)$ *such that each simple closed geodesic of* c *is of length* $l \leq L_{g+b}$*, where* $L_{g+b} = 4(3(g+b)-3)\log(\frac{8\pi(g+b-1)}{3(g+b)-3})$.

If a becomes small, we see that the upper bound becomes big, but we are also able to show that for $0 \leq n < b$, σ_n is bounded above by $\frac{1}{\arctan(\frac{1}{\sinh \frac{a}{2}})} \leq \frac{2}{\pi}$.

An important tool for obtaining our results is estimating isoperimetric constants in an improved statement of a lower bound given by Jammes for the first non-zero Steklov eigenvalue. The strategy of estimating isoperimetric constants has been used in the past for obtaining lower bounds for the first non-zero Laplace eigenvalue on closed surfaces (see e.g. [Bus79] and [SWY80]). We also use comparisons with mixed problems.

4.2 Cheeger-type estimates and mixed problems

4.2.1 Steklov eigenvalues

Let (M, g) be a compact connected Riemannian manifold with Lipschitz boundary ∂M. The Steklov problem on M is the eigenvalue problem

$$\begin{cases} \triangle u = 0 \\ \frac{\partial u}{\partial \nu} = \sigma u \end{cases}$$

where σ is the spectral parameter. The Stekov eigenvalues form a sequence $0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \nearrow$. They can be characterized variationally as follows:

$$\sigma_k(M) = \min_{E \in V_k} \max_{0 \neq u \in E} R(u),$$

where V_k is the set of all $k+1$ dimensional subspaces of the Sobolev space $H^1(M)$, and $R(u)$ is the Rayleigh quotient associated to the Steklov problem,

$$R(u) = \frac{\int_M |\nabla u|^2 dv_g}{\int_{\partial M} u^2 dS_g}.$$

There is a connection between Steklov eigenvalues of a Riemannian manifold (M, g) with boundary ∂M, and eigenvalues of mixed problems on a Lipschitz open subset $A \subset M$ containing ∂M. Given a Lipschitz open subset $A \subset M$ such that $\partial M \subset A$, the mixed Steklov-Neumann problem on A is

$$\begin{cases} \triangle u = 0 & \text{in } A, \\ \frac{\partial u}{\partial \nu} = \sigma u & \text{on } A \cap \partial M, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial A; \end{cases}$$

the mixed Steklov-Dirichlet problem on A is

$$\begin{cases} \triangle u = 0 & \text{in } A, \\ \frac{\partial u}{\partial \nu} = \sigma u & \text{on } A \cap \partial M, \\ u = 0 & \text{on } \partial A. \end{cases}$$

The eigenvalues of the mixed Steklov-Neumann problem form a discrete sequence $0 = \sigma_0^N(A) \leq \sigma_1^N(A) \leq \sigma_2^N(A) \leq \cdots \nearrow$ and the eigenvalues of the mixed Steklov-Dirichlet problem form a discrete sequence $0 < \sigma_0^D(A) \leq \sigma_1^D(A) \leq \sigma_2^D(A) \leq \cdots \nearrow$.

The eigenvalues satisfy

$$\sigma_k^N(A) \leq \sigma_k(M) \leq \sigma_k^D(A). \tag{4.2.1}$$

The proof of this inequality follows from a comparison between the Rayleigh quotients of these problems, see [CGG19] for more details.

4.2.2 Cheeger-type estimates

In 1969, J. Cheeger [Che70] gave a lower bound in term of an isoperimetric constant for the first non-zero Laplace eigenvalue of a compact Riemannian manifold. A similar estimate for the first non-zero Steklov eigenvalue was shown by P. Jammes in 2015 [Jam15]. We give an improvement of this result that we use to obtain the explicit lower bounds presented in this article.

Definition 4.4. *We define the following geometric constants:*

1.

$$h_1(M) := \inf_{|D| \leq \frac{|M|}{2}} \frac{|\partial D|}{|D|},$$

2.

$$h_2(M) := \inf_{|D| \leq \frac{|M|}{2}} \frac{|\partial D|}{|D \cap \partial M|},$$

where in both cases, D is taken among the domains of M satisfying $D \cap \partial M \neq \emptyset$, and such that $M \setminus D$ is also connected and intersects ∂M.

Remark 4.1. *The set ∂D is the topological boundary of the open subset D of the manifold with boundary M; this set does not contain $D \cap \partial M$.*

Remark 4.2. *Jammes defines two constants in a similar way but the domain D is only required to satisfy $|D| \leq \frac{|M|}{2}$.*

Proposition 4.1. *Let (M, g) be a compact Riemannian manifold with boundary ∂M. We have*

$$\sigma_1(M) \geq \frac{h_1(M) \cdot h_2(M)}{4}.$$

This is the result of Jammes but with slightly modified constants. It is obtained by modifying the conclusion of Jammes's proof. Example 4.1 below shows that in dimension 2 this inequality is stronger than the one given by Jammes where D is only required to satisfy $|D| \leq \frac{|M|}{2}$ in the isoperimetric constants. Another situation where the constants h_1 and h_2 will not go to zero while the constants of Jammes do is Example 4.5 of [CGGS22].

Example 4.1. *Let C be a 2-dimensional right cylinder in $\mathbb{R}^3$ whose base contains a line segment. We consider the surfaces obtained by gluing a surface of revolution containing a thin collapsing cylinder on the middle of the flat part of C, as shown in Figure 4.1.. These surfaces are all Steklov isospectral to C (see [CGG19], Appendix A, and [Bri19] for more details). However,* *Jammes's constants tend to zero as the thin passage collapses. In contrast, the constants h_1 and h_2 that we use remain bounded (see Lemma 4.2).*

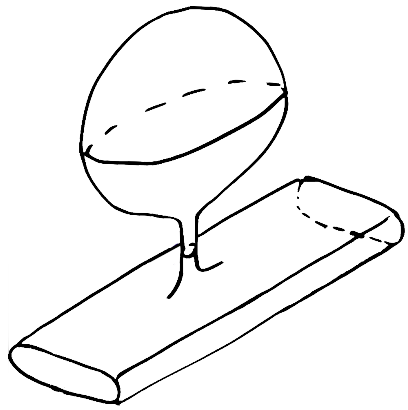

Figure 4.1. – A cylinder on which we have glued a surface of revolution.

Proof of Proposition 4.1. Let u be an eigenfunction associated to the first non-zero Steklov eigenvalue on M. We define

$$D(t) := \{x \in M, u(x) > t\}.$$

Without loss of generality, we can assume $|D(t)| \leq \frac{|M|}{2}$ for all $t \geq 0$. From the proof of Jammes's result, which is similar to the classical proof of Cheeger, we have

$$\sigma_1(M) \geq \frac{1}{4} \min_{t\geq 0} \frac{|\partial D(t)|}{|D(t)|} \cdot \min_{t\geq 0} \frac{|\partial D(t)|}{|D(t) \cap \partial M|}.$$

Since u is harmonic and not constant, it follows from the maximum principle that each connected component of $M \setminus \{u^{-1}(t)\}$ intersects ∂M. Therefore, the inequalities $\min_{t\geq 0} \frac{|\partial D(t)|}{|D(t)|} \geq \inf_{|D|\leq \frac{|M|}{2}} \frac{|\partial D|}{|D|}$ and $\min_{t\geq 0} \frac{|\partial D(t)|}{|D(t)\cap\partial M|} \geq \inf_{|D|\leq \frac{|M|}{2}} \frac{|\partial D|}{|D\cap\partial M|}$ are true if the infima are taken among all sets D such that each connected component of D and of $M \setminus D$ intersects ∂M. Finally, as observed by S. T. Yau in [Yau75], we can assume that both D and $M \setminus D$ are connected. □

Remark 4.3. *It has been showed (see e.g. [Bus80], Theorem 1.14) that the lower bound of Cheeger for the first non-zero Laplace eigenvalue is sharp. It would be interesting to know if the lower bound of Jammes is sharp too.*

Remark 4.4. *Given a domain A in M such that $A \cap \partial M \neq \emptyset$, we define the constants $h_1(A)$ and $h_2(A)$ in the same way as for M, by replacing M by A and ∂M by $\partial M \cap A$ in the conditions that D has to satisfy. The same proof as for Proposition 4.1 shows that $\sigma_1^N(A) \geq \frac{h_1(A)\cdot h_2(A)}{4}$.*

In the construction described in Example 4.1, if we glue two surfaces of revolution of equal volume on C instead of one and let grow the volume of these surfaces of revolution, we see that h_1 tends to zero by choosing a domain that contains one of the two surfaces of revolution. This shows that in this case, the estimate of Proposition 2 is not equivalent to the first non-zero Steklov eigenvalue, which is constant since the surfaces obtained are isospectral to C.

The following proposition shows a way of improving Proposition 4.1.

Proposition 4.2. *Let (M, g) be a compact Riemannian manifold with boundary ∂M. For any domain A in M such that $\partial M \subset A$, we have*

$$\sigma_1(M) \geq \frac{h_1(A) \cdot h_2(A)}{4}.$$

Proof. The proof follows from the comparison (4.2.1) between Steklov and mixed Steklov-Neumann eigenvalues and Remark 4.4. □

This estimate is interesting if we can find domains such that h_1 and h_2 are bounded below. Finally, having in mind Question 4.6 of [CGGS22], we remark that by taking the supremum over the domains A, a new constant is defined. It is more acurate than the product $h_1(M) \cdot h_2(M)$ but difficult to calculate.

4.3 Explicit estimates for Steklov eigenvalues

4.3.1 Lower bounds for σ_1 of surfaces with cylindrical boundary components

We recall the following estimate for Steklov eigenvalues of surfaces with cylindrical boundary components, which follows directly from the comparison (4.2.1) with eigenvalues of mixed Steklov-Neumann and Steklov-Dirichlet problems on the union of the cylindrical boundary nieghborhoods, and the explicit calculation of these.

Lemma 4.1. *Let (M, g) be a compact 2-dimensional Riemannian manifold with $b \geq 1$ boundary components having length a. Assume that the boundary $\partial M = \Sigma_1 \cup \cdots \cup \Sigma_b$ has a neighborhood $V(\partial M)$ which is isometric to the union of disjoint right cylinders $\cup_{i=1}^{b} \Sigma_i \times [0, L)$. The Steklov eigenvalues σ_k of M satisfy*

$$0 \leq \sigma_k \leq \frac{1}{L}$$

if $k < b$, and

$$\frac{2\pi j}{a} \tanh(\frac{2\pi j}{a} L) \leq \sigma_k \leq \frac{2\pi j}{a} \coth(\frac{2\pi j}{a} L)$$

if $(2j - 1)b \leq k < (2j + 1)b$, where $j \in \mathbb{N}^*$.

We see that if $b = 1$, σ_1 is bounded below by $\frac{2\pi}{a} \tanh(\frac{2\pi}{a} L)$, but if $b > 1$ this lemma does not give a lower bound for σ_1. Therefore, our results concern only the case $b \geq 2$ which is interesting.

Theorem 4.1 is in fact a particular case of a result involving domains of M containing the cylindrical neighborhood of ∂M. We note that given such a domain A, we can define $\mathrm{l}(A)$ in the same way as we have defined $\mathrm{l}(M)$ in Definition 4.1 by considering curves that divide A into two connected components without intersecting ∂M. Instead of proving Theorem 4.1, we prove the following result.

Theorem 4.4. *Let (M, g) be a compact connected 2-dimensional Riemannian manifold with a boundary having $b \geq 2$ components of length a. Assume that the boundary $\partial M = \Sigma_1 \cup \cdots \cup \Sigma_b$ has* *a neighborhood $V(\partial M)$ which is isometric to the union of disjoint right cylinders $\cup_{i=1}^{b}\Sigma_i \times [0, L)$. For any domain A in M such that $V(\partial M) \subset A$ (possibly $A = M$), we have*

$$\sigma_1(M) \geq \frac{\min\{\mathrm{l}(A), L\}^2}{2(b-1)a|A|}.$$

The proof of Theorem 4.4 involves estimating the constants h_1 and h_2 of compact connected 2-dimensional manifolds with cylindrical boundary.

Lemma 4.2. *Let (M, g) be a compact connected 2-dimensional Riemannian manifold with $b \geq 2$ boundary components having length a. Assume that the boundary $\partial M = \Sigma_1 \cup \cdots \cup \Sigma_b$ has a neighborhood $V(\partial M)$ which is isometric to the union of disjoint right cylinders $\cup_{i=1}^{b}\Sigma_i \times [0, L)$. Let A be a domain in M such that $V(\partial M) \subset A$ (me may have $A = M$). We have the following estimates of h_1 and h_2:*

$$h_1(A) \geq \frac{2\min\{\mathrm{l}(A), L\}}{|A|},$$
$$h_2(A) \geq \frac{\min\{\mathrm{l}(A), L\}}{(b-1)a}.$$

Proof. We recall that

$$h_1(A) = \inf \frac{|\partial D|}{|D|}$$

where the infimum is taken among all domains satisfying $|D| \leq \frac{|A|}{2}$, $D \cap \partial M \neq \emptyset$ and such that $A \setminus D$ is also connected and intersects ∂M. Given such a domain D the following situations can happen.

1. ∂D intersects a boundary component Σ_i and is contained in the cylindrical neighborhood of Σ_i. If $|\partial D| \geq L$, the fact that $aL < |A|$ gives $\frac{|\partial D|}{|D|} \geq \frac{|\partial D|}{aL} \geq \frac{L}{aL} = \frac{1}{a} > \frac{L}{|A|}$. If $|\partial D| < L$, we know from the isoperimetric inequality that the domain D minimising $\frac{|\partial D|}{|D|}$ is the half-disk with radius $r = \frac{|\partial D|}{\pi}$ and area $\frac{|\partial D|^2}{2\pi}$. This gives $\frac{|\partial D|}{|D|} \geq |\partial D| \cdot \frac{2\pi}{|\partial D|^2} = \frac{2\pi}{|\partial D|} > \frac{2\pi}{L} > \frac{2\pi a}{|A|} \geq \frac{2\pi l(A)}{|A|} > \frac{2\,\mathrm{l}(A)}{|A|}$.
2. ∂D intersect a boundary component Σ_i but D is not contained in the cylindrical neighbourhood of Σ_i. The length of the curve ∂D between its extremity in Σ_i and the point where it leaves the cylindrical neighborhood is greater or equal to L. Hence, we have $\frac{|\partial D|}{|D|} \geq \frac{2L}{|A|}$.
3. ∂D contains a curve of $C(A)$. Since $\mathrm{l}(A)$ is the minimal length of such a curve, $|\partial D| \geq \mathrm{l}(A)$. Moreover, D satisfies $|D| \leq \frac{|A|}{2}$. Hence we have $\frac{|\partial D|}{|D|} \geq \frac{2\,\mathrm{l}(A)}{|A|}$.

In each case, we have either $\frac{|\partial D|}{|A|} \geq \frac{2\,\mathrm{l}(A)}{|A|}$ or $\frac{|\partial D|}{|A|} \geq \frac{2L}{|A|}$. Since we have considered all possible cases, we conclude that $h_1 \geq \frac{2\min\{\mathrm{l}(A), L\}}{|A|}$.

We now estimate $h_2(A)$. We recall that

$$h_2(A) := \inf \frac{|\partial D|}{|D \cap \partial M|}$$

where the infimum is taken among all domains satisfying $|D| \leq \frac{|A|}{2}$, $D \cap \partial M \neq \emptyset$ and such that $A \setminus D$ is also connected and intersects ∂M. Given such a domain D the following situations can happen.

1. ∂D intersects a boundary component Σ_i and D is contained in the cylindrical neighborhood of Σ_i. Since the the complement of D in M is connected, ∂D is homotopic to $D \cap \Sigma_i$. Since $D \cap \Sigma_i$ is a geodesic arc and the cylindrical neighborhood has zero curvature, $|\partial D| \geq |D \cap \Sigma_i| = |D \cap \partial M|$ and finally $\frac{|\partial D|}{|D \cap \partial M|} \geq 1$.
2. ∂D intersects a boundary component Σ_i but D is not contained in the cylindrical neighborhood of Σ_i. The length of the curve ∂D between its extremity in Σ_i and the point where it leaves the cylindrical neighborhood is greater or equal to L. Hence, we have $\frac{|\partial D|}{|D\cap\partial M|} \geq \frac{2L}{ba} \geq \frac{L}{(b-1)a}$.
3. ∂D contains a curve of $C(A)$. Since $\mathrm{l}(A)$ is the minimal length of such a curve, $|\partial D| \geq \mathrm{l}(A)$. Moreover, D cannot contain all the connected components of ∂M, which implies $|D \cap \partial M| \leq (b-1)a$. Hence, we have $\frac{|\partial D|}{|D\cap\partial M|} \geq \frac{\mathrm{l}(A)}{(b-1)a}$.

We have considered all possible cases. To conclude, we observe that $\mathrm{l}(A) \leq a$ since the curves $\Sigma_i \times \{L\}$ belong to $C(A)$. Hence, we have $1 \geq \frac{\mathrm{l}(A)}{a} \geq \frac{\mathrm{l}(A)}{(b-1)a}$. Since $\frac{|\partial D|}{|D\cap\partial M|} \geq \frac{\mathrm{l}(A)}{(b-1)a}$ or $\frac{|\partial D|}{|D\cap\partial M|} \geq \frac{L}{(b-1)a}$ for all possible D, we have $h_2(A) \geq \frac{\min\{\mathrm{l}(A),L\}}{(b-1)a}$. □

We note that in higher dimensions, similar estimates cannot be obtained because in the second situation, the volume of ∂D cannot be bounded below by L.

Proof of Theorem 4.4. Theorem 4.4 follows from Lemma 4.2 and Proposition 4.2. □

The exponent of the geometric quantities involved in the estimate given in Theorem 4.1 cannot be improved. This is obtained by showing that σ_1 and the lower bound are equivalent for families of surfaces for which all geometric quantities involved in the lower bound except one are fixed. We recall that the Steklov eigenvalues of right cylinders can be computed.

Proposition 4.3. *The Steklov eigenvalues of the right cylinder $S^1_R \times [-T, T]$, where S^1_R denotes the circle of radius R, are*

$$0, \frac{1}{T}, \frac{k}{R}\tanh(\frac{k}{R}T) < \frac{k}{R}\coth(\frac{k}{R}T), \quad k \in \mathbb{N}^*.$$

We note that if $\frac{T}{R} \geq \rho$, where $\rho \approx 1,19968$ is the positive root of $1 = x\tanh(x)$, the first non-zero eigenvalue is $\frac{1}{T}$.

Example 4.2. *Consider the sequence $\{M_n\}_{n\geq 1}$ where M_n are right cylinders that have height $4\pi n$ and whose bases are unit circles. Since $2\pi n \geq \rho\ \forall n \geq 1$, $\sigma_1(M_n) = \frac{1}{2\pi n}$. Hence, we have $\frac{4\pi}{|M_n|} = \frac{1}{2\pi n} = \sigma_1(M_n) \geq \frac{\mathrm{l}(M_n)^2}{2(b-1)a|M_n|} = \frac{\pi}{|M_n|}$.*

Example 4.3. *Consider a surface M_ϵ with two boundary components of length 1, having a cylindrical neighborhood of length L and connected by a thin cylinder C_ϵ of circumference $\epsilon < L$ and of length $\frac{1}{\epsilon}$ (see Figure 4.3). Consider the function taking the value -1 on one side of C_ϵ, 1 on the other side, and extended continuously to a linear function on C_ϵ, that is, on $C_\epsilon = S^1_{\frac{\epsilon}{2\pi}} \times [-\frac{1}{2\epsilon}, \frac{1}{2\epsilon}]$, we have $f(s,t) = 2\epsilon t$. Its Dirichlet energy is zero except on C_ϵ where it is*

$$\int_{C_\epsilon} |\nabla f|^2 dv_g = \int_0^\epsilon \int_{\frac{-1}{2\epsilon}}^{\frac{1}{2\epsilon}} 4\epsilon^2 dt ds = \int_0^\epsilon 4\epsilon ds = 4\epsilon^2.$$

Since the restriction of f to the boundary is orthogonal to a constant function and $\int_{\partial M} f^2 dS_g = \int_{\partial M} 1 dS_g = 2$, *we obtain*

$$\sigma_1(M) = \min\left\{R(u) : u \in H^1(M), \int_{\partial M} u = 0\right\} \leq R(f) = \frac{4\epsilon^2}{2} = 2\epsilon^2.$$

We note that if L is small enough, the volume of M_ϵ *satisfies* $|M_\epsilon| \leq 2$. *Hence, we have* $2\,\mathrm{l}(M_\epsilon)^2 = 2\epsilon^2 \geq \sigma_1(M_\epsilon) \geq \frac{\mathrm{l}(M_\epsilon)^2}{2(b-1)a|M_\epsilon|} \geq \frac{\mathrm{l}(M_\epsilon)^2}{4}$.

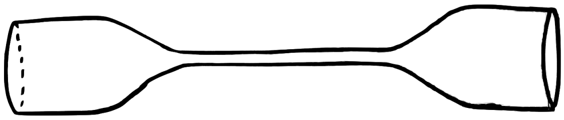

Figure 4.2. – A surface with two cylindrical boundary neighborhoods connected by a thin cylinder.

Since we have shown that the exponent of $\min\{\mathrm{l}(M), L\}$ and $|M|$ cannot be changed, we can deduce that the exponent of a must be -1 from the fact that the degree of homogeneity of the lower bound has to be consistent with the degree of homogeneity of σ_1. We conclude that, up to a constant, we cannot have a better lower bound for σ_1 depending on these geometric quantities (however, we may have different geometric quantities).

A lower bound is optimal if we can show that it goes to zero if and only if σ_1 goes to zero. Using the same strategy as in Example 4.1, it is easy to construct a family of surfaces such that σ_1 is constant but the volume goes to infinity and therefore the lower bound given in Theorem 4.1 tends to zero. This example shows that the volume of the manifold seems not to be an optimal quantity for estimating σ_1. Theorem 4.2 is an improvement of Theorem 4.1 for surfaces whose Gaussian curvature is bounded below, which does not involve the volume of the manifold.

Proof of Theorem 4.2. For $2 \leq i \leq b$, we let γ_i be a geodesic minimising the distance between Σ_1 and Σ_i. Around each γ_i, we consider the tube

$$T_i = \{x \in M, \text{ there exists a geodesic } \xi \text{ of length } l(\xi) < \mathrm{inj}_{\partial M}(M) \text{ from } x \text{ meeting } \gamma_i \text{ orthogonally}\}.$$

Since γ_i meets ∂M orthogonally, $T_i = \{x \in M, d(x, \gamma_i) < \mathrm{inj}_{\partial M}(M)\}$. We define

$$A = \cup_{i=1}^{b}(\Sigma_i \times [0, \mathrm{inj}_{\partial M}(M))) \cup (\cup_{i=2}^{b} T_i).$$

We approximate the volume of A by using a Bishop-Günther inequality for tubes (Theorem 8.16, point ii, in [Gra04]). In the particular case of a tube T of radius r around a geodesic γ in a surface whose Gaussian curvature is bounded from below by $\kappa < 0$, this comparison result says that

$$|T| \leq \frac{2l(\gamma)\sinh(\sqrt{-\kappa} r)}{\sqrt{-\kappa}}.$$

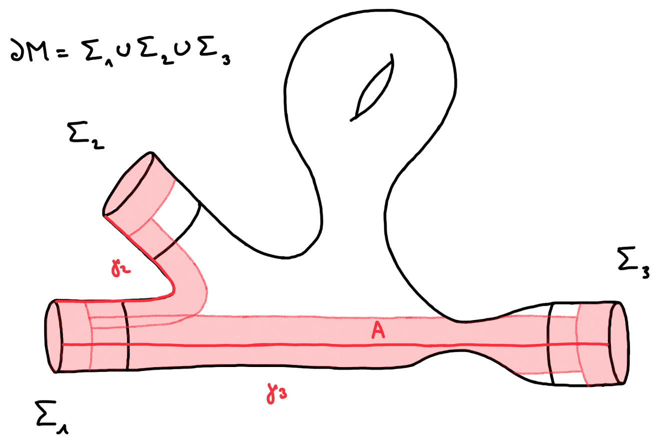

Figure 4.3. – The domain $A = \cup_{i=1}^{b}(\Sigma_i \times [0, \mathrm{inj}_{\partial M}(M))) \cup (\cup_{i=2}^{b} T_i)$.

By applying this inequality to estimate the volume of the tubes T_i, we obtain

$$\begin{aligned} |A| &= |\cup_{i=1}^{b} (\Sigma_i \times [0, \mathrm{inj}_{\partial M}(M))) \cup (\cup_{i=2}^{b} T_i)| \\ &\leq ab\, \mathrm{inj}_{\partial M}(M) + \sum_{i=2}^{b} |(T_i)| \\ &\leq ab\, \mathrm{inj}_{\partial M}(M) + \sum_{i=2}^{b} \frac{2l(\gamma_i)\sinh(\sqrt{-\kappa}\, \mathrm{inj}_{\partial M}(M))}{\sqrt{-\kappa}} \\ &\leq ab\, \mathrm{inj}_{\partial M}(M) + \frac{2(b-1)\,\mathrm{diam}_M(\partial M)\sinh(\sqrt{-\kappa}\, \mathrm{inj}_{\partial M}(M))}{\sqrt{-\kappa}}. \end{aligned}$$

The set A can be approximated by smooth domains in the following way (for more details, see [Dan08], Section 8.2). We define $V_n := \{x \in A : d(x, \partial A) > \frac{1}{n}\}$ and consider ϕ_n a bump function for $\overline{V_n}$ supported in V_{n+1} (on bump functions, see [Lee13], Proposition 2.25). By Sard's Theorem, there exists $t_n \in (0,1)$ such that $E_n := \{x \in M : \phi_n(x) > t_n\}$ is a smooth domain. Since $E_1 \subset E_2 \subset \ldots$ and $\cup_{n\in\mathbb{N}^*} E_n = A$, $|E_n| \to |A|$ as n tends to infinity. Let n_0 be such that $\frac{1}{n_0} < \frac{\mathrm{inj}_{\partial M}(M)}{8}$. By taking $\tilde{A} = E_{n_0}$, we have that $\tilde{A}$ contains a cylindrical neighborhood of length $\frac{\mathrm{inj}_{\partial M}(M)}{2}$ of ∂M, $|A| \geq |\tilde{A}|$ and $\mathrm{l}(\tilde{A}) \geq \frac{\mathrm{inj}_{\partial M}(M)}{2}$. This last statement follows from the fact that a curve c which divides $\tilde{A}$ into two connected components, each containing at least one connected component of ∂M, must intersect a geodesic γ_i at a point x and cannot be contained in the ball $B_{\frac{\mathrm{inj}_{\partial M}(M)}{2}}(x) \subset \tilde{A}$.

Hence, by Theorem 4.1, we have

$$\sigma_1(M) \geq \frac{\min\{\mathrm{l}(\tilde{A}), \frac{\mathrm{inj}_{\partial M}(M)}{2}\}^2}{2(b-1)a|\tilde{A}|} \geq \frac{\mathrm{inj}_{\partial M}(M)^2}{8(b-1)a|A|}.$$

By combining the above inequality with the approximation of the volume of A, we obtain

$$\sigma_1(M) \geq \frac{\mathrm{inj}_{\partial M}(M)^2}{8(b-1)a(ab\, \mathrm{inj}_{\partial M}(M) + \frac{2(b-1)\,\mathrm{diam}_M(\partial M)\sinh(\sqrt{-\kappa}\, \mathrm{inj}_{\partial M}(M))}{\sqrt{-\kappa}})}.$$

Since we assume that $L \leq 1$, we have $\mathrm{inj}_{\partial M}(M) \leq 1$. Using the Taylor-Lagrange formula, we obtain that $\frac{\sinh(\sqrt{-\kappa}\,\mathrm{inj}_{\partial M}(M))}{\sqrt{-\kappa}} \leq \cosh(\sqrt{-\kappa})\,\mathrm{inj}_{\partial M}(M)$. Hence, we have

$$\sigma_1(M) \geq \frac{\mathrm{inj}_{\partial M}(M)}{8(b-1)a(ab + 2(b-1)\,\mathrm{diam}_M(\partial M)\cosh(\sqrt{-\kappa}))}$$

We note that this inequality is interesting in itself because it shows clearly how the different geometric quantities affect the lower bound. If we assume that $a \leq \mathrm{diam}_M(\partial M)$, we obtain

$$\begin{aligned}\sigma_1(M) &\geq \frac{\mathrm{inj}_{\partial M}(M)}{8(b-1)a(\mathrm{diam}_M(\partial M)b + 2(b-1)\,\mathrm{diam}_M(\partial M)\cosh(\sqrt{-\kappa}))} \\ &\geq \frac{\mathrm{inj}_{\partial M}(M)}{16b^2\cosh(\sqrt{-\kappa})a\,\mathrm{diam}_M(\partial M)} \\ &= C(\kappa, b)\frac{\mathrm{inj}_{\partial M}(M)}{a\,\mathrm{diam}_M(\partial M)},\end{aligned}$$

where $C(\kappa, b) = \frac{1}{16b^2\cosh(\sqrt{-\kappa})}$. □

Remark 4.5. *The exponent of the geometric quantities involved in Theorem 4.2 cannot be improved. To show this, we first observe that the exponent of the diameter of the boundary cannot be improved because the family of right cylinders of fixed base and growing height $\{M_n\}_{n\geq 1}$ of Example 4.2 satisfy $\sigma_1(M_n) = \frac{1}{2\pi n} = \frac{2}{\mathrm{diam}_M(\partial M_n)}$. We consider now the family of right cylinders $\{M_a\}_{a\geq 1}$ of height 2 and growing base of length a. We have $\sigma_1(M_a) \leq \frac{4\pi}{a^2}$, which shows that the exponent of a is also optimal. For obtaining that the exponent of the injectivity radius if optimal, we note that in Example 4.3 we can construct the surfaces M_ϵ so that their Gaussian curvature is bounded from below. This can be done by joining the inner cylinder and the two cylindrical neighborhoods of the boundary by a cylinder of constant Gaussian curvature equal to -1 and smoothing the joints. Hence, from Example 4.3, we have $\sigma_1(M_\epsilon) \leq 2\epsilon^2 \leq 8\,\mathrm{inj}_{\partial M_\epsilon}(M_\epsilon)^2$. On the other hand, by construction, $\mathrm{diam}_M(\partial M_\epsilon)$ is of the same order as $\frac{1}{\mathrm{inj}_{\partial M_\epsilon}(M_\epsilon)}$ as ϵ goes to zero. This implies that there exists a constant c such that $\sigma_1(M_\epsilon) \geq c\,\mathrm{inj}_{\partial M_\epsilon}(M_\epsilon)^2$.*

We remark that if a goes to zero, $\mathrm{l}(M)$ and $\mathrm{inj}_{\partial M}(M)$ also go to zero. Therefore, Theorems 4.1 and 4.2 do not say that σ_1 goes to infinity as a goes to zero, which is not true, as shown by the following example. We consider the sequence of right cylinders $\{S^1_{\frac{1}{n}} \times [-1, 1]\}_{n\geq 1}$. Proposition 4.3 shows that if $n \geq 2$, $\sigma_1 = 1$. By taking the sequence $\{S^1_{\frac{1}{n}} \times [-n, n]\}_{n\geq 1}$, we even have that σ_1 tends to zero as the length of the boundary tends to zero. This is in contrast to the case of surfaces with one cylindrical boundary component where Lemma 4.1 shows that σ_1 goes to infinity as the length of the boundary goes to zero.

4.3.2 Geometric bounds on the low Steklov eigenvalues of a compact hyperbolic surface with geodesic boundary

A compact hyperbolic surface of signature (g, b) is a compact 2-dimensional Riemannian manifold of constant Gaussian curvature equal to -1 with genus g and a geodesic boundary having b connected components. An important property of hyperbolic surfaces is that they are isometric to a warped product around simple closed geodesics.

Proposition 4.4. *Let M be a closed hyperbolic surface of genus $g \geq 2$ and let $\gamma_1, \ldots, \gamma_m$ be pairwise disjoint simple closed geodesics on M. Then $m \leq 3g - 3$ and there exist simple* *closed geodesics* $\gamma_{m+1}, \ldots, \gamma_{3g-3}$ *which, together with* $\gamma_1, \ldots, \gamma_m$*, decompose M into surfaces of signature* $(0, 3)$*. Moreover, the collars*

$$K(\gamma_i) = \{p \in M, dist(p, \gamma_i) \leq w(\gamma_i)\}$$

where

$$w(\gamma_i) = \operatorname{arcsinh}(\frac{1}{\sinh(\frac{1}{2}l(\gamma_i))})$$

are pairwise disjoint and each collar $K(\gamma_i)$ *is isometric to the cylinder* $S^1 \times [-w(\gamma_i), w(\gamma_i)]$ *with the metric* $g(s,t) = \frac{l^2(\gamma_i)\cosh^2(t)}{(2\pi)^2} g_{S^1}(s) + dt^2$ *where* g_{S^1} *is the canonical metric on* S^1*.*

For a proof of this result, we refer to [Bus10], Theorem 4.1.1. A direct consequence is that a hyperbolic surface with geodesic boundary has a boundary neighborhood which is isometric to a union of disjoint warped products. This implies the following approximation of the Steklov eigenvalues.

Lemma 4.3. *Let M be a hyperbolic surface with* $b \geq 2$ *geodesic boundary components of length a. Then, the Steklov eigenvalues* σ_k *of M satisfy*

$$0 \leq \sigma_k \leq \frac{1}{\arctan(\frac{1}{\sinh \frac{a}{2}})}$$

if $k < b$*, and*

$$\frac{2\pi j}{a} \tanh(\frac{2\pi j}{a} \arctan(\frac{1}{\sinh(\frac{a}{2})})) \leq \sigma_k \leq \frac{2\pi j}{a} \coth(\frac{2\pi j}{a} \arctan(\frac{1}{\sinh(\frac{a}{2})}))$$

if $(2j-1)b \leq k < (2j+1)b$*, where* $j \in \mathbb{N}^*$*.*

Proof. Let $\Sigma_1, ..., \Sigma_b$ be the b boundary components, where $l(\Sigma_1) = ... = l(\Sigma_b) = a$. By gluing a hyperbolic surface of signature $(1, 1)$ to each boundary component, we obtain a closed hyperbolic surface of genus $g \geq 2$. Theorem 4.4 says that the collars $K(\Sigma_i) = \{p \in M, dist(p, \Sigma_i) \geq w(\Sigma_i)\}$, where $w(\Sigma_i) = \operatorname{arcsinh}(\frac{1}{\sinh(\frac{a}{2})})$, are disjoint and isometric to cylinders $S^1 \times [0, w(\gamma_i)]$ with the metric $g(s,t) = \frac{a^2\cosh^2(t)}{(2\pi)^2} g_{S^1}(s) + dt^2$. Let $A = \cup_i K(\Sigma_i)$ be the union of these collars. We consider the mixed Steklov-Neumann and Steklov-Dirichlet problems on A. From Equation 4.2.1, we have

$$\sigma_i^N(A) \leq \sigma_i(M) \leq \sigma_i^D(A).$$

Since the $K(\Sigma_i)$ are warped products, the eigenvalues of these mixed problems can be explicitly calculated. This calculation leads to the result.

□

A classical result due to L. Bers says that every closed hyperbolic surface of genus $g \geq 2$ admits a decomposition into surfaces of signature $(0, 3)$ such that the length of the separating geodesics is controlled by a constant depending on the genus. We give a statement of this result due to P. Buser (see [Bus10], Theorem 5.2.3) which is convenient to deduce an analog result for surfaces with geodesic boundary of controlled length.

Proposition 4.5. *Let M be a closed hyperbolic surface of genus* $g \geq 2$ *and let* $\gamma_1, ..., \gamma_m$ *be the set of all distinct simple closed geodesics of length* $l \leq 2\operatorname{arcsinh}(1)$*. This system is extendable* *to a partition* $\gamma_1, \dots, \gamma_{3g-3}$ *satisfying*

$$l(\gamma_k) \le 4k \log(\frac{8\pi(g-1)}{k}), \quad k = 1, ..., 3g-3.$$

Corollary 4.1. *There exists a constant* L_{g+b}*, depending only on* g *and* b*, such that every hyperbolic surface* M *of genus* g *with* $b \ge 2$ *geodesic boundary components of length* $l \le 2\,\mathrm{arcsinh}(1)$ *can be decomposed into surfaces of signature* $(0,3)$ *by simple closed geodesics* $\gamma_1, ..., \gamma_{3g-3+b}$ *satisfying*

$$l(\gamma_i) \le L_{g+b}, \quad i = 1, ..., 3g-3+b.$$

Proof. Let $\gamma_1, ..., \gamma_b$ be the geodesic boundary components of M. By gluing a hyperbolic surface of signature $(1,1)$ to each boundary component, we obtain a closed hyperbolic surface M' of genus $g+b \ge 2$ and $\gamma_1, \dots, \gamma_b$ are closed geodesics of M' of length $l \le 2\,\mathrm{arcsinh}(1)$. We add to this set all distinct simple closed geodesics on M' of length $l \le 2\,\mathrm{arcsinh}(1)$. From Bers'Theorem, the resulting set $\gamma_1, \dots, \gamma_m$ can be extended to a partition $\gamma_1, ..., \gamma_{3(g+b)-3}$ of simple closed geodesics satisfying $l(\gamma_k) \le 4k \log(\frac{8\pi(g+b-1)}{k})$ for $k = 1, ..., 3(g+b)-3$. In particular, there exists a constant $L_{g+b} = 4(3(g+b)-3)\log(\frac{8\pi(g+b-1)}{3(g+b)-3})$ such that $l(\gamma_k) \le L_{g+b}$ for $k = 1, ..., 3(g+b)-3$. Among this family of geodesics, we have the b geodesics $\gamma_1, ..., \gamma_b$ of the boundary of M and we also have b simple closed geodesics that divide the surfaces of signature $(1,1)$ glued at each boundary to make them surfaces of signature $(0,3)$. The $3g-3+b$ remaining geodesics decompose M into surfaces of signature $(0,3)$ and their length is bounded by L_{g+b}. □

We are now able to give the proof of Theorem 4.3. The strategy is the same as the strategy used in [SWY80] for obtaining a result for Laplace eigenvalues.

Proof of Theorem 4.3.

Step 1: $\mathrm{l}_n \le \beta_1$ **where** β_1 **is a constant depending only on** g **and** b**.** From Corollary 4.1 there exists a family of simple closed geodesics of length $l \le L_{g+b}$, dividing M into $3g-3+b$ surfaces of signature $(0,3)$. Since we assume that M is connected, each of this surfaces of signature $(0,3)$ contains at most two components of ∂M. Hence, by choosing a subset of these geodesics, we can obtain for $1 \le n < \lceil \frac{b}{2} \rceil$ a division of M into $n+1$ connected components, each one containing at least one component of ∂M. Let γ denote the curve consisting of the union of these geodesics. Because γ consists of at most $3g-3+b$ geodesics of length $l \le L_{g+b}$, there exists a constant β_1, depending only on g and b and such that $l(\gamma) \le \beta_1$. Let $c \in C_n(M)$ be a curve satisfying $l(c) = \mathrm{l}_n$. Since $\gamma \in C_n(M)$, we have $\mathrm{l}_n \le l(\gamma) \le \beta_1$. For $\lceil \frac{b}{2} \rceil \le n < b$, the assumption says that there exists $c \in C_n(M)$ such that each simple closed geodesic of c is of length $l \le L_{g+b}$. Since c consists of at most $3g-3+b$ geodesics, we have $\mathrm{l}_n \le l(c) \le \beta_1$.

Step 2: $\sigma_n \le C_2 \frac{\mathrm{l}_n}{a}$**.** If $\mathrm{l}_n > 1$, we obtain from the combination of Lemma 4.3 and the hypothesis that $a \le 2\,\mathrm{arcsinh}(1)$ that $\sigma_n \le \frac{1}{\arctan(\frac{1}{\sinh \frac{a}{2}})} < \frac{8\,\mathrm{arcsinh}(1)\,\mathrm{l}_n}{\pi a}$. Now assume that $\mathrm{l}_n \le 1$. Let $c \in C_n(M)$ be the curve from step 1 satisfying $l(c) = \mathrm{l}_n$. This curve decompose M into $n+1$ connected components $M_1, \dots, M_{n+1}$, each one containing at least one boundary component. We suppose $c = \gamma_1 \cup \dots \cup \gamma_p$ where the γ_i are simple closed geodesics on M. From Proposition 4.4, we know that there exist disjoint collars $K(\gamma_1), \dots, K(\gamma_p)$ about the geodesics $\gamma_1, \dots, \gamma_p$. If $K_j \cap M_i \neq \emptyset$,

$K_j \cap \overline{M_i}$ is isometric to $S^1 \times [0, w(\gamma_j)]$ with the metric $g(s,t) = \frac{l^2(\gamma_i)\cosh^2(t)}{(2\pi)^2} g_{S^1}(s) + dt^2$, and $K_j \cap \partial M_i$ corresponds to $S^1 \times \{0\}$. The upper bound is obtained by using test functions. We define

$$\phi_i(x) = \begin{cases} 1 & \text{if } x \in M_i \setminus \cup_{j=1}^{p} K_j; \\ \phi_{i,j}(x) & \text{if } x \in M_i \cap K_j \text{ for a } j = 1, \ldots, p; \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\begin{aligned} \phi_{i,j} : S^1 \times [0, w(\gamma_j)] &\to \mathbb{R} \\ (s,t) &\mapsto \frac{\arctan(\sinh(t))}{\arctan(\frac{1}{\sinh(\frac{l(\gamma_i)}{2})})}. \end{aligned}$$

The Dirichlet energy of this function on the half-collar $M_i \cap K_j$ is

$$\int_{S^1 \times [0, w(\gamma_i)]} |\nabla \phi_{i,j}|^2 dv = \frac{l(\gamma_j)}{\arctan(\frac{1}{\sinh(\frac{l(\gamma_j)}{2})})}.$$

Let E be the set of indices j such that $M_i \cap K_j \neq \emptyset$. The total Dirichlet energy of ϕ_i satisfies

$$\begin{aligned} \int_{M_i} |\nabla \phi_i|^2 dv &= \sum_{j \in E} \frac{l(\gamma_j)}{\arctan(\frac{1}{\sinh(\frac{l(\gamma_j)}{2})})} \\ &\leq \frac{\sum_{j \in E} l(\gamma_j)}{\arctan(\frac{1}{\sinh(\frac{\sum_{j=1}^{r} l(\gamma_j)}{2})})} \\ &\leq \frac{\mathrm{l}_n}{\arctan(\frac{1}{\sinh(\frac{\mathrm{l}_n}{2})})}. \end{aligned}$$

We also have

$$\int_{\partial M_i} \phi_i^2 dS = m \times a \geq a,$$

where m is the number of boundary components included in M_i. Hence the Rayleigh quotient of ϕ_i satisfy

$$R(\phi_i) \leq \frac{\mathrm{l}_n}{a \arctan(\frac{1}{\sinh(\frac{\mathrm{l}_n}{2})})}.$$

Since $\mathrm{l}_n \leq 1$, we have $\frac{1}{\arctan(\frac{1}{\sinh(\frac{\mathrm{l}_n}{2})})} < \frac{1}{\arctan(\frac{1}{\sinh(\frac{1}{2})})} =: \beta_2$ and

$$R(\phi_i) \leq \beta_2 \frac{\mathrm{l}_n}{a}.$$

Let V be the linear span of $\phi_1, \ldots, \phi_{n+1}$ in $H^1(M)$. Since the functions ϕ_i have disjoint support, V is an $(n+1)$-dimensional vector space and we have

$$\max\{R(u), u \in V\} = \max\{R(\phi_1), ..., R(\phi_{n+1})\}.$$

Since $R(\phi_i) \leq \beta_2 \frac{l_n}{a}$ for $i = 1, ..., n+1$, we have $\max\{R(\phi_1), ..., R(\phi_{n+1})\} \leq \beta_2 \frac{l_n}{a}$. Using the variational characterization $\sigma_n(M) = \min_{V \in V_k} \max_{0 \neq u \in V} R(u)$, where V_k is the set of all $(k+1)$-dimensional linear subspace of $H^1(M)$, we obtain

$$\sigma_n(M) \leq \beta_2 \frac{l_n}{a}.$$

Because we have obtained the desired result both when $l_n > 1$ and when $l_n \leq 1$, we have

$$\sigma_n(M) \leq C_2 \frac{l_n}{a},$$

where $C_2 = \max\{\frac{8 \operatorname{arcsinh}(1)}{\pi}, \beta_2\}$ is a universal constant.

Step 3: $C_1 l_n^2 \leq \sigma_n$**.** Since $l(c) = l_n$, one of the p components γ_i of c must satisfy $l(\gamma_i) \geq \frac{l_n}{p}$; we call it $\gamma_{\max}$. The geodesic $\gamma_{\max}$ is contained in the boundary of two sets M_j and M_k. We let $\Omega_1 = M_j \cup M_k \cup (\partial M_j \cap \partial M_k)$ and $\Omega_2, \ldots, \Omega_n$ be the remaining M_i. Let $A = \cup_{i=1}^n \Omega_i$. On each Ω_i, we consider the mixed Steklov-Neumann problem with Steklov condition on $\Omega_i \cap \partial M$ and Neumann condition on $\partial\Omega_i$. Since the Ω_i are disjoint, we have

$$\sigma_n^N(A) = \min\{\sigma_1^N(\Omega_1), ..., \sigma_1^N(\Omega_n)\}$$

and since A contains all boundary components of M, we have

$$\sigma_k(M) \geq \sigma_k^N(A).$$

Therefore, the proof will be finished if we can show that $\sigma_1^N(\Omega_i) \geq \alpha_1 l_n^2$ for $i = 1, \ldots, n$. If Ω_i contains only one boundary component Σ_i, we consider the mixed Steklov-Neumann problem on the half-collar $K(\Sigma_i)$. By comparing the Rayleigh quotients, we see that $\sigma_1^N(\Omega_i) \geq \sigma_1^N(K(\Sigma_i))$. We have already mentioned that a calculation shows that $\sigma_1^N(K(\Sigma_i)) = \frac{2\pi}{a} \tanh(\frac{2\pi}{a} \arctan(\frac{1}{\sinh(\frac{a}{2})}))$. Since $a \leq 2 \operatorname{arcsinh}(1)$, by letting $\beta_3 = \frac{\pi}{\operatorname{arcsinh}(1)} \tanh(\frac{\pi}{\operatorname{arcsinh}(1) \arctan(1)})$, we obtain $\sigma_1^N(K) \geq \beta_3$.

If Ω_i contains several boundary components, we obtain the result by estimating the constants $h_1(\Omega_i)$ et $h_2(\Omega_i)$ and using Proposition 4.1 and Remark 4.4.

Estimation of $h_1(\Omega_i)$. We recall that

$$h_1(\Omega_i) := \inf \frac{|\partial D|}{|D|}$$

where the infimum if taken among all domains D of Ω_i satisfying $|D| \leq \frac{|\Omega_i|}{2}$, $D \cap \partial M \neq \emptyset$, and such that $M \setminus D$ is also connected and intersects ∂M. Given such a domain D, we have the following possibilities that are illustrated in Figure 4.4..

1. ∂D intersects a component Σ_i of ∂M and D is contained in the collar neighborhood $K(\Sigma_i)$. From the isoperimetric inequality for simply connected domains in the hyperbolic plane we know that $|D| \leq |\partial D|$. So we have $\frac{|\partial D|}{|D|} \geq \frac{|\partial D|}{|\partial D|} = 1$.
2. ∂D intersects a boundary component Σ_i but D is not contained in $K(\Sigma_i)$. Since $w(\Sigma_i) \geq \operatorname{arcsinh}(1)$, we have $|\partial D| \geq w(\Sigma_i) \geq \operatorname{arcsinh}(1)$. Therefore $\frac{|\partial D|}{|D|} \geq \frac{\operatorname{arcsinh}(1)}{|M|} = \frac{\operatorname{arcsinh}(1)}{2\pi(2g-2+b)} =: \beta_4$. We see that β_4 only depends on g and b.

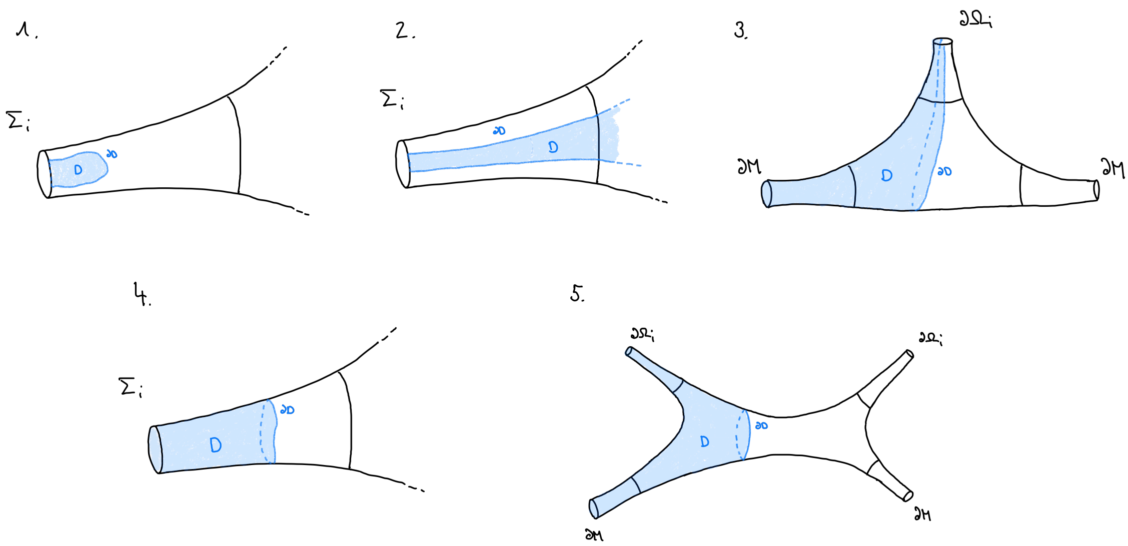

Figure 4.4. – A schematic representation of possible configurations of D in Ω_i corresponding to each of the five cases.

3. ∂D intersects a boundary geodesic γ_i of $\partial\Omega_i$. Since both D and $\Omega_i \setminus D$ have to intersect ∂M, ∂D cannot be contained in $K(\gamma_i)$. Since $l(\gamma_i) \leq L_{g+b}$, $|\partial D| \geq w(\gamma_i) \geq \beta_5$ where β_5 is a constant depending only on g and b. Thus $\frac{|\partial D|}{|D|} \geq \frac{\beta_5}{|M|} = \frac{\beta_5}{2\pi(2g-2+b)} =: \beta_6$.
4. ∂D does not intersect ∂M and a component Γ of ∂D is freely homotopic to a boundary component Σ_i. In this case $\Gamma \cup \Sigma_i$ bounds an annulus and since the Gaussian curvature is negative and Σ_i is a geodesic, $|\Gamma| \geq l(\Sigma_i)$. If $|\Gamma| \geq 1$, we have $\frac{|\partial D|}{|D|} \geq \frac{1}{|M|} = \frac{1}{2\pi(2g-2+b)}$. If $|\Gamma| < 1$, we deduce that $\frac{|\partial D|}{|D|} \geq \frac{1}{10}$ from the isoperimetric inequality given in Theorem 3 of [Yau75] and the fact that $|\Gamma| \geq l(\Sigma_i)$. Thus we have $\frac{|\partial D|}{|D|} \geq \min\{\frac{1}{2\pi(2g-2+b)}, \frac{1}{10}\} = \frac{1}{2\pi(2g-2+b)} \geq \beta_4$.
5. ∂D does not intersect ∂M and none of its components is freely homotopic to a boundary component. We note that each component of ∂D is freely homotopic to a simple closed geodesic of Ω_i. Let Γ be the union of these geodesics. We have $|\partial D| \geq |\Gamma|$. Γ divides Ω_i into two connected components, each of them containing at least one connected component of ∂M. We recall that the geodesics $\gamma_1, ..., \gamma_p$ divide M into $n+1$ regions and that a subset of these geodesics divides M into n regions $\Omega_1, ..., \Omega_n$. Let $\tilde{\gamma}$ be the union of the geodesics that do not belong to this subset. We have $\gamma_{max} \in \tilde{\gamma}$. If $|\Gamma|$ were smaller than $l(\tilde{\gamma})$ there would be a family of geodesics of M, dividing M into $n+1$ regions and their total length would be smaller than l_n, which is a contradiction. Hence we have $|\Gamma| \geq l(\tilde{\gamma}) \geq l(\gamma_{\max}) \geq \frac{\mathrm{l}_n}{p}$ and since $3g-3+b$ is the maximal number of these geodesics, $|\Gamma| \geq \frac{\mathrm{l}_n}{3g-3+b}$. Therefore, $\frac{|\partial D|}{|D|} \geq \frac{|\Gamma|}{|M|} \geq \frac{\mathrm{l}_n}{(3g-3+b)2\pi(2g-2+b)} = \frac{\mathrm{l}_n}{\beta_7}$ where β_7 is a constant depending only on g and b.

Since we have considered all possibilities for ∂D, we have

$$h_1(\Omega_i) \geq \min\{1, \beta_4, \beta_6, \frac{\mathrm{l}_n}{\beta_7}\}.$$

Since $\mathrm{l}_n \leq \beta_1$, $h_1(\Omega_i) \geq \beta_8 \, \mathrm{l}_n$ where $\beta_8 = \min\{\beta_1^{-1}, \beta_4\beta_1^{-1}, \beta_6\beta_1^{-1}, \frac{1}{\beta_7}\}$ is a constant depending only on g and b.

Estimation of $h_2(\Omega_i)$. We recall

$$h_2(\Omega_i) := \inf \frac{|\partial D|}{|D \cap \partial M|}$$

where the infimum if taken among all domains D of Ω_i satisfying $|D| \leq \frac{|\Omega_i|}{2}$, $D \cap \partial M \neq \emptyset$, and such that $M \setminus D$ is also connected and intersects ∂M. Given such a domain D, we have the following possibilities that are illustrated in Figure 4.4..

1. ∂D intersects a component Σ_i of ∂M and D is contained in the collar neighborhood $K(\Sigma_i)$. Since the Gaussian curvature is negative and Σ_i is a geodesic, $|\partial D| \geq l(D \cap \Sigma_i)$. Thus, we have $\frac{|\partial D|}{|D\cap\partial M|} \geq \frac{|D\cap\Sigma_i|}{|D\cap\Sigma_i|} = 1$.
2. ∂D intersects a boundary component Σ_i but D is not contained in $K(\Sigma_i)$. Since $l(\Sigma_i) \leq 2\operatorname{arcsinh}(1)$, we have $|\partial D| \geq w(\Sigma_i) \geq \operatorname{arcsinh}(1)$, which implies $\frac{|\partial D|}{|D\cap\partial M|} \geq \frac{\operatorname{arcsinh}(1)}{ba} \geq \frac{1}{2b}$.
3. ∂D intersects a boundary geodesic γ_i of $\partial\Omega_i$. Since both D and $\Omega_i \setminus D$ have to intersect ∂M, ∂D cannot be contained in $K(\gamma_i)$. Since $l(\gamma_i) \leq L_{g+b}$, $|\partial D| \geq w(\gamma_i) \geq \beta_5$ where β_5 is a constant depending only on g and b. Thus $\frac{|\partial D|}{|D\cap\partial M|} \geq \frac{\beta_5}{ba} = \frac{\beta_5}{2\operatorname{arcsinh}(1)b} =: \beta_9$.
4. ∂D does not intersect ∂M and a component Γ of ∂D is freely homotopic to a boundary component Σ_i. Since the Gaussian curvature is negative and Σ_i is a geodesic, $|\partial D| \geq l(\Sigma_i)$. Therefore, we have $\frac{|\partial D|}{|D\cap\partial M|} \geq \frac{|\Sigma_i|}{|\Sigma_i|} = 1$.
5. ∂D does not intersect ∂M and none of its components is freely homotopic to a boundary component. We note that each component of ∂D is freely homotopic to a simple closed geodesic of Ω_i. Let Γ be the union of these geodesics. We have $|\partial D| \geq |\Gamma|$. Γ divides Ω_i into two connected components, each of them containing at least one connected component of ∂M. We recall that the geodesics $\gamma_1, ..., \gamma_p$ divide M into $n+1$ regions and that a subset of these geodesics divides M into n regions $\Omega_1, ..., \Omega_n$. Let $\tilde{\gamma}$ be the union of the geodesics that do not belong to this subset. We have $\gamma_{max} \in \tilde{\gamma}$. If $|\Gamma|$ were smaller than $l(\tilde{\gamma})$ there would be a family of geodesics of M, dividing M into $n+1$ regions and their total length would be smaller than l_n, which is a contradiction. Hence we have $|\Gamma| \geq l(\tilde{\gamma}) \geq l(\gamma_{\max}) \geq \frac{\mathrm{l}_n}{p}$ and since $3g-3+b$ is the maximal number of these geodesics, $|\Gamma| \geq \frac{\mathrm{l}_n}{3g-3+b}$. Therefore, $\frac{|\partial D|}{|D\cap\partial M|} \geq \frac{|\Gamma|}{ab} \geq \frac{\mathrm{l}_n}{(3g-3+b)(2\operatorname{arcsinh}(1)b)} = \beta_{10}\,\mathrm{l}_n$ and β_{10} is a constant depending only on g and b.

Since we have considered all possibilities for ∂D, we have

$$h_2(\Omega_i) \geq \min\{1, \frac{1}{2b}, \beta_9, \beta_{10}\,\mathrm{l}_n\}.$$

Since $\mathrm{l}_n \leq \beta_1$, $h_2(\Omega_i) \geq \beta_{11}\,\mathrm{l}_n$, where $\beta_{11} := \min\{\beta_1^{-1}, \frac{1}{2b}\beta_1^{-1}, \beta_9\beta_1^{-1}, \beta_{10}\}$ is a constant depending only on g and b.

If Ω_i has several boundary components, we have shown that $\sigma_1^N(\Omega_i) \geq \frac{h_1(\Omega_i)h_2(\Omega_i)}{4} \geq \frac{\beta_8\beta_{11}\,\mathrm{l}_n^2}{4} = \beta_{12}\,\mathrm{l}_n^2$ where β_{12} is a constant depending only on g and b.

We conclude that $\sigma_1^N(\Omega_i) \geq \min\{\beta_3, \beta_{12}\,\mathrm{l}_n^2\}$. Since $\mathrm{l}_n \leq \beta_1$, $\sigma_1^N(\Omega_i) \geq \beta_{13}\,\mathrm{l}_n^2$ where $\beta_{13} = \min\{\beta_3\beta_1^{-2}, \beta_{12}\}$. Since it is true for all Ω_i, we obtain

$$\sigma_n(M) \geq \min\{\sigma_1^N(\Omega_1), ..., \sigma_1^N(\Omega_n)\} \geq \beta_{13}\,\mathrm{l}_n^2,$$

where β_{13} is a constant depending only on g and b. □

Remark 4.6. *We have seen that the presence of the area of M in the denominator of the lower bound of Theorem 4.1 can make this estimate inaccurate. In Theorem 4.3 the weight of the area of M is hidden in the constant since it depends only on the signature of the hyperbolic surface.*

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