

Counting Reeb Chords on Spherizations

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Summary

In classical physics, one is interested in finding solutions of the Newtonian equations of motion. If there is a certain number of bodies which attract each others and if one assumes an initial configuration of these masses, then one would like to understand the time evolution of this system according to Newton's equations, i.e. the change of position and momentum of all these bodies as functions of time. But already in the case of three bodies – say, the moon, the sun and the earth – one knows only very little and this question remains essentially unanswered.

Rewriting the Newtonian equations of motion in an equivalent way leads to Hamilton's equations. Solutions of Hamilton's equations are paths – in physical terms – in phase space, whereas in mathematical terms one calls this space the cotangent bundle. So classical physical evolution takes mathematically place in cotangent bundles.

Symplectic geometry is a new and prominent subject within differential geometry, one of the few basic branches of mathematics. The cotangent bundle is probably the most famous representative of a so-called symplectic manifold. It holds true that the old physical questions got via the steps explained above a new and strong geometrical interpretation.

Floer homology is a powerful tool to study solutions of Hamilton's equations. It gives the possibility to use topological information about the cotangent bundle to obtain qualitative and quantitative results on solutions of Hamilton's equations.

The energy is a property of a physical system which remains constant during evolution of time. Therefore, it is natural to look for solutions of Hamiltonian systems on surfaces as certain subsets – called energy hypersurfaces – of cotangent bundles which are characterized by the fact that the energy function takes for all points of these surfaces the same value. Roughly speaking, solutions of Hamilton's equations along energy

hypersurfaces are called Reeb chords. The spectrum of such an energy hypersurface is simply the set of all times needed to move along the paths which are solutions of Hamilton's equations. So it is the set of times required to walk along the Reeb chords of a given energy hypersurface. In particular, the counting function associated to an energy hypersurface is studied. This function calculates the number of solutions whose times are shorter than a given value.

In this thesis, steps are taken towards an understanding of the time spectrum of fiberwise starshaped hypersurfaces in cotangent bundles. The base manifold is throughout assumed to be a closed connected Riemannian manifold. It is shown that under the additional assumption of exponential- resp. polynomial growth of the fundamental group of the base manifold, the counting function grows at least exponentially resp. at least polynomially in time. Generally, for every fiberwise starshaped hypersurface over a closed connected Riemannian manifold, the associated counting function grows at least linearly in time. These are asymptotic results. Afterwards the question of understanding fast Reeb chords is considered. An estimate for the time of the fastest resp. of the second fastest Reeb chord is given. More specifically, this question is addressed by choosing special base manifolds, or configuration spaces, such as Lie groups or generally (Riemannian) symmetric spaces. Estimates for the times of the k fastest Reeb chords are deduced. These estimates depend on the geometry of the base manifold only. Another attempt is of group theoretic nature. If the fundamental group of the base manifold is of order k , then there are at least k Reeb chords satisfying an upper time bound k times the diameter of the (compact) base manifold. Finally, some results concerning the stability of the time of the fastest Reeb chord are presented.

Raphael Wullschleger

Keywords. Hamiltonian Dynamics; Symplectic Geometry; Lagrangian Floer Homology; Contact Geometry; Reeb Dynamics

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Notations and Symbols

$\mathbb{N} := \{1, 2, 3, \dots\}$	The natural numbers
$\mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$	The natural numbers with 0
$\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$	The integers
$\mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$	The strictly positive real numbers
$\mathbb{R}_0^+ := \{x \in \mathbb{R} \mid x \geq 0\}$	The non-negative real numbers
\mathbb{F}_p	A series of coefficient fields, see Section 2.2.3
$\mathbb{Z}_2 := \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$	The set of integers \mathbb{Z} modulo 2
$M, (M, g)$	A closed (i.e. compact without boundary) connected finite-dimensional smooth manifold. Usually furnished with a Riemannian metric g , we speak of the Riemannian manifold (M, g)
q, q'	A pair of points in M . The point q is the starting point of a geodesic segment, whereas q' is the point where this curve ends
$d := \text{diam } M$	The diameter of the manifold (M, g)
T^*M	The cotangent bundle associated with M

T_x^*M	The fiber of the cotangent bundle T^*M over the base point $x \in M$
Σ	A fiberwise starshaped hypersurface in T^*M
Σ_x	For $x \in M$: $\Sigma_x := \Sigma \cap T_x^*M$, see Section 2.1.2
H	A function $H: T^*M \rightarrow \mathbb{R}$, called the Hamiltonian, see Section 2.1.1
$\mathcal{A}_H, \mathcal{A}, \mathcal{T}$	The Hamiltonian action functional associated with H and the reduced action or the time, see Section 2.1.1
$\mathcal{P}(H, q, q'), \mathcal{P}^b(H, q, q')$	The set of solutions of Hamilton's equations starting in T_q^*M and ending in $T_{q'}^*M$; b indicates an upper action-bound
G_-, K, G_+	Three special Hamiltonians, see Section 2.2
$U(q, H), V(q)$	$q \in M$ and $H: T^*M \rightarrow \mathbb{R}$ a Hamiltonian function, see Definitions (2.11), (2.12)
$\text{CF}_{q, q', \Sigma}$	The counting function counts Reeb chords from Σ_q to $\Sigma_{q'}$, defined in (2.9)
CF_k	Floer chain groups of Conley–Zehnder index k
\mathcal{C}_p	$\mathcal{C}_p := \{q \in M \mid q \text{ conjugate to } p\}$ is the set of conjugate points of $p \in M$

Abstract Mathematics is about “interesting structures”. What makes a structure interesting is an abundance of interesting problems; we study a structure by solving these problems.

M. Gromov, [20]

Chapter 1

Introduction

The first chapter shall describe in detail the questions addressed in this thesis and the results obtained. Starting from basic physical principles, we will focus on the naturality and the importance of the problems considered, and we will place the topic in the broader context of mathematical research. By doing so, we follow the books of Arnol’d [7] and of Hofer–Zehnder [25]. Finally, we give an overview of the results obtained.

1.1 The questions of this thesis

The roots of the questions of this thesis lie in physics. Classical mechanics is the first analytic approach to describe physical phenomena. This theory focuses mainly on understanding the time evolution of the positions of physical bodies which exert forces on each others. The definitions and principles were introduced in the Mathematical Principles of Natural Philosophy (*Philosophiæ Naturalis Principia Mathematica*) by I. Newton in the year 1687.

Let us consider the motion of a certain number n of point-mass particles in three-dimensional real space \mathbb{R}^3 . The totality of these n particles forms the *physical system*. More precisely, if we fix one of these n particles, say P_i , where $i \in \{1, \dots, n\}$ specifies our choice, the time evolution of the position of P_i can be described by a coordinate map,

$$\begin{aligned} x_i : \mathbb{R} &\longrightarrow \mathbb{R}^3, \\ t &\longmapsto x_i(t). \end{aligned}$$

The variable t refers to time. These coordinate maps shall assumed to be at least twice continuously differentiable mappings. As an illustration we can consider the so-called world lines traced by these points under time evolution, see Figure 1.1.

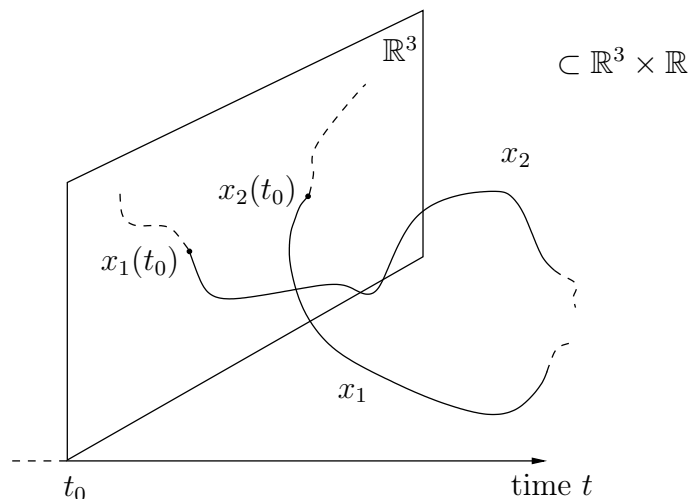


Figure 1.1: This figure shows the world lines (as paths in “space–time” $\mathbb{R}^3 \times \mathbb{R}$) of two particles P_1 and P_2 . The horizontal axis stands for the time parameter of the two coordinate maps. Observe that, possibly, there is a (collision) time t' for which $x_1(t') = x_2(t')$ holds true and in addition, that the vector $x(t) := (x_1(t), x_2(t)) \in \mathbb{R}^6$ is a point in six-dimensional real space.

So far, we treated the n particles, or the n bodies, separately. In physics, one is interested in interactions between the particles or in understanding the forces exerted on a given body by the other bodies. Therefore, it is natural to define a mapping depending on time and describing the positions of these n particles at once,

$$\begin{aligned} x : \mathbb{R} &\longrightarrow \mathbb{R}^{3n}, \\ t &\longmapsto x(t) := (x_1(t), \dots, x_n(t)), \end{aligned}$$

where \mathbb{R}^{3n} stands for the direct product of n copies of \mathbb{R}^3 , just because all the n particles can move freely in three-dimensional space. Since the x_i are differentiable mappings, we can consider their derivatives

$$\dot{x}_i(t_0) = \left. \frac{dx_i}{dt} \right|_{t=t_0} \in \mathbb{R}^3,$$

called the velocity vectors at the time t_0 , as well as the acceleration vectors at t_0

$$\ddot{x}_i(t_0) = \left. \frac{d^2 x_i}{dt^2} \right|_{t=t_0} \in \mathbb{R}^3.$$

Newton's principle of determinacy, one of the principles of classical mechanics, implies that the initial positions $(x_0)_i$ of all n point-masses and their initial velocities $(v_0)_i$ uniquely determine the motion of the system. So, the vector of initial positions $x_0 = ((x_0)_1, \dots, (x_0)_n)$ and the initial velocities $v_0 = ((v_0)_1, \dots, (v_0)_n)$ must determine also the acceleration of any body at any time. Mathematically, this means that there exists a map F (F for force)

$$F : \Omega \subset \mathbb{R}^{3n} \times \mathbb{R}^{3n} \times \mathbb{R} \longrightarrow \mathbb{R}^{3n}$$

such that

$$\ddot{x}(t) = F(x(t), \dot{x}(t), t). \quad (1.1)$$

The equations (1.1) are the well-known *Newtonian equations of motion* of classical mechanics. Observe that we set all masses m_i equal to one. The map F introduced above is found by experimental means. Forces are measurable. Note that the domain Ω of F is often a strict subset of $\mathbb{R}^{3n} \times \mathbb{R}^{3n} \times \mathbb{R}$. This is the case for example if one considers two bodies and the forces exerted on them due to gravity. One has to exclude the points of collision, since for these F is not defined.

Newtons equations of motion (1.1) form a system of ordinary differential equations for the time evolution of the positions x_i , or the trajectories, of the n point-masses. As explained at the beginning, we are interested in finding these trajectories to predict the time evolution of the considered physical system.

A physical system is said to be *conservative* if there exists a continuously differentiable function U

$$U : \Omega \subset \mathbb{R}^{3n} \longrightarrow \mathbb{R}$$

such that

$$F(x(t), \dot{x}(t), t) = -\nabla U(x(t)). \quad (1.2)$$

This function U is called the *potential* or the *potential function*. As it is apparent from

equation (1.2), the function U evaluated at a point is formally an energy and describes the energy of a particle according to its position relative to the others. For such a system of n particles we introduce its *total kinetic energy* (which is the sum of the individual kinetic energies of the n particles):

$$K(\dot{x}(t)) := \frac{1}{2} \sum_{i=1}^n \dot{x}_i(t)^2.$$

We assumed that all the n bodies are point-mass particles, so they do not have any spatial extent what could lead to rotational energy or similar. Then we can speak of the *total energy* of the considered system:

$$E(x(t), \dot{x}(t)) := K(\dot{x}(t)) + U(x(t)). \quad (1.3)$$

The following theorem points out the very important property of conservative systems. For a proof, see Theorem 1.1.4.

Theorem 1.1.1 (Energy conservation). *Let x be a solution of equation (1.2), then the total energy E is constant along this solution x , i.e.*

$$\forall t_0 \in \mathbb{R} : \left. \frac{dE}{dt}(x(t), \dot{x}(t)) \right|_{t=t_0} = 0.$$

So far, we derived Newton's equations of motion and we tried to point out why we are interested in finding their solutions. The next step is to show that the solutions of a conservative system can be determined via a variational principle, "Hamilton's principle of least action". The calculus of variations is concerned with the extremals of functions, or functionals, whose domain is an infinite-dimensional space, the space of all curves from one point to another one.

Let us consider the *Lagrangian* or the *Lagrange function* of a conservative system, defined by

$$L(\gamma(t), \dot{\gamma}(t)) := K(\dot{\gamma}(t)) - U(\gamma(t)), \quad (1.4)$$

and the associated *action functional*

$$\Phi(\gamma) = \int_{t_0}^{t_1} L(\gamma(t), \dot{\gamma}(t), t) dt, \quad (1.5)$$

where $\gamma : [t_0, t_1] \rightarrow \mathbb{R}^{3n}$ is a C^2 -path in \mathbb{R}^{3n} . A theorem says that the extremals, formally given by $d\Phi(\gamma) = 0$, see Arnold [7] for details, coincide with the solutions of Newton's equations of motions,

$$\ddot{x} = -\nabla U(x).$$

This is *Hamilton's principle of least action*. Suppose that $\gamma : [t_0, t_1] \rightarrow \mathbb{R}^{3n}$ is an extremal of the functional Φ . Then it is necessary and sufficient that γ satisfies the so-called *Euler-Lagrange equations*

$$\frac{d}{dt} \nabla_{\dot{x}} L(\gamma(t), \dot{\gamma}(t)) - \nabla_x L(\gamma(t), \dot{\gamma}(t)) = 0. \quad (1.6)$$

This variational principle opens the door to the vast area of the calculus of variations. But the drawback from the point of view of mathematics is still there: the Euler-Lagrange equations (1.6) are in fact a system of n second-order ordinary differential equations.

The Euler-Lagrange equations (1.6) are evaluated at points lying on paths x defined on a time interval having values in the space \mathbb{R}^{3n} , for example $x : [0, 1] \rightarrow \mathbb{R}^{3n}$. In particular, this holds true for very short segments of this path x . These segments can be viewed as short segments of paths in a chart of some manifold M . Therefore, we can consider our situation in the more general setting of manifolds. This means formally that the Lagrangian is a smooth function on the tangent bundle TM of M ,

$$\begin{aligned} L : TM &\longrightarrow \mathbb{R}, \\ (x, v_x) &\longmapsto L(x, v_x). \end{aligned} \quad (1.7)$$

Note that we omit here an explicit time-dependence of L – see below for an explanation – and also that we allow all possible dimensions $\dim M = n$ for the manifold M . Assumed constraints on the physical space reduce the dimension of M by one, two or higher. Denote by T^*M the cotangent bundle of M and define $L_x(\cdot) := L(x, \cdot) : T_x M \rightarrow \mathbb{R}$.

Definition 1.1.2 (Fiberwise Legendre transform map). *Let M be a smooth manifold and L a Lagrangian. The fiberwise Legendre transform map of L , or Legendre transform map of L_x , is the map*

$$\begin{aligned} \mathcal{L} : TM &\longrightarrow T^*M, \\ (x, v_x) &\longmapsto \mathcal{L}(x, v_x) := (x, dL_x(v_x)). \end{aligned}$$

Note that the differential $dL_x(\cdot) : T_xM \rightarrow T_x^*M$ evaluated at the point v_x gives $dL_x(v_x) \in T_x^*M$. Hence, $(x, dL_x(v_x)) \in T^*M$. We define the *fiberwise Legendre transform* of L by

$$\begin{aligned} H : T^*M &\longrightarrow \mathbb{R}, \\ (x, a_x) &\longmapsto H(x, a_x) := \sup_{v \in T_xM} (a_x(v) - L(x, v)). \end{aligned}$$

This function H is usually called the *Hamiltonian* or the *Hamilton function* associated to L . Note that if L is of the form $K - U$, then H is of the form $K + U$. So in this case H coincides with the total energy E of the system.

Let $U \subset TM$ be an open set, choose coordinate functions q_i, w_i , on U , $i \in \{1, \dots, n\}$. If L is fiberwise strictly convex, i.e. if the fiberwise Hessian of L is positive definite

$$\det \left(\frac{\partial^2 L}{\partial w_i \partial w_j} \right) > 0,$$

then the fiberwise Legendre transform map \mathcal{L} of L is a diffeomorphism. For a given path $\gamma : \mathbb{R} \rightarrow M$, consider its analog in the cotangent bundle, given by the Legendre transform map

$$\mathcal{L}(\gamma)(t) := (\gamma(t), p_\gamma(t)) := \mathcal{L}(\gamma(t), \dot{\gamma}(t)).$$

Theorem 1.1.3. *The path $\gamma : \mathbb{R} \rightarrow M$ is a solution of the Euler-Lagrange equations (1.6) if and only if the path $\mathcal{L}(\gamma) : \mathbb{R} \rightarrow T^*M$ satisfies the so-called Hamilton equations,*

$$\frac{d}{dt} p_i = -\frac{\partial H}{\partial q_i}, \quad \frac{d}{dt} q_i = \frac{\partial H}{\partial p_i},$$

where q_i, p_i , $i \in \{1, \dots, n\}$, are local coordinates on T^*M .

One can write down the Hamilton equations in the following compact form

$$\dot{z}(t) = J\nabla H(z(t)),$$

where

$$J := \begin{pmatrix} 0 & -\text{Id}_n \\ \text{Id}_n & 0 \end{pmatrix}.$$

Hamilton's equations build a system of $2n$ first-order differential equations. This is in contrast to the system of n second-order differential equations given by the Euler-Lagrange equations, (1.6). We will make the abbreviation $X_H := J\nabla H$ and call it the Hamiltonian vector field, see below for the details. An *energy hypersurface* of the cotangent bundle is a regular level set of the Hamiltonian $H : T^*M \rightarrow \mathbb{R}$. This energy hypersurface will be denoted by Σ . In this thesis, we will study the solutions of the dynamical system

$$\dot{\gamma}(t) = X_H(\gamma(t)), \quad \gamma : \mathbb{R} \rightarrow T^*M$$

on energy hypersurfaces. Solutions of Hamilton's equations must lie on such energy hypersurfaces, see Theorem 1.1.4 below.

Let us describe the setting of our questions. Consider the cotangent bundle T^*M of a n -dimensional closed connected Riemannian manifold (M, g) , where g is the Riemannian metric on M . A *symplectic structure* on a smooth manifold P is a non-degenerate closed 2-form $\omega \in \Omega^2(P)$, see Definition 2.1.1. Choose a point $\bar{x} = (x, \xi_x) \in T^*M$ and consider the standard projection $\pi : T^*M \rightarrow M$ defined by $(x, \xi_x) \mapsto x$. We then are able to define globally a differential 1-form,

$$\lambda(\bar{x})(v_{\bar{x}}) := \xi_x(d\pi(\bar{x})(v_{\bar{x}})), \quad v_{\bar{x}} \in T_{\bar{x}}(T^*M). \quad (1.8)$$

This is the so-called Liouville or tautological 1-form, see Hofer–Zehnder [25]. The cotangent bundle T^*M equipped with the 1-form (1.8) leads to a symplectic manifold $(T^*M, d\lambda)$. Choose a hypersurface Σ in T^*M which is *fiberwise starshaped* with respect to the origin, i.e. $\Sigma_x := \Sigma \cap T_x^*M$ is strictly starshaped with respect to the zero element $0_x \in T_x^*M$. A *contact form* on Σ is a 1-form η on Σ with $\eta \wedge (d\eta)^{n-1} > 0$

everywhere. An example is the 1-form $\lambda|_{\Sigma}$. A contact form η determines a *contact structure* on Σ , the oriented hyperplane field $\xi := \ker(\eta) \subset T\Sigma$. In our case this shall be $\xi_{\Sigma} := \ker(\lambda|_{\Sigma})$. Choose a smooth function $H : T^*M \rightarrow \mathbb{R}$ which is fiberwise homogeneous of degree two such that Σ is a regular level set of H , so Σ is an energy hypersurface. The *Hamiltonian vector field* X_H belonging to the function H is defined by $d\lambda(X_H, \cdot) = dH(\cdot)$. Since all levels are compact, this vector field has a flow φ^t satisfying Hamilton's equations

$$\frac{d}{dt}\varphi^t(x) = X_H(\varphi^t(x)), \quad x \in T^*M. \quad (1.9)$$

We explained above that the solutions of the Euler-Lagrange equations (1.6) coincide with the extremals of the action functional Φ (1.5). A main object of this thesis is the Hamiltonian action functional which generalizes (1.5) to the setting of cotangent bundles: Let $\Omega_{q,q'}^1 T^*M$ be the space of paths of $W^{1,2}$ -Sobolev type in T^*M on the unit interval $[0, 1] \subset \mathbb{R}$ from the point $\gamma(0) \in \Sigma_q$ to the point $\gamma(1) \in \Sigma_{q'}$. Then we can introduce the *Hamiltonian action functional*

$$\begin{aligned} \mathcal{A}_H : \Omega_{q,q'}^1 T^*M &\longrightarrow \mathbb{R}, \\ \gamma &\longmapsto \mathcal{A}_H(\gamma) := \int_{\gamma} \lambda - \int_0^1 H(\gamma(t)) dt. \end{aligned}$$

In accordance with what we said concerning the Lagrangian action functional, it is well-known that the solutions of (1.9) are the critical points of the Hamiltonian action functional \mathcal{A}_H . In the field of symplectic geometry, Floer homology is a powerful tool to study the critical points of the Hamiltonian action functional. Floer homology is a Morse theory for this functional. So it provides the possibility to use topological information about the cotangent bundle T^*M to get qualitative and quantitative results concerning the solutions of Hamilton's equations (1.9). We will pursue this approach to get answers to our questions.

The following result generalizes Theorem 1.1.1.

Theorem 1.1.4 (Flow invariance of H). *If φ^t is the flow of the Hamiltonian vector field X_H , then it holds true that for all $x \in T^*M$ and for all $t \in \mathbb{R}$ (for which the flow is*

defined)

$$H(\varphi^t(x)) = H(x).$$

Proof. Indeed,

$$\begin{aligned} \frac{d}{dt}H(\varphi^t(x)) &= dH(\varphi^t(x)) \cdot \frac{d}{dt}\varphi^t(x) = dH(\varphi^t(x)) \cdot X_H(\varphi^t(x)) \\ &= d\lambda(X_H(\varphi^t(x)), X_H(\varphi^t(x))) \\ &= 0, \end{aligned}$$

by antisymmetry of $d\lambda$. □

Recall that we assumed that the Lagrangian L (1.7) does not depend explicitly on the time t . This implies via the Legendre transform the same for the Hamiltonian H . If H depends explicitly on time, then this invariance property does not hold.

The contact form $\lambda|_\Sigma$ determines by the following two conditions the unique *Reeb vector field* R on $T\Sigma$ by

$$d(\lambda|_\Sigma)(R, \cdot) \equiv 0, \quad \lambda|_\Sigma(R) \equiv 1.$$

The associated flow is called the *Reeb flow* φ_R of R . One can show that the Reeb flow φ_R of $\ker(\lambda|_\Sigma)$ is a reparametrization of the flow $\varphi_H|_\Sigma$ of X_H restricted to Σ , we refer to Lemma 4.8.4. A *Reeb chord* is a flow line of φ_R .

Question A. *Is it true that for any two points $q, q' \in M$, there exists a Reeb chord from Σ_q to $\Sigma_{q'}$?*

This is a version of the Arnol'd Chord Conjecture for fibers of a starshaped hypersurface in the cotangent bundle. Let (C, ξ) be a $(2\ell + 1)$ -dimensional contact manifold and let L be an integral submanifold of ξ . If $\dim L = \ell$ then the submanifold L is called *Legendrian*, see [31]. The Arnol'd Chord Conjecture stated in [6] asks in the case of a contact manifold C for a Reeb chord which starts and ends in a given Legendrian submanifold L of C .

Consider the length spectrum of the Riemannian manifold (M, g) given by

$$\sigma_{q,q'}(g) := \{\text{lengths of all geodesic segments from } q \text{ to } q'\}.$$

In this definition, the length is induced by the Riemannian metric g . The study of the length spectrum of a Riemannian manifold is an important problem, with many results and many open questions, see Berger [11, Chapter 10] and Paternain [42, Chapter 5].

In this thesis we are interested in a more general problem: We study the spectrum of fiberwise starshaped hypersurfaces Σ of the cotangent bundle T^*M . Define the set

$$\mathcal{S}(\Sigma, q, q') := \{\mathcal{T}(\gamma) \mid \gamma \text{ a Reeb chord on } \Sigma \text{ from } \Sigma_q \text{ to } \Sigma_{q'}\},$$

where the number $\mathcal{T}(\gamma)$ is the *time* needed by the Reeb chord γ to go from Σ_q to $\Sigma_{q'}$. It is given by

$$\mathcal{T}(\gamma) = \int_{\gamma} \lambda,$$

and where the numbers $\mathcal{T}(\gamma)$ are listed with multiplicities. Knowing the set $\sigma_{q,q'}(\Sigma)$ is equivalent to knowint the *counting function*

$$\text{CF}_{q,q',\Sigma}(T) := \#\{\tau \in \mathcal{S}(\Sigma, q, q') \mid \tau \leq T\}. \quad (1.10)$$

If one fixes a time T , then the counting function gives the number $\text{CF}_{q,q',\Sigma}(T)$ of Reeb chords starting in the fiber Σ_q and ending in some point of $\Sigma_{q'}$ before or at the time T .

Question B. *Is it possible to find a function $f_{\Sigma} : \mathbb{R} \rightarrow \mathbb{R}$ with $f_{\Sigma}(T) \rightarrow +\infty$ ($T \rightarrow +\infty$) such that*

$$\text{CF}_{q,q',\Sigma}(T) \geq f_{\Sigma}(T) > 0$$

independently of the points $q, q' \in M$?

1.2 Summary of the results

Let (M, g) be an n -dimensional closed connected Riemannian manifold and denote by $d := \text{diam } M$ the diameter of M . Consider a fiberwise starshaped hypersurface $\Sigma \subset T^*M$ in the cotangent bundle T^*M and a Hamiltonian function $K : T^*M \rightarrow \mathbb{R}$ homogeneous of degree two. Assume that $\frac{1}{2}$ is a regular value of K such that $\Sigma = K^{-1}(\frac{1}{2})$. We refer to Section 2.2 for details.

In this section, we state the main results of the thesis and explain the connections between them. The main tools for analyzing the spectrum of spherizations are Morse theory, Floer homology, and results on the growth of finitely generated groups and of the homology of based loop spaces. Morse theory for the energy functional gives lower bounds for the number of geodesic paths between two non-conjugate points in terms of the homology of the based loop space of M .

Assume first that our Reeb flow is a geodesic flow on the Riemannian manifold (M, g) . Assume that q, q' are non-conjugate. We would like to understand the function $\text{CF}_{q,q',\Sigma}(t)$ counting geodesics from q to q' of length $\leq t$. Since we are looking for lower bounds of this number that are “true for all metrics g ”, we only look for homologically visible geodesics (Definition 4.1.18), and hence use Morse theory: Consider the energy functional

$$\mathcal{E}_g(\gamma) := \frac{1}{2} \int_0^1 g(\dot{\gamma}(t), \dot{\gamma}(t)) dt \quad (1.11)$$

on the space of candidates $\Omega_{q,q'}^1 M$ of $W^{1,2}$ -paths $\gamma: [0, 1] \rightarrow M$ with $\gamma(0) = q$ and $\gamma(1) = q'$. The critical points of \mathcal{E}_g are precisely the geodesics from q to q' . If we denote by $\mathcal{E}_g^a(q, q')$ the sublevel set $\{\gamma \in \Omega_{q,q'}^1 M \mid \mathcal{E}_g(\gamma) \leq a\}$, and notice that for a geodesic, twice the energy equals the length squared (Lemma C.3.4), the classical Morse-inequalities tell us that

$$\text{CF}_{q,q',\Sigma}(t) \geq \dim \text{H}_* \left(\mathcal{E}_g^{t^2/2}(q, q'), \mathbb{F} \right) = \sum_{j=0}^{\infty} \dim \text{H}_j \left(\mathcal{E}_g^{t^2/2}(q, q'), \mathbb{F} \right) \quad (1.12)$$

provided that q, q' are non-conjugate. Indeed, this condition is equivalent to saying that \mathcal{E}_g is Morse. On the right hand side, H_* denotes singular homology, and coefficients are taken in a field \mathbb{F} . The sum on the right hand side is finite, since $\mathcal{E}_g^{t^2/2}(q, q')$ is homotopy equivalent to a finite dimensional CW-complex, see Milnor [32]. The inequality (1.12) looks wonderful, since it seems to translate our geometric-dynamical problem into a topological one. However, two questions arise:

Question 1. How can we understand $\dim \text{H}_* \left(\mathcal{E}_g^a(q, q'), \mathbb{F} \right)$?

Question 2. And what if q, q' are conjugate ?

Let us first address the choice of the coefficient field \mathbb{F} . To get the best possible

estimate in (1.12), one should take the supremum over all fields \mathbb{F} . By the universal coefficient theorem, it suffices to consider only one field per characteristic, say \mathbb{Q} and the finite fields \mathbb{F}_p for p prime. Note that in Chapter 4 we will prove related results for special choices of \mathbb{F} . For example we will choose $\mathbb{F} = \mathbb{Z}_2$ in Section 4.2.3.

The numbers $\dim H_* (\mathcal{E}_g^a(q, q'), \mathbb{F})$ are in general too hard to compute. One reason is that they may depend rather irregularly on a , g and q, q' . In particular, the function $a \mapsto \dim H_* (\mathcal{E}_g^a(q, q'), \mathbb{F})$ may not be monotone increasing. To remedy for these irregularities, we consider the numbers $\dim \iota_*^a H_* (\mathcal{E}_g^a(q, q'), \mathbb{F})$ instead. Here $\iota_*^a: \mathcal{E}_g^a(q, q') \rightarrow \Omega_{q, q'}^1 M$ is the inclusion. The number $\dim \iota_*^a H_* (\mathcal{E}_g^a(q, q'), \mathbb{F})$ is the dimension of the part of the homology of $\Omega_{q, q'}^1 M$ that can be represented by cycles in $\mathcal{E}_g^a(q, q')$. A cycle in $\mathcal{E}_g^a(q, q')$ is still a cycle in $\Omega_{q, q'}^1 M$, while it may be bounded in $\Omega_{q, q'}^1 M$ but not in $\mathcal{E}_g^a(q, q')$. Hence

$$\dim H_* (\mathcal{E}_g^a(q, q'), \mathbb{F}) \geq \dim \iota_*^a H_* (\mathcal{E}_g^a(q, q'), \mathbb{F}) .$$

The functions $b_{g, q, q'}(a) := \dim \iota_*^a H_* (\mathcal{E}_g^a(q, q'), \mathbb{F})$ are much better behaved: Clearly, they are monotone increasing in a . A more fundamental reason that we are interested in geodesic chords that are homologically visible in the total path space $\Omega_{q, q'}^1 M$, and not just in the sublevel $\mathcal{E}_g^a(q, q')$, is the following: We shall find Reeb chords by sandwiching the sublevel of Σ between two sublevels of \mathcal{E}_g , and this will lead to a lower bound of $\text{CF}_{q, q', \Sigma}(t)$ in terms of $b_{g, q, q'}(a)$, but not in terms of $\dim H_* (\mathcal{E}_g^a(q, q'), \mathbb{F})$. The following theorem will provide lower bounds of $\text{CF}_{q, q', \Sigma}$ in terms of the homology $H_* (\Omega_{q, q'}^{1, a} M, \mathbb{F})$. It is proven in Section 2.2.3. See the different parts of Section 2.2 for the definitions and notions used in the statement.

Proposition 1.2.1. *Fix two points $q, q' \in M$ such that q' is not Σ -conjugate to q . Let g be a Riemannian metric on M such that q, q' are not g -conjugate. Scale g such that $G \leq F$. Let $a < b$ and $\sigma \geq \sigma_g \geq 1$ be such that*

$$a, b, a/\sigma, \sigma b \notin \mathcal{S}(G, q, q') \quad \text{and} \quad a/\sigma, b \notin \mathcal{S}(F, q, q') .$$

Then for any field \mathbb{F} the number of Reeb chords from Σ_q to $\Sigma_{q'}$ in class α with action in $(\sqrt{2a/\sigma}, \sqrt{2b}]$ is bounded from below by the rank of the homomorphism induced by

inclusion

$$\tilde{H}_* (\mathcal{QE}^{(a/\sigma, b]}(q, q', \alpha); \mathbb{F}) \rightarrow \tilde{H}_* (\mathcal{QE}^{(a, \sigma b]}(q, q', \alpha); \mathbb{F}).$$

By \tilde{H}_* we denote reduced homology. The Proposition 1.2.1 is a consequence of the Abbondandolo–Schwarz isomorphism, see [3], from the Floer homology groups of T^*M to the homology groups of the based loop space $\Omega_{q, q'}^1 M$.

1.2.1 Exponential and polynomial growth of the number of solutions

We give an overview on asymptotic results of the counting function $\text{CF}_{q, q', \Sigma}$. By doing so, we answer Question B in special cases.

Since M is a closed manifold, its fundamental group $\pi_1(M, q)$, for $q \in M$, is a finitely presented group. Choose a finite set S of generators of $\pi_1(M, q)$. For each positive integer m the function $\gamma_S(m)$ counts the number of distinct elements in $\pi_1(M, q)$ which can be written as words with at most m letters from $S \cup S^{-1}$. If the following limit is strictly positive, we say that $\pi_1(M, q)$ has *exponential growth*,

$$\lim_{m \rightarrow +\infty} \frac{\log \gamma_S(m)}{m} \in [0, +\infty). \quad (1.13)$$

Note that this limit exists, but depends on S .

Similarly, the *polynomial growth* of $\pi_1(M, q)$ is defined by

$$\gamma(G) := \limsup_{m \rightarrow +\infty} \frac{\log \gamma_S(m)}{\log m} \in [0, +\infty]. \quad (1.14)$$

Note that $\gamma(G)$ does not depend on S .

See Section 2.5 for details and examples of spaces with fundamental groups of exponential resp. polynomial growth.

Theorem 1.2.2 (Exponential and polynomial growth of the number of solutions). *Let $q \in M$. If $\pi_1(M, q)$ has exponential growth, then for every $q' \in M$ the number of orbits of the flow $\varphi_K^t|_\Sigma$ from Σ_q to $\Sigma_{q'}$ grows at least exponentially in time,*

$$\text{CF}_{q, q', \Sigma}(t) \succcurlyeq e^t.$$

Analogously, if $\pi_1(M, q)$ has polynomial growth k , it follows that for every $q' \in M$ the counting function $\text{CF}_{q, q', \Sigma}$ grows at least polynomially in time,

$$\text{CF}_{q, q', \Sigma}(t) \succcurlyeq t^k.$$

To give a partial answer to Question A and to Question B, let the fundamental group of M be finite.

Theorem 1.2.3 (Linear growth of Reeb chords). *If $\pi_1(M)$ is finite, then it follows that*

$$\text{CF}_{q, q', \Sigma}(t) \succcurlyeq t.$$

The proof uses the well-known result of Serre [46] that for any simply-connected manifold M there is a sequence of integers $(k_\ell)_{\ell \in \mathbb{N}}$ for which the Betti numbers $b_{k_\ell}(\Omega M, \mathbb{F})$ of the corresponding based loop space ΩM are not zero.

1.2.2 Time bounds for the first and the second Reeb chord

In Section 1.2.1 about exponential and polynomial growth we gave asymptotic results on the counting function $\text{CF}_{q, q', \Sigma}$. In particular, the constants appearing in the expressions for the lower bounds are not well-understood. This issue shall be addressed next.

Convention 1.2.4. *Let (M, g) be furnished with a Riemannian metric g such that the Hamiltonian functions F, G defined in Section 2.2.1 satisfy*

$$F \geq G.$$

Denote by $d := \text{diam}(M, g)$ the diameter of (M, g) .

Chapter 5 covers the details on how to deduce a concrete upper bound on the times of the first two shortest Reeb chords from Σ_q to $\Sigma_{q'}$.

Due to basic geometric facts, we know that for every Riemannian metric g satisfying the Convention 1.2.4 the following estimate for the first Reeb chord holds. Denote by

$\text{dist}(q, q')$ the distance of q and q' with respect to g . Then,

$$\mathcal{T}_1(\Sigma, q, q') \leq \text{dist}(q, q').$$

Given a simply-connected manifold M , let k_0 be the smallest integer k such that $H_k(M; \mathbb{F}) \neq 0$. Then $k_0 \in \{2, \dots, \dim M\}$.

Theorem 1.2.5. *Let Σ be a fiberwise starshaped hypersurface in T^*M with M simply-connected. Assume that g satisfies the Convention 1.2.4. If q' is not Σ -conjugate to q , then*

$$\begin{aligned} \mathcal{T}_2(\Sigma, q, q') &\leq 8k_0^2d + (2k_0^2 - 1)3d(\sqrt{\sigma_g} - 1) \\ &< 2k_0^2d(4 + 3\sqrt{\sigma_g}). \end{aligned}$$

Recall that n is the dimension of M and d refers to the diameter $d = \text{diam}(M, g)$. The existence of the short Reeb chords with the given time bound relies on results of Nabutovsky–Rotman, see [34] and for more details the sections in Chapter 5.

1.2.3 Time bounds for the first k Reeb chords – via Morse theory

Fix $k \in \mathbb{N}$. In Chapter 4 we explain two approaches to get concrete upper bounds on the times of the first k Reeb chords on Σ .

We start with Morse theory in infinite dimensions under the assumption that the energy functional (1.11) is a so-called *perfect* Morse function on the space of candidates $\Omega_{q,q'}^1 M$ (for almost all pairs q, q') with respect to \mathbb{F} . The notion of a perfect Morse function is explained in Section 6.1. This allows us to interpret k geodesic segments (on M) with given length bounds as homologically visible, see the Definition 4.1.18. Their existence is guaranteed by the work of Nabutovsky–Rotman [37]. Via Proposition 1.2.1 we then get

Proposition 1.2.6. *Let M be n -dimensional [2-dimensional]. If for $q \in M$ and almost every q' the energy functional \mathcal{E}_g is an \mathbb{F} -perfect Morse function on $\Omega_{q,q'}^1 M$, then for every pair of points $q, q' \in M$ and for every $k \in \mathbb{N}$ there exist at least k Reeb chords \tilde{x}_ℓ*

from Σ_q to $\Sigma_{q'}$ satisfying the time bound

$$\begin{aligned} \mathcal{T}(\tilde{x}_\ell) &\leq 2n(k+1)^2d, \\ [\mathcal{T}(\tilde{x}_\ell) &\leq (22k-21)d]. \end{aligned}$$

The expression in square brackets accounts for the 2-dimensional case. In general it is a very hard problem to understand whether \mathcal{E}_g is a perfect Morse function. This question leads deeply into the field of algebraic topology.

Work done by Bott-Samelson [14] yields that \mathcal{E}_g is perfect with respect to \mathbb{Z}_2 -coefficients on $\Omega_{q,q'}^1 M$ (for almost all pairs q, q') if M is a compact Riemannian symmetric space. This fact and Proposition 1.2.6 imply the following result:

Let (G, g_{bi}) be an n -dimensional compact connected Lie group carrying a bi-invariant Riemannian metric g_{bi} and let $H \subset G$ be a closed connected subgroup. Note that we can scale g_{bi} such that this metric satisfies the Convention 1.2.4 and is still bi-invariant.

Theorem 1.2.7. *Let G/H be a compact symmetric space with the induced Riemannian metric. Further, let $q, q' \in G/H$ be two arbitrary points and fix $k \in \mathbb{N}$. Then there exist k Reeb chords \tilde{x}_ℓ on $T^*(G/H)$ from Σ_q to $\Sigma_{q'}$ satisfying the time bound*

$$\begin{aligned} \mathcal{T}(\tilde{x}_\ell) &\leq \sqrt{2 \left(2(n(k+1)^2d)^2 + d \right)}, \\ \left[\mathcal{T}(\tilde{x}_\ell) &\leq \sqrt{2 \left(\frac{(22(k-1)d + \text{dist}(q, q'))^2}{2} + d \right)} \right]. \end{aligned}$$

An analogous statement holds true if M is a compact simply-connected Lie group, see Proposition 4.2.20, but with respect to any coefficient field \mathbb{F} . Therefore, we treat this special case individually.

Moreover, Chapter 4 consists of other results concerning the Conley–Zehnder index of the k Reeb chords \tilde{x}_ℓ and manifolds M of non-positive curvature.

1.2.4 Time bounds for the first k Reeb chords – via topology

For the next step we pursue a group-theoretic approach: We forgo the last geometric restrictions on M and say something about quantitative existence results of Reeb chords

on Σ under topological assumptions on the order of the fundamental group $\pi_1(M, q)$. An important ingredient is a beautiful proposition due to Gromov (Proposition 4.4.2) and further work done by Nabutovsky–Rotman [37]. This all together yields

Theorem 1.2.8. *If $\pi_1(M)$ has infinite or finite order $\geq k$, then for every pair $q, q' \in M$ there exist at least k Reeb chords \tilde{x}_ℓ from Σ_q to $\Sigma_{q'}$ satisfying the time bound*

$$\mathcal{T}(\tilde{x}_\ell) \leq kd.$$

1.2.5 Stability of the minimal time of Reeb chords

Denote by $\mathcal{T}_1(\Sigma)$ the minimal time, or the smallest element of the time spectrum of a given fiberwise starshaped hypersurface Σ . We then consider the C^0 -stability of the minimal time $\mathcal{T}_1(\Sigma)$: Let $\{\Sigma_k\}_{k \in \mathbb{N}}$ be a sequence of fiberwise starshaped hypersurfaces of the cotangent bundle T^*M .

Definition 1.2.9 (C^0 -Convergence of fiberwise starshaped hypersurfaces). *The sequence $(\Sigma_k)_{k \in \mathbb{N}}$ converges in the C^0 -sense to Σ if for every $\varepsilon > 0$ there exists N_ε such that*

$$(1 - \varepsilon)D^*\Sigma \subset D^*\Sigma_k \subset (1 + \varepsilon)D^*\Sigma, \quad \text{for all } k \geq N_\varepsilon.$$

We want to understand what happens with the sequence $(\mathcal{T}_1(\Sigma_k))_{k \in \mathbb{N}}$ if $(\Sigma_k)_{k \in \mathbb{N}}$ converges in the C^0 -sense to Σ .

Proposition 1.2.10. *Fix $q \in M$ and suppose that $\{\Sigma_k\}_{k \in \mathbb{N}}$ converges in the C^0 -sense to Σ . Then for all $q' \in M$ it holds true that*

$$\mathcal{T}_1(\Sigma_k) \rightarrow \mathcal{T}_1(\Sigma)$$

as $k \rightarrow \infty$.

Chapter 2

Methods and Spaces

This chapter shall explain the basic notions and definitions which will be used later on. We also outline the main methods and theorems that we will apply.

2.1 Background and Setting

Throughout this thesis, let (M, g) be an n -dimensional closed connected Riemannian manifold of diameter $d := \text{diam}(M)$, where $n \in \mathbb{N}$ is a natural number. Let $|\cdot|_g$ be the norm on the fibers of the tangent bundle TM induced by g . Consider the cotangent bundle T^*M which is isomorphic to the tangent bundle TM via the isomorphism

$$\begin{aligned} T : \quad TM &\rightarrow T^*M \\ (q, v) &\mapsto (q, \alpha_q), \quad \text{where } \alpha_q(w) = g_q(v, w). \end{aligned}$$

Using T , define a Riemannian metric g^* on the fibers of the cotangent bundle T^*M by

$$g_q^*(\alpha, \beta) := g_q(T^{-1}(q, \alpha), T^{-1}(q, \beta)), \quad \forall \alpha, \beta \in T_q^*M, q \in M.$$

Denote the canonical coordinates on T^*M by (q, p) .

Definition 2.1.1 (Symplectic manifold, [31]). *Let P be C^∞ -smooth manifold. A symplectic structure on P is a non-degenerate closed 2-form $\omega \in \Omega^2(P)$, i.e. if*

1. $d\omega = 0$, and if

2. for $p \in P$ and $v \in T_p P$ it follows that $\forall w \in T_p P : \omega_p(v, w) = 0 \Rightarrow v = 0$.

The pair (P, ω) is then called a symplectic manifold.

Note that this definition implies that a symplectic manifold is of even dimension and orientable, see [31].

The cotangent bundle T^*M with the standard Liouville 1-form $\lambda = pdq$ is the basic example of a symplectic manifold $(T^*M, d\lambda)$. Choose $\beta > 0$ and introduce the *Riemannian Hamiltonian* function $G : T^*M \rightarrow \mathbb{R}$,

$$G(q, p) := \beta g_q^*(p, p), \quad (2.1)$$

which will be denoted by $G(q, p) =: \beta |p|_{g^*}^2 =: \beta |p|^2$. Let $q, q' \in M$, and consider the space of paths $\gamma : [0, 1] \rightarrow M$ of Sobolev class-(1, 2) from q to q' ,

$$\Omega_{q, q'}^1 M = \{ \gamma \in W^{1,2}([0, 1], M) \mid \gamma(0) = q, \gamma(1) = q' \}, \quad (2.2)$$

as well as the space of continuous paths from q to q'

$$\Omega_{q, q'} M := \{ \gamma \in C([0, 1], M) \mid \gamma(0) = q, \gamma(1) = q' \}.$$

Recall the definition of the energy functional $\mathcal{E} : \Omega_{q, q'}^1 M \rightarrow \mathbb{R}$,

$$\mathcal{E}(\gamma) := \frac{1}{2} \int_0^1 |\dot{\gamma}(t)|_g^2 dt \quad (2.3)$$

and of the length $\mathcal{L} : \Omega_{q, q'}^1 M \rightarrow \mathbb{R}$ of such a path,

$$\mathcal{L}(\gamma) := \int_0^1 |\dot{\gamma}(t)|_g dt. \quad (2.4)$$

For $a > 0$ we consider the sublevel sets

$$\mathcal{E}^a(q, q') := \{ \gamma \in \Omega_{q, q'}^1 M \mid \mathcal{E}(\gamma) \leq a \} \quad (2.5)$$

as well as

$$\Omega_{q,q'}^{1,a}M := \{\gamma \in \Omega_{q,q'}^1M \mid \mathcal{L}(\gamma) \leq a\}.$$

2.1.1 The action functional and the time

A Hamiltonian function is a smooth function on a smooth manifold. Choose a Hamiltonian function $H : T^*M \rightarrow \mathbb{R}$ on the symplectic manifold $(T^*M, d\lambda)$. The functional $\mathcal{A}_H : \Omega_{q,q'}^1T^*M \rightarrow \mathbb{R}$,

$$\mathcal{A}_H(\gamma) := \int \gamma^*\lambda - \int_0^1 H(\gamma(t)) dt \quad (2.6)$$

is called the *action*, and the *reduced action* or the *time* is defined by

$$\mathcal{T}(\gamma) := \mathcal{A}(\gamma) := \int \gamma^*\lambda. \quad (2.7)$$

We look for paths $\gamma : [0, 1] \rightarrow T^*M$ with $\gamma(0) \in T_q^*M$ and $\gamma(1) \in T_{q'}^*M$ solving *Hamilton's equations*

$$\dot{\gamma} = J\nabla H(\gamma(t)) = X_H(\gamma(t)), \quad X_H := J\nabla H, \quad (2.8)$$

and denote by $\mathcal{P}(H, q, q')$ the set of all such solutions. The vector field $X_H : T^*M \rightarrow T(T^*M)$ is called the *Hamiltonian vector field* of H . This vector field has a flow called the *Hamiltonian flow* φ_H^t . The action functional (2.6) is C^∞ -smooth, and its critical points are precisely the elements of the space $\mathcal{P}(H, q, q')$ of C^∞ -smooth paths $\gamma : [0, 1] \rightarrow T^*M$ solving (2.8). If we specify an action bound $\mathcal{A}_H(\gamma) \leq C \in \mathbb{R}$, we denote the set of solutions which satisfy this action bound by $\mathcal{P}^C(H, q, q') \subset \mathcal{P}(H, q, q')$.

Convention 2.1.2. *Throughout this thesis we will consider a proper Hamiltonian function $K : T^*M \rightarrow \mathbb{R}$; then its Hamiltonian flow φ_K^t exists for all times.*

2.1.2 The spherization and fiberwise starshaped hypersurfaces

We are interested in finding solutions of the Hamiltonian equations (2.8). These solutions are Hamiltonian flow lines lying in certain hypersurfaces of the cotangent bundle. Let us describe these so-called fiberwise starshaped hypersurfaces.

The positive real numbers $c \in \mathbb{R}_+$ freely act on the cotangent bundle T^*M by $\nu_c : T^*M \rightarrow T^*M, (q, p) \mapsto (q, cp)$. On T^*M there is the Liouville 1-form $\lambda = pdq$ and we have $\nu_c^*(\lambda) = c\lambda$, so λ does not descend to the quotient $S^*M := T^*M/\mathbb{R}_+$, but the kernel does since $\ker(c\lambda) = \ker(\lambda) =: \xi$. The contact manifold (S^*M, ξ) is called the *spherization* of the cotangent bundle T^*M . The choice of a nowhere vanishing 1-form α on S^*M with $\ker(\alpha) = \xi$ (called the *contact form*) defines a vector field R_α , called the *Reeb vector field* of α , by the two conditions

$$d\alpha(R_\alpha, \cdot) = 0, \quad \alpha(R_\alpha) = 1.$$

Its flow φ_α^t is called the *Reeb flow* of α . A *Reeb chord* is a flow line of φ_α^t .

To give a more concrete description of the manifold (S^*M, ξ) and the flows φ_α^t , consider a fiberwise starshaped hypersurface Σ of T^*M . We think of it as a smooth hypersurface which is fiberwise starshaped with respect to the zero-section: For every $q \in M$ the set $\Sigma_q = \Sigma \cap T_q^*M$ bounds a set D_q in T^*M , i.e. $\partial D_q = \Sigma_q$, that is strictly starshaped with respect to the origin $0_q \in T_q^*M$. The restriction $\lambda|_\Sigma$ of the Liouville 1-form on T^*M to Σ is a contact form for the contact structure $\xi_\Sigma = \ker(\lambda|_\Sigma)$ on Σ , that gives it the structure of a contact manifold. The diffeomorphism $\Psi : \Sigma \rightarrow S^*M, (q, p) \mapsto \left(q, \frac{p}{\|p\|} \cdot \mathbb{R}_+\right)$ obtained by radial projection is a contactomorphism, so (Σ, ξ_Σ) and (S^*M, ξ) are contactomorphic. One can show that there is a bijection from the set of Reeb flows on (S^*M, ξ) to the set of Reeb flows φ_Σ^t on the set of fiberwise starshaped hypersurfaces Σ in T^*M . This equivalence gives two ways to study the counting function $\text{CF}_{q, q', \Sigma}$ introduced in the Section 2.1.3 from a dynamical point of view.

On the other hand, one can describe our problem in a more geometrical way. We follow Hofer and Zehnder [25, Chapter 4]. Let $\Sigma \subset T^*M$ be a submanifold of the cotangent bundle of codimension one. The cotangent bundle T^*M together with the standard symplectic structure $\omega := d\lambda$ is a symplectic manifold. If we restrict ω to vector fields in $T\Sigma \subset T(T^*M)$, i.e. if we restrict the 2-form ω to the odd-dimensional subspaces $T_x\Sigma \subset T_x(T^*M)$ for $x \in \Sigma$, then ω is necessarily degenerate. The kernel of

this restriction is therefore of dimension one. This defines a line bundle, $\mathcal{L}_\Sigma \subset T\Sigma$,

$$\mathcal{L}_\Sigma = \{(x, \xi) \in T_x\Sigma \mid \omega_x(\xi, \eta) = 0, \forall \eta \in T_x\Sigma\}.$$

The line bundle \mathcal{L}_Σ is called the *characteristic line bundle* of the hypersurface Σ . This line bundle gives the direction of every Hamiltonian vector field X_H having Σ as a regular energy surface, i.e. if $H : T^*M \rightarrow \mathbb{R}$ is constant on Σ and $dH \neq 0$ on Σ then $X_H(x) \in \mathcal{L}_\Sigma(x)$ for $x \in \Sigma$. Note that \mathcal{L}_Σ is determined by the hypersurface Σ and by the symplectic structure ω , hence by geometric quantities. Note that if Σ is interpreted as a contact manifold as above it holds true that the associated Reeb vector field R lies also in \mathcal{L}_Σ and satisfies trivially $\lambda|_\Sigma(R) = 1$. A *characteristic* of Σ , or a solution of $\dot{x}(t) = X_H(x(t))$, going from one point to another one *on* Σ , is an embedded open interval $I \subset \Sigma$ satisfying

$$TI \subset \mathcal{L}_\Sigma|_I.$$

The set of characteristics of Σ agrees with the set of unparameterized solutions solving Hamilton's equations for every Hamiltonian vector field X_H on Σ having Σ as a regular energy surface. So these characteristics agree also with the traces of Reeb chords on the hypersurface Σ .

For more details and examples of fiberwise starshaped hypersurfaces, we refer to Section 2.3.

2.1.3 The counting function $\text{CF}_{q,q',\Sigma}$

Consider the length spectrum of a Riemannian manifold (M, g) given by

$$\sigma_{q,q'}(g) = \{\text{lengths of all geodesic segments from } q \text{ to } q'\}.$$

In this definition the length is induced by the Riemannian metric g . The study of the length spectrum of a Riemannian manifold is an important problem, with many results and many open questions, see Berger [11, Chapter 10] and Paternain [42, Chapter 5].

Here we are interested in a more general problem: We study the spectrum of a fiberwise starshaped hypersurface $\Sigma \subset T^*M$ of the cotangent bundle T^*M by interpreting

it as a contact manifold. Consider the set (the time spectrum of Σ , see Section 2.2.2)

$$\mathcal{S}(\Sigma, q, q') := \{\mathcal{T}(\gamma) \mid \gamma \text{ a path on } \Sigma \text{ from } \Sigma_q \text{ to } \Sigma_{q'} \text{ solving } \dot{\gamma} = R(\gamma)\},$$

where the number $\mathcal{T}(\gamma)$ is the time (or the reduced action of γ), see (2.7), needed by the Reeb chord γ to go from Σ_q to $\Sigma_{q'}$ and R is the Reeb vector field of the contact manifold $(\Sigma, \lambda|_{\Sigma})$. Consider the *counting function*

$$\text{CF}_{q,q',\Sigma}(T) = \#\{\tau \in \mathcal{S}(\Sigma, q, q') \mid \tau \leq T\}. \quad (2.9)$$

If one fixes a time \bar{T} , then the counting function gives the number $\text{CF}_{q,q',\Sigma}(\bar{T})$ of Reeb chords starting in the fiber Σ_q and ending in some point of $\Sigma_{q'}$ before or with the time \bar{T} .

Notation (Growth type). Given functions $f, g: [0, \infty) \rightarrow [0, \infty) \cup \{+\infty\}$, we write $f \succcurlyeq g$ if there exist constants C, c such that $f(t) \geq g(Ct) + c$ for all $t \geq 0$. Moreover, we write $f \approx g$ if $f \succcurlyeq g$ and $g \succcurlyeq f$.

We say that f has *linear growth* if $f(t) \approx t$, that f has *polynomial growth* if $p \succcurlyeq f$ for some polynomial p , and that f has *exponential growth* if $f(t) \approx e^t$. We say that f and g have the same growth type if $f \approx g$. \diamond

In the subsequent chapters we will study the growth type of the counting function $\text{CF}_{q,q',\Sigma}$ for different choices of base manifolds M (of T^*M). This is done by deriving lower bounds for $\text{CF}_{q,q',\Sigma}$ from *homological visible* geodesic segments of (M, g) , see Definition 4.1.18.

2.2 Lagrangian Floer Homology and the Sandwiching method

In this section we give a short summary of those parts of Lagrangian Floer homology used later on. We follow Macarini–Schlenk [30] and modify the ideas slightly to get a suitable formulation for our purposes.

2.2.1 The Hamiltonians $G_- \leq K \leq G_+$

Let $\Sigma \subset T^*M$ be a fiberwise starshaped hypersurface. This property allows one to define a function $F: T^*M \rightarrow \mathbb{R}$ by the two requirements

$$F|_{\Sigma} \equiv \frac{1}{2}, \quad F(q, sp) = s^2 F(q, p) \quad \text{for all } s \geq 0 \text{ and } (q, p) \in T^*M.$$

This function is fiberwise homogenous of degree 2. (If Σ was not fiberwise starshaped, homogeneity of F would not make sense.) Further, F is of class C^1 and moreover smooth off the zero section. To make it smooth, we introduce another smooth function $f: \mathbb{R} \rightarrow \mathbb{R}$, where ε shall be fixed appropriately later on. See Figure 2.1.

$$\begin{cases} f(r) = 0, & r \leq \varepsilon^2 \\ f(r) = r, & r \geq \varepsilon \\ f'(r) > 0, & r > \varepsilon^2 \\ 0 \leq f'(r) \leq 2, & \forall r \in \mathbb{R}. \end{cases}$$

Then $f \circ F: T^*M \rightarrow \mathbb{R}$ is smooth. Let $G: T^*M \rightarrow \mathbb{R}$ be the usual Riemannian

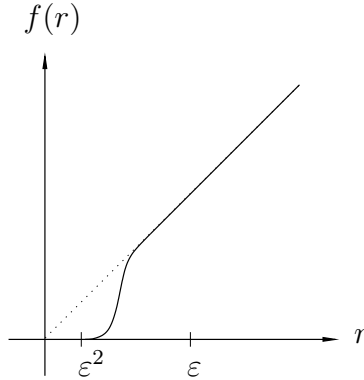


Figure 2.1: The “cut off” function f .

Hamiltonian $G(q, p) = \frac{1}{2}|p|^2$. Multiply the Riemannian metric g of M by a positive constant, so the value $G(q, p) \in \mathbb{R}$ gets scaled independently of (q, p) . Hence, we can assume $F(q, p) \geq G(q, p)$, $\forall (q, p) \in T^*M$, where the inequality shall be sharp, i.e. there exists a point $(\bar{q}, \bar{p}) \in T^*M$ such that $F(\bar{q}, \bar{p}) = G(\bar{q}, \bar{p})$. We abbreviate this by $F \geq G$.

Then we can choose a positive constant $\sigma_g \geq 1$ such that $\sigma_g G \geq F$. The constant $\sigma_g = \sigma_g(\Sigma)$ is called the *module of starshapedness of Σ* . We refer to Section 2.2.4 for more on this geometric quantity.

To construct Lagrangian Floer homology, we need the following definitions, see [30] for details. Consider the r -disc

$$D(r) = \{(q, p) \in T^*M \mid |p| \leq r\},$$

and fix $b > 0$. Choose a smooth function $\tau : \mathbb{R} \rightarrow \mathbb{R}$

$$\begin{cases} \tau(r) = 0 & \text{if } r \leq \sqrt{2b}, \\ \tau(r) = 1 & \text{if } r \geq 2\sqrt{2b}, \\ \tau'(r) \geq 0 & \text{for all } r \in \mathbb{R}. \end{cases}$$

Moreover, we define the following three Hamiltonians $G_-, K, G_+ : T^*M \rightarrow \mathbb{R}$

$$\begin{aligned} G_+(q, p) &= \sigma_g G(q, p), \\ K(q, p) &= (1 - \tau(|p|))(f \circ F)(q, p) + \tau(|p|)G_+(q, p), \\ G_-(q, p) &= (1 - \tau(|p|))(f \circ G)(q, p) + \tau(|p|)G_+(q, p). \end{aligned} \tag{2.10}$$

So $G_- \leq K \leq G_+$ and $K = f \circ F$ and $G_- = f \circ G$ on $\{G \leq b\}$. Since $\{F \leq b\} \subset \{G_- \leq b\}$, we in particular have

$$K = f \circ F \quad \text{on } \{F \leq b\}.$$

Moreover,

$$G_- = K = G_+ \quad \text{outside } \{G \geq 4b\}.$$

Figure 2.2 illustrates these Hamiltonians. Consider the space of Hamiltonian functions

$$\mathcal{H}_{\sqrt{8b}}(G_+) := \left\{ H : T^*M \rightarrow \mathbb{R} \mid H \in C^\infty(T^*M), H = G_+ \text{ on } T^*M \setminus D(\sqrt{8b}) \right\}.$$

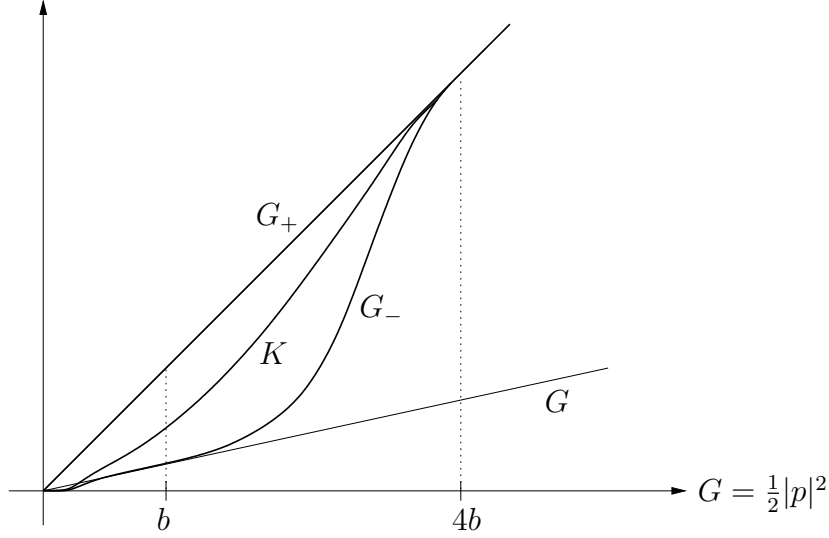


Figure 2.2: The functions $G_- \leq K \leq G_+$, schematically.

Fix now $q \in M$. Define

$$D_q(\sqrt{8b}) := \{p \in T_q^*M \mid |p| \leq \sqrt{8b}\} \subset D(\sqrt{8b}) \cap T^*M.$$

Given $H \in \mathcal{H}_{\sqrt{8b}}(G_+)$ let

$$U(q, H) := \left\{ q' \in M \mid \varphi_H^1 \left(D_q(\sqrt{8b}) \right) \text{ and } D_{q'}(\sqrt{8b}) \text{ intersect transversely.} \right\} \quad (2.11)$$

The set $U(q, H)$ is open and of full measure in M , see [30]. Let us define the following set which is also of full measure in M , again taken from [30]:

$$V(q) := U(q, G_-) \cap U(q, K) \cap U(q, G_+). \quad (2.12)$$

Definition 2.2.1 (Σ -conjugate points). *Fix $q \in M$ and assume $\Sigma = H^{-1}(\beta)$ for a regular value β of the Hamiltonian H . We say that $q' \in M$ is Σ -conjugate to q if $q' \notin U(H, q)$. (Then it follows trivially that $q' \notin V(q)$.)*

2.2.2 Hamiltonian action spectra

The action spectrum $\mathcal{S}(H, q, q')$ of a (proper) Hamiltonian function $H: T^*M \rightarrow \mathbb{R}$ is the set of critical values of $\mathcal{A}_H: \Omega_{q, q'}^1 T^*M \rightarrow \mathbb{R}$,

$$\mathcal{S}(H, q, q') := \{\mathcal{A}_H(x) \mid x \in \mathcal{P}(H, q, q')\}.$$

For $b \in \mathbb{R}$ define the subsets $\mathcal{S}^b(H, q, q') := \mathcal{S}(H, q, q') \cap (-\infty, b]$.

Let again $F: T^*M \rightarrow \mathbb{R}$ be the function with $F^{-1}(\frac{1}{2}) = \Sigma$ that is fiberwise homogeneous of degree 2, and denote by $\pi: T^*M \rightarrow M$ the projection along the fibers. We denote by $\mathcal{R}(\Sigma, q, q')$ the set of Reeb chords from Σ_q to $\Sigma_{q'}$. Now we define the *time spectrum* of Σ :

$$\mathcal{S}(\Sigma, q, q') := \{\mathcal{A}(\gamma) \mid \gamma \in \mathcal{R}(\Sigma, q, q')\}.$$

Similar to the action spectrum, we set $\mathcal{S}^b(\Sigma, q, q') := \mathcal{S}(\Sigma, q, q') \cap (-\infty, b]$.

Lemma 2.2.2. *Fix $\gamma \in \mathcal{P}(F, q, q')$.*

(i) $\mathcal{A}_F(\gamma) = F(\gamma)$.

(ii) *The time of the unique Reeb chord $\tilde{\gamma} \in \mathcal{R}(\Sigma, q, q')$ for which the trace of $\pi \circ \tilde{\gamma}$ equals the trace of $\pi \circ \gamma$ is $\mathcal{A}(\tilde{\gamma}) = \sqrt{2F(\gamma)}$.*

(iii) *In particular, $\mathcal{S}^{\sqrt{2b}}(\Sigma, q, q') = \mathcal{S}^b(F, q, q')$ for every $b > 0$.*

Proof. For point (i), see the proof of Lemma 3.1. of [30] with $h: x \mapsto x$. Concerning point (ii), Proposition 4.8.4 implies

$$\gamma^t(x) = \tilde{\gamma}^{\sigma(t, x)}(x),$$

where the function $\sigma(t, x)$ is of the form $\sigma(t, x) = s(x)t$ with $s(x) > 0$ constant in the t variable, see the proof of Proposition 4.8.4. Let us calculate,

$$\begin{aligned} \mathcal{A}(\tilde{\gamma}) &= \int_0^1 \lambda(\dot{\tilde{\gamma}}(t)) dt \\ &= \frac{1}{s(x)} \int_0^1 \lambda(\dot{\tilde{\gamma}}(s(x)t)) dt \end{aligned}$$

$$= \frac{1}{s(x)}.$$

The second step is a change of variables and the third step follows because $\tilde{\gamma}$ is a Reeb chord. On the other hand, we know that F is fiberwise homogeneous of degree two, and in addition that $F|_{\Sigma} = \frac{1}{2}$. Calculate,

$$\frac{1}{2} = F(\tilde{\gamma}(t)) = F((q(t), s(x)p(t))) = s(x)^2 F(\gamma),$$

what implies that $s(x) = \frac{1}{\sqrt{2F(\gamma)}}$, therefore $\mathcal{A}(\tilde{\gamma}) = \sqrt{2F(\gamma)}$.

The last point (iii) is a direct consequence of point (ii). \square

Now fix $a < b$ (where $a \leq 0$ is not excluded). We can choose $\varepsilon > 0$ in the definition of the function f so small that for every non-constant $\gamma \in \mathcal{P}^b(f \circ F, q, q')$ we have $(f \circ F)(\gamma) \geq \varepsilon$, and for every non-constant $\gamma \in \mathcal{P}^b(F, q, q')$ we have $F(\gamma) \geq \varepsilon$. Since $f(r) = r$ for $r \geq \varepsilon$, we then have

$$\mathcal{S}^{(a,b)}(f \circ F, q, q') = \mathcal{S}^{(a,b)}(F, q, q'). \quad (2.13)$$

Furthermore, Proposition 3.2 in [30] shows that $\gamma \in \mathcal{S}^b(K, q, q')$ if and only if $\gamma \subset \{F \leq b\}$. Since $K = f \circ F$ on $\{F \leq b\}$, we conclude with Lemma 2.2.2 (iii) and (2.13) that

Lemma 2.2.3. $\mathcal{S}^{(\sqrt{2a}, \sqrt{2b})}(\Sigma, q, q') = \mathcal{S}^{(a,b)}(F, q, q') = \mathcal{S}^{(a,b)}(K, q, q')$ for all $a, b \in \mathbb{R}$.

This lemma generalizes to our Hamiltonians F and K the well-known fact that a geodesic path of length ℓ has energy $\frac{1}{2}\ell^2$, see Lemma C.3.4.

2.2.3 From the Homology of $\Omega_{q,q'}^1 M$ to the time spectrum of Σ

Let $\Sigma \subset T^*M$ be a fiberwise starshaped hypersurface, and let $K: T^*M \rightarrow \mathbb{R}$ be the function constructed in Section 2.2.1. Denote by $\Omega_{q,q',\alpha}^1 M$ the set of $W^{1,2}$ -paths $q: [0, 1] \rightarrow M$ with $q(0) = q$ and $q(1) = q'$ that lie in the homotopy class α . We often drop q, q' from the notation. Given a Riemannian metric g on M , the energy functional $\mathcal{E}_g: \Omega_{q,q',\alpha}^1 M \rightarrow \mathbb{R}$ is defined by

$$\mathcal{E}_g(q) = \frac{1}{2} \int_0^1 g(\dot{q}(t), \dot{q}(t)) dt.$$

For $a < b$ consider the subsets

$$\mathcal{E}^a(q, q', \alpha) = \{q \in \Omega_{q, q', \alpha}^1 M \mid \mathcal{E}_g(q) \leq a\}$$

and the quotient space

$$\mathcal{QE}^{(a, b]}(q, q', \alpha) = \mathcal{E}^b(q, q', \alpha) / \mathcal{E}^a(q, q', \alpha).$$

Proposition 2.2.4. *Fix two points $q, q' \in M$ such that q' is not Σ -conjugate to q . Let g be a Riemannian metric on M such that q, q' are not g -conjugate. Scale g such that $G \leq F$. Let $a < b$ and $\sigma \geq \sigma_g \geq 1$ be such that*

$$a, b, a/\sigma, \sigma b \notin \mathcal{S}(G, q, q') \quad \text{and} \quad a/\sigma, b \notin \mathcal{S}(F, q, q').$$

Then for any field \mathbb{F} the number of Reeb chords from Σ_q to $\Sigma_{q'}$ in class α with action in $(\sqrt{2a/\sigma}, \sqrt{2b}]$ is bounded from below by the rank of the homomorphism induced by inclusion

$$\tilde{H}_*(\mathcal{QE}^{(a/\sigma, b]}(q, q', \alpha); \mathbb{F}) \rightarrow \tilde{H}_*(\mathcal{QE}^{(a, \sigma b]}(q, q', \alpha); \mathbb{F}).$$

By \tilde{H}_* we denote reduced homology.

Remark. Given a, b, σ, g we find a', b', σ', g' as close to a, b, σ, g as we like and such that a', b', σ', g' meet the hypothesis of the proposition. Indeed, the complement of $\mathcal{S}^b(G, q, q') \cup \mathcal{S}^b(F, q, q')$ in \mathbb{R} is open and dense. \diamond

Proof of Proposition 2.2.4. We throughout fix $q, q' \in M$, a, b, σ and g as in the proposition, and also fix the field \mathbb{F} . The proof is based on Floer homology for Lagrangian intersections. We only recall those properties of Lagrangian Floer homology that we use in the proof, and refer to Section 4 of [30] and the references therein for more details.

Let $K: T^*M \rightarrow \mathbb{R}$ be the function constructed in Section 2.2.1. (It depends on a, b, σ_g, g .) The Floer chain group $\text{CF}^b(K, \alpha)$ is the \mathbb{F} -vector space freely generated by the chords in $\mathcal{P}^b(K, q, q', \alpha)$. The Conley–Zehnder index (see Section 6.2) of these chords (normalized such that it agrees with the Morse index in case of a non-degenerate geodesic chord) gives this vector space a grading $*$. The Floer boundary operator on

$\text{CF}_*^b(K, \alpha)$ is a map of degree -1 . Its homology is the Floer homology $\text{HF}_*^b(K, \alpha)$. Since the boundary operator maps $\text{CF}_*^b(K, \alpha)$ to itself, it descends to the quotient groups $\text{CF}_*^{(a,b]}(K, \alpha) = \text{CF}_*^b(K, \alpha) / \text{CF}_*^a(K, \alpha)$. The resulting homology is denoted $\text{HF}_*^{(a,b]}(K, \alpha)$. The Floer homology of the functions G_-, G_+ is defined in the same way. There is a commutative diagram

$$\begin{array}{ccccc}
\text{HF}_*^b(G_-, \alpha) & \xrightarrow{\Phi_{G_-G_+}} & \text{HF}_*^{b/\sigma}(G_+, \alpha) & \xrightarrow{\text{ASM}} & \text{H}_*(\mathcal{E}^b(\alpha)) \\
\Phi_{G_-K} \downarrow & & \downarrow & & \downarrow \\
\text{HF}_*^b(K, \alpha) & \xrightarrow{\Phi_{KG_+}} & \text{HF}_*^b(G_+, \alpha) & \xrightarrow{\text{ASM}} & \text{H}_*(\mathcal{E}^{\sigma b}(\alpha)) .
\end{array} \tag{2.14}$$

Here, the three maps Φ between Floer homologies are Floer continuation maps, and $\Phi_{G_-G_+}$ is an isomorphism. The upper map ASM is the composition

$$\text{HF}_*^{b/\sigma}(G_+, \alpha) \xrightarrow{\text{AS}} \text{HM}_*^{b/\sigma}(L, \alpha) \xrightarrow{\text{AM}} \text{H}_*(\mathcal{E}^b(\alpha))$$

of the Abbondandolo–Schwarz isomorphism from Floer homology to the Morse homology [3] of the Legendre transform L of G_+ with the Abbondandolo–Mayer isomorphism from this homology to the homology of $\Omega_\alpha^{1,b}M$ [2]. Finally, the two unlabeled vertical arrows are induced by inclusion.

For the left part of this diagram it is important that the boundaries of the action windows do not belong to the spectrum of the Hamiltonian functions. This is guaranteed by our assumptions: The definition (2.10) of G_- implies that $\mathcal{A}_{G_-}(\gamma) > b$ if $\gamma \in \mathcal{P}(G_-, q, q')$ lies outside $\{G \leq b\}$. Since $G_- = f \circ G$ on $\{G \leq b\}$, we thus have $\mathcal{S}^b(G_-, q, q') = \mathcal{S}^b(G, q, q')$ and hence $\mathcal{S}(G_-, q, q')$. Moreover, $b \notin \mathcal{S}(F, q, q')$ by assumption, whence $b \notin \mathcal{S}(K, q, q')$ by Lemma 2.2.3. Finally, $b/\sigma, b \notin \mathcal{S}(G_+, q, q')$ because $b, \sigma b \notin \mathcal{S}(G, q, q')$ and $\mathcal{S}(G_+, q, q') = \mathcal{S}(\sigma G, q, q') = \frac{1}{\sigma}\mathcal{S}(G, q, q')$.

The above diagram holds true with b replaced by any $c \leq b$, provided that again the boundaries of the action intervals do not belong to the spectrum of the Hamiltonian functions. This is clear for $c \leq 0$, and it holds for $c \in (0, b]$ in view of the computation in the proof of Lemma 3.3 of [30], provided we choose $\varepsilon = \varepsilon(c)$ in the the definition of f small enough. In particular, our assumptions imply that the diagram holds with b replaced by a/σ . Since the homomorphisms in the above diagram are all defined at the

chain level, we then obtain the commutative diagram

$$\begin{array}{ccccc}
\mathrm{HF}_*^{(a/\sigma, b]}(G_-, \alpha) & \xrightarrow{\cong} & \mathrm{HF}_*^{(a/\sigma^2, b/\sigma]}(G_+, \alpha) & \xrightarrow{\cong} & \tilde{\mathrm{H}}_* (\mathcal{QE}^{(a/\sigma, b]}(\alpha)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{HF}_*^{(a/\sigma, b]}(K, \alpha) & \longrightarrow & \mathrm{HF}_*^{(a/\sigma, b]}(G_+, \alpha) & \longrightarrow & \tilde{\mathrm{H}}_* (\mathcal{QE}^{(a, \sigma b]}(\alpha)) .
\end{array} \tag{2.15}$$

By $\tilde{\mathrm{H}}_*$ we denote reduced homology. It follows that the cardinality of $\mathcal{P}^{(a/\sigma, b]}(K, \alpha)$ is at least the rank of the right vertical map. The theorem follows together with Lemma 2.2.3. \square

For later reference, we state the “absolute case” separately:

Proposition 2.2.5. *Fix two points $q, q' \in M$ such that q' is not Σ -conjugate to q . Let g be a Riemannian metric on M such that q, q' are not g -conjugate. Scale g such that $G \leq F$. Let $b > 0$ be such that*

$$b \notin \mathcal{S}(G, q, q') \cup \mathcal{S}(F, q, q') .$$

Then for any field \mathbb{F} the number of Reeb chords from Σ_q to $\Sigma_{q'}$ in class α with action in $[0, \sqrt{2b}]$ is bounded from below by the rank of the homomorphism induced by inclusion

$$\mathrm{H}_* (\mathcal{E}^b(q, q', \alpha); \mathbb{F}) \rightarrow \mathrm{H}_* (\mathcal{E}^{\sigma b}(q, q', \alpha); \mathbb{F}) .$$

Remark. In Proposition 2.2.4 we assumed in addition, this in contrast to the situation in Proposition 2.2.5, that $b/\sigma \notin \mathcal{S}(G, q, q')$. Note that this condition is implicitly satisfied as it follows directly from Macarini–Schlenk [30, Proposition 3.3]. \diamond

2.2.4 The module of starshapedness σ_g

In Section 2.2.1 the *module of starshapedness of Σ* was introduced. We give a geometric interpretation. By assumption it holds true that $\Sigma = F^{-1}(\frac{1}{2})$. Comparing this set to $G^{-1}(\frac{1}{2})$ and to $(\sigma_g G)^{-1}(\frac{1}{2})$ shows that these hypersurfaces are nested or “sandwiched”: First, Σ lies – by touching its boundary at least at one point – in the bounded part of $G^{-1}(\frac{1}{2})$, while $(\sigma_g G)^{-1}(\frac{1}{2})$ is enclosed by Σ . If $S_r^*M = \{(q, p) \in T^*M \mid |p| = r\}$ is the r -

co-sphere bundle, we can see $G^{-1}\left(\frac{1}{2}\right) = \{(q, p) \in T^*M \mid (q, p') \in S_1^*M \text{ and } p = r_{\text{outer}} p'\}$, so $G^{-1}\left(\frac{1}{2}\right) = S_{r_{\text{outer}}}^*M$, and similarly we get $(\sigma_g G)^{-1}\left(\frac{1}{2}\right) = S_{r_{\text{inner}}}^*M$ for the minimal $r_{\text{outer}} > 0$ and the maximal $r_{\text{inner}} > 0$. So, for $x = (q, r_{\text{outer}} p') \in S_{r_{\text{outer}}}^*M$ we have $G(x) = \frac{1}{2}r_{\text{outer}}^2 = \frac{1}{2}\left|r_{\text{outer}}\frac{r_{\text{inner}}}{r_{\text{inner}}}p'\right|^2 = \left(\frac{r_{\text{outer}}}{r_{\text{inner}}}\right)^2 \frac{1}{2}|r_{\text{inner}}p'|^2 = \sigma_g G(\cdot, r_{\text{inner}} p')$. Therefore, $\sigma_g = \left(\frac{r_{\text{outer}}}{r_{\text{inner}}}\right)^2$. If a result does not depend on the actual choice of the Riemannian metric g , one can choose σ_g to be independent of g by considering the following (smaller) constant of starshapedness: $\sigma := \inf_g \left\{ \left(\frac{r_{\text{outer}}}{r_{\text{inner}}}\right)^2 \right\}$. See Figure 2.3 for an illustration.

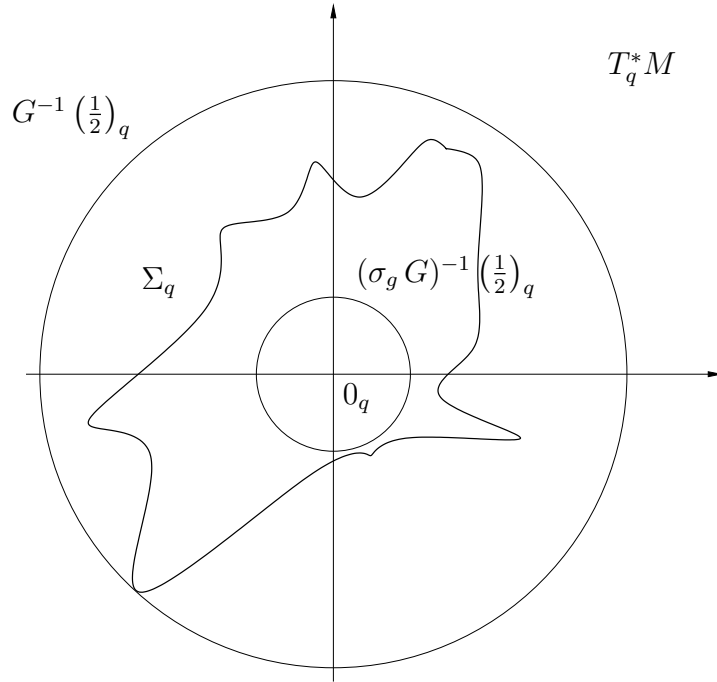


Figure 2.3: The sandwiching of Σ by the two co-sphere bundles $(\sigma_g G)^{-1}\left(\frac{1}{2}\right)$ and $G^{-1}\left(\frac{1}{2}\right)$ restricted to the fiber T_q^*M . Note that Σ_q must intersect $(\sigma_g G)^{-1}\left(\frac{1}{2}\right)_q$ and independently $G^{-1}\left(\frac{1}{2}\right)_q$ for at least one $q \in M$.

We next give a class of examples of fiberwise starshaped hypersurfaces for which one can calculate the module of starshapedness.

Examples. (Physical Hamiltonians). Consider a *physical Hamiltonian*

$$\begin{aligned} H_{\text{phys}}: T^*M &\longrightarrow \mathbb{R}, \\ (q, p) &\longmapsto H_{\text{phys}}(q, p) := \beta|p|^2 + V(q). \end{aligned}$$

One can understand this Hamiltonian as the sum of a Riemannian Hamiltonian (2.1)

and a smooth function $V: M \rightarrow \mathbb{R}$. If $V(q) < \frac{1}{2}$, then $\Sigma := H_{\text{phys}}^{-1}(\frac{1}{2})$ is fiberwise convex w.r.t. $0_q \in T_q^*M$. Since M is a closed manifold, V attains its maximum and minimum. Denote them by $V_{\max} := \max_{x \in M} V(x) < \frac{1}{2}$ and $V_{\min} := \min_{x \in M} V(x)$. For the maximal and the minimal momenta we compute

$$|p|_{\max}^2 = \frac{\frac{1}{2} - V_{\min}}{\beta}, \quad |p|_{\min}^2 = \frac{\frac{1}{2} - V_{\max}}{\beta}.$$

Hence,

$$\sigma_g = \left(\frac{r_{\text{outer}}}{r_{\text{inner}}} \right)^2 = \left(\frac{|p|_{\max}}{|p|_{\min}} \right)^2 = \frac{\frac{1}{2} - V_{\min}}{\frac{1}{2} - V_{\max}}.$$

(Co-sphere bundles). If the fiberwise starshaped hypersurface $\Sigma \subset T^*M$ is a co-sphere bundle, i.e. if it is given by $\Sigma = \{(q, p) \in T^*M \mid g_q(p, p) = r\}$ for some $r \in \mathbb{R}$, then $\sigma_g(\Sigma) = 1$. \diamond

2.3 Fiberwise Starshaped Hypersurfaces

This section is devoted to examples of fiberwise starshaped hypersurfaces in the cotangent bundle of a closed connected and finite-dimensional smooth manifold M .

Examples. We give three classes of fiberwise starshaped hypersurfaces on some base manifolds M .

(Co-sphere bundles). Let (M, g) be a closed connected Riemannian manifold. If the fiberwise starshaped hypersurface $\Sigma \subset T^*M$ is a co-sphere bundle, i.e. if it is given by $\Sigma = \{(q, p) \in T^*M \mid g_q(p, p) = r\}$ for some $r \in \mathbb{R}$, then $\Sigma_q = \Sigma \cap T_q^*M$ is a circle of radius \sqrt{r} in T_q^*M for every q .

(Level sets of physical Hamiltonians). Let H_{phys} be a physical Hamiltonian

$$\begin{aligned} H_{\text{phys}}: T^*M &\longrightarrow \mathbb{R}, \\ (q, p) &\longmapsto H_{\text{phys}}(q, p) := \beta|p|^2 + V(q). \end{aligned}$$

as it introduced in Section 2.2.4. Consider a regular level set Σ of H_{phys} . The set $\Sigma_q = \Sigma \cap T_q^*M$ is as before a circle in every fiber, but to the contrary, the radius of the circle depends smoothly on the base point q , $r = r(q)$.

(Fiberwise convex hypersurfaces). Let (M, F) be a closed connected Finsler manifold. The Finsler structure F leads to a Minkowski norm, sometimes also called an asymmetric norm, on every tangent space $T_q M$,

$$\begin{aligned} F_q: T_q M &\longrightarrow \mathbb{R}, \\ (q, v) &\longmapsto F(q, v). \end{aligned}$$

see for example Shen [47] for an exposition of Finsler geometry. In particular, F_q is a convex function since it is an asymmetric norm, meaning since $F_q(v) \neq F_q(-v)$ in general. According to Álvarez Paiva [4], one can consider the dual normed vector space $(T_q^* M, F_q^*)$, where

$$\begin{aligned} F_q^*: T_q^* M &\longrightarrow \mathbb{R}, \\ (q, p) &\longmapsto F_q^*(p) := \sup_{v \in T_q M} \{|p(v)| \mid F_q(v) \leq 1\}. \end{aligned}$$

Then, he defines a Hamiltonian function $H_F: T^* M \rightarrow \mathbb{R}$ by

$$\begin{aligned} H_F: T^* M &\longrightarrow \mathbb{R}, \\ (q, p) &\longmapsto H_F(q, p) := F_q^*(p). \end{aligned}$$

Level sets of H_F are fiberwise convex since level sets of convex functions are convex sets. Therefore, for any regular α the related level set $H_F^{-1}(\alpha)$ is fiberwise starshaped. \diamond

A norm of a closed fiberwise starshaped hypersurface. Denote by $FSH(T^* M)$ the set of closed fiberwise starshaped hypersurfaces of the cotangent bundle $T^* M$. Introduce the following multiplication on $FSH(T^* M)$. For a $r \in \mathbb{R}$ and $A \in FSH(T^* M)$, $\cdot : (r, A) \mapsto r \cdot A := \{(q, rp) \in A \mid (q, p) \in A\}$. Observe $r \cdot A \in FSH(T^* M)$. An addition is given as follows. Take $A, A' \in FSH(T^* M)$ and define $+$: $(A, A') \mapsto A + A' := \{(q, \{(s_i + r_i)e_i\}_{i=1}^n) \mid (q, \{s_i e_i\}_{i=1}^n) \in A, (q, \{r_i e_i\}_{i=1}^n) \in A'\}$, where the addition in the fibre variables componentwise. Again, $A + A' \in FSH(T^* M)$ due to definition. The “zero vector” is the null section M of $T^* M$. Therefore, $FSH(T^* M)$ admits the structure of an n -dimensional \mathbb{R} -vector space. A basis can be given by

$E_i := \{(q, p) \in T^*M \mid p_i = e_i\} \in FSH(T^*M)$, $i \in \{1, \dots, n\}$. Consider the quotient

$$QFSH(T^*M) := FSH(T^*M) / \{r\text{-cosphere bundles, } r \in \mathbb{R}\}.$$

On this vector space $QFSH(T^*M)$, the map

$$\begin{aligned} \mathcal{N}: QFSH(T^*M) &\longrightarrow \mathbb{R} \\ \Sigma &\longmapsto \mathcal{N}(\Sigma) := \max_{(q,p) \in \Sigma} |p| - \min_{(q,p) \in \Sigma} |p| \end{aligned}$$

is a norm. It is formally analogous to the Hofer norm introduced in the book by Hofer and Zehnder [25, Page 146]. Let $\Pi \in QFSH(T^*M)$. If Π has a certain norm, say δ , this means that Π can be “sandwiched” inbetween two spherical hypersurfaces such that the difference of their radii is equal to δ . By a spherical hypersurface of radius r we understand the following set: $S_r := \{(q, p) \in T^*M \mid |p| = r\} \in FSH(T^*M)$. It might be possible to deduce non-trivial relations relating the norm $\mathcal{N}(\Sigma)$ to its module of starshapedness $\sigma(\Sigma)$.

2.4 Action windows for limit solutions

Let us define the set of action bounded solutions. Fix $a, b \in \mathbb{R} \cup \{-\infty\}$ with $b > a$, $q, q' \in M$ and let $H: T^*M \rightarrow \mathbb{R}$ be a Hamiltonian satisfying the Convention 2.1.2.

$$\mathcal{P}_a^b(H, q, q') := \{\eta \in \mathcal{P}(H, q, q') \mid a \leq \mathcal{A}_H(\eta) \leq b\}. \quad (2.16)$$

If $a = -\infty$ we simply write $\mathcal{P}^b(H, q, q')$.

Notation (Action window). We call a closed interval $[a, b] \subset \mathbb{R}$ an *action window* for H and q, q' , if $a, b \in \mathbb{R} \setminus \mathcal{S}(H, q, q')$. \diamond

Define for $A \in \mathbb{R}$ the set of action bounded solutions of Hamilton’s equations associated to H :

$$Sol_A(M, H) := \{x \in W^{1,2}([0, 1], T^*M) \mid \mathcal{A}_H(x) \leq A \text{ and } \dot{x} = X_H(x(t))\}. \quad (2.17)$$

See Section C.3.1 for Definitions of the Sobolev spaces. Obviously, $\mathcal{P}^A(H, q, q') \subset \text{Sol}_A(M, H)$. Similarly,

$$\text{Sol}_{[A,B]}(M, H) := \{x \in W^{1,2}([0, 1], T^*M) \mid A \leq \mathcal{A}_H(x) \leq B \text{ and } \dot{x} = X_H(x(t))\} .$$

It is important to notice that we understand the space

$$C^0([0, 1], T^*M) := \{f \in C^0([0, 1], \mathbb{R}^{2K}) \mid f \subset \Phi(T^*M) \subset \mathbb{R}^{2K}\}$$

as the space $(C^0([0, 1], T^*M), \|\cdot\|_{C^0})$ equipped with the norm

$$\|f\|_{C^0} := \|f\|_{C^0([0,1], T^*M)} := \|f\|_{C^0([0,1], \mathbb{R}^{2K})} := \max_{t \in [0,1]} |f(t)|_{\mathbb{R}^{2K}} .$$

We refer to the discussion given in Section C.3.1. Let K be the Hamiltonian introduced in (2.10).

Lemma 2.4.1. *Fix $a, b \in \mathbb{R} \cup \{-\infty\}$ with $b > a$ and $q \in M$. Let $(q_n)_{n \in \mathbb{N}}$ be a sequence of points in M with $q_n \rightarrow q' \in M$ if $n \rightarrow \infty$. Further, let $(\gamma_n)_{n \in \mathbb{N}}$ be a sequence satisfying $\gamma_n \in \mathcal{P}_a^b(K, q, q_n)$ for all n . Then, there exists a $\gamma \in \mathcal{P}_a^b(K, q, q')$ and a subsequence $(\gamma_{n_m})_{m \in \mathbb{N}}$ such that $\gamma_{n_m} \rightarrow \gamma$ in the C^0 -sense as $m \rightarrow \infty$.*

Proof. By assumption, we have $\forall n \in \mathbb{N}$ a smooth path γ_n from Σ_q to Σ_{q_n} solving Hamilton's equations,

$$\dot{\gamma}_n(t) = X_K(\gamma_n(t)), \forall t \in [0, 1] \text{ and } a \leq \mathcal{A}_K(\gamma_n) \leq b .$$

Observe that $(\gamma_n)_{n \in \mathbb{N}} \subset \text{Sol}_{[a,b]}(M, K)$. So we can apply the Arzelà-Ascoli Theorem C.1.1 to the space $\text{Sol}_{[a,b]}(M, K)$, which is due to the Lemmas 4.8.2, 4.8.3 bounded and equicontinuous, hence relatively compact. Therefore there exists a path $\gamma \in C^0([0, 1], T^*M)$ such that for a subsequence $(\gamma_{n_m})_{m \in \mathbb{N}} \subset (\gamma_n)_{n \in \mathbb{N}}$ we can conclude $\gamma_{n_m} \rightarrow \gamma$ if $m \rightarrow \infty$. Now we have to show that $\gamma(0) \in \Sigma_q$ and $\gamma(1) \in \Sigma_{q'}$, that $\gamma \in C^\infty([0, 1], T^*M)$ and that it satisfies Hamilton's equations (2.8).

We know that there exists $\gamma \in C^0([0, 1], T^*M)$ such that $\lim_{m \rightarrow \infty} \gamma_{n_m} = \gamma$. Equiva-

lently, $\lim_{m \rightarrow \infty} \|\gamma_{n_m} - \gamma\|_{C^0} = 0$, which means by definition that

$$\lim_{m \rightarrow \infty} \max_{t \in [0,1]} |\gamma_{n_m}(t) - \gamma(t)|_{\mathbb{R}^{2K}} = 0,$$

so $\gamma_{n_m}(t)$ converges uniformly to $\gamma(t)$ on $[0, 1]$. This explains why $\gamma(0) \in \Sigma_q$ and $\gamma(1) \in \Sigma_{q'}$. Furthermore, the limit solution γ satisfies

$$a \leq \mathcal{A}_K(\gamma) \leq b,$$

this follows by contradiction. We know that X_K is a smooth vector field, hence it is continuous and therefore the sequence $X_K(\gamma_{n_m}(t))$ converges uniformly on $[0, 1]$ (because $\gamma_{n_m}(t)$ does and thus $\dot{\gamma}_{n_m}(t)$ too). For every $m \in \mathbb{N}$, we have

$$\dot{\gamma}_{n_m}(t) = X_K(\gamma_{n_m}(t)),$$

so, because of uniform convergence

$$\lim_{m \rightarrow \infty} \dot{x}_m(t) = \dot{x}_\infty(t),$$

and by continuity

$$\lim_{m \rightarrow \infty} X_{nK}(x_m(t)) = X_K(x_\infty(t)),$$

Therefore

$$\dot{\gamma}(t) = X_K(\gamma(t)).$$

We conclude that γ is in fact one time continuously differentiable. By bootstrapping we see even that $\gamma \in C^\infty([0, 1], T^*M)$, this means $\gamma \in \mathcal{P}_a^b(K, q, q')$. \square

Lemma 2.4.2. *Let $q \in M$, $q' \in U(q, K)$. Suppose that there exist functions $f_\Sigma, h: \mathbb{R} \rightarrow \mathbb{R}$ with $f_\Sigma(t) \rightarrow +\infty$ if $t \rightarrow +\infty$ such that*

$$\dim \bigoplus_{k \in \mathbb{Z}} \text{HF}_k^{h(t)}(K, q, q') \asymp f_\Sigma(t).$$

Fix $r \in \mathbb{R} \setminus \mathcal{S}(K, q, q')$. Then there exist numbers $a, b \in \mathbb{R} \setminus \mathcal{S}(K, q, q')$ satisfying

$b > a \geq r$ such that

$$\dim \bigoplus_{k \in \mathbb{Z}} \mathrm{HF}_k^{(a,b]}(K, q, q') > 0.$$

Proof. Fix $r \in \mathbb{R} \setminus \mathcal{S}(K, q, q')$. Recall that $\dim \mathrm{HF}_*^a(K, q, q')$ is finite for $a \in \mathbb{R} \setminus \mathcal{S}(K, q, q')$ because $\mathcal{P}^a(K, q, q')$ is a finite set. We refer to [30] and [3]. Then we know that $\dim \bigoplus_{k \in \mathbb{Z}} \mathrm{HF}_k^r(K, q, q')$ is finite. Due to assumption we know that there exist functions such that $\dim \bigoplus_{k \in \mathbb{Z}} \mathrm{HF}_k^{h(t)}(K, q, q') \asymp f_\Sigma(t)$ with $f_\Sigma(t) \rightarrow \infty$ for $(t \rightarrow \infty)$ and the claim follows. \square

Remark. Note that Lemma 2.4.2 can be used to deduce the existence of action windows: Due to [25] we know that the action spectrum $\mathcal{S}(K, q, q')$ is compact and nowhere dense in \mathbb{R} . Therefore, if we have found an interval $(a, b]$ as in the lemma, we know that there exists an $a' \in \mathbb{R} \setminus \mathcal{S}(K, q, q')$ with $a' > a$ such that for all Floer-Homology classes $[\gamma] \in \bigoplus_{k \in \mathbb{Z}} \mathrm{HF}_k^{(a,b]}(K, q, q')$ it follows that $\mathcal{A}_K(\gamma) \in [a', b]$. The latter interval is an action window for K, q and q' . \diamond

Lemma 2.4.3. *Let H be a Hamiltonian and consider for $k \in \mathbb{N}$ and for $\ell \in \{1, \dots, k\}$ k sequences $(x_\ell^i)_{i \in \mathbb{N}}$ of solutions of Hamilton's equations (4.19). Suppose there exists an $N \in \mathbb{N}$ such that for all $\ell_1, \ell_2 \in \{1, \dots, k\}$ with $\ell_1 \neq \ell_2$ it holds true that $\mathcal{A}_H(x_{\ell_1}^N) \neq \mathcal{A}_H(x_{\ell_2}^N)$. Then there exist k mutually disjoint action windows $[a_\ell, b_\ell]$, i.e. if $\ell_1 \neq \ell_2$ it follows that $[a_{\ell_1}, b_{\ell_1}] \cap [a_{\ell_2}, b_{\ell_2}] = \emptyset$, k subsequences $(x_\ell^{i_m})_{m \in \mathbb{N}}$ and k solutions γ_ℓ such that*

$$\mathcal{A}_H(x_\ell^{i_m}) \in [a_\ell, b_\ell]$$

for all $i_m \geq N$ with $x_\ell^{i_m} \rightarrow \gamma_\ell$ (for $m \rightarrow \infty$) and $\mathcal{A}_H(\gamma_\ell) \in [a_\ell, b_\ell]$.

Proof. The proof follows from the finiteness of k , the continuity of \mathcal{A}_H and the previous lemmas. \square

2.5 The growth of the fundamental group $\pi_1(M)$

Since M is a closed manifold, its fundamental group $\pi_1(M)$ is a finitely presented group. Consider, more generally, a finitely generated group G . Choose a finite set S of generators of G . For each positive integer m , let $\gamma_S(m)$ be the number of distinct elements

in G which can be written as words with at most m letters from $S \cup S^{-1}$. Hence $\gamma_S(m)$ is the number of vertices of the Cayley graph of G with respect to S that lie in the closed m -ball centered at the neutral element.

According to Milnor [33] we know that the following limit exists, but depends on the choice of S . It is called the *exponential growth*.

$$\nu(G, S) := \lim_{m \rightarrow +\infty} \frac{\log \gamma_S(m)}{m} \in [0, +\infty). \quad (2.18)$$

We say that G has *exponential growth* if $\nu(G, S) > 0$, i.e. if $\gamma_S(m) \asymp e^m$.

The *polynomial growth* of G is defined as

$$\gamma(G) := \limsup_{m \rightarrow +\infty} \frac{\log \gamma_S(m)}{\log m} \in [0, +\infty]. \quad (2.19)$$

This limit indeed does not exist necessarily, but it is independent of the set of generators S . One says that G has *polynomial growth* if $\gamma(G) < +\infty$. By Gromov's theorem from [21], a finitely generated group G has polynomial growth if and only if G has a nilpotent subgroup G_1 of finite index (that is, G is virtually nilpotent). The growth $\gamma(G)$ of these groups is highly computable: The equality $\gamma(G) = \gamma(G_1)$ follows from Corollary [18, Corollary IV.24] and the fact that the growth of groups is a quasi-isometry invariant. Let $(G_k)_{k \geq 1}$ be the lower central series of G_1 inductively defined by $G_{k+1} = [G_1, G_k]$. Then the Bass–Guivarc'h formula

$$\gamma(G_1) = \sum_{k \geq 1} k \dim((G_k/G_{k+1}) \otimes_{\mathbb{Z}} \mathbb{Q}) \quad (2.20)$$

holds true. We in particular have that $\gamma(G)$ is an integer.

Examples. If a closed manifold M admits a Riemannian metric of negative sectional curvature, then $\pi_1(M)$ has exponential growth, i.e. $\nu(\pi_1(M)) > 0$. We refer to Milnor [33, Theorem 2] for details. The fundamental group of the 2-torus $T^2 := \mathbb{R}^2/\mathbb{Z}^2$ is given by $\pi_1(T^2) = \mathbb{Z}^2$, therefore $\gamma_{\{(1,0),(0,1)\}}(m) = (m+1)^2 + 1$ and hence T^2 has polynomial growth $\gamma(\pi_1(T^2)) = 2$. Similarly for the d -torus $T^d := \mathbb{R}^d/\mathbb{Z}^d$ we get $\gamma(\pi_1(T^d)) = d$.

Let M be a manifold whose fundamental group is given by $\pi_1(M, \cdot) = \{\text{Id}, a\}$ which is torsion free, i.e. there exists no natural number ℓ such that $a^\ell = \text{Id}$. Then it follows that $\gamma(\pi_1(M, \cdot)) = 1$. ◇

Chapter 3

Asymptotic Results

In this chapter we state the asymptotic results and prove them. We will show exponential and polynomial growth of the growth function $CF_{q,q',\Sigma}$ under certain topological conditions imposed on the base manifold M . Further, we will prove linear growth of $CF_{q,q',\Sigma}$ without additional assumptions.

Let (M, g) be an n -dimensional closed connected Riemannian manifold and denote by $d := \text{diam } M$ the diameter of M . Consider a fiberwise starshaped hypersurface $\Sigma \subset T^*M$ in the cotangent bundle T^*M and a Hamiltonian function $F: T^*M \rightarrow \mathbb{R}$ homogeneous of degree two. As it was done for the explanation of the setting in Section 2.2, assume that $\frac{1}{2}$ is a regular value of F such that $\Sigma = F^{-1}(\frac{1}{2})$.

3.1 Exponential and polynomial growth of the number of solutions

If the fundamental group of M has exponential resp. polynomial growth, then we can show that the number of solutions of Hamilton's equations grows exponentially resp. polynomially in time, independently of the starting and the end point. For the notion of growth of groups we refer to Section 2.5.

Theorem 3.1.1 (Exponential and polynomial growth of the number of solutions). *Let $q \in M$. If $\pi_1(M, q)$ has exponential growth, then for every $q' \in M$ the number of orbits*

of the flow $\varphi_K^t|_\Sigma$ from Σ_q to $\Sigma_{q'}$ grows at least exponentially in time,

$$\text{CF}_{q,q',\Sigma}(t) \succcurlyeq e^t .$$

Analogously, if $\pi_1(M, q)$ has polynomial growth k , it follows that for every $q' \in M$ the counting function $\text{CF}_{q,q',\Sigma}$ grows at least polynomially in time,

$$\text{CF}_{q,q',\Sigma}(t) \succcurlyeq t^k .$$

Proof. Assume first that q' is not Σ -conjugate to q . Choose a Riemannian metric g on M such that $G \leq F$ and such that q, q' are not g -conjugate. Let $\beta_1^\pm, \dots, \beta_j^\pm$ be a set of generators of $\pi_1(M, q)$. Let ℓ_i be the length of a shortest curve in M representing β_i , and set $\ell = \max \{\ell_1, \dots, \ell_j\}$. Fix a path c from q to q' of length $\leq \text{diam}(M, g)$. Composition with c induces a bijection

$$\pi_1(M, q) \longrightarrow \pi_0(\Omega_{q,q'}^1 M) .$$

For $m \in \mathbb{N}$ let $\gamma_S(m)$ be the number of elements in $\pi_1(M, q)$ that can be written as words of length $\leq m$ in the generators $\beta_1^\pm, \dots, \beta_j^\pm$. Then we find $\gamma_S(m)$ elements α in different components of $\pi_0(\Omega_{q,q'}^1 M)$ that can be represented by geodesics of length $\leq \ell m + \text{diam}(M, g)$. Set $b = \frac{1}{2} (\ell m + \text{diam}(M, g))^2$. Increasing b slightly if necessary, we can assume that $b \notin \mathcal{S}(G, q, q') \cup \mathcal{S}(F, q, q')$. By Lemma 3.2.3, the homomorphism induced by inclusion

$$\text{H}_0(\mathcal{E}^b(q, q', \alpha); \mathbb{Q}) \longrightarrow \text{H}_0(\mathcal{E}^{\sigma b}(q, q', \alpha); \mathbb{Q}) .$$

has rank at least one. In view of Proposition 2.2.5 we thus find at least $\gamma_S(m)$ elements α in different components of $\pi_0(\Omega_{q,q'}^1 M)$ that can be represented by Reeb chords of action $\leq \ell m + \text{diam}(M, g)$. This implies the proposition in the case that q' is not Σ -conjugate to q .

Assume now that q' is Σ -conjugate to q . Choose a sequence of points q'_n in M that are not Σ -conjugate to q and such that $q'_n \rightarrow q'$. For n large we can canonically identify $\pi_0(\Omega_{q,q'_n}^1 M)$ with $\pi_0(\Omega_{q,q'}^1 M)$. For $\alpha \in \pi_0(\Omega_{q,q'}^1 M)$ of word-length $\leq m$ we find as

above Reeb chords γ_n^α of action $\leq \ell m + \text{diam}(M, g)$ from q to q'_n . By the Arzelà–Ascoli theorem, a subsequence of γ_n^α converges to a Reeb chord γ^α from Σ_q to $\Sigma_{q'}$ of action $\leq \ell m + \text{diam}(M, g)$. Since γ^α lies in class α , the Reeb chords γ^α are all different.

The steps above can be done for every $m \in \mathbb{N}$. Knowing that the function $\gamma_S(m)$ grows exponentially resp. polynomially in m , we can conclude that the counting function $\text{CF}_{q,q',\Sigma}$ grows exponentially resp. polynomially. This concludes the proof. \square

3.2 Existence of Reeb chords and linear growth

If M is simply-connected, then we can prove that the function $\text{CF}_{q,q',\Sigma}$ grows at least linearly. This conclusion implies that there are infinitely many solutions of Hamilton’s equations (2.8) from Σ_q to $\Sigma_{q'}$.

Theorem 3.2.1. *Let M be simply-connected and choose points $q, q' \in M$. Then the counting function $\text{CF}_{q,q',\Sigma}$ grows at least linearly,*

$$\text{CF}_{q,q',\Sigma}(t) \gtrsim t.$$

Remark. We consider for this proof singular homology with respect to coefficients in a field \mathbb{F}_p . This we do for convenience. The result due to Serre, which we will use below, admits more general coefficient groups. \diamond

Proof. Let G_+ and K be the Hamiltonians introduced in (2.10) and recall that $\Sigma = K^{-1}(\frac{1}{2})$. Choose $q \in M$ and $q' \in V(q)$.

Let us restate a result of Serre’s thesis [46]: For any simply-connected space M there is an infinite number of integers k for which the Betti numbers $b_k(\Omega M, \mathbb{F}_p)$ of the corresponding (based) loop space ΩM are not zero, so

$$\#\{k \in \mathbb{Z} \mid b_k(\Omega M, \mathbb{F}_p) \neq 0\} = +\infty.$$

Let $\dim M = n$ and choose a subsequence k_ℓ , $\ell \in \mathbb{N}$, of those k ’s for which $b_k(\Omega M, \mathbb{F}_p) \neq 0$ such that

$$2n < k_1 < k_2 < \dots$$

and

$$|k_\ell - k_{\ell-1}| > 2n. \quad (3.1)$$

Let $a \in \mathbb{R}_0^+$. Due to Proposition A.0.9 we know that there exists a constant $c = c(g)$ such that for all $k \geq 1$ it holds

$$H_k(\mathcal{E}^a(q, q'); \mathbb{F}_p) = 0 \quad \text{for } k^2 c(g) > a.$$

According to Serre we know that

$$\dim H_{k_\ell}(\Omega_{q, q'}^1 M) > 0,$$

and due to Lemma 4.8.6 (which follows from Gromov's theorem A.0.6)

$$\dim H_{k_\ell}(\mathcal{E}^{a_\ell}(q, q')) > 0, \quad (3.2)$$

for an a_ℓ satisfying

$$\frac{1}{2} (\overline{C} k_\ell)^2 \geq a_\ell \geq c(g) k_\ell^2.$$

Note that this intermediate result is independent of the points q, q' . For ℓ' large enough we can suppose,

$$c(g) k_{\ell'}^2 > \frac{1}{2} \left(\overline{C} \left(k_\ell + \frac{\sqrt{c(g)} (\sqrt{2})^3}{\overline{C}} \dim M \right) \right)^2. \quad (3.3)$$

We then re-choose the sequence $(k_\ell)_{\ell \in \mathbb{N}}$ from above such that

$$c(g) k_{\ell+1}^2 > \frac{1}{2} \left(\overline{C} \left(k_\ell + \frac{\sqrt{c(g)} (\sqrt{2})^3}{\overline{C}} \dim M \right) \right)^2. \quad (3.4)$$

So for every ℓ we have,

$$k_\ell \in \left[\sqrt{\frac{a_{\ell-1}}{c(g)}} + 2 \dim M, \sqrt{\frac{a_\ell}{c(g)}} \right]. \quad (3.5)$$

Now let $(q'_r)_{r \in \mathbb{N}}$ be a sequence such that $q'_r \in V(q)$ for all r and $q'_r \rightarrow q' \notin V(q)$ as

$r \rightarrow \infty$. Since (3.2) does not depend on q, q' , the estimates (3.2) hold for every r .

Fix $a = c(g)k_\ell^2$ and $b = \frac{1}{2}(\overline{C}k_\ell)^2 + \varepsilon$ with $\varepsilon > 0$ such that $a < b$ if necessary. Replace a by a sufficiently close a_0 such that $a_0 < a$ resp. replace b by a sufficiently close b_0 satisfying $b_0 > b$, if a resp. b should be an element of the action spectrum of F or of G . Choose $\sigma \geq \sigma_g \geq 1$ such that $a/\sigma, \sigma b$ lie also not in the action spectrum of F and of G . Because of (3.2) and Lemma 3.2.4,

$$\tilde{\mathbb{H}}_k(\mathcal{QE}^{(a,b]}(q, q'_r); \mathbb{F}_p) \cong \mathbb{H}_k(\mathcal{E}^b(q, q'_r); \mathbb{F}_p) \neq 0,$$

for all $a' \leq a$ due to construction. Choose $a' = a/\sigma \leq a$. Then,

$$\tilde{\mathbb{H}}_k(\mathcal{QE}^{(a/\sigma, b]}(q, q'_r); \mathbb{F}_p) \cong \mathbb{H}_k(\mathcal{E}^b(q, q'_r); \mathbb{F}_p) \rightarrow \mathbb{H}_k(\mathcal{E}(q, q'_r); \mathbb{F}_p),$$

because of the choice of b and Gromov's theorem A.0.6 the map is a surjective homomorphism. Therefore, the rank of the homomorphism in Proposition 2.2.4 does not vanish: So there is for all r and for all ℓ a non-trivial Floer homology class

$$[\gamma_r^\ell] \in \text{HF}_{k_\ell}^{(a/\sigma, b]}(K, q, q'_r).$$

Finally, we apply Lemma 2.4.1 to the sequence γ_r^ℓ . So there is a subsequence of γ_r^ℓ which converges to a solution γ^ℓ . Because of (3.1), of (3.5) and via Lemma 6.2.4 we know that the Conley–Zehnder index k_∞^ℓ of γ^ℓ satisfies $k_\infty^\ell \neq k_\infty^{\ell'}$ if $\ell \neq \ell'$. According to the estimate (3.4) we see that the interval between k_ℓ and $k_{\ell+1}$ is constant. Therefore, the counting function $\text{CF}_{q, q', \Sigma}$ grows for all pairs of points at least linearly. This concludes the proof. \square

Theorem 3.2.2. *For any fiberwise starshaped hypersurface Σ over M and for any two points $q, q' \in M$ the function $\text{CF}_{q, q', \Sigma}$ grows at least linearly,*

$$\text{CF}_{q, q', \Sigma}(t) \succcurlyeq t.$$

Proof. If $\pi_1(M)$ is infinite, the theorem follows from Theorem 3.1.1, since then $\pi_1(M)$ grows at least linearly. Indeed, if $\pi_1(M)$ grows only polynomially, Gromov's theorem

in [21] implies that $\pi_1(M)$ has a nilpotent subgroup Γ of finite index. Its growth agrees with the one of $\pi_1(M)$ in view of the following fact: It follows from Corollary [18, Corollary IV.24] and the fact that the growth of groups is a quasi-isometry invariant. By the Bass–Guivarc’h formula (2.20) $\gamma(\Gamma)$ is a non-negative integer. If $\gamma(\Gamma) = 0$, then all the quotients Γ_k/Γ_{k+1} are finitely generated Abelian groups that are torsion, and hence finite. Thus $\Gamma = \Gamma_1$ is finite too.

So assume that $\pi_1(M)$ is finite. Let $p: \widetilde{M} \rightarrow M$ be the universal covering space. Since $\pi_1(M)$ is finite, \widetilde{M} is compact. Let $\widetilde{\Sigma} \subset T^*\widetilde{M}$ be the fiberwise starshaped hypersurface covering $\Sigma \subset T^*M$. Given $q, q' \in M$ choose lifts $\tilde{q}, \tilde{q}' \in \widetilde{M}$. Reeb chords from $\widetilde{\Sigma}_{\tilde{q}}$ to $\widetilde{\Sigma}_{\tilde{q}'}$ project to Reeb chords from Σ_q to $\Sigma_{q'}$ of the same action. We can thus assume that M is simply connected. To conclude the proof we apply Theorem 3.2.1. \square

Lemma 3.2.3. *Let X be a topological space and $Y \subset X$ a subspace. Then the map $\iota_0: H_0(Y) \rightarrow H_0(X)$ induced by inclusion does not vanish.*

Proof. Fix a point $y \in Y$. Then $[y] \in H_0(Y)$, and $\iota_0([y]) = [y] \in H_0(X)$ does not vanish. \square

Since we want to apply Theorem 2.2.4, the following lemma is very important for our purposes. In particular we will use it to prove that the rank of the homomorphism induced by inclusion does not vanish. By \widetilde{H}_k we denote reduced homology.

Lemma 3.2.4. *Let $q, q' \in M$. Fix $a < b$ and $k \in \mathbb{N}$. Assume*

$$H_k(\mathcal{E}^a(q, q'); \mathbb{F}_p) = 0.$$

Then

$$H_k(\mathcal{E}^b(q, q'); \mathbb{F}_p) \cong \widetilde{H}_k(\mathcal{QE}^{(a,b)}(q, q'); \mathbb{F}_p).$$

Proof. The proof follows by the long exact sequence of the pair $\mathcal{E}^a(q, q') \subset \mathcal{E}^b(q, q')$

$$0 \longrightarrow H_k(\mathcal{E}^b(q, q'); \mathbb{F}_p) \longrightarrow H_k(\mathcal{E}^b(q, q'), \mathcal{E}^a(q, q'); \mathbb{F}_p) \longrightarrow 0,$$

and the isomorphism $H_k(\mathcal{E}^b(q, q'), \mathcal{E}^a(q, q'); \mathbb{F}_p) \cong \widetilde{H}_k(\mathcal{QE}^{(a,b)}(q, q'); \mathbb{F}_p)$. \square

Chapter 4

Special Configuration Spaces (M, g)

In the following paragraphs we will focus on special configuration spaces (M, g) . Symmetric spaces, in particular Lie groups. and in addition configuration spaces (M, g) with assumptions on their fundamental groups $\pi_1(M, \cdot)$ will be the examples. It is the goal to find many Reeb chords starting in Σ_q and ending in $\Sigma_{q'}$. One way to achieve this is to use existing results concerning the length spectrum of (M, g) . We will use this information and other geometric properties of (M, g) to count Reeb chords on the hypersurface Σ . In order to do this, we have to apply infinite-dimensional Morse theory for the energy functional. By providing definitions, lemmas and proofs, we go on step by step and present the details of this vast and technically rather involved subject.

Again as in Chapter 3, let (M, g) be an n -dimensional closed connected Riemannian manifold. Then the hypersurface $\Sigma \subset T^*M$ shall be fiberwise starshaped and we choose the 2-homogeneous Hamiltonian function $F : T^*M \rightarrow \mathbb{R}$ as constructed in Section 2.2. Assume that $\frac{1}{2}$ is a regular value of F such that $\Sigma = F^{-1}(\frac{1}{2})$. In addition we introduce the following convention.

Convention 4.0.5. *Let (M, g) be furnished with a Riemannian metric g such that the Hamiltonian functions F, G defined in Section 2.2.1 satisfy*

$$F \geq G.$$

Denote by $d := \text{diam}(M, g)$ the diameter of (M, g) .

4.1 Towards a finer quantitative understanding

As explained in the introduction of this thesis, one is interested in describing quantitative existence results of Reeb chords. These results should be as concrete as possible. The next proposition, which is qualitatively a similar result as Theorem 3.2.1, should be seen as a starting point for more concrete results concerning this direction. The proof uses different ideas and results than the statements presented later on. Namely, we apply ideas of Schwarz [50], for more details see Appendix B.

Proposition 4.1.1. *Let M be simply-connected. Then for every pair of points $q, q' \in M$ and for every $k \in \mathbb{N}$ there exist at least k Reeb chords $\tilde{x}_\ell: [0, 1] \rightarrow T^*M$, $\ell \in \{1, 2, \dots, k\}$, from Σ_q to $\Sigma_{q'}$ satisfying the time bound*

$$\mathcal{T}(\tilde{x}_\ell) \leq \underline{C}k(n-1).$$

Hence $\text{CF}_{q,q',\Sigma}(g(k)) \geq k$, where $g(k) := \underline{C}k(n-1)$.

This proposition proves linear growth of the counting function $\text{CF}_{q,q',\Sigma}$, similar to Theorem 3.2.2. Whereas the constants of this case are much harder to understand: The constant \underline{C} is Gromov's constant found in the proof of Theorem A.0.5. It might be possible to get bounds on \underline{C} . Nabutovsky and Rotman considered this point of view in [37, Chapter 8]. The proof of Proposition 4.1.1 is given in Section 4.6. From Proposition 4.1.1 the case of finite fundamental group follows at once by passing to the universal cover (cf. Remark 7.8 in Gromov [22]).

Remark. By assuming further conditions on the base manifold M , one gets a better upper bound. For example – see the discussion above – if M is a simply-connected and closed manifold satisfying $H_2(M, \mathbb{Q}) \neq 0$ then the time estimates in Proposition 4.1.1 can be improved to

$$\mathcal{T}(x_\ell) \leq \underline{C}k,$$

because of $H_1(\Omega M, \mathbb{Q}) \cong H_2(M, \mathbb{Q}) \not\cong 0$, see [49]. Further, it is readily seen that Proposition 4.1.1 implies the existence of infinitely many solutions of Hamilton's equation starting in some and ending in any possibly other point of M . For geodesic flows, this result was already known due to Serre [46]. \diamond

4.1.1 Applying infinite-dimensional Morse theory

We will use infinite-dimensional Morse theory to get more precise quantitative existence results of Reeb chords on Σ . We describe this approach in the following paragraphs.

For the next steps we follow Chang [16] and Palais [40]. Let W be a C^2 -Hilbert manifold.

Definition 4.1.2 (Palais-Smale condition, [16]). *Given $f \in C^1(W; \mathbb{R})$ and $c \in \mathbb{R}$, we say that f satisfies the $(PS)_c$ condition if any sequence $(x_n)_{n \in \mathbb{N}} \subset W$ along which $f(x_n) \rightarrow c$ and $df(x_n) \rightarrow 0$ (for $n \rightarrow \infty$) possesses a convergent subsequence. We say that f satisfies the (PS) condition if it satisfies $(PS)_c$ for all $c \in \mathbb{R}$.*

If f is a proper C^1 -function $f : Q \rightarrow \mathbb{R}$ on some topological space Q (i.e. $f^{-1}([a, b])$ is compact in Q for all closed intervals $[a, b]$) then f satisfies automatically the (PS) condition. But since W is infinite-dimensional, i.e. not locally compact, it is impossible that f is proper on W . Nevertheless, such an f can fulfill the Palais-Smale condition.

Define the *critical set* K of $f \in C^1(W; \mathbb{R})$ by

$$K := \{x \in W \mid df(x) = 0\}. \quad (4.1)$$

Lemma 4.1.3. *Let $[a, b] \subset \mathbb{R}$. If $f \in C^1(W; \mathbb{R})$ satisfies $(PS)_c$ for all $c \in [a, b]$, the critical set $K_{[a, b]} := K \cap f^{-1}([a, b])$ is compact.*

Proof. Choose a sequence $(x_n)_{n \in \mathbb{N}}$ such that $(x_n)_{n \in \mathbb{N}} \subset K_{[a, b]}$, then $f(x_n) \in [a, b]$ and $df(x_n) = 0$ for all n . The $(PS)_c$ condition for $c \in [a, b]$ implies the existence of a convergent subsequence $(x_{n_j})_{j \in \mathbb{N}}$ and of a $c^* = c^*((x_n)_{n \in \mathbb{N}}) \in [a, b]$ such that $f(x_{n_j}) \rightarrow c^*$ if $(j \rightarrow \infty)$. Hence $K_{[a, b]}$ is compact. \square

Corollary 4.1.4 ([16]). *If $f \in C^1(W; \mathbb{R})$ satisfies the Palais-Smale condition $(PS)_c$ for $c \in \mathbb{R}$, then $K_c := K \cap f^{-1}(c)$ is compact.*

Proof. If $(x_n)_{n \in \mathbb{N}} \subset K_c$, then $f(x_n) = c$ and $df(x_n) = 0$ for all n . The $(PS)_c$ condition implies the existence of a convergent subsequence and hence K_c is compact. \square

Lemma 4.1.5 ([16]). *If $f \in C^1(W; \mathbb{R})$ satisfies the Palais-Smale condition $(PS)_c$ for all $c \in [a, b]$ and if $K \cap f^{-1}([a, b]) = \emptyset$, then $\exists \varepsilon_0, \delta_0 > 0$ such that*

$$\|df(x)\| \geq \varepsilon_0 \forall x \in f^{-1}([a - \delta_0, b + \delta_0]).$$

Proof. If not, there exists a sequence $x_n \in f^{-1}([a - \frac{1}{n}, b + \frac{1}{n}])$, $n = 1, 2, \dots$, satisfying $df(x_n) \rightarrow 0$. According to $(PS)_c$ for all $c \in [a, b]$, there exists a convergent sub-sequence $x_{n_i} \rightarrow x^*$, which implies $x^* \in K \cap f^{-1}([a, b])$. This is a contradiction. \square

Proposition 4.1.6 ([40, Page 307]). *If p is a critical point of $f \in C^2(W; \mathbb{R})$ then there is a uniquely determined continuous symmetric bilinear form $H(f)_p$ on $T_p W$, called the Hessian of f at p , with the following property: If φ is any chart at p*

$$H(f)_p(v, w) = d^2(f \circ \varphi^{-1})_{\varphi(p)}(\varphi(v), \varphi(w)) \quad (4.2)$$

Given a Banach space V and a bounded symmetric bilinear form B on V we say that B is *non-degenerate* if the linear map

$$\begin{aligned} T : V &\longrightarrow V^* \\ v &\longmapsto T(v)(\cdot) := B(v, \cdot) \end{aligned}$$

is a linear isomorphism from V to V^* , otherwise B is called *degenerate*. Thus there is a dichotomy of critical points of f into non-degenerate and degenerate critical points. We define the *index* of B to be the supremum of the dimensions of subspaces Q of V on which B is negative definite. Denote by ν_T the *nullity* of T , i.e. the dimension of the subspace $N \subset V$ such that $\forall w \in N : T(w) = 0$. This means that if $\nu_T > 0$, then T is not an isomorphism.

Definition 4.1.7 ((Non-)Degeneracy and Morse-index, [40]). *Let f be a C^2 -real valued function on a C^2 -Hilbert manifold W and p a critical point of f . Then p is (non-)degenerate if the Hessian $H(f)_p$ of f at p is (non-)degenerate. The Morse index of f at p is defined as the index of the Hessian $H(f)_p$ of f at p .*

Proposition 4.1.8. *Suppose that the function $f \in C^2(W; \mathbb{R})$ satisfies the Palais-Smale condition $(PS)_c$ for all $c \in [a, b] \subset \mathbb{R}$ and has only non-degenerate critical points*

in $f^{-1}([a, b])$. Then the set

$$K_{[a,b]}(f) := \{x \in W \mid f(x) \in [a, b] \text{ and } df(x) = 0\}$$

is finite.

Proof. This is a consequence of the fact that non-degenerate critical points are isolated and of Lemma 4.1.3. \square

Recall from Klingenberg [28] that the path space $\Omega_{q,q'}^1 M$ carries the structure of a smooth Hilbert manifold. This means that the tangent spaces of this manifold are Hilbert spaces. Let us consider as f the energy functional on $\Omega_{q,q'}^1 M$ given by

$$\begin{aligned} \mathcal{E}: \Omega_{q,q'}^1 M &\longrightarrow \mathbb{R} \\ \gamma &\longmapsto \mathcal{E}(\gamma) = \int_0^1 g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) dt. \end{aligned}$$

This functional is of class C^2 on $\Omega_{q,q'}^1 M$ and satisfies the Palais–Smale condition, see Abbondandolo–Figalli [1, Propositions 4.1 and 4.2]. According to Jost [26, Chapter 5.1] the differential $d\mathcal{E}(\gamma)$ in a point $\gamma \in \Omega_{q,q'}^1 M$ vanishes (i.e. γ is a critical point of \mathcal{E}) if and only if γ is a geodesic segment:

Lemma 4.1.9 ([26]). *If $\gamma \in \Omega_{q,q'}^1 M$, then γ is a critical point of the energy functional \mathcal{E} if and only if γ is a geodesic segment from q to q' .*

Proof. See Lemma 5.1.1 in [26] and the remarks below. \square

Note that a geodesic segment is necessarily parametrized proportionally to arc-length, [32, 26]. This means by Lemma C.3.4 that a geodesic segment is also a critical point of the length functional \mathcal{L} . Since \mathcal{L} is invariant under reparametrizations, if one is interested in finding critical points of \mathcal{E} , one could look for critical points of \mathcal{L} in the class of paths parametrized proportionally to arc-length. But the functional \mathcal{E} has better analytic properties. (For instance, \mathcal{L} is not differentiable at paths of length 0.)

Moreover, for such a geodesic segment γ , it is explained in [26, Chapter 5.1] how to compute the second variation of \mathcal{E} at the point γ . This leads to the definition of the

so-called *index form* of the geodesic segment γ ,

$$I_\gamma(X, Y) := \int_0^1 \left(g_{\gamma(t)} \left(\nabla_{\frac{d}{dt}} X, \nabla_{\frac{d}{dt}} Y \right) - g_{\gamma(t)} (R(\dot{\gamma}, X)Y, \dot{\gamma}) \right) dt \quad (4.3)$$

for $X, Y \in T_\gamma \Omega_{q, q'}^1 M$. Using the symmetries of the curvature tensor R as well as the properties of the Riemannian connection ∇ , one concludes that $I_\gamma(X, Y)$ is bilinear and symmetric in X and Y . Integrating by parts yields

$$\begin{aligned} I_\gamma(X, Y) &= g_{\gamma(t)} \left(\nabla_{\frac{d}{dt}} X, Y \right) \Big|_0^1 \\ &\quad - \int_0^1 \left(g_{\gamma(t)} \left(\nabla_{\frac{d}{dt}} \nabla_{\frac{d}{dt}} X, Y \right) - g_{\gamma(t)} (R(\dot{\gamma}, X)Y, \dot{\gamma}) \right) dt. \end{aligned} \quad (4.4)$$

Applying now the Cauchy-Schwarz inequality, we get

$$|I_\gamma(X, Y)| \leq C(X) \|Y\|_\gamma,$$

where

$$\|Y\|_\gamma := \left(\int_0^1 \left(g_{\gamma(t)} \left(\nabla_{\frac{d}{dt}} Y, \nabla_{\frac{d}{dt}} Y \right) + g_{\gamma(t)} (Y, Y) \right) dt \right)^{\frac{1}{2}} \quad \text{for } Y \in T_\gamma \Omega_{q, q'}^1 M.$$

Note that if one chooses local coordinates in a neighborhood of $\gamma(t)$, $t \in [0, 1]$, then elements $X_{\gamma(t)}$ are of the form $X_{\gamma(t)} = X_{\gamma(t)}^i \partial_i$; hence $\|X\|_\gamma$ coincides locally with the standard Sobolev norm of $W^{1,2}((a, b), \mathbb{R}^n)$. So $I_\gamma(X, \cdot)$ is a bounded linear functional on the Hilbert space $T_\gamma \Omega_{q, q'}^1 M$ and hence continuous. The same argument applies to the X -variable. Due to the Lax-Milgram theorem, see for example [51, Aufgabe V.6.18], it follows that I_γ is a continuous bounded bilinear symmetric operator on $T_\gamma \Omega_{q, q'}^1 M \times T_\gamma \Omega_{q, q'}^1 M$. Therefore, by Proposition 4.1.6 we know that the Hessian of the energy functional \mathcal{E} in γ is of the form $H_\gamma(\mathcal{E}) = I_\gamma$. For the understanding of the Morse-index resp. of the nullity of the geodesic segment γ , we follow [26, Sections 5.2, 5.3].

Definition 4.1.10. *Let $\gamma : [0, 1] \rightarrow M$ be a geodesic segment. A vector field X along γ is called a Jacobi field if*

$$\nabla_{\frac{d}{dt}} \nabla_{\frac{d}{dt}} X + R(X, \dot{\gamma})\dot{\gamma} = 0.$$

The following lemma follows at once from (4.4).

Lemma 4.1.11 (Lemma 5.2.1, [26]). *A vector field X along a geodesic segment $\gamma : [0, 1] \rightarrow M$ is a Jacobi field if and only if the index form I_γ satisfies*

$$I_\gamma(X, Y) = 0$$

for all vector fields Y along γ with $Y(0) = Y(1) = 0$.

The space of Jacobi fields is not empty: Given $v, w \in T_{\gamma(0)}M$, there exists a unique Jacobi field X satisfying $X(0) = v$ and $\nabla_{\frac{d}{dt}}X = w$, see Lemma 5.2.3 in [26].

Definition 4.1.12 (Conjugate points along a geodesic). *Let $\gamma : [0, 1] \rightarrow M$ be a geodesic segment. For $t_0, t_1 \in [0, 1]$, $\gamma(t_0)$ and $\gamma(t_1)$ are called conjugate along γ if there exists a Jacobi field $X(t)$ along γ that does not vanish identically and satisfies*

$$X(t_0) = 0 = X(t_1).$$

Let \mathcal{V}_γ be the space of vector fields along the geodesic segment $\gamma : [0, 1] \rightarrow M$. Further, denote by $\mathcal{V}_\gamma^\circ \subset \mathcal{V}_\gamma$ the subspace consisting of all vector fields V along γ satisfying $V(0) = V(1) = 0$.

Lemma 4.1.13. *Let $\gamma : [0, 1] \rightarrow M$ be a geodesic segment. Then there is no pair of conjugate points along γ if and only if the index form I_γ is positive definite on \mathcal{V}_γ° .*

See the proof of Lemma 5.3.1 in [26].

In accordance with Definition 4.1.7 we introduce the Morse-index of a geodesic segment.

Definition 4.1.14 (Morse-index of a geodesic segment). *The Morse-index $\mu(\gamma)$ of a geodesic segment $\gamma : [0, 1] \rightarrow M$ is the dimension of the largest subspace of \mathcal{V}_γ° on which I_γ is negative definite. The nullity $\nu(\gamma)$ of γ is the dimension of the largest subspace of \mathcal{V}_γ° on which I_γ is zero.*

For the development of Morse theory in the infinite-dimensional setting the following lemma is fundamental.

Lemma 4.1.15 (Lemma 5.3.2, [26]). *For a geodesic segment $\gamma : [0, 1] \rightarrow M$ the numbers $\mu(\gamma)$ and $\nu(\gamma)$ are finite and*

$$\nu(\gamma) = \dim \{X \in \mathcal{V}_\gamma^\circ \mid X \text{ a Jacobi field}\}.$$

Proof. See the proofs of Lemmas 5.2.1 and 5.3.2 in [26]. □

Lemma 4.1.16. *Let $\gamma : [0, 1] \rightarrow M$ be a geodesic segment.*

$$\nu(\gamma) = 0 \iff \text{the points } \gamma(0) \text{ and } \gamma(1) \text{ are non-conjugate.}$$

Proof. The proof follows easily from the definitions. □

According to Definition 4.1.7 and the previous explanations, we see that if q and q' are non-conjugate along every geodesic segment γ , then the energy functional \mathcal{E} is a Morse function on $\Omega_{q,q'}^1 M$ since all its critical points are non-degenerate. For given $q \in M$, the points $p \in M$ that are non-conjugate to q lie dense in M and form a set of full measure with respect to the Riemannian measure on M . This follows from the fact [29, Chapter 10] that the exponential map $\exp_q : T_q M \rightarrow M$ is a local diffeomorphism at $V \in T_q M$ if and only if q is not conjugate to $p = \exp_q(V)$ along the geodesic $\gamma(t) := \exp_q(tV)$, $t \in [0, 1]$, and from Sard's theorem.

Corollary 4.1.17. *Fix a real number $a > 0$ and $q \in M$. Then for almost all $p \in M$ the critical points of the energy functional \mathcal{E} with energy at most a*

$$K_{[0,a]}(\mathcal{E}) \subset \mathcal{E}^a(q, p)$$

build a finite set.

Proof. Immediate application of Proposition 4.1.8. □

We would like to develop the ideas and the reasons why we are interested in understanding the circumstances which allow the energy functional \mathcal{E} (on some path space) to be a so-called perfect Morse function. This involves the development of some other

parts of (infinite-dimensional) Morse theory. For an explanation of this class of Morse functions we refer to Chapter 6.

Define for a geodesic segment $\gamma \in \Omega_{q,q'}^1 M$ as a critical point of \mathcal{E} with $\mathcal{E}(\gamma) = c$ according to Chang [16] the so called *r-th critical group* with coefficient field \mathbb{F}

$$C_r(\mathcal{E}, \gamma) := H_r(\mathcal{E}^c(q, q') \cap U_\gamma, (\mathcal{E}^c(q, q') \setminus \{\gamma\}) \cap U_\gamma; \mathbb{F}),$$

where $r \in \mathbb{N}_0$ and $U_\gamma \subset \Omega_{q,q'}^1 M$ is an open neighborhood of γ such that γ is the only critical point of \mathcal{E} in U_γ . This means $K \cap U_\gamma = \{\gamma\}$ for $K = \{\eta \in \Omega_{q,q'}^1 M \mid d\mathcal{E}(\eta) = 0\}$. Recall the definition of $\mathcal{E}^c(q, q')$ in (2.5). The group $C_r(\mathcal{E}^c(q, q'), \gamma)$ does not depend on the U_γ chosen, as follows from the excision property of homology. We know that if q' is non-conjugate to q the critical values of \mathcal{E} lie isolated on the real line and have finitely many associated critical points. Indeed, suppose the critical values of \mathcal{E} accumulate near $b \in (0, a)$, so $K_{[0,a]}$ builds an infinite set and therefore q, q' are conjugate in view of Lemma 4.1.17. So, there is an ordering of critical values as

$$0 \leq c_0 < c_1 < c_2 < \dots$$

because \mathcal{E} has values only on the closed positive real half line. Define

$$K_{c_i} = \{z_j^i\}_{j=1}^{m_i} \subset K, \quad i = 0, 1, 2, \dots,$$

such that $\mathcal{E}(z_j^i) = c_i, \forall j \in \{1, \dots, m_i\}$. Choose $0 < \varepsilon_0 < c_1$ and for every i choose

$$0 < \varepsilon_i < \min\{c_{i+1} - c_i, c_i - c_{i-1}\}, \quad i = 1, 2, 3, \dots$$

For a pair of regular values $a, b \in \mathbb{R}$ with $a < b$, define the *r-th Morse type number* of \mathcal{E} with respect to (a, b) , $r \in \mathbb{N}_0$:

$$M_r(a, b) := \sum_{a < c_i < b} \dim H_r(\mathcal{E}^{c_i + \varepsilon_i}(q, q'), \mathcal{E}^{c_i - \varepsilon_i}(q, q'); \mathbb{F}). \quad (4.5)$$

Suppose a non-degenerate critical point γ of \mathcal{E} has Morse index $\text{ind}(\gamma) = j \in \mathbb{N}$.

Then there is the following connection to the critical groups, [16, Theorem 4.1]:

$$C_r(\mathcal{E}, \gamma) = \begin{cases} \mathbb{F}, & \text{if } r = j \\ 0, & \text{if } r \neq j. \end{cases}$$

One can prove (see again Chang [16]) that for an (isolated) critical value c of \mathcal{E} and $K_c = \{z_j\}_{j=1}^m$ and for sufficiently small $\varepsilon > 0$,

$$H_* (\mathcal{E}^{c+\varepsilon}(q, q'), \mathcal{E}^{c-\varepsilon}(q, q'); \mathbb{F}) \cong H_* (\mathcal{E}^c(q, q'), \mathcal{E}^c(q, q') \setminus K_c; \mathbb{F}) \cong \bigoplus_{j=1}^m C_*(\mathcal{E}, z_j).$$

This implies the following connection between the critical groups and the Morse type numbers of \mathcal{E} :

$$M_r(a, b) = \sum_{a < c_i < b} \sum_{j=1}^{m_i} \dim C_r(\mathcal{E}, z_j^i), \quad r \in \mathbb{N}_0.$$

By assuming the finiteness of the coefficients of the following series, we get the following implicit statement of the Morse inequalities for \mathcal{E} on $\Omega_{q, q'}M$, see Theorem 4.3 of Chang [16],

$$\sum_{r=0}^{\infty} M_r t^r = \sum_{r=0}^{\infty} \beta_r t^r + (1+t)Q(t), \quad (4.6)$$

where Q is a formal series with nonnegative coefficients, $M_r = \sum_{j=1}^l \dim C_r(\mathcal{E}, z_l)$, $\{z_1, \dots, z_l\} = K \cap \mathcal{E}^{-1}([a, b])$ and where $\beta_r = \beta_r(a, b) = \dim H_r(\mathcal{E}^b(q, q'), \mathcal{E}^a(q, q'); \mathbb{F})$ for $r \in \mathbb{N}_0$.

As an application, let $(\gamma_j)_{j \in \mathbb{N}}$ be a sequence of isolated local minima of \mathcal{E} . Then

$$C_r(\mathcal{E}, \gamma_j) = \begin{cases} \mathbb{F} & r = 0 \\ 0 & r \neq 0, \end{cases}$$

by definition of singular homology.

Consider for a regular value $D > 0$ of \mathcal{E} the space $\mathcal{E}^D(q, q')$ and choose a real number

$$b > \max_{\eta \in K_{[0, D]}(\mathcal{E})} \mathcal{E}(\eta)$$

which exists since $K_{[0,D]}(\mathcal{E})$ consists only of finitely many critical points and analogously $a < \min_{\eta \in K_{[0,D]}(\mathcal{E})} \mathcal{E}(\eta)$. We replace in the exposition above $\Omega_{q,q'}^1 M$ by $\mathcal{E}^D(q, q')$. Then $M_r = M_r(a, b)$ is the number of critical points of \mathcal{E} on $\mathcal{E}^D(q, q')$ of Morse index r , while $\beta_r = \beta_r(a, b)$ is the r -th Betti number of the space $\mathcal{E}^D(q, q')$. For this latter space the coefficients in (4.6) are finite and the series exist.

Recall the notion of a perfect Morse function. In this particular setting it means the following: The energy functional is an \mathbb{F} -perfect Morse function on $\mathcal{E}^D(q, q')$ if it holds true that

$$\sum_{r=0}^{\infty} M_r t^r = \sum_{r=0}^{\infty} \beta_r t^r.$$

Equivalently, $Q \equiv 0$. This is also equivalent to

$$M_r = \beta_r, \forall r \in \mathbb{N}_0,$$

i.e. the number of critical points of f of Morse index r equals the r -th Betti number of $\mathcal{E}^D(q, q')$.

More generally, if E is a Hilbert manifold and $F \subset E$ a compact subspace, a Morse function f on F is called \mathbb{F} -perfect if the associated Morse series and Poincaré series are equal. See Section 6.1 for more details concerning perfect Morse functions.

Consider the Morse homology for the energy functional on the path space $\Omega_{q,q'}^1 M$ of a Riemannian manifold (M, g) with q' non-conjugate to q . We denote these homology groups by $H_* (\{CM_*(\mathcal{E}_g), \partial_*\}; \mathbb{F})$, following the notation used in [3]. This version of Morse homology for Hilbert manifolds is explained in [2, 3].

Definition 4.1.18 (Homological Visibility). *Let (M, g) be a closed Riemannian manifold, and let $q, q' \in M$ be non-conjugate. A critical point γ of the energy functional $\mathcal{E}_g : \Omega_{q,q'}^1 M \rightarrow \mathbb{R}$ is called homologically visible, if*

$$\partial_* \gamma = 0 \text{ and } [\gamma] \neq 0 \text{ in } H_* (\{CM_*(\mathcal{E}_g), \partial_*\}; \mathbb{F}).$$

If we have given a manifold F admitting an \mathbb{F} -perfect Morse function f , then we know that every non-degenerate critical point p of f is homologically visible. If we suppose that γ is the only *global* minimum of \mathcal{E} , then the Morse inequalities imply

$\beta_0(a, b) = M_0(a, b) = 1$, i.e. this global minimum is homologically visible.

Nabutovsky and Rotman discover – see for example Theorem 4.7.1 – a collection of geodesic segments on an n -dimensional Riemannian manifold satisfying some length bound. Note that they suppose $n > 1$. We wish to be able to see that these geodesics are homologically visible. Then we can use the Abbondandolo–Majer respectively the Abbondandolo–Schwarz isomorphisms to find the same number of Reeb chords on the starshaped hypersurface $\Sigma \subset T^*M$ with the given length bound which is directly related to the action bound. By doing so, we get some insight into Gromov’s constant \underline{C} . Therefore, in our setting, we need to prove that the energy functional $\mathcal{E} : \Omega_{q,q'}^1 M \rightarrow \mathbb{R}$ is an \mathbb{F} -perfect Morse function on $\Omega_{q,q'}^1 M$ for some coefficient field \mathbb{F} .

Remark. An other idea to achieve that every geodesic which is found by Nabutovsky and Rotman is homologically visible could go along the following question: Is there a Riemannian metric on M such that every geodesic on M is non-degenerate (as a critical point of \mathcal{E}) and homologically visible? But the author has no idea how to go in the direction of an understanding of this question. If the Riemannian metric of g on M is bumpy, then there are finitely many geodesics, hence they lie isolated and are non-degenerate. But this does not imply homological visibility. Consider for example the height function f on the “heart shaped” sphere in two dimensions as in Figure 4.1. Then f has four non-degenerate critical points, but only the minimum is homologically visible.

◇

The steps explained above shall be carried out in some detail. Let (M, g) be an n -dimensional closed connected Riemannian manifold. Suppose that the Riemannian metric g satisfies the Convention 4.0.5. Moreover, denote by $d := \text{diam}(M, g)$ the diameter of (M, g) . Choose two distinct points $q, q' \in M$ such that q' is not conjugate to q . Let $\Omega_{q,q'}^1 M$ be the associated path space. Assume that there is some coefficient field \mathbb{F} such that the energy functional \mathcal{E} is an \mathbb{F} -perfect Morse function on $\Omega_{q,q'}^1 M$. (See the Section 4.2 for statements when this assumption holds.) Fix a natural number $k \in \mathbb{N}$. Then according to Theorem 4.7.1 there are k geodesic segments x_ℓ , $\ell \in \{1, \dots, k\}$, connecting q with q' in M , satisfying the length bound $\mathcal{L}(x_\ell) \leq 2n(k+1)^2 d \leq 22(k-1)d + \text{dist}(q, q')$, $\forall \ell \in \{1, \dots, k\}$. The expression in

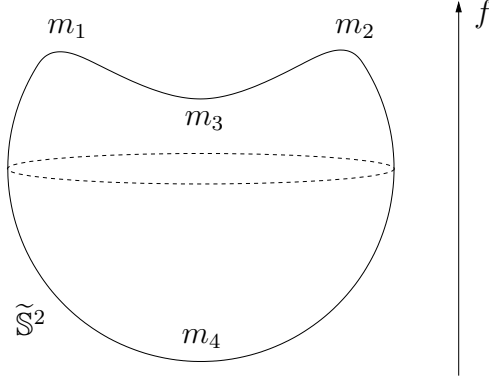


Figure 4.1: A representation of the mentioned “heart shaped” sphere $\tilde{\mathbb{S}}^2$. The points m_1 and m_2 are local maxima and a linear combination of them is homologically visible, i.e. represents the generator of $H_2(\tilde{\mathbb{S}}^2; \mathbb{F})$. Whereas m_3 is a local minima and m_4 is the global minima, which represents zeroth degree singular homology $H_0(\tilde{\mathbb{S}}^2; \mathbb{F})$.

square brackets refers to the 2-dimensional case, see Section 4.7. Since the x_ℓ are geodesic segments, one interpretes them as smooth paths connecting q with q' parameterized by arclength. By Lemma C.3.4 we can conclude $\mathcal{L}^2(x_\ell) = 2\mathcal{E}(x_\ell)$, $\forall \ell \in \{1, \dots, k\}$. Therefore we have the energy bound in n -dimensions [2-dimensions],

$$\begin{aligned} \mathcal{E}(x_\ell) &\leq 2(n(k+1)^2d)^2, \forall \ell \in \{1, \dots, k\}, \\ \left[\mathcal{E}(x_\ell) &\leq \frac{(22(k-1)d + \text{dist}(q, q'))^2}{2}, \forall \ell \in \{1, \dots, k\} \right] \end{aligned} \quad (4.7)$$

Define the real number $\mathcal{D}(k, d, q, q') := 2(n(k+1)^2d)^2 \left[:= \frac{(22(k-1)d + \text{dist}(q, q'))^2}{2} \right]$.

Consider for $q' \in M \setminus ((V(q))^c \cup \mathcal{C}_q)$ the homomorphism induced by inclusion

$$\iota_i: H_i\left(\mathcal{E}^{\mathcal{D}(k, d, q, q') + \varepsilon}(q, q'); \mathbb{F}\right) \rightarrow H_i\left(\mathcal{E}^{\sigma \mathcal{D}(k, d, q, q') + \varepsilon}(q, q'); \mathbb{F}\right), \quad (4.8)$$

where $\mathcal{C}_q := \{p \in M \mid q' \text{ conjugate to } q\}$. For the definition of the set $V(q)$ see Section 2.2. We add the summand $\varepsilon > 0$ to guarantee that $\mathcal{D}(k, d, q, q')$ and $\sigma \mathcal{D}(k, d, q, q')$, for $\sigma \geq \sigma_g \geq 1$, do not belong to the action spectrum of F or G . This can be achieved because the action spectrum is a compact and nowhere dense subset of \mathbb{R} . Note that the union $(V(q))^c \cup \mathcal{C}_q$ has zero Lebesgue measure in (M, g) . Due to the assumption that \mathcal{E} is an \mathbb{F} -perfect Morse function on $\Omega_{q, q'}^1 M$, this implies that each of the k geodesic

segments found by Nabutovsky and Rotman as explained in Section 4.7 represent different nontrivial homology classes in $H_* \left(\mathcal{E}^{\mathcal{D}(k,d,q,q')+\varepsilon}(q, q'); \mathbb{F} \right)$. All these classes must exist also in $H_* \left(\mathcal{E}^{\sigma\mathcal{D}(k,d,q,q')+\varepsilon}(q, q'); \mathbb{F} \right)$ due to perfection of \mathcal{E} . Therefore, the rank of the homomorphism (4.8) satisfies

$$\dim \iota_i H_* \left(\mathcal{E}^{\mathcal{D}(k,d,q,q')+\varepsilon}(q, q'); \mathbb{F} \right) = \dim H_* \left(\mathcal{E}^{\sigma\mathcal{D}(k,d,q,q')+\varepsilon}(q, q'); \mathbb{F} \right).$$

We are interested in measuring the time or the reduced action of Reeb chords on the prescribed hypersurface $\Sigma = F^{-1} \left(\frac{1}{2} \right)$. Thus we use the homomorphism (4.8) to interpret the geodesic segments x_ℓ via Proposition 2.2.5 as nontrivial Floer homology classes of $\text{HF}_r^{\mathcal{D}(k,d,q,q')+\varepsilon} (K, q, q'; \mathbb{F})$. This gives the existence of k Reeb chords (as reparameterized restrictions of the Hamiltonian flow lines of K to Σ , see Lemma 4.8.4), denoted by $\tilde{x}_\ell: [0, 1] \rightarrow \Sigma \subset T^*M$ for $\ell \in \{1, \dots, k\}$ starting in $\tilde{x}_\ell(0) \in \Sigma_q$ and ending in $\tilde{x}_\ell(1) \in \Sigma_{q'}$. More precisely, this follows from

$$\dim \text{HF}_r^{\mathcal{D}(k,d,q,q')+\varepsilon} (K, q, q'; \mathbb{F}) \leq \dim \text{CF}_r^{\mathcal{D}(k,d,q,q')+\varepsilon} (K, q, q'; \mathbb{F}).$$

Hence, by using Lemma 2.2.2 we can estimate the time for \tilde{x}_ℓ :

$$\mathcal{A}(\tilde{x}_\ell) = \sqrt{2 \mathcal{A}_K(x_\ell)} \leq \sqrt{2 (\mathcal{D}(k, d, q, q') + \varepsilon)},$$

which yields for all $\ell \in \{1, \dots, k\}$

$$\begin{aligned} \mathcal{A}(\tilde{x}_\ell) &\leq 2n(k+1)^2 d & (4.9) \\ [\mathcal{A}(\tilde{x}_\ell) &\leq 22(k-1)d + \text{dist}(q, q')] \end{aligned}$$

because $\varepsilon > 0$ is arbitrary small. From the discussion above it follows for $i, j \in \{1, \dots, k\}$ with $i < j$ that $\mathcal{A}_K(x_i) < \mathcal{A}_K(x_j)$. This can be done for all $q' \in V(q)$. So choose a sequence $q_m \rightarrow q' \notin V(q)$, hence we get k sequences of solutions $(x_j^m)_{m \in \mathbb{N}}$ satisfying $x_j^m(0) \in T_q^*M$ and $x_j^m(1) \in T_{q_m}^*M$. We know that for $i < j$ that $\mathcal{A}_K(x_i^m) < \mathcal{A}_K(x_j^m)$. Observe that we can choose all the b_ℓ smaller or equal than the expressions given in (4.7). Therefore we find by Lemma 2.4.3 k different Reeb chords with $\gamma_j(0) \in T_q^*M$ and

$\gamma_j(1) \in T_{q'}^*M$ satisfying the time bounds (4.9). So we can state:

Proposition 4.1.19. *Choose a natural number $n > 1$. Let M be n -dimensional [2 -dimensional]. If, for $q \in M$ and $q' \in M \setminus ((V(q))^c \cup \mathcal{C}_q)$ the energy functional \mathcal{E} is an \mathbb{F} -perfect Morse function on $\Omega_{q,q'}^1 M$, then for every pair of points $q, q' \in M$ and for every $k \in \mathbb{N}$ there exist at least k Reeb chords $\tilde{x}_\ell: [0, 1] \rightarrow \Sigma \subset T^*M$, $\ell \in \{1, 2, \dots, k\}$, from Σ_q to $\Sigma_{q'}$ satisfying the time bound*

$$\begin{aligned} \mathcal{T}(\tilde{x}_\ell) &\leq 2n(k+1)^2d, \\ [\mathcal{T}(\tilde{x}_\ell) &\leq 22(k-1)d + \text{dist}(q, q') \leq (22k-21)d] \end{aligned}$$

where $\ell \in \{1, 2, \dots, k\}$. Therefore $\text{CF}_{q,q',\Sigma}(g(k)) \geq k$ if we define

$$g(k) := 2n(k+1)^2d \quad [:= 22(k-1)d + \text{dist}(q, q')] .$$

Let us do the following remark for later use in the Section 4.2.

Remark. If a given lower bound of $\dim H_*(\mathcal{E}^a(q, q''); \mathbb{F})$ is due to a construction which uses a certain *fixed* pair of points $q, q'' \in M$, the homomorphism (4.8) needs to be adjusted. For this case we have according to Lemma C.2.1

$$H_*(\mathcal{E}^a(q, q''); \mathbb{F}) \cong H_*(\mathcal{E}^{a+d}(q, q'); \mathbb{F})$$

for some other pair of points $q, q' \in M$ the additional term d for the energy upper bound. This leads to the following adjustments of the time bounds for k Reeb chords \tilde{x}_ℓ in Proposition 4.1.19

$$\begin{aligned} \mathcal{A}(\tilde{x}_\ell) &\leq \sqrt{2(2(n(k+1)^2d)^2 + d)}, \\ \left[\mathcal{A}(\tilde{x}_\ell) &\leq \sqrt{2\left(\frac{(22(k-1)d + \text{dist}(q, q'))^2}{2} + d\right)} \right]. \end{aligned} \tag{4.10}$$

◇

In Proposition 4.1.19 we have found k Reeb chords on Σ satisfying a certain time bound. Unfortunately, we do not know anything about their Conley–Zehnder indices.

At least we can give an upper bound in terms of the geometry of M .

Corollary 4.1.20. *The Conley–Zehnder indices r_ℓ of the k Reeb chords \tilde{x}_ℓ found in Proposition 4.1.19 above must satisfy*

$$r_\ell \leq \frac{\dim M}{\rho} \sqrt{(\dim M(k+1)^2 d)^2 + \dim M}$$

$$\left[r_\ell \leq \frac{1}{\rho} \sqrt{(22(k-1)d + \text{dist}(q, q'))^2 + 8} \right]$$

where $\ell \in \{1, \dots, k\}$ and $\rho = \rho(M, g) > 0$ is the injectivity radius of (M, g) .

Observe the following simplification of the first estimate.

$$r_\ell \leq \frac{\dim M}{\rho} \sqrt{(\dim M(k+1)^2 d)^2 + \dim M}$$

$$\leq \frac{(\dim M)^2}{\rho} \sqrt{((k+1)^2 d)^2 + 1}$$

$$\leq \frac{(\dim M)^2}{\rho} ((k+1)^2 d + 1) .$$

Proof. Fix $a > 0$. Due to Proposition A.0.9 we know that there exists a constant $c = c(g)$ such that for all $k \geq 1$ it holds

$$H_k(\mathcal{E}^a(q, q'); \mathbb{F}_p) = 0 \quad \text{for } k^2 c(g) > a .$$

The constant $c(g)$ is given by $c(g) = 2 \left(\frac{\rho}{\dim M} \right)^2$ where ρ is the injectivity radius of (M, g) . Choose $q' \in V(q)$. By using the notation of the proof of Proposition 4.1.19, we consider the k Floer homology classes in $\text{HF}_*^{\frac{1}{\sigma g} \mathcal{D}(k, d, q, q') + \varepsilon}(G_+, q, q'; \mathbb{F})$ which gave rise to the existence of the k Reeb chords x_ℓ on Σ . These k classes can be interpreted via the isomorphism for $q' \in M \setminus ((V(q))^c \cup \mathcal{C}_q)$

$$\text{HF}_*^{\frac{1}{\sigma g} \mathcal{D}(k, d, q, q') + \varepsilon}(G_+, q, q'; \mathbb{F}) \longrightarrow H_* \left(\mathcal{E}^{\mathcal{D}(k, d, q, q') + \varepsilon}(q, q'); \mathbb{F} \right) , \quad (4.11)$$

where $\mathcal{C}_q := \{p \in M \mid q' \text{ conjugate to } q\}$, as representatives of non-trivial singular

homology classes in $H_{r_\ell}(\mathcal{E}^{\mathcal{D}(k,d,q,q')+\varepsilon}(q,q');\mathbb{F})$ of some degree r_ℓ . Therefore,

$$r_\ell^2 \leq \frac{1}{c(g)} \mathcal{D}(k,d,q,q').$$

For the case when $q' \notin V(q)$, one chooses as above k sequences, invokes once again Lemma 2.4.3, applies afterwards Lemma 6.2.4 to get bounds on the Conley–Zehnder indices of the k limit solutions and has hence an additional correction summand:

$$r_\ell^2 \leq \frac{1}{c(g)} (\mathcal{D}(k,d,q,q') + 2 \dim M).$$

Finally, $\mathcal{D}(k,d,q,q')$ is chosen to be equal to the expressions given in (4.7):

$$\begin{aligned} r_\ell^2 &\leq \frac{(\dim M)^2}{\rho^2} \left((\dim M(k+1)^2 d)^2 + \dim M \right) \\ \left[r_\ell^2 &\leq \frac{2}{\rho^2} \left(\frac{(22(k-1)d + \text{dist}(q,q'))^2}{2} + 4 \right) \right] \end{aligned}$$

□

4.2 If M is a Lie group or a symmetric space

We are going to prove results concerning the existence of short Reeb chords on Lie groups and symmetric spaces. Essentially, it will be used that the energy functional is a perfect Morse function on the respective path spaces. To show these facts, we apply results of Bott [12] and of Bott–Samelson [14] who use for their work a version of the Lacunary Principle 6.1.3 and in the case of symmetric spaces a version of the Completion Principle 6.1.4. Because of this and since the two approaches lead to different results and insights, we treat them individually.

4.2.1 Variationally complete Lie group actions and orbits of maximal dimension

In this section we essentially follow Bott–Samelson [14]. Let \mathcal{K} be a compact Lie group, (W, g) a Riemannian manifold and π a smooth map

$$\begin{aligned} \pi : \mathcal{K} \times W &\longrightarrow W \\ (k, p) &\longmapsto \pi(k)p := \pi(k, p). \end{aligned}$$

called the *left action* of \mathcal{K} on W . Denote by

$$\mathcal{O}(p) = \pi(\mathcal{K})p = \{\pi(k, p) \mid k \in \mathcal{K}\}$$

the \mathcal{K} -orbit of $p \in W$ in W under the action of \mathcal{K} . We write also $\mathcal{O}_H(x)$ to precise the group H acting on x . Let $V \subset W$ be the \mathcal{K} -orbit of a point $p \in W$. Then V is a smooth submanifold of W . It is well known that W can be given a Riemannian metric g_{bi} such that $\pi(k, \cdot) : W \rightarrow W$ acts by isometries, i.e. $\forall a, b \in W, k \in \mathcal{K} : d_{g_{bi}}(a, b) = d_{g_{bi}}(\pi(k, a), \pi(k, b))$. Let us do the

Convention 4.2.1. *Throughout this section we choose $(W, g) := (W, g_{bi})$.*

Denote by \mathfrak{k} the Lie algebra of \mathcal{K} identified with the tangent space to \mathcal{K} at its neutral element. The action of \mathcal{K} on W induces a representation of the Lie algebra \mathfrak{k} of \mathcal{K} by vector fields on W . These vector fields can be defined as follows. Let X be an element of \mathfrak{k} and let $h : \mathbb{R} \rightarrow \mathcal{K}$ be the corresponding 1-parameter subgroup satisfying $\dot{h}(0) = X$. The unique homomorphism h is given by $t \mapsto \exp(tX)$, see [23, Chapter 2].

Definition 4.2.2 (Infinitesimal \mathcal{K} -motions). *For $x \in W$, let $h_x : \mathbb{R} \rightarrow W$ be the curve defined by $h_x(\alpha) := \pi(h(\alpha))x$ for any $\alpha \in \mathbb{R}$. The assignment $x \mapsto \dot{h}_x(0)$ defines a vector field \overline{X} on W , which is called the infinitesimal \mathcal{K} -motion corresponding to X . More precisely, the map $\dot{h}_x(\beta) : T_\beta\mathbb{R} \rightarrow T_{h(\beta)}W$ is given by $\dot{h}_x(\beta) = d\pi(h(\beta), x)\dot{h}(\beta)$.*

The tangent space to the \mathcal{K} -orbit $\mathcal{O}(x)$ at $x \in W$ is given by the values of all possible infinitesimal \mathcal{K} -motions at x : $T_x\mathcal{O}(x) = \{\overline{X}(x) \mid X \in \mathfrak{k}\}$. On the other hand, the *stabilizer* of x given by $\mathcal{K}_x = \{k \in \mathcal{K} \mid \pi(k, x) = x\}$ is a Lie subgroup of \mathcal{K} and its Lie algebra is $\mathfrak{s}_x := \{X \in \mathfrak{k} \mid \overline{X}(x) = 0\}$.

Definition 4.2.3 (Defect and Maximal Dimension). *Let $p \in W$. The defect $\delta(p)$ of the point p is the difference between the maximum of the dimensions of \mathcal{K} -orbits in W and the dimension of the \mathcal{K} -orbit of p ,*

$$\delta(p) := \sup_{x \in W} \dim T_x \mathcal{O}(x) - \dim T_p \mathcal{O}(p).$$

We say that a point $p \in W$ has maximal dimension if its defect is zero, i.e. $\delta(p) = 0$.

As an immediate application we get the following statement: We know that for $x \in W$ it holds true that $n = \dim T_x W = \dim T_x \mathcal{O}(x) + \dim \mathfrak{s}_x$. Therefore, if $p \in W$ is of maximal dimension, it follows for x that

$$\delta(x) = \dim \mathfrak{s}_x - \dim \mathfrak{s}_p.$$

Definition 4.2.4. *A geodesic γ of W is called \mathcal{K} -transversal if for each $t \in \mathbb{R}$ the tangent vector $\dot{\gamma}(t)$ is orthogonal to the \mathcal{K} -orbit of the point $\gamma(t)$, this means that $\dot{\gamma}(t)$ is orthogonal to the tangent space of the submanifold $\mathcal{O}(\gamma(t))$ of W at $\gamma(t)$.*

Proposition 4.2.5. *The geodesic γ is \mathcal{K} -transversal if there exists a $t_0 \in \mathbb{R}$ such that $\dot{\gamma}(t_0)$ is orthogonal to $\mathcal{O}(\gamma(t_0))$.*

See [14, Page 973] for the proof. Therefore, there are a lot of such \mathcal{K} -transversal geodesics, provided that \mathcal{K} does not act transitively on W . If so, there are none since $\mathcal{O}(\gamma(t_0)) = W$.

Example. Consider the $\mathcal{K} = \mathbb{S}^1$ -action on $W = \mathbb{S}^2 \subset \mathbb{R}^3$ (with the induced metric from \mathbb{R}^3) given by the rotation of the sphere around the vertical axis.

$$R : \begin{array}{ccc} \mathbb{S}^1 \times \mathbb{S}^2 & \longrightarrow & \mathbb{S}^2 \\ \left(\alpha, \begin{pmatrix} \cos(\theta) \sin(\varphi) \\ \sin(\theta) \sin(\varphi) \\ \cos(\theta) \end{pmatrix} \right) & \longmapsto & \begin{pmatrix} \cos(\theta) \sin(\varphi + \alpha) \\ \sin(\theta) \sin(\varphi + \alpha) \\ \cos(\theta) \end{pmatrix} \end{array}.$$

This action is not transitive, but acts by isometries on \mathbb{S}^2 . All geodesics (great circles) of \mathbb{S}^2 which are \mathbb{S}^1 -transversal pass through the north and the south pole of \mathbb{S}^2 . This

means that there are infinitely many \mathbb{S}^1 -transversal geodesics. \diamond

Recall that a geodesic variation $\{V_\alpha\}$ of a geodesic γ in W is a smooth map $V : \mathbb{R} \times I \rightarrow W$ where $I \subset \mathbb{R}$ is an open interval containing 0, such that for each $\alpha \in I$ the map $V_\alpha : \mathbb{R} \rightarrow W$, defined by $V_\alpha(t) = V(t, \alpha)$ is a geodesic and $V_0 = \gamma$. Denote by J_γ the set of Jacobi fields along γ .

Definition 4.2.6. *Suppose γ is a \mathcal{K} -transversal geodesic. A Jacobi field is called transversal if it is derived from a geodesic variation $\{V_\alpha\}$ of γ in which all V_α are \mathcal{K} -transversal geodesics.*

Proposition 4.2.7. *If γ is a \mathcal{K} -transversal geodesic, then the restriction of any infinitesimal \mathcal{K} -motion to γ is a transversal Jacobi field.*

Again, see [14] for the proof.

Definition 4.2.8. *Let γ be a \mathcal{K} -transversal geodesic and let $t_0 \in \mathbb{R}$. We denote by $J_\gamma^\pi(t_0)$ the subspace of J_γ of those transversal Jacobi fields which at t_0 are tangent to the \mathcal{K} -orbit of $\gamma(t_0)$.*

Let \bar{X} be an infinitesimal \mathcal{K} -motion, which therefore restricts to a transversal Jacobi field η along the \mathcal{K} -transversal geodesic γ . Because of $T_x\mathcal{O}(x) = \{\bar{X}(x) \mid X \in \mathfrak{k}\}$ we know that η is tangent to the orbit at every point of γ , hence $\eta \in J_\gamma^\pi(t)$ for all t . Since we will be interested in the behaviour of transversal Jacobi fields at the two end points of a geodesic segment s (of some underlying geodesic γ), we introduce the following notion.

Definition 4.2.9 (Variational Completeness). *The action of \mathcal{K} on W is called variationally complete if every transversal Jacobi field η , which is tangent to the \mathcal{K} -orbits for two different points of the underlying geodesic γ (i.e. for which there exist $t_0, t_1 \in \mathbb{R}$, $t_0 \neq t_1$, such that $\eta \in J_\gamma^\pi(t_0) \cap J_\gamma^\pi(t_1)$) is induced by \mathcal{K} , i.e. is the restriction to γ of an infinitesimal \mathcal{K} -motion.*

We will introduce later on the Morse index of a geodesic segment s . If the action of \mathcal{K} on W is variationally complete, then the Morse index of s equals $\sum_{0 \leq t < 1} \delta(s(t))$ which is heavily used to show the ‘‘Completeness Condition 8.4’’ of [14, Pages 979, 980].

We want to introduce now the Morse index for geodesic segments and to relate it to the notions discussed in this section so far. Therefore, we need to consider path spaces.

Definition 4.2.10 ([12]). *If A, B are subsets of W , then $\Omega(W; A, B)$ denotes the metric space of all piecewise smooth curves u of the unit interval $[0, 1]$ in W from A to B , i.e. with $u(0) \in A$, $u(1) \in B$, parametrized proportionally to arc-length, with the distance between two curves u and u' defined as*

$$d_{\Omega(W;A,B)}(u, u') = \sup_{t \in [0,1]} d(u(t), u'(t)) + |\mathcal{L}(u) - \mathcal{L}(u')| .$$

Here d is the distance in W , and \mathcal{L} denotes the length of the curves.

Notice that $\Omega(W; A, B)$ is a subspace of the space of all continuous curves from A to B . If the latter space is topologized by the compact-open topology, then the inclusion map is continuous and induces an isomorphism in singular homology, [14, Page 968], or in [32, Chapter 17]. This fact will be used to establish the isomorphism from singular homology to Floer homology.

Let $x \in W$ and let $V = \mathcal{O}(p) \subset W$ be the \mathcal{K} -orbit of a point $p \in W$ such that $x \notin V$ to avoid trivial complications.

Notation (Transversal geodesic segments). Denote by $S = S(W; x, V) \subset \Omega(W; x, V)$ be the set of \mathcal{K} -transversal geodesic segments, parameterized on $[0, 1]$, with initial point x and with terminal point on the submanifold V . \diamond

Similarly, we denote by $S(W; V, x)$ the \mathcal{K} -transversal segments starting in V and ending in x .

For any $t_0 \in \mathbb{R}$, let $\Lambda_\gamma(t_0)$ be the subspace of J_γ consisting of those Jacobi fields vanishing at t_0 , i.e.

$$\Lambda_\gamma(t_0) := \{\eta \in J_\gamma \mid \eta(t_0) = 0\} . \tag{4.12}$$

We next define the Morse index of a geodesic segment $s \in S(W; x, V)$.

Definition 4.2.11 (Morse index). *The index λ_s (relative to V) of the geodesic segment*

$s \in S(W; x, V)$ is given by

$$\lambda_s = \sum_{0 \leq t < 1} \dim J_\gamma^\pi(1) \cap \Lambda_\gamma(t).$$

In words, λ_s is the following: Bott and Samelson call a value $t_0 \in [0, 1)$ and the point $s(t_0)$ *focal* (for V), if there exists a non-trivial Jacobi field along γ (the underlying geodesic of s) that is tangent to V for $t = 1$ and vanishes for $t = t_0$, see [14, Page 977]; this means if the space $J_\gamma^\pi(1) \cap \Lambda_\gamma(t_0)$ is of positive dimension. Hence, the Morse index λ_s equals the number of focal points of s (for V). The segment s is called *non-degenerate* if $\dim J_\gamma^\pi(1) \cap \Lambda_\gamma(0) = 0$, i.e. if $t = 0$ is not a focal value. Also in [14] it is proven that the Morse index of a geodesic segment is always a finite number, this implies that the points of maximal dimensions lie dense in W . The notion of density is understood with respect to the topology of W .

Remark. We will consider the situation when the submanifold $V \subset W$ just consists of a point $V = \{pt\}$. In this case, the tangent space of V is zero-dimensional, which implies that the condition above for a point being focal simplifies to the statement that the transversal Jacobi field must vanish in both end points, i.e. in V and in x . Then this focal point is a conjugate point, see the Definition 4.1.12. \diamond

The following statement shows the importance of the introduced notions:

Proposition ([14, Proposition 9.1, Page 979]). *Let the point x lie on a \mathcal{K} -orbit of maximal dimension and suppose the action of \mathcal{K} is variationally complete. Then all geodesic segments making up the set $S = S(W; x, V)$ are non-degenerate.*

The following proposition is a very important result for the coming steps.

Proposition 4.2.12 ([12, Proposition 4.1, Page 259]). *Let $x \in W$ be a point such that for all $s \in S(W; x, V)$ it holds true that $x = s(0)$ is not focal for V . Then the set $S(W, x, V)$ contains only a finite number of segments of length less than a given number.*

The next proposition is a direct consequence of Definition 4.2.1.

Proposition 4.2.13. *Let $p, x \in W$ be two distinct points. If $p \in W$ is a fixed point for the action of \mathcal{K} on W , i.e. if*

$$\mathcal{O}_{\mathcal{K}}(p) = \{p\},$$

then all the geodesic segments starting in x and ending in p are \mathcal{K} -transversal and therefore elements of $S(W; x, p)$.

Another consequence is that such a geodesic segment $s \in S(W; x, p)$ must be a non-degenerate critical point of the energy functional \mathcal{E} on the associated path space.

4.2.2 An excursion into symmetric spaces

In this section we recall basic properties of symmetric spaces that we shall use later on. We follow Jost [26].

Let (W, g) be a Riemannian manifold. Recall that a map $\sigma : W \rightarrow W$ is called an *isometry* if σ is a diffeomorphism of W onto itself which satisfies $\sigma^*g = g$. An isometry σ preserves distances, i.e. $d_g(\sigma(a), \sigma(b)) = d_g(a, b) \forall a, b \in W$.

Definition 4.2.14 (Symmetric Space). *A Riemannian manifold (W, g) is called symmetric if for every $p \in W$ there exists an isometry $\sigma_p : W \rightarrow W$ with*

$$\begin{aligned}\sigma_p(p) &= p, \\ d\sigma_p(p) &= -\text{Id}_{T_p W}.\end{aligned}$$

Such an isometry is also called an involution.

Lemma 4.2.15. *An involution $\sigma_p : W \rightarrow W$ of a symmetric space reverses the geodesics through p . Thus, if $\gamma : (-\varepsilon, \varepsilon) \rightarrow W$ is a geodesic with $\gamma(0) = p$ (as always parametrized by arc-length), then $\sigma_p(\gamma(t)) = \gamma(-t)$.*

Proof. The isometry σ_p maps geodesics to geodesics. If γ is a geodesic through p (with $\gamma(0) = p$), then

$$d\sigma_p(p)(\dot{\gamma}(0)) = -\dot{\gamma}(0).$$

The claim follows since a geodesic is uniquely determined by its initial point and initial direction. □

Lemma 4.2.16. *Let γ be a geodesic in the symmetric space W , $\tau \in \mathbb{R}$ and $\gamma(0) = p$, $\gamma(\tau) = x$. Then,*

$$\sigma_x(\sigma_p(\gamma(t))) = \gamma(t + 2\tau)$$

for all t for which $\gamma(t)$ and $\gamma(t + 2\tau)$ are defined. For $v \in T_{\gamma(t)}W$, $d\sigma_x(d\sigma_p(v)) \in T_{\gamma(t+2\tau)}W$ is the vector at $\gamma(t + 2\tau)$ obtained by parallel transport of v along γ .

Proof. Let $\bar{\gamma}(t) := \gamma(t + \tau)$. Then, $\bar{\gamma}$ is a geodesic with $\bar{\gamma}(0) = x$. It follows from the preceding lemma that

$$\begin{aligned} \sigma_x(\sigma_p(\gamma(t))) &= \sigma_x(\gamma(-t)) \\ &= \sigma_x(\bar{\gamma}(-t - \tau)) \\ &= \bar{\gamma}(t + \tau) \\ &= \gamma(t + 2\tau). \end{aligned}$$

Let $v \in T_pW$ and let V be the parallel vector field along γ with $V(p) = v$. Since σ_p is an isometry, $d\sigma_p(p)(V(\cdot))$ is also parallel. Moreover, $d\sigma_p(p)(V(p)) = -V(p)$. Hence, as before,

$$\begin{aligned} d\sigma_p(\gamma(t))(V(\gamma(t))) &= -V(\gamma(-t)) \\ d\sigma_x \circ d\sigma_p(\gamma(t))(V(\gamma(t))) &= V(\gamma(t + 2\tau)). \end{aligned}$$

□

A corollary of the Lemma 4.2.16 is that a symmetric space is geodesically complete, because every geodesic can be indefinitely extended. Moreover, the Hopf-Rinow theorem implies that in a symmetric space any two points can be connected by a geodesic, see [26, Section 6.3].

Let G be a connected Lie group and K a closed subgroup, and denote by (G, K) their *(Riemannian) symmetric pair*, for precise details we refer to [23, Chapter IV]. It is well known that a symmetric space is a manifold of the form G/K , where (G, K) is some symmetric pair.

Examples. The complex projective space $(G, K) = \mathbb{C}\mathbb{P}^n$ equipped with the Fubini-

Study metric is a symmetric space, in addition a Kähler manifold.

The n -spheres $(G, K) = \mathbb{S}^n$, $n \geq 2$, are symmetric spaces, in particular this is true for \mathbb{S}^2 which *does not* carry the structure of a Lie group. The isometry group of \mathbb{S}^n acts transitively on \mathbb{S}^n , so it suffices to provide an involution σ_N at the north pole $N = (0, \dots, 0, 1)$. It is given by the restriction of the map $\sigma_N : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$,

$$\sigma_N(x_1, x_2, \dots, x_{n+1}) = (-x_1, -x_2, \dots, -x_n, x_{n+1}).$$

Clearly, $\sigma_N(N) = N$ and $d\sigma_N(N)|_{\mathbb{S}^n} = -\text{Id}_{T_N\mathbb{S}^n}$.

In the case of \mathbb{R}^n equipped with the Euclidean metric, the involution at $p \in \mathbb{R}^n$ is given by the map

$$\sigma_p(x) = 2p - x.$$

Hence this space is also symmetric. ◇

4.2.3 Application to symmetric spaces

After this excursion into symmetric spaces we turn to the main questions of this thesis. Let G be a compact connected Lie group and $H \subset G$ a closed connected subgroup. We introduce on G a bi-invariant Riemannian metric g_{bi} whose existence is granted by standard Lie group theory. Then (G, H) is a symmetric pair and G/H with the induced metric (from G) is a symmetric space.

Theorem (R. Hermann, [24]). *Let \mathcal{K} be a connected subgroup of the group of isometries of a compact connected symmetric space M . Then, \mathcal{K} acts in a variationally complete manner on M .*

This theorem is a generalization of a result of Bott [12] according to which the action of G acting on itself by conjugation is variationally complete, and of Bott–Samelson [14] who prove that the action of H on G/H is variationally complete:

$$\begin{aligned} *_H : H \times G/H &\longrightarrow G/H \\ (h, gH) &\longmapsto h *_H gH := hgH \end{aligned}$$

As in Section 4.2.1, let $S(M; V, q')$ be the set of \mathcal{K} -transversal geodesic segments

on the manifold M with initial point on the submanifold $V \subset M$ and terminal point $q' \in M \setminus V$. Denote by $S_*(M; V, q')$ the \mathbb{Z}_2 -vector space generated by the elements of $S(M; V, q')$ as a basis and graded by their Morse index.

Theorem 4.2.17 (Bott–Samelson [14, Page 976]). *Let V be a \mathcal{K} -orbit of some point of M . If the action of \mathcal{K} on M is variationally complete and $q' \notin V$ lies on an orbit of maximal dimension, then there is the isomorphism*

$$H_*(\Omega(M, V, q'), \mathbb{Z}_2) \cong S_*(M; V, q').$$

Recall that the set of points $q' \in M$ satisfying $\delta(q') = 0$ – i.e. those points lying on an orbit of maximal dimension – is dense in M . The notations used are $H_*(A, \cdot) = \bigoplus_{r=0}^{\infty} H_r(A, \cdot)$, and analogously for $S_*(M; V, q')$.

Denote the neutral element of the Lie group G by e_G . This element is a fixed point of the action of G on itself by conjugation. Let

$$\begin{aligned} P : G &\longrightarrow G/H \\ g &\longmapsto P(g) := gH \end{aligned}$$

be the natural projection. Now, the image of e_G under P is a fixed point $e_{G/H} := P(e_G)$ of $*_H$: $h *_H P(e_G) = h *_H H = hH = H, \forall h \in H$. Choose in the Theorem 4.2.17 of Bott–Samelson $M = G/H$, $\mathcal{K} = H$, further as $V = \{e_{G/H}\}$ and a point $q' \neq e_{G/H}$ lying on an orbit of maximal dimension. Then we have

$$H_*(\Omega(G/H, e_{G/H}, q'), \mathbb{Z}_2) \cong S_*(G/H; e_{G/H}, q'). \quad (4.13)$$

This implies that all the geodesic segments from $e_{G/H}$ to q' are non-degenerate and homologically visible. Therefore, the energy functional \mathcal{E} is a \mathbb{Z}_2 -perfect Morse function on $\Omega(G/H; e_{G/H}, q')$ and necessarily this space is free of torsion, see Section 6.1. So, by the reasoning given in Section 4.1.1 we get for a fixed $k \in \mathbb{N}$ the time bound (4.9) for at least k Reeb chords from $\Sigma_{e_{G/H}}$ to $\Sigma_{q'}$. To get the statement for all pairs of points we

need the following isomorphism (for $D \geq 0$)

$$H_* (\mathcal{E}^D (e_{G/H}, q'); \mathbb{Z}_2) \longrightarrow H_* (\mathcal{E}^{D+d} (q, q'); \mathbb{Z}_2) \quad (4.14)$$

for $q \in G/H \setminus ((V(q'))^c \cup \{w \in G \mid \delta(w) \neq 0\})$ (recall $F^{-1}(\{1\}) = \Sigma$), to show that the previously found time-bounded Reeb chords exist for almost all pairs $(q, q') \in G/H \times G/H$. To get this isomorphism we applied Lemma C.2.1 and note that the index $*$ of homology in the isomorphism (4.14) stands for a non-negative integer, in contrast to its use in (4.13). Then we apply the isomorphism to Floer homology. Finally, we do the same steps as it was used in the proof of Proposition 4.1.19 to get the same time estimates as in (4.10) for every pair $(q, q') \in G/H \times G/H$, more precisely, we get for a fixed $k \in \mathbb{N}$ the time bound (4.10) for at least k Reeb chords from Σ_q to $\Sigma_{q'}$.

Let (G, g_{bi}) be an n -dimensional compact connected Lie group carrying a bi-invariant Riemannian metric g_{bi} and let $H \subset G$ be a closed connected subgroup of G . Recall that compactness of the symmetric space G/H implies compactness of G . Recall that we need for the sandwiching method that the Riemannian metric on G/H satisfies the Convention 4.0.5. The Riemannian metric g_{bi} can always be scaled such that it satisfies the convention and is still bi-invariant.

Theorem 4.2.18. *Let G/H be a compact symmetric space with the induced Riemannian metric satisfying the Convention 4.0.5. Further, let $q, q' \in G/H$ be two arbitrary points and fix $k \in \mathbb{N}$. Then there exist k Reeb chords $\tilde{x}_\ell : [0, 1] \rightarrow T^*(G/H)$, $\ell \in \{1, \dots, k\}$ from Σ_q to $\Sigma_{q'}$ satisfying the time bound given in (4.10) with $\mathbb{F} = \mathbb{Z}_2$.*

Note that in this proposition we did *not* assume the manifold to be simply connected. Furthermore, this proposition leads to the interest to know concrete values of $d = \text{diam}(M, g)$. We refer to Yang [52] for results on this question.

Corollary 4.2.19. *Let M be a compact symmetric space. Then we have the following lower bound for the counting function $\text{CF}_{\Sigma, q, q'}(t)$,*

$$\text{CF}_{\Sigma, q, q'}(t) \geq \frac{1}{\sqrt{2nd}} \sqrt[4]{\frac{t^2 - d}{2}} - 1, \\ \left[\text{CF}_{\Sigma, q, q'}(t) \geq \frac{\sqrt{t^2 - 2d} - \text{dist}(q, q')}{22d} + 1 \right],$$

for all $t \geq \sqrt{d}$ [$\geq \sqrt{2d}$] and for all $q, q' \in M$.

This corollary is a consequence of the results of Nabutovsky and Rotman, we refer to Section 4.7.

4.2.4 Examples

As already discussed in Section 4.2.2, examples of compact symmetric spaces are \mathbb{S}^n and $\mathbb{C}\mathbb{P}^n$. A more complicated one is the the four-dimensional symmetric space $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric $g_{\mathbb{S}^2 \times \mathbb{S}^2}$. The isometry group of $\mathbb{S}^2 \times \mathbb{S}^2$ is given by

$$\mathcal{IG}_{\mathbb{S}^2 \times \mathbb{S}^2} = \left\{ \begin{array}{l} \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) \in O(6) \\ \left. \begin{array}{l} A, D \in O(3) \text{ and } B = C = 0 \\ \text{or } B, D \in O(3) \text{ and } A = D = 0 \end{array} \right\} \subset O(6), \end{array} \right.$$

a subgroup of the group of six-dimensional orthogonal matrices $O(6)$, see the paper of Castro–Urbano [15]. Set $N = (0, 0, 1, 0, 0, 1)$. Consider the map

$$\begin{aligned} \sigma_N : \mathbb{S}^2 \times \mathbb{S}^2 &\longrightarrow \mathbb{S}^2 \times \mathbb{S}^2 \\ x &\longmapsto \sigma_N(x) := (-x_1, -x_2, x_3, -x_4, -x_5, x_6), \end{aligned}$$

where $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{S}^2 \times \mathbb{S}^2$. Obviously, if $x = N = (0, 0, 1, 0, 0, 1)$ it holds true that $\sigma_N(x) = x$ and by the transitive action of the connected component of the neutral element of the isometry group $\mathcal{IG}_{\mathbb{S}^2 \times \mathbb{S}^2}$ we can construct an involution for all points $x \in \mathbb{S}^2 \times \mathbb{S}^2$. For the details we refer to the example \mathbb{S}^n discussed in Section 4.2.2.

We can calculate lower bounds for the counting function $\text{CF}_{\Sigma, q, q'}$ of a fiberwise star-shaped hypersurface $\Sigma \subset T^*(\mathbb{S}^2 \times \mathbb{S}^2)$:

$$\text{CF}_{\Sigma, q, q'}(t) \geq \frac{1}{4\sqrt{\pi}} \sqrt[4]{\frac{t^2 - 2\pi}{2}} - 1,$$

for all $t \geq \sqrt{2\pi}$ and for all $q, q' \in \mathbb{S}^2 \times \mathbb{S}^2$.

4.2.5 The case of Lie groups

We now consider the special case of Lie groups. In this case, the results of the previous section are true over *all* coefficient fields \mathbb{F} .

Let \mathcal{K} be a compact connected Lie group, W a smooth manifold, and recall the basic notions introduced in Section 4.2.1. First, we explain the setting of Bott in [12]. Let s be a geodesic segment starting on a submanifold $V \subset W$. We consider the subset J_s of Jacobi fields $t \mapsto Y_t$ along s such that there exists a smooth variation V_α of s through geodesics with $V_0 = s$ satisfying

$$\begin{cases} Y_t f = \frac{\partial}{\partial \alpha} f(V_\alpha(t))|_{\alpha=0}, & t \in \mathbb{R}, f \in C^\infty(W) \\ V_\alpha(0) \in V, \\ \frac{\partial}{\partial t} f(V_\alpha(t))|_{t=0} \in (T_q V)^\perp, \quad q = V_\alpha(0). \end{cases}$$

A point $\bar{q} \in W$ is called a *regular* point [12] if there is no geodesic segment s (of a geodesic γ in W) starting in V and ending in q' such that at least one element η of J_s vanishes in q' . To the contrary, if there is a Jacobi field η vanishing in q' , then q' is also called a *focal point*, see [12]. Therefore, focal points generalize the notion of conjugate points given in Definition 4.1.12, i.e. these notions coincide if the submanifold V is a point, $V = \{p\}$, and because $\frac{\partial}{\partial t} f(V_\alpha(t))|_{t=0}$ must vanish.

Denote by \mathfrak{k} the Lie algebra of \mathcal{K} identified with the tangent space to \mathcal{K} at its neutral element. Choose a regular point $q' \in W \setminus \{q\}$. A geodesic segment is called *π -transversal* for the map π – as it is given in Section 4.2.1, see [12] – if its initial direction is perpendicular to the orbit $V = \mathcal{O}_{\mathcal{K}}(s(0))$ of its initial point, i.e. if the tangent vectors are orthogonal to each other. For such a π -transversal segment s one can consider the restriction of $\dot{\pi}(X)$ to vector fields *along* s , denoted by $\dot{\pi}_s(\mathfrak{k})$. Denote by $\Lambda^\pi(s)$ the set of Jacobi fields which vanish at the end-point of s . The action of \mathcal{K} on W via π is called *variationally complete* [12] if for any π -transversal geodesic segment s

$$\Lambda^\pi(s) \subset \dot{\pi}_s(\mathfrak{k}) \subset \dot{\pi}(\mathfrak{k}).$$

Notice, that this definition of variational completeness is equivalent to the one given in

Definition 4.2.9, see [14]. Let $c(p)$ be the subspace of \mathfrak{k} whose image under $\hat{\pi}$ vanishes at $p \in W$. Then we can associate to the orbit $\mathcal{O}(p)$ of p a dimension [12, 14], namely

$$\dim \mathcal{O}(p) = \dim \mathfrak{k} - \dim c(p).$$

We will say a \mathcal{K} -orbit $\mathcal{O}(p)$ has *maximal dimension* [12] if $\dim \mathcal{O}(p) = \dim \mathfrak{k}$. Let $S = S(W; V, q')$ be the set of π -transversal geodesic segments from V to q' , i.e. with initial point on V and with terminal point q' . Compare this to the Definition 4.2.1. Then we can prove the analog of Proposition 4.2.1:

Proposition ([12, Proposition 6.1, Page 261]). *Let the action of \mathcal{K} on W be variationally complete, and let V be the orbit of any point p of W under \mathcal{K} , $\mathcal{O}(p) = V$. If a regular point, $q' \notin V$ is chosen on an orbit of maximal dimension, the index $\mu_V(s)$, of any segment $s \in S(W; V, q')$ is given by*

$$\mu_V(s) = \sum_{0 < t \leq a} \Lambda^\pi \left(s|_{[0,t]} \right)$$

where $s = g|_{[0,a]}$.

So also in this slightly different setting we have the consequence that such a geodesic segment $s \in S(W; V, q')$ must be a non-degenerate critical point of the energy functional \mathcal{E} on the associated path space. In Proposition 3.2 [12] it is shown, that the Morse index $\mu_V(s)$ of s is always a finite integer, what implies directly that the regular points $q' \notin V$ lie dense.

4.2.6 Application to Lie groups

Let M be a closed and simply-connected Lie group. In the notation introduced above, define $W := M$ and $\mathcal{K} := M$, fix $\pi : M \times M \rightarrow M$ to be the adjoint action of M on M given by $(g, h) \mapsto g \cdot h \cdot g^{-1}$ and set $q' \in M$. Choose a submanifold $V = \mathcal{O}(p)$, where $q' \neq p \in M$. Bott's version of the Morse series

$$\mathcal{M}(M, V, q'; t) = \sum_{s \in S(M, V, q')} t^{\mu_V(s)}$$

runs over all geodesic segments $s \in S(M, V, q')$ by adding up $t^{\mu_V(s)}$. This formal series is equivalent to the left hand side in (4.6). Note that to consider this series makes sense only if all the coefficients are finite numbers. But there are only a finite number of geodesic segments in $S(M, V, q')$ of length less than a given number, see [12, Proposition 4.1]. Let $H_k(\Omega(M, V, q'); \mathbb{F})$ denote the singular homology group of degree k with respect to some coefficient field \mathbb{F} , then the *Poincaré series* of $\Omega(M, V, q')$ relative to \mathbb{F} is given as follows:

$$\mathcal{P}(\Omega(M, V, q'); \mathbb{F}; t) := \sum_{k=0}^{\infty} \dim H_k(\Omega(M, V, q'); \mathbb{F}) t^k.$$

In [12, Section 10], Bott proves the following theorem.

Theorem (Bott [12]). *The space of paths $\Omega(M, V, q')$ is free of torsion. Its odd Betti numbers vanish, and the Poincaré series of $\Omega(M, V, q')$ coincides with the Morse series of $\Omega(M, V, q')$.*

The maximal orbits build an open set in M , so q' can always be chosen to lie on a maximal orbit, see the remark after Proposition 6.1 in [12]. Note that the theorem by Bott holds for *all* coefficient fields \mathbb{F} . The theorem follows from the Morse inequalities, which say that the Morse series dominate the Poincaré series, from the Lacunary Principle, see Theorem 6.1.3, and from the fact that the adjoint action of M on M is variationally complete, see [12, Sections 7 and 8].

Choose a *bi-invariant* Riemannian metric g on M (as the Lie group above). Recall that on every compact Lie group there exists such a bi-invariant Riemannian metric. This choice of Riemannian metric is made by Bott in his proof of variational completeness of the adjoint action, in particular to have the geodesics in the form of translates of one-parameter subgroups of M .

Let $N = \{e\}$ be the orbit of the unit element $e \in M$ under the adjoint action of M on M , meaning that we consider the space of paths $\Omega(M, e, q')$. The point q' is a regular point, so it is not conjugate to the neutral element e . So on the space $\Omega(M, e, q')$ the energy functional \mathcal{E} is an \mathbb{F} -perfect Morse function, because q' was assumed to lie on an orbit of maximal dimension and the adjoint action is variationally complete. This in contrast to the result for symmetric spaces in Section 4.2.3. Therefore, by the reasoning in the last section we get for a fixed $k \in \mathbb{N}$ the time bound (4.9) for at least k Reeb

chords from Σ_e to $\Sigma_{q'}$. To get the general statement we need the isomorphism (for $D \geq 0$)

$$H_* (\mathcal{E}^D(e, q'); \mathbb{F}) \longrightarrow H_* (\mathcal{E}^{D+d}(q, q'); \mathbb{F})$$

for $q \in M \setminus ((V(q))^c \cup \mathcal{C}_q)$, to show that the previously found time-bounded Reeb chords exist for almost all pairs $(q, q') \in M \times M$, then we apply the isomorphisms to the Floer homology groups. Finally, we do the same steps as it was used in the proof of Proposition 4.1.19 to get the same time estimates as in (4.10) for every pair $(q, q') \in M \times M$, more precisely, we get for a fixed $k \in \mathbb{N}$ the time bound (4.10) for at least k Reeb chords from Σ_q to $\Sigma_{q'}$. So we can state the next

Proposition 4.2.20. *Let (M, g_{bi}) be an n -dimensional, compact simply-connected Lie group carrying a bi-invariant Riemannian metric g_{bi} satisfying the Convention 4.0.5. (Recall the remarks done before Theorem 4.2.18.) Further, let $q, q' \in M$ be two arbitrary points and fix $k \in \mathbb{N}$. Then there exist at least k Reeb chords from Σ_q to $\Sigma_{q'}$ satisfying the time bound given in (4.10) with arbitrary coefficient field \mathbb{F} .*

4.3 Spheres, CROSSes and their products

Let $q, q' \in M$ and denote by $\mathcal{T}_k(\Sigma, q, q')$ the time of the k -th fastest Reeb chord from Σ_q to $\Sigma_{q'}$ for any $q' \in M$. For more details and the proofs of the propositions, we refer to Schlenk–Wulschleger [45].

4.3.1 Spheres and CROSSes

Let (M, g) be a round sphere S^d , $d \geq 2$, or a CROSS (compact rank one symmetric space), namely one of $\mathbb{R}P^d$, $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{C}aP^2$. We scale the symmetric Riemannian metric such that $\text{diam}(M, g) = \pi$. Then $\mathbb{C}P^1 = S^2$ and $\mathbb{H}P^1 = S^4$. We thus agree that $n \geq 2$. Let $\Sigma \subset T^*M$ be a fiberwise starshaped hypersurface. For convenience, we scale Σ (instead of g) such that $D\Sigma \subset D\Sigma_g$ (which is equivalent to Convention 4.0.5). Fix $q \in M$.

Proposition 4.3.1. (i) If $q' \neq q$ is not Σ -conjugate to q , then

$$\mathcal{T}_k(\Sigma, q, q') \leq k\pi \text{ for all } k \geq 1. \quad (4.15)$$

(ii) If q' is Σ -conjugate to q , then the estimate (4.15) still holds true for S^d and $\mathbb{R}P^d$, while for $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{C}aP^2$ we have

$$\mathcal{T}_k(\Sigma, q, q') \leq (2k - 1)\pi \text{ for all } k \geq 1.$$

All geodesics emanating from q are closed of length 2π . The first conjugate point is at distance π , besides for $\mathbb{R}P^d$ where this point is q itself, and its multiplicity is

S^d	$\mathbb{R}P^d$	$\mathbb{C}P^n$	$\mathbb{H}P^n$	$\mathbb{C}aP^2$
$d - 1$	$d - 1$	1	3	7

(4.16)

Assume that q' is not g -conjugate to q . Then $\delta := \text{dist}_g(q, q') \in (0, \pi)$. The lengths of the geodesics from q to q' are

$$\delta, 2\pi - \delta, 2\pi + \delta, 4\pi - \delta, 4\pi + \delta, \dots$$

In particular, the length of the k 'th geodesic is $< k\pi$. In view of Table (4.16) the Morse indices of these geodesics are

S^d	$\mathbb{R}P^d$	$\mathbb{C}P^n$	$\mathbb{H}P^n$	$\mathbb{C}aP^2$
$(d - 1)j$	$(d - 1)j$	$dj, dj + 1$	$(d + 2)j, (d + 2)j + 3$	$22j, 22j + 7$

(4.17)

where $j \geq 0$. On $\mathbb{R}P^d$ there are two geodesics of index $(d - 1)j$, and they lie in different components of $\Omega \mathbb{R}P^d$. The table is readily compiled by recalling that $\dim \mathbb{C}P^n = 2n$, $\dim \mathbb{H}P^n = 4n$, $\dim \mathbb{C}aP^2 = 16$.

Lemma 4.3.2. $H_k(\Omega M; \mathbb{Z}_2) = \mathbb{Z}_2$ if k appears in Table 4.17, and $H_k(\Omega M; \mathbb{Z}_2) = 0$ otherwise.

Proof. For all M different from S^2 and $\mathbb{C}P^n$ there are no consecutive numbers $k, k + 1$ in Table 4.17, and so the claim follows from the Morse Lacunary principle, Theorem 6.1.3.

To cover also the cases S^2 and $\mathbb{C}P^n$, we appeal to the Bott–Samelson Theorem 4.2.17, according to which – as it is explained in Section 4.2.3 – the energy functional \mathcal{E}_g on the based loop space of a compact symmetric space is \mathbb{Z}_2 -perfect, i.e., each index k critical point of \mathcal{E}_g gives a \mathbb{Z}_2 -summand of $H_k(\Omega M; \mathbb{Z}_2)$. \square

4.3.2 Products of Symmetric spaces

Let (M_1, g_1) , (M_2, g_2) be two closed Riemannian manifolds. Endow the product $M = M_1 \times M_2$ with the Riemannian metric $g = g_1 \oplus g_2$. Geodesics on this product are of the form $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ with geodesics γ_i on (M_i, g_i) . Their length is $\mathcal{L}_g(\gamma) = \sqrt{\mathcal{L}_{g_1}^2(\gamma_1) + \mathcal{L}_{g_2}^2(\gamma_2)}$. A point $q = (q_1, q_2)$ is not g -conjugate to $q' = (q'_1, q'_2)$ if and only if q_i is not g_i -conjugate to q'_i for both $i = 1, 2$, and in this case the Morse index of a geodesic γ from q to q' is the sum of the Morse indices of γ_1 and γ_2 .

Products of symmetric spaces are still symmetric spaces, and so by [14] their energy functional is still a perfect Morse function for non-conjugate points. Given a starshaped hypersurface $\Sigma \subset T^*M$ with $D\Sigma \subset D\Sigma_g$ we can thus again estimate $\mathcal{T}_k(\Sigma, q, q')$ from above in terms of the length-spectrum of g .

Proposition 4.3.3. (i) *If $q' \neq q$ is not Σ -conjugate to q , then $\mathcal{T}_k(\Sigma, q, q') \leq \ell_k$, where ℓ_k is the k -th number in the sequence obtained from arranging the numbers $\sqrt{m^2 + n^2} \pi$ with $m, n \in \mathbb{N}$ in increasing order. In particular,*

$$\limsup_{k \rightarrow \infty} \mathcal{T}_k(\Sigma, q, q') / \sqrt{k\pi} \leq 1.$$

(ii) *If q' is Σ -conjugate to q , then $\mathcal{T}_k(\Sigma, q, q') \leq k\sqrt{2}\pi$.*

4.4 Group-theoretic considerations

So far, we considered the closed and connected Riemannian manifold (M, g) to be simply-connected, or we restricted g to be a bi-invariant Riemannian metric. But note that we assume g to satisfy the Convention 4.0.5. Recall the results on symmetric spaces and Lie groups. In this section we forgo this geometric restrictions and we try to say something about quantitative existence results of Reeb chords on Σ under certain

topological assumptions. More precisely, we will use information about the order of the fundamental group $\pi_1(M, \cdot)$.

Theorem 4.4.1. *Let $q \in M$. If $\pi_1(M, q)$ has infinite or finite order $\geq k \in \mathbb{N}$, then for every $q' \in M$ there exist at least k Reeb chords \tilde{x}_ℓ , $\ell \in \{1, \dots, k\}$, from Σ_q to $\Sigma_{q'}$ satisfying the time bound*

$$\mathcal{T}(\tilde{x}_\ell) \leq kd.$$

Proof of Theorem 4.4.1. We use the following theorem by Nabutovsky and Rotman:

Theorem (Nabutovsky–Rotman, see Theorem 4.7.3). *If the fundamental group $\pi_1(M)$ of a closed Riemannian manifold M of diameter d is either infinite or finite of order $\geq k$, then for every pair of points $q, q' \in M$ there exist at least k geodesics connecting q and q' of length $\leq kd$ that lie in different homotopy classes.*

To prove this theorem, Nabutovsky and Rotman use Gromov’s beautiful result that we recall as Proposition 4.4.2 below.

Scale g such that $G \leq F$ (Convention 4.0.5). We know,

$$\dim H_0 \left(\mathcal{E}^{\frac{1}{2}r^2}(q, q'); \mathbb{F}_p \right) = \#\pi_0 \left(\mathcal{E}^{\frac{1}{2}r^2}(q, q') \right) = \Pi_{\mathcal{L}}^r(q, q'),$$

where $\Pi_{\mathcal{L}}^r(q, q')$ is the set of homotopy classes of $(1, 2)$ -Sobolev paths $x: [0, 1] \rightarrow M$ from q to q' which can be represented by a path of length at most r . So,

$$\#\pi_0 \left(\mathcal{E}^{\frac{1}{2}(kd)^2}(q, q') \right) = \Pi_{\mathcal{L}}^{kd}(q, q') \geq k$$

since every geodesic segment guaranteed by the theorem of Nabutovsky–Rotman is a path of $(1, 2)$ -Sobolev type.

Let $q \in M$ and $q' \in U_\Sigma(q)$ be not g -conjugate. Assume that $\frac{1}{2}(kd)^2 \notin \mathcal{S}(F, q, q') \cup \mathcal{S}(G, q, q')$ (if not, we add an $\varepsilon > 0$).

So the rank of

$$\iota_0: H_0 \left(\mathcal{E}^{\frac{1}{2}(kd)^2}(q, q', \alpha); \mathbb{F}_p \right) \longrightarrow H_0 \left(\mathcal{E}^{\sigma_g \frac{1}{2}(kd)^2}(q, q', \alpha); \mathbb{F}_p \right)$$

is ≥ 1 for $\alpha \in \Pi_{\mathcal{L}}^{kd}(q, q')$ which follows from Lemma 3.2.3. Hence

$$\dim \text{HF}_0^{\frac{1}{2}(kd)^2}(K, q, q', \alpha; \mathbb{F}_p) \geq 1. \quad (4.18)$$

Now, fix $q' \notin U_{\Sigma}(q)$ or q' g -conjugate to q . Let $(q'_n)_{n \in \mathbb{N}}$ be a sequence such that $q'_n \in U_{\Sigma}(q)$, not g -conjugate to q for all n and $q'_n \rightarrow q'$ as $n \rightarrow \infty$.

Define the space of action bounded solutions lying in homotopy class α .

$$\text{Sol}_{\frac{1}{2}(kd)^2}(K, M, \alpha) := \left\{ \gamma \in W_{\alpha}^{1,2}([0, 1], T^*M) \mid \mathcal{A}_K(\gamma) \leq \frac{1}{2}(kd)^2, \dot{\gamma} = X_K(\gamma(t)) \right\}$$

By the steps above, in particular, because the right hand side of (4.18) does not depend on the point q' , we find for all n a solution $\gamma_n \in \text{Sol}_{\frac{1}{2}(kd)^2}(K, M, \alpha)$ satisfying $\gamma_n(0) \in T_q^*M$ and $\gamma_n(1) \in T_{q'_n}^*M$.

The space $\text{Sol}_{\frac{1}{2}(kd)^2}(K, M, \alpha)$ is bounded in the C^0 -norm due to Lemma 4.8.2 and thanks to Lemma 4.8.3 equicontinuous. By the (continuous) Sobolev embedding

$$W^{1,2}([0, 1], T^*M) \longrightarrow C^0([0, 1], T^*M)$$

we find in every equivalence class $\gamma \in \text{Sol}_{\frac{1}{2}(kd)^2}(K, M, \alpha)$ a continuous representative, denoted by γ_{cont} . The set of all those γ_{cont} is by the Arzelà–Ascoli theorem relatively compact in $C^0([0, 1], T^*M)$, so we know that there exists a solution $\gamma_{\infty} \in \text{Sol}_{\frac{1}{2}(kd)^2}(K, M, \alpha)$ satisfying $\gamma_{\infty}(0) \in T_q^*M$ and $\gamma_{\infty}(1) \in T_{q'}^*M$. Finally, we restrict all these flows to Σ and reparametrize them, to get the k different Reeb chords. \square

As already stated one could give here an estimation of the Conley–Zehnder indices, analogous to Corollary 4.1.20.

We close this section with a discussion of the first part of M. Gromov’s proposition [22, Proposition 3.22] mentioned above. This wonderful result is essential for the purpose of this section and it gives a characterization of the fundamental group of a compact Riemannian manifold.

Proposition 4.4.2 (Gromov [22, Proposition 3.22]). *Let (M, g) be a compact Riemannian manifold. For each $q \in M$, the fundamental group $\pi_1(M, q)$ is generated by a finite*

number of classes represented by loops based at q having length at most $2 \operatorname{diam}(M, g)$.

Proof. For each element α of the fundamental group $\pi_1(M, q)$, we choose a representative loop a based at $q \in M$. For $\varepsilon > 0$ we divide each loop a into pieces of length less than ε . Connect by following a minimizing geodesic each of the endpoints of these segments with q , so if the endpoint of some segment is called x , then we denote by c_x the minimizing geodesic going from q to x , see Figure 4.2. Then a is homotopic to the *product of the loops* $c_x a|_{[x, x']} c_{x'}^{-1}$ based at q , where x, x' denotes the consecutive pair of endpoints (bounding a subsegment of a from x to x' of length less than ε) along a . Therefore, we

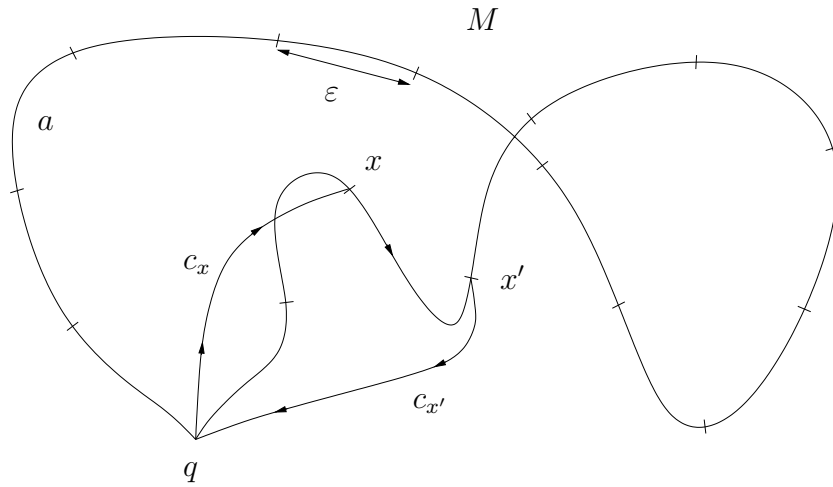


Figure 4.2: The idea of Gromov's construction: The path along a from x to x' is $a|_{[x, x']}$. So if one starts in q , travels along c_x , then $a|_{[x, x']}$ and via $c_{x'}$ back to q , one traversed a closed loop of length smaller than $2 \operatorname{diam}(M) + \varepsilon$. The figure is taken from [22].

have found a system of generators for $\pi_1(M, q)$ represented by loops of length less than $2 \operatorname{diam}(M) + \varepsilon$. It follows from the Arzelà–Ascoli Theorem that for each $a > 0$ there are only finitely many elements of $\pi_1(M, q)$ that can be represented by a geodesic of length $\leq a$. Hence the set of lengths of geodesics that are minimal in these homotopy class is discrete. So, for sufficiently small ε , the interval $(2 \operatorname{diam}(M), 2 \operatorname{diam}(M) + \varepsilon)$ contains no length of a closed minimizing geodesic based at q . Consequently, each of the loops $c_x a|_{[x, x']} c_{x'}^{-1}$ we have constructed is homotopic to a loop of length at most $2 \operatorname{diam}(M)$ and all the homotopy classes in $\pi_1(M, q)$ have representatives that can be generated by products of those loops $c_x a|_{[x, x']} c_{x'}^{-1}$. \square

4.5 Manifolds of non-positive curvature

It is well-known that geodesics on Riemannian manifolds (M, g) of non-positive curvature have no conjugate points. This means that the Morse indices of all geodesics are zero. Therefore by Morse homology they all are representants of (different) homology classes of $H_0(\{CM_*(\mathcal{E}_g), \partial_*\})$. So every geodesic is homologically visible and hence \mathcal{E} is an \mathbb{F} -perfect Morse function on $\Omega_{q,q'}^1 M$ for all pairs of points $q, q' \in M$.

4.6 Other results

Proposition 4.6.1 (Existence of Reeb chords). *If M is simply-connected and $q, q' \in M$, then there are infinitely many Reeb chords from Σ_q to $\Sigma_{q'}$.*

Remark. For almost all $q' \in M$, technically if q' lies in a dense subset $V(q)$ (2.11) of M , see Section 2.2, the Abbondandolo–Schwarz and the Abbondandolo–Majer isomorphisms, see [3] and [2], establish immediately infinitely many Reeb chords from Σ_q to $\Sigma_{q'}$. Via a part of the steps which will be explained in the proof below. \diamond

Proof. Recall the result of Serre’s thesis, see [46]: For any simply-connected space W there is an infinite number of integers k for which the Betti numbers $b_k(\Omega W, \mathbb{F}_p)$ of the corresponding (based) loop space ΩW are not zero, so

$$\#\{k \in \mathbb{Z} \mid b_k(\Omega W, \mathbb{F}_p) \neq 0\} = +\infty.$$

Hence, this holds for $W = M$. Let $\dim M = n$ and choose a subsequence k_ℓ , $\ell \in \mathbb{N}$, of those k ’s for which $b_k(\Omega M, \mathbb{F}_p) \neq 0$ such that

$$2n < k_1 < k_2 < \dots$$

and

$$|k_\ell - k_{\ell-1}| > 2n.$$

Fix an element k_ℓ of this subsequence. Take $q \in M$ and $q' \notin V(q)$, see (2.10) and (2.11). We know that there is the homotopy equivalence from the space of (based) loops

$\Omega_{p,p}M = \Omega M$ on M (we omit the base point, since M is simply-connected) to the space of paths from p to p' :

$$\Omega M \rightarrow \Omega_{p,p'}M.$$

By Lemma [27, Page 15] we know that the inclusion

$$\Omega_{p,p'}^1 M \hookrightarrow \Omega_{p,p'}M$$

is a homotopy equivalence, what implies that their homology groups are isomorphic, so because of $b_{k_\ell}(\Omega M, \mathbb{F}_p) \neq 0$ we have

$$0 \neq b_{k_\ell}(\Omega M, \mathbb{F}_p) = b_{k_\ell}(\Omega_{q,q'}M, \mathbb{F}_p) = b_{k_\ell}(\Omega_{q,q'}^1 M, \mathbb{F}_p).$$

Therefore, we know that there exists an element given by

$$0 \neq \alpha \in H_{k_\ell}(\Omega_{q,q'}^1 M, \mathbb{F}_p).$$

Now, choose a sequence $q_n \rightarrow q' \notin V(q)$ with $q_n \in V(q), \forall n \in \mathbb{N}$. So we find for all n nontrivial cycles α_n

$$0 \neq \alpha_n \in H_{k_\ell}(\Omega_{q,q_n}^1 M, \mathbb{F}_p).$$

The theorem of Gromov, see Theorem A.0.5, implies the Lemma 4.8.6 (here we use that M is assumed to be simply-connected). This means that we can consider the cycle α_n as an element in $H_{k_\ell}(\mathcal{E}^{B(k_\ell)}(q, q_n), \mathbb{F}_p)$ for all $n \in \mathbb{N}$. Now, since every $q_n \in V(q)$, the Floer homology groups $\text{HF}_*^a(K, q, q_n; \mathbb{F}_p)$ are defined, see Section 2.2. For every $n \in \mathbb{N}$ there is the homomorphism induced by inclusion

$$\iota_{k_\ell} : H_{k_\ell}(\mathcal{E}^{B(k_\ell)+\varepsilon}(q, q_n), \mathbb{F}_p) \longrightarrow H_{k_\ell}(\mathcal{E}^{\sigma B(k_\ell)+\varepsilon}(q, q_n), \mathbb{F}_p),$$

($B(k_\ell)$ is a strictly monotonically increasing function $B: \mathbb{Z} \rightarrow \mathbb{R}$ and $\varepsilon > 0$ is added if $B(k_\ell)$ or $\sigma B(k_\ell)$ should be an element of the action spectrum of F or G). By the same arguments as in the proof of Theorem 3.2.1, we conclude that the rank of ι_{k_ℓ} does not vanish. Hence by Proposition 2.2.5 it follows that there exists a solution

$\gamma_n: [0, 1] \rightarrow \Sigma \subset T^*M$ of Hamilton's equations

$$\dot{x} = X_K(x) \tag{4.19}$$

starting from Σ_q and ending in Σ_{q_n} with Conley–Zehnder index $\mu(\gamma_n) = k_\ell$ and action $\mathcal{A}_K(\gamma_n) < B(k_\ell)$. So we end up with a sequence of solutions of (4.19) with bounded action and Conley–Zehnder index k_ℓ . It follows trivially that

$$\gamma_n \in \text{Sol}_{B(k_\ell)}(M, K), \forall n \in \mathbb{N}.$$

We apply Lemma 2.4.1 with $a = -\infty$ and $b = B(k_\ell)$. Then there exists a limit solution $\gamma \in \mathcal{P}^{B(k_\ell)}(K, q, q')$.

According to Proposition 6.2.4 the Conley–Zehnder index of γ satisfies

$$\mu(\gamma) \in [k_\ell - 2n, k_\ell + 2n].$$

But we have assumed the k_ℓ 's to satisfy $|k_\ell - k_{\ell-1}| > 2n$. Therefore, if we do the whole process for every k_ℓ chosen, we get infinitely many different solutions of the Hamiltonian equations starting in T_q^*M and ending in $T_{q'}^*M$. Finally, we restrict all the solutions γ to Σ and reparameterize them according to Lemma 4.8.4. So we find infinitely many Reeb chords starting in Σ_q and ending in $\Sigma_{q'}$. This concludes the proof. \square

Proof of Proposition 4.1.1. Let $q \in M$ be the base point of the based loop space $\Omega_q M := \Omega_{q,q} M$. By Schwarz's Lemma B.0.11, there is for a certain $r \in \mathbb{N}_0$ a homology class $\alpha \in H_r(\Omega_q M, \mathbb{F}_p)$ such that all its Pontryagin products α^k , $k \in \mathbb{N}$, are non zero elements in $H_{kr}(\Omega_q M, \mathbb{F}_p)$. (One could give here directly an estimation of the length of the images of α^k , but this is not needed.)

Let G_+ be the Riemannian Hamiltonian $G_+(q, p) = \sigma_g \frac{1}{2} |p|^2$ introduced in (2.10) and recall that we had $\Sigma = K^{-1}(\frac{1}{2})$. Choose a pair $q, q' \in M$ such that $q' \in V(q)$. We know due to the proof of Proposition 4.6.1 that

$$\alpha^k \in H_{kr}(\Omega_q M, \mathbb{F}_p) \cong H_{kr}(\Omega_{q,q}^1 M, \mathbb{F}_p) \cong H_{kr}(\Omega_{q,q'}^1 M, \mathbb{F}_p) .$$

Therefore, by Lemma 4.8.6 $\alpha^k \in \mathbb{H}_{kr} \left(\mathcal{E}^{\frac{1}{2}(\underline{C}kr)^2}(q, q'), \mathbb{F}_p \right)$.

We have the homomorphism induced by inclusion

$$\iota_{kr} : \mathbb{H}_{kr} \left(\mathcal{E}^{\frac{1}{2}(\underline{C}kr)^2 + \varepsilon}(q, q'), \mathbb{F}_p \right) \rightarrow \mathbb{H}_{kr} \left(\mathcal{E}^{\sigma \frac{1}{2}(\underline{C}kr)^2 + \varepsilon}(q, q'), \mathbb{F}_p \right),$$

where $\varepsilon > 0$ and $\sigma \geq \sigma_g \geq 1$ are chosen such that $\frac{1}{2}(\underline{C}kr)^2, \sigma \frac{1}{2}(\underline{C}kr)^2$ are not elements of the action spectrum of F and G . (Note that we could also consider the homomorphism $\iota_{kr} : \mathbb{H}_{kr} \left(\mathcal{E}^{\frac{1}{2}(\underline{C}kr)^2 + \varepsilon}(q, q'), \mathbb{F}_p \right) \rightarrow \mathbb{H}_{kr} \left(\mathcal{E}(q, q'), \mathbb{F}_p \right)$.) The rank of ι_{kr} satisfies

$$\text{rank}(\iota_{kr}) \geq \dim \mathbb{H}_{kr} \left(\mathcal{E}^{\frac{1}{2}(\underline{C}kr)^2 + \varepsilon}(q, q'), \mathbb{F}_p \right) \geq 1.$$

This allows us to invoke Proposition 2.2.5 which implies the existence of k Reeb chords (according with Lemma 4.8.4 as reparameterized restrictions of the Hamiltonian flow lines of K to Σ), denoted by $\tilde{x}_\ell : [0, 1] \rightarrow \Sigma \subset T^*M$ for $\ell \in \{1, 2, \dots, k\}$, starting in a point $\tilde{x}_\ell(0) \in \Sigma_q$ and ending in another point $\tilde{x}_\ell(1) \in \Sigma_{q'}$. We then apply Lemma 2.2.2 and get

$$\mathcal{A}(\tilde{x}_\ell) = \sqrt{2 \mathcal{A}_K(x_\ell)} \leq \underline{C}kr + 2\varepsilon$$

for an appropriate $\varepsilon > 0$ and all $\ell \in \{1, 2, \dots, k\}$. The latter estimation gives an upper bound for the reduced action of the k Reeb chords from Σ_q to $\Sigma_{q'}$ for $q' \in V(q)$. By using the Lemma 2.4.3 as it was done in the proof of Theorem 3.2.1, one gets the statement for every pair of points $q, q' \in M$.

Fix $\mathbb{F}_p = \mathbb{Q}$. Sullivan explains for this choice of coefficients in [49, Page 45] how the knowledge of the minimal model of M has as a consequence the following relation between the generators of the first algebra, called γ_i , and those of the minimal model of the associated based loop space of M denoted by $\bar{\gamma}_i$: $\dim \gamma_i = \dim \bar{\gamma}_i + 1$. Hence, we know that the natural number r chosen above must satisfy $r \leq \dim(M) - 1 = n - 1$. Therefore,

$$\mathcal{A}(\tilde{x}_\ell) \leq \underline{C}k(n - 1),$$

for all $\ell \in \{1, 2, \dots, k\}$. This concludes the proof. \square

4.7 On the results of Nabutovsky and Rotman

In Section 4.1.1 we introduced the notion of a homological visible geodesic segment. Later on, we give an explanation of perfect Morse functions in chapter 6. If the path space $\Omega_{q,q'}^1 M$ of a given manifold M admits the energy functional as a perfect Morse function, then we can use infinite-dimensional Morse theory to understand all the geodesic segments from q to q' (on M) as homologically visible. Therefore, we are primarily interested in existence results of geodesic segments from q to q' (and later on we will wonder whether the energy functional is a perfect Morse function on $\Omega_{q,q'}^1 M$). In this section we give a short overview of some existing results concerning the first question.

A. Nabutovsky and R. Rotman have studied deeply the following question: Is there a function $f : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{R}_0^+$ such that for every n -dimensional closed Riemannian manifold M of diameter d and every pair of points $q, q' \in M$ there exist at least k geodesic segments from q to q' of length at most $f(k, n)d$? See for example [35], [36] or [37]. An example explained in [10] shows that one cannot take $f(k, n) = k$ as it is possibly suggested by the round two sphere, by starting in the north-pole going along one of the great-circles to the south-pole and afterwards back by following the same great-circle to the north-pole and so on.

The most general answer to this question until today is given in [37] by $f(k, n) = 4nk^2$, where q and q' are not required to be distinct. But first we focus on the following result which concerns the two-dimensional case:

Theorem 4.7.1 (Nabutovsky–Rotman [36]). *Let M be a smooth Riemannian manifold of diameter d diffeomorphic to S^2 . Then for each pair of points $q, q' \in M$ there exist at least k geodesic segments connecting the points q and q' of length at most $(22k - 22)d + \text{dist}(q, q') \leq (22k - 21)d < 22kd$.*

This means that in the two-dimensional case one can take $f(k, n) = 22k$. In fact, this theorem implies the same upper bound for all closed Riemann surfaces. Generally, for all closed n -dimensional Riemannian manifolds ($n > 1$) they have the following result.

Theorem 4.7.2 (Nabutovsky–Rotman [37]). *Let M be a closed n -dimensional Riemannian manifold with diameter d . Then for each pair of points $q, q' \in M$ there exist at*

least k geodesic segments starting at q and ending in q' of length at most

$$((2n - 1.5)k^2 + (2n - 3.5)k - (1 - (-1)^k))d + (2n - 1.5)k \operatorname{dist}(q, q'), \text{ if } k \text{ is even,}$$

and on the other hand, if k is odd

$$((2n - 1.5)k^2 + (2n - 3.5)k - (1 - (-1)^k))d + (2n - 1.5)(k + 1) \operatorname{dist}(q, q').$$

Remark. Nabutovsky–Rotman mention in [37] that both of these length estimations for k odd and k even can be majorized by

$$((2n - 1.5)k^2 + (4n - 5)k - (2n - 3.5))d < 2n(k + 1)^2d < 4nk^2d. \quad (4.20)$$

Note that for the two shortest geodesic segments they have the better upper bound $2nd (< 4nk^2d)$ on their lengths, see [34]. \diamond

A different approach is pursued by Nabutovsky and Rotman in the proof of the following theorem:

Theorem 4.7.3 (Nabutovsky–Rotman [37]). *If the fundamental group $\pi_1(M)$ of a closed Riemannian manifold M of diameter d is either infinite or finite of order $\geq k$, then for every pair of points $q, q' \in M$ there exist at least k geodesics connecting q and q' of length $\leq kd$ that lie in different homotopy classes.*

This theorem gives under the assumption on the fundamental group $\pi_1(M)$ very precise and nice results on the length of geodesic segments from q to q' on M . With this approach Nabutovsky and Rotman find the desired form of the function f . Indeed, $f(k, n) = k$. To prove this theorem, they use essentially the nice Proposition 4.4.2 on the fundamental group of a compact Riemannian manifold M given by Gromov and a notion called “complexity” which measures the minimal length of a word in terms of generators of the fundamental group.

In a series of results concerning the existence of geodesic segments, Nabutovsky and Rotman considered also estimations of the two shortest ones. This result is remarkable and nice in its simplicity. The constants which arise are for our purposes very convenient.

Theorem 4.7.4 (Nabutovsky–Rotman [34]). *Let M be a closed n -dimensional Riemannian manifold, $T = \min_i \{i \mid \pi_i(M) \neq 0\}$, and let d be the diameter of (M, g) . Then for each pair of points $q, q' \in M$ there exist at least two distinct geodesics from q to q' of length not exceeding $2Td$ ($\leq 2nd$).*

4.8 Tools and technical lemmas

We will need to do estimations and calculations. For this purpose we would like to understand the behavior of the Hamiltonian function K introduced in (2.10) for large momenta, i.e. for $|p|_{g^*}$ large.

Recall that the set $D(4b) \subset T^*M$ is compact and that K is smooth. Then the following two quantities are welldefined.

$$K_{\max} := \max_{(q,p) \in D(4b)} |K(q,p)| \quad (4.21)$$

$$G_{\max,q} := \max_{(q,p) \in D(4b)} |\nabla_q K(q,p)| \quad (4.22)$$

$$G_{\max,p} := \max_{(q,p) \in D(4b)} |\nabla_p K(q,p)| \quad (4.23)$$

$$G_{\max} := \max \{G_{\max,q}, G_{\max,p}\} . \quad (4.24)$$

By ∇_q resp. ∇_p we denote the gradient of K with respect to the base- resp. the fiber variables $q = (q_1, \dots, q_n)$ and $p = (p_1, \dots, p'_n)$. Further, we have shown in Section 2.2.1 that the Hamiltonians (2.10) satisfy $G_- \leq K \leq G_+$. Since we know that G_- and the Riemannian Hamiltonian G_+ grow quadratically in the fiber variables for p larger than $4b$, see Figure 2.2, we can deduce the following estimates.

$$dK(q,p)(\eta) - K(q,p) \geq |p|^2 - K_{\max} \quad (4.25)$$

$$|\nabla_q K(q,p)| \leq G_{\max,q} (|p|^2 + 1) \quad (4.26)$$

$$|\nabla_p K(q,p)| \leq G_{\max,p} (|p| + 1) \quad (4.27)$$

Note that $\eta = \sum_{i=1}^n p_i \frac{\partial}{\partial p_i}$ is the Liouville vector field. The estimate (4.25) follows from the fact $dK(q,p)(\eta) = \langle \nabla K(q,p), \eta(q,p) \rangle = \langle \nabla_p K(q,p), \eta(q,p) \rangle$: For $|p| \geq 4b$, we can

conclude that $\langle \nabla_p K(q, p), \eta(q, p) \rangle = |p|^2$. We then subtract the term K_{\max} to correct the estimate for the term $-K(q, p)$. The estimates (4.26) and (4.27) are deduced similarly.

Lemma 4.8.1. *The Hamiltonian function K introduced in (2.10) satisfies the following estimate,*

$$|X_K(q, p)|_{T(T^*M)} = |\nabla K(q, p)|_{T(T^*M)} \leq \sqrt{5} G_{\max} \left(1 + |p|_{T_q^*M}^2 \right).$$

Observe that this lemma is useful since it can be used for example to estimate the usual length used in Riemannian geometry of a solution $\gamma \in \mathcal{P}(K, q, q')$. In particular, we would like to point out that $|p|$ appears to the power of two which is convenient when one recalls that the action functional is defined for $W^{1,2}$ -Sobolev functions.

Proof of Lemma 4.8.1.

$$\begin{aligned} |\nabla K(q, p)|_{T(T^*M)}^2 &= |\nabla_q K(q, p)|_{T(T^*M)}^2 + |\nabla_p K(q, p)|_{T(T^*M)}^2 \\ &\leq G_{\max}^2 (1 + |p|_{T_q^*M}^2)^2 + G_{\max}^2 (1 + |p|_{T_q^*M}^2)^2 \\ &\leq G_{\max}^2 (1 + |p|_{T_q^*M}^2)^2 + 4 G_{\max}^2 (1 + |p|_{T_q^*M}^2)^2 \\ &= 5 G_{\max}^2 (1 + |p|_{T_q^*M}^2)^2 \\ \Rightarrow |\nabla K(q, p)|_{T(T^*M)} &\leq \sqrt{5} G_{\max} (1 + |p|_{T_q^*M}^2). \end{aligned}$$

In the third step we used the fact that for $x \in \mathbb{R}$ it holds that

$$1 + x \leq 2(1 + x^2).$$

□

Consider the space

$$C^0([0, 1], T^*M) := \{f \in C^0([0, 1], \mathbb{R}^{2K}) \mid f \subset \Phi(T^*M) \subset \mathbb{R}^{2K}\}$$

as the space $(C^0([0, 1], T^*M), \|\cdot\|_{C^0})$ equipped with the norm

$$\|f\|_{C^0} := \|f\|_{C^0([0,1], T^*M)} := \|f\|_{C^0([0,1], \mathbb{R}^{2K})} := \max_{t \in [0,1]} |f(t)|_{\mathbb{R}^{2K}}. \quad (4.28)$$

We refer to the discussion given in Section C.3.1. Define for a Hamiltonian function $H: T^*M \rightarrow \mathbb{R}$ satisfying the Convention 2.1.2 and for $A \in \mathbb{R}$ the set of action bounded solutions of Hamilton's equations associated to H :

$$Sol_A(M, H) := \{x \in W^{1,2}([0, 1], T^*M) \mid \mathcal{A}_H(x) \leq A \text{ and } \dot{x} = X_H(x(t))\} . \quad (4.29)$$

Obviously, $\mathcal{P}^A(H, q, q') \subset Sol_A(M, H)$. By the continuous Sobolev embedding

$$W^{1,2}([0, 1], T^*M) \subset C^0([0, 1], T^*M)$$

we know that every element of the set (4.29) (being an equivalence class of Sobolev functions) can be represented by a continuous function.

The following Lemma can be proven in a more general setting than it is done here, we refer to the paper of Abbondandolo–Schwarz [3, Lemma 1.10].

Lemma 4.8.2. *Again, let K be the Hamiltonian defined in (2.10). Then for every $q, q' \in M$ and for every action bound $A \in \mathbb{R}$ the set $Sol_A(M, K)$ is bounded in the C^0 -norm $\|\cdot\|_{C^0}$.*

Note that this lemma implies trivially that the space of solutions with fixed end-points $\mathcal{P}^A(K, q, q')$ is also bounded in the C^0 -norm.

Proof. Let $x = (q, p) \in Sol_A(M, K)$. Then let us calculate,

$$\begin{aligned} A &\geq \mathcal{A}_K(x) = \int_0^1 (\lambda(\dot{x}) - K(x(t))) dt \\ &= \int_0^1 (d\lambda(\eta, X_K(x(t))) - K(x(t))) dt \\ &= \int_0^1 (dK(q, p)(\eta) - K(x(t))) dt \\ &\geq \int_0^1 |p|^2 dt - K_{max} . \end{aligned}$$

Due to Lemma 4.8.1 and from the estimation above we know that $\|\dot{x}\|_{L^1([0,1], T^*M)}$ is bounded:

$$\|\dot{x}\|_{L^1([0,1], T^*M)} = \int_0^1 |\dot{x}|_{T^*M} dt$$

$$\begin{aligned}
&= \int_0^1 |\nabla K(q, p)|_{T(T^*M)} dt \\
&\leq \sqrt{5} G_{\max} \int_0^1 \left(1 + |p|_{T_q^*M}^2\right) dt \\
&\leq \sqrt{5} G_{\max} (1 + A + K_{\max}) .
\end{aligned}$$

So in short,

$$\|\dot{x}\|_{L^1([0,1],T(T^*M))} \leq \sqrt{5} G_{\max} (1 + A + K_{\max}) . \quad (4.30)$$

This result will be used a couple of times. Finally, we want to have an estimate for the $W^{1,1}$ -norm of x . Let us do the following calculation. See the Appendix C.3.1 for definitions of the Sobolev spaces used here.

$$\begin{aligned}
\|x\|_{W^{0,2}([0,1],T^*M)}^2 &= \|x\|_{W^{0,2}([0,1],\mathbb{R}^{2K})}^2 \\
&= \|x_q\|_{W^{0,2}([0,1],\mathbb{R}^K)}^2 + \|x_p\|_{W^{0,2}([0,1],\mathbb{R}^K)}^2 \\
&\leq \max_{x_q \in M} (\|x_q\|_{\mathbb{R}^K}^2) + \int_0^1 (\|x_p(t)\|_{\mathbb{R}^K}^2) dt \\
&\leq \text{const}(M) + (A + K_{\max}) \\
&< +\infty .
\end{aligned}$$

It is well-known that $L^2 \equiv W^{0,2} \subseteq L^1 \equiv W^{0,1}$, so x is also an element of $L^1([0, 1], T^*M)$, therefore,

$$x \in W^{1,1}([0, 1], T^*M), \quad \forall x \in \mathcal{P}^A(K, q, q') . \quad (4.31)$$

Then by Sobolev embedding the result follows,

$$\begin{aligned}
\|x\|_{C^0([0,1],T^*M)} &\leq C \|x\|_{W^{1,1}([0,1],T^*M)} \\
&= C (\|x\|_{L^1([0,1],\mathbb{R}^K)} + \|\dot{x}\|_{L^1([0,1],\mathbb{R}^K)}) \leq \text{const.} < +\infty .
\end{aligned} \quad (4.32)$$

□

Lemma 4.8.3. *Let K be the Hamiltonian function defined in (2.10). Then the space $Sol_A(M, K)$ is equicontinuous.*

Proof. Let $\gamma \in Sol_A(M, K)$. Let $t, s \in \mathbb{R}$ with $t > s$ and define $B := A + K_{\max}$. We

start in accordance with (4.28) by estimating the distance between $\gamma(t)$ and $\gamma(s)$ (the first inequality follows simply from triangle inequality in \mathbb{R}^{2K}),

$$\begin{aligned}
|\gamma(t) - \gamma(s)|_{\mathbb{R}^{2K}} &\leq \int_s^t |\dot{\gamma}|_{T(T^*M)} d\tau \\
&= \int_s^t |X_K(\gamma(\tau))|_{T(T^*M)} d\tau \\
&\leq \sqrt{5} G_{\max} \int_s^t \left(1 + |p|_{T_q^*M}^2\right) d\tau \\
&= \sqrt{5} G_{\max} \left((t-s) + \int_s^t \left(|p_q(\tau)|_{T_q^*M}^2 - B + B\right) d\tau \right) \\
&\leq \sqrt{5} G_{\max} \left((t-s) + B \int_s^t d\tau \right) \\
&= \sqrt{5} G_{\max} (1+B)(t-s) \\
&= \sqrt{5} G_{\max} (1+A+K_{\max})(t-s),
\end{aligned}$$

The first estimate comes from the fact that the metric d_{T^*M} is induced by the Riemannian metric on T^*M and measures the length of the shortest geodesic between $\gamma(t)$ and $\gamma(s)$, therefore, if one measures the length between the latter two points along the curve γ , it is certainly at least longer than the one given by measuring the distance along the shortest geodesic. Because of Lemma 4.8.2 we get

$$B \geq \int_0^1 |p(\tau)|_{T_q^*M}^2 d\tau \geq \int_s^t |p(\tau)|_{T_q^*M}^2 d\tau \geq 0,$$

what means

$$\int_s^t |p(\tau)|_{T_q^*M}^2 d\tau - B \leq 0.$$

So γ is equicontinuous. □

The Arzelà–Ascoli theorem C.1.1 implies that the set of solutions with bounded action $Sol_A(M, K)$ is relatively compact in $C([0, 1], T^*M)$ with respect to the norm (4.28), since it is bounded and equicontinuous.

Remark. Let us introduce for $A, B \in \mathbb{R}$, with $B \leq A$, the set of two sided action

bounded paths.

$$\text{Sol}_{[A,B]}(M, H) := \{x \in \text{Sol}_A(M, H) \mid B \leq \mathcal{A}_H(x) \leq A\}. \quad (4.33)$$

The space (4.33) is relatively compact in $C([0, 1], T^*M)$. \diamond

Lemma 4.8.4. *Let $\Sigma \subset T^*M$ be a fiberwise starshaped hypersurface. If Σ is the level set of a Hamiltonian function $H : T^*M \rightarrow \mathbb{R}$ with $H^{-1}(\alpha) = \Sigma$ such that α is a regular value of H , then the Reeb flow φ_R of $\ker(\lambda|_\Sigma)$ is a reparametrization of the Hamiltonian flow $\varphi_H|_\Sigma$ restricted to Σ .*

Remark. It follows from Lemma 4.8.4 that if there is another hamiltonian $G : T^*M \rightarrow \mathbb{R}$ such that the fiberwise starshaped hypersurface Σ satisfies also $\Sigma = G^{-1}(\alpha')$, where $\alpha' \in \mathbb{R}$ is a regular value of G , then the flow $\varphi_H^t|_\Sigma$ is a time reparametrization of the flow $\varphi_G^t|_\Sigma$. So, the orbits do not depend on the Hamiltonian defining Σ . \diamond

Proof. The symplectic form $\omega = d\lambda$ is a volume form on T^*M , hence the restriction to the hypersurface Σ is degenerate and of rank $2n - 2$, recall $\dim(M) = n$. So its kernel is 1-dimensional. We have by definitions of the Reeb vector field R on Σ

$$\begin{aligned} \iota_R \lambda &\equiv 1 \\ \iota_R d\lambda|_{T\Sigma} &\equiv 0 \end{aligned}$$

and of the Hamiltonian vector field – because Σ is a level set of H the gradient is perpendicular to Σ –

$$\iota_{X_H} \omega|_{T\Sigma} = -dH|_{T\Sigma} = 0.$$

These are two expressions defining the kernel of $\omega|_{T\Sigma}$. Therefore,

$$X_H(x) = a(x)R(x), \forall x \in \Sigma,$$

for a positive smooth function $a : \Sigma \rightarrow \mathbb{R}$, so we have

$$\varphi_H^t(x_0) = \varphi_R^{\sigma(x_0, t)}(x_0), x_0 \in \Sigma,$$

where $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth and strongly monotone function. And this function is a reparametrization of the restricted Hamiltonian flow. Now, we want to find this function σ , see [53]. It is given by the solution of the following Cauchy initial value problem, where $x_0 \in \Sigma$.

$$\begin{cases} \frac{d\sigma}{dt} = \frac{1}{a(\varphi_R^\sigma(x_0))} > 0 \\ \sigma(0, x_0) = 0. \end{cases}$$

(The > 0 is incorporated to emphasize strong monotonicity of the solution.) To see this, observe the following calculation.

$$\begin{aligned} \frac{d}{dt} \varphi_R^{\sigma(t, x_0)}(x_0) &= \frac{d}{d\sigma} \varphi_R^\sigma(x_0) \frac{d\sigma}{dt}(t, x_0) \\ &= R(\varphi_R^\sigma(x_0)) \frac{d\sigma}{dt}(t, x_0) \\ &= a(\varphi_R^\sigma(x_0)) X_H(\varphi_R^\sigma(x_0)) \frac{d\sigma}{dt}(t, x_0) \\ &= a(\varphi_R^\sigma(x_0)) X_H(\varphi_R^\sigma(x_0)) \frac{1}{a(\varphi_R^\sigma(x_0))} \\ &= X_H(\varphi_R^\sigma(x_0)). \end{aligned}$$

Because the flow of a vector field is unique it follows that $\varphi_H^t(x_0) = \varphi_R^{\sigma(t, x_0)}(x_0)$ with $\sigma(0, x_0) = 0$. \square

Lemma 4.8.5. *Let $\Sigma \subset T^*M$ be a fiberwise starshaped hypersurface. If Σ is the level set of a Hamiltonian function $H : T^*M \rightarrow \mathbb{R}$ with $H^{-1}(\alpha) = \Sigma$ such that α is a regular value of H , then the reduced action of the restricted Hamiltonian flow $\varphi_H|_\Sigma$ such as the one of the Reeb flow φ_R of $\ker(\lambda|_\Sigma)$ equal the time needed by φ_R to go from $\varphi_H^0(x_0) = x_0 \in \Sigma$ to $\varphi_H^1(x_0) \in \Sigma$, i.e.*

$$\mathcal{A}_0(\varphi_H|_\Sigma) = \mathcal{A}_0(\varphi_R) = \sigma(x_0, 1),$$

where we used the notations introduced in Lemma 4.8.4.

Proof.

$$\begin{aligned} \int_{\varphi_H} \lambda &= \int_0^1 \lambda(\dot{\varphi}_H^t(x_0)) dt \\ &= \int_0^1 \lambda\left(\frac{d}{dt} \varphi_R^{\sigma(t, x_0)}(x_0)\right) dt \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \lambda \left(\frac{d}{d\sigma} \varphi_R^{\sigma(t,x_0)}(x_0) \frac{d\sigma}{dt}(t, x_0) \right) dt \\
&= \int_0^1 \lambda \left(\frac{d}{d\sigma} \varphi_R^{\sigma(t,x_0)}(x_0) \right) \frac{d\sigma}{dt}(t, x_0) dt \\
&= \int_0^{\sigma(1,x_0)} \lambda \left(\frac{d}{d\sigma} \varphi_R^{\sigma(t,x_0)}(x_0) \right) d\sigma \quad \left(= \int_{\varphi_R} \lambda \right) \\
&= \int_0^{\sigma(1,x_0)} d\sigma = \sigma(1, x_0).
\end{aligned}$$

For the second step we used Lemma 4.8.4 and for the fifth one the property that it holds true that $\lambda \left(\frac{d}{d\sigma} \varphi_R^{\sigma(t,x_0)}(x_0) \right) = 1$ for Reeb vector fields. \square

Lemma 4.8.6. *There exists a constant $\bar{C} > 0$ depending only on (M, g) such that each element of $H_j(\Omega_{q,q'}^1, M)$ can be represented by a cycle whose image lies in $\mathcal{E}^{\frac{1}{2}(\bar{C}j)^2}(q, q')$.*

Proof. See [19] and Theorem A.0.5 for the proof. \square

Chapter 5

The Lower Part of the Time Spectrum

In this chapter we estimate the times of the first two Reeb chords of the fiberwise starshaped hypersurface Σ in T^*M . This is done by topological arguments and by using existing results concerning the length spectrum of (M, g) .

After being focused on the two fastest Reeb chords, we consider stability of the first one: The time of the fastest Reeb chord is stable under continuous deformations of the hypersurface Σ .

5.1 The times of the first two Reeb chords

We want to estimate the times of the first two Reeb chords of Σ . Due to basic geometric facts, we know that for every Riemannian metric g satisfying the Convention 4.0.5 the following estimate for the first Reeb chord holds. Denote by $\text{dist}(q, q')$ the distance of q and q' with respect to g . Then,

$$\mathcal{T}_1(\Sigma, q, q') \leq \text{dist}(q, q').$$

In the next step we focus on the second Reeb chord of Σ . Given a simply-connected manifold M , let k_0 be the smallest integer k such that $H_k(M; \mathbb{F}) \neq 0$. Then $k_0 \in \{2, \dots, \dim M\}$. This follows from Hurewicz's theorem. For convenience, we introduce the following definition $\mathcal{L}^a := \mathcal{L}_{g, q, q'}^a := \Omega_{q, q'}^{1, a} M$ for $a \geq 0$.

Theorem 5.1.1. *Let Σ be a fiberwise starshaped hypersurface in T^*M with M simply-connected. Assume that g satisfies the Convention 4.0.5. Abbreviate $d = \text{diam}(M, g)$. If q' is not Σ -conjugate to q , then*

$$\begin{aligned} \mathcal{T}_2(\Sigma, q, q') &\leq 8k_0^2d + (2k_0^2 - 1)3d(\sqrt{\sigma_g} - 1) \\ &< 2k_0^2d(4 + 3\sqrt{\sigma_g}). \end{aligned}$$

Remark. The theorem is for M simply-connected, since if *not* it is known that

$$\mathcal{T}_2(\Sigma) \leq 2d.$$

See Theorem 4.7.3. ◇

Proof. We abbreviate the length functional by $\mathcal{L} = \mathcal{L}_{g,q,q'}$ and the energy functional by $\mathcal{E} = \mathcal{E}_{g,q,q'}$, and set $\delta = \text{dist}_g(q, q')$. We also set $S = 3d(\sqrt{\sigma_g} - 1)$, fix $\varepsilon > 0$ and write $S_\varepsilon = S + \varepsilon$. We will show the statement in the theorem for S replaced by S_ε , which suffices since $\varepsilon > 0$ is arbitrary. Let γ_1 be a shortest geodesic from q to q' . Its length is δ .

By Theorem 7.3 of [36] with $(p, q, x) = (q, q', q')$ and $k = 1$ one of the following assertions takes place:

- (1) There exists a geodesic γ_2 from q to q' of length $\ell \in (\delta, \delta + 2d]$ which is a local minimum of \mathcal{L} and is such that the connected components of γ_1 and γ_2 in \mathcal{L}^L are disjoint for all $L \in [\ell, \ell + S_\varepsilon]$. See Figure 5.1.
- (2) There exists a field \mathbb{F} and a non-trivial class in $H_{k_0}(\Omega_{q,q'}^1 M; \mathbb{F})$ that can be represented by a cycle in \mathcal{L}^L where $L = (6k_0 - 1)d + (2k_0 + 1)\delta + (2k_0 - 1)S_\varepsilon$.

In Case (2) the theorem (with S replaced by $S_{2\varepsilon}$) follows from Proposition 2.2.5 from the explanations in [36] by considering Schwarz' spherical cycles. The factor k_0 appears with a power of two after maximizing the length upper bound of the paths which represent Schwarz' cycle.

In Case (1) the connected components of γ_1 and γ_2 in \mathcal{E}^a are disjoint for all $a \in [\frac{1}{2}\ell^2, \frac{1}{2}(\ell + S_\varepsilon)^2]$. The definition of S shows that $\sqrt{\sigma_g}3d = 3d + S$. Since $\sqrt{\sigma_g} \geq 1$ and

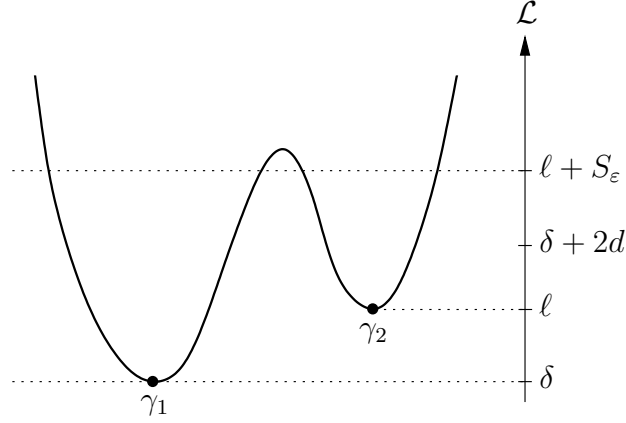


Figure 5.1: An intuitive image of the first assertion: The geodesic segment γ_1 has length δ ; the local minimum γ_2 of \mathcal{L} is of length $\ell \in (\delta, \delta + 2d]$.

$\ell \leq 3d$, it follows that $\sqrt{\sigma_g} \ell \leq \ell + S$. Choose a regular value $\ell_+ > \ell$ of \mathcal{L} such that $\ell_+ \leq \ell + d$ and

$$\sqrt{\sigma_g} \ell_+ < \ell + S_\varepsilon.$$

With $b := \frac{1}{2} \ell_+^2$ we then have $\sigma_g b < \frac{1}{2} (\ell + S_\varepsilon)^2$. Hence the map

$$H_0(\mathcal{E}^b(q, q'); \mathbb{Q}) \longrightarrow H_0(\mathcal{E}^{\sigma_g b}(q, q'); \mathbb{Q})$$

has rank ≥ 2 because γ_1, γ_2 are local minima and so their indices are zero. Proposition 2.2.5 shows that $\mathcal{T}_2(\Sigma, q, q') \leq \ell_+ \leq 4d$. \square

5.2 On the C^0 -stability of the minimal time

Let us first introduce some notion. Let $(\Sigma_k)_{k \in \mathbb{N}}$ be a sequence of fiberwise starshaped hypersurfaces and Σ another fiberwise starshaped hypersurface of the cotangent bundle T^*M .

Definition 5.2.1 (C^0 -Convergence of fiberwise starshaped hypersurfaces). *The sequence $(\Sigma_k)_{k \in \mathbb{N}}$ converges in the C^0 -sense to Σ if the following holds true:*

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : \forall m \geq N \Rightarrow \sup_{x \in M} \left(\sup_{p_x \in T_x^* \Sigma} d_{T^*M}((x, \alpha_m \cdot p_x), (x, p_x)) \right) < \varepsilon,$$

where $(x, \alpha_m \cdot p_x) \in T_x^* \Sigma_m$ and $(x, p_x) \in T_x^* \Sigma$. The previous point

$$(x, \alpha_m \cdot p_x) = (x, \alpha_m(p_x, \Sigma_m) \cdot p_x) \in T_x^* \Sigma_m$$

is just the point of intersection of Σ_m with the linear line (or subvector space) going through the two points $(x, 0_x) \in T_x^* M$ and $(x, p_x) \in T_x^* \Sigma$, where α_m is an element of the positive real numbers \mathbb{R}_+ . This construction is possible because the considered hypersurfaces are fiberwise starshaped.

Observe that this definition is equivalent to the one given in Section 1.2.5.

Let $\Sigma \subset T^* M$ be a fiberwise starshaped hypersurface. Define for $n \in \mathbb{N}$ and ε_n with $1 > \varepsilon_n > 0$ and $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ the “fiberwise shrunked” (fiberwise starshaped) hypersurfaces

$$\underline{\Sigma}_n := (1 - \varepsilon_n) \cdot \Sigma, \quad (5.1)$$

where the operation “ \cdot ” stands for fiberwise multiplication, i.e. $\delta \cdot \Sigma := \{(q, \delta p) \mid (q, p) \in \Sigma\} \subset T^* M$ if $\delta \in \mathbb{R}_+$. It is easy to see that $\underline{\Sigma}_n \rightarrow \Sigma$ for $n \rightarrow \infty$ in the C^0 -sense. (Equivalently, one can consider $\bar{\Sigma}_n := (1 + \varepsilon_n) \cdot \Sigma$.) If we fix two points $q_1, q_2 \in M$ then $\mathcal{T}_1(\Sigma_k)$ shall denote the shortest time – the smallest reduced action – needed by the Reeb flow to go from Σ_{k, q_1} to Σ_{k, q_2} . We want to understand that $\mathcal{T}_1(\underline{\Sigma}_n) \rightarrow \mathcal{T}_1(\Sigma)$ (or $\mathcal{T}_1(\bar{\Sigma}_n) \rightarrow \mathcal{T}_1(\Sigma)$) as $n \rightarrow \infty$.

Lemma 5.2.2. *Let $(\Sigma_k)_{k \in \mathbb{N}}$ be a sequence of fiberwise starshaped hypersurfaces, each of them given as a regular level set of a Hamiltonian function $H : T^* M \rightarrow \mathbb{R}$. So there is a sequence $(a_k)_{k \in \mathbb{N}}$ of regular values of H , i.e. this means that we have for all k $H^{-1}(a_k) = \Sigma_k$. In addition, let $\Sigma = H^{-1}(a)$ be another fiberwise starshaped hypersurface and suppose that $a_n \rightarrow a$ for $n \rightarrow \infty$. Then $(\Sigma_k)_{k \in \mathbb{N}}$ converges in the C^0 -sense to Σ .*

Proof. Because M and $T_x^* \Sigma$ are compact sets, one finds a N such that

$$d_{T^* M}((x, \alpha_m(p_x, \Sigma_m) \cdot p_x), (x, p_x)) < \varepsilon$$

for all $m \geq N$. □

As in Section 2.2 we choose $K : T^* M \rightarrow \mathbb{R}$ to be the smooth Hamiltonian with the

property $\Sigma = K^{-1}(\frac{1}{2})$ where $\frac{1}{2}$ is a regular value of K . According to the theorem of Sard we know that the singular values of a smooth function are rare, so we can assume the existences of a sequence $(a_n)_{n \in \mathbb{N}}$ and of an ε satisfying $1 > \varepsilon > 0$ such that every $a_n \in (-\varepsilon + 1, 1 + \varepsilon)$ is a regular value of K for $n \in \mathbb{N}$ and that $a_n \rightarrow \frac{1}{2}$ as $n \rightarrow \infty$.

Define the Hamiltonians

$$K_n(x) := \frac{1}{a_n}K(x), \quad n \in \mathbb{N}, \quad (5.2)$$

and set $\Sigma_n := K_n^{-1}(\frac{1}{2}) = K^{-1}(a_n)$. Note that Σ_n converges to Σ in the C^0 -sense because of Lemma 5.2.2.

Lemma 5.2.3. *Fix $q \in M$. Then for almost all $q' \in M$ it holds true that*

$$\mathcal{T}_1(\Sigma_n) \rightarrow \mathcal{T}_1(\Sigma)$$

as $n \rightarrow \infty$.

Proof. Since M is closed it is according to the Hopf-Rinow theorem a complete metric space with respect to the metric induced by the Riemannian metric g . We know that the sets $U(K_n, q)$ are open and of full measure in M , see [30]. Due to the theorem of Baire, the set

$$V(q) := \bigcap_{n=1}^{\infty} U(K_n, q)$$

is dense in M . In the following, we consider an easily modified version of the Langrangian Floer Homology explained in [30]: Therein, the k^{th} Floer chain group $\text{CF}_k^a(H, q, q'; \mathbb{F}_p)$ is defined as the finite-dimensional \mathbb{F}_p -vector space freely generated by the solutions $x : [0, 1] \rightarrow T^*M$ of $\dot{x} = X_H(x)$ of Conley–Zehnder index k . We define the Floer chain groups $\text{CF}_k^{A,a}(H, q, q'; \mathbb{F}_p)$ as the finite-dimensional \mathbb{F}_p -vector spaces freely generated by the solutions $x : [0, A] \rightarrow T^*M$ of $\dot{x} = X_H(x)$ of Conley–Zehnder index k . This means that we consider solutions of Hamilton’s equations on this modified time interval of Conley–Zehnder index k satisfying the following action bound

$$\mathcal{A}_H^A(x) := \int_0^A p(x(t))dq(\dot{x}(t))dt - \int_0^A H(x(t))dt < a.$$

We denote the reduced action by $\mathcal{A}_0^A(x) = \int_0^A p(x(t))dq(\dot{x}(t))dt$, trivially $\mathcal{A}_0^1(x) = \mathcal{A}_0(x)$. Further, we have in the same sense $\text{CF}_k^{1,a}(H, q, q'; \mathbb{F}_p) = \text{CF}_k^a(H, q, q'; \mathbb{F}_p)$.

Let us fix the point $q' \in V(q)$. Due to the Abbondandolo-Schwarz isomorphism (we use the existence of geodesics from q to q' on M) and the use of the Sandwiching method (Section 2.2) we can conclude for all n the existence of a non-trivial Floer homology class

$$0 \neq [\gamma_n] \in \text{HF}_0^{a_n, B_n + \varepsilon_n}(K_n, q, q'; \mathbb{F}_p), \quad \varepsilon_n > 0, \quad (5.3)$$

with

$$\mathcal{A}_{K_n}^{a_n}(\gamma_n) = B_n.$$

Analogously,

$$0 \neq [\gamma] \in \text{HF}_0^{B + \varepsilon_\infty}(K, q, q'; \mathbb{F}_p), \quad \varepsilon_\infty > 0,$$

and

$$\mathcal{A}_K(\gamma) = B.$$

Notice that we have found for every $n \in \mathbb{N}$ a solution

$$\gamma_n : [0, a_n] \rightarrow \Sigma_n \subset T^*M$$

of $\dot{x} = X_{K_n}(x)$ on Σ_n in the time interval $[0, a_n]$. By a change of variables we can interpret all the γ_n as solutions $x : [0, 1] \rightarrow \mathbb{R}$ of Hamilton's equations $\dot{x} = X_K(x)$, i.e. elements of $\mathcal{P}(K, q, q')$: Consider the maps $\alpha_n : [0, 1] \rightarrow \Sigma_n \subset T^*M$ defined by $\alpha_n(t) := \gamma_n(a_n t)$. It is easily seen that $\dot{\alpha}_n(t) = X_K(\alpha_n(t))$, therefore, $\alpha_n \in \mathcal{P}(K, q, q')$ for all $n \in \mathbb{N}$. A small calculation gives the following identity,

$$\mathcal{A}_{K_n}^{a_n}(\gamma_n) = a_n^2 \mathcal{A}_K(\alpha_n), \quad n \in \mathbb{N}. \quad (5.4)$$

We assumed that Σ_n converges in the C^0 -sense to Σ . Therefore, we know that γ_n converges uniformly to $\gamma : [0, 1] \rightarrow \Sigma \subset T^*M$. Define $\alpha \equiv \gamma : [0, 1] \rightarrow \Sigma$. So $\alpha_n \rightarrow \alpha$ for $n \rightarrow \infty$. It follows by the continuity of the action that $\mathcal{A}_K(\alpha_n) \rightarrow \mathcal{A}_K(\alpha)$ for $n \rightarrow \infty$. This implies that $\{\mathcal{A}_K(\alpha_n), n \in \mathbb{N}\}$ is a bounded set, say, bounded by $C > 0$. (Recall the definition (4.29) and the following two lemmas afterwards.) This means that if we

consider the compact set $Sol_C(M, K)$ we can conclude $\gamma = \alpha \in Sol_C(M, K)$.

Finally, let us calculate the limit of the shortest time $\mathcal{T}_1(\Sigma_n)$ for $n \rightarrow \infty$ where we use just the definitions and the identity (5.4):

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mathcal{T}_1(\Sigma_n) &= \lim_{n \rightarrow \infty} \mathcal{A}^{a_n}(\tilde{\gamma}_n) \\
&= \lim_{n \rightarrow \infty} \sqrt{2 \mathcal{A}_{K_n}^{a_n}(\gamma_n)} \\
&= \lim_{n \rightarrow \infty} \sqrt{2 a_n^2 \mathcal{A}_K(\alpha_n)} \\
&= \lim_{n \rightarrow \infty} \left(a_n \sqrt{2 \mathcal{A}_K(\alpha_n)} \right) \\
&= \sqrt{2 \mathcal{A}_K(\alpha)} \\
&= \mathcal{A}(\alpha).
\end{aligned}$$

This last expression must be equal to the shortest time of the Reeb flow on Σ from Σ_q to $\Sigma_{q'}$, because the shortest times $\mathcal{T}_1(\Sigma_n)$ of Reeb flows on Σ_n are given by $a_n \sqrt{2 \mathcal{A}_K(\alpha_n)}$, see Lemma 2.2.2, and the action spectrum $\mathcal{A}_K(\cdot)$ is discrete, hence

$$\mathcal{A}_0(\alpha) = \mathcal{T}_1(\Sigma).$$

□

Observe that the proof of Lemma 5.2.3 can easily be modified such that one gets the statement that if Σ_n converges to Σ in the C^0 -sense, then for almost all pairs of points of M the time of the k^{th} shortest Reeb chord on Σ_n converges to the one of the k^{th} shortest Reeb chord on Σ . (Because the action spectrum is discrete.) This is not true for all pairs of points in M . But for the shortest time the following holds true:

Proposition 5.2.4. *Fix $q \in M$. Then for all $q' \in M$ it holds true that*

$$\mathcal{T}_1(\Sigma_n) \rightarrow \mathcal{T}_1(\Sigma)$$

when $n \rightarrow \infty$.

Proof. Fix $q \in M$ and $q' \notin V(q)$. The situation is the same as in the beginning of the proof of Lemma 5.2.3, up to the following: Choose a sequence $\{q_n\}_{n \in \mathbb{N}}$ such that for all

n one has $q_n \in V(q)$ and that $\lim_{n \rightarrow \infty} q_n = q'$. Replace in equation (5.3) the endpoint q' by q_n , i.e. we consider

$$0 \neq [\gamma_n] \in \text{HF}_0^{a_n, B_n + \varepsilon_n}(K_n, q, q_n; \mathbb{F}_p), \quad \varepsilon_n > 0,$$

as in the proof above. Also therein, we argued why the set $\text{Sol}_C(M, K)$ is compact. Consider now the following closed subset of $\text{Sol}_C(M, K)$:

$$\text{Sol}_{\Sigma, C}(M, K) := \{x \in C([0, 1], \Sigma) \mid \mathcal{A}_K(x) \leq C \text{ and } \dot{x} = X_K(x)\}$$

It is closed and hence compact in $C([0, 1], T^*M)$. Therefore, the action \mathcal{A}_K attains on $\text{Sol}_{\Sigma, C}(M, K)$ its minimum. Denote this value by $\mathcal{M}(K, \Sigma, q, q') = \mathcal{M}$. As in the proof above, because the action is continuous we have that $\mathcal{A}_K(\alpha_n) \rightarrow \mathcal{M}$ if $n \rightarrow \infty$. So this sequence of minimal actions converges to the minimal action on $\text{Sol}_{\Sigma, C}(M, K)$, although the action spectrum of solutions with starting points in Σ_q and endpoints in $\Sigma_{q'}$ may not be discrete. This concludes the proof. \square

Corollary 5.2.5. *Fix $q \in M$. Then for all $q' \in M$ it holds true that*

$$\mathcal{T}_1(\underline{\Sigma}_n) \rightarrow \mathcal{T}_1(\Sigma)$$

when $n \rightarrow \infty$. The same is true for $\{\mathcal{T}_1(\overline{\Sigma}_n)\}_{n \in \mathbb{N}}$.

Proof. Consider the Hamiltonians in (5.2): Choose $a_n := K(x)$ for any $x \in \underline{\Sigma}_n$. So, $a_n \rightarrow 1$ for $n \rightarrow \infty$ and it holds true that $\underline{\Sigma}_n = K_n^{-1}(\frac{1}{2})$. Afterwards one invokes Proposition 5.2.4. \square

Chapter 6

Perfect Morse Functions and the Conley–Zehnder Index

In this chapter we will first address the class of so-called perfect Morse functions. Roughly speaking, a perfect Morse function has the distinguished property that the Morse inequalities are in fact equalities. Then we are able to make use of the very convenient consequence that knowledge about the Betti numbers of the manifold leads directly to geometrical insights, and vice-versa. Unfortunately, perfect Morse functions are rare.

Afterwards we will give an explanation of the Conley–Zehnder index for symplectic arcs. Before we close this chapter, we will focus on a few properties of this index which we use in other chapters.

6.1 Perfect Morse functions

A Morse function $f : E \rightarrow \mathbb{R}$ on a (real) C^2 -Hilbert manifold E is a C^2 -function whose Hessian is non-degenerate at all critical points, so they are isolated. For the definitions of these notions see Section 4.1.1 on Morse theory in infinite dimensions. Define the spaces

$$E^{\leq a} := \{x \in E \mid f(x) \leq a\}$$

and

$$E^{<a} := \{x \in E \mid f(x) < a\},$$

as well as

$$K_c := \{x \in E \mid f(x) = c, df(x) = 0\}$$

for a critical value c of f .

Consider for a critical value c the triple $(E^{\leq c}, E^{\leq c} \setminus K_c, E^{<c})$ and note that

$$E^{<c} \subset E^{\leq c} \setminus K_c \subset E^{\leq c}.$$

The associated long exact sequence in singular homology with respect to any coefficient field \mathbb{F} is

$$\begin{aligned} \dots & \xrightarrow{\partial_n} \mathrm{H}_n(E^{\leq c} \setminus K_c, E^{<c}; \mathbb{F}) \longrightarrow \mathrm{H}_n(E^{\leq c}, E^{<c}; \mathbb{F}) \\ & \longrightarrow \mathrm{H}_n(E^{\leq c}, E^{\leq c} \setminus K_c; \mathbb{F}) \xrightarrow{\partial_{n-1}} \mathrm{H}_{n-1}(E^{\leq c} \setminus K_c, E^{<c}; \mathbb{F}) \dots \end{aligned}$$

The following definition is a special case of the notion of a “perfect function” given by A. Oancea in his talk on “Completing manifolds in Morse theory”, see [39].

Definition 6.1.1 (Perfect Morse Function). *Let E be a C^2 -Hilbert manifold and \mathbb{F} a coefficient field. A Morse function $f : E \rightarrow \mathbb{R}$ on E is called \mathbb{F} -perfect if for all critical values c of f and for all $n \in \mathbb{N}_0$ it holds true that the connecting homomorphisms are trivial, i.e. $\partial_n \equiv 0$. This means that the long exact sequence considered above splits*

$$\begin{aligned} 0 \longrightarrow \mathrm{H}_n(E^{\leq c} \setminus K_c, E^{<c}; \mathbb{F}) & \longrightarrow \mathrm{H}_n(E^{\leq c}, E^{<c}; \mathbb{F}) \\ & \longrightarrow \mathrm{H}_n(E^{\leq c}, E^{\leq c} \setminus K_c; \mathbb{F}) \longrightarrow 0. \end{aligned}$$

Note that it is not necessary to assume f to be Morse. It is enough that f is differentiable, as mentioned by Oancea in his talk.

The following steps are due to Chang [16, Section 1.4]. Let a and b be two regular values of the function f and suppose $a < b$. In addition, let f satisfy the Palais-Smale condition 4.1.2 on $[a, b]$. Define by $B_k := \dim(\mathrm{H}_k(E^{\leq b}, E^{\leq a}; \mathbb{F}))$ the k -th Betti number of the pair $(E^{\leq b}, E^{\leq a})$ and by C_k the number of critical points of f of index k in $f^{-1}([a, b])$.

Lemma 6.1.2 (Morse Equalities). *Let $f : E \rightarrow \mathbb{R}$ be an \mathbb{F} -perfect Morse function on E satisfying the Palais-Smale condition (PS) on $[a, b]$. Then for all $M \in \mathbb{N}_0$*

$$\sum_{k=0}^M (-1)^{M-k} C_k = \sum_{k=0}^M (-1)^{M-k} B_k$$

and in particular

$$\sum_{k=0}^{\infty} (-1)^k C_k = \sum_{k=0}^{\infty} (-1)^k B_k.$$

Moreover,

$$B_k = C_k, \quad \forall k \in \mathbb{N}_0.$$

Proof. We do a special case of Theorem 4.3. in Chang [16]. He considers for a triple of (topological) spaces $Z \subset Y \subset X$ the long exact sequence

$$\dots \xrightarrow{\partial_n} H_n(Y, Z; \cdot) \longrightarrow H_n(X, Z; \cdot) \longrightarrow H_n(X, Y; \cdot) \xrightarrow{\partial_{n-1}} H_{n-1}(Y, Z; \cdot) \dots$$

As a further step, we choose $(Z, Y, X) = (E^{\leq c}, E^{\leq c} \setminus K_c, E^{< c})$ and compute

$$\begin{aligned} \dim H_n(E^{\leq c} \setminus K_c, E^{< c}; \mathbb{F}) &= \dim \operatorname{Im} \iota_* + \dim \operatorname{Im} \partial_n \\ \dim H_n(E^{\leq c}, E^{< c}; \mathbb{F}) &= \dim \operatorname{Im} j_* + \dim \operatorname{Im} \iota_* \\ \dim H_n(E^{\leq c}, E^{\leq c} \setminus K_c; \mathbb{F}) &= \dim \operatorname{Im} \partial_{n-1} + \dim \operatorname{Im} j_*, \end{aligned}$$

and

$$\begin{aligned} H_n(E^{\leq c}, E^{\leq c} \setminus K_c; \mathbb{F}) &+ \dim H_n(E^{\leq c} \setminus K_c, E^{< c}; \mathbb{F}) && (6.1) \\ &- \dim H_n(E^{\leq c}, E^{< c}; \mathbb{F}) \\ &= \dim \operatorname{Im} \partial_n + \dim \operatorname{Im} \partial_{n-1}. \end{aligned}$$

Observe that the right hand side of (6.1) is zero due to the fact that f was supposed to be \mathbb{F} -perfect. This and the proof of Theorem 4.3. in [16] concludes the lemma, while the last assertion follows by induction over k . Due to Proposition 4.1.8 we know that the considered series are finite. \square

We want to address the question when a given Morse function f on a C^2 -(Hilbert) manifold E is \mathbb{F} -perfect. Very good references concerning this section are [8] and [13]. For compact surfaces, some work was done by Andrica, see [5]. If we look for an \mathbb{F} -perfect Morse function on E then the homology groups $H_k(E; \mathbb{F})$ must be torsion free, to state a necessary condition. We do not know of other necessary conditions. At least, there is an indication given by Smale, that for a finite-dimensional E non-trivial torsion could be the only obstruction to the existence of a perfect Morse function: *If E is a simply-connected closed manifold of dimension greater than five with no torsion in the homology of E (w.r.t. some coefficient field \mathbb{F}), then there is a non-degenerate function f on E with the Morse type numbers equal to the Betti numbers of E .* Then f is an \mathbb{F} -perfect Morse function on E . See [48, Theorem 6.3]. What is known about sufficient conditions?

Let

$$\mathcal{M}(t; f) := \sum_{p \in \text{Crit}(f)} t^{\text{ind}(p)}$$

be the Morse series (a formal power series in t) of some Morse function f on E . The sum ranges over the critical points $p \in \text{Crit}(f)$ of f and $\text{ind}(p)$ indicates the Morse index of $p \in \text{Crit}(f)$. One can rewrite the series $\mathcal{M}(t; f)$ in the following easy way $\mathcal{M}(t; f) = \sum_{k=0}^{\infty} m_k(f) t^k$. Then one calls $m_k(f)$ the k -th Morse type number of f . Recall that we introduced them already in (4.5) for the case when f is the energy functional on $E = \Omega_{q,q'}^1 M$. Further, there is the well-known (formal) Poincaré series

$$\mathcal{P}(t; E) := \sum_{k=0}^{\infty} \dim H_k(E; \mathbb{F}) t^k.$$

The group H_k denotes the usual k -th singular homology group of E with respect to a coefficient field \mathbb{F} . The number $\beta_k := \dim H_k(E; \mathbb{F})$ is the k -th Betti number of E . Also here we refer to the (more special) definitions made in Section 4.1.1. Suppose that $\mathcal{M}(t; f)$ and $\mathcal{P}(t; E)$ exist. Then, as explained in the book by Chang [16], the following holds:

$$\mathcal{M}(t; f) = \mathcal{P}(t; E) + (1+t)Q(t; f),$$

where $Q(t; f) = \sum_{k=0}^{\infty} q_k t^k$ is a formal power series in t with non-negative coefficients

$q_k \geq 0$. Instead of this equation often one writes the so called Morse inequalities,

$$\mathcal{M}(t; f) \geq \mathcal{P}(t; E).$$

Due to Bott [13] there is the so called Morse Lacunary principle:

Theorem 6.1.3 (Morse Lacunary Principle). *Suppose no two consecutive powers of t in $\mathcal{M}(t; f)$ occur. Then $Q(t; f) \equiv 0$ and hence f is an \mathbb{F} -perfect Morse function on E for every coefficient field \mathbb{F} . In particular, E (with respect to coefficients in the field \mathbb{F}) is then torsion-free.*

The Lacunary Principle is used in the proof of Proposition 4.2.20.

Proof. Let m_j be the coefficients of the Morse series $\mathcal{M}(t; f)$ and denote by p_j those of the Poincaré series $\mathcal{P}(t; f)$. The Morse inequalities imply the following relations:

$$m_0 - p_0 = q_0 \quad \text{and} \quad m_j - p_j = q_j + q_{j-1} \quad \forall j \in \{1, 2, \dots\}.$$

We already know that $q_j \geq 0$. Now, if $m_{2j} = 0$ (or $m_{2j+1} = 0$), $j \in \mathbb{N}_0$, we get $-p_{2j} = q_{2j} + q_{2j-1} \geq 0$, hence $p_{2j} = 0$ and $q_{2j} + q_{2j-1} = 0$. Therefore $q_j = 0$ for all $j \in \mathbb{N}_0$, i.e. $Q(t; f) \equiv 0$.

From the step above, if $p_k = \dim H_k(E; \mathbb{F}) \neq 0$, then it must hold that $p_{k-1} = \dim H_{k-1}(E; \mathbb{F}) = 0$, hence from the universal coefficient theorem

$$0 \cong H_{k-1}(E; \mathbb{F}) \cong (H_{k-1}(E; \mathbb{Z}) \otimes \mathbb{F}) \oplus \text{Tor}(H_{k-2}(E; \mathbb{Z}), \mathbb{F})$$

it follows that

$$H_{k-1}(E; \mathbb{Z}) \cong 0 \quad \text{and} \quad \text{Tor}(H_{k-2}(E; \mathbb{Z}), \mathbb{F}) \cong 0.$$

Let us compute,

$$H_k(E; \mathbb{F}) \cong (H_k(E; \mathbb{Z}) \otimes \mathbb{F}) \oplus \text{Tor}(H_{k-1}(E; \mathbb{Z}), \mathbb{F}) \cong H_k(E; \mathbb{Z}) \otimes \mathbb{F}.$$

So the singular homology groups of E with coefficients in \mathbb{F} are torsion-free. □

For example, if we take the two-torus $E = T^2$ in \mathbb{R}^3 (standing on one side on the plane and slightly tilted) and choose the height function of T^2 as a Morse function on T^2 , then we see that the height function is – for example for $\mathbb{F} = \mathbb{Z}$ – an \mathbb{F} -perfect Morse function. Consider the n -sphere \mathbb{S}^n and again the height function $f : \mathbb{S}^n = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 = 1\} \rightarrow \mathbb{R}$ defined by $f(x_1, \dots, x_n) = x_n$. We get the two critical points $(0, \dots, 0, \pm 1) \in \mathbb{R}^n$, where the one with $x_n = -1$ corresponds to the minimum of f and the other one to the maximum. The Morse index of the first is zero and the one of the maximum is n . By the Morse Lacunary Principle we get for $n > 1$ that f is for all coefficient fields \mathbb{F} an \mathbb{F} -perfect Morse function on \mathbb{S}^n . Another easy way to see this is just by writing down the Morse and the Poincaré series of f , in this case for all $n \in \mathbb{N}$. On the other hand, the height function on the “heart shaped” 2-sphere $\tilde{\mathbb{S}}^2$ is not a perfect Morse function, see remark 4.1.1 and figure 4.1. A reference for a less trivial example and a further application of Morse’s Lacunary Principle is R. Botts “favorite example” of an \mathbb{F} -perfect Morse function on the n -dimensional complex projective space $\mathbb{C}\mathbb{P}^n$, see [13, Page 338].

A further sufficient condition is the so-called Completion Principle. We prove it for a finite-dimensional manifold E (this restriction is not necessary, see Oancea [39]). Suppose that $p \in E$ is a non-degenerate critical point of f at level $c \in \mathbb{R}$ and of Morse index $\text{ind}(p) = k$. The Morse lemma implies that in a suitable coordinate system x_1, \dots, x_n centered at p , the function f has near p the form

$$f(x_1, \dots, x_n) = c - x_1^2 - x_2^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2.$$

Define the k -disc

$$\nu_p := \{x \mid x_1^2 + x_2^2 + \dots + x_k^2 \leq \epsilon, x_{k+1} = \dots = x_n = 0\}.$$

The set ν_p is then a disc near p , whose boundary $\partial\nu_p$ is a $(k - 1)$ -sphere in the space

$$E_{c-\epsilon} := \{x \in E \mid f(x) \leq c - \epsilon\}.$$

We call p *completable* if this sphere $\partial\nu_p$ bounds a singular chain (w.r.t. coefficients in

some field \mathbb{F}) in $E_{c-\epsilon}$ for an $\epsilon > 0$ small enough.

Theorem 6.1.4 (Completion Principle [8]). *If f is non-degenerate and all its critical points are completable, then f is an \mathbb{F} -perfect Morse function.*

The Completion Principle is used in the proof of Theorem 4.2.18.

Proof. Let p be a non-degenerate critical point of f which is completable and with $f(p) = c$. Recall Definition 6.1.1. Consider the following parts (for all n) of the long exact sequence in homology of the triple $(E^{\leq c}, E^{\leq c} \setminus K_c, E^{< c})$:

$$\cdots \longrightarrow H_n(E^{\leq c}, E^{\leq c} \setminus K_c; \mathbb{F}) \xrightarrow{\partial_{n-1}} H_{n-1}(E^{\leq c} \setminus K_c, E^{< c}; \mathbb{F}) \longrightarrow \cdots .$$

We know that p is completable (w.r.t. \mathbb{F}), i.e. there is a singular k -chain α (with image lying in $E^{< c}$) whose boundary is $\partial\nu_p$. It is the boundary of the disc ν_p which exists since p is non-degenerate. This means that the $(k-1)$ -sphere $\partial\nu_p$ can be taken as the representative of a homologically trivial cycle in $H_{n-1}(E^{\leq c} \setminus K_c, E^{< c}; \mathbb{F})$. Since this is true for all the critical points of f one sees that the connecting homomorphisms ∂_n are all trivial and therefore the Morse function f is \mathbb{F} -perfect. \square

As an example consider the round two-sphere \mathbb{S}^2 . As the function f we take the height function from above. The north-pole is a non-degenerate critical point of f of index 2. The boundary of an ε -disc D_N centered at the north-pole is an \mathbb{S}^1 . It is bounded by the closure of $\mathbb{S}^2 \setminus D_N$. So the north-pole N is completable. For the south-pole S this follows trivially.

Then, as a second example, consider the ‘‘heart shaped’’ $\tilde{\mathbb{S}}^2$ which was introduced in Section 4.1.1. Again we choose as the function f the height function as drawn in Figure 6.1. Let us start by choosing an ε -disc D_{m_1} around the non-degenerate critical point m_1 of index 2. Deform D_{m_1} such that its boundary (an \mathbb{S}^1) lies in a level set of f , i.e. there is a regular value α of f with $\partial D_{m_1} \subset f^{-1}(\alpha)$. The disc D_{m_1} is bounded by the closure of $\tilde{\mathbb{S}}^2 \setminus D_{m_1}$. So there exists only one singular chain as a candidate which could complete the point m_1 . But this chain does not lie in the sublevel of α since it represents also the point m_2 , see the figure. Hence m_1 is not completable.

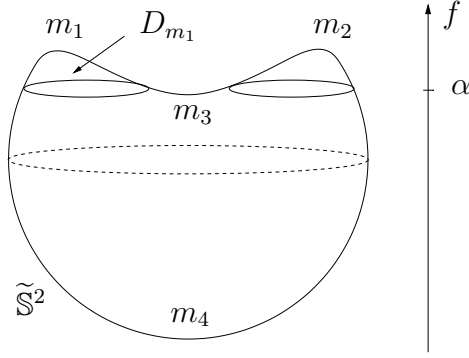


Figure 6.1: A representation of the “heart shaped” sphere $\tilde{\mathbb{S}}^2$ with the level set $f^{-1}(\alpha)$. The points m_1 and m_2 are local maxima. The ε -disc D_{m_1} centered at m_1 is bounded by the closure of $\tilde{\mathbb{S}}^2 \setminus D_{m_1}$ which itself lies *not* in the sublevel set $\{x \in \tilde{\mathbb{S}}^2 \mid f(x) < \alpha\}$.

6.2 The Conley–Zehnder index for symplectic arcs

Let $(W := T^*M, d\lambda)$ be the cotangent bundle. Choose according to Convention 2.1.2 a Hamiltonian function $H: W \rightarrow \mathbb{R}$ and fix $a > 0$.

Floer chain groups are generated by non-degenerate critical points of the action functional associated to a Hamiltonian function $H: M \rightarrow \mathbb{R}$ on a compact smooth manifold M . To get a grading of these groups, one introduces the so-called Conley–Zehnder index by associating to a solution $x \in \mathcal{P}(H, q, q')$ of Hamilton’s equations $\dot{x}(t) = X_H(x(t))$ an integer $\mu(x)$. The Floer chain group $\text{CF}_k^a(H, q, q')$ is then the finite-dimensional vector space freely generated by the elements of $\mathcal{P}^a(H, q, q')$ of Maslov index k .

Given a non-degenerate solution $x \in \mathcal{P}(H, q, q')$ of the Hamilton equation, we follow Audin–Damian [9, Chapter 7] and choose a symplectic basis of the tangent space $T_{x(t)}W$. We call it

$$Z(t) = (Z_1(t), \dots, Z_{2n}(t)).$$

Write the solution in different notation $x(t) = \varphi^t(x_0)$, with $x(0) =: x_0$. We know that the flow φ^t preserves the symplectic form $(\varphi^t)^* \omega = \omega$. This means that the linearization $T_{x(t)}\varphi^t$ of φ^t in the base $Z(t)$ is a symplectic matrix $A(t) \in \text{Sp}(2n, \mathbb{R})$. So, the map $t \mapsto A(t)$ is a path of symplectic matrices with $A(0) = \text{Id}$ and $A(1)$ such that 1 is not in

its spectrum, because x is assumed to be non-degenerate. Let us define for $0 < \tau \in \mathbb{R}$

$$\mathcal{SP}(\tau) := \{\psi : [0, \tau] \rightarrow \mathrm{Sp}(2n, \mathbb{R}) \mid \psi(0) = \mathrm{Id}, \det(\mathrm{Id} - \psi(\tau)) \neq 0\}$$

and denote by $U(n)$ the unitary matrices. The determinant map

$$\det : U(n) \rightarrow S^1$$

induces an isomorphism of fundamental groups

$$\det_* : \pi_1(U(n)) \rightarrow \pi_1(S^1) \cong \mathbb{Z}.$$

This can be seen by computing the homotopy exact sequence of the fibration

$$SU(n) \hookrightarrow U(n) \rightarrow S^1$$

and by using the fact that $SU(n)$ is simply connected. The quotient $\mathrm{Sp}(2n, \mathbb{R})/U(n)$ is contractible, so $\pi_1(\mathrm{Sp}(2n, \mathbb{R})) \cong \mathbb{Z}$. This isomorphism can be represented by a continuous map

$$\rho : \mathrm{Sp}(2n, \mathbb{R}) \rightarrow S^1$$

which restricts to the determinant map on $\mathrm{Sp}(2n, \mathbb{R}) \cap O(2n) \simeq U(n)$, where $O(2n)$ denotes the set of orthogonal $2n \times 2n$ matrices. Write $\phi \in \mathrm{Sp}(2n, \mathbb{R})$ in the form $\phi = PQ$, where P is symmetric positive definite and $Q \in O(2n)$ is an orthogonal matrix. Then one constructs a retraction of $\mathrm{Sp}(2n, \mathbb{R})$ onto $U(n)$, see McDuff–Salamon [31, Pages 45 and 46] for details.

Lemma 6.2.1 (Salamon–Zehnder [43]). *The space $\mathrm{Sp}(2n, \mathbb{R})^* := \{A \in \mathrm{Sp}(2n, \mathbb{R}) \mid \det(\mathrm{Id} - A) \neq 0\}$ has two connected components*

$$\mathrm{Sp}(2n, \mathbb{R})^\pm := \{A \in \mathrm{Sp}(2n, \mathbb{R}) \mid \pm \det(\mathrm{Id} - A) > 0\}.$$

Moreover, every loop in $\mathrm{Sp}(2n, \mathbb{R})^$ is contractible in $\mathrm{Sp}(2n, \mathbb{R})$.*

For the proof, see [43, Page 1316]. As examples consider the two matrices

$$W^+ := -\text{Id} \in \text{Sp}(2n, \mathbb{R})^+,$$

and

$$W^- := \text{diag} \left(2, -1, \dots, -1, \frac{1}{2}, -1, \dots, -1 \right) \in \text{Sp}(2n, \mathbb{R})^-.$$

In the proof of this lemma, Salamon and Zehnder construct n continuous (and periodic) functions (because there are n eigenvalues “of the first kind”, see [43, Page 1315] for details)

$$\alpha_\nu : \text{Sp}(2n, \mathbb{R})^+ \rightarrow [0, 2\pi], \quad \nu = 1, \dots, n$$

satisfying

$$\exp \left(i \sum_{\nu=1}^n \alpha_\nu(A) \right) = \rho(A), \quad A \in \text{Sp}(2n, \mathbb{R})^+$$

where

$$0 \leq \alpha_1(A) \leq \dots \leq \alpha_n(A) \leq 2\pi$$

such that

$$\exp(i\alpha_\nu(A)) = \pm 1,$$

that is,

$$\alpha_\nu(A) \in \{0, \pi, 2\pi\}. \tag{6.2}$$

Further, they choose the α_ν such that there is the same number of ν 's with $\alpha_\nu(A) = 0$ and with $\alpha_\nu(A) = 2\pi$. We return to the construction of the Conley–Zehnder index.

For any path $\gamma : [0, \tau] \rightarrow \text{Sp}(2n, \mathbb{R})$ we choose a function $\alpha : [0, \tau] \rightarrow \mathbb{R}$ such that

$$\rho(\gamma(t)) = e^{i\alpha(t)}. \tag{6.3}$$

Define

$$\Delta_\tau(\gamma) := \frac{\alpha(\tau) - \alpha(0)}{\pi}. \tag{6.4}$$

For $A \in \text{Sp}(2n, \mathbb{R})^*$ take a path $\gamma_A : [0, 1] \rightarrow \text{Sp}(2n, \mathbb{R})^*$ with $\gamma_A(0) = A$ and $\gamma_A(1) \in \{W^+, W^-\}$. Then it follows from the lemma above that $\Delta_1(\gamma_A)$ is independent of the

choice of this path γ_A , because $\mathrm{Sp}(2n, \mathbb{R})^\pm$ is path connected and $\Delta_1(\gamma_A)$ just depends on the fixed endpoints of γ_A . Define

$$r(A) := \Delta_1(\gamma_A), \quad A \in \mathrm{Sp}(2n, \mathbb{R})^*. \quad (6.5)$$

The Conley–Zehnder index of a path $\psi \in \mathcal{SP}(\tau)$ is defined by

$$\mu(\psi) := \Delta_\tau(\psi) + r(\psi(\tau)), \quad (\text{Conley–Zehnder index}). \quad (6.6)$$

So this index can roughly be described as a mean winding number for the linearized flow along $x(t)$ or the number of times an eigenvalue crosses 1. To get the hands on this number, let us convince ourselves that the Conley–Zehnder index is really an integer.

Proposition 6.2.2. *The Conley–Zehnder index is an integer.*

Proof. We follow Salamon–Zehnder [43]. We extend $\psi \in \mathcal{SP}(\tau)$ to a smooth path $\gamma : [0, \tau + 1] \rightarrow \mathrm{Sp}(2n, \mathbb{R})$ which agrees with ψ on $[0, \tau]$ and satisfies $\gamma(t) \in \mathrm{Sp}(2n, \mathbb{R})^*$ for $\tau \leq t \leq \tau + 1$ with $\gamma(\tau + 1) \in \{W^+, W^-\}$. Since $\mathrm{Sp}(2n, \mathbb{R})^\pm$ is path-connected this extension exists. By definition, it follows that

$$\begin{aligned} \mu(\psi) &= \Delta_\tau(\psi) + \Delta_1(\psi(\tau)) \\ &= \Delta_{\tau+1}(\gamma). \end{aligned}$$

So, let us calculate $\alpha(\tau + 1)$.

$$\begin{aligned} \rho(W^+) = \rho(\gamma(\tau + 1)) &= \det(X + iY) \\ &= \det(X) \\ &= \det(-\mathrm{Id}_n) = (-1)^n \end{aligned}$$

$$\Rightarrow e^{i\alpha(\tau+1)} = (-1)^n$$

$$\Rightarrow \alpha(\tau + 1) = \pi n + 2\pi l, \quad l \in \mathbb{Z}.$$

Here we used the fact that $-\text{Id} = W^+ \in U(n)$ is of the form

$$B = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$

what means that $X = -\text{Id}_n$ and $Y = 0$. It follows from the determinant property of ρ that $\rho(B) = \det(X + iY)$. Now, we focus on the other quantity $\alpha(0)$: Similarly,

$$\rho(\gamma(0)) = \rho(\psi(0)) = \rho(\text{Id}) = 1^{2n} \Rightarrow \alpha(0) = 2\pi m, \quad m \in \mathbb{Z}.$$

This leads to

$$\Delta_{\tau+1}(\gamma) = \frac{\pi n + 2\pi l - 2\pi m}{\pi} = n + 2l - 2m \in \mathbb{Z}.$$

Notice that $\rho(W^-) = (-1)^{n-1}$ which leads to the same conclusion. □

Proposition 6.2.3 (Salamon–Zehnder [43]). *For $A \in \text{Sp}(2n, \mathbb{R})^*$ we have*

$$|r(A)| \leq n.$$

Proof. If $\gamma_A(1) = W^+$, then

$$\begin{aligned} |r(A)| &= \left| \frac{\alpha(1) - \alpha(0)}{\pi} \right| = \left| \frac{1}{\pi} \sum_{\nu=1}^n (\alpha_\nu(\gamma_A(1)) - \alpha_\nu(\gamma_A(0))) \right| \\ &\leq \frac{1}{\pi} \sum_{\nu=1}^n |\pi - \alpha_\nu(\gamma_A(0))| \\ &\leq \frac{1}{\pi} \sum_{\nu=1}^n |\pi| = \frac{1}{\pi} n\pi = n. \end{aligned}$$

If $\gamma_A(1) = W^-$, then

$$\begin{aligned} |r(A)| &= \left| \frac{\alpha(1) - \alpha(0)}{\pi} \right| = \left| \frac{1}{\pi} \sum_{\nu=1}^n (\alpha_\nu(\gamma_A(1)) - \alpha_\nu(\gamma_A(0))) \right| \\ &\leq \frac{1}{\pi} \left(2\pi + \sum_{\nu=1, \nu \neq k \in [1, n]}^n |\pi - \alpha_\nu(\gamma_A(0))| \right) \\ &\leq \frac{1}{\pi} (2\pi + (n-1)\pi) = n. \end{aligned}$$

Both estimates use (6.2), and in the second case there is one $\mu = k$ such that $\alpha_k(W^-) \in \{0, 2\pi\}$, because $\frac{1}{2}$ is an eigenvalue of the first kind, hence the estimate with 2π . So, we can conclude from the latter two calculations that $|r(A)| \leq n$. \square

Proposition 6.2.4. *If $\gamma_m \in \mathcal{P}(H, q, q')$ with $\mu(\gamma_m) = k \in \mathbb{Z}$ for all $m \in \mathbb{N}$ and $\lim_{m \rightarrow \infty} \gamma_m = \gamma \in \mathcal{P}(H, q, q')$ in C^∞ , then it holds that*

$$\mu(\gamma) \in [k - 2n, k + 2n].$$

Proof. Let $m \in \mathbb{N}$.

$$\begin{aligned} |\mu(\gamma_m) - k| &= |\Delta_\tau(\gamma_m) + r(\gamma_m(\tau)) - k| \\ &\leq |\Delta_\tau(\gamma_m) - k| + |r(\gamma_m(\tau))| \\ &= 2|r(\gamma_m(\tau))| \\ &\leq 2n. \end{aligned}$$

We have used triangle inequality and Proposition 6.2.3. \square

Appendices

Appendix A

A Theorem of Gromov

Let (M, g) be a smooth simply-connected and closed Riemannian manifold. For $q, q' \in M$, let $\Omega_{q,q'}M = \{\gamma \in C_{pw}^\infty([0, 1], M) \mid \gamma(0) = q \text{ and } \gamma(1) = q'\}$ be the set of piece-wise smooth paths in M from q to q' , and let $\Omega_{q,q'}^a M = \{\gamma \in \Omega_{q,q'}M \mid \mathcal{L}_g(\gamma) \leq a\}$, where \mathcal{L}_g is the length given by the Riemannian metric g . In this chapter we consider singular homology with respect to a fixed coefficient group Γ . We omit Γ from the notation.

Theorem A.0.5 (Gromov). *There exists a constant $\underline{C} > 0$ depending only on g such that given any pair of points $q, q' \in M$ and any positive integer i , any element in $H_i(\Omega_{q,q'}M)$ can be represented by a cycle whose image lies in $\Omega_{q,q'}^{\underline{C}i}M$.*

Remark. Theorem A.0.5 can be restated as follows: There is a constant $\underline{C} = \underline{C}(g) > 0$ such that for every $i \in \mathbb{N}$ the homomorphism

$$H_i\left(\Omega_{q,q'}^{\underline{C}i}M\right) \hookrightarrow H_i(\Omega_{q,q'}M)$$

induced by the inclusion

$$\Omega_{q,q'}^{\underline{C}i}M \hookrightarrow \Omega_{q,q'}M$$

is surjective. ◇

Theorem A.0.5 is half of the following, for our purposes very convenient theorem. For $i \in \mathbb{N}$ let $L(i)$ be the *smallest* real number T such that the homomorphism

$$H_i\left(\Omega_{q,q'}^T M\right) \hookrightarrow H_i(\Omega_{q,q'}M)$$

induced by inclusion

$$\Omega_{q,q'}^T M \hookrightarrow \Omega_{q,q'} M$$

is surjective.

Theorem A.0.6 (Gromov [22], Section 7.3). *If M is a compact simply-connected Riemannian manifold with path space ΩM , then there are constants $\underline{C}, \overline{C}$ such that*

$$\underline{C}i \leq L(i) \leq \overline{C}i.$$

Remark. The result is usually stated for simply connected manifolds. From this the case of finite fundamental group follows at once by passing to the universal cover (cf. Remark 7.8 in [22]). \diamond

Due to Serre [46] we know that the loop space ΩM of any compact simply-connected manifold M admits an infinite number of integers k for which the Betti numbers $b_k(\Omega M)$ are non-zero. Recall that the spaces ΩM and $\Omega_{q,q'} M$ are homotopy equivalent. Therefore, if we fix $i \in \mathbb{N}$ such that $b_i(\Omega M) \neq 0$, we know that there is a non-trivial homology class $[\alpha] \in H_i(\Omega_{q,q'}^{L(i)} M)$ which can be represented by a cycle whose image lies in $\Omega_{q,q'}^{L(i)} M$. Again according to Serre we know that there is a $j \in \mathbb{N}$, with $j > i$, satisfying $|\underline{C}j - \overline{C}i| > 0$ what implies $L(j) > L(i)$ and with $b_j(\Omega_{q,q'} M) \neq 0$. Since $L(j)$ is the smallest real number such that the homomorphism

$$H_j(\Omega_{q,q'}^{L(j)} M) \hookrightarrow H_j(\Omega_{q,q'} M)$$

is surjective, we can conclude that there must be a non-trivial element

$$[\alpha'] \in H_j(\Omega_{q,q'}^{L(j)} M)$$

which can be represented by a cycle α' whose image restricted to a non-empty subset $A \subset [0, 1]^j$ (because of $L(j) > L(i)$) lies in $\Omega_{q,q'}^{L(j)} M \setminus \Omega_{q,q'}^{L(i)} M$, meaning

$$\alpha' : A \subset [0, 1]^j \rightarrow \Omega_{q,q'}^{L(j)} M \setminus \Omega_{q,q'}^{L(i)} M.$$

Assume, such an α' with this property does not exist. Then one can choose $L(j)$ smaller what leads to a contradiction. This means that the lengths of the images $\alpha'(A)$ are strictly longer than the lengths of all the images of all the cycles representing homology classes in $H_i\left(\Omega_{q,q'}^{L(i)}M\right)$.

Proof of Theorem A.0.6. Let us first prove the existence of the constant $\bar{C} = \bar{C}(M)$, the basic ideas are out of [22]. Fix an $i \in \mathbb{N}$ and then choose the smallest number $L(i)$ such that $H_i\left(\Omega_{q,q'}^{L(i)}M\right) \hookrightarrow H_i(\Omega_{q,q'}M)$ is surjective. According to Milnor [32, Chapter 16] we know that the space $\Omega_{q,q'}^{L(i)}M$ is homotopy equivalent to a $d(i)$ -dimensional (smooth) manifold, since every element can be homotoped to a broken geodesic. According to Serre [46] we know that for every i there exists an $k \in (0, i) \subset \mathbb{N}$ such that $H_{i+k}(\Omega_{q,q'}M) \neq 0$. So, let us increase i by such a k , define $i' := i + k$. Now we look for the smallest number $L(i')$ such that $H_{i'}\left(\Omega_{q,q'}^{L(i')}M\right) \hookrightarrow H_{i'}(\Omega_{q,q'}M)$ is surjective. Due to the construction of Milnor this means that $L(i) < L(i')$ because of $i < i'$ and because we consider i' -cycles instead of i -cycles. The former need strictly longer paths to get realized. We know that $L(i)$ has to grow as a function in i , otherwise this would contradict Serre's Theorem. Also due to construction of Milnor this growth has to be linear in i , because d grows linear in i and there is a constant $D = D(M)$ such that $L(\ell) = D(M)d(\ell)$ for ℓ such that $H_\ell\left(\Omega_{q,q'}^{L(\ell)}M\right) \simeq H_\ell(X^{d(\ell)}) \neq 0$. So $L(i) \leq \bar{C}i$ for another constant $\bar{C} = \bar{C}(M) > 0$.

For the proof of the existence of the constant \underline{C} we follow [41]. Let $\{V_\alpha\}$ be a finite covering of M by convex open sets. Recall that a set is convex if any two points in it are joined by a unique geodesic that stays in the set. Choose a triangulation T of M that is fine enough so that each closed simplex of T lies in one of the V_α . We shall also assume that the 1-skeleton of T consists of geodesic segments. For each point $p \in M$, let $T(p)$ be the closed face of T of minimum dimension that contains p and let $O(p)$ be the union of all maximal simplices of T that contain p .

Given a positive integer k , we define open subsets $(\Omega_{q,q'}M)_k$ of $\Omega_{q,q'}M$ in the following way. We shall say that $\omega \in (\Omega_{q,q'}M)_k$ if for each integer $j = 1, 2, \dots, 2^k$ the image under

ω of each subinterval $[(j-1)/2^k, j/2^k]$ lies in one of the sets V_α and

$$O(\omega((j-1)/2^k)) \cup O(\omega(j/2^k)) \subset V_\alpha$$

lies in the same V_α .

Let $B_k(M, q, q') \subset (\Omega_{q, q'} M)_k$ be the space of broken geodesics γ such that $\gamma \in (\Omega_{q, q'} M)_k$ and the restriction to each subinterval $[(j-1)/2^k, j/2^k]$ is a geodesic parameterized at constant speed. Each $\gamma \in B_k(M, q, q')$ determines a sequence of points $\{p_j = \gamma(j/2^k)\}$ which has the properties

1. $p_0 = x$ and $p_{2^k} = y$
2. $O(p_{j-1}) \cup O(p_j)$ lies in a single V_α , $\forall j = 1, \dots, 2^k$

Conversely any sequence $\{p_j\}$, $0 \leq j \leq 2^k$, with these properties determines a broken geodesic in $B_k(M, q, q')$. Because if one has two neighbouring points p_{j-1}, p_j there is – due to the second property – a unique geodesic connecting them. Moreover, the correspondence between broken geodesics in $B_k(M, q, q')$ and sequences of points is bijective. Because the parameterization will distinguish different geodesics with the same trace. Observe that this correspondence induces on $B_k(M, q, q')$ a cell decomposition: A cell that contains $\gamma \in B_k(M, q, q')$ is given by:

$$T(p_1) \times T(p_2) \times \cdots \times T(p_{2^k-1}).$$

Observe that p_0 and p_{2^k} were kept fixed. For example, if one had the following cell decomposition of a broken geodesic with three different legs $T(p_1) \times T(p_2)$ and $\dim(T(p_1)) = \dim(T(p_2)) = 2$, then we have six faces.

Therefore we can think of $B_k(M, q, q')$ as a finite cell complex. Due to the fact that the dimension of $B_k(M, q, q')$ is

$$\dim(B_k(M, x, y)) = (2^k - 1) \dim(M).$$

(Take the highest stratum, or just maximize a cell decomposition with respect to dimension.) By using Milnor's trick, see [32], one can show that $(\Omega_{q, q'} M)_k$ deformation

retracts onto $B_k(M, q, q')$.

Since M is simply connected, there exists a smooth map $f : M \rightarrow M$ such that f collapses the 1-skeleton of the triangulation T to a point and f is smoothly homotopic to the identity. One starts by constructing this f from the identity, so it is homotopic to it. Observe also that f naturally induces a map on $\Omega_{q, q'} M$

$$\begin{aligned} \hat{f} : \Omega_{q, q'} M &\rightarrow \Omega_{f(q), f(q')} M \\ \gamma &\mapsto f \circ \gamma. \end{aligned}$$

Now we prove the following intermediate lemma.

Lemma A.0.7. *There exists a constant $C' = C'(g) > 0$ such that for any $\mathbb{N} \ni k \geq 1$, we have*

$$\hat{f}(i\text{-skeleton of } B_k(M, q, q')) \subset \Omega_{f(q), f(q')}^{C'i} M$$

for all $i \leq \dim B_k(M, q, q')$.

Proof. Consider a cell

$$K = T(p_1) \times T(p_2) \times \cdots \times T(p_{2^k-1})$$

with $\dim(K) = \sum_{i=1}^{2^k-1} \dim(T(p_i)) = i \leq \dim B_k(M, q, q')$. Take a path $\gamma \in K$. Then γ is a broken geodesic, each leg of which lies in one of the sets V_α . Since f sends the 1-skeleton of the triangulation to a point we can estimate the length of the path γ afterwards:

$$\begin{aligned} \mathcal{L}_g(\hat{f}(\gamma)) &= \int_0^1 \left| \frac{d}{dt} \hat{f}(\gamma(t)) \right| dt = \int_0^1 |df(\gamma(t)) \dot{\gamma}(t)| dt \\ &\leq \max_{x \in M} \|df(x)\| \int_0^1 |\dot{\gamma}(t)| dt \\ &\leq \max_{x \in M} \|df(x)\| \cdot d \cdot N(\gamma), \end{aligned}$$

where $N(\gamma)$ denotes the number of legs of γ that do not lie in the 1-skeleton (due to collapsing) and $d := \max_\alpha(\text{diam}(V_\alpha))$. Now, we go on calculating $N(\gamma)$. The 1-skeleton is made up of geodesic segments. Moreover, the leg of a broken geodesic γ that joins

$T(p_j)$ and $T(p_{j+1})$ must lie in the 1-skeleton if $1 \leq j < 2^k - 1$ and $\dim(T(p_j)) = \dim(T(p_{j+1})) = 0$. Hence, the only legs of γ that could fail to lie in the 1-skeleton are the initial leg, which starts at x , the final leg, which ends at y , and legs that begin or end in a $T(p_j)$ with nonzero dimension. For example, if one leg starts or ends in a $T(p_j)$ with nonzero dimension, there could be – for every $j \in \{1, \dots, 2^k - 1\}$ and at most – a second leg joining the first one in this $T(p_j)$, so we can roughly estimate

$$N(\gamma) \leq 2 + 2 \cdot \sum_{i=1}^{2^k-1} \dim(T(p_i)) = 2 + 2i \leq 4i,$$

where the last estimation comes from the fact that in the theorem A.0.5 we assumed i to be a positive integer. Now, if we set $C' := 4 \cdot \max_{x \in M} \|df(x)\| \cdot d$, we obtain the result

$$\mathcal{L}_g(\hat{f}(\gamma)) \leq C'i.$$

□

We shall show that for all $q, q' \in M$, any $\eta \in H_i(\Omega_{f(q), f(q')}M)$ can be represented by a cycle whose image lies in $\Omega_{f(q), f(q')}^{C'i}M$, where C' is the constant given in lemma A.0.7. Since f is homotopic to the identity, the naturally induced morphism on homology \hat{f}_* is an isomorphism, so there exists a $\mu \in H_i(\Omega_{q, q'}M)$ such that $\hat{f}_*(\mu) = \eta$. Observe,

$$\Omega_{q, q'}M = \bigcup_{k=1}^{\infty} (\Omega_{q, q'}M)_k.$$

If we are given a cycle that represents μ , this cycle will have an image that lies by the last observation in $(\Omega_{q, q'}M)_k$ for some k . Retract $(\Omega_{q, q'}M)_k$ onto $B_k(M, q, q')$. Then our cycle can be moved by a homotopy into the i -skeleton of $B_k(M, q, q')$. (This can be achieved by using a standard theorem of algebraic topology which says that if one is given a CW complex W and j dimensional cycle, then the latter cycle can be homotopically deformed into the j -skeleton of W .) This means that we have deformed the cycle homotopically in a homologous way, so if h is this homotopy we have $[\mu'] = [h \circ \mu']$, where μ' is a representative of the class μ . Now, by lemma A.0.7 \hat{f} maps all points in the i -skeleton of $B_k(M, q, q')$ to points in $\Omega_{f(q), f(q')}^{C'i}M$ and hence $\hat{f}_*(\mu) = \eta$ can be represented by a

cycle whose image lies in $\Omega_{f(q),f(q')}^{C'i}M$, concretely:

$$\hat{f}_*[\mu'] = \hat{f}_*[h \circ \mu'] = [\hat{f} \circ h \circ \mu'] = [\eta'] = \eta,$$

where η' has its image in $\Omega_{f(q),f(q')}^{C'i}M$.

We know that the spaces $\Omega_{q,q'}M$ and $\Omega_{x',y'}M$, for $x', y' \in M$, are homotopy equivalent, hence $H_i(\Omega_{q,q'}M) \simeq H_i(\Omega_{x',y'}M)$. Now, we take $x' = f(q)$ and $y' = f(q')$. Then $\eta \in H_i(\Omega_{q,q'}M)$ can be represented as a cycle having an image in $\Omega_{q,q'}^{C'i+2 \cdot \text{diam}(M)}M \subset \Omega_{q,q'}^{(C'+2 \cdot \text{diam}(M))i}M$, where the path is possibly longer than the one in $\Omega_{f(q),f(q')}M$ and the latter estimate comes from the fact that $\mathbb{N} \ni i > 0$. Finally, we define $\underline{C} := C' + 2 \cdot \text{diam}(M)$ and conclude the theorem. \square

Proposition A.0.8 (Milnor's lemma). *Consider a closed n -dimensional Riemannian manifold (M, g) . Then there exists a constant $c = c(g)$ such that for every $k \geq 1$,*

$$H_k(\Omega_{q,q'}^{<a}M) = 0 \quad \text{for } a \leq ck \text{ and for all } q, q' \in M.$$

As the proof will show, one can take $c = \rho^2/(2n)$, where ρ is the injectivity radius of (M, g) .

Proof. We follow [32, Section 16]. Fix $a > 0$. We can assume that $\Omega_{q,q'}^{<a}M \neq \emptyset$. Let ρ be the injectivity radius of (M, g) . Set $h = \lceil 2a/\rho^2 \rceil$, and consider the equidistant subdivision $T = \{0 = t_0 < t_1 < \dots < t_h = 1\}$ of $[0, 1]$, i.e. $t_j = j/h$. Then $t_j - t_{j-1} = 1/h \leq \rho^2/(2a)$. Then for every $\gamma \in \Omega_{q,q'}^{<a}M$ the distance between the points $\gamma(t_{j-1}), \gamma(t_j)$ is $< \rho$. Hence there is a unique geodesic from $\gamma(t_{j-1})$ to $\gamma(t_j)$ of length $< \rho$. We can thus define the set $\Omega_{q,q'}^{<a}(T, M) \subset \Omega_{q,q'}^{<a}M$ of broken geodesics. This space is homotopy equivalent to $\Omega_{q,q'}^{<a}M$. Moreover, a broken geodesic $\gamma \in \Omega_{q,q'}^{<a}(T, M)$ is uniquely determined by the $(h-1)$ -tuple

$$((\gamma(t_1), \gamma(t_2), \dots, \gamma(t_{h-1}))) \in \times_{h-1}M.$$

Hence $\Omega_{q,q'}^{<a}(T, M)$ is homeomorphic to an open subset of $\times_{h-1}M$. In particular,

$$\dim \Omega_{q,q'}^{<a}(T, M) = n(h-1) < \frac{2n}{\rho^2}a.$$

Hence $H_k(\Omega_{q,q'}^{<a}, M) = H_k(\Omega_{q,q'}^{<a}(T, M)) = 0$ if $k \geq \frac{2n}{\rho^2}a$, i.e., $H_k(\Omega_{q,q'}^{<a}, M) = 0$ if $a \leq \frac{\rho^2}{2n}k$. \square

Proposition A.0.9. *Consider a closed n -dimensional Riemannian manifold (M, g) . Then there exists a constant $c = c(g)$ such that for every $k \geq 1$,*

$$H_k(\mathcal{E}^a(q, q')) = 0 \quad \text{for } a < ck^2 \text{ and for all } q, q' \in M .$$

As the proof shows, one can take $c = c(g) = 2 \left(\frac{\rho}{n}\right)^2$, where ρ is the injectivity radius of (M, g) .

Proof. The proof follows from Gromov [22]. \square

Appendix B

An Article by Schwarz

In this chapter we collect results from Albert Schwarz' paper [50] which only exists in Russian. In this article, Schwarz gives the following quantitative version of the result of Serre [46]:

Theorem B.0.10 (Schwarz, [50]). *Let M be a closed and simply-connected Riemannian manifold and $q, q' \in M$. Then there exists a sequence of numbers c_1, c_2, \dots appearing as the lengths of geodesic segments from q to q' with $c_k < c_{k+1}$ for all k . More precisely, there exists a number $d \geq 0$ such that for all $k \in \mathbb{N}$ it holds that $c_k \leq kd$ and $\{c_k\}_{k \in \mathbb{N}}$ forms an arithmetic progression.*

The following lemma which he uses to prove this theorem is taken from “Mathematical Reviews on the web” by the American Mathematical Society.

Lemma B.0.11 (Schwarz, [50]). *In the space of loops $\Omega_q M$, there is a homology class x and a cohomology class ξ such that $\langle x^k, \xi^k \rangle = k!$ for $k = 1, 2, 3, \dots$. Here x^k is the k -fold Pontryagin product of x and ξ^k is the k -fold cup-product of ξ .*

From this lemma he concludes the following proposition.

Proposition B.0.12 (Schwarz, [50]). *One can choose homology classes*

$$x_1, x_2, \dots, x_k, \dots \in H_*(\Omega_{q,q'}M)$$

such that there exists a number d such that for all k the homology class x_k can be realized

in $\Omega_{q,q'}^{kd}M$. Further, there is a cohomology class ξ such that

$$\langle x_k, \xi^k \rangle \neq 0, \forall k \in \mathbb{N}.$$

Finally, Schwarz makes the following remark:

Remark. (Schwarz, [50]) The theorem holds also for closed Riemannian manifolds with finite fundamental group. Compare this to Gromov's remark 7.8 in [22]. \diamond

Remark. The family name of Schwarz is sometimes written as Švarc, Svarc or also Shvartz. \diamond

For proofs and explanations of Schwarz' results mentioned above we refer to the work of Nabutovsky–Rotman [37] and [35]. They carry out the proofs and give sufficient information concerning Schwarz' ideas.

Appendix C

Topology and Functional Analysis

C.1 The Arzelà–Ascoli theorem

We prove the Arzelà–Ascoli theorem for continuous maps on general *compact* metric spaces. For the proof, we follow [44].

Let (X, d_X) be a compact metric space, (Y, d_Y) a general metric space and denote by

$$C(X, Y) := \{f : X \rightarrow Y \mid f \text{ continuous}\}$$

the space of continuous functions from X to Y . It is an easy exercise to see that the function $d : C(X, Y) \times C(X, Y) \rightarrow \mathbb{R}$ defined by

$$d(f, g) := \max_{x \in X} d_Y(f(x), g(x)), \forall f, g \in C(X, Y) \tag{C.1}$$

is a metric on the space of continuous functions $C(X, Y)$.

Theorem C.1.1. *Let $K \subset C(X, Y)$. K is compact with respect to the metric d if and only if K is closed, bounded and equicontinuous.*

Proof. “ \Rightarrow ”. K is compact, so it is sequentially compact, which implies that K is closed and bounded. Now we show the equicontinuity of K . Let $\varepsilon > 0$. K is a compact metric space, so there are finitely many elements $f_1, \dots, f_n \in K$ such that

$$\bigcup_{i=1}^n B_{\frac{\varepsilon}{3}}(f_i) = K. \tag{C.2}$$

For every $i \in \{1, \dots, n\}$ there is a $\delta_i > 0$ such that for every $x, y \in X$

$$d_X(x, y) < \delta_i \Rightarrow d_Y(f_i(x), f_i(y)) < \frac{\varepsilon}{3}, \quad (\text{C.3})$$

because X is compact and so the functions f_i are uniformly continuous. Define

$$\delta := \min_{i \in \{1, \dots, n\}} \delta_i.$$

Choose $x, y \in X$ with $d_X(x, y) < \delta$ and $f \in K$. By (C.2) there exists $i \in \{1, \dots, n\}$ such that

$$d(f, f_i) < \frac{\varepsilon}{3}. \quad (\text{C.4})$$

Hence for all $x, y \in X$,

$$d_Y(f(x), f(y)) \leq \underbrace{d_Y(f(x), f_i(x))}_{< \frac{\varepsilon}{3} \text{ (C.4)}} + \underbrace{d_Y(f_i(x), f_i(y))}_{< \frac{\varepsilon}{3} \text{ (C.3)}} + \underbrace{d_Y(f_i(y), f(y))}_{< \frac{\varepsilon}{3} \text{ (C.4)}} < \varepsilon.$$

“ \Leftarrow ”. For this part of the proof, there will be four steps A, B, C and D.

A.

Lemma C.1.2. *There exists a sequence $(x_k)_{k \in \mathbb{N}} \subset X$ such that for all $\delta > 0$ there exist $m(\delta) \in \mathbb{N}$ with $\bigcup_{k=1}^{m(\delta)} B_\delta(x_k) = X$.*

Proof. We do this by an inductive construction.

First step: For $\delta = 1$ there are finitely many points $x_1, \dots, x_{m(1)} \in X$ such that

$$\bigcup_{k=1}^{m(1)} B_{\delta=1}(x_k) = X.$$

Second step: For $\delta = \frac{1}{2}$ there are $x_{m(1)+1}, \dots, x_{m(2)} \in X$ such that

$$X = \bigcup_{k=m(1)+1}^{m(2)} B_{\delta=\frac{1}{2}}(x_k) = \bigcup_{k=1}^{m(2)} B_{\delta=\frac{1}{2}}(x_k).$$

(n+1)-th step: For $\delta = \frac{1}{n+1}$ there are $x_{m(n)+1}, \dots, x_{m(n+1)} \in X$ such that

$$X = \bigcup_{k=m(n)+1}^{m(n+1)} B_{\delta=\frac{1}{n+1}}(x_k) = \bigcup_{k=1}^{m(n+1)} B_{\delta=\frac{1}{n+1}}(x_k).$$

This means that the sequence $(x_k)_{k \in \mathbb{N}}$ is constructed for $\delta > 0$: Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$ and fix $m(\delta) := m(n)$, then

$$X \supset \bigcup_{k=1}^{m(\delta)} B_{\delta}(x_k) \supset \bigcup_{k=1}^{m(n)} B_{\frac{1}{n}}(x_k) = X.$$

□

B.

Lemma C.1.3. *Let $(f_n)_{n \in \mathbb{N}} \subset K$ be a sequence. Then there exists a subsequence $(f_{n_i})_{i \in \mathbb{N}}$ such that the limit*

$$y_k := \lim_{i \rightarrow \infty} f_{n_i}(x_k)$$

exist for every $k \in \mathbb{N}$.

Proof. Let $k = 1$, then the sequence $(f_n(x_1))_{n \in \mathbb{N}}$ of real numbers is bounded. Hence has due to the theorem of Bolzano–Weierstrass a convergent subsequence. In other words, there exists a strictly monotonically growing function $g_1: \mathbb{N} \rightarrow \mathbb{N}$ such that the limit

$$y_1 := \lim_{i \rightarrow \infty} f_{g_1(i)}(x_1)$$

exists. Then also the sequence $(f_{g_1(i)}(x_2))_{i \in \mathbb{N}}$ is bounded and has therefore a convergent subsequence. Hence there exists a strictly monotonically increasing function $g_2: \mathbb{N} \rightarrow \mathbb{N}$ such that the limit

$$y_2 := \lim_{i \rightarrow \infty} f_{g_1 \circ g_2(i)}(x_2)$$

exists.

Inductively, we find a sequence of strictly monotonically increasing functions $g_k: \mathbb{N} \rightarrow \mathbb{N}$ such that the limit

$$y_k := \lim_{i \rightarrow \infty} f_{g_1 \circ g_2 \circ \dots \circ g_k(i)}(x_k)$$

exists. Let

$$n_i := g_1 \circ g_2 \circ \dots \circ g_i(i)$$

for $i \in N$. Then $(f_{n_i(i)}(x_k))_{i \geq k}$ is a subsequence of $(f_{g_1 \circ g_2 \circ \dots \circ g_k(i)}(x_k))_{k \in \mathbb{N}}$ and converges to y_k . This holds for all k , which implies the lemma. □

C.

Lemma C.1.4. *Let $(f_n)_{n \in \mathbb{N}} \subset K$ be a sequence. Then $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to the metric d .*

Proof. Let $\varepsilon > 0$. Because of equicontinuity of K , there is $\delta > 0$ such that for all $f \in K$ and for all $x, y \in X$

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \frac{\varepsilon}{3}. \quad (\text{C.5})$$

By step A, there exists $m(\delta) =: \bar{m} \in \mathbb{N}$ with

$$\bigcup_{k=1}^{\bar{m}} B_\delta(x_k) = X. \quad (\text{C.6})$$

Due to Step B and by Cauchy's criteria of convergence the following holds: $\forall k \in \{1, \dots, \bar{m}\}$ there exists $N(k) \in \mathbb{N}$ such that $\forall r, s \geq N(k)$ we have

$$d_Y(f_{n_r}(x_k), f_{n_s}(x_k)) < \frac{\varepsilon}{3}. \quad (\text{C.7})$$

If we define $N := \max_{k \in \{1, \dots, \bar{m}\}} N(k)$ then (C.7) holds for all x_k with $r, s \geq N$. □

Lemma C.1.5. *In the notation used above, we have for all $r, s \geq N$*

$$d(f_{n_r}, f_{n_s}) = \max_{x \in X} d_Y(f_{n_r}(x), f_{n_s}(x)) < \varepsilon.$$

Proof. Let $r, s \geq N$ and $x \in X$. Due to (C.6) there exists a $k \in \{1, \dots, \bar{m}\}$ such that $d_X(x, x_k) < \delta$. Hence, $\forall x \in X$, we have

$$d_Y(f_{n_r}(x), f_{n_s}(x))$$

$$\leq d_Y(f_{n_r}(x), f_{n_r}(x_k)) + d_Y(f_{n_r}(x_k), f_{n_s}(x_k)) + d_Y(f_{n_s}(x_k), f_{n_s}(x)) < \varepsilon.$$

$\stackrel{(C.5)}{< \frac{\varepsilon}{3}}$
 $\stackrel{(C.7)}{< \frac{\varepsilon}{3}}$
 $\stackrel{(C.5)}{< \frac{\varepsilon}{3}}$

□

D.

Lemma C.1.6. *There exists $f \in K$ such that $f_n \xrightarrow{n \rightarrow \infty} f$ with respect to the metric d .*

Proof. The metric space $(C(X, Y), d)$ is complete, see Lemma C.1.7. Therefore, there exists $f \in C(X, Y)$ satisfying $f_n \xrightarrow{n \rightarrow \infty} f$ with respect to the metric d . Since K is closed $f \in K$.

□

This last step concludes the theorem.

□

Lemma C.1.7 (Completeness). *The metric space $(C(X, Y), d)$ is complete.*

Proof. Take a Cauchy sequence $(g_n) \subset C(X, Y)$. Fix $\varepsilon > 0$.

$$\begin{aligned} &\Rightarrow \exists N(\varepsilon) : \forall n, m \geq N(\varepsilon), d(g_n, g_m) < \varepsilon \\ &\Rightarrow \forall n, m \geq N(\varepsilon) \forall x \in X d_Y(g_n(x), g_m(x)) < \varepsilon \\ &\Rightarrow g_n \text{ uniformly convergent} \\ &\Rightarrow \exists g \in C(X, Y) : g_n \xrightarrow{n \rightarrow \infty} g \\ &\Rightarrow C(X, Y) \text{ is complete.} \end{aligned}$$

□

C.2 Some Topology

Lemma C.2.1. *Let $a \geq 0$. Let (M, g) be a n -dimensional closed connected Riemannian manifold and let $q, q', q'' \in M$ be three different points, define $d := \text{diam}(M, g)$. Then there is a homotopy equivalence*

$$\mathcal{E}^a(q, q') \rightarrow \mathcal{E}^{a+d}(q, q'').$$

Proof. Let $\gamma \in \mathcal{E}^a(q, q')$ and choose a minimal geodesic c from q' to q'' . Consider the homotopy,

$$H(t, s) := \begin{cases} \gamma(t(1+s)), & t \in [0, 1 - \frac{s}{2}] \\ c(t(1+s) - 1), & t \in [1 - \frac{s}{2}, 1] \end{cases}.$$

Observe that $H(\cdot, 0) = \gamma \in \mathcal{E}^a(q, q')$ and $H(\cdot, 1) \in \mathcal{E}^{a+d}(q, q'')$. \square

Lemma C.2.2. *Let $\{x_n : [0, 1] \rightarrow T^*M\}_{n \in \mathbb{N}}$ be a sequence of solutions of Hamilton's equations (2.8). If x_n converges uniformly to another solution $x_\infty : [0, 1] \rightarrow T^*M$ of (2.8), then there exists $N \in \mathbb{N}$ such that for all $m \geq N$ it holds that x_m and x_∞ are homotopic.*

Proof. (Idea of the proof is due to [17].) Notice that the set $x_\infty([0, 1]) \subset T^*M$ is an embedded submanifold of the cotangent bundle T^*M . Due to the Tubular Neighborhood theorem we know that there exists an open neighborhood $U \subset T^*M$ of $x_\infty([0, 1]) \subset U$ such that U is diffeomorphic to the normal bundle N of $x_\infty([0, 1])$. We assumed that the solutions x_m converge uniformly to x_∞ in the metric (4.28). This implies that there exists $N \in \mathbb{N}$ such that for all $m \geq N$ it follows that $x_m([0, 1]) \subset U$. Via the diffeomorphism

$$B: U \subset T^*M \rightarrow N \subset T(T^*M)$$

we find that every $x_m([0, 1])$ (as an image of B) is a section of the normal bundle of $B(x_\infty([0, 1])) = x_\infty([0, 1])$ in T^*M . The normal bundle is a vector bundle and hence fiberwise contractible. Therefore, there is a homotopy $H: [0, 1] \times N \rightarrow N$ pulling each $B(x_m)$ onto x_∞ . Now, if we precompose the latter homotopy with the diffeomorphism B given above, we get finally the homotopy $H': [0, 1] \times U \rightarrow U$ defined by $H(t, B(x)) \in U$ for $x \in U$. This proves the lemma. \square

C.3 Some functional analysis

C.3.1 The Sobolev spaces

We want to define the Sobolev spaces $W^{k,p}([0, 1], T^*M)$ whose elements are equivalence classes of functions $f : [0, 1] \rightarrow M$, where (M, g) is an n -dimensional Riemannian

manifold and $k, p \in \mathbb{N}_0$.

According to the Nash embedding theorem [38], there exists $K \in \mathbb{N}$ such that we can isometrically embed M into \mathbb{R}^K , $\varphi: M \rightarrow \mathbb{R}^K$. Hence T^*M isometrically embeds into $T^*\mathbb{R}^K = \mathbb{R}^{2K}$:

$$\begin{aligned} \Phi: T^*M &\longrightarrow TM &\longrightarrow T^*\mathbb{R}^K \equiv \mathbb{R}^{2K} \\ (q, p) &\longmapsto T^{-1}(q, p) = (q, v) &\longmapsto (\varphi(q), d\varphi(q)v) \end{aligned}$$

The map $T: T^*M \rightarrow TM$ was introduced in Section 2.1. We then are able to give a definition of the (k, p) -Sobolev space of functions (on $[0, 1]$) into the cotangent bundle T^*M :

$$W^{k,p}([0, 1], T^*M) := \{\gamma \in W^{k,p}([0, 1], \mathbb{R}^{2K}) \mid \gamma \subset \Phi(T^*M)\}. \quad (\text{C.8})$$

Let $\gamma \in W^{k,p}([0, 1], T^*M)$, then we can define the (k, p) -Sobolev norm

$$\|\gamma\|_{W^{k,p}([0,1], T^*M)} := \|\gamma\|_{W^{k,p}([0,1], \mathbb{R}^{2K})}. \quad (\text{C.9})$$

More generally,

$$W^{k,p}([0, 1], M) := \{\gamma \in W^{k,p}([0, 1], \mathbb{R}^K) \mid \gamma \subset \varphi(M)\}, \quad (\text{C.10})$$

and similarly as above, for $\gamma \in W^{k,p}([0, 1], M)$ we define the (k, p) -Sobolev norm

$$\|\gamma\|_{W^{k,p}([0,1], M)} := \|\gamma\|_{W^{k,p}([0,1], \mathbb{R}^K)}. \quad (\text{C.11})$$

C.3.2 On the involved functionals

Lemma C.3.1. *Let $B \rightarrow M$ be a vector bundle, M a smooth compact manifold. If $F: B \rightarrow \mathbb{R}$ is a fiberwise strictly convex function which is also fiberwise homogeneous of degree k , for $k \in \mathbb{N}$, then there exists a constant $K = K(M)$ such that*

$$\frac{1}{K}|v|_q^k \leq F(q, v) \leq K|v|_q^k.$$

($|\cdot|_q$ is the norm on the fiber B_q over q .)

Proof. Suppose $F(q, v) > 0$, what implies that $v \neq 0$. Then,

$$\min_{h \in B_q, |h|_q=1} F(q, h) \cdot |v|_q^k \leq F(q, v) = F\left(q, \frac{v}{|v|_q}\right) \cdot |v|_q^k \leq \max_{h \in B_q, |h|_q=1} F(q, h) \cdot |v|_q^k,$$

with

$$c(q) := \min_{h \in B_q, |h|_q=1} F(q, h) > 0 \quad \text{and} \quad C(q) := \max_{h \in B_q, |h|_q=1} F(q, h) > 0,$$

because of our assumption $F(q, v) > 0$. Define the quantity $D(q) := \max\left\{\frac{1}{c(q)}, C(q)\right\}$. Since $C(q)$ depends continuously on q , we introduce the sequence of continuous functions $G_n : M \rightarrow \mathbb{R}$ defined by $G_n(q) := C(q) + n$, $n \in \mathbb{N}$. Then, there exists the smallest N such that $G_N(q) \geq D(r) \geq C(r) > 0$, what implies $0 < \frac{1}{G_N(q)} \leq c(r)$. Define $\max_{q \in M} G_N(q) =: K(M) =: K$. All this implies:

$$\frac{1}{K}|v|_q^k \leq F(q, v) \leq K|v|_q^k.$$

□

Lemma C.3.2. *Let T^*M be the cotangent bundle of a compact manifold M , λ the standard Liouville form and $x \in \Omega_{q, q'}^1 T^*M$, for $q, q' \in M$. If $H_1, H_2 : T^*M \rightarrow \mathbb{R}$ are two Hamiltonian functions with the property that for all $t \in [0, 1]$ it holds that*

$$H_2(x(t)) \geq H_1(x(t)),$$

then we have

$$\mathcal{A}_{H_1}(x) \geq \mathcal{A}_{H_2}(x).$$

Proof.

$$\begin{aligned} \mathcal{A}_{H_1}(x) - \mathcal{A}_{H_2}(x) &= \int_0^1 (H_2(x(t)) - H_1(x(t))) dt \\ &\geq \int_0^1 0 dt = 0 \end{aligned}$$

$$\Rightarrow \mathcal{A}_{H_1}(x) \geq \mathcal{A}_{H_2}(x).$$

□

Remark. It would be enough to ask $H_2(x(t)) \geq H_1(x(t))$ for almost all $t \in [0, 1]$, but since by Sobolev embedding theorem x is also a continuous function this distinction is not of importance here. \diamond

Lemma C.3.3. *Let M be a smooth and connected manifold. The functionals $\mathcal{L}, \mathcal{E} : W^{1,2}([0, 1], M) \rightarrow \mathbb{R}$ are continuous with respect to the norm of $W^{1,2}([0, 1], M)$.*

Proof. According to Section C.3.1, the Sobolev norm on $W^{1,2}([0, 1], M)$ implies by definition

$$\|x\|_{W^{1,2}([0,1],M)}^2 = \|x\|_{W^{0,2}([0,1],M)}^2 + \|\dot{x}\|_{W^{0,2}([0,1],M)}^2.$$

Hence as being a norm, the expression for $\|x\|_{W^{1,2}([0,1],M)}^2$ just given is a continuous function on $W^{1,2}([0, 1], M)$ and so the difference $\|x\|_{W^{1,2}([0,1],M)}^2 - \|x\|_{W^{0,2}([0,1],M)}^2$ too, what means that the energy functional $\|\dot{x}\|_{W^{0,2}([0,1],M)}^2 = \mathcal{E}(x)$ is continuous. Let $x_n \in W^{1,2}([0, 1], M)$ such that $x_n \rightarrow x$ in $W^{1,2}([0, 1], M)$. So,

$$0 \leq |\mathcal{L}(x_n) - \mathcal{L}(x)| \leq \left| \sqrt{\mathcal{E}(x_n)} - \sqrt{\mathcal{E}(x)} \right| \rightarrow 0 \quad (n \rightarrow \infty),$$

which implies that the length functional \mathcal{L} is also a continuous function for elements in the Sobolev space $W^{1,2}([0, 1], M)$. \square

Lemma C.3.4. *Let $\gamma \in W^{1,2}([0, 1], M)$. Then*

$$\mathcal{L}^2(\gamma) \leq 2 \mathcal{E}(\gamma),$$

with equality if and only if for a given constant $C \in \mathbb{R}$ it holds true that $|\dot{\gamma}| = C$ almost everywhere.

Proof. We apply the Cauchy-Schwarz inequality to \mathcal{L} ,

$$\mathcal{L}(\gamma) \leq \left(\int_0^1 1 dt \right)^{\frac{1}{2}} \left(\int_0^1 |\dot{\gamma}|^2 dt \right)^{\frac{1}{2}} = \sqrt{2} \left(\frac{1}{2} \int_0^1 |\dot{\gamma}|^2 dt \right)^{\frac{1}{2}} = \sqrt{2} \mathcal{E}^{\frac{1}{2}}(\gamma).$$

The second part follows from the proof of the Cauchy-Schwarz inequality. \square

There is a one-to-one correspondence between the elements of $\mathcal{P}(H, q, q')$ – the space of C^∞ -smooth paths $x : [0, 1] \rightarrow T^*M$ solving $\dot{x}(t) = X_H(x(t))$, $x(j) \in T_{\pi(x(j))}^*M$ – and

the set $\phi_H^1(D_q(\Sigma)) \cap D_{q'}(\Sigma)$ given by the evaluation map

$$\begin{aligned} EV : \mathcal{P}(H, q, q') &\rightarrow \phi_H^1(D_q(\Sigma)) \cap D_{q'}(\Sigma) \\ x &\mapsto x(1) \end{aligned}$$

Lemma C.3.5. *The map EV is bijective.*

Proof. Assume there are $x, x' \in \mathcal{P}(H, q, q') : x \neq x'$ with $x(1) = x'(1)$. Then,

$$\begin{aligned} \Rightarrow \phi_H^1(x(0)) &= \phi_H^1(x'(0)) \\ \Rightarrow \phi_H^{-t} \phi_H^1(x(0)) &= \phi_H^{-t} \phi_H^1(x'(0)), t \in [0, 1] \\ \Rightarrow \phi_H^s(x(0)) &= \phi_H^s(x'(0)), s \in [0, 1] \\ \Rightarrow x(s) &= x'(s), \forall s \in [0, 1] \end{aligned}$$

Therefore the contradiction hence EV is injective. Take an element of the target space $x(1) \in \phi_H^1(D_q(\Sigma)) \cap D_{q'}(\Sigma)$.

$$\begin{aligned} \Rightarrow \phi_H^1(x(0)) &= x(1) \\ \Rightarrow \phi_H^{1-t}(x(0)) &= x(1-t), t \in [0, 1] \\ \Rightarrow \phi_H^s(x(0)) &= x(s), s \in [0, 1] \\ \Rightarrow x &\in \mathcal{P}(H, q, q'). \end{aligned}$$

So EV is surjective. □

C.3.3 A metric on the space of Sobolev functions

Sometimes we need to have a metric on the space of Sobolev functions from the unit interval to the cotangent bundle. It can be constructed as follows. According to the Nash embedding theorem [38] there exists $K \in \mathbb{N}$ such that we can isometrically embed (the Riemannian manifold) M into \mathbb{R}^K , $\varphi: M \rightarrow \mathbb{R}^K$. Hence T^*M isometrically embeds into $T^*\mathbb{R}^K = \mathbb{R}^{2K}$:

$$\phi: T^*M \longrightarrow \mathbb{R}^{2K}.$$

So, let us define the distance between two curves $x_1, x_2 \in W^{1,2}([0, 1], T^*M)$.

$$\begin{aligned}
 d(x_1, x_2)^2 &:= \|\phi \circ x_1 - \phi \circ x_2\|_{W^{1,2}([0,1], \mathbb{R}^{2K})}^2 & (C.12) \\
 &= \|\phi \circ x_1 - \phi \circ x_2\|_{L^2([0,1], \mathbb{R}^{2K})}^2 \\
 &\quad + \left\| \frac{d}{dt} \phi \circ x_1 - \frac{d}{dt} \phi \circ x_2 \right\|_{L^2([0,1], \mathbb{R}^{2K})}^2 .
 \end{aligned}$$

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