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Jérémy Blanc, Susanna Zimmermann

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TOPOLOGICAL SIMPLICITY OF THE CREMONA GROUPS

By JÉRÉMY BLANC and SUSANNA ZIMMERMANN

Abstract. The Cremona group is topologically simple when endowed with the Zariski or Euclidean topology, in any dimension ≥ 2 and over any infinite field. Two elements are always connected by an affine line, so the group is path-connected.

1. Introduction. Fixing a field k and an integer n , the *Cremona group of rank n over k* can be described algebraically as the group of automorphisms of the k -algebra $\text{Cr}_n(k) = \text{Aut}_k(k(x_1, \dots, x_n))$ or geometrically as the group $\text{Bir}_{\mathbb{P}^n}(k)$ of birational transformations of \mathbb{P}^n that are defined over the field k .

In an open problem session held at the international congress (see [Mumfo1974]), D. Mumford asked the following: “Let $G = \text{Aut}_{\mathbb{C}} \mathbb{C}(X, Y)$ be the Cremona group [...]. The problem is to topologize G [...] Is G simple?”.

As described in [Serre2010] (see section 2.1 below), one can endow the Cremona group with a natural *Zariski topology*, which is induced by *morphisms* $A \rightarrow \text{Bir}_{\mathbb{P}^n}$, where A is an algebraic variety (see Section 2). In [Blanc2010], it is shown that the group $\text{Bir}_{\mathbb{P}^2}(k)$ is topologically simple when endowed with this topology (i.e., it does not contain any non-trivial closed normal strict subgroup), when k is algebraically closed. In this text, we generalise this result and give a simple proof of the following:

THEOREM 1. *For each infinite field k and each $n \geq 1$, the group $\text{Bir}_{\mathbb{P}^n}(k)$ is topologically simple when endowed with the Zariski topology (i.e., it does not contain any non-trivial closed normal strict subgroup).*

Remark 1.1. For each field k , the group $\text{Bir}_{\mathbb{P}^2}(k)$ is not simple as an abstract group [CanLam2013, Lonjo2015]. If $k = \mathbb{R}$, it contains normal subgroups of index 2^m for each $m \geq 1$ [Zimme2015]. For each $n \geq 3$ and each field k , deciding whether the abstract group $\text{Bir}_{\mathbb{P}^n}(k)$ is simple or not is a still wide open question.

Remark 1.2. If k is a finite field, the Zariski topology on $\text{Bir}_{\mathbb{P}^n}(k)$ is the discrete topology (see Lemma 2.8), so the topological simplicity is equivalent to the simplicity as an abstract group, and is therefore false for $n = 2$, and open for $n \geq 3$. For $n = 1$, this is true if and only if $k = \mathbb{F}_{2^a}$, $a \geq 2$ (see Lemma 2.14).

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Recall that a local field is a locally compact topological field with respect to a non-discrete valuation. All examples are \mathbb{R} , \mathbb{C} and finite extensions of \mathbb{Q}_p and $\mathbb{F}_q((t))$. If k is a local field then there exists a natural topology on $\text{Bir}_{\mathbb{P}^n}(k)$, which makes it a Hausdorff topological group, and whose restriction on any algebraic subgroup (for instance on $\text{Aut}_{\mathbb{P}^n}(k) = \text{PGL}_{n+1}(k)$ and $(\text{PGL}_2(k))^n \subset \text{Aut}_{(\mathbb{P}^1)^n}(k)$) is the Euclidean topology (the classical topology given by distances between matrices) [BlaFur2013, Theorem 3]. This topology was called *Euclidean topology* of $\text{Bir}_{\mathbb{P}^n}(k)$. We will show the following analogue of Theorem 1, for this topology:

THEOREM 2. *For each local field k and each $n \geq 2$, the topological group $\text{Bir}_{\mathbb{P}^n}(k)$ is simple when endowed with the Euclidean topology (i.e., it does not contain any non-trivial closed normal strict subgroup).*

Remark 1.3. The result is, of course, false for $n = 1$, since $\text{PSL}_2(\mathbb{R})$ is a non-trivial normal strict subgroup of $\text{PGL}_2(\mathbb{R})$, closed for the Euclidean topology.

In the 1000-th Bourbaki Seminar [Serre2010], J.-P. Serre asked whether the group $\text{Bir}_{\mathbb{P}^n}(k)$ is connected with respect to the Zariski topology. When k is algebraically closed, a positive answer is given in [Blanc2010, Théorème 5.1]. We generalise this result (and give a simpler proof of it) as follows:

THEOREM 3. *For each infinite field k , each $n \geq 2$ and each $f, g \in \text{Bir}_{\mathbb{P}^n}(k)$, there is a morphism $\rho: \mathbb{A}^1 \rightarrow \text{Bir}_{\mathbb{P}^n}$, defined over k , such that $\rho(0) = f$ and $\rho(1) = g$. In particular, the group $\text{Bir}_{\mathbb{P}^n}(k)$ is connected with respect to the Zariski topology.*

The second property is also true for $n = 1$, although the first one is false.

For each $n \geq 2$, the groups $\text{Bir}_{\mathbb{P}^n}(\mathbb{R})$ and $\text{Bir}_{\mathbb{P}^n}(\mathbb{C})$ are path-connected, and thus connected with respect to the Euclidean topology.

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2. Preliminaries.

2.1. The families of birational maps and the Zariski topology induced.

In [Demaz1970], M. Demazure introduced the following functor (that he called Psaut, for pseudo-automorphisms, the name he gave to birational transformations):

Definition 2.1. Let k be an algebraically closed field, X be an irreducible algebraic variety and A a noetherian scheme, both defined over k . We define

$$\text{Bir}_X(A) = \left\{ \begin{array}{l} A\text{-birational transformations of } A \times X \text{ inducing an} \\ \text{isomorphism } U \longrightarrow V, \text{ where } U, V \text{ are open subsets} \\ \text{of } A \times X, \text{ whose projections on } A \text{ are surjective} \end{array} \right\},$$

$$\text{Aut}_X(A) = \{A\text{-automorphisms of } A \times X\} = \text{Bir}_X(A) \cap \text{Aut}(A \times X).$$

Remark 2.2. When $A = \text{Spec}(\mathbf{k})$, we see that $\text{Bir}_X(A)$ corresponds to the group of birational transformations of X defined over \mathbf{k} , which we will denote by $\text{Bir}_X(\mathbf{k})$. Similarly, $\text{Aut}_X(\mathbf{k})$ corresponds to the group of automorphisms of X defined over \mathbf{k} .

For each field k over which X is defined, we will similarly denote by $\text{Bir}_X(k)$ and $\text{Aut}_X(k)$ the group of birational transformations and automorphisms of X defined over k .

Definition 2.1 implicitly gives rise to the following notion of families, or morphisms $A \rightarrow \text{Bir}_X(\mathbf{k})$ (as in [Serre2010, Blanc2010, BlaFur2013]):

Definition 2.3. Taking A, X as above, an element $f \in \text{Bir}_X(A)$ and a \mathbf{k} -point $a \in A(\mathbf{k})$, we obtain an element $f_a \in \text{Bir}_X(\mathbf{k})$ given by $x \dashrightarrow p_2(f(a, x))$, where $p_2: A \times X \rightarrow X$ is the second projection.

The map $a \mapsto f_a$ represents a map from A (more precisely from the $A(\mathbf{k})$ -points of A) to $\text{Bir}_X(\mathbf{k})$, and will be called a **\mathbf{k} -morphism** (or morphism defined over \mathbf{k}) from A to Bir_X . If moreover $f \in \text{Aut}_X(A)$, then f also yields a morphism from A to Aut_X .

If $k \subset \mathbf{k}$ is a subfield over which X, A and f are defined, we will also say that the \mathbf{k} -morphism above is a *k -morphism*.

Remark 2.4. (1) If X, Y are two irreducible algebraic varieties and $\psi: X \dashrightarrow Y$ is a birational map, all of them defined over an algebraically closed field \mathbf{k} , the two functors Bir_X and Bir_Y are isomorphic, via ψ . In other words, morphisms $A \rightarrow \text{Bir}_X$ corresponds, via ψ , to morphisms $A \rightarrow \text{Bir}_Y$. The same holds with Aut_X and Aut_Y , if ψ is an isomorphism. We further get a bijection between k -morphisms to Bir_X and Bir_Y if X, Y and ψ are defined over a subfield $k \subset \mathbf{k}$.

(2) If X is projective, the connected component Aut_X° of Aut_X is an algebraic group, so there is a natural notion of morphism from A to Aut_X in this case, and this one coincides with the above definition.

(3) Just like with morphisms of algebraic varieties, for any field extension $k \subset k'$, any k -morphism $A \rightarrow \text{Bir}_X$ is also a k' -morphism, and thus yields a map $A(k') \rightarrow \text{Bir}_X(k')$.

Even if Bir_X is not representable by an algebraic variety or an ind-algebraic variety in general [BlaFur2013], we can define a topology on the group $\text{Bir}_X(k)$, given by this functor. This topology is called *Zariski topology* by J.-P. Serre in [Serre2010]:

Definition 2.5. Let X be an irreducible algebraic variety defined over a field k . A subset $F \subseteq \text{Bir}_X(k)$ is *closed in the Zariski topology* if for any k -algebraic variety A and any k -morphism $A \rightarrow \text{Bir}_X$ the preimage of F in $A(k)$ is closed.

Remark 2.6. In this definition one can of course replace “any algebraic variety A ” with “any *irreducible* algebraic variety A ”.

Endowed with this topology, $\text{Bir}_{\mathbb{P}^n}(\mathbf{k})$ is connected for each $n \geq 1$, and $\text{Bir}_{\mathbb{P}^2}(\mathbf{k})$ is topologically simple for each algebraically closed field \mathbf{k} [Blanc2010].

Let us make the following observation, whose statement and proof is analogue to classical statements for algebraic varieties:

LEMMA 2.7. *Let k be a field and X a geometrically irreducible algebraic variety defined over k . The Zariski topology on $\text{Bir}_X(k)$ is finer than the topology on $\text{Bir}_X(k)$ induced by the Zariski topology of $\text{Bir}_X(\mathbf{k})$, where \mathbf{k} is the algebraic closure of k .*

Proof. We show that for each closed subset $F' \subset \text{Bir}_X(\mathbf{k})$, the set $F = F' \cap \text{Bir}_X(k)$ is closed with respect to the Zariski topology.

To do this, we need to show that the preimage of F by any k -morphism $\rho: A \rightarrow \text{Bir}_X$ is closed. By definition of the Zariski topology of $\text{Bir}_X(\mathbf{k})$, the set $C = \{a \in A(\mathbf{k}) \mid \rho(a) \in F'\}$ is Zariski closed in $A(\mathbf{k})$. The closure R of $C \cap A(k)$ in $A(\mathbf{k})$ is defined over k [Sprin2009, Lemma 11.2.4]. Since $R(\mathbf{k}) \subset C(\mathbf{k})$, we have $R \cap A(k) = R(k) \subset C \cap A(k) \subset R \cap A(k)$, so $C \cap A(k) = R(k)$ is closed in $A(k)$.

It remains to observe that the equality $F = F' \cap \text{Bir}_X(k)$ implies that $C \cap A(k) = \{a \in A(k) \mid \rho(a) \in F'\} = \{a \in A(k) \mid \rho(a) \in F\} = \rho^{-1}(F)$. □

LEMMA 2.8. *Let k be a finite field and X be an algebraic variety defined over k . The Zariski topology on $\text{Bir}_X(k)$ is the discrete topology.*

Proof. Let us show that any subset $F \subset \text{Bir}_X(k)$ is closed. For this, we take a k -algebraic variety A , a k -morphism $\rho: A \rightarrow \text{Bir}_X$, and observe that $\rho^{-1}(F)$ is finite in A , hence is closed. □

2.2. The varieties H_d . The following algebraic varieties are useful to study morphisms to $\text{Bir}_{\mathbb{P}^n}$.

Definition 2.9. [BlaFur2013, Definition 2.3] Let d, n be positive integers.

(1) We define W_d to be the projective space parametrising, for each field k , equivalence classes of non-zero $(n + 1)$ -uples (h_0, \dots, h_n) of homogeneous polynomials $h_i \in k[x_0, \dots, x_n]$ of degree d , where (h_0, \dots, h_n) is equivalent to $(\lambda h_0, \dots, \lambda h_n)$ for any $\lambda \in k^*$. The equivalence class of (h_0, \dots, h_n) will be denoted by $[h_0 : \dots : h_n]$.

(2) We define $H_d \subseteq W_d$ to be the set of elements $h = [h_0 : \dots : h_n] \in W_d$ such that the rational map $\psi_h: \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ given by

$$[x_0 : \dots : x_n] \dashrightarrow [h_0(x_0, \dots, x_n) : \dots : h_n(x_0, \dots, x_n)]$$

is birational. We denote by π_d the map $H_d(k) \rightarrow \text{Bir}_{\mathbb{P}^n}(k)$ which sends h onto ψ_h .

PROPOSITION 2.10. *Let d, n be positive integers.*

(1) *The set H_d is locally closed in the projective space W_d and thus inherits the structure of an algebraic variety;*

(2) The map π_d corresponds to a morphism $H_d \rightarrow \text{Bir}_{\mathbb{P}^n}$, defined over any field. For each field k , the image of the corresponding map $H_d(k) \rightarrow \text{Bir}_{\mathbb{P}^n}(k)$ consists of all birational maps of degree $\leq d$.

Proof. Follows from [BlaFur2013, Lemma 2.4]. □

2.3. The Euclidean topology. Suppose that k is a local field.

The Euclidean topology of $\text{Bir}_{\mathbb{P}^n}(k)$ described in [BlaFur2013, Section 5] is defined as follows: on $W_d(k) \simeq \mathbb{P}^{(n+1)\binom{n+d}{d}-1}(k)$ we put the classical Euclidean topology, on $H_d(k) \subset W_d(k)$ the induced topology and on $\pi_d(H_d(k)) = \{f \in \text{Bir}_{\mathbb{P}^n}(k) \mid \deg(f) \leq d\}$ the quotient topology induced by π_d . The Euclidean topology on $\text{Bir}_{\mathbb{P}^n}(k)$ is then the inductive limit topology induced by the inclusions

$$\{f \in \text{Bir}_{\mathbb{P}^n}(k) \mid \deg(f) \leq d\} \longrightarrow \{f \in \text{Bir}_{\mathbb{P}^n}(k) \mid \deg(f) \leq d + 1\}.$$

LEMMA 2.11. *Let k be a local field, let A be an algebraic variety defined over k , and let $\rho: A \rightarrow \text{Bir}_{\mathbb{P}^n}$ be a k -morphism. Then the map*

$$A(k) \longrightarrow \text{Bir}_{\mathbb{P}^n}(k)$$

is continuous for the Euclidean topologies.

Proof. There exists an open affine covering $(A_i)_{i \in I}$ of A , with respect to the Zariski topology, with the following property: for each $i \in I$ there exists an integer d_i and a morphism of algebraic varieties $\rho_i: A_i \rightarrow H_{d_i}$, such that the restriction of ρ to A_i is $\pi_{d_i} \circ \rho_i$ [BlaFur2013, Lemma 2.6]. It follows from the construction that the A_i and ρ_i can be assumed to be defined over k .

We take a subset $U \subset \text{Bir}_{\mathbb{P}^n}(k)$, open with respect to the Euclidean topology, and want to show that $\rho^{-1}(U) \subset A(k)$ is open with respect to the Euclidean topology. As all $A_i(k)$ are open in $A(k)$, it suffices to show that $\rho^{-1}(U) \cap A_i(k)$ is open in $A_i(k)$ for each i . This follows from the fact that $\rho|_{A_i} = \pi_{d_i} \circ \rho_i$ and that both π_{d_i} and ρ_i are continuous with respect to the Euclidean topology. □

2.4. The projective linear group. Note that $\text{Bir}_{\mathbb{P}^n}(k)$ contains the algebraic group $\text{Aut}_{\mathbb{P}^n}(k) = \text{PGL}_{n+1}(k)$ and that the restriction of the Zariski topology to this subgroup corresponds to the usual Zariski topology of the algebraic variety $\text{PGL}_{n+1}(k)$, which can be viewed as the open subset of $\mathbb{P}^{(n+1)^2-1}(k)$, more precisely as complement of the hypersurface given by the vanishing of the determinant.

Let us make the following two observations:

LEMMA 2.12. *If k is an infinite field and $n \geq 2$, then $\text{PSL}_n(k)$ is dense in $\text{PGL}_n(k)$ with respect to the Zariski topology. Moreover, every non-trivial normal subgroup of $\text{PGL}_n(k)$ contains $\text{PSL}_n(k)$. In particular, $\text{PGL}_n(k)$ does not contain*

any non-trivial normal strict subgroup which is closed with respect to the Zariski topology.

Proof. (1) Observe that the group homomorphism $\det: \text{GL}_n(\mathbf{k}) \rightarrow \mathbf{k}^*$ yields a group homomorphism

$$\det: \text{PGL}_n(\mathbf{k}) \longrightarrow (\mathbf{k}^*)/\{f^n \mid f \in \mathbf{k}^*\},$$

whose kernel is the group $\text{PSL}_n(\mathbf{k})$. We consider the morphism

$$\begin{aligned} \rho: \mathbb{A}^1(\mathbf{k}) \setminus \{0\} &\longrightarrow \text{PGL}_n(\mathbf{k}) \\ t &\longmapsto \begin{pmatrix} t & 0 \\ 0 & I \end{pmatrix} \end{aligned}$$

where I is the identity matrix of size $(n - 1) \times (n - 1)$, and observe that $\rho^{-1}(\text{PSL}_n(\mathbf{k}))$ contains $\{t^n \mid t \in \mathbb{A}^1(\mathbf{k})\}$, which is an infinite subset of $\mathbb{A}^1(\mathbf{k})$ and is therefore dense in $\mathbb{A}^1(\mathbf{k})$. The closure of $\text{PSL}_n(\mathbf{k})$ contains thus $\rho(\mathbb{A}^1(\mathbf{k}) \setminus \{0\})$. As every element of $\text{PGL}_n(\mathbf{k})$ is equal to some $\rho(t)$ modulo $\text{PSL}_n(\mathbf{k})$, we obtain that $\text{PSL}_n(\mathbf{k})$ is dense in $\text{PGL}_n(\mathbf{k})$.

(2) Let $N \subset \text{PGL}_n(\mathbf{k})$ be a normal subgroup with $N \neq \{\text{id}\}$, and let $f \in N$ be a non-trivial element. We want to show that N contains $\text{PSL}_n(\mathbf{k})$. Since the center of $\text{PGL}_n(\mathbf{k})$ is trivial, one can replace f with $\alpha f \alpha^{-1} f^{-1}$, where $\alpha \in \text{PGL}_n(\mathbf{k})$ does not commute with f , and assume that $f \in N \cap \text{PSL}_n(\mathbf{k})$. Then, as $\text{PSL}_n(\mathbf{k})$ is a simple group [Dieud1971, Chapitre II, Section 2], we obtain $\text{PSL}_n(\mathbf{k}) \subset N$.

(1) and (2) imply that $\text{PGL}_n(\mathbf{k})$ does not contain any non-trivial normal strict subgroup which is closed with respect to the Zariski topology. □

Remark 2.13. Lemma 2.12 does not work for the Euclidean topology. For instance, for each $n \geq 1$, the group $\text{PSL}_{2n}(\mathbb{R}) = \{A \in \text{PGL}_{2n}(\mathbb{R}) \mid \det(A) > 0\}$ is a normal strict subgroup of $\text{PGL}_{2n}(\mathbb{R})$ which is closed with respect to the Euclidean topology.

LEMMA 2.14. *Let \mathbf{k} be a finite field. Then*

- (1) $\text{PGL}_2(\mathbf{k}) = \text{PSL}_2(\mathbf{k})$ if and only if $\text{char}(\mathbf{k}) = 2$,
- (2) $\text{PGL}_2(\mathbf{k})$ is a simple group if and only if $\mathbf{k} = \mathbb{F}_{2^a}$, $a \geq 2$.

Proof. (1) As explained before, $\text{PSL}_2(\mathbf{k}) = \text{PGL}_2(\mathbf{k})$ if and only if every element of \mathbf{k}^* (or equivalently of \mathbf{k}) is a square. As \mathbf{k} is finite, the group homomorphism

$$\begin{aligned} \mathbf{k}^* &\longrightarrow \mathbf{k}^* \\ x &\longmapsto x^2 \end{aligned}$$

is surjective if and only if it is injective, and this corresponds to ask that the characteristic of \mathbf{k} is 2.

(2) If $\text{char}(k) \neq 2$, then $\text{PSL}_2(k) \subsetneq \text{PGL}_2(k)$ is a non-trivial normal subgroup. If $\text{char}(k) = 2$, then $\text{PGL}_2(k) = \text{PSL}_2(k)$ is a simple group if and only if $k \neq \mathbb{F}_2$ [Dieud1971, Chapitre II, Section 2]. \square

3. Proof of the results.

3.1. The construction associated to fixed points. Let us explain the following simple construction that will be often used in the sequel.

Example 3.1. Let $f \in \text{Bir}_{\mathbb{P}^n}(k)$ be an element fixing the point $p = [1 : 0 : \dots : 0]$ and that induces a local isomorphism at p .

In the chart $x_0 = 1$, we can write f locally as

$$x = (x_1, \dots, x_n) \mapsto \left(\frac{p_{1,1}(x) + \dots + p_{1,m}(x)}{1 + q_{1,1}(x) + \dots + q_{1,m}(x)}, \dots, \frac{p_{n,1}(x) + \dots + p_{n,m}(x)}{1 + q_{n,1}(x) + \dots + q_{n,m}(x)} \right),$$

where the $p_{i,j}, q_{i,j} \in k[x_1, \dots, x_n]$ are homogeneous of degree j . For each $t \in k \setminus \{0\}$, the element

$$\theta_t : (x_1, \dots, x_n) \mapsto (tx_1, \dots, tx_n)$$

extends to a linear automorphism of $\mathbb{P}^n(k)$ fixing p . Then the map $t \mapsto (\theta_t)^{-1} \circ f \circ \theta_t$ gives rise to a morphism $F : \mathbb{A}^1 \setminus \{0\} \rightarrow \text{Bir}_{\mathbb{P}^n}(k)$ whose image contains only conjugates of f by linear automorphisms.

Writing F locally, we can observe that F extends to a morphism $\mathbb{A}^1 \rightarrow \text{Bir}_{\mathbb{P}^n}(k)$ such that $F(0)$ is linear. Indeed, $F(t)$ can be written locally as follows:

$$F(t)(x) = F(t)(x_1, \dots, x_n) = \left(\frac{p_{1,1}(x) + tp_{1,2}(x) + \dots + t^{m-1}p_{1,m}(x)}{1 + tq_{1,1}(x) + \dots + t^mq_{1,m}(x)}, \dots, \frac{p_{n,1}(x) + tp_{n,2}(x) + \dots + t^{m-1}p_{n,m}(x)}{1 + tq_{n,1}(x) + \dots + t^mq_{n,m}(x)} \right),$$

and $F(0)$ corresponds to the derivative (linear part) of F at p , which is locally given by

$$(x_1, \dots, x_n) \mapsto (p_{1,1}(x), \dots, p_{n,1}(x))$$

and which is an element of $\text{Aut}_{\mathbb{P}^n}(k) \subset \text{Bir}_{\mathbb{P}^n}(k)$ since f was chosen to be a local isomorphism at p .

Using the example above, one can construct k -morphisms $\mathbb{A}^1 \rightarrow \text{Bir}_{\mathbb{P}^n}$.

PROPOSITION 3.2. *Let k be a field, $n \geq 1$, let $g \in \text{Bir}_{\mathbb{P}^n}(k)$ and $p \in \mathbb{P}^n(k)$ be a point such that g fixes p and induces a local isomorphism at p . Then there exist*

k -morphisms $\nu: \mathbb{A}^1 \setminus \{0\} \rightarrow \text{Aut}_{\mathbb{P}^n}$ and $\rho: \mathbb{A}^1 \rightarrow \text{Bir}_{\mathbb{P}^n}$ such that the following hold:

(1) For each field extension $k \subset k'$ and each $t \in \mathbb{A}^1(k') \setminus \{0\}$, we have

$$\rho(t) = \nu(t)^{-1} \circ g \circ \nu(t).$$

Moreover, $\nu(1) = \text{id}$, so $\rho(1) = g$.

(2) The element $\rho(0)$ belongs to $\text{Aut}_{\mathbb{P}^n}(k)$. It is the identity if and only if the action of g on the tangent space $T_p(\mathbb{P}^n)$ is trivial.

Proof. Conjugating by an element of $\text{Aut}_{\mathbb{P}^n}(k)$, we can assume that $p = [1 : 0 : \dots : 0]$. We then choose ν to be given by

$$\nu(t): [x_0 : x_1 : \dots : x_n] \mapsto [x_0 : tx_1 : \dots : tx_n],$$

and define $\rho: \mathbb{A}^1 \setminus \{0\} \rightarrow \text{Bir}_{\mathbb{P}^n}$ by $\rho(t) = \nu(t)^{-1} \circ g \circ \nu(t)$. As it was shown in Example 3.1, the k -morphism ρ extends to a k -morphism $\mathbb{A}^1 \rightarrow \text{Bir}_{\mathbb{P}^n}$ such that $\rho(0) \in \text{Aut}_{\mathbb{P}^n}(k)$. Moreover, this element is trivial if and only if the action of g on the tangent space $T_p(\mathbb{P}^n)$ is trivial. \square

3.2. Closed normal subgroups of the Cremona groups. As a consequence of Proposition 3.2, we obtain the following result:

PROPOSITION 3.3. *Let k be an infinite field. Let n be a positive integer. Let $N \subset \text{Bir}_{\mathbb{P}^n}(k)$ be a normal subgroup. If N is closed with respect to the Zariski topology or to the Euclidean topology (if k is a local field), then $N \cap \text{Aut}_{\mathbb{P}^n}(k)$ is not the trivial group.*

Proof. We can assume that $n \geq 2$, as the result is trivial for $n = 1$ (in which case $\text{Bir}_{\mathbb{P}^n}(k) = \text{Aut}_{\mathbb{P}^n}(k)$). Let us choose a non-trivial element $f \in N$. As f is a birational transformation, it induces an isomorphism $U \rightarrow V$, where $U, V \subset \mathbb{P}^n$ are two non-empty open subsets defined over k . Since k is infinite, $U(k)$ and $V(k)$ are not empty, so we can find $p \in U(k)$, and $q = f(p) \in V(k)$. We can moreover choose $p \neq q$, since $\{p \in U \mid f(p) \neq p\}$ is open and non-empty in U . Let us take an element $\alpha \in \text{Aut}_{\mathbb{P}^n}(k)$ that fixes p and q . The element $g = \alpha^{-1} f^{-1} \alpha f \in N$ fixes p and is a local isomorphism at this point. We can choose α such that the derivative $D_p(g)$ of g at this point is not trivial, since

$$D_p(g) = D_p(\alpha^{-1}) \circ D_q(f^{-1}) \circ D_q(\alpha) \circ D_p(f).$$

Indeed, changing coordinates one can assume that $q = [1 : 0 : \dots : 0]$, $p = [0 : 1 : 0 : \dots : 0]$ and can for instance choose $\alpha: [x_0 : \dots : x_n] \mapsto [x_0 + \xi x_2 : x_1 : x_2 : \dots : x_n]$, for some $\xi \in k$. This choice yields $D_q(\alpha) = \text{id}$ and gives infinitely many possibilities for $D_p(\alpha^{-1})$.

By Proposition 3.2, there exists a k -morphism $\rho: \mathbb{A}^1 \rightarrow \text{Bir}_{\mathbb{P}^n}$ such that $\rho(0) \in \text{Aut}_{\mathbb{P}^n}(k) \setminus \{\text{id}\}$ and such that $\rho(t) \in N$ for each $t \in \mathbb{A}^1(k) \setminus \{0\}$. Since N is closed

(with respect to the Zariski or to the Euclidean topology), $\rho^{-1}(N) \subset \mathbb{A}^1(k)$ is closed (with respect to the Zariski or to the Euclidean topology respectively, see Lemma 2.11 in the latter case) and contains $\mathbb{A}^1(k) \setminus \{0\}$. For the Zariski topology, one uses the fact that k is infinite to get $\rho^{-1}(N) = \mathbb{A}^1(k)$. For the Euclidean topology, one uses the fact that k is non-discrete to get the same result. In each case, we find that $\rho(0) \in N \cap \text{Aut}_{\mathbb{P}^n}(k)$. \square

LEMMA 3.4. *Let k be an infinite field, $n \geq 2$ an integer and $N \subset \text{Bir}_{\mathbb{P}^n}(k)$ be a normal subgroup, with $N \cap \text{Aut}_{\mathbb{P}^n}(k) \neq \{\text{id}\}$. Then $\text{PGL}_{n+1}(k) = \text{Aut}_{\mathbb{P}^n}(k) \subset N$.*

Proof. The group $N \cap \text{Aut}_{\mathbb{P}^n}(k)$ is a non-trivial normal subgroup of $\text{Aut}_{\mathbb{P}^n}(k) = \text{PGL}_{n+1}(k)$, so contains $\text{PSL}_{n+1}(k)$ by Lemma 2.12.

For each $a \in k^*$, we define $g_a \in N$ and $h \in \text{Bir}_{\mathbb{P}^n}(k)$ by

$$g_a : [x_0 : \cdots : x_n] \mapsto \left[x_0 : ax_1 : \frac{1}{a}x_2 : x_3 : \cdots : x_n \right]$$

$$h : [x_0 : \cdots : x_n] \dashrightarrow \left[x_0 : x_1 : x_2 \cdot \frac{x_1}{x_0} : x_3 : \cdots : x_n \right].$$

Then, $g'_a = hg_a h^{-1} \in N$ is given by

$$g'_a : [x_0 : \cdots : x_n] \mapsto [x_0 : ax_1 : x_2 : x_3 : \cdots : x_n].$$

As every element of $\text{PGL}_n(k)$ is equal to some g'_a modulo $\text{PSL}_{n+1}(k)$, we obtain that $\text{PGL}_{n+1}(k) \subset N$. \square

PROPOSITION 3.5. *Let k be an infinite field, $n \geq 2$ an integer and consider $\text{Bir}_{\mathbb{P}^n}(k)$ endowed with the Zariski topology or the Euclidean topology (if k is a local field). Then the normal subgroup of $\text{Bir}_{\mathbb{P}^n}(k)$ generated by $\text{Aut}_{\mathbb{P}^n}(k)$ is dense in $\text{Bir}_{\mathbb{P}^n}(k)$.*

In particular, $\text{Bir}_{\mathbb{P}^n}(k)$ does not contain any non-trivial closed normal strict subgroup.

Proof. (1) Let $f \in \text{Bir}_{\mathbb{P}^n}(k)$, $f \neq \text{id}$. It induces an isomorphism $U \rightarrow V$, where $U, V \subset \mathbb{P}^n$ are two non-empty open subsets, defined over k . Since k is infinite, we can find $p \in U(k)$. There exist $\alpha_1, \alpha_2 \in \text{Aut}_{\mathbb{P}^n}(k)$ such that $g := \alpha_1 f \alpha_2$ fixes p , is a local isomorphism at this point and such that $D_p(g)$ is not trivial. By Proposition 3.2, there exist k -morphisms $\nu : \mathbb{A}^1 \setminus \{0\} \rightarrow \text{Aut}_{\mathbb{P}^n}(k)$ and $\rho_1 : \mathbb{A}^1 \rightarrow \text{Bir}_{\mathbb{P}^n}(k)$ such that $\rho_1(t) = \nu(t)^{-1} \circ g^{-1} \circ \nu(t)$ for each $t \in \mathbb{A}^1(k) \setminus \{0\}$ and $\rho_1(0) \in \text{Aut}_{\mathbb{P}^n}(k)$. We define a k -morphism

$$\rho_2 : \mathbb{A}^1 \longrightarrow \text{Bir}_{\mathbb{P}^n}(k), \quad \rho_2(t) = \alpha_1^{-1} \circ g \circ \rho_1(t) \circ \rho_1(0)^{-1} \circ \alpha_2^{-1}.$$

Since $\alpha_1, \alpha_2, \rho_1(0), \nu(t) \in \text{Aut}_{\mathbb{P}^n}(k)$ for all $t \in \mathbb{A}^1 \setminus \{0\}$, the map

$$\rho_2(t) = \alpha_1^{-1} \circ (g \circ \nu(t)^{-1} \circ g^{-1}) \circ \nu(t) \circ \rho_1(0)^{-1} \circ \alpha_2^{-1}$$

is contained in the normal subgroup of $\text{Bir}_{\mathbb{P}^n}(\mathbf{k})$ generated by $\text{Aut}_{\mathbb{P}^n}(\mathbf{k})$, for each $t \in \mathbb{A}^1 \setminus \{0\}$. Therefore, $f = \rho_2(0)$ is contained in the closure of the normal subgroup of $\text{Bir}_{\mathbb{P}^n}(\mathbf{k})$ generated by $\text{Aut}_{\mathbb{P}^n}(\mathbf{k})$.

(2) Let $\{\text{id}\} \neq N \subset \text{Bir}_{\mathbb{P}^n}(\mathbf{k})$ be a closed normal subgroup (with respect to the Zariski or to the Euclidean topology). It follows from Proposition 3.3 and Lemma 3.4 that $\text{Aut}_{\mathbb{P}^n}(\mathbf{k}) \subset N$. Since N is closed, it contains the closure of the normal subgroup generated by $\text{Aut}_{\mathbb{P}^n}(\mathbf{k})$, which is equal to $\text{Bir}_{\mathbb{P}^n}(\mathbf{k})$. \square

Note that Proposition 3.5, together with Lemma 2.12 (for dimension 1 in the case of the Zariski topology), yields Theorems 1 and 2.

3.3. Connectedness of the Cremona groups. The group $\text{Bir}_{\mathbb{P}^n}$ is connected with respect to the Zariski topology [Blanc2010]. More precisely, we have the following:

PROPOSITION 3.6. [Blanc2010, Théorème 5.1] *Let \mathbf{k} be an algebraically closed field and $n \geq 1$. For each $f, g \in \text{Bir}_{\mathbb{P}^n}(\mathbf{k})$ there is an open subset $U \subset \mathbb{A}^1(\mathbf{k})$ that contains 0 and 1, and a morphism $\rho: U \rightarrow \text{Bir}_{\mathbb{P}^n}(\mathbf{k})$ such that $\rho(0) = f$ and $\rho(1) = g$.*

This corresponds to saying that $\text{Bir}_{\mathbb{P}^n}(\mathbf{k})$ is “rationally connected”. We will generalise this for any field \mathbf{k} , and provide a morphism from the whole \mathbb{A}^1 (Proposition 3.11 below), showing then that $\text{Bir}_{\mathbb{P}^n}(\mathbf{k})$ is “ \mathbb{A}^1 -uniruled”.

Let us recall the following classical fact:

LEMMA 3.7. *For each field \mathbf{k} and each integer $n \geq 2$, there is an integer m and a \mathbf{k} -morphism $\rho: \mathbb{A}^m \rightarrow \text{SL}_n$ such that $\rho(\mathbb{A}^m(\mathbf{k})) = \text{SL}_n(\mathbf{k})$.*

Proof. Using Gauss-Jordan elimination, every element of $\text{SL}_n(\mathbf{k})$ is a product of a diagonal matrix and r elementary matrices of the first kind: matrices of the form $I + \lambda e_{i,j}$, $\lambda \in \mathbf{k}$, $i \neq j$, where $(e_{i,j})_{i,j=1,\dots,n}$ is the canonical basis of the vector space of $n \times n$ -matrices. Moreover, the number r can be chosen to be the same for all elements of $\text{SL}_n(\mathbf{k})$. We then observe that

$$\begin{pmatrix} 1 & \lambda - 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda^{-1} - 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for each $\lambda \in \mathbf{k}^*$. Using finitely many such products, we obtain then all diagonal elements. This gives the existence of $s \in \mathbb{N}$, only dependent on n , such that every element of $\text{SL}_n(\mathbf{k})$ is a product of s elementary matrices of the first kind.

Denoting by $\nu_{i,j}: \mathbb{A}^1 \rightarrow \text{SL}_n(\mathbf{k})$ the \mathbf{k} -morphism sending λ to $I + \lambda e_{i,j}$, this shows that every element of $\text{SL}_n(\mathbf{k})$ is in the image of a product morphism $\mathbb{A}^m \rightarrow \text{SL}_n(\mathbf{k})$ of finitely many $\nu_{i,j}$. The number of such maps being finite, we can enlarge m and obtain one morphism for all maps. \square

COROLLARY 3.8. *For each field k , each integer $n \geq 2$ and all $f, g \in \text{PSL}_n(k)$, there exists a k -morphism $\nu: \mathbb{A}^1 \rightarrow \text{PSL}_n$ such that $\nu(0) = f$ and $\nu(1) = g$.*

Proof. It suffices to take a morphism $\rho: \mathbb{A}^m \rightarrow \text{SL}_n$ as in Lemma 3.7, to choose $v, w \in \mathbb{A}^m(k)$ such that $\rho(v) = f, \rho(w) = g$ in $\text{PSL}_n(k)$, and to define $\nu(t) = \rho(v + t(w - v))$. □

Remark 3.9. By construction, Corollary 3.8 also works for $\text{SL}_n(k)$, but is in fact false for $\text{GL}_n(k)$. Indeed, every k -morphism $\nu: \mathbb{A}^1 \rightarrow \text{GL}_n$ gives rise to a morphism $\det \circ \nu: \mathbb{A}^1 \rightarrow \mathbb{A}^1 \setminus \{0\}$, which is necessarily constant. As every morphism $\mathbb{A}^1 \rightarrow \text{PGL}_n$ lifts to a morphism $\mathbb{A}^1 \rightarrow \text{GL}_n$, the same holds for PGL_n .

Example 3.10. Let k be a field, $n \geq 2$ and $\lambda \in k^*$. We consider $g \in \text{Bir}_{\mathbb{P}^n}(k)$ given by

$$g: [x_0 : \cdots : x_n] \mapsto \left[\frac{x_0(x_1 + \lambda x_2) + x_1 x_2}{x_1 + x_2} : x_1 : \cdots : x_n \right].$$

We observe that $p_1 = [0 : 1 : 0 : \cdots : 0]$ and $p_2 = [0 : 0 : 1 : 0 : \cdots : 0]$ are both fixed by g . In local charts $x_1 = 1$ and $x_2 = 1$, the map g becomes:

$$\begin{aligned} [x_0 : 1 : x_2 : x_3 : \cdots : x_n] &\mapsto \left[\frac{x_0(1 + \lambda x_2) + x_2}{x_2 + 1} : 1 : x_2 : x_3 : \cdots : x_n \right] \\ [x_0 : x_1 : 1 : x_3 : \cdots : x_n] &\mapsto \left[\frac{x_0(x_1 + \lambda) + x_1}{x_1 + 1} : x_1 : 1 : x_3 : \cdots : x_n \right]. \end{aligned}$$

Applying Proposition 3.2 to the two fixed points, we get two k -morphisms $\rho_1, \rho_2: \mathbb{A}^1 \rightarrow \text{Bir}_{\mathbb{P}^n}$ such that $\rho_1(1) = g = \rho_2(1)$ and $\rho_1(0), \rho_2(0) \in \text{Aut}_{\mathbb{P}^n}(k)$. The two elements are provided by the construction Example 3.1. Choosing for this one the affine coordinates $x_1 \neq 0$ and $x_2 \neq 0$ using permutations of the coordinates, we obtain the following maps corresponding to the linear parts in these affine spaces:

$$\begin{aligned} \rho_1(0): [x_0 : x_1 : x_2 : x_3 : \cdots : x_n] &\mapsto [x_0 + x_2 : x_1 : x_2 : x_3 : \cdots : x_n], \\ \rho_2(0): [x_0 : x_1 : x_2 : x_3 : \cdots : x_n] &\mapsto [x_0 \lambda + x_1 : x_1 : x_2 : x_3 : \cdots : x_n]. \end{aligned}$$

We can now give the following generalisation of [Blanc2010, Théorème 5.1] (Proposition 3.6):

PROPOSITION 3.11. *For each infinite field k , each integer $n \geq 2$ and all $f, g \in \text{Bir}_{\mathbb{P}^n}(k)$, there exists a k -morphism $\nu: \mathbb{A}^1 \rightarrow \text{Bir}_{\mathbb{P}^n}$ such that $\nu(0) = f$ and $\nu(1) = g$.*

Proof. Multiplying the morphism with f^{-1} , we can assume that $f = \text{id}$. We denote by $N \subset \text{Bir}_{\mathbb{P}^n}(\mathbb{k})$ the subset given by

$$N = \left\{ g \in \text{Bir}_{\mathbb{P}^n}(\mathbb{k}) \mid \begin{array}{l} \text{there exists a } \mathbb{k}\text{-morphism } \nu: \mathbb{A}^1 \longrightarrow \text{Bir}_{\mathbb{P}^n} \\ \text{such that } \nu(0) = \text{id and } \nu(1) = g \end{array} \right\}.$$

If $f, g \in N$ are associated to \mathbb{k} -morphisms ν_f, ν_g , we define a \mathbb{k} -morphism $\nu_{fg}: \mathbb{A}^1 \rightarrow \text{Bir}_{\mathbb{P}^n}$ by $\nu_{fg}(t) = \nu_f(t)\nu_g(t)$, which satisfies $\nu_{fg}(0) = \text{id}$ and $\nu_{fg}(1) = fg$. For each $h \in \text{Bir}_{\mathbb{P}^n}(\mathbb{k})$, we can also define a morphism $t \mapsto h\nu_f(t)h^{-1}$. Thus, N is a normal subgroup of $\text{Bir}_{\mathbb{P}^n}(\mathbb{k})$ and it contains $\text{PSL}_{n+1}(\mathbb{k})$ by Corollary 3.8. As N is a priori not closed, we cannot apply Theorem 1. However, we will apply Proposition 3.2 and Example 3.10 to obtain the result.

First, taking $\lambda, g, \rho_1, \rho_2$ as in Example 3.10, the morphisms $t \mapsto \rho_i(t) \circ \rho_i(0)^{-1}$, $i = 1, 2$, show that $g \circ (\rho_1(0))^{-1}, g \circ (\rho_2(0))^{-1} \in N$, which implies that $\rho_1(0) \circ (\rho_2(0))^{-1} \in N$. Since $\rho_1(0) \in \text{PSL}_{n+1}(\mathbb{k}) \subset N$, this implies that

$$\rho_2(0): [x_0 : x_1 : x_2 : x_3 : \cdots : x_n] \longmapsto [x_0\lambda + x_1 : x_1 : x_2 : x_3 : \cdots : x_n]$$

belongs to N , for each $\lambda \in \mathbb{k}^*$. Hence, $\text{Aut}_{\mathbb{P}^n}(\mathbb{k}) = \text{PGL}_{n+1}(\mathbb{k}) \subset N$.

Second, we take any $g \in \text{Bir}_{\mathbb{P}^n}(\mathbb{k})$ of degree $d \geq 2$, take a point $p \in \mathbb{P}^n(\mathbb{k})$ such that g induces a local isomorphism at p , choose $\alpha \in \text{PSL}_{n+1}(\mathbb{k})$ such that $\alpha \circ g$ fixes p . Proposition 3.2 yields the existence of a \mathbb{k} -morphism $\rho: \mathbb{A}^1 \rightarrow \text{Bir}_{\mathbb{P}^n}$ with $\rho(1) = \alpha \circ g$ and $\rho(0) \in \text{Aut}_{\mathbb{P}^n}(\mathbb{k})$. Choosing $\rho': \mathbb{A}^1 \rightarrow \text{Bir}_{\mathbb{P}^n}$ given by $\rho'(t) = \rho(t) \circ \rho(0)^{-1}$, we obtain that $\rho'(1) = \alpha \circ g \circ \rho(0)^{-1} \in N$. Since $\alpha, \rho(0) \in \text{Aut}_{\mathbb{P}^n}(\mathbb{k}) \subset N$, this shows that $g \in N$ and concludes the proof. \square

COROLLARY 3.12. *For each infinite field \mathbb{k} and each $n \geq 1$, the group $\text{Bir}_{\mathbb{P}^n}(\mathbb{k})$ is connected with respect to the Zariski topology.*

Proof. For $n = 1$, the result follows from the fact that $\text{Bir}_{\mathbb{P}^1} = \text{Aut}_{\mathbb{P}^1} = \text{PGL}_2$ is an open subvariety of \mathbb{P}^3 . For $n \geq 2$, this follows from Proposition 3.11. \square

COROLLARY 3.13. *For each $n \geq 2$, the groups $\text{Bir}_{\mathbb{P}^n}(\mathbb{R})$ and $\text{Bir}_{\mathbb{P}^n}(\mathbb{C})$ are path-connected, and thus connected with respect to the Euclidean topology.*

Proof. Let us fix $\mathbb{k} = \mathbb{R}$ or $\mathbb{k} = \mathbb{C}$. For each $f, g \in \text{Bir}_{\mathbb{P}^n}(\mathbb{k})$ there is a \mathbb{k} -morphism $\nu: \mathbb{A}^1 \rightarrow \text{Bir}_{\mathbb{P}^n}$ such that $\nu(0) = f$ and $\nu(1) = g$ (Proposition 3.11). The corresponding map $\mathbb{k} = \mathbb{A}^1(\mathbb{k}) \rightarrow \text{Bir}_{\mathbb{P}^n}(\mathbb{k})$ is continuous with respect to the Euclidean topologies (Lemma 2.11). The restriction of this map to the interval $[0, 1] \subset \mathbb{R} \subset \mathbb{C}$ yields a map $[0, 1] \rightarrow \text{Bir}_{\mathbb{P}^n}(\mathbb{k})$, continuous with respect to the Euclidean topologies and sending 0 to f and 1 to g . \square

Theorem 3 is now proven, as a consequence of Proposition 3.11 and Corollaries 3.12 and 3.13.

UNIVERSITÄT BASEL, SPIEGELGASSE 1, CH-4051 BASEL, SWITZERLAND

E-mail: jeremy.blanc@unibas.ch

LAREMA, UNIVERSITÉ D'ANGERS, 2 BD LAVOISIER, 49045 ANGERS, FRANCE

E-mail: susanna.zimmermann@univ-angers.fr

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