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Selfextensional Logics in Abstract Algebraic Logic: a Brief Survey [1]

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1 Introduction

The name 'selfextensional logic' is due to Ryszard Wójcicki, who was the first to study the class of selfextensional logics in general in the papers [48, 49, 50] and the monographs [51] and [52]. Since 1996 the interest in selfextensional logics among researchers working in the field of Abstract Algebraic Logic (AAL) has been increasing, for the class of selfextensional logics cuts across the widely studied Leibniz hierarchy of logics which constitutes the core of the present day AAL. Therefore, the study of selfextensional logics can provide interesting new insights in this field.

A *selfextensional logic* S is any propositional logic with the following replacement property: if two formulas φ and ψ are interderivable (i.e. $\varphi \dashv_S \vdash \psi$), then for every formula δ and every propositional variable p, the formulas $\delta(p/\varphi)$ and $\delta(p/\psi)$ are also interderivable (i.e. $\delta(p/\varphi) \dashv_S \vdash \delta(p/\psi)$), where $\delta(p/\varphi)$ and $\delta(p/\psi)$ are the formulas obtained by substituting φ for p and ψ for p in δ respectively. In algebraic terms this means that the interderivability relation between formulas is a congruence relation of the formula algebra. Typical examples of selfextensional logics are classical propositional logic and intuitionistic propositional logic. These two logics have a stronger replacement property: for any set of formulas Γ, any formulas φ,

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ψ, δ and any variable p

if $\Gamma, \varphi \vdash_{\mathcal{S}} \psi$ and $\Gamma, \psi \vdash_{\mathcal{S}} \varphi$, then $\Gamma, \delta(p/\varphi) \vdash_{\mathcal{S}} \delta(p/\psi)$ and $\Gamma, \delta(p/\varphi) \vdash_{\mathcal{S}} \delta(p/\psi)$.

The logics with this stronger replacement property are called Fregean logics. Accordingly, selfextensional logics can also be called weakly Fregean logics, but we will use Wójcicki's terminology in this paper. For information on Fregean logics and the reasons why they are so named we refer the reader to [14, 17, 18, 39] and the survey [25]. Many selfextensional logics are not Fregean; one example is the deducibility relation defined by the Hilbert-style calculus whose axioms are the theorems of a normal modal logic L and modus ponens is taken as the sole inference rule. The logics defined in this way from the theorems of a modal logic are frequently called the local consequence relations of the modal logics, and we will use this term in this paper.

In AAL the concept of logic that has proven fruitful is the following. A *logic* (or deductive system) is a pair $\mathcal{S} = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}} \rangle$ where $\boldsymbol{Fm}$ is the algebra of the formulas of some set of connectives (or algebraic similarity type) and a denumerable set of variables $\mathcal{L}_{\mathcal{S}}$, and $\vdash_{\mathcal{S}}$ is a consequence relation on the universe Fm of $\boldsymbol{Fm}$, i.e. a relation between subsets of the set of formulas Fm and elements of Fm with the following three properties

1. if $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathcal{S}} \varphi$,
2. if $\Gamma \vdash_{\mathcal{S}} \varphi$ for all $\varphi \in \Delta$ and $\Delta \vdash_{\mathcal{S}} \psi$, then $\Gamma \vdash_{\mathcal{S}} \psi$,
3. if $\Gamma \subseteq \Delta$ and $\Gamma \vdash_{\mathcal{S}} \varphi$, then $\Delta \vdash_{\mathcal{S}} \varphi$,

that in addition is *substitution invariant*[2], that is, it satisfies that for every substitution σ (i.e. every endomorphism σ of $\boldsymbol{Fm}$)

4. if $\Gamma \vdash_{\mathcal{S}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathcal{S}} \sigma(\varphi)$.

A logic is *finitary* if the consequence relation $\vdash_{\mathcal{S}}$ is finitary, i.e. if for every Γ, φ

5. $\Gamma \vdash_{\mathcal{S}} \varphi$ iff there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\mathcal{S}} \varphi$.

[2]Frequently this condition is called the structurality condition, and a consequence relation that satisfies it is said to be structural. Due to the increasing interest in the substructural logics, we prefer to use the term 'substitution invariant' introduced by Pigozzi in order to avoid misleading mental associations.

This concept of logic stems from Tarski's notion of consequence operation given in [43]. The algebra $\boldsymbol{Fm}$ is called the *algebra of formulas* of type $\mathcal{L}_{\mathcal{S}}$, or the $\mathcal{L}_{\mathcal{S}}$-algebra of formulas.

We will say that a pair $\langle \Gamma, \varphi \rangle$, where Γ is a set of formulas and φ is a formula, is a *rule* of $\mathcal{S}$ if $\Gamma \vdash_{\mathcal{S}} \varphi$. Thus, if $\langle \Gamma, \varphi \rangle$ is a rule of $\mathcal{S}$, then for every substitution σ, $\langle \sigma[\Gamma], \sigma(\varphi) \rangle$ is also a rule of $\mathcal{S}$.

Notice that the notion of logic just introduced encompasses both logics defined by syntactic means and logics defined by semantic means, and it departs from the notion of logic common in some quarters like the modal logic field where a logic is usually taken to be merely a set of formulas containing a set of axioms and closed under certain inference rules ([2, 10, 37]).

Besides classical logic, intuitionistic logic and the local consequences of the normal modal logics, other examples of selfextensional logics recently studied[3] are Visser's logic (called Basic logic by Visser) [45, 9], the strict implication fragments of the local consequences of the normal modal logics, in particular some subintuitionistic logics, [9], positive modal logic, [20, 8], Belnap's four-valued logic, [22], and the system of relevance logic WR, [27].

Some examples of non-selfextensional logics are the global consequences of the normal modal logics (they are defined in the same way as the local consequences but now, in addition to modus ponens, the rule of necessity is also taken as an inference rule), the system R of relevance logic, the classical and the intutitionistic linear logics without exponentials, and Łukasiewicz infinite-valued logic.

2 Generalities on AAL

Abstract Algebraic Logic can be described by saying that it studies the process of algebraization of logics rather than the algebraization of the particular logics in which one is interested. It intends to find the right concepts and discover the theorems that best explain the connections between properties of the algebras (or other algebra-related structures) associated with particular logics and their metalogical properties. For an overview of AAL we refer the reader to [25], where an extensive list of references is also given; for a detailed presentation of its main core, [14] is the best source.

As we will see in this paper, the study of selfextensional logics brings in new perspectives for the development of the theory of the algebraization of

[3] The list is not exhaustive.

logics that AAL seeks.

One of the main goals in the AAL agenda is to find general and useful criteria to select in a canonical way and for any arbitrary logic $\mathcal{S}$ the class of algebras that best encodes or reflects the metalogical properties of $\mathcal{S}$. We shall start by describing the class of algebras that nowadays is taken to fulfill this role.

2.1 The class of algebras of a logic

The set of connectives of a logic $\mathcal{S}$ can be regarded as an algebraic similarity type. As usual in this context we adopt the convention of identifying formulas with terms.

In AAL some consensus has emerged on the canonical class of algebras that should be associated with a given logic. Different researchers have arrived at the same class despite starting from different perspectives. This gives some stability to the notion and provides some ground for the claim that we are on the right track. One example of this stability will be shown in a theorem below. The class of algebras of a logic can, therefore, be defined in several ways; the one we present here is the best suited for our purposes in this paper.

Let $\boldsymbol{A}$ be an algebra of type $\mathcal{L}_{\mathcal{S}}$. A set $F \subseteq A$ is an *$\mathcal{S}$-filter* if it is "closed under the rules of $\mathcal{S}$": formally speaking, if for every valuation v from the algebra of formulas into $\boldsymbol{A}$, whenever $\Gamma \vdash_{\mathcal{S}} \varphi$ and $v[\Gamma] \subseteq F$, $v(\varphi) \in F$. We denote the set of all $\mathcal{S}$-filters of $\boldsymbol{A}$ by $\mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}$. Notice that the $\mathcal{S}$-filters of the formula algebra are just the theories of $\mathcal{S}$, that is, the sets of formulas closed under the rules of $\mathcal{S}$. We denote the set of theories of $\mathcal{S}$ by $\mathrm{Th}(\mathcal{S})$.

A congruence θ of an algebra $\boldsymbol{A}$ is said to be *compatible* with a subset F of its universe if F is a union of equivalence classes, i.e. if $a\theta b$ and $a \in F$, then $b \in F$. There always exists the greatest congruence of $\boldsymbol{A}$ compatible with F; this is denoted by $\boldsymbol{\Omega}_{\boldsymbol{A}}F$ and it is called the *Leibniz congruence* of F. The operator that maps each subset of $\boldsymbol{A}$ to its Leibniz congruence is called the *Leibniz operator*.

Much of the work on AAL before 1996 centred on the behaviour of the Leibniz operator on $\mathcal{S}$-filters. A *local perspective* on $\mathcal{S}$-filters was taken and conditions such as monotonicity, continuity, commutation with arbitrary inverse homomorphisms, etc. of the Leibniz operator on the $\mathcal{S}$-filters of the algebras have been very useful in building the general theory of the algebraization of logics. But a *global perspective* on the family of $\mathcal{S}$-filters turns out to be even better suited for the development of this general theory.

This perspective in AAL was systematically explored for the first time in [23].

Let us associate with a given algebra $\boldsymbol{A}$ the greatest congruence which is compatible with all the $\mathcal{S}$-filters of $\boldsymbol{A}$; it is denoted by $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}\boldsymbol{A}$ and it is called the *Tarski congruence* of the pair $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\rangle$. It can be described as the intersection of the Leibniz congruences of the $\mathcal{S}$-filters of $\boldsymbol{A}$, i.e.

$$\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}\boldsymbol{A} = \bigcap_{F \in \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}} \boldsymbol{\Omega}_{\boldsymbol{A}}F.$$

The class of algebras which from the global perspective on $\mathcal{S}$-filters is associated canonically with a logic $\mathcal{S}$ can then be defined as follows

$$\mathsf{Alg}\mathcal{S} = \{\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}\boldsymbol{A} \text{ is the identity}\}.$$

This class of algebras is the class that is considered in [23] to be the canonical class of algebras of $\mathcal{S}$. It does not always coincide with the class of algebras which from the local perspective on $\mathcal{S}$-filters has been standardly associated with a logic $\mathcal{S}$. This last class of algebras is the class

$$\mathsf{Alg}^*\mathcal{S} = \{\boldsymbol{A} : (\exists F \in \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A})\ \boldsymbol{\Omega}_{\boldsymbol{A}}(F) \text{ is the identity}\}.$$

It is not difficult to see that $\mathsf{Alg}\mathcal{S}$ is the closure of $\mathsf{Alg}^*\mathcal{S}$ under subdirect products.

A class of algebras K is an *algebraic semantics*, or provides a complete algebraic semantics, for a logic $\mathcal{S}$ if there is a set of equations in one variable $E(p)$ (for many logics it is the singleton of the equation $p \approx 1$) such that, if we denote for any algebra $\boldsymbol{A}$ the set of solutions on $\boldsymbol{A}$ of the equations in $E(p)$ by $E(\boldsymbol{A})$, that is, if

$$E(\boldsymbol{A}) = \{a \in A : \delta^{\boldsymbol{A}}(a) = \varepsilon^{\boldsymbol{A}}(a), \forall \delta(p) \approx \varepsilon(p) \in E(p)\},$$

then

$$\Gamma \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad \text{for every} \boldsymbol{A} \in \mathsf{K} \text{ and every valuation } v \text{ on } \boldsymbol{A}, \tag{1}$$

$$\text{if } v[\Gamma] \subseteq E(\boldsymbol{A}), \text{ then } v(\varphi) \in E(\boldsymbol{A}),$$

Many logics have an algebraic semantics in this sense and in particular for many logics the classes of algebras $\mathsf{Alg}\mathcal{S}$ or $\mathsf{Alg}^*\mathcal{S}$ are algebraic semantics, but for other logics no algebraic semantics exists. Thus to obtain a uniform way of giving a completeness theorem for every logic $\mathcal{S}$ using the algebras in $\mathsf{Alg}\mathcal{S}$ or in $\mathsf{Alg}^*\mathcal{S}$, the consideration of the algebras is not enough. The problem can be overcome by adding more structure to the algebras. We will discuss the two approaches that are mainly considered in AAL: the addition to the algebras in $\mathsf{Alg}\mathcal{S}$ or $\mathsf{Alg}^*\mathcal{S}$ of a set of distinguished elements that basically plays the role of the set $E(\boldsymbol{A})$ but it is not necessarily a definable subset of the algebra, and the addition of a non-empty family of such sets. The first approach gives us completeness theorems closer in their form to the algebraic completeness theorems of type (1), when these exist, and it can be considered directly inspired by them, while the second is not based on these theorems.

2.2 Logical matrices and atlases

Before expounding the details of these two approaches for adding more structure to the algebras in the classes $\mathsf{Alg}\mathcal{S}$ and $\mathsf{Alg}^*\mathcal{S}$ of any given logic $\mathcal{S}$ in order to obtain (at least) a correct and complete semantics for $\mathcal{S}$, we should make two introductory remarks.

First, notice that the class $\mathsf{Alg}^*\mathcal{S}$ is obtained from pairs of the form $\langle \boldsymbol{A}, F\rangle$ where $\boldsymbol{A}$ is an algebra and F is one of its $\mathcal{S}$-filters. The objects of this form, namely an algebra and a subset of its domain, called the set of distinguished elements, are known as *logical matrices*. The local perspective on $\mathcal{S}$-filters leads immediately to the idea that the logical matrices are the natural candidates for providing any propositional logic with a complete semantics. This was the approach mainly used by the Polish logicians to attain this goal; the books [52] and [14] together constitute a compendium of this approach. The second book also expounds the developments of AAL that are grounded on the local perspective on $\mathcal{S}$-filters.

By contrast, the class of algebras $\mathsf{Alg}\mathcal{S}$ is obtained from pairs of the form $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\rangle$. From the global perspective on $\mathcal{S}$-filters these objects, which we call *basic full models* of $\mathcal{S}$, are essentially the natural candidates for a complete semantics for $\mathcal{S}$. This is one of the main ideas of [23]. For reasons that will be explained later it is useful to move to structures of the form $\langle \boldsymbol{A}, \mathcal{B}\rangle$, where $\boldsymbol{A}$ is an algebra and $\mathcal{B}$ a family of $\mathcal{S}$-filters closed under intersections of arbitrary subfamilies. These structures are the *abstract logics* of Brown and Suzko [7], and they are used in [23] to develop a general algebraic semantics for propositional logics and to obtain some results on selfextensional logics. Sometimes it is advisable to move to an even more general kind of structures of this form where $\mathcal{B}$ is just a family of $\mathcal{S}$-filters.

They are called *generalized matrices* by R. Wójcicki in [47] and *atlases* by J.M. Dunn and G.M. Hardegree in [21]. We will use the latter terminology in this paper. Notice that one can associate an abstract logic with every atlas $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B} \rangle$ by closing $\mathcal{B} \cup \{A\}$ under intersections of arbitrary non-empty families.

Before 1996 the approach of considering atlases or abstract logics to obtain correct and complete semantics for logics was less frequently and sytematically explored than the logical matrix approach. Even so several authors, coming from different perspectives, did consider abstract logics either in the form defined above or in the equivalent forms of a closure operation or of a consequence operation on the domain of an algebra (or even simply on a set without any algebraic structure, in some of the most abstract approaches) in their logical studies; in addition to those mentionad in the previous paragraph we should cite (the list is not exhaustive) Beziau, Cleave, Koslow, Magari, Mangani, Martin and Pollard. Clearly, from the purely mathematical point of view, atlases encompass both logical matrices (identify a matrix with the atlas with the same algebra and the singleton of the set of distinguished elements of the matrix) and basic full models.

One of the advantages of abstract logics and atlases over logical matrices is that abstract logics, and therefore atlases, are good structures for modelling metalogical properties. The reason is that an abstract logic can be described by considering the dual closure operator (or the dual consequence operation) instead of the closure system. Thus the properties of the consequence relation of a logic can be translated into properties of abstract logics. We will come to this in detail in Section 5.

In the rest of this subsection we introduce the basic concepts of both the semantics of logical matrices and the semantics of atlases.

A logical matrix $\langle \boldsymbol{A}, F \rangle$ is a *model* of a logic $\mathcal{S}$ when $\boldsymbol{A}$ is an $\mathcal{L}_{\mathcal{S}}$-algebra and F is an $\mathcal{S}$-filter; that is, if whenever $\Gamma \vdash_{\mathcal{S}} \varphi$, then for every valuation v on $\boldsymbol{A}$ such that $v[\Gamma] \subseteq F$, $v(\varphi) \in F$. A logical matrix $\langle \boldsymbol{A}, F \rangle$ is said to be *reduced* if $\boldsymbol{\Omega}_{\boldsymbol{A}}(F)$ is the identity relation on A.

Every logic $\mathcal{S}$ is complete relative to the class of its logical matrix models and with respect to the class of its reduced logical matrix models. That is, if $\mathsf{M}(\mathcal{S})$ is the class of the logical matrices which are a model of $\mathcal{S}$, then

$$\Gamma \vdash_{\mathcal{S}} \varphi \quad \text{iff} \quad \forall \langle \boldsymbol{A}, F \rangle \in \mathsf{M}(\mathcal{S})\, \forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A}),\ \text{if } v[\Gamma] \subseteq F, \text{ then } v(\varphi) \in F,$$

where $\mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ denotes the set of valuations on $\boldsymbol{A}$, namely the set of homomorphisms from $\boldsymbol{Fm}$ into $\boldsymbol{A}$, and similarly we have a completeness theorem when instead of $\mathsf{M}(\mathcal{S})$ we take the class of reduced logical matrix models of $\mathcal{S}$ in the statement displayed above.

Let $\mathcal{S}$ be a logic. An atlas $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B}\rangle$ of type $\mathcal{L}_{\mathcal{S}}$ is a *model* of $\mathcal{S}$ if $\mathcal{B} \subseteq \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}$; that is, if for every $F \in \mathcal{B}$, the logical matrix $\langle \boldsymbol{A}, F\rangle$ is a model of $\mathcal{S}$. Since the intersection of any family of $\mathcal{S}$-filters is an $\mathcal{S}$-filter, an atlas is a model of a logic iff its associated abstract logic is also a model.

The *Tarski congruence* of an atlas $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B}\rangle$ is the greatest congruence compatible with all the elements of $\mathcal{B}$; it will be denoted by $\widetilde{\Omega}\mathbb{A}$. Thus,

$$\widetilde{\Omega}\mathbb{A} = \bigcap_{F\in\mathcal{B}} \Omega_{\boldsymbol{A}}F.$$

An atlas $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B}\rangle$ is *reduced* if its Tarski congruence is the identity on A. If a basic full model $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\rangle$ is reduced (as an atlas) we will say that it is a *reduced basic full model* of $\mathcal{S}$. Given an atlas $\mathbb{A}$, the quotient by its Tarski congruence, which is defined in the natural way[4], is a reduced atlas and it is called the *reduction* of $\mathbb{A}$; it will be denoted by $\mathbb{A}^*$.

An atlas $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B}\rangle$ of type $\mathcal{L}$ induces a *substitution invariant consequence relation* on the $\mathcal{L}$-algebra of formulas (i.e. a logic), the *consequence of* $\mathbb{A}$, denoted $\vdash_{\mathbb{A}}$, and defined by

$$\Gamma \vdash_{\mathbb{A}} \varphi \quad \text{iff} \quad \forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}), \forall F \in \mathcal{B} \text{ if } v[\Gamma] \subseteq F, \text{ then } v(\varphi) \in F.$$

Similarly a class of atlases K induces a substitution invariant consequence relation on the $\mathcal{L}$-algebra of formulas, the *consequence of* K, denoted $\vdash_{\mathsf{K}}$, and defined by

$$\Gamma \vdash_{\mathsf{K}} \varphi \quad \text{iff} \quad \forall \mathbb{A} \in \mathsf{K} \;\; \Gamma \vdash_{\mathbb{A}} \varphi.$$

The consequence relation induced by an atlas and the consequence relation induced by the associated abstract logic are the same. Thus from the logical point of view an atlas and its associated abstract logic behave in a similar way. Therefore, for many purposes to consider a semantics of atlases or a semantics of abstract logics makes no essential difference.

The logic induced by a class of atlases and the logic induced by the class of the reductions of its members are the same; in particular the logic induced by an atlas and the logic induced by its reduction coincide. Since reduced atlases encode the necessary logical information and their algebras do not have logically redundant elements, the reduced atlases and the reduced

[4]It is defined as the atlas whose algebra is $\mathcal{A}/\widetilde{\;\;}\mathbb{A}$ and whose family of sets is the family of the sets of the form $\{a/\widetilde{\;\;}\mathbb{A} : a \in X\}$ for an $X \in \mathcal{B}$.

abstract logics are the objects that play a privileged role in the global perspective on $\mathcal{S}$-filters on the semantics of propositional logics.

Every logic $\mathcal{S}$ is complete relative to the class of its atlas models and with respect to the class of its reduced atlas models. Moreover, every logic is complete relative to the class of its basic full models and with respect to the class of its reduced basic full models. That is, if K is any of the classes of atlases just mentioned, then $\vdash_{\mathsf{K}} = \vdash_{\mathcal{S}}$.

The connection between the semantics for a logic $\mathcal{S}$ given so far with the classes of algebras $\mathsf{Alg}\mathcal{S}$ and $\mathsf{Alg}^*\mathcal{S}$ is the following: on the one hand $\mathsf{Alg}\mathcal{S}$ is both the class of the algebras of the reduced basic full models of $\mathcal{S}$ and the class of algebras of the reduced atlas models of $\mathcal{S}$, and on the other hand $\mathsf{Alg}^*\mathcal{S}$ is the class of the algebras of the reduced logical matrix models of $\mathcal{S}$.

2.3 The Leibniz hierarchy

The local perspective on the logical filters taken by many AAL works has been very fruitful. It has enabled AAL to arrived at what is known as the *Leibniz hierarchy of logics*. This hierarchy maps the landscape of logics according to the behaviour of the Leibniz operator on the $\mathcal{S}$-filters of the algebras in $\mathsf{Alg}\mathcal{S}$. First of all it divides the landscape of logics into two disjoint classes: protoalgebraic and non-protoalgebraic logics. Then it divides the class of protoalgebraic logics into several subclasses. The main ones that have proven useful to isolate and study are the regularly algebraizable logics, the algebraizable logics, the weakly algebraizable logics and the equivalential logics. We will describe them briefly in this subsection.

A logic $\mathcal{S}$ is said to be *protoalgebraic* if for every $\mathcal{L}_{\mathcal{S}}$-algebra $\boldsymbol{A}$ the Leibniz operator is $\subseteq$-monotone on the $\mathcal{S}$-filters of $\boldsymbol{A}$. Protoalgebraic logics were introduced by Blok and Pigozzi in [3], and in a different but (essentially) equivalent way by Czelakowski in [12]. Protoalgebraic logics can be characterized as the logics with a generalized implication connective. They are the logics $\mathcal{S}$ with a set of formulas in two variables $(p \Rightarrow q)$ that satisfies the generalized modus ponens:

$$p, (p \Rightarrow q) \vdash_{\mathcal{S}} q,$$

and the generalized identity:

$$\vdash_{\mathcal{S}} \varphi(p,p), \text{ for every formula } \varphi(p,q) \in (p \Rightarrow q).$$

A set of formulas with these two properties is called a *set of implication formulas* for $\mathcal{S}$.

For any protoalgebraic logic $\mathcal{S}$ the classes of algebras $\mathsf{Alg}\mathcal{S}$ and $\mathsf{Alg}^*\mathcal{S}$ coincide; thus the global and the local perspectives on $\mathcal{S}$-filters coincide as to the class of algebras they associate with a protoalgebraic logic. Moreover the semantics of logical matrices works mathematically very well for protoalgebraic logics in the sense that many of the results of universal algebra on varieties and quasivarieties generalize to results for the classes of logical matrices of these logics.

A logic $\mathcal{S}$ is *equivalential* if it is protoalgebraic and the Leibniz operator commutes with inverse homomorphisms between $\mathcal{L}_\mathcal{S}$-algebras, which means that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are $\mathcal{L}_\mathcal{S}$-algebras, then for every homomorphism $h : \boldsymbol{A} \to \boldsymbol{B}$ and every $\mathcal{S}$-filter F of $\boldsymbol{B}$, $\langle a, b\rangle \in \boldsymbol{\Omega}_{\boldsymbol{A}}(h^{-1}[F])$ iff $\langle h(a), h(b)\rangle \in \boldsymbol{\Omega}_{\boldsymbol{B}}(F)$, for any $a, b \in A$. The original definition of equivalential logic given by Prucnal and Wroński in [40] is syntactical. A logic $\mathcal{S}$ is equivalential iff there is a set of formulas in two variables $(p \Leftrightarrow q)$ such that it satisfies the generalized modus ponens: $p, (p \Leftrightarrow q) \vdash_\mathcal{S} q$, and the generalized congruence rules, namely, for every formula $\varphi(p, q) \in (p \Leftrightarrow q)$

1. $\vdash_\mathcal{S} \varphi(p,p)$
2. $(p \Leftrightarrow q) \vdash_\mathcal{S} \varphi(q,p)$
3. $(p \Leftrightarrow q) \cup (q \Leftrightarrow r) \vdash_\mathcal{S} \varphi(p,r)$
4. $(p_0 \Leftrightarrow q_0)\cup\ldots\cup(p_{n-1} \Leftrightarrow q_{n-i}) \vdash_\mathcal{S} \varphi(\star(p_0,\ldots,p_{n-1}), \star(q_0,\ldots,q_{n-1}))$, for every connective $\star$ of $\mathcal{L}_\mathcal{S}$, where n is its arity.

A set of formulas with these properties is called a *set of equivalence formulas* for $\mathcal{S}$. If $\mathcal{S}$ has a finite set of equivalence formulas it is said to be *finitely equivalential.* After the definition was given by Prucnal and Wroński in 1974, Czelakowski was the first to study equivalential logics in depth in [11]; to some extent this 1981 publication can be considered one of the starting points of AAL.

A logic $\mathcal{S}$ is *weakly algebraizable* iff it is protoalgebraic and the Leibniz operator is injective on the $\mathcal{S}$-filters of every $\mathcal{L}_\mathcal{S}$-algebra. Weakly algebraizable logics are studied in [16]. If in addition $\mathcal{S}$ is equivalential it is said to be *algebraizable.* The notion of algebraizable logic for finitary and finitely equivalential logics is due to Blok and Pigozzi [4], who presented it in 1989, and it was extended to arbitrary equivalential logics by Herrman in [31, 32] and by Czelakowski in [13, 14].

Algebraizable logics can be characterized syntactically in a similar way to equivalential logics, but with the extra condition that, in addition to the set of equivalence formulas $(p \Leftrightarrow q)$, there is a set of pairs of formulas, or equations, in one variable $E(r) = \{\langle \delta_i(r), \varepsilon_i(r) \rangle : i \in I\}$, called a *set of defining equations*, such that for every $i \in I$ and every $\varphi(p,q) \in (p \Leftrightarrow q)$,

5. $p \vdash_{\mathcal{S}} \varphi(\delta_i(p), \varepsilon_i(p))$ and $\{(\delta_i(p) \Leftrightarrow \varepsilon_i(p)) : i \in I\} \vdash_{\mathcal{S}} p$.

Finally, a logic $\mathcal{S}$ is *regularly algebraizable* if it is algebraizable and

6. $p, q \vdash_{\mathcal{S}} \varphi(p,q)$, for every $\varphi \in (p \Leftrightarrow q)$.

Every algebraizable logic $\mathcal{S}$ is complete with respect to its class of algebras $\mathsf{Alg}\mathcal{S}$, in the sense mentioned in Subsection 2.2, using any of the sets $E(p)$ of defining equations for $\mathcal{S}$, that is, if $\mathcal{S}$ is algebraizable and $E(p)$ is one of these sets, then

$$\Gamma \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad \text{for every } \boldsymbol{A} \in \mathsf{Alg}\mathcal{S} \text{ and every valuation } v \text{ on } \boldsymbol{A},$$

$$\text{if } v[\Gamma] \subseteq E(\boldsymbol{A}) \text{ then } v(\varphi) \in E(\boldsymbol{A}).$$

Moreover, they also have a kind of *inverse completeness theorem.* Let Π be any set of equations and let $\varphi \approx \psi$ be any equation. Define the consequence relation between sets of equations and equations modulo $\mathsf{Alg}\mathcal{S}$ as follows:

$$\Pi \models_{\mathsf{Alg}\mathcal{S}} \varphi \approx \psi \quad \text{iff} \quad \text{for every } \boldsymbol{A} \in \mathsf{Alg}\mathcal{S} \text{ and every valuation } v \text{ on } \boldsymbol{A},$$

$$\text{if } \boldsymbol{A} \models \Pi[v], \text{ then } \boldsymbol{A} \models \varphi \approx \psi[v].$$

If $\mathcal{S}$ is algebraizable and $(p \Leftrightarrow q)$ is any one of its sets of equivalence formulas, then

$$\Pi \models_{\mathsf{Alg}\mathcal{S}} \varphi \approx \psi \quad \text{iff} \quad \bigcup\{(\delta \Leftrightarrow \varepsilon) : \delta \approx \varepsilon \in \Pi\} \vdash_{\mathcal{S}} \gamma(\varphi,\psi), \forall \gamma \in (p \Leftrightarrow q)$$

Moreover, if $E(r)$ is any one of its sets of defining equations, for every $\delta \approx \varepsilon \in E(r)$ and every $\gamma \in (p \Leftrightarrow q)$, then

$$p \approx q \models_{\mathsf{Alg}\mathcal{S}} \delta(\gamma(p,q)) \approx \varepsilon(\gamma(p,q)),$$

and

$$\{\delta(\gamma(p,q)) \approx \varepsilon(\gamma(p,q)) : \delta \approx \varepsilon \in E(r), \gamma \in (p \Leftrightarrow q)\} \models_{\mathsf{Alg}\mathcal{S}} p \approx q.$$

In fact the existence for a logic $\mathcal{S}$ of a set of formulas $(p \Leftrightarrow q)$ in two variables and a set of equations $E(r)$ in one variable with the completeness and inverse completeness theorems above and with the two last properties characterize the algebraizable logics.

If an algebraizable logic $\mathcal{S}$ is finitary and one of its sets of equivalence formulas is finite, then its class of algebras $\mathsf{Alg}\mathcal{S}$ is a quasivariety. The algebraizable logics with these two properties are the logics called algebraizable by Blok and Pigozzi in [4]. For these logics $\mathsf{Alg}\mathcal{S}$ is known as its *equivalent quasivariety semantics.*

We now give some examples of logics in each class of the Leibniz hierarchy. There are protoalgebraic logics which are not equivalential, such as the local consequence of the classical modal logic E. There are equivalential logics which are not algebraizable such as the local consequences of the normal modal logics; some, such as the local consequence of the modal logic K, are not finitely equivalential, but others, such as the local consequence of $S4$, are finitely equivalential. There are algebraizable logics such as classical linear logic without the exponentials or the relevance logic R, which are algebraizable but not regularly algebraizable. Finally, there are regularly algebraizable logics such as classical logic, intuitionistic logic and the global consequences of the normal modal logics.

3 Selfextensional logics

The class of selfextensional logics is orthogonal to the Leibniz hierarchy in the sense that there are interesting selfextensional logics in each hierarchy's class. Some examples of selfextensional but non-protoalgebraic logics are: positive modal logic ([34]), some subintuitionistic logics ([9]) and the system of relevance logic WR ([27]). Among the selfextensional and protoalgebraic logics we find protoalgebraic but not equivalential selfextensional logics, such as the logic $\mathcal{G}_1$ studied in [6] and defined by a restricted version of the deduction theorem. Also we find equivalential but not algebraizable selfextensional logics, such as the local consequences of the normal modal logics, and algebraizable selfextensional logics, such as classical logic and intuitionistic logic.

My view is that the study of selfextensional logics deserves to be developed in full for at least the following three reasons:

1. The most developed part of the general theory of the algebraization of logics is the theory of the protoalgebraic logics, but a truly general theory

should also encompass the non-protoalgebraic logics. Since several of the selfextensional logics are non-protoalgebraic, the study of selfextensional logics provides insights for the development of the desired theory. The monograph [23] is one of the first attempts to build such theory and in that work the results on selfextensional logics are central.

2. The study of selfextensional logics introduces a point of view into the study of the algebraization of logics which differs from the one taken by the more standard studies in AAL. Thus it can highligth some hitherto unnoticed phenomena even in the protoalgebraic logics family whose study can help to articulate the general theory.

3. Selfextensional logics can be used in the study of non-selfextensional logics. An example of this use will be given in the last part of this survey where we address one of the important open questions in AAL: Why does the class of algebras $\mathsf{Alg}\mathcal{S}$ of many algebraizable logics in the sense of Blok and Pigozzi turn out to be a variety while, in general, according to Blok and Pigozzi's theory of the algebraizable logics, one can only say that it is a quasivariety? Some results on selfextensional logics explain why for some algebraizable and selfextensional logics $\mathcal{S}$, $\mathsf{Alg}\mathcal{S}$ is a variety. By associating a selfextensional companion with the same class of algebras to some algebraizable but non-selfextensional logics we can also explain why for some algebraizable but non-selfextensional logics $\mathcal{S}$, $\mathsf{Alg}\mathcal{S}$ is a variety.

3.1 Selfextensional logics and referential semantics

R. Wójcicki characterized the selfextensional logics as the logics which admit a local referential semantics ([48, 50, 51, 52]). Referential semantics is an abstraction of the different Kripke style semantics encountered in the literature and we will expound in the subsequent subsection how it is connected with the semantics of atlases for selfextensional logics. Here we expound the tools of referential semantics and Wójcicki's characterization.

Let $\mathcal{L}$ be an algebraic similarity type. A *$\mathcal{L}$-referential algebra* is a structure $\mathcal{F} = \langle W, \mathcal{A}\rangle$ where W is a non-empty set, whose elements are called points, reference points, indices or states, and $\mathcal{A}$ is an algebra of type $\mathcal{L}$ whose universe is a set of subsets of W that we denote by A. We say that $\langle W, \mathcal{A}\rangle$ is a referential algebra on W. Referential algebras are an abstraction of the relational general frames of modal logic and similar models for other kinds of logics, not necessarily distributive, like orthologics. The labels 'referential algebra' and 'referential semantics' are due to the fact that each formula φ is interpreted in a referential algebra $\langle W, \mathcal{A}\rangle$ as an element

X of A, which can be considered as its intension, and that then the formula obtains at each point $w \in W$ a reference, which is its truth value at the point: `true`, if $w \in X$ and `false`, if $w \notin X$. The elements of W can therefore be called reference points because it is on them that the interpreted formulas obtain their reference. Precisely speaking, given a $\mathcal{L}$-referential algebra $\mathcal{F} = \langle W, \mathcal{A} \rangle$, an *interpretation* is an homomorphism from the algebra of $\mathcal{L}$-formulas to $\mathcal{A}$, and given a point $w \in W$, a formula φ is *true at* w under the interpretation h if $w \in h(\varphi)$; otherwise it is *false at* w.

An $\mathcal{L}$-referential algebra $\mathcal{F} = \langle W, \mathcal{A} \rangle$ induces two very natural substitution invariant consequence relations, $\vdash^l_{\mathcal{F}}$ and $\vdash^g_{\mathcal{F}}$, on the $\mathcal{L}$-algebra of formulas, called respectively the *local* and the *global* consequence induced by $\mathcal{F}$; they are defined for every set of formulas Γ and every formula φ by

$$\Gamma \vdash^l_{\mathcal{F}} \varphi \text{ iff } \forall h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A}), \bigcap_{\psi \in \Gamma} h(\psi) \subseteq h(\varphi)$$

and

$$\Gamma \vdash^g_{\mathcal{F}} \varphi \text{ iff } \forall h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A}), \text{ if } \bigcap_{\psi \in \Gamma} h(\psi) = W, \text{ then } h(\varphi) = W.$$

Thus $\Gamma \vdash^l_{\mathcal{F}} \varphi$ iff in any interpretation φ is true at a point whenever all the formulas in Γ are true at that point, and $\Gamma \vdash^g_{\mathcal{F}} \varphi$ iff whenever all the formulas in Γ are true at every point, φ is also true at every point. Similarly, a class F of $\mathcal{L}$-referential algebras induces two substitution invariant consequence relations on the $\mathcal{L}$-algebra of formulas, the *local consequence*, denoted by $\vdash^l_{\mathsf{F}}$, and the *global consequence*, denoted by $\vdash^g_{\mathsf{F}}$. They are defined as follows:

$$\Gamma \vdash^l_{\mathsf{F}} \varphi \text{ iff } \forall \mathcal{F} \in \mathsf{F}\ \Gamma \vdash^l_{\mathcal{F}} \varphi \quad \text{ and } \quad \Gamma \vdash^g_{\mathsf{F}} \varphi \text{ iff } \forall \mathcal{F} \in \mathsf{F}\ \Gamma \vdash^g_{\mathcal{F}} \varphi,$$

for every set of formulas Γ and every formula φ.

We say that an $\mathcal{L}$-referential algebra $\mathcal{F} = \langle W, \mathcal{A} \rangle$ is a *local model* of a logic $\mathcal{S}$ if $\vdash_{\mathcal{S}} \subseteq \vdash^l_{\mathcal{F}}$, i.e., if whenever $\Gamma \vdash_{\mathcal{S}} \varphi$ and $h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A})$, then $\bigcap_{\psi \in \Gamma} h(\psi) \subseteq h(\varphi)$. Also, we say that $\mathcal{F} = \langle W, \mathcal{A} \rangle$ is a *global model* of a logic $\mathcal{S}$ if $\vdash_{\mathcal{S}} \subseteq \vdash^g_{\mathcal{F}}$, i.e. if whenever $\Gamma \vdash_{\mathcal{S}} \varphi$, $h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A})$ and $\bigcap_{\psi \in \Gamma} h(\psi) = W$, then $h(\varphi) = W$.

A class of $\mathcal{L}$-referential algebras $\mathcal{F}$ is a *complete local referential semantics* for a logic $\mathcal{S}$ if $\vdash_{\mathcal{S}} = \vdash^l_{\mathcal{F}}$.

Theorem 1 (Wójcicki) *A logic $\mathcal{S} = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}} \rangle$ of type $\mathcal{L}$ is selfextensional iff it is the local consequence relation of some class of $\mathcal{L}$-referential algebras, i.e. iff it admits a local referential semantics.*

3.2 Duality between referential algebras and atlases

For selfextensional logics the semantics of referential algebras and the semantics given by atlases are essentially dual to each other. The details of this duality were worked out in the M.A. thesis of Alessandra Palmigiano and also appear in [35]. We present its main traits here.

We will consider two categories which are dually equivalent, one whose objects are referential algebras and the other whose objects are atlases. We assume throughout this subsection that the atlases and the referential algebras are all of the same algebraic similarity type.

Frege-reduced atlases and atlas morphisms

An atlas $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B} \rangle$ is said to have *the congruence property* if the relation $\Lambda_{\boldsymbol{A}}(\mathcal{B})$ that relates two elements if and only if they belong to the same sets in $\mathcal{B}$, formally

$$\langle a, b \rangle \in \Lambda_{\boldsymbol{A}}(\mathcal{B}) \quad \text{iff} \quad (\forall X \in \mathcal{B})(a \in X \Leftrightarrow b \in X),$$

is a congruence relation of $\boldsymbol{A}$. If this is the case it coincides with $\widetilde{\Omega}_{\boldsymbol{A}}(\mathcal{B})$. We call the relation $\Lambda_{\boldsymbol{A}}(\mathcal{B})$ the *Frege relation* of the atlas $\langle \boldsymbol{A}, \mathcal{B} \rangle$. If $\Lambda_{\boldsymbol{A}}(\mathcal{B})$ is the identity relation on A we say that the atlas $\mathbb{A}$ is *Frege-reduced*. It is easy to see that an atlas $\mathbb{A}$ has the congruence property iff its reduction $\mathbb{A}^*$ is Frege-reduced. Thus the reduced atlases with the congruence property are exactly the Frege-reduced atlases. Notice that a logic $\mathcal{S}$ is selfextensional iff the atlas $\langle \boldsymbol{Fm}, \mathrm{Th}\mathcal{S} \rangle$ has the congruence property.

Let $\mathbb{A}_1 = \langle \boldsymbol{A}_1, \mathcal{B}_1 \rangle$ and $\mathbb{A}_2 = \langle \boldsymbol{A}_2, \mathcal{B}_2 \rangle$ be two $\mathcal{L}$-atlases. A map $h : A_1 \to A_2$ is an *atlas morphism* from $\mathbb{A}_1$ into $\mathbb{A}_2$ if

1. h is an (algebra) homomorphism from $\boldsymbol{A}_1$ into $\boldsymbol{A}_2$;
2. $\{h^{-1}[Y] : Y \in \mathcal{B}_2\} \subseteq \mathcal{B}_1$.

An atlas morphism h from $\mathbb{A}_1 = \langle \boldsymbol{A}_1, \mathcal{B}_1 \rangle$ into $\mathbb{A}_2 = \langle \boldsymbol{A}_2, \mathcal{B}_2 \rangle$ is said to be *strict* if in addition it satisfies

3. $\mathcal{B}_1 = \{h^{-1}[Y] : Y \in \mathcal{B}_2\}$.

A typical example of strict atlas morphism is the projection of an atlas onto its reduction.

Proposition 2 *Let h be an atlas morphism from $\mathbb{A}_1$ into $\mathbb{A}_2$.*

1. *If h is a strict and $\mathbb{A}_1$ is Frege-reduced, then h is injective.*

2. *if h is strict, then $\vdash_{\mathbb{A}_2} \subseteq \vdash_{\mathbb{A}_1}$;*

3. *if h is onto, then $\vdash_{\mathbb{A}_1} \subseteq \vdash_{\mathbb{A}_2}$;*

4. *if h is strict and onto, then $\mathbb{A}_1$ is a model of a logic $\mathcal{S}$ iff $\mathbb{A}_2$ is a model of $\mathcal{S}$.*

Reduced referential algebras and referential algebra morphisms

Given a referential algebra $\mathcal{F} = \langle W, \mathcal{A}\rangle$ we can consider the relation $\theta(\mathcal{F})$ defined by

$$\langle u, v\rangle \in \theta(\mathcal{F}) \quad \text{iff} \quad (\forall X \in \mathcal{A})(u \in X \Leftrightarrow v \in X)$$

This relation is clearly an equivalence relation. If it is the identity, that is if every pair of different points are separated by some set in the universe of $\mathcal{A}$, the referential algebra is said to be *reduced*, *differentiated*, or simply T_0.

Each referential algebra $\mathcal{F} = \langle W, \mathcal{A}\rangle$ has its reduction $\mathcal{F}^*$ which is defined by identifying the elements of W that belong to the same elements of $\mathcal{A}$. It is the referential algebra $\mathcal{F}^* = \langle W/\theta(\mathcal{F}), \mathcal{A}/\theta(\mathcal{F})\rangle$ where the universe of the algebra $\mathcal{A}/\theta(\mathcal{F})$ is the set $\{\pi[X] : X \in \mathcal{A}\}$, where $\pi[X] = \{w/\theta(\mathcal{F}) : w \in X\}$, and the operations are the naturally induced ones by the operations of $\mathcal{A}$. Formally speaking, if $f \in \mathcal{L}$ is n-ary,

$$f^{\mathcal{A}/\theta(\mathcal{F})}(\pi[X_1], \dots \pi[X_n]) = \pi[f^{\mathcal{A}}(X_1, \dots, X_n)],$$

for every $X_1, \dots, X_n \in \mathcal{A}$. This definition is sound because from the definition of $\theta(\mathcal{F})$ it easily follows that for every $X, Y \in \mathcal{A}$, $\pi[X] = \pi[Y]$ iff $X = Y$.

It is easy to see that:

1. The local consequence of $\mathcal{F}$ and the local consequence of its reduction $\mathcal{F}^*$ are the same.

2. $\theta(\mathcal{F}^*)$ is the identity, that is $\mathcal{F}^*$ is reduced.

From the logical point of view, it is natural then to restrict ourselves to referential algebras with the last property.

Let $\mathcal{F}_1 = \langle W_1, \mathcal{A}_1\rangle$ and $\mathcal{F}_2 = \langle W_1, \mathcal{A}_1\rangle$ be two $\mathcal{L}$-referential algebras. A map $f : W_1 \to W_2$ is a *morphism (of referential algebras)* from $\mathcal{F}_1 = \langle W_1, \mathcal{A}_1\rangle$ into $\mathcal{F}_2 = \langle W_1, \mathcal{A}_1\rangle$ if

1. $f^{-1}[Y] \in A_1$ for every $Y \in A_2$;
2. the map $f^{-1} : A_2 \rightarrow A_1$ is an homomorphism from $\mathcal{A}_2$ into $\mathcal{A}_1$.

A morphism f from $\mathcal{F}_1 = \langle W_1, \mathcal{A}_1 \rangle$ into $\mathcal{F}_2 = \langle W_1, \mathcal{A}_1 \rangle$ is said to be *strict* if in addition it satisfies

3. $A_1 = \{f^{-1}[Y] : Y \in A_2\}$.

A typical example of a strict morphism of referential algebras is the projection π from a referential algebra $\mathcal{F} = \langle W, \mathcal{A} \rangle$ onto its reduction $\mathcal{F}^*$.

Proposition 3 *Let f be a morphism from $\mathcal{F}_1 = \langle W_1, \mathcal{A}_1 \rangle$ into $\mathcal{F}_2 = \langle W_1, \mathcal{A}_1 \rangle$.*

1. *If f is a strict and $\mathcal{F}_1$ is differentiated, then f is injective.*
2. *if f is strict, then $\vdash_{\mathcal{F}_2} \subseteq \vdash_{\mathcal{F}_1}$;*
3. *if f is onto, then $\vdash_{\mathcal{F}_1} \subseteq \vdash_{\mathcal{F}_2}$;*
4. *if f is strict and onto, then $\mathcal{F}_1$ is a model of a logic $\mathcal{S}$ iff $\mathcal{F}_2$ is a model of $\mathcal{S}$.*

The functor $(.)^+$ from referential algebras to atlases

Let $\mathcal{F} = \langle W, \mathcal{A} \rangle$ be a referential algebra. The dual atlas of $\mathcal{F}$ is defined as the pair $\mathcal{F}^+ = \langle \mathcal{A}, W^+ \rangle$ with

$$W^+ = \{\varepsilon(v) : v \in W\}$$

where ε is the map from W to the powerset of the universe A of $\mathcal{A}$ defined by

$$\varepsilon(v) = \{Y \in A : w \in Y\}$$

for every $v \in W$.

For every referential algebra $\mathcal{F}$, its dual $\mathcal{F}^+$ is a Frege-reduced atlas. Moreover the local consequence relation of $\mathcal{F}$ and the consequence relation of $\mathcal{F}^+$ are the same. Thus, for every referential algebra $\mathcal{F}$, $\mathcal{F}$ is a local model of a logic $\mathcal{S}$ iff its dual $\mathcal{F}^+$ is a model of $\mathcal{S}$. These facts show that to find the dual category of referential algebras given by the construction $(.)^+$ we have to restrict ourselves to Frege-reduced atlas.

The dual of a referential algebra morphism is defined as follows. Let f be a referential algebra morphism from $\mathcal{F}_1 = \langle W_1, \mathcal{A}_1 \rangle$ into $\mathcal{F}_2 = \langle W_2, \mathcal{A}_2 \rangle$. The dual of f is the function $f^+ : A_2 \to A_1$ defined by

$$f^+(Y) = f^{-1}[Y]$$

for every $Y \in A_2$.

Proposition 4 *Let f be a referential algebra morphism from $\mathcal{F}_1 = \langle W_1, \mathcal{A}_1 \rangle$ into $\mathcal{F}_2 = \langle W_2, \mathcal{A}_2 \rangle$. Then, f^+ is an atlas morphism from $(\mathcal{F}_2)^+$ into $(\mathcal{F}_1)^+$, which is strict if f is onto, and which is onto if f is strict.*

As a corollary we have:

Corollary 5 *Let $\mathcal{F} = \langle W, \mathcal{A} \rangle$ be a referential algebra and let $\mathcal{F}^*$ be its reduction, which is differentiated. Then $\mathcal{F}^+$ and $(\mathcal{F}^*)^+$ are isomorphic.*

The corollary shows that we can concentrate on reduced referential algebras. Thus the category of referential algebras we will consider has the reduced referential algebras as objects and the referential algebra morphisms between them as arrows. It is easy to check that it is indeed a category. We denote it by **RRA**.

The functor $(.)_+$ from Frege-reduced atlases to referential algebras

Let $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B} \rangle$ be a Frege-reduced atlas. We define the dual referential algebra of $\mathbb{A}$ as the structure $\mathbb{A}_+ = \langle \mathcal{B}, \overline{\boldsymbol{A}} \rangle$ that we describe below. Let us first define the map η from A into the power set of $\mathcal{B}$ by

$$\eta(a) = \{X \in \mathcal{B} : a \in X\}$$

for every $a \in A$. Then $\mathbb{A}_+$ is defined by defining the algebra $\overline{\boldsymbol{A}}$ as follows:

1. the universe of the algebra $\overline{\boldsymbol{A}}$ is the set $\overline{A} = \{\eta(a) : a \in A\}$
2. for every n-ary symbol $f \in \mathcal{L}$ we define the n-ary operation on $\overline{A}$ by declaring

   $$f^{\overline{\boldsymbol{A}}}(\eta(a_1), \ldots, \eta(a_n)) = \eta(f^{\boldsymbol{A}}(a_1, \ldots, a_n))$$

   for every $a_1, \ldots, a_n \in A$.

The fact that $\mathbb{A}$ is Frege-reduced guarantees that for every $f \in \mathcal{L}$ the operation $f^{\overline{\boldsymbol{A}}}$ is well defined. Moreover, the definition of the algebra $\overline{\boldsymbol{A}}$ guarantees that η is an homomorphism from $\boldsymbol{A}$ onto $\overline{\boldsymbol{A}}$.

If $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B} \rangle$ is a Frege-reduced atlas, its dual $\mathbb{A}_+$ is a reduced referential algebra. Moreover, the consequence relation of $\mathbb{A}$ and the local consequence relation of $\mathbb{A}_+$ are the same; therefore $\mathbb{A}$ is a model of a logic $\mathcal{S}$ iff $\mathbb{A}_+$ is a local model of $\mathcal{S}$. Finally, the map η is an isomorphism between $\boldsymbol{A}$ and $\overline{\boldsymbol{A}}$.

Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be Frege-reduced atlases and let h be an atlas morphism from $\mathbb{A}_1$ into $\mathbb{A}_2$. The dual function $h_+ : \mathcal{B}_2 \to \mathcal{B}_1$ is defined by

$$h_+(Y) = h^{-1}[Y]$$

for every $Y \in \mathcal{B}_2$.

Proposition 6 *Let h be an atlas morphism from an atlas $\mathbb{A}_1$ into an atlas $\mathbb{A}_2$, both Frege-reduced. Then, h_+ is a referential algebra morphism from $(\mathbb{A}_2)_+$ into $(\mathbb{A}_1)_+$ which is strict if h is onto and onto if h is strict.*

The duality theorem

Let **FRAt** be the category whose objects are the Frege-reduced atlases of type $\mathcal{L}$ and whose arrows are the atlas morphisms between them, and let **RRA** be the category whose objects are the reduced referential algebras of type $\mathcal{L}$ and whose arrows are the referential algebra morphism between them. We consider the functors $(.)_+$ from **FRAt** into **RRA** and $(.)^+$ from **RRA** into **FRAt** defined above.

Theorem 7 (Palmigiano) *The functors $(.)_+$ and $(.)^+$ establish a dual equivalence between the category* **FRA** *and the category* **RRA**.

4 Fully selfextensional logics

For every selfextensional logic $\mathcal{S}$ it holds (by definition) that the Frege relation of the atlas $\langle \boldsymbol{Fm}, \mathrm{Th}(\mathcal{S}) \rangle$, namely $\Lambda_{\boldsymbol{Fm}}(\mathrm{Th}(\mathcal{S}))$ (which is the interderivability relation of $\mathcal{S}$), is a congruence relation. One of the open questions in [23] was whether for every selfextensional logic $\mathcal{S}$ and every algebra $\boldsymbol{A}$, the atlas $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A} \rangle$ has the congruence property.

Questions such as the above are typical of AAL. They are called *transfer problems.* Given a metalogical property Φ that is applicable to logics and can be abstractly formulated for basic full models, we say that Φ transfers for a logic $\mathcal{S}$ with the property Φ if any one of its basic full models $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\rangle$ also has the property Φ. We say that Φ transfers (in general) if it transfers for every logic with the property Φ.

A logic $\mathcal{S}$ is said to be *fully selfextensional* if for every algebra $\boldsymbol{A}$, the atlas $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\rangle$ has the congruence property, that is, if the "selfextensionality" property transfers for $\mathcal{S}$.

The transfer problem for selfextensionality has been solved in the negative by Babyonishev [1]. Nevertheless the only logic which is known to be selfextensional but not fully selfextensional was defined by Babyonishev in an ad hoc way just to show that the concepts selfextensional and fully selfextensional are not coextensive and also that the concepts of Fregean logic and fully Fregean logic (stronger than selfextensional and fully selfextensional respectively), which we do not discuss here, are not coextensive either.

In [23] it is proved that the selfextensional logics with a conjunction and the selfextensional logics with an implication that satisfies the modus ponens and the deduction theorem are fully selfextensional. These results show clearly that it will be hard to find natural selfextensional logics which are not fully selfextensional. To be precise, a logic $\mathcal{S}$ has the *property of conjunction* (PC) if there is a binary term $x \wedge y$ such that the three rules

$$\varphi, \varphi \vdash \varphi \wedge \psi \qquad \varphi \wedge \psi \vdash \varphi \qquad \varphi \wedge \psi \vdash \psi$$

are rules of $\mathcal{S}$. In this situation we say that $\mathcal{S}$ has (PC) relative to $\wedge$. A logic $\mathcal{S}$ has the *uniterm deduction-detachment property* (u-DDP) if there is a binary term $x \to y$ such that:

$$\Gamma, \varphi \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} \varphi \to \psi.$$

In this situation we say that $\mathcal{S}$ has the u-DDP relative to $\to$.

Theorem 8 (Font, Jansana) *If $\mathcal{S}$ is selfextensional with (PC) or the (u-DDDT), then $\mathcal{S}$ is fully selfextensional and moreover* $\mathsf{Alg}\mathcal{S}$ *is a variety.*

Fully selfextensional logics can be characterized inside the class of selfextensional logics by properties of their complete local referential semantics of reduced referential algebras. We expound briefly two characterizations given in [35]. One of them shows that the fully selfextensional logics are the selfextensional logics $\mathcal{S}$ for which the duality between the categories **RRA** and **FRAt** specializes in a very good way to a duality between the category with objects the elements of $\mathsf{Alg}\mathcal{S}$ and with arrows the homomorphisms between the elements of $\mathsf{Alg}\mathcal{S}$ and a subcategory of the reduced referential algebra models of $\mathcal{S}$.

Let $\mathcal{S}$ be a logic and let us consider the full subcategories of **RRA** and **FRAt** whose objects are respectively the reduced referential algebras which are a model of $\mathcal{S}$ and the Frege-reduced atlases which are a model of $\mathcal{S}$. We denote these two categories by $\mathbf{RRAM}_{\mathcal{S}}$ and $\mathbf{FRAt}_{\mathcal{S}}$. Moreover let us consider the category of atlases whose objects are the atlases of the form $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\rangle$ with $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$ (i.e., the basic full models of $\mathcal{S}$ whose algebra is in $\mathsf{Alg}\mathcal{S}$) and whose arrows are the atlas morphisms between them. If $\mathcal{S}$ is fully selfextensional the objects of this category are all Frege-reduced atlases and the category is a full subcategory of $\mathbf{FRAt}_{\mathcal{S}}$. We denote it by $\mathbf{FRBFM}_{\mathcal{S}}$.

Let $\mathcal{S}$ be a fully selfextensional logic. Since the inverse image of an $\mathcal{S}$-filter by an algebra homomorphism is an $\mathcal{S}$-filter, the category $\mathbf{FRBFM}_{\mathcal{S}}$ is isomorphic to the category $\mathbf{ALG}_{\mathcal{S}}$ whose objects are the elements of $\mathsf{Alg}\mathcal{S}$ and whose arrows are the algebra homomorphisms. The duality theorem implies that $\mathbf{FRBFM}_{\mathcal{S}}$ is dually equivalent to a full subcategory of the category $\mathbf{RRAM}_{\mathcal{S}}$. Thus the category $\mathbf{ALG}_{\mathcal{S}}$ is dually equivalent to a full subcategory $\mathbf{C}$ of $\mathbf{RRAM}_{\mathcal{S}}$. This statement hides important information given by its proof. If we move from the subcategory $\mathbf{C}$ of $\mathbf{RRAM}_{\mathcal{S}}$ by the functor $(.)^{+}$ to its dually equivalent category $\mathbf{FRBFM}_{\mathcal{S}}$ of Frege-reduced atlases, then the category $\mathsf{Alg}\mathcal{S}$ is obtained by the "forgetful" functor that maps each Frege-reduced atlas $\langle \boldsymbol{A}, \mathcal{B}\rangle$ to its algebra $\boldsymbol{A}$. The statement can be turned into a characterization of the fully selfextensional logics.

Theorem 9 *A logic $\mathcal{S}$ is fully selfextensional iff the category $\mathbf{ALG}_{\mathcal{S}}$ is dually equivalent to a full subcategory $\mathbf{C}$ of the category $\mathbf{RRAM}_{\mathcal{S}}$ and the composition of the functor $(.)^{+}$ with the forgetful "functor" is the functor of the equivalence from $\mathbf{C}$ onto $\mathsf{Alg}\mathcal{S}$.*

Another interesting characterization of the selfextensional logics which are fully selfextensional is given by the theorem below. The direction from left to right is an abstract *representation theorem* for the class of algebras $\mathsf{Alg}\mathcal{S}$ of any fully selfextensional logic $\mathcal{S}$. It can be considered as the abstract general framework for most of the well-known representation theorems for the classes of algebras associated with specific logics because many of them, even if not selfextensional, have an associated fully selfextensional logic companion with the same class of algebras. In the last section of the paper we will define this companion for a certain class of algebraizable logics.

Let

$$\mathsf{AlgRef}\mathcal{S} = \{\mathcal{A} : (\exists W)\ \langle W, \mathcal{A}\rangle \text{ is a reduced referential algebra model of } \mathcal{S}\},$$

that is, $\mathsf{AlgRef}\mathcal{S}$ is the class of the algebraic reducts of the reduced referential algebras which are a model of $\mathcal{S}$. In general it holds that $\mathsf{AlgRef}\mathcal{S} \subseteq \mathsf{Alg}\mathcal{S}$.

Theorem 10 *For any logic $\mathcal{S}$, $\mathcal{S}$ is fully selfextensional iff* $\mathsf{Alg}\mathcal{S} = \mathbf{I}(\mathsf{AlgRef}\mathcal{S})$.

The theorem above provides one of the reasons to consider $\mathsf{Alg}\mathcal{S}$ as the canonical class of algebras of a logic. Selfextensional logics are the logics with a local referential semantics. Among them, fully selfextensional logics are the logics whose class of algebras coincides up to isomorphisms with the class of algebras provided by its local referential semantics. Thus they are the selfextensional logics for which there is a good match between the two approaches to their semantics: the purely algebraic and the referential algebraic.

5 Abstract logics and Gentzen style rules

One of the features of abstract logics, and more generally of atlases, that make the global perspective on $\mathcal{S}$-filters fruitful is that they serve, as we already mentioned, as models of metalogical properties, in particular of the metalogical properties that can be encoded in Gentzen-style rules.

The perspective on atlases that enables one to consider them as possible models of Gentzen-style rules arises when we associate a closure operation with them. Let $\langle \boldsymbol{A}, \mathcal{B}\rangle$ be an atlas. Consider the closure system $\mathcal{C}_{\mathcal{B}}$ generated by $\mathcal{B}$, that is the set $\{\bigcap X : X \subseteq \mathcal{B}\}$, where $\bigcap\emptyset = A$. As a closure system $\mathcal{C}_{\mathcal{B}}$ has its corresponding closure operation $C_{\mathcal{B}}$, which is the map $C_{\mathcal{B}} : \mathcal{P}(A) \to \mathcal{P}(A)$ defined by

$$C_{\mathcal{B}}(X) = \bigcap\{Z \in \mathcal{C} : X \subseteq Z\}$$

for every $X \subseteq A$.

The properties of $C_{\mathcal{B}}$ that make it a closure operation are

1. $X \subseteq C_{\mathcal{B}}(X)$;

2. if $X \subseteq Y$, then $C_{\mathcal{B}}(X) \subseteq C_{\mathcal{B}}(Y)$;
3. $C_{\mathcal{C}}(C_{\mathcal{B}}(X)) \subseteq C_{\mathcal{B}}(X)$.

Moreover, we have the consequence relation $\vdash_{\mathcal{B}}$ (between subsets of A and elements of A) associated with $C_{\mathcal{B}}$. It is defined by

$$X \vdash_{\mathcal{B}} a \quad \textit{iff} \quad a \in C_{\mathcal{B}}(X)$$

for every $X \subseteq A$ and every $a \in A$. This relation will be called the consequence relation of the atlas $\langle A, \mathcal{B} \rangle$. The properties that make $\vdash_{\mathcal{B}}$ into a consequence relation are

1. if $a \in X$, $X \vdash_{\mathcal{B}} a$ (identity);
2. if $X \subseteq Y$ and $X \vdash_{\mathcal{B}} a$, then $Y \vdash_{\mathcal{B}} a$ (monotonicity);
3. if for every $a \in Y$, $X \vdash_{\mathcal{B}} a$ and $Y \vdash_{\mathcal{B}} b$, then $X \vdash_{\mathcal{B}} b$ (Abstract General Cut)

These conditions are the *abstract versions* of the conditions that define a logic, which for finitary logics are usually encoded together in the three Gentzen style rules

$$\frac{}{\Gamma, \varphi \vdash \varphi} \qquad \frac{\Gamma \vdash \varphi}{\Gamma, \Delta \vdash \varphi} \qquad \frac{\Gamma \vdash \varphi, \quad \Gamma, \varphi \vdash \psi}{\Gamma \vdash \psi}$$

A *sequent* is a pair (Γ, φ) where Γ is a finite set of formulas and φ is a formula. A sequent is usually written in the form

$$\Gamma \vdash \varphi$$

A *Gentzen style rule*, G-rule for short, is a pair $(\overline{\Pi}, \Gamma \vdash \varphi)$, where $\overline{\Pi}$ is a finite set of sequents, the sequents to which the rule applies, and (Γ, φ) is a sequent, the sequent produced by the rule. If $\overline{\Pi} = \{\Gamma_i \vdash \varphi_i : i \leq n\}$, the Gentzen rule is usually written as

$$\frac{\Gamma_i \vdash \varphi_i : i \leq n}{\Gamma \vdash \varphi} \tag{2}$$

We can generalize the notion of Gentzen style rule to admit rules where $\overline{\Pi}$ is an infinite set.

Given a Gentzen style rule (2), a substitution instance of (2) is a rule

$$\frac{\sigma[\Gamma_i] \vdash \sigma[\varphi_i] : i \leq n}{\sigma[\Gamma] \vdash \sigma[\varphi]}$$

for an arbitrary substitution σ.

An atlas, or an abstract logic, $\langle \boldsymbol{A}, \mathcal{B} \rangle$ will be said to be a *model of a Gentzen style rule* (2) if for every valuation v on $\boldsymbol{A}$, whenever $v(\varphi_i) \in C_{\mathcal{B}}(v[\Gamma_i])$ for every $i \leq n$, $v(\varphi) \in C_{\mathcal{B}}(v[\Gamma])$. We will also say that the rule holds in the atlas or the abstract logic.

In particular, given a logic $\mathcal{S}$ we say that a Gentzen style rule (2) is *valid of* $\mathcal{S}$ if $\langle \boldsymbol{Fm}, \mathrm{Th}(\mathcal{S}) \rangle$ is a model of (2), that is, if for every substitution σ such that $\sigma[\Gamma_i] \vdash_{\mathcal{S}} \sigma(\varphi_i)$ for every $i \leq n$, $\sigma[\Gamma] \vdash_{\mathcal{S}} \sigma(\varphi)$.

6 Full Models

From the global perspective on the $\mathcal{S}$-filters of a logic $\mathcal{S}$ we are interested mainly in the properties of the basic full models of $\mathcal{S}$. Nevertheless each basic full model is "logically indiscernible" from a proper class of abstract logics related to it by the relativeness relation we describe below. Therefore for many purposes it is better to deal with this wider class.

Let $\langle \boldsymbol{A}, \mathcal{C} \rangle$ and $\langle \boldsymbol{B}, \mathcal{D} \rangle$ be abstract logics. A *bilogical morphism* between $\langle \boldsymbol{A}, \mathcal{C} \rangle$ and $\langle \boldsymbol{B}, \mathcal{D} \rangle$ is any strict atlas morphism from $\langle \boldsymbol{A}, \mathcal{C} \rangle$ onto $\langle \boldsymbol{B}, \mathcal{D} \rangle$.

An abstract logic $\langle \boldsymbol{A}, \mathcal{C} \rangle$ is said to be a *relative* of an abstract logic $\langle \boldsymbol{B}, \mathcal{D} \rangle$ if it belongs to the smallest class of abstract logics that contains $\langle \boldsymbol{B}, \mathcal{D} \rangle$ and is closed under images and inverse images by bilogical morphisms.

The property of being a model of a Gentzen style rule is preserved under images and inverse images by bilogical morphisms, that is, if h is a bilogical morphism from $\langle \boldsymbol{A}, \mathcal{C} \rangle$ onto $\langle \boldsymbol{B}, \mathcal{D} \rangle$ and $\mathcal{G}$ is a Gentzen rule, then $\langle \boldsymbol{A}, \mathcal{C} \rangle$ is a model of $\mathcal{G}$ iff $\langle \boldsymbol{B}, \mathcal{D} \rangle$ is a model of $\mathcal{G}$. This fact shows that if we are interested in a class of abstract logics that are models of a logic $\mathcal{S}$, it seems reasonable that if we accept one model we must also accept all its relatives.

Since, as we already said, our main candidates as models of $\mathcal{S}$ are the basic full models of $\mathcal{S}$, we shall be interested in the class of abstract logics whose elements are the basic full models of $\mathcal{S}$ and all their relatives. We call the elements of this class *full models* of $\mathcal{S}$. In fact it can be shown that an abstract logic $\langle \boldsymbol{A}, \mathcal{C} \rangle$ is a full model of $\mathcal{S}$ iff there is a bilogical morphism from $\langle \boldsymbol{A}, \mathcal{C} \rangle$ onto a basic full model of $\mathcal{S}$.

By the preservation of the property of being a model of a Gentzen style rule by bilogical morphisms and inverses of bilogical morphisms, we have that for any Gentzen style rule R, the full models of a logic $\mathcal{S}$ are models of R iff the basic full models of $\mathcal{S}$ are models of R. In particular, since every logic is complete relative to the class of its basic full models, every logic is complete relative to the class of its full models and with respect to the class of its reduced full models. Moreover, to study which Gentzen rules hold in every full model of a logic $\mathcal{S}$ it is enough to see which ones hold in every one of its reduced basic full models.

From the perspective of the closure operators we can also say that an atlas $\langle \boldsymbol{A}, \mathcal{B}\rangle$ is *finitary* if the associated closure operator $C_{\mathcal{B}}$ is finitary, that is, if it holds that for every $X \subseteq A$ and every $a \in A$, such that $a \in C_{\mathcal{B}}(X)$, then there is a finite set $Y \subseteq X$ such that $a \in C_{\mathcal{B}}(Y)$. The basic full models of a finitary logic are all finitary and the property of being finitary is preserved by images and inverse images by bilogical morphisms. Thus every full model of a finitary logic is a finitary abstract logic.

7 Fully adequate Gentzen systems

From an abstract point of view we define a *Gentzen calculus* as just a set of Gentzen style rules. In the same way that a Hilbert-style axiomatic system defines a consequence relation on the set of formulas that is substitution invariant (namely, the relation $\vdash$ defined by: $\Gamma \vdash \varphi$ iff there is a proof of φ in the calculus using premises in Γ) a Gentzen calculus defines a consequence relation between sets of sequents and sequents which is invariant under substitutions in the natural way. Formally, given a Gentzen calculus $\mathbf{G}$, we denote by

$$\mid\!\sim_{\mathbf{G}}$$

the relation between sets of sequents and sequents generated by $\mathbf{G}$. That is,

$$\{\Gamma_i \vdash \varphi_i : i \in I\} \mid\!\sim_{\mathbf{G}} \Gamma \vdash \varphi \text{ iff there is a proof of } \Gamma \vdash \varphi \text{ from the sequents}$$

$$\text{in } \{\Gamma_i \vdash \varphi_i : i \in I\} \text{ using the substitution instances of the rules of } \mathbf{G}.$$

The consequence relations defined in this way by Gentzen calculus are called *Gentzen systems*, see [42, 30, 25]. Since atlases and abstract logics serve as models of Gentzen style rules, they also serve as models of Gentzen calculus and of Gentzen systems.

It is easy to show that an atlas $\langle \boldsymbol{A}, \mathcal{B}\rangle$ is a *model of* a Gentzen calculus **G** iff for every set of sequents $\{\Gamma_i \vdash \varphi_i : i \in I\}$ and every sequent $\Gamma \vdash \varphi$ such that $\{\Gamma_i \vdash \varphi_i : i \in I\} \mid\!\sim_{\mathbf{G}} \Gamma \vdash \varphi$ it holds that $\langle \boldsymbol{A}, \mathcal{B}\rangle$ is a model of the corresponding 'infinite' Gentzen style rule

$$\frac{\Gamma_i \vdash \varphi_i : i \in I}{\Gamma \vdash \varphi}.$$

Given a finitary logic $\mathcal{S}$ the following question was investigated in [23]: Is there a Gentzen calculus such that its finitary abstract logic models are exactly the full models of $\mathcal{S}$? If such a Gentzen calculus exists then its Gentzen system captures exactly all the Gentzen style rules which hold in every full model of $\mathcal{S}$, and therefore these rules are the Gentzen style rules which are valid of $\mathcal{S}$ and transfer to every one of its full models.

Given a finitary logic $\mathcal{S}$ with theorems we say that a Gentzen system is *fully adequate* if its finitary abstract logic models are exactly the full models of $\mathcal{S}$; if $\mathcal{S}$ does not have theorems we say that a Gentzen system is *fully adequate* if its finitary abstract logic models $\langle \boldsymbol{A}, \mathcal{C}\rangle$ with the property that $\emptyset \in \mathcal{C}$ are exactly the full models of $\mathcal{S}$. If a finitary logic $\mathcal{S}$ has a fully adequate Gentzen system, it is unique. Thus the question above is the question whether any finitary logic has a fully adequate Gentzen system. In [23] the following result is obtained:

Theorem 11 *If a finitary selfextensional logic $\mathcal{S}$ has (PC) or the (u-DDDT), then it has a fully adequate Gentzen system.*

It is also known that not every finitary logic has a fully adequate Gentzen system. In [24] it is shown that a finitary and weakly algebraizable logic has a fully adequate Gentzen system iff it has the multiterm deduction detachment-theorem (which is like the u-DDT but instead of a single formula $p \to q$ one has a set of formulas in two variables that collectively behave as the formula $p \to q$). There are algebraizable logics without the multiterm deduction-detachment theorem, for instance the global consequence of the least normal modal logic K; thus there are logics without a fully adequate Gentzen system. Moreover, there are examples of selfextensional logics without a fully adequate Gentzen system, for instance the $\Box$-fragment of the local consequence of the normal modal logic K.

8 Selfextensional logics and algebraizable logics

One of the important questions that it is not still fully answered in AAL is this: why do the class of algebras $\mathsf{Alg}\mathcal{S}$ of many finitary algebraizable logics turn out to be a variety when in general, according to the theory of algebraizable logics of Blok and Pigozzi one can only say that it is a quasivariety?

The results in [23] provide a partial answer, given in the theorem below which is implied by Theorem 8.

Theorem 12 *If $\mathcal{S}$ is an algebraizable logic which is finitary and selfextensional and has (PC) or has (u-DDT), then its equivalent algebraic semantics $\mathsf{Alg}\mathcal{S}$ is a variety.*

This result can be used to explain why other algebraizable logics wich have(PC) but are non-selfextensional have a variety as their equivalent algebraic semantics. We report some results of [36].

Let K be any class of algebras and $\wedge$ a binary term that defines a meet-semilattice operation on each algebra in K. Define the finitary logic $\mathcal{S}^{\leq}_{\mathsf{K}}$ as follows: for every $\varphi_0, \ldots, \varphi_{n-1}, \varphi$,

$$\varphi_0, \ldots, \varphi_{n-1} \vdash_{\mathcal{S}^{\leq}_{\mathsf{K}}} \varphi \quad \text{iff} \quad \forall \boldsymbol{A} \in \mathsf{K}\ \forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$$

$$v(\varphi_0) \wedge^{\boldsymbol{A}} \ldots \wedge^{\boldsymbol{A}} v(\varphi_{n-1}) \leq^{\boldsymbol{A}} v(\varphi)$$

and

$$\vdash_{\mathcal{S}^{\leq}_{\mathsf{K}}} \varphi \quad \text{iff} \quad \forall \boldsymbol{A} \in \mathsf{K}\ \forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})\ \forall a \in A, a \leq^{\boldsymbol{A}} v(\varphi),$$

where $\leq^{\boldsymbol{A}}$ is the order associated with the meet operation $\wedge^{\boldsymbol{A}}$. We say that a finitary logic $\mathcal{S}$ is *semilattice-based* if there is a class of algebras K and a binary term that defines a meet-semilattice operation on every element of K such that $\mathcal{S} = \mathcal{S}^{\leq}_{\mathsf{K}}$. Every semilattice-based logic is selfextensional.

Let $\mathcal{S}$ be from now on a finitary algebraizable logic with (PC) relative to $\wedge$ such that for every algebra $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, $\langle A, \wedge^{\boldsymbol{A}} \rangle$ is a semilattice. We define the *semilattice-based companion* of $\mathcal{S}$ as the logic $\mathcal{S}^{\leq}_{\mathsf{Alg}\mathcal{S}}$, which we simply denote by $\mathcal{S}^{\leq}$.

Under these conditions:

Proposition 13

1. *The logic $\mathcal{S}$ is an extension of $\mathcal{S}^{\leq}$;*

2. *$\mathcal{S}^{\leq}$ is fully selfextensional;*

3. $\mathsf{Alg}\mathcal{S} \subseteq \mathsf{Alg}^{*}\mathcal{S}^{\leq} \subseteq \mathsf{Alg}\mathcal{S}^{\leq}$.

Assume moreover that $(p \Leftrightarrow q)$ is a set of equivalence formulas and $E(p)$ a set of defining equations for $\mathcal{S}$ (see Section 2.3). Recall that we denote by $E(\boldsymbol{A})$ the set of solutions of the equations in $E(p)$ in the algebra $\boldsymbol{A}$.

Theorem 14 *If $\mathcal{S}$ has an implication set of formulas $(p \Rightarrow q)$ (see Section 2.3) such that*

1. *for every $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, $a \leq^{\boldsymbol{A}} b$ iff $(a \Rightarrow^{\boldsymbol{A}} b) \subseteq E(\boldsymbol{A})$,*

and in every algebra $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}^{\leq}$,

2. *the set $E(\boldsymbol{A})$ is an $\mathcal{S}$-filter;*

3. *for every $a, b \in A$, if $(a \Leftrightarrow^{\boldsymbol{A}} b) \subseteq E(\boldsymbol{A})$, then $a = b$,*

then $\mathsf{Alg}\mathcal{S} = \mathsf{Alg}\mathcal{S}^{\leq}$ and, therefore, $\mathsf{Alg}\mathcal{S}$ is a variety.

Notice that condition (1) says that the semilattice order of each algebra in $\mathsf{Alg}\mathcal{S}$ is definable in the stated way using the implication set of formulas.

Theorem 14 applies to several important algebraizable logics like the systems of relevance logic R and R_t, Łukasiewicz's infinite-valued logic and the different systems of linear logic without the exponentials, which are algebraizable but non-selfextensional, and explains why they have a variety as its equivalent algebraic semantics. It is worth mentioning that for the relevance systems R and R_t as well as for the linear logics, their semilattice-based companions do not have theorems, and so they are non-protoalgebraic. This is not the case for the Łukasiewicz infinite-valued logic, whose semilattice-based companion has theorems but is also non-protoalgebraic.

When the semilattice-based companion is protoalgebraic we have a simpler theorem.

Theorem 15 *Let $\mathcal{S}$ be an algebraizable logic with (PC) relative to $\wedge$ such that for an implication set $(p \Rightarrow q)$ for $\mathcal{S}$ and a set $E(p)$ of defining equations,*

1. *for every* $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, $\langle A, \wedge^{\boldsymbol{A}} \rangle$ *is a semilattice, whose ordering we denote by* $\leq^{\boldsymbol{A}}$;

2. *for every* $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, $a \leq b$ *iff* $(a \Rightarrow^{\boldsymbol{A}} b) \subseteq E(\boldsymbol{A})$.

Assume moreover that in any algebra $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}^{\leq}$ *the least* $\mathcal{S}^{\leq}$*-filter is an* $\mathcal{S}$*-filter and assume also that* $\mathcal{S}^{\leq}$ *is protoalgebraic. Then* $\mathsf{Alg}\mathcal{S} = \mathsf{Alg}\mathcal{S}^{\leq}$ *and therefore* $\mathsf{Alg}\mathcal{S}$ *is a variety.*

This theorem is applicable to the several logics of a modal type known in the literature. In particular if we restrict ourselves to the standard modal language, we have the local consequence relation of a normal modal logic L and its global consequence relation. Usually the global consequence relation $\mathcal{S}^g_L$ is a non-selfextensional and algebraizable logic with a variety as its equivalent algebraic semantics. But the local consequence relation $\mathcal{S}^l_L$ is selfextensional, non algebraizable but protoalgebraic and $(\mathcal{S}^g_L)^{\leq} = \mathcal{S}^l_L$. This, together with the theorem, explains why the equivalent algebraic semantics of $\mathcal{S}^g_L$ is a variety.

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