A NOTE ON ASYMPTOTIC PSEUDO TRAJECTORIES FOR SET-VALUED DYNAMICS

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Abstract. The purpose of this note is to correct an inaccuracy in Section 4.1 of [2] which was pointed to us by Scholom Schechtman.

1. Introduction

With the definition of *Asymptotic Pseudo Trajectories* as given in section 4.1 of [2] page 432, the equivalence stated Theorem 4.1 is false for differential inclusions, although it holds for locally Lipchitz ordinary differential equations. We provide below the right definition (4.1) and a corrected version.

2. Section 4.1 of [2], corrected version

The translation flow Θ: $C^0(\mathbb{R}, \mathbb{R}^m) \times \mathbb{R} \to C^0(\mathbb{R}, \mathbb{R}^m)$ is the flow defined by

$$\Theta^t(\mathbf{x})(s) = \mathbf{x}(s+t)$$

A continuous function $\mathbf{z} : \mathbb{R}^+ \to \mathbb{R}^m$ is an asymptotic pseudotrajectory (APT) for Φ if

$$\lim_{t\to\infty} \mathbf{D}(\Theta^t(\mathbf{z}), S) = 0 \tag{4.1}$$

where $S = \bigcup_{x\in\mathbb{R}^m} S_x$ denotes the set of all solutions of (I)).

Alternatively, for all T

$$\lim_{t\to\infty} \inf_{\mathbf{x}\in S} \sup_{0\leq s\leq T} \|\mathbf{z}(t+s) - \mathbf{x}(s)\| = 0.$$

In other words, for each fixed T, the curve

$$[0,T] \to \mathbb{R}^m : s \to \mathbf{z}(t+s)$$

shadows some solution over the interval $[0,T]$ with arbitrary accuracy for sufficiently large t. Hence $\mathbf{z}$ has a forward trajectory under Θ attracted by S. As usual, one extends $\mathbf{z}$ to $\mathbb{R}$ by letting $\mathbf{z}(t) = \mathbf{z}(0)$ for $t < 0$.

The next result is a natural extension of Benaïm and Hirsch [1, Theorem 7.2]. Note nevertheless that in this ODE set-up an equivalent definition of APT is

$$\lim_{t\to\infty} \mathbf{D}(\Theta^t(\mathbf{z}), S_{\mathbf{z}(t)}) = 0$$

while it is strictly stronger in the current framework.

THEOREM 4.1. (characterization of APT) *Assume* $\mathbf{z}$ *is bounded. Then there is equivalence between the following statements:*
(i) $\mathbf{z}$ *is an APT for* Φ.
(ii) $\mathbf{z}$ *is uniformly continuous, and any limit point of* $\{\Theta^t(\mathbf{z})\}$ *is in* S.
In both cases the set $\{\Theta^t(\mathbf{z});\ t \geq 0\}$ *is relatively compact.*

Proof. By hypothesis, $K = \overline{\{\mathbf{z}(t);\ t \geq 0\}}$ is compact, so is $K_1 = \{x \in \mathbb{R}^m, d(x,K) \leq 1\}$.

For any $1 \geq \varepsilon > 0$, there exists $\eta > 0$ such that $\|z - x\| < \varepsilon/3$, for any $x \in K_1$, any $z \in \Phi_s(x)$, and any $|s| < \eta$, using property (d) of the dynamical system.

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$\mathbf{z}$ being an APT, there exists T such that $t > T$ implies

$$d(\mathbf{z}(t+s), \Phi_s(x)) < \frac{\varepsilon}{3} \quad \forall |s| < \eta;$$

for some $x \in K_1$. Hence for some $z \in \Phi_s(x)$

$$\|\mathbf{z}(t+s) - \mathbf{z}(t)\| \leq \|\mathbf{z}(t+s) - z\| + \|z - x\| + \|\mathbf{z}(t) - x\| \leq \varepsilon$$

and $\mathbf{z}$ is uniformly continuous. Clearly any limit point belongs to S by the condition (4.1) above.

Conversely, if $\mathbf{z}$ is uniformly continuous, then the family of functions $\{\Theta^t(\mathbf{z});\ t \geq T\}$ is equicontinuous and hence (K being compact) relatively compact by Ascoli's theorem. Since any limit point belongs to S, property (4.1) follows. □

References

[1] M. Benaïm and M.W. Hirsch, (1996) Asymptotic pseudotrajectories and chain recurrent flows, with applications, *J. Dynam. Differential Equations*, **8**, 141-176.

[2] M. Benaïm, J. Hofbauer and S. Sorin, (2005) Stochastic approximations and differential inclusions, *SIAM Journal on Control and Optimization*, **44**, 328-348.

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